Relations between Kendall distributions and families of bivariate Values at Risk in exchangeable survival models*

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Abstract

We consider the two concepts of *Multivariate Value at Risk* and *Kendall distribution function*. Attention will be restricted to the case of $n = 2$ non-negative, exchangeable random variables; furthermore, conditions are assumed under which, for any *level* $v \in (0, 1)$, the bivariate Value at Risk (VaR) is a curve within the quadrant $\mathbb{R}^2_+$.

As main purposes, we study the relations between the two concepts; in particular, we focus attention on the survival models simultaneously admitting a specified Kendall distribution and a specified family of bivariate VaR's. Furthermore we obtain some conditions of qualitative type in terms of the notions of IFRA and PKD dependence.

For our results we combine two different methods. On the one hand we use a slight extension of a transformation result proved by Genest and Rivest (2001); on the other hand we resort to the bivariate aging functions introduced by Bassan and Spizzichino (2001) to represent the family of the level curves of a bivariate survival function.

Different aspects of our results will be outlined by considering in some detail the cases of Archimedean and Cuadras-Augé survival copulas.

Key Words: Bivariate VaR Curves, Kendall Distribution, Survival Copulas, Bivariate Aging Functions, Semicopulas.

1 Introduction

Let X, Y be a pair of non-negative random variables and let $\overline{F}(x, y)$ denote their *joint survival function*, namely, for $x \geq 0, y \geq 0$,

$$\overline{F}(x, y) = P\{X > x, Y > y\}.$$

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Related to $\overline{F}$, we analyze the relations between two different concepts that emerged in the literature about multivariate models: the Bivariate Value at Risk and the Kendall Distribution. In what immediately follows, we recall the definitions of these concepts, by adapting them to our framework, language and notation.

Definition 1 *Let* $v \in [0,1]$ *and*

$$A_v := \{(x,y) \in \mathbb{R}_+^2 : \overline{F}(x,y) \leq 1 - v\}.$$

The Bivariate **upper-orthant Value at Risk** *at a probability level* v *is defined as*

$$\overline{VaR}_v\left(\overline{F}\right) := \partial A_v,$$

where ∂A *denotes the boundary of a subset* $A \subset \mathbb{R}_+^2$.

We assume some regularity conditions on the sets ∂A_v, and in particular we assume that they can be represented by means of continuous curves (see also Section 2), and therefore in the sequel we refer to these sets also as level curves. The symbol $\mathcal{D}$ will denote the family of the Bivariate upper-orthant Values at Risk:

$$\mathcal{D} := \{\partial A_v; v \in (0,1)\}.$$

This family determines in a natural way a set of curves; with a slight abuse of notation, we continue to denote this set by $\mathcal{D}$. For references and applied interest of the concept of the Bivariate Value at Risk see e.g. [7], see also [12].

We recall that the random variable $F(X,Y)$ is known as the *Bivariate Probability Integral Transformation* (BIPIT) (see e.g. [9]) and that its distribution function K is called the *Kendall distribution function* (see [11] and references therein for basic details about Kendall distributions). In [9] and [11] several aspects of the Kendall distribution K have been analyzed, based on the fact that it only depends on the connecting copula of X, Y. These two papers provide an essential background for the present paper.

For our purposes we need the following related concepts, using the term "upper-orthant" in order to distinguish them from the standard ones.

Definition 2 *The random variable* $Z := \overline{F}(X,Y)$ *is called the* **upper-orthant Bivariate Probability Integral Transformation**. *The (univariate) distribution function of* Z *is called the* **upper-orthant Kendall distribution function**, *and is denoted by* $\widehat{K}$*: i.e., for* $v \in [0,1]$,

$$\widehat{K}(v) := P\{\overline{F}(X,Y) \leq v\} = P\{(X,Y) \in A_{1-v}\}. \tag{1}$$

Here we concentrate our attention on the case of exchangeable strictly positive random variables X, Y; for this case we will denote by $\widehat{C}$ the *survival copula* and by $\overline{G}$ the common univariate marginal survival function of X and Y. We also assume $\overline{F}$ to be continuous, strictly positive and strictly 1−decreasing, i.e., for any $x \geq 0$, we assume the function $\overline{F}_x(\cdot) := \overline{F}(x,\cdot) = \overline{F}(\cdot,x)$ to be strictly decreasing. Therefore $\overline{G}$ will be continuous, strictly positive and strictly decreasing all over the half-line $[0, +\infty)$, with $\overline{G}(0) = 1$. We shall then write, for $0 \leq u, v \leq 1$,

$$\widehat{C}(u, v) = \overline{F}\left(\overline{G}^{-1}(u), \overline{G}^{-1}(v)\right), \tag{2}$$

or, equivalently,

$$\overline{F}(x, y) = \widehat{C}\left(\overline{G}(x), \overline{G}(y)\right). \tag{3}$$

In order to describe the family $\mathcal{D}$ we will make use of the notion of *bivariate aging function* B (see (13) for the definition), considered by Bassan and Spizzichino in [3], [4] and [5]; in this respect it is worthwhile mentioning that, along with the familiar representation (3), $\overline{F}$ can also be alternatively described in terms of the pair $\left(B, \overline{G}\right)$.

Where needed, we shall add an index F to object such as G, C, K and an index $\overline{F}$ to objects such as $\overline{G}, \widehat{C}, \widehat{K}, B, \mathcal{D}$, etc... in order to point out that they correspond to random variables X, Y, with given joint distribution function F and survival function $\overline{F}$, so that, e.g., for $x \in \mathbb{R}$ and $u, v \in [0, 1]$,

$$\overline{G}_{\overline{F}}(x) = 1 - G_F(x),\ \widehat{C}_{\overline{F}}(u, v) = u + v - 1 + C_F(1 - u, 1 - v).$$

Roughly speaking, one main purpose here is to describe "compatibility conditions" under which a one-dimensional distribution function $\widehat{K}$ and a family of level curves $\mathcal{D}$ can be seen as the Kendall distribution and as the family of bivariate Values at Risk curves, associated to a same survival function $\overline{F}$, respectively. In particular it will be shown that, when $\mathcal{D}$ and $\widehat{K}$ are compatible, the family of survival functions $\overline{F}$ such that $\widehat{K}_{\overline{F}} = \widehat{K}$, $\mathcal{D}_{\overline{F}} = \mathcal{D}$ contains at most ∞^1 elements. Furthermore, we shall analyze some specific aspects of the relations between $\mathcal{D}_{\overline{F}}$ and $\widehat{K}_{\overline{F}}$. More precisely, the plan of the paper is as follows. Sections 2 and 3 will be devoted to some basic developments and results about (upper-orthant) bivariate VaR curves and Kendall distributions that are needed in our context. Basic aspects of the family $\mathcal{D}_{\overline{F}}$ will be studied in Section 2, where we also recall the definition of B and show in which sense $\mathcal{D}_{\overline{F}}$ can be described in terms of B; in Section 3 we analyze a few aspects of $\widehat{K}_{\overline{F}}$, similar to those obtained for K in [9]; in particular we analyze the form under which $\widehat{K}_{\overline{F}}$ depends on the survival copula $\widehat{C}_{\overline{F}}$ of $\overline{F}$. Our results about the compatibility problem and some related examples will be presented in Section 4. In particular we show (Theorem 9) the form that $\overline{F}$ must have in order to be such that $\widehat{K}_{\overline{F}} = K$; $\mathcal{D}_{\overline{F}} = \mathcal{D}$, for given K and $\mathcal{D}$. In Section 5 we present a short discussion including the analysis of some special cases and some concluding remarks.

2 On the representation of $\mathcal{D}$

In view of our assumptions on $\overline{F}$, $\overline{VaR}_\alpha(\overline{F})$ is the image of a function $\gamma_\alpha : [0, 1] \to \mathbb{R}_+ \times \mathbb{R}_+$, connecting the Cartesian axes, i.e. such that $\gamma_\alpha(0) = (x_\alpha, 0)$ and $\gamma_\alpha(1) = (0, y_\alpha)$, for some strictly positive x_α and y_α.

Concerning x_α and y_α and the curve γ_α, we can state the following

Lemma 3
(a) $x_\alpha = y_\alpha = G^{-1}(\alpha) = \overline{G}^{-1}(1-\alpha)$.
(b) *The curve* γ_α *can be parameterized as a function* $y(\alpha, x)$ *of* x, *with* $x \in [0, x_\alpha]$, $y(\alpha, 0) = x_\alpha$, $y(\alpha, x_\alpha) = 0$, *and, for* $x \in [0, x_\alpha]$,

$$y(\alpha, x) = \overline{F}_x^{-1}\big(1-\alpha\big) = h^{-1}_{\overline{F},x}\big(G^{-1}(\alpha)\big), \tag{4}$$

where $h_{\overline{F}}(x, y)$ *is the* x*-component of* $\gamma_{1-\overline{F}(x,y)}(0)$, *i.e.*

$$h_{\overline{F}}(x, y) := x_{1-\overline{F}(x,y)},$$

and $h_{\overline{F},x}(\cdot) := h_{\overline{F}}(x, \cdot)$.

Before giving the proof we observe that $h_{\overline{F}}$ determines the family $\mathcal{D}_{\overline{F}}$ of level curves, while the marginal G determines the values of the level on each level curve.
Proof. Clearly, by the exchangeability assumption on X and Y, $y_\alpha = x_\alpha$, and moreover a similar parameterization for $\overline{VaR}_\alpha(\overline{F})$ is possible w.r.t. y.

Observe that obviously (x, y) belongs to $\partial A_{1-\overline{F}(x,y)}$, and therefore $\overline{F}(x, y) = \overline{F}\big((x_{1-\overline{F}(x,y)}, 0)\big)$, i.e., it holds

$$\overline{F}(x, y) = \overline{G}\left(h_{\overline{F}}(x, y)\right) \tag{5}$$

or equivalently

$$h_{\overline{F}}(x, y) = \overline{G}^{-1}\left(\overline{F}(x, y)\right), \tag{6}$$

where the equivalence is due to the strict monotonicity of $\overline{G}$. Clearly it is necessary that

$$h_{\overline{F}}(x, y(\alpha, x)) = h_{\overline{F}}(x_\alpha, 0) = x_\alpha, \tag{7}$$

so that the value of x_α is uniquely determined by taking $x = x_\alpha$ and $y = 0$ in (5), i.e.,

$$1 - \alpha = \overline{G}\,(x_\alpha) \quad \Leftrightarrow \quad x_\alpha = G^{\,-1}(\alpha),$$

and point ***(a)*** is proved. Then point ***(b)*** follows by (7) and the observation that $h_{\overline{F}}(x, \cdot)$ is invertible, as can immediately deduced by (6). ■

Remark 1 *From the Eq.* (7), *under obvious regularity conditions, the function* $y(\alpha, x)$ *solves*

$$\dot{y}(x) = -\frac{\frac{\partial}{\partial x} h(x, y(x))}{\frac{\partial}{\partial y} h(x, y(x))} \qquad x \in [0, G^{-1}(\alpha)] \tag{8}$$

$$y(G^{-1}(\alpha)) = 0, \tag{9}$$

with $h = h_{\overline{F}}$.

In order to fix ideas it is useful to consider the families of level curves, or equivalently the upper-orthant Bivariate VaR's, of three special cases, described by means of the corresponding survival function, and that will have a basic role in the subsequent analysis.

Example 1 *The first case is* **perfect dependence**, *i.e.* $P\{X=Y\}=1$. *Then* $\overline{F}(x,y)=\overline{G}\,(x\vee y)$ *and, for* $0\leq v\leq 1$,

$$\overline{VaR}_v(\overline{F})=\{(x,y):x\vee y=G^{-1}\,(v)\}. \tag{10}$$

For $\overline{G}$ *continuous and strictly decreasing, with* $\overline{G}(0)=1$, *the level curves can be represented as* $\gamma_\alpha(t)=\big(G^{-1}(\alpha),2tG^{-1}(\alpha)\big)$ *for* $t\in[0,\frac{1}{2}]$ *and* $\gamma_\alpha(t)=\big(2(t-\frac{1}{2})G^{-1}(\alpha),G^{-1}(\alpha)\big)$ *for* $t\in[\frac{1}{2},1]$. *However the representation of the level curves as a function* $y(\alpha,x)$ *of* x *fails in this case. In fact the strict monotonicity assumption on* $\overline{F}_x$ *fails.*

Then we consider the case of **Schur-constant** $\overline{F}$, *i.e. the case when* $\overline{F}\,(x,y)=\overline{G}\,(x+y)$ *where* $\overline{G}$ *is a univariate continuous, convex, strictly positive and strictly decreasing survival function on* $[0,+\infty)$. *It is immediate to check that the function* $\overline{G}$ *has also the role of univariate marginal; furthermore*

$$\overline{VaR}_v(\overline{F})=\{(x,y):x+y=G^{-1}\,(v)\}, \tag{11}$$

and the level curves are parameterized as $y(\alpha,x)=G^{-1}\,(\alpha)-x$. *Note that* (11) *is actually a characterization of the Schur-constant case.*

Finally, when X,Y *are* **i.i.d. random variables**, *i.e. when* $\overline{F}\,(x,y)=\overline{G}(x)\cdot\overline{G}(y)$, *we can write*

$$\overline{VaR}_v(\overline{F})=\{(x,y):\Lambda(x)+\Lambda(y)=-\log(1-v)\}, \tag{12}$$

by setting $\Lambda(x):=-\log\overline{G}\,(x)$. *Under our assumptions, the function* Λ *is strictly increasing and continuous, and thus,* $y(\alpha,x)=\Lambda^{-1}\big(-\log(1-\alpha)-\Lambda(x)\big)$.

We now let (see e.g. [3] or [5])

$$B(u,v):=\exp\{-\overline{G}^{-1}\,\big(\overline{F}(-\log u,-\log v)\big)\} \tag{13}$$

be the *bivariate aging function* of X,Y.

Then $B(u,v)=B(v,u)$, and

$$\overline{F}(x,y)=\overline{G}\,\big(-\log B\,\big(e^{-x},e^{-y}\big)\big)\,. \tag{14}$$

Notice that Eq. (14) provides the already mentioned representation of $\overline{F}(x,y)$ in terms of the pair $\big(B,\overline{G}\big)$; this is in a sense analogous to (but different from) the representation (3).

Remark 2 *Under our assumptions* $\overline{G}$ *is invertible. However the definition* (13) *makes generally sense by interpreting* $\overline{G}^{-1}$ *as a generalized inverse. As to* (14), *we notice that it holds whenever* $\overline{G}(\overline{G}^{-1})$ *coincides with the identity on* $[0,1]$.

The form of the bivariate aging function B for the basic cases in Example 1 is displayed in the example below.

Example 2 *In the perfect dependence case, we easily get that B is the maximal copula*

$$B(u,v) = u \wedge v. \tag{15}$$

As to the Schur-constant case, we observe that condition (11) *holds if and only if B is the product copula:*

$$B(u,v) = u \cdot v. \tag{16}$$

Finally $\overline{F}\,(x,y) = \overline{G}(x) \cdot \overline{G}(y)$ if and only if the following two conditions hold

$$B(u,v) = Q^{-1}[Q(u) + Q(v)], \tag{17}$$

$$Q(u) := -\log \overline{G}\,(-\log u) = \Lambda(-\log u), \tag{18}$$

where we understand that $Q(0) = Q(0+) = +\infty$ and $Q^{-1}(\infty) = 0$. Observe that $Q : (0,1] \to [0,+\infty)$ is strictly decreasing, with $Q(1) = 0$.

It is immediate to check that the function $B : [0,1] \times [0,1] \to [0,1]$ defined by (13) is strictly increasing in each variable and also shares with a copula the property to be *grounded*, i.e.

$$B(u,0) = B(0,u) = 0, \quad B(u,1) = B(1,u) = u, \quad \forall\, 0 \le u \le 1.$$

Though (15) and (16) are copulas, generally the aging function $B(u,v)$ may not be a copula, since in some cases it can happen that is not *two-increasing*. For example we may take $\overline{F}(x,y) = \left(e^{-\beta x} + e^{-\beta y} - 1\right)^+$, with $\beta > 0$. Though $\overline{F}$ is not strictly positive all over $\mathbb{R}^2_+$, B can still be defined by (13) and furthermore (14) holds (see Remark 2). Then, setting $b = -\beta$, one gets

$$B(u,v) = \left[(u^\beta + v^\beta - 1)^+\right]^{\frac{1}{\beta}} = \left[(u^{-b} + v^{-b} - 1)^+\right]^{-\frac{1}{b}}. \tag{19}$$

When $\beta > 1$, there exists a u_β such that $\frac{1}{2} < u_\beta \le \left(\frac{1}{2}\right)^{1/\beta}$, and therefore

$$\begin{aligned} B(1,1) - B(u_\beta,1) - B(1,u_\beta) + B(u_\beta,u_\beta) &= 1 - 2u_\beta + \left(\left(2\,(u_\beta)^\beta - 1\right)^+ \right)^{1/\beta} \\ &= 1 - 2\,u_\beta < 0. \end{aligned}$$

Following [5] and [6], we use the term *semicopula* to describe functions $S : [0,1] \times [0,1] \to [0,1]$ that are grounded and increasing in each variable. From the above definition it follows that the function B defined by (13) is a semicopula; actually, under our assumptions B is a continuous semicopula.

An *Archimedean semicopula* will be a bivariate function of the form

$$S^\varphi(u,v) = \varphi^{-1}[\varphi(u) + \varphi(v)], \tag{20}$$

where φ, the *generator* of the semicopula, is a decreasing function, with $\varphi(1) = 0$, as in the independent case with $\varphi = Q$. When $\varphi(0+) = \infty$, then the generator φ is said to be *strict*. It is clear that if φ is a generator, then $\theta\,\varphi$ is also a generator, for any $\theta > 0$. As it is well known, when the generator is a convex function ϕ, an Archimedean semicopula is then an Archimedean copula; in this case we will use the notation

$$C^{\phi}(u,v) = \phi^{-1}[\phi(u) + \phi(v)]. \tag{21}$$

With the above notation, when $\beta \in (0,1]$ (or equivalently $b \in [-1,0)$), the aging function in (19) is the Archimedean copula C^{ϕ}, with $\phi(u) = 1 - u^{\beta}$. Furthermore, setting $\varphi(u) = (1-u^{\beta})\mathrm{sign}(\beta) = (u^{-b}-1)\mathrm{sign}(b)$ one can consider the Archimedean semicopulas S^{φ} with generator φ for all $b \neq 0$; then $\varphi^{-1}(w) = \left[(1-\mathrm{sign}(\beta)w)^{+}\right]^{\frac{1}{\beta}} = \left[(1+\mathrm{sign}(b)w)^{+}\right]^{-\frac{1}{b}}$, and the corresponding Archimedean semicopulas S^{φ} are still expressed by (19). Clearly, for $b < -1$, φ is not convex, while this is the case when $b \geq -1$, $b \neq 0$. In the latter case (19) defines a Clayton copula. Therefore we refer to (19) as *Clayton semicopulas*. Finally we note that the generator φ is strict if and only if $b > 0$.

Concerning the role of B in the representation of the family $\mathcal{D}$, a comparison between (14) and (6), immediately gives

$$h_{\overline{F}}(x,y) = -\log B\left(e^{-x}, e^{-y}\right). \tag{22}$$

In view of (22), and taking into account (4) and Remark 1, the family $\mathcal{D}$ of the bivariate upper-orthant Values at Risk will be described from now on in terms of B; more in details, setting $B_u(\cdot) := B(u,\cdot)$, Eq. (4) can be rewritten as

$$y(\alpha,x) = -\log\left(B^{-1}_{e^{-x}}(e^{-G^{-1}(\alpha)})\right).$$

For two different bivariate survival functions $\overline{J}$ and $\overline{H}$ satisfying our assumptions, the two sets $\mathcal{D}_{\overline{J}}$ and $\mathcal{D}_{\overline{H}}$ do coincide if and only if $B_{\overline{J}} = B_{\overline{H}}$. Furthermore, as it is easy to check (see e.g. [4]), the condition $B_{\overline{J}} = B_{\overline{H}}$ holds if and only if there exists a continuous, strictly increasing function $\psi : [0,1] \to [0,1]$, such that

$$\overline{J}(x,y) = \psi\left(\overline{H}(x,y)\right),$$

with

$$\psi(x) = \overline{G}_{\overline{J}}\left[\overline{G}^{-1}_{\overline{H}}(x)\right]. \tag{23}$$

Independently on the function B being or not a copula, the relations between B and $\widehat{C}$ involve the marginal survival function $\overline{G}$; in fact we can write, from (2), (3), (13) and (14),

$$\begin{cases} B(u,v) = \exp\left\{-\overline{G}^{-1}\left(\widehat{C}(-\log\overline{G}(u), -\log\overline{G}(v))\right)\right\}, \\ \widehat{C}\,(u,v) = \overline{G}\left(-\log B\left(e^{-\overline{G}^{-1}(u)}, e^{-\overline{G}^{-1}(v)}\right)\right), \end{cases}$$

or

$$\begin{cases} B(u,v) = \gamma\left(\widehat{C}\left(\gamma^{-1}(u),\ \gamma^{-1}(v)\right)\right), \\ \widehat{C}(u,v) = \gamma^{-1}\left(B\left(\gamma(u),\ \gamma(v)\right)\right) \end{cases} \tag{24}$$

where $\gamma, \gamma^{-1} : [0,1] \to [0,1]$ are the increasing functions defined by

$$\gamma(w) = \exp\{-\overline{G}^{-1}(w)\}, \quad \gamma^{-1}(u) = \overline{G}(-\log u). \tag{25}$$

Some consequences of the relations between the semicopula B and the survival copula $\widehat{C}$ in the three basic cases of Examples 1 and 2 are presented in the example below.

Example 3 *In the perfect dependence case it is, trivially,*

$$\widehat{C}(u,v) = B(u,v) = u \wedge v. \tag{26}$$

Notice that the maximal copula $u \wedge v$ is a fixed point of both the transformations in (24)*, insensitively of the marginal $\overline{G}$.*

As to the Schur-constant case, with given marginal $\overline{G}$, we observe that $B(u,v) = u \cdot v$ holds if and only if it is

$$\widehat{C}(u,v) = \overline{G}\left(\overline{G}^{-1}(u) + \overline{G}^{-1}(u)\right),$$

i.e., using the notation (21)*, $\widehat{C} = C^{\overline{G}^{-1}}$.*

Finally, we consider the case of independence, with given marginal $\overline{G}$. In this case, using the notation (20)*, the necessary condition* (17) *for B, can be written as $B = S^Q$, with Q given by* (18)*. We observe that the independence condition $\widehat{C}(u,v) = u \cdot v$ is not a consequence of B being an Archimedean semicopula, unless we just impose that its generator be Q, i.e. we impose that* (18) *holds.*

Example 4 (Exponential marginals) *When $\overline{G}(x) = e^{-\beta x}$, then γ and γ^{-1} in* (25) *are given respectively by $\gamma(w) = w^{\frac{1}{\beta}}$ and $\gamma^{-1}(u) = u^{\beta}$, and* (24) *becomes*

$$B(u,v) = \left(\widehat{C}\left(u^{\beta},\ v^{\beta}\right)\right)^{\frac{1}{\beta}}, \quad \textit{and} \quad \widehat{C}(u,v) = \left(B\left(u^{\frac{1}{\beta}},\ v^{\frac{1}{\beta}}\right)\right)^{\beta}. \tag{27}$$

Note that if we take the copula $\widehat{C}(u,v) = (u+v-1)^+$, the lower Frechét-Hoeffding bound, then we recover for B the Clayton semicopula (19).

The condition

$$\overline{G}(x) = e^{-x} \tag{28}$$

implies $\gamma(w) = w$, and therefore

$$B(u,v) = \widehat{C}(u,v), \tag{29}$$

and we find that if B is compatible with the standard exponential marginal distribution, i.e. if there exists a survival function $\overline{F}$ such that $(\overline{G}, B) =$

$(\overline{G}_{\overline{F}}, B_{\overline{F}})$, then necessarily B is a copula (on this point see also Remark 5). However (28) is not necessary for (29) as we see, for instance, by considering the perfect dependence case (26). Another case is the one considered in the next Example 5, where the marginal distributions are exponential, but non-necessarily standard exponential.

Example 5 (Marshall-Olkin models and Cuadras-Augé copulas) *Consider three independent, exponentially distributed, non-negative variables A, B, τ, where $A, B \sim Exp(\lambda), \tau \sim Exp(\mu)$ and set*

$$X = \min(A, \tau), \quad Y = \min(B, \tau).$$

The joint survival function of X, Y is

$$\overline{F}(x, y) = \exp\{-\lambda\,(x + y) - \mu\,(x \vee y)\}.$$

In this case it is $\overline{G}(x) = e^{-(\lambda+\mu)x}$ and $\overline{G}^{-1}(u) = -\frac{\log u}{\lambda+\mu}$, and thus, by (2), and by (27) in Example 4, with $\beta = \lambda + \mu$,

$$\widehat{C}(u, v) = B(u, v) = (uv)^{\alpha}\,(u \wedge v)^{1-\alpha}\,, \tag{30}$$

with $\alpha = \frac{\lambda}{\lambda+\mu}$, i.e. $\widehat{C} = B$ is a Cuadras-Augé copula.

We conclude this Section with the following remarks and preliminary results, the role of which will emerge in the next Section 4.

Remark 3 *In view of (24), B is Archimedean with an invertible generator if and only if $\widehat{C}$ is such; furthermore, since generators of Archimedean semicopulas are determined up to a constant, if $\widehat{\phi}$ is a convex generator of $\widehat{C}$ and φ is a generator of B, then*

$$\theta\,\varphi(u) = \widehat{\phi}\big(\gamma^{-1}(u)\big) = \widehat{\phi}\big(\overline{G}(-\log u)\big), \tag{31}$$

for some constant $\theta > 0$. From the above relation we see that, in our framework, the generator φ is continuous. An Archimedean copula C is associative, i.e. $C(C(u, v), w) = C(u, C(u, w))$ holds for every u, v and w in $[0, 1]$. The property of associativity can be immediately extended to semicopulas. Similarly to the Archimedean property above, it can also be derived from (24) that B is associative if and only if $\widehat{C}$ is such. Finally we observe that the Cuadras-Augé copula $\widehat{C}$ defined in (30) is not associative, and therefore is not Archimedean, as it can be easily seen by observing that $\widehat{C}\big(u, \widehat{C}(v, w)\big) \neq \widehat{C}\big(\widehat{C}(u, v), w\big)$, for $v = w$ and $v^{1+\alpha} < u < v$.

In view of the previous Remark 3 we state a result concerning (31).

Lemma 4 ***(a)*** *Assume that* $\theta > 0$*,* $\widehat{\phi}$*, and* $\overline{G}$ *are given, with* $\widehat{\phi}$ *a strict and convex generator, and* $\overline{G}$ *a strictly positive survival function with* $\overline{G}(0) = 1$*. Then* φ *defined in* (31) *is a strict generator of a semicopula. Furthermore if* $\widehat{\phi}$ *and* $\overline{G}$ *are strictly decreasing, then the same holds for* φ*.*
(b) *Assume that* $\theta > 0$*,* $\widehat{\phi}$*, and* φ *are given, with* $\widehat{\phi}$ *a strict and convex generator and* φ *a strict generator. Assume furthermore that* $\widehat{\phi}$ *and* φ *are strictly decreasing, and that* $\overline{G}_\theta$ *is defined by means of* (31)*, i.e.*

$$\overline{G}_\theta(x) := \widehat{\phi}^{-1}\big(\theta\,\varphi(e^{-x})\big). \tag{32}$$

Then $\overline{G}_\theta$ *is a strictly decreasing survival function, with* $\overline{G}_\theta(0) = 1$*.*
(c) *Assume that* $\widehat{\phi}$ *and* φ *satisfy the same properties of point* ***(b)****, except the strictness property, i.e.* $\widehat{\phi}(0) = \widehat{\phi}(0^+)$ *and* $\varphi(0) = \varphi(0^+)$ *are finite. Then* (32) *defines a survival function* $\overline{G}_\theta$ *if and only if* $\theta \geq \theta_0$*, where* $\theta_0 := \varphi(0)/\widehat{\phi}(0)$*. Furthermore* $\overline{G}_\theta$ *is strictly decreasing on* $[0, \infty)$ *if and only if* $\theta = \theta_0$*.*

Proof. The proof of part ***(a)*** is a simple verification: Since $\theta > 0$, $\widehat{\phi}$, $\overline{G}$ and $-\log$ are (strictly) decreasing, then φ is (strictly) decreasing. Since $\lim_{u\to 0^+} -\log(u) = +\infty$, $\lim_{x\to+\infty} \overline{G}(x) = 0$ and $\lim_{u\to 0^+} \widehat{\phi}(u) = +\infty$, then $\lim_{u\to 0^+} \varphi(u) = +\infty$. Finally it is $\varphi(1) = 0$, since $-\log(1) = 0$, $\overline{G}(0) = 1$, and $\widehat{\phi}(1) = 0$.
The proof of part ***(b)*** is similar: Since $\widehat{\phi}$, φ and e^{-x} are strictly decreasing, then $\widehat{\phi}^{-1}$ and therefore $\overline{G}_\theta$ are also strictly decreasing. Since $\varphi(1) = 0$, $\widehat{\phi}^{-1}(0) = 1$, then $\overline{G}_\theta(0) = \widehat{\phi}^{-1}\big(\theta\,\varphi(1)\big) = 1$. Finally, since $\widehat{\phi}$ and φ are strict generators, then $\lim_{x\to\infty} \overline{G}_\theta(x) = \lim_{y\to 0^+} \widehat{\phi}^{-1}\big(\theta\,\varphi(y)\big) = \lim_{x\to\infty} \widehat{\phi}^{-1}\big(\theta\,x\big) = 0$.
The proof of part ***(c)*** is due to the observation that $\widehat{\phi}^{-1}\big(\theta\,x\big) = 0$ if and only if $\theta\,x \geq \widehat{\phi}(0)$. ∎

Note that in the proof of Lemma 4 we have not used the fact that $\widehat{\phi}$ is convex, while the convexity property has a key role in the following Corollary; the latter is a simple consequence of the previous Lemma 4, and its proof is left to the reader.

Corollary 5 *Assume the same hypotheses of Lemma 4 part* ***(b)****. Assume furthermore that* φ *is continuous and that* $\overline{G}_\theta$ *is defined by* (32)*. Then, for every* $\theta > 0$*, the bivariate function*

$$\widehat{\phi}^{-1}\left(\theta\left(\varphi(e^{-x}) + \varphi(e^{-y})\right)\right) \tag{33}$$

coincides with the bivariate survival function defined by

$$\overline{F}_\theta(x, y) := \widehat{\phi}^{-1}\Big(\widehat{\phi}\big(\overline{G}_\theta(x)\big) + \widehat{\phi}\big(\overline{G}_\theta(y)\big)\Big). \tag{34}$$

Furthermore $\overline{F}_\theta(x, y)$ *satisfies our standing hypotheses, with* $\widehat{C}_{\overline{F}_\theta} = C^{\widehat{\phi}}$ *and* $B_{\overline{F}_\theta} = S^{\varphi}$ *Archimedean with generators* $\widehat{\phi}$ *and* φ *respectively, and marginal* *distribution $\overline{G}_\theta$. Viceversa, if $\overline{F}$ is a bivariate survival function, $\widehat{C}_{\overline{F}} = C^{\widehat{\phi}}$ and $B_{\overline{F}} = S^{\varphi}$ are Archimedean with generators $\widehat{\phi}$ and φ respectively, then $\overline{F} = \overline{F}_\theta$ and $\overline{G}_{\overline{F}} = \overline{G}_\theta$, for some $\theta > 0$.*

Example 6 *Let $\widehat{\phi}(t)$ be the Clayton copula's generator $sign(\beta)(1 - t^{\widehat{\beta}}) = sign(b)(t^{-\widehat{b}} - 1)$, for $\widehat{\beta} \le 1$, $\widehat{\beta} \ne 0$ ($\widehat{b} \ge -1$, $b \ne 0$), and $\varphi(t)$ be the Clayton semicopula's generator $(1 - t^{\beta})sign(\beta)$ $\big(= (t^{-b} - 1)sign(b)\big)$, $\beta \ne 0$ ($b \ne 0$). Then, whenever $\widehat{\beta}$ and β are strictly negative, $\widehat{\phi}$ and φ are strict generators and Corollary 5 applies.*

When instead β is strictly positive, then it must also be $0 < \widehat{\beta} \le 1$, in order to guarantee that $\widehat{\phi}$ be a copula; furthermore, it is necessary that θ is greater or equal to 1. Indeed if this is not the case, then $\overline{G}_\theta(x) = \big[\big(1 - \theta(1 - e^{-\beta x})\big)^+\big]^{1/\widehat{\beta}}$ is not a one-dimensional survival function (see point (c) of Lemma 4). We notice that the support of $\overline{G}_\theta(x)$ is the whole half-line only for $\theta = 1$.

Under these conditions $\overline{F}_\theta$ in (34) is a true bivariate survival function, the support of which is strictly contained in $\mathbb{R}^2_+$, even when $\theta = 1$. The latter observation shows that $\overline{F}_\theta$ does not fit our assumptions. Nevertheless it still turns out that (34) and (33) define the same bivariate survival function

$$\begin{aligned}\overline{F}_\theta(x, y) &= \widehat{\phi}^{-1}\left(\theta\left(\varphi(e^{-x}) + \varphi(e^{-y})\right)\right)\\ &= \left[\left(1 - 2\,\theta + \theta\, e^{-\beta\, x} + \theta\, e^{-\beta\, y}\right)^+\right]^{1/\widehat{\beta}},\end{aligned}$$

with $B_{\overline{F}_\theta} = S^{\varphi}$.

In view of Remark 2, these conclusions may be extended to the case when $\widehat{\phi}$ and φ satisfy all the conditions of Corollary 5, except strictness, provided θ is sufficiently large (see Lemma 4 ***part (c)****).*

The following simple remarks are also relevant for our purposes.

Remark 4 *Let $(\overline{G}, B) = (\overline{G}_{\overline{F}}, B_{\overline{F}})$ be the survival function and the semicopula obtained from some joint survival function $\overline{F}$ by means of the Eq. (13). Assume that F has a jointly continuous and strictly positive density f, so that $G = G_F$ has a a strictly positive density $g = g_F \in C^1$ and $B = B_F \in C^2$. Then necessarily*

$$1 + \frac{g'(x)}{g(x)}\Big|_{x = -\log B(u,v)} \le \frac{B(u, v)\, \frac{\partial^2}{\partial u \partial v} B(u, v)}{\frac{\partial}{\partial u} B(u, v)\, \frac{\partial}{\partial v} B(u, v)} \tag{35}$$

or equivalently

$$1 + \frac{d}{dx}\log g(x)\Big|_{x = -\log B(u,v)} \le \frac{\frac{\partial}{\partial u}\log \frac{\partial}{\partial v} B(u, v)}{\frac{\partial}{\partial u}\log B(u, v)} \tag{36}$$

Note that when $\overline{G}(x) = e^{-x}$, then condition (35) may be satisfied if and only if $\frac{\partial^2}{\partial u \partial v} B(u, v) \ge 0$, so that B has to be a copula, in order to be compatible with *a standard exponential. Similar considerations hold when* $\overline{G}(x) = e^{-\theta x}$*, with* $\theta \geq 1$*.*

Remark 5 *Let* $(\overline{G}, B) = (\overline{M}, S)$*, where* $\overline{M}$ *is an arbitrary continuous, strictly decreasing and strictly positive survival function over* $[0, +\infty)$*, and* S *is a semicopula. The bivariate function defined by*

$$\overline{G}\left(-\log B\left(e^{-x}, e^{-y}\right)\right) = \overline{M}\left(-\log S\left(e^{-x}, e^{-y}\right)\right)$$

is a bivariate survival function (with aging function $B = S$ *and marginal survival function* $\overline{G} = \overline{M}$*) provided appropriate compatibility conditions hold.*

For instance, if we assume that M *has a strictly positive density* $m \in C^1$ *and that* $B = S = B_{\overline{F}}$*, with* F *admitting a jointly continuous and strictly positive density* f *(and therefore* $B \in C^2$*), then the necessary and sufficient compatibility conditions are given by* (35) *with* $g = m$*. As a consequence, in this case, a sufficient condition is given by*

$$\frac{m'(x)}{m(x)} \leq \frac{g_F'(x)}{g_F(x)} \quad \Leftrightarrow \quad \frac{d}{dx}\log m(x) \leq \frac{d}{dx}\log g_F(x), \tag{37}$$

which implies that

$$1 + \frac{m'(x)}{m(x)}\Big|_{x=-\log B(u,v)} \leq 1 + \frac{g_F'(x)}{g_F(x)}\Big|_{x=-\log B(u,v)} \leq \frac{B(u,v)\,\frac{\partial^2}{\partial u \partial v}B(u,v)}{\frac{\partial}{\partial u}B(u,v)\,\frac{\partial}{\partial v}B(u,v)}$$

As a particular case, consider

$$g_F(x) = k\,\exp\{-A_F(x)\}$$

with A_F *positive and increasing. Then, for*

$$m(x) = m_\theta(x) := k_\theta\,\exp\{-\theta A_F(x)\}$$

with $\theta \geq 1$*, the compatibility condition* (37) *is satisfied:*

$$\frac{d}{dx}\log m_\theta(x) = -\theta A_F'(x) \leq -A_F'(x) = \frac{d}{dx}\log g_F(x).$$

The exponential case considered in Remark 4 corresponds to $A_F(x) = x$*.*

On the other hand, when S *is a copula, we certainly can find a bivariate survival function* $\overline{F}$ *such that Eq.* (13) *is satisfied with* $(\overline{G}, B) = (\overline{M}, S)$*, for a suitable choice of* $\overline{G} = \overline{M}$*: It suffices e.g. to consider*

$$\overline{F}(x, y) = S(e^{-x}, e^{-y}) = B(e^{-x}, e^{-y}).$$

In this case the survival copula coincides with the aging function, the marginal distribution is a standard exponential and therefore condition (36) *is always satisfied, with* $g(x) = m(x) = e^{-x}$ *(see* (28)*,* (29)*, and the previous Remark 4). When the function* S *is continuous and* $S_u(\cdot) := S(u, \cdot)$ *is strictly increasing for any* $u \in [0, 1]$*, then* $\overline{F}$ *satisfies our standing assumptions.*

3 Survival copulas and upper-orthant Kendall distributions

In this Section we concentrate our attention on the upper-orthant Kendall distribution $\widehat{K}$ defined in (1).

We start by introducing the bivariate function

$$\overline{\Phi}(x, y) := \widehat{K}\big(\overline{F}(x, y)\big), \tag{38}$$

which has the interesting meaning:

$$\overline{\Phi}(x, y) = P\{(X, Y) \in A_{1-\overline{F}(x,y)}\}. \tag{39}$$

Since $\overline{G}(0) = 1$ and $\widehat{K}(0) = 0$, $\overline{\Phi}(x, 0) = \widehat{K}\left(\overline{G}(x)\right)$ is a univariate survival function, with $\overline{\Phi}(0, 0) = 1$; moreover, since $\widehat{K}$ is strictly increasing, the function $\overline{\Phi}(x, 0)$ is strictly decreasing. When $\overline{\Phi}(x, y)$ turns out to be a bivariate survival function, then it is

$$\overline{G}_{\overline{\Phi}}(x) = \widehat{K}\left(\overline{G}(x)\right), \quad B_{\overline{\Phi}}(u, v) = B_{\overline{F}}(u, v).$$

We notice that

$$\overline{F}(x, y) \leq \overline{\Phi}(x, y) = \widehat{K}\left(\overline{F}(x, y)\right), \tag{40}$$

which follows immediately by (39), observing that

$$E_{x,y} := \{(x', y') : \ x' > x, y' > y\} \subseteq A_{1-\overline{F}(x,y)} = \{(x', y') : \ \overline{F}(x', y') \leq \overline{F}(x, y)\},$$

and that $\overline{F}(x, y) = P\{(X, Y) \in E_{x,y}\}$.

For the Kendall distribution it is well known that

$$K_F(t) := P\{F(X, Y) \leq t\} = P\big\{C_F\big(G_F(X), G_F(Y)\big) \leq t\big\}, \tag{41}$$

and that

$$K_F(u) \geq u \quad \text{for all } u \in [0, 1].$$

Similarly to (41), by the definition (1) of the upper-orthant Kendall distribution, it is

$$\widehat{K}(t) = \widehat{K}_{\overline{F}}(t) = P\big\{\widehat{C}_{\overline{F}}\big(\overline{G}_{\overline{F}}(X), \overline{G}_{\overline{F}}(Y)\big) \leq t\big\}, \tag{42}$$

since

$$(\widehat{U}, \widehat{V}) = \big(\overline{G}_{\overline{F}}(X), \overline{G}_{\overline{F}}(Y)\big) \tag{43}$$

have joint distribution $\widehat{C}_{\overline{F}}$. Furthermore the following properties hold.

Lemma 6 *For any bivariate survival function $\overline{F}$ it holds*

$$\widehat{K}(u) \geq u, \quad \forall\, u \in [0,1], \tag{44}$$

and

$$\widehat{K}(t-) = \widehat{K}(t) \quad \text{if and only if} \quad P\big(\overline{F}(X,Y) = t\big) = 0, \tag{45}$$

i.e. if and only if $P\big((X,Y) \in \partial A_{1-t}\big) = 0$.

Finally, under our standing assumptions on $\overline{F}$*, the inequality in* (44) *is strict for all* $u \in (0,1)$*, namely*

$$\widehat{K}(u) > u, \quad \forall\, u \in (0,1). \tag{46}$$

Proof. Inequality (44) is an immediate consequence of (42). The proof of (45) is based on the observation that $\widehat{K}(t-) = P\big(\overline{F}(X,Y) < t\big)$. Finally, the strict inequality (46), can be proved by observing that the condition

$$\widehat{K}(t) = t, \quad \text{for some } t \in (0,1), \tag{47}$$

implies

$$P\big(X > \overline{G}^{-1}(t),\, Y \leq \overline{G}^{-1}(t)\big) = P\big(X \leq \overline{G}^{-1}(t),\, Y > \overline{G}^{-1}(t)\big) = 0. \tag{48}$$

Indeed, since the event $\{\overline{F}(X,Y) \leq t\}$ contains both the events $\{\overline{G}(X) \leq t\}$ and $\{\overline{G}(Y) \leq t\}$, it is sufficient to show that

$$P\big(\overline{G}(X) > t,\, \overline{G}(Y) \leq t\big) \leq P\big(\overline{G}(X) > t,\, \overline{F}(X,Y) \leq t\big) = 0.$$

The last equality is immediate, since (47) implies that

$$t = P\big(\overline{F}(X,Y) \leq t\big) \leq P\big(\overline{G}(X) \leq t\big) = t.$$

The condition (48) implies that $\overline{F}\big(x, \overline{G}^{-1}(t)\big) = t$ for every $x \in [0, \overline{G}^{-1}(t)]$. Therefore, $\overline{F}$ being strictly 1-increasing by assumption, the inequality in (44) is strict for all $u \in (0,1)$. ■

Remark 6 *It is interesting to note that if* (47) *holds for every* $t \in (t_1, t_2)$*, then* (48) *holds in the same interval, and the latter property implies*

$$P\big(X = Y \big| X \in (t_1,t_2)\big) = 1. \tag{49}$$

Moreover we note that the continuity condition (45) *is not automatically fulfilled, under our assumptions; as a counterexample we can take* $(X,Y) = (X_1, Y_1)$ *with probability* $\lambda \in (0,1)$*, and* $(X,Y) = (U, 1-U)$ *with probability* $1-\lambda$*,* *where* (X_1, Y_1) *are independent and standard exponential, and* U *is uniformly distributed in* $(0,1)$*, then*

$$\begin{aligned}\overline{F}(x,y) &= P(X > x,\ Y > y)\\ &= \lambda\, P(X_1 > x,\ Y_1 > y) + (1-\lambda)\, P(U > x,\ 1-U > y)\\ &= \lambda\, e^{-(x+y)} + (1-\lambda)\,(1-x-y)^+ = \overline{G}(x+y).\end{aligned}$$

Then $\overline{F}$ *is a Schur-constant survival function, satisfying all our standing assumptions, nonetheless*

$$P\big(\overline{F}(X,Y) = \lambda\, e^{-1}\big) = 1 - \lambda > 0.$$

Under natural regularity conditions, the Kendall distribution can be obtained analytically from the connecting copula, as it is shown by Genest and Rivest in [9]. Similarly, $\widehat{K}$ can be obtained analytically from the survival copula $\widehat{C}$; more precisely, we can easily formulate a result, that is stated in (52) below and is based on the one provided in [9]. To this end, we define an operator $\mathcal{K}$ on the bivariate semicopulas S, such that $S_v\,(\cdot) := S\,(\cdot, v)$ is a strictly increasing and continuous function for every v:

$$\mathcal{K}S(t) := t + \int_t^1 \frac{\partial S}{\partial u}(u,v)|_{v=v^S_{u,t}}\, du, \qquad \text{with } v^S_{u,t} = S_u^{-1}(t), \tag{50}$$

i.e. $S_{\overline{F}}(u, v^S_{u,t}) = t$.

Then the result in [9] can be stated as

$$K_F(t) = \mathcal{K}C_F(t) \tag{51}$$

whenever $S = C_F$ is strictly 1-increasing and with S_v continuous. An analogous representation connects the upper-orthant Kendall distribution function and the survival copula $\widehat{C} = \widehat{C}_{\overline{F}}$, when $\widehat{C}$ satisfies the above conditions; more in detail, by using the arguments in [9], i.e. the fact that $P(\widehat{V} \leq v | \widehat{U} = u) = \frac{\partial \widehat{C}}{\partial u}(u,v)$, with $\widehat{H}$ and $\widehat{V}$ as in (43), it can be checked that

$$\widehat{K}_{\overline{F}}(t) = \mathcal{K}\widehat{C}_{\overline{F}}(t). \tag{52}$$

For the copulas C_F and $\widehat{C}_{\overline{F}}$, the functions $\mathcal{K}C_F$ and $\mathcal{K}\widehat{C}_{\overline{F}}$ are distribution functions, while, generally, $\mathcal{K}S$ is not a distribution function, and therefore we shall use the term *Kendall pseudo-distribution function* for $\mathcal{K}S$. For example, when $S(u,v) = B(u,v)$ is the Clayton semicopula (19) with $\beta = 2$, then

$$\mathcal{K}B(t) = t + \frac{1-t^2}{2t}, \qquad t \in (0,1).$$

Clearly $\mathcal{K}B(0^+) = \lim_{t\to 0^+} \mathcal{K}B(t) = \infty$, and therefore $\mathcal{K}B$ is not a distribution function.

For a given bivariate survival function $\overline{F}$ the negative function $t - K_{\overline{F}}(t)$ depends only on the survival copula $\widehat{C}_{\overline{F}}$, because $K_{\overline{F}} = K_{\widehat{C}_{\overline{F}}} = \mathcal{K}\widehat{C}_{\overline{F}}$. Extending the notation used, e.g. in [9], we generally set, for a semicopula S as above

$$\lambda_S(t) := t - \mathcal{K}S(t) = -\int_t^1 \frac{\partial S}{\partial u}(u,v)|_{v=v^S_{u,t}}\, du,$$

with $v^S_{u,t} = S_u^{-1}(t)$.

When $B = B_F$ is a strictly 1-increasing semicopula, with B_v continuous, we can therefore define $\mathcal{K}B$ and λ_B. Below (see Proposition 7) we will see the relation between $\mathcal{K}B$, $\mathcal{K}\widehat{C}$ and $\overline{G}$, for $(B, \widehat{C}, \overline{G}) = (B_{\overline{F}}, \widehat{C}_{\overline{F}}, \overline{G}_{\overline{F}})$.

In the next Example 7 we compute the upper-orthant Kendall distribution functions $\mathcal{K}\widehat{C}$ and the functions $\lambda_{\widehat{C}}$, together with $\mathcal{K}B$ and λ_B, for the basic cases considered in Examples 1, 2, and 3, replacing the Schur-constant case with the more general Archimedean case. Afterwards, in Example 8, we consider the model of Example 5; note that the corresponding $\overline{F}$ is not absolutely continuous, as well as in the basic perfect dependence case.

Example 7 *In the perfect dependence case, in agreement with* (49) *of Remark 6, it holds*

$$\widehat{K}_{\overline{F}}(t) = \mathcal{K}\widehat{C}(t) = t, \quad \lambda_{\widehat{C}}(t) = t - t = 0,$$

since $\overline{F}(x,y) = \overline{G}(x \vee y)$*, and* $\widehat{U} = \overline{G}(X)$ *is a uniform random variable on* $(0,1)$*. This corresponds to the fact that* $\overline{F}(x,y) = \overline{\Phi}(x,y)$*. Furthermore* $\mathcal{K}B(t) = t$ *and* $\lambda_B(t) = 0$*, since* $B = \widehat{C}$*.*

In the case of an **Archimedean survival copula** *with convex generator* $\widehat{\phi}$*, i.e. (using the notation* (21)*)*

$$\widehat{C}(u,v) = C^{\widehat{\phi}}(u,v) = \widehat{\phi}^{-1}\left(\widehat{\phi}(u) + \widehat{\phi}(v)\right), \tag{53}$$

it is (see e.g. [10] p.102)

$$\mathcal{K}\widehat{C}(t) = \mathcal{K}C^{\widehat{\phi}}(t) = t - \frac{\widehat{\phi}(t)}{\widehat{\phi}'(t+)}, \quad \lambda_{C^{\hat{\phi}}}(t) = \frac{\widehat{\phi}(t)}{\widehat{\phi}'(t+)}. \tag{54}$$

As we already know, in this case the aging function is an Archimedean semicopula, i.e. (using the notation (20)*)* $B = S^{\varphi} = \varphi^{-1}\big(\varphi(u) + \varphi(v)\big)$*, with* φ *satisfying* (31)*. As a consequence it is not difficult to get that*

$$\mathcal{K}B(t) = \mathcal{K}S^{\varphi}(t) = t - \frac{\varphi(t)}{\varphi'(t+)}, \quad \lambda_B(t) = \lambda_{S^{\varphi}}(t) = \frac{\varphi(t)}{\varphi'(t+)}. \tag{55}$$

Finally, for the product copula $\widehat{C}(u,v) = u \cdot v$*, it holds* $\mathcal{K}\widehat{C}(t) = t - t\log t$ *(see e.g. [9], p. 394); so that it is*

$$\lambda_{\widehat{C}}(t) = t\log t.$$

The above relations can also be obtained by the previous Archimedean case, observing that $\widehat{C} = C^{\widehat{\phi}_0}$*, with* $\widehat{\phi}_0 = -\log t$*. This observation allows also to compute immediately* $\mathcal{K}B$ *and* λ_B*, by means of* (55)*, with* $\varphi = Q$*, and* Q *defined as in* (18)*.*

Example 8 (Marshall-Olkin model-II) *By means of some computation, it can be shown that, in the case* (30)*,* $\frac{\partial \widehat{C}}{\partial u}(u,v)|_{v=v^{\widehat{C}}_{u,t}} = \frac{t}{u}$*, for* $u \in (t, t^{\frac{1}{1+\alpha}})$*, and* $\frac{\partial \widehat{C}}{\partial u}(u,v)|_{v=v^{\widehat{C}}_{u,t}} = \alpha\,\frac{t}{u}$*, for* $u \in (t^{\frac{1}{1+\alpha}}, 1)$*; therefore, taking into account that* $B = \widehat{C}$ *it is*

$$\widehat{K}_{\overline{F}}(t) = \mathcal{K}\widehat{C} = \mathcal{K}\widehat{B} = t - \frac{2\,\alpha}{1+\alpha}\,t\log t, \quad \lambda_{\widehat{C}}(t) = \lambda_B(t) = \frac{2\,\alpha}{1+\alpha}\,t\log t.$$

Remark 7 *For what follows it is important to note that* (44) *characterizes upper-orthant Kendall distribution functions: indeed the condition* $K(t) \geq t$ *in* $[0,1]$ *characterizes Kendall distribution functions, since, for such a given distribution function* K*, there exists a (generally infinite) family* $\mathcal{C}_K$ *of copulas* C *such that* $\mathcal{K}C = K$*, or equivalently* $\lambda_C(t) = t - K(t)$*. The family* $\mathcal{C}_K$ *is an equivalence class in the set* $\mathcal{C}$ *of all copulas and contains one and only one associative copula (see [11] and the references therein); the latter turns out to be Archimedean (see Genest and Rivest [8]) if and only if it is*

$$K(t^-) > t. \tag{56}$$

Note that, under the assumptions considered here, (56) *for* $\widehat{K}$ *is granted if the equivalent conditions in* (45) *hold (see also* (46) *).*

If (56) *holds, then (see [8]) there exists an Archimedean copula* C *with convex generator given by*

$$\phi_\theta(t) = \theta\phi(t) = \theta\,\exp\left\{\int_{t_0}^{t} \frac{1}{\lambda(s)}\,ds\right\}, \quad \theta > 0, \tag{57}$$

where t_0 *is a fixed number in* $(0,1)$ *and* $\lambda(t) = t - K(t)$*.*

Note that the Archimedean copula C *coincides with* C^{ϕ_θ} *(see* (21) *for the notation), and does not depend on the choice of the constant* θ*. Nevertheless this constant will play a role in the compatibility conditions in the next section.*

In view of the result (57) it is natural to introduce the following notations:

We will refer to a distribution function K satisfying (56) as to an ***Archimedean Kendall distribution function***. For such a distribution function, denoting by $\Upsilon_{t_0,K} : [0,1] \to \mathbb{R}_+$ the function

$$\Upsilon_{t_0,K}(t) := \exp\left\{ \int_{t_0}^{t} \frac{1}{s - K(s)}\, ds \right\}, \quad \text{with } t_0 \in (0,1), \tag{58}$$

the generators in (57) are given by $\phi_\theta(t) = \theta\, \Upsilon_{t_0,K}(t)$. Furthermore we will call K a ***strict Archimedean Kendall distribution function*** when the generators ϕ_θ defined above are strict.

The operator (58) can be defined also for functions which are not necessarily Archimedean Kendall distributions, under the (obvious) condition that the right hand side of (58) makes sense.

We end this section with the announced relations between $\mathcal{K}\widehat{C}$ and $\mathcal{K}B$, the Kendall distribution and pseudo-distribution functions, respectively, in terms of the marginal survival function.

Proposition 7 *Assume that $\widehat{C} = \widehat{C}_{\overline{F}}$ is a survival copula such that $B = B_{\overline{F}}$ is the corresponding aging function. Assume that the marginal distribution $G = G_F$ admits a strictly positive density $g = g_F$. Then $\gamma^{-1}(u) = \overline{G}\,(-\log u)$ is differentiable, bijective from $[0,1]$ to $[0,1]$, and*

$$\mathcal{K}B(t) = t + \frac{1}{(\gamma^{-1})'(t)}\, \lambda_{\widehat{C}}\big(\gamma^{-1}(t)\big). \tag{59}$$

If furthermore $\widehat{K} = \mathcal{K}\widehat{C}$ is Archimedean, then $\Upsilon_{t_0,\mathcal{K}B}$ is well defined, with

$$\Upsilon_{t_0,\mathcal{K}B}(t) = \exp\left\{ \int_{t_0}^{t} \frac{1}{s - \mathcal{K}B(s)}\, ds \right\} = \exp\left\{ \int_{\gamma^{-1}(t_0)}^{\gamma^{-1}(t)} \frac{1}{s - \mathcal{K}\hat{C}(s)}\, ds \right\}, \tag{60}$$

and there exists a positive constant θ such that the following relation holds

$$\theta\, \Upsilon_{t_0,\mathcal{K}B}(t) = \Upsilon_{t_0,\widehat{K}}\big(\overline{G}(-\log(t))\big). \tag{61}$$

Before giving the proof of Proposition 7 we direct the reader's attention to the following Theorem 8, given by Genest and Rivest in [9].

Theorem 8 *If C and C^* are copulas that are related via relation*

$$C^*(u,v) = \gamma^{-1}\left(C\ \left(\gamma(u),\ \gamma(v)\right)\right)$$

by a strictly increasing, differentiable bijection γ, then

$$\lambda_{C^*}(v) = \frac{\lambda_C(\gamma(v))}{\gamma'(v)}, \qquad 0 < v < 1.$$

The proof of this result can be extended in a rather straightforward way to the case of semicopulas and the latter extension can be applied to (24) in order to prove the first part of Proposition 7, i.e. Eq. (59). In what follows, however, we sketch the proof of Eq. (59), in order to be self-contained; afterwards we focus on the proof of the remaining parts.

Proof of Proposition 7. The proof of Eq. (59) is based on the observation that, since $B(u,v) = \gamma\left(\widehat{C}\left(\gamma^{-1}(u),\ \gamma^{-1}(v)\right)\right)$, it holds

$$\mathcal{K}B(t) = t + \int_t^1 \frac{1}{(\gamma^{-1})'(t)} \frac{\partial \widehat{C}}{\partial u'}\big(\gamma^{-1}(u), \gamma^{-1}(B_u^{-1}(t))\big)\,(\gamma^{-1})'(u)\,du. \tag{62}$$

Then Eq. (59) is achieved by observing that $B(u,v) = t$ if and only if $\widehat{C}\left(\gamma^{-1}(u),\ \gamma^{-1}(v)\right) = \gamma^{-1}(t)$, which is equivalent to $\gamma^{-1}\big(B_u^{-1}(t)\big) = \widehat{C}_u^{-1}\big(\gamma^{-1}(t)\big)$, i.e. $\gamma^{-1}\big(v_{u,t}^{B}\big) = v_{u,\gamma^{-1}(t)}^{\widehat{C}}$. Relation (60) is a direct consequence of Eq. (59), while relation (61) follows by observing that

$$\Upsilon_{t_0,\mathcal{K}B}(t) = \Upsilon_{\gamma^{-1}(t_0),\mathcal{K}\widehat{C}}\big(\gamma^{-1}(t)\big) = \widehat{\theta}\,\Upsilon_{t_0,\widehat{K}}\big(\gamma^{-1}(t)\big),$$

where $\widehat{\theta} = \exp\left\{\int_{\gamma^{-1}(t_0)}^{t_0} \frac{1}{s-\mathcal{K}\hat{C}(s)}\,ds\right\}$. ■

If $\widehat{K}_{\overline{F}}$ is an Archimedean Kendall distribution, then the unique Archimedean copula in $\mathcal{C}_{\widehat{K}_{\overline{F}}}$ is the copula $C^{\widehat{\phi}}$ that admits, as a generator, the convex function

$$\widehat{\phi}(t) := \Upsilon_{t_0,\widehat{K}_{\overline{F}}}(t).$$

Then, by letting

$$\varphi(t) := \Upsilon_{t_0,\mathcal{K}B_{\overline{F}}}(t),$$

the relation (61) becomes exactly the relation (31), which concerns models with Archimedean survival copulas. Therefore (61) can be thought of as an extension of (31). If furthermore the convex generator $\widehat{\phi}$ is strict, then (61) necessarily implies that

$$\overline{G}_{\overline{F}}(x) = \widehat{\phi}^{-1}\big(\theta\,\varphi(e^{-x})\big), \tag{63}$$

for some θ, which again is the generalization of a property concerning models with Archimedean survival copulas (see the last statement in Corollary 5).
The above observations are not surprising: set

$$\overline{H}(x,y) := C^{\widehat{\phi}}\big(\overline{G}_{\overline{F}}(x), \overline{G}_{\overline{F}}(y)\big); \tag{64}$$

we know that the corresponding aging function $B_{\overline{H}}$ is an Archimedean semicopula (see Remark 3); by (61), together with the obvious observation that $\widehat{K}_{\overline{H}} = \widehat{K}_{\overline{F}}$ and $\overline{G}_{\overline{H}} = \overline{G}_{\overline{F}}$, it is

$$\Upsilon_{t_0,\mathcal{K}B_{\overline{H}}}(t) = \Upsilon_{t_0,\mathcal{K}B_{\overline{F}}}(t)\big(= \varphi(t)\big),$$

which immediately implies $B_{\overline{H}}(u,v) = S^{\varphi}(u,v) = \varphi^{-1}\big(\varphi(u) + \varphi(v)\big)$.

4 Compatibility conditions between Kendall distributions and families of upper-orthant VaR curves

The arguments developed in the previous Sections will now be used to reformulate and, subsequently, to face the compatibility problem described in the Introduction. Later on, by using the notion of PKD dependence, we shall also present conditions of qualitative type concerning compatibility. Our main problem becomes

Given $\widehat{K}$, a distribution function over $[0,1]$, and B, a semicopula, does there exists a survival function $\overline{F}$, satisfying our standing assumptions, such that $\widehat{K}_{\overline{F}} = \widehat{K}$ and $B_{\overline{F}} = B$?

The following two remarks are immediate, but crucial.
(i) In view of Eq. (14), if a solution $\overline{F}$ to the above problem exists, then, in order to determine it, it only remains to specify the marginal $\overline{G}_{\overline{F}}$.
(ii) Some necessary conditions arise: first of all it is necessary that $\widehat{K}(u) \geq u$ in $[0,1]$, secondly we need that B actually determines a set of level curves for some survival function; i.e. we need that $B = B_{\overline{J}}$ for a given survival function $\overline{J}$.

Besides the above necessary conditions, we require

(A1) $\widehat{K}$ is a strict Archimedean Kendall distribution;

(A2) The bivariate survival $\overline{J}$ satisfies our standing assumptions, i.e. $\overline{J}$ is strictly 1-decreasing, and jointly continuous, furthermore the associated upper-orthant Kendall distribution $\widehat{K}_{\overline{J}} = \mathcal{K}\widehat{C}_{\overline{J}}$ is a strict Archimedean Kendall distribution;

(A3) the function $\gamma_{\overline{J}}^{-1}(u) = \overline{G}_{\overline{J}}(-\log u)$ is differentiable.

Note that **(A3)** is obviously true when $\overline{G}_{\overline{J}}$ admits a strictly positive and continuous density.

With the above notations we can rephrase the above compatibility problem as follows: *Does there exists a bivariate survival function $\overline{F}$ such that $B_{\overline{F}} = B_{\overline{J}}$ and $\widehat{K}_{\overline{F}} = \widehat{K}$?*

Proposition 7 helps us in giving a necessary condition concerning the possible candidates for the marginal survival function $\overline{G}_{\overline{F}}$:
$\widehat{\phi}(t) := \Upsilon_{t_0,\widehat{K}}(t)$ and $\varphi_B(t) := \Upsilon_{t_0,\mathcal{K}B}(t)$ be strict generators; then, by (63),

the possible candidates for $\overline{G}_{\overline{F}}$ necessarily are of the form

$$\overline{G}_\theta(x) = \widehat{\phi}^{-1}\big(\theta\,\varphi_B(e^{-x})\big) \qquad \text{for some } \theta > 0. \tag{65}$$

The latter considerations put us in a position to exploit the arguments developed in Sections 2 and 3 in order to formulate the result announced in the Introduction. More precisely we obtain necessary conditions on the form of the solutions $\overline{F}$ to the compatibility problem.

Theorem 9 *Under assumptions **(A1)**, **(A2)**, and **(A3)** the univariate functions $\overline{G}_\theta$ defined in* (65) *are survival functions for any $\theta > 0$, and the only possible solutions $\overline{F}$ are of the form*

$$\overline{F}_\theta(x,y) = \overline{G}_\theta\big(-\log B(e^{-x},e^{-y})\big) = \widehat{\phi}^{-1}\left(\theta\,\varphi_B\left(B\left(e^{-x},e^{-y}\right)\right)\right). \tag{66}$$

Proof. Provided that $\overline{G}_\theta$ is a one-dimensional survival function, the form (66) for $\overline{F}_\theta(x,y)$ is immediately obtained by combining the Eq. (65) with Eq. (14). We then have only to check that $\overline{G}_\theta$ are survival functions. First of all note that φ_B is well defined since, for some $\theta_B > 0$,

$$\theta_B\, \Upsilon_{t_0,\mathcal{K}B}(t) = \Upsilon_{t_0,\widehat{K}_{\overline{J}}}\big(\overline{G}_{\overline{J}}(-\log t)\big),$$

as immediately follows from assumption **(A2)** and (61), applied to $\overline{J}$.

By assumption **(A2)**, $\widehat{\phi}_{\overline{J}} := \Upsilon_{t_0,\widehat{K}_{\overline{J}}}$ is a strict and convex generator and $\overline{G}_{\overline{J}}$ is strictly decreasing, strictly positive. Then φ_B is a strict generator, and the result follows by Lemma 4, part ***(b)***. ∎

Up to now we do not know whether the bivariate functions $\overline{F}_\theta$ are actually survival functions, for a given choice of $B = B_{\overline{J}}$ and $\widehat{K}$.

Under suitable regularity assumptions on $\overline{J}$ (and therefore on $B = B_{\overline{J}}$), and on $\widehat{K}$, we may assume that $g_\theta = -\overline{G}_\theta'$ is a C^1 probability density, and we may use the compatibility conditions of Remark 5 (see also Remark 4): When B is C^2, a necessary and sufficient condition is given by (35) with g replaced by g_θ, and a sufficient condition is given by

$$\frac{g_\theta'(x)}{g_\theta(x)} \le \frac{g_J'(x)}{g_J(x)} \quad \Leftrightarrow \quad \frac{d}{dx}\log g_\theta(x) \le \frac{d}{dx}\log g_J(x), \tag{67}$$

as immediately follows by (37).

Actually the above conditions are not often satisfied. In a number of cases, however, it can be checked directly that the bivariate functions $\overline{F}_\theta$ in (66) are bivariate survival functions and therefore they are solutions of our compatibility problem. A relevant special case of this kind arises when, besides the assumptions of Theorem 9, we add the condition that B is Archimedean. Even though this remark is nothing but an application of Corollary 5, we prefer in the following example to summarize some details about the Archimedean cases and to briefly discuss a few related aspects.

Example 9 *Here we consider the special case of exchangeable survival models characterized by the condition that the survival copula $\widehat{C}_{\overline{F}}$ is Archimedean with a (convex) invertible generator $\widehat{\phi}$. In this case $B_{\overline{F}}$ is Archimedean with a continuous and invertible generator φ. Concerning the role of $\widehat{\phi}$, we first recall Eq.* (53) *and also write*

$$\overline{F}(x,y) = \widehat{\phi}^{-1}\left(\widehat{\phi}\left(\overline{G}_{\overline{F}}(x)\right) + \widehat{\phi}\left(\overline{G}_{\overline{F}}(y)\right)\right).$$

By (13)*, it is easy to check that it is $B_{\overline{F}} = S^{\varphi}$, for $\varphi(u) = \widehat{\phi}\left(\overline{G}_{\overline{F}}(-\log u)\right)$. Thus, in terms of $\widehat{\phi}$ and φ, $\overline{G}_{\overline{F}}$ is given by Eq.* (32)*, and it must be*

$$\overline{F}(x,y) = \widehat{\phi}^{-1}\left(\varphi(e^{-x}) + \varphi(e^{-y})\right).$$

In the latter formulas, $\overline{G}_{\overline{F}}(x)$ and $\overline{F}(x,y)$ respectively are a one-dimensional and a two-dimensional survival function, for any choice of the generators $\widehat{\phi}$ and φ, under our assumptions; then the compatibility problems admits solutions. Of course, for $\theta > 0$, $\theta\,\varphi$ is also a generator of $B_{\overline{F}}(u,v)$ and, correspondingly, any bivariate survival function having the form (33)*, i.e.*

$$\widehat{\phi}^{-1}\left(\theta\left(\varphi(e^{-x}) + \varphi(e^{-y})\right)\right)$$

is still a solution of the same compatibility problem.

Two remarks are here in order.

First, Eq. (66) *can be seen as the natural generalization of Eq.* (33)*. However Eq.* (33) *can be derived directly, just by taking into account* (24) *and* (32)*; the derivation of Eq.* (66) *on the contrary stands on the appropriate extension of Theorem 8.*

Second, the condition that $B_{\overline{F}}$ is Archimedean with strict generator φ characterizes the bivariate models such that the family $\mathcal{D}_{\overline{F}}$ is the same as in the case of stochastic independence; in fact (see also [3] and (17) *in Example 2) a pair of independent, identically distributed, non-negative variables with marginal survival function $e^{-\theta\,\varphi(e^{-x})}$ has S^{φ} as its aging function. We can then conclude by saying that any strict Archimedean Kendall distribution is compatible with any family of bivariate (upper-orthant) VaR curves associated to an independent model, provided that the marginal of the latter is strictly positive, strictly decreasing, and continuous all over* $[0;+\infty)$.

Theorem 9 can also be used in cases where B is neither C^2 nor Archimedean: we end this part by showing a characterization of relevant classes of exchangeable, not absolutely continuous models. Let us start with the celebrated Marshall-Olkin model.

Example 10 *Let $B = B_{\overline{J}}$ be the Cuadras-Augé copula* (30)*, which is not Archimedean (see Remark 3). The bivariate aging function being a copula,* *we may consider the compatibility problem with $\widehat{K}(t) = \mathcal{K}B(t)$ (i.e. $\widehat{K}(t) = t - \widehat{\rho}\, t \log t$, with $\widehat{\rho} = \frac{2\,\alpha}{1+\alpha}$, see the Example 8); $\widehat{K}$ is a strict Archimedean Kendall distribution since*

$$\varphi_B(t) = \widehat{\phi} = \exp\left\{\int_{t_0}^{t} \frac{1}{\widehat{\rho}}\,\frac{1}{\log u}\, d\log u\right\} = const\,|\log t|^{1/\widehat{\rho}}.$$

Therefore (65) *shows that the compatible marginal survival functions have to be exponential. These are exactly the marginal survival functions of exchangeable Marshall-Olkin models. Summarizing, the choice of B as the Cuadras-Augé copula, and $\widehat{K} = \mathcal{K}B$, characterizes the exchangeable Marshall-Olkin models.*

Example 11 *Different models, still with a Cuadras-Augé survival copula $\widehat{C}$, but with non-exponential marginals, have been considered in the literature; for instance it is interesting to consider the cases with Weibull marginals. In our framework, the latter models can be characterized by imposing*

$$B(u,v) = \exp\left\{-\left[\alpha\,(x^\beta + y^\beta) + (1-\alpha)\,(x^\beta \vee y^\beta)\right]^{1/\beta}\right\}\Big|_{x=-\log u,\, y=-\log v},$$

and

$$\widehat{K}(t) = t - \widehat{\rho}\, t \log t,$$

with $\widehat{\rho} = \frac{2\,\alpha}{1+\alpha}$, $\alpha \in (0,1)$, and $\beta > 0$ (since $\beta = 1$ corresponds to the case considered in the previous Example 10, we here assume that $\beta \neq 1$). In fact $\mathcal{K}B = t - \rho\, t \log t$, and $\varphi_B(t) = const|\log t|^{1/\rho}$, with $\rho = \widehat{\rho}/\beta$ and $\widehat{\phi}(t) = const|\log t|^{1/\widehat{\rho}}$; therefore (65) *yields*

$$\overline{G}_\theta(x) = \exp\{-\theta\, x^{\rho/\widehat{\rho}}\} = \exp\{-\theta\, x^{\beta}\}.$$

Still considering the problem of compatibility for a pair $\left(B, \widehat{K}\right)$, we can obtain some necessary conditions of qualitative type in terms of the notions of IFRA and PKD dependence.

By definition, a copula C is PKD (see e.g. [1] or [11]) if $K_C(v) \geq v - v \log v$ or, equivalently, $\lambda_C(v) \geq v \log v$. In view of the position (50), this notion can be extended to the case of semicopulas; by reversing the above inequality, we can also define, for a copula or for semicopula, the condition of NKD. The notions of PKD and NKD give, respectively, rise to conditions of positive and negative dependence, that are in some way related to the concepts of IFRA and DFRA. A univariate survival function $\overline{G}$ is IFRA if and only if $-\frac{\log \overline{G}(x)}{x}$ is an increasing function of x (see e.g. [2]); $\overline{G}$ is DFRA if and only if $-\frac{\log \overline{G}(x)}{x}$ is a non-decreasing function. For the case when $\widehat{\phi} := \overline{G}^{-1}$ is a convex and strict generator, it has been shown in [1] that the Archimedean copula $C^{\widehat{\phi}}$ is PKD if and only if $\overline{G}$ is DFRA.

Similarly, in the specific cases of bivariate models with B an Archimedean copula or semicopula, the condition of PKD for B has been interpreted in [5] [6] as a property of bivariate IFRA; here we can extend arguments therein to the case when B is a (non-necessarily Archimedean) semicopula, and formulate the following compatibility result of qualitative type.

Proposition 10
(a) *If $\widehat{C}$ is PKD and $\overline{G}$ is IFRA then B is PKD*
(b) *If B is PKD and $\overline{G}$ is DFRA then $\widehat{C}$ is PKD*
(c) *If B is PKD and $\widehat{C}$ is NKD then $\overline{G}$ is IFRA*

Proof. The proof of ***(a)***, ***(b)***, and ***(c)*** immediately follows by taking into account the following facts.

In view of (59), with $t = e^{-x}$, it is

$$\lambda_B\left(e^{-x}\right) = -e^{-x}\frac{\lambda_{\widehat{C}}\left[\overline{G}(x)\right]}{\overline{G}'(x)};$$

then the condition B is PKD can also be written in terms of $\lambda_{\widehat{C}}$ and $\overline{G}$ as

$$\lambda_{\widehat{C}}\left[\overline{G}(x)\right] \geq x\overline{G}'(x).$$

The conditions $\widehat{C}$ is PKD and $\widehat{C}$ is NKD respectively mean

$$\lambda_{\widehat{C}}\left[\overline{G}(x)\right] \geq \overline{G}(x)\log\overline{G}(x),$$

$$\lambda_{\widehat{C}}\left[\overline{G}(x)\right] \leq \overline{G}(x)\log\overline{G}(x);$$

while the conditions $\overline{G}$ is IFRA and $\overline{G}$ is DFRA respectively mean

$$-\frac{\overline{G}'(x)}{\overline{G}(x)} \geq -\frac{\log\overline{G}(x)}{x},$$

$$-\frac{\overline{G}'(x)}{\overline{G}(x)} \leq -\frac{\log\overline{G}(x)}{x}.$$

■

Example 12 *As in the Examples 8 and 10, we consider here bivariate models such that B is the Cuadras-Augé copula* (30)*. In these cases B is PKD. In fact it is $\lambda_B(t) = \frac{2\alpha}{1+\alpha}t\log t$, $0 < \alpha < 1$ (see Example 8). In the case of Marshall-Olkin bivariate models, i.e. when the marginal distribution is exponential, it is immediate that $\widehat{C}$ is PKD as well; in fact it is $\widehat{C} = B$. When the marginal G is DFRA, but not exponential, though $\widehat{C}$ does not coincides with B, the part (b) of Proposition 10 shows that $\widehat{C}$ has to be still PKD.*

Example 13 *The models considered in Example 11 provide instances where (a) and (b) of Proposition 10 apply. Indeed $\widehat{C}$ is always PKD, since $\widehat{\rho} \in (0, 1]$, and, furthermore:*
(a) By taking $\beta > 1$ we obtain that $\overline{G}$ is IFR (and then IFRA) and that B is PKD, since $\rho = \widehat{\rho}/\beta < \widehat{\rho} < 1$.
(b) By taking $\beta < \widehat{\rho} \leq 1$ we obtain that B is PKD and $\overline{G}$ is DFR (and then DFRA).

Finally we observe that, by taking $\widehat{\rho} < \beta \leq 1$, we obtain an example of a situation "dual" of (c), i.e. an example where $\widehat{C}$ is PKD, B is NKD and $\overline{G}$ is DFRA.

5 Summary and concluding remarks

Our attention has been focused on exchangeable bivariate survival models characterized by joint survival functions $\overline{F}$, that satisfy a convenient set of regularity conditions.

For any such model, we considered the (upper-orthant) Kendall distribution $\widehat{K}_{\overline{F}}$ and the family $\mathcal{D}_{\overline{F}}$ of (upper-orthant) bivariate VaR curves. From a *geometric* viewpoint, the relation between these two objects becomes immediately clear when considering the family of the level sets A_v $(0 < v < 1)$ of $\overline{F}$ (see Definition 2).

For a given Kendall distribution $\widehat{K}$ and a given family $\mathcal{D}$ of bivariate VaR curves, it is then natural to wonder whether they are compatible, i.e. whether it there exists $\overline{F}$ such that $\widehat{K}_{\overline{F}} = \widehat{K}$, $\mathcal{D}_{\overline{F}} = \mathcal{D}$. In our paper we developed a method that allows us to find sufficient or necessary conditions for compatibility.

We described the family $\mathcal{D}_{\overline{F}}$ in terms of the "aging function" $B_{\overline{F}}$ and this allowed us to rephrase that problem in an *analytical* form, as follows: for a given $\widehat{K}$ and a given aging function B does there exist $\overline{F}$ such that $\widehat{K}_{\overline{F}} = \widehat{K}$, $B_{\overline{F}} = B$?

The advantage offered by this formulation is based on the possibility to use (a slight extension of) the transformation result presented in [9] and here recalled as Theorem 8. In the present context this result plays a key role in describing the relations between the (semi-)copula B and the copula $\widehat{C}$ (see (24)). Furthermore we also relied on (a slight extension of) the representation (58), which provides the generators of the unique Archimedean (semi-)copula having a given strict Archimedean (pseudo-)Kendall distribution. The above-mentioned extensions were summarized in Proposition 7.

As an application of such results, we could determine the form that possible solutions to the system of equations $\widehat{K}_{\overline{F}} = \widehat{K}$, $B_{\overline{F}} = B$ must have. To this purpose, the strategy followed is simply summarized as follows: since, by Eq. (14), $\overline{F}$ is determined from the knowledge of both $B_{\overline{F}}$ and the marginal $\overline{G}_{\overline{F}}$, then, in view of the condition $B_{\overline{F}} = B$, the condition $\widehat{K}_{\overline{F}} = \widehat{K}$ is to be used to obtain $\overline{G}_{\overline{F}}$. For this reason we first found that the possible marginals are of the form displayed in (65), i.e.

$$\overline{G}_{\theta}(x) = \widehat{\phi}^{-1}\left(\theta \cdot \varphi_B\left(e^{-x}\right)\right),$$

where $\widehat{\phi}$ is the generator of the unique Archimedean copula with Kendall distribution $\widehat{K}$, and, analogously, φ_B is the generator of the Archimedean semicopula $B_{\overline{H}}$ associated to the Archimedean model $\overline{H}$ in (64), and such that $\mathcal{K}B_{\overline{H}} = \mathcal{K}B$. Then, by applying (14) in Theorem 9, we obtained the family of "candidate" joint survival functions

$$\overline{F}_{\theta}(x, y) = \widehat{\phi}^{-1}\left(\theta \cdot \varphi_B\left(B\left(e^{-x}, e^{-y}\right)\right)\right).$$

Whenever $\overline{F}_{\theta}$ turns out to be a joint survival function, then it actually is a solution to the compatibility problem. As we discussed, this can happen when some suitable further conditions are satisfied. We can easily exhibit however simple cases where this does not happen; for instance, if $\hat{\phi}$ and φ_B are not strict, we can find values of θ for which even $\overline{G}_{\theta}(x) = \widehat{\phi}^{-1}(\theta \cdot \varphi_B(e^{-x}))$ is not a one-dimensional survival function (see point (c) of Lemma 4). However, under the assumptions of Theorem 9, we proved that $\overline{G}_{\theta}(x)$ is a one-dimensional survival function.

One basic assumption that we used is that $\widehat{K}$ is a strict Archimedean Kendall distribution (i.e. condition (56) holds) so that there exists a unique Archimedean copula in the equivalence class determined by $\widehat{K}$. This yields that $\overline{G}_{\theta}(x)$ coincides with the marginal survival function of the Archimedean model $\overline{H}$ recalled above. Furthermore we note that $\varphi = \Upsilon_{t_0, K_{B_{\overline{H}}}}$, the generator of the Archimedean semicopula $B_{\overline{H}}$, is strict if and only if the convex generator $\widehat{\phi} = \Upsilon_{t_0, \widehat{K}}$ is such. This observation leads us to another source of incompatibility: B and $\widehat{K}$ are certainly not compatible whenever $\widehat{K}$ is a strict Archimedean Kendall distribution, and the generator φ_B is not strict.

The above arguments point out the interest of the fact that any equivalence class associated to a strict Archimedean Kendall distribution contains one and only one Archimedean copula (see [11] and the references therein).

The assumption that $\widehat{K}$ is a strict Archimedean Kendall distribution actually allows for the possibility that $\overline{F}_{\theta}(x, y)$ in (66) are solutions to the compatibility problem for infinite different values of the parameter θ. An example of this situation is provided by the family of Marshall-Olkin models, that is obtained by taking B as a Cuadras-Augé copula (see (30)) and $\widehat{K} = \mathcal{K}B = t - \widehat{\rho}\, t \log t$, with $\widehat{\rho} = \frac{2\alpha}{1+\alpha}$. In fact, in this case we have that all the exponential marginals are compatible (see Example 10). A further example is provided by the models considered in Example 11.

For the sake of simplicity, we assumed that $\overline{F}$ is strictly 1-decreasing all over $\mathbb{R}_+^2$, and this was actually used in Theorem 9. However there are examples (see Example 6) of solutions to the compatibility problem where this assumption does not hold. In Proposition 7, we strictly used the assumption that γ^{-1} is a differentiable bijection, i.e. that $\overline{G}$ admits a strictly positive density.

In the present paper we focused on models admitting "strict Archimedean" (upper orthant) Kendall distributions. As already remarked, in these cases a central role for the analysis of the compatibility problem is played by the unique Archimedean copula with the given strict Archimedean Kendall distribution. We can conjecture that, in the more general case when $\widehat{K}(t^-) = t$ for some t, a similar role would be played by the unique associative copula in the class of $\widehat{K}$.

Finally we mention an open problem that, in our setting, arises as a natural one: to characterize the semicopulas B that are "aging functions", i.e. that can be obtained by applying a continuous transformation as in the first line of (24) on a bivariate copula $\widehat{C}$. For the Archimedean case Corollary 5 gives at least a sufficient condition: any Archimedean semicopula with continuous, strict, and strictly decreasing generator φ is an aging function. This statement may be extended to the case of non-strict generators φ, as sketched in the concluding remark of Example 6.

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