On the moments of the modulus of continuity of Itô diffusions

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1 Introduction

A typical trajectory of standard Brownian motion is Hölder continuous of any order less than one half. If such a trajectory is evaluated at two different time points $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \leq h$ small, then the difference between the values at t_1 and t_2 is not greater than a multiple of $\sqrt{h \ln(\frac{1}{h})}$, where the proportionality factor depends on the trajectory (and on the time horizon, here equal to 1), but not on the choice of the time points. This is a consequence of Lévy's uniform modulus of continuity for Brownian motion.

If $f : [0,\infty) \to \mathbb{R}^d$ is a function and $T > 0$, then let us call *modulus of continuity* of f on the interval $[0,T]$ the function $\mathbf{w}_f(.,T)$ defined by

$$[0,\infty) \ni h \mapsto \mathbf{w}_f(h,T) := \sup_{t,s\in[0,T],|t-s|\leq h} |f(t) - f(s)| \ \in [0,\infty],$$

where $|.|$ denotes Euclidean distance of appropriate dimension. Recall that f is continuous on $[0,T]$ if and only if $\mathbf{w}_f(h,T)$ tends to zero as the mesh size h goes to zero. Let f be any function. Then, by definition, $\mathbf{w}_f(0,T) = 0$ for all $T > 0$, $\mathbf{w}_f(h_1,T) \leq \mathbf{w}_f(h_2,T)$ for all $0 \leq h_1 \leq h_2$, all $T > 0$, and $\mathbf{w}_f(h,T_1) \leq \mathbf{w}_f(h,T_2)$ for all $0 < T_1 \leq T_2$, $h \geq 0$.

The modulus of continuity of a stochastic process is a random element for any fixed mesh size $h > 0$. The following results show that the moments of the modulus of continuity of Brownian motion and, more generally, of Itô diffusions whose coefficients satisfy suitable integrability conditions allow for upper bounds of the form

$$\text{(1)} \qquad \mathbf{E}\left(\left(\mathbf{w}_Y(h,T)\right)^p\right) \ \leq \ C_p \left(h \ln(\tfrac{T}{h})\right)^{\frac{p}{2}} \quad \text{for all } h \in (0, \alpha \cdot T],$$

where C_p is a finite positive number depending on the moment p, the fraction $\alpha \in (0,1)$ and the coefficients of the Itô diffusion Y. For Brownian motion we will also show that the order in h and T of the p-th moments of the modulus of continuity as given in Inequality (1) is exact.

The result that Inequality (1) holds for some finite constant in case $T = 1$ and Y is a d-dimensional Itô diffusion with uniformly bounded coefficients can be found in the literature (see e.g. Ritter (1990), Slominski (1994), Pettersson (1995), Słomiński (2001), Victoir and Friz (2005)).

We too start from the special case of one-dimensional Brownian motion. We not only prove the existence of a finite constant C_p, but derive a bound in terms of the moment p and make explicit the dependence on the time horizon T, see Lemma 1 in Section 2. This allows us to prove Inequality (1) also for Itô diffusions with possibly unbounded coefficients, see Lemma 2 in Section 3.

In Section 4 we give an alternative proof of the fact that the p-th moment of the modulus of continuity of Brownian motion is of order $(h\ln(\frac{T}{h}))^{p/2}$ in the time horizon T and the mesh size h provided $\frac{h}{T}$ is small enough. The proof is based on results from the theory of extreme values.

In Section 5 we consider an application of the previous results to stochastic diffusion processes with delay. The latter processes have been used as models in many applications. A closed form solution can be achieved only in a few specific cases, and therefore the problem of the approximation of the solution naturally arises. The results obtained in this paper allow to get strong uniform convergence of a piece-wise linear Euler approximation scheme to the solution of a stochastic delay differential equation: the main result is an upper bound for the strong uniform rate of convergence for a piece-wise linear Euler-scheme. Similar and/or related results can be found in the literature, concerning the approximation for solution of ordinary stochastic differential equations (see e.g. Faure (1992), Hofmann et alii (2000), Müller-Gronbach (2002)) or for solutions of stochastic differential equations with reflection (see e.g. Slominski (1994), Pettersson (1995), Słomiński (2001)).

2 Upper bounds for one-dimensional Brownian motion

Lemma 1. *Let W be a standard one-dimensional Brownian motion living on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then there is a constant $K > 0$ such that for every $p \geq 1$, every $T > 0$,*

$$\mathbf{E}\left(\left(\mathbf{w}_W(h,T)\right)^p\right) \;\leq\; K^p \cdot p^{\frac{p}{2}} \cdot \left(h\ln(\tfrac{T}{h})\right)^{\frac{p}{2}} \quad \text{for all } h \in (0, \tfrac{T}{e}]. \tag{2}$$

The approach we take in proving the lemma should be compared to the derivation of Lévy's exact modulus of continuity for Brownian motion described in Exercise 2.4.8 of Stroock and Varadhan (1979). The main ingredient is an inequality due to Garsia, Rodemich, and Rumsey Jr., see Theorem 2.1.3 in Stroock and Varadhan (1979: p. 47) and the original paper by Garsia et al. (1970). Their inequality allows us to get an upper bound for $|W(t,\omega) - W(s,\omega)|^p$ in terms of T, the distance $|t-s|$ and $\xi(\omega)$, where ξ is a suitable random variable.

Proof of Lemma 1. Let $p \geq 1$. Let us first suppose that $T = 1$. The inequality for $T \neq 1$ will be derived from the self-similarity of Brownian motion. In order to prepare for the application of the Garsia-Rodemich-Rumsey lemma, define on $[0,\infty)$ the strictly increasing functions Ψ and μ by

$$\Psi(x) := \exp\left(\tfrac{x^2}{2}\right) - 1, \qquad \mu(x) := \sqrt{c\,x}, \qquad x \in [0,\infty),$$

where $c > p$. Clearly,

$$\Psi(0) = 0 = \mu(0), \quad \Psi^{-1}(y) = \sqrt{2\ln(y+1)} \quad \text{for all } y \geq 0, \quad d\mu(x) = \mu(dx) = \frac{\sqrt{c}}{2\sqrt{x}}\,dx.$$

Define the $\mathcal{F}$-measurable random variable $\xi = \xi_c$ with values in $[0,\infty]$ by letting

$$\xi(\omega) := \int_0^1 \int_0^1 \Psi\left(\frac{|W(t,\omega) - W(s,\omega)|}{\mu(|t-s|)}\right) ds\,dt, \quad \omega \in \Omega. \tag{3}$$

Notice that μ and ξ depend on the choice of the parameter c. Since $\frac{W(t)-W(s)}{\sqrt{t-s}}$ has standard normal distribution $N(0,1)$, we see that

$$\begin{aligned}
\mathbf{E}\left(\xi^p\right) &\leq \mathbf{E}\left((\xi+1)^p\right) = \mathbf{E}\left(\left(\int_0^1\int_0^1 \exp\left(\frac{|W(t)-W(s)|^2}{2c|t-s|}\right) ds\,dt\right)^p\right) \\
&\leq \mathbf{E}\left(\int_0^1\int_0^1 \exp\left(\frac{|W(t)-W(s)|^2}{2c|t-s|}\right)^p ds\,dt\right) \\
&= \int_0^1\int_0^1 \mathbf{E}\left(\exp\left(\frac{p}{2c}\left(\frac{W(t)-W(s)}{\sqrt{t-s}}\right)^2\right)\right) ds\,dt = \sqrt{\frac{c}{c-p}}\,.
\end{aligned}$$

In particular, $\xi(\omega) < \infty$ for $\mathbf{P}$-almost all $\omega \in \Omega$. The Garsia-Rodemich-Rumsey inequality now implies that for all $\omega \in \Omega$, all $t, s \in [0,1]$,

$$\big|W(t,\omega) - W(s,\omega)\big| \leq 8\int_0^{|t-s|} \Psi^{-1}\left(\frac{4\xi(\omega)}{x^2}\right)\mu(dx) = 8\int_0^{|t-s|} \sqrt{2\ln\left(\tfrac{4\xi(\omega)}{x^2}+1\right)}\,\frac{\sqrt{c}}{2\sqrt{x}}\,dx.$$

Notice that if $\xi(\omega) = \infty$ then the above inequality is trivially satisfied. With $h \in (0, \frac{1}{e}]$, we have

$$\sup_{t,s\in[0,1],|t-s|\leq h} \big|W(t,\omega) - W(s,\omega)\big| \leq 4\sqrt{2c}\int_0^h \sqrt{\ln\big(4\xi(\omega)+x^2\big) + 2\ln(\tfrac{1}{x})}\,\frac{dx}{\sqrt{x}} \tag{4}$$

$$\begin{aligned}
&\leq 4\sqrt{2c}\left(\sqrt{\ln(4\xi(\omega)+1)}\int_0^h \frac{dx}{\sqrt{x}} + \sqrt{2}\int_0^h\left(\sqrt{\ln(\tfrac{1}{x})} - \frac{1}{\sqrt{\ln(\frac{1}{x})}} + \frac{1}{\sqrt{\ln(\frac{1}{x})}}\right)\frac{dx}{\sqrt{x}}\right) \\
&= 8\sqrt{c}\left(\sqrt{2h}\sqrt{\ln(4\xi(\omega)+1)} + 2\sqrt{h\ln(\tfrac{1}{h})} + \int_0^h \frac{dx}{\sqrt{x\ln(\frac{1}{x})}}\right) \\
&\leq 8\sqrt{2c}\left(\sqrt{\ln(4\xi(\omega)+1)} + 2\sqrt{2}\right)\sqrt{h\ln(\tfrac{1}{h})} \\
&\leq 32\sqrt{c}\left(\sqrt{\xi(\omega)}+1\right)\sqrt{h\ln(\tfrac{1}{h})}.
\end{aligned}$$

Consequently, for all $h \in (0, \frac{1}{e}]$,

$$\mathbf{E}\left(\sup_{t,s\in[0,1],|t-s|\le h} |W(t)-W(s)|^p\right) \le 32^p \cdot c^{\frac{p}{2}} \cdot \mathbf{E}\left(\left(\sqrt{\xi}+1\right)^p\right)\left(h\ln(\tfrac{1}{h})\right)^{\frac{p}{2}}$$
$$\le 64^p \cdot c^{\frac{p}{2}} \cdot \left(\sqrt{\mathbf{E}(\xi^p)}+1\right)\left(h\ln(\tfrac{1}{h})\right)^{\frac{p}{2}}.$$

Choosing the parameter c to be equal to $\frac{9}{8}p$, we obtain for all $h \in (0, \frac{1}{e}]$,

$$\mathbf{E}\left(\left(\mathbf{w}_W(h,1)\right)^p\right) = \mathbf{E}\left(\sup_{t,s\in[0,1],|t-s|\le h} |W(t)-W(s)|^p\right) \tag{5}$$
$$\le (96/\sqrt{2})^p \cdot p^{\frac{p}{2}} \cdot (\sqrt{3}+1)\left(h\ln(\tfrac{1}{h})\right)^{\frac{p}{2}} < 192^p \cdot p^{\frac{p}{2}} \cdot \left(h\ln(\tfrac{1}{h})\right)^{\frac{p}{2}}.$$

The asserted inequality thus holds for any $K \ge 192$ in case $T = 1$. To derive the assertion for arbitrary $T > 0$, recall that by letting $\tilde{W}(t) := \frac{1}{\sqrt{T}} W(T\cdot t)$, $t \ge 0$, we obtain a second standard one-dimensional Wiener process $\tilde{W}$. Therefore,

$$\mathbf{E}\left(\left(\mathbf{w}_W(h,T)\right)^p\right) = \mathbf{E}\left(\sup_{t,s\in[0,T],|t-s|\le h} |W(t)-W(s)|^p\right)$$
$$= \mathbf{E}\left(\sup_{t,s\in[0,T],|t-s|\le h} \left|\sqrt{T}\tilde{W}(\tfrac{t}{T}) - \sqrt{T}\tilde{W}(\tfrac{s}{T})\right|^p\right)$$
$$= T^{\frac{p}{2}}\,\mathbf{E}\left(\sup_{t,s\in[0,1],|t-s|\le \frac{h}{T}} |\tilde{W}(t)-\tilde{W}(s)|^p\right) = T^{\frac{p}{2}}\,\mathbf{E}\left(\left(\mathbf{w}_{\tilde{W}}(\tfrac{h}{T},1)\right)^p\right).$$

Since W and $\tilde{W}$ have the same distribution, estimate (5) implies that for all $h \in (0, \frac{T}{e}]$,

$$\mathbf{E}\left(\left(\mathbf{w}_W(h,T)\right)^p\right) \le T^{\frac{p}{2}} \cdot 192^p \cdot p^{\frac{p}{2}} \cdot \left(\tfrac{h}{T}\ln(\tfrac{T}{h})\right)^{\frac{p}{2}},$$

which yields Inequality (2). □

The assertion of Lemma 1 remains valid with h from the interval $(0, \alpha\cdot T]$ for any $\alpha \in (\frac{1}{e}, 1)$, but the constants K such that Inequality (2) holds will be different.

From the chain of inequalities (4) in the proof of Lemma 1 it is easy to see that higher than polynomial moments of the Wiener modulus of continuity exist. More specifically, let $\xi = \xi_c$ be the random variable defined by (3), and let $\lambda > 0$. By the second but last line in (4) we have for all $h \in (0, \frac{T}{e}]$,

$$\mathbf{E}\left(\exp\left(\lambda(\mathbf{w}_W(h,T))^2\right)\right) \le \mathbf{E}\left(\left(e\cdot(4\xi+1)\right)^{2048c\,\lambda\, h\ln(\frac{1}{h})}\right).$$

The expectation on the right-hand side above is finite if $c > 2048c\,\lambda\, h\ln(\frac{1}{h})$, that is, the above exponential-quadratic moment exists if $\lambda\, h\ln(\frac{1}{h}) < \frac{1}{2048}$. The situation here should be compared to the case of standard Gaussian random variables. The constant $\frac{1}{2048}$ is, of course, not optimal.

3 Upper bounds for Itô diffusions

Lemma 2. *Let W be a d_1-dimensional Wiener process adapted to a filtration $(\mathcal{F}_t)$ satisfying the usual assumptions and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $Y = (Y^{(1)}, \ldots, Y^{(d)})^\mathsf{T}$ be an Itô diffusion of the form*

$$Y(t) \;=\; y_0 + \int_0^t b(s)ds \;+\; \int_0^t \sigma(s)dW(s), \quad t \geq 0,$$

where $y_0 \in \mathbb{R}^d$ and b, σ are $(\mathcal{F}_t)$-adapted processes with values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d_1}$, respectively. Let $T > 0$, and let $\zeta_1 = \zeta_{1,T}$, $\zeta_2 = \zeta_{2,T}$ be $[0,\infty]$-valued $\mathcal{F}_T$-measurable random variables such that for all $\omega \in \Omega$,

$$\zeta_1(\omega) \;\geq\; \max_{i\in\{1,\ldots,d\}} \operatorname{ess\,sup}_{t\in[0,T]} |b_i(t,\omega)|, \quad \zeta_2(\omega) \;\geq\; \max_{i\in\{1,\ldots,d\}, j\in\{1,\ldots,d_1\}} \operatorname{ess\,sup}_{t\in[0,T]} \sigma_{i,j}^2(t,\omega).$$

Let $p \geq 1$, and assume that

(H1) $\mathbf{E}(\zeta_1^p) < \infty$,

(H2) there is $\varepsilon > 0$ such that

$$\mathbf{E}\left(\zeta_2^{\frac{p}{2}+\varepsilon}\right) < \infty.$$

Then there is a finite constant $\tilde{C}_p > 0$ such that

$$\mathbf{E}\left(\left(\mathbf{w}_Y(h,T)\right)^p\right) \;\leq\; \tilde{C}_p \left(h \ln(\tfrac{T}{h})\right)^{\frac{p}{2}} \quad \textit{for all } h \in (0, \tfrac{T}{e}]. \tag{6}$$

Proof. With $T > 0$, $p \geq 1$, it holds for all $t, s \in [0,T]$,

$$\left|Y(t) - Y(s)\right|^p \quad\leq\quad d^{\frac{p}{2}} \left(\left|Y^{(1)}(t) - Y^{(1)}(s)\right|^p + \ldots + \left|Y^{(d)}(t) - Y^{(d)}(s)\right|^p\right),$$

and for the i-th component we have

$$\begin{aligned} \left|Y^{(i)}(t) - Y^{(i)}(s)\right|^p \quad &= \quad \Big|\int_s^t b_i(\tilde{s})d\tilde{s} \;+\; \sum_{j=1}^{d_1} \int_s^t \sigma_{ij}(\tilde{s})dW^j(\tilde{s})\Big|^p \\ \leq \quad (d_1{+}1)^p &\left(\zeta_1^p\,|t{-}s|^p \;+\; \sum_{j=1}^{d_1}\Big|\int_s^t \sigma_{ij}(\tilde{s})dW^j(\tilde{s})\Big|^p\right). \end{aligned}$$

Hence, by Hypothesis (H1), for $h \in (0, \frac{T}{e}]$,

$$\begin{aligned} \mathbf{E}\left(\left(\mathbf{w}_Y(h,T)\right)^p\right) \quad &= \quad \mathbf{E}\left(\sup_{t,s\in[0,T],|t-s|\leq h} \left|Y(t) - Y(s)\right|^p\right) \\ \leq \quad d^{\frac{p}{2}}(d_1{+}1)^p &\left(d\cdot \mathbf{E}\left(\zeta_1^p\right)\cdot h^p \;+\; \sum_{i=1}^{d}\sum_{j=1}^{d_1} \mathbf{E}\left(\sup_{t,s\in[0,T],|t-s|\leq h}\Big|\int_s^t \sigma_{ij}(\tilde{s})dW^j(\tilde{s})\Big|^p\right)\right). \end{aligned}$$

To prove the assertion, it is enough to show that the $d \cdot d_1$ expectations on the right-hand side of the last inequality are finite and of the right order in h. Let $i \in \{1, \ldots, d\}$, $j \in \{1, \ldots, d_1\}$, and define the one-dimensional process $M = M^{(i,j)}$ by

$$M(t) := \begin{cases} \int_0^t \sigma_{ij}(\tilde{s})\, dW^j(\tilde{s}) & \text{if } t \in [0,T], \\ M(T) + W^j(t) - W^j(T) & \text{if } t > T. \end{cases}$$

Since $\int_0^T \sigma_{ij}^2 ds < \infty$ $\mathbf{P}$-almost surely as a consequence of Hypothesis (H2), the process M is a (continuous) local martingale vanishing at zero and can be represented as a time-changed Brownian motion. More precisely, by the Dambis-Dubins-Schwarz theorem, see for instance Theorem 3.4.6 in Karatzas and Shreve (1991: p. 174), there is a standard one-dimensional Brownian motion $\tilde{W}$ living on $(\Omega, \mathcal{F}, \mathbf{P})$ such that, $\mathbf{P}$-almost surely,

$$M(t) \;=\; \tilde{W}(\langle M\rangle_t) \quad \text{for all } t \geq 0,$$

where $\langle M\rangle$ is the quadratic variation process associated with M, that is,

$$\langle M\rangle_t \;=\; \begin{cases} \int_0^t \sigma_{ij}^2(\tilde{s})\, d\tilde{s} & \text{if } t \in [0,T], \\ \int_0^T \sigma_{ij}^2(\tilde{s})\, d\tilde{s} + (t-T) & \text{if } t > T. \end{cases}$$

Consequently, it holds $\mathbf{P}$-almost surely that

$$\begin{aligned} &\sup_{t,s\in[0,T],|t-s|\leq h} \Big| \int_s^t \sigma_{ij}(\tilde{s}) dW^{(j)}(\tilde{s}) \Big|^p \;=\; \sup_{t,s\in[0,T],|t-s|\leq h} \big| M(t) - M(s) \big|^p \\ =\; &\sup_{t,s\in[0,T],|t-s|\leq h} \big| \tilde{W}(\langle M\rangle_t) - \tilde{W}(\langle M\rangle_s) \big|^p \\ \leq\; &\sup \Big\{ \big| \tilde{W}(u) - \tilde{W}(v) \big|^p \;;\; u, v \in [0, \langle M\rangle_T], |u-v| \leq \sup_{t\in[h,T]} \langle M\rangle_t - \langle M\rangle_{t-h} \Big\} \\ \leq\; &\big(\mathbf{w}_{\tilde{W}}(\delta_h, \tau) \big)^p, \end{aligned}$$

where $\tau := \langle M\rangle_T$ and δ_s, $s \in [0,T]$, are the random elements defined by

$$\delta_s(\omega) := \sup_{t\in[s,T]} \langle M\rangle_t(\omega) - \langle M\rangle_{t-s}(\omega), \quad \omega \in \Omega.$$

Notice that $\tau = \langle M\rangle_T = \delta_T$, and $\delta_h \leq \langle M\rangle_T$ for all $h \in [0,T]$. By the monotonicity of the modulus of continuity and Hypothesis (H2), for $\mathbf{P}$-almost all $\omega \in \Omega$ it holds that

$$\mathbf{w}_{\tilde{W}(\omega)}\big(\delta_h(\omega), \tau(\omega)\big) \;\leq\; \mathbf{w}_{\tilde{W}(\omega)}\big(\zeta_2(\omega)h, \tau(\omega)\big) \;\leq\; \mathbf{w}_{\tilde{W}(\omega)}\big(\zeta_2(\omega)h, \zeta_2(\omega)T\big)$$

Let $\alpha > 1$. Then, by Hölder's inequality and Lemma 1, for every $h \in (0, \frac{T}{e}]$ it holds that

$$
\begin{aligned}
\mathbf{E}\left((\mathbf{w}_{\tilde{W}}(\delta_h,\tau))^p\right) \quad &\leq \quad \mathbf{E}\left((\mathbf{w}_{\tilde{W}}(\zeta_2\, h, \zeta_2\, T))^p\right) \\
\leq \quad & \sum_{n=1}^{\infty} \mathbf{E}\left(\mathbf{1}_{\{\zeta_2 \in (n-1,n]\}} (\mathbf{w}_{\tilde{W}}(nh, nT))^p\right) \\
\leq \quad & \sum_{n=1}^{\infty} \mathbf{P}\{\zeta_2 \in (n-1,n]\}^{\frac{\alpha-1}{\alpha}}\, \mathbf{E}\left((\mathbf{w}_{\tilde{W}}(nh, nT))^{\alpha\cdot p}\right)^{\frac{1}{\alpha}} \\
\leq \quad & \left(\sum_{n=1}^{\infty} \mathbf{P}\{\zeta_2 \in (n-1,n]\}^{\frac{\alpha-1}{\alpha}} \cdot n^{\frac{p}{2}}\right) (\sqrt{\alpha}K)^p \cdot p^{\frac{p}{2}} \cdot h^{\frac{p}{2}} \cdot \ln\left(\tfrac{T}{h}\right)^{\frac{p}{2}} \\
\leq \quad & \left(\sum_{n=1}^{\infty} \mathbf{P}\{\zeta_2 \in (n-1,n]\} \cdot n^{\frac{(p+\varepsilon)\alpha}{2(\alpha-1)}}\right)^{\frac{\alpha-1}{\alpha}} \left(\sum_{n=1}^{\infty} n^{-\frac{\varepsilon\cdot\alpha}{2}}\right)^{\frac{1}{\alpha}} (\sqrt{\alpha}K)^p \cdot p^{\frac{p}{2}} \cdot h^{\frac{p}{2}} \cdot \ln\left(\tfrac{T}{h}\right)^{\frac{p}{2}} \\
\leq \quad & \mathbf{E}\left((\zeta_2+1)^{\frac{(p+\varepsilon)\alpha}{2(\alpha-1)}}\right)^{\frac{\alpha-1}{\alpha}} \left(\sum_{n=1}^{\infty} n^{-\frac{\varepsilon\cdot\alpha}{2}}\right)^{\frac{1}{\alpha}} (\sqrt{\alpha}K)^p \cdot p^{\frac{p}{2}} \cdot h^{\frac{p}{2}} \cdot \ln\left(\tfrac{T}{h}\right)^{\frac{p}{2}},
\end{aligned}
$$

where $\varepsilon > 0$ is as in Hypothesis (H2). If we choose α greater than $\max\{\frac{2}{\varepsilon}, \frac{p}{\varepsilon}+2\}$, then Hypothesis (H2) implies that the expectation and the infinite sums in the last two lines above are finite. The assertion follows. □

4 Moments of the modulus of continuity and extreme values theory

The main aim of this section is to prove that for T positive and for all $h \leq T$

$$\mathbf{E}\left[(\mathbf{w}_W(h,T))^p\right] \geq c(p)\left(h\,\ln\frac{2T}{h}\right)^{p/2}, \quad \text{for all } p > 0, \tag{7}$$

where $c(p)$ are strictly positive constant and can be explicitly computed.

The proof is based on classical results in extreme value theory and will give as a byproduct also the upper bound

$$\mathbf{E}\left[(\mathbf{w}_W(h,T))^p\right] \leq C(p)\left(h\,\ln\frac{2T}{h}\right)^{p/2}, \tag{8}$$

for all $0 < h/T \leq 1$.

Unfortunately the method does not allow to get the precise value of the constants $c(p)$ and $C(p)$, nor to compute exactly the interval in which the lower bound is valid.

We start this section by introducing another modulus of continuity, which we will call for brevity the Euler modulus of continuity:

Definition 1 (Euler modulus of continuity)**.** Let f be a deterministic function, the Euler modulus of continuity of mesh h in the interval $[0,T]$ is defined as

$$\mathbf{w}_f^E(h,T) := \sup_{t\in[0,T]} |f(t) - f(h\lfloor t/h\rfloor)|$$

With the above definition it is immediate to check that

$$\mathbf{w}_f^E(h,T) \le \mathbf{w}_f(h,T) \le 3\,\mathbf{w}_f^E(h,T), \tag{9}$$

and therefore we concentrate on the Euler modulus of continuity.

Lemma 3. *Let W be a standard Brownian motion and $T = nh$, with $n \in \mathbb{N}$. Let Z_i, $i \ge 1$, be a sequence of independent random variables with standard Gaussian distribution.*

Then, for every $p > 0$

$$\mathbf{E}\left[\max_{i=1,\cdots n} |Z_i|^p\right] h^{p/2} \le \mathbf{E}\left[\left(\mathbf{w}_W^E(h,T)\right)^p\right] \le 2\,\mathbf{E}\left[\max_{i=1,\cdots n} |Z_i|^p\right] h^{p/2}, \tag{10}$$

Proof. It is clear that, for any continuous function f,

$$\begin{aligned}\mathbf{w}_f^E(h,T) &= \max_{i=1,\ldots,n} \sup_{t\in[(i-1)h,ih]} |f(t) - f((i-1)\,h)| \\ &= \max_{i=1,\ldots,n} \max\left(\Delta_f(i,h),\, \Delta_{-f}(i,h)\right),\end{aligned}$$

where we have used the following notation

$$\Delta_f(i,h) = \sup\left\{f(t) - f((i-1)\,h);\ t \in [(i-1)h,\, ih]\right\}. \tag{11}$$

Furthermore one easily gets that

$$\big(\max_{i=1,\ldots,n} \Delta_f(i,h)\big)^p \le \left(\mathbf{w}_f^E(h,T)\right)^p \le \big(\max_{i=1,\ldots,n} \Delta_f(i,h)\big)^p + \big(\max_{i=1,\ldots,n} \Delta_{-f}(i,h)\big)^p.$$

Then, by the symmetry of Brownian motion, we immediately get

$$\mathbf{E}\left[\big(\max_{i=1,\cdots n} \Delta_W(i,h)\big)^p\right] \le \mathbf{E}\left[\left(\mathbf{w}_W^E(h,T)\right)^p\right] \le 2E\left[\big(\max_{i=1,\cdots n} \Delta_W(i,h)\big)^p\right]. \tag{12}$$

Then, by means of a rescaling argument, we immediately get that

$$\mathbf{E}\left[\big(\max_{i=1,\cdots n} \Delta_W(i,h)\big)^p\right] = h^{p/2}\,\mathbf{E}\left[\big(\max_{i=1,\cdots n} \Delta_W(i,1)\big)^p\right].$$

Finally, the well known fact that the random variables

$$\Delta_W(i,1) = \sup\left\{W(t) - W(i);\ t \in [i-1,i]\right\} \tag{13}$$

are independent, with the same distribution of $|Z_i|$, ends the proof of the inequalities (10). □

The next step is then to prove the following result

Lemma 4. *Let Z_i be a sequence of standard Gaussian independent random variables. Then, for every $p > 0$*

$$\lim_{n\to\infty} \mathbf{E}\left[\left(\frac{\max_{i=1,\cdots n}|Z_i|}{\sqrt{2\ln(2n)}}\right)^p\right] = 1. \tag{14}$$

Before giving the proof of the above result, we show how to use it for the proof of (7) and (8). Indeed by (14) we can find a nondecreasing sequence $c_0(p,n)$ and a nonincreasing $C_0(p,n)$, both converging to 1, such that

$$c_0(p,n)\ (2\ln(2n))^{p/2} \le \mathbf{E}\left[\max_{i=1,\cdots n}|Z_i|^p\right] \le C_0(p,n)\ (2\ln(2n))^{p/2}. \tag{15}$$

Clearly, for n sufficiently large, $c_0(p,n) > 0$. Therefore, taking into account that

$$\mathbf{w}_f^E(h,\, h\lfloor \tfrac{T}{h}\rfloor) \le \mathbf{w}_f^E(h,T) \le \mathbf{w}_f^E(h,\, h\lceil \tfrac{T}{h}\rceil),$$

and that, for $x \ge 1$, $\frac{1}{2}\ \ln(2x) \le \ln(2\lfloor x\rfloor) \le \ln(2\lceil x\rceil) \le 2\ \ln(2x)$

$$\frac{1}{2}\, c_0(p,n_0)\, \big(h\ \ln \tfrac{2T}{h}\big)^{p/2} \le \mathbf{E}\left[\big(\mathbf{w}_W^E(h,T)\big)^p\right] \le 2\, C_0(p,1)\, \big(h\ \ln \tfrac{2T}{h}\big)^{p/2}, \tag{16}$$

where the upper bound is valid for for all $h/T \le 1$, while the lower bound is valid for $h/T > 0$ sufficiently small. The inequalities in (7) and (8) then follow by (9).

The proof of Lemma 4 is based on some classical results of extreme values theory, concerning the sequence

$$M_n := \big(\max_{i=1,\cdots,n} X_i\big),$$

where $\{X_i,\ i \ge 1\}$ is a sequence of i.i.d. random variables with common distribution function F.

The first result we recall can be found in Gnedenko (1943), while the second one is due to Pickands (1968), and we recall it in a simplified version.

Proposition 1 (Gnedenko (1943))**.** *Assume that $F(x) < 1$ for every x and that the sequence β_n converges to infinity. Then the sequence $\frac{M_n}{\beta_n}$ converges to 1 in probability if and only if and for any $\epsilon > 0$*

$$\lim_{n\to\infty} n\Big[1 - F\big(\beta_n(1+\epsilon)\big)\Big] = 0$$
$$\lim_{n\to\infty} n\Big[1 - F\big(\beta_n(1-\epsilon)\big)\Big] = +\infty$$

Proposition 2 (Theorem 3.2 in Pickands (1968), Exercise 2.1.3 in Resnick (1987))**.** *Assume that $F(x) < 1$ for every x, and $E\big[(X_1)_-^p\big] < \infty$, with $p > 0$. Assume furthermore that the sequence $\frac{M_n}{\beta_n}$ converges to 1 in probability. Then*

$$\mathbf{E}\left[\left(\frac{M_n}{\beta_n}\right)^p\right] \longrightarrow_{n\to\infty} 1.$$

We are now ready to prove Lemma 4

Proof of Lemma 4. In our case $X_i = |Z_i|$, and, if $\Phi(x)$ denotes the distribution function of Z_i, then the proof of (14) follows by Proposition 2 since
(i) $1 - F(x) = 2\big(1 - \Phi(x)\big) < 1$, for every x,
(ii) $M_n/\sqrt{2\,\ln(2\,n)} \to 1$ in probability,
(iii) the negative part of X_1 is zero.

Properties (i) and (iii) are obvious, and we only need to check property (ii). Let $\varphi(x) = \Phi'(x)$ denote the density of Z_i, so that asymptotically $1 - \Phi(x) \sim \varphi(x)/x$. Then, for any $\alpha = 1 \pm \epsilon > 0$ we have

$$\begin{aligned} n\,\big(1 - F\big(\beta_n\,\alpha\big)\big) &= n\,2\big(1 - \Phi(\beta_n\,\alpha)\big) \sim 2\,n\,\frac{\varphi\big(\beta_n\,\alpha\big)}{\beta_n\,\alpha} \\ &= \sqrt{\frac{2}{\alpha^2\,\pi}}\,\frac{n}{\sqrt{\ln 2n}}\,\exp\{-\tfrac{1}{2}\,2\,\alpha^2\,\ln(2\,n)\} \\ &= \sqrt{\frac{2}{\alpha^2\,\pi}}\,\frac{n}{\sqrt{\ln 2n}}\,(2\,n)^{-\alpha^2} = C(\alpha)\,\frac{n^{1-\alpha^2}}{\sqrt{\ln 2n}}. \end{aligned}$$

The sequence $\frac{n^{1-\alpha^2}}{\sqrt{\ln 2n}}$ converge to zero or to infinity as $\alpha = 1 + \epsilon > 1$ or $\alpha = 1 - \epsilon < 1$, and therefore one can apply Proposition 1, in order to get the relative stability of M_n.

□

5 Euler approximation for SDDE and the modulus of continuity

In this section we assume that the process $\boldsymbol{X} = \big(X(t)\big)_{t\in[-\tau,T]}$ satisfies the stochastic delay differential equation on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$

$$(17)\qquad \begin{cases} X(t) = \eta(t), & -\tau \le t \le 0, \\ \\ X(t) = \eta(0) + \int_0^t \mu(u, \Pi_u X)du + \int_0^t \sigma(u, \Pi_u X)dW(u), & 0 \le t \le T, \end{cases}$$

where τ is a positive constant, $\big(\Pi_t X\big)_{t\in[0,T]}$ is a $C([-\tau, 0], \mathbb{R})$ random valued process defined by

$$\Pi_t X(s) = X(t + s) \qquad -\tau \le s \le 0,$$

$\boldsymbol{W} = \big(W(t)\big)_{t\in[0,T]}$ is a standard Brownian motion, and $\boldsymbol{\eta} = \big(\eta(s)\big)_{s\in[-\tau,0]}$ is a $C([-\tau, 0], \mathbb{R})$ valued random variable.

As an example the functions $\mu(t, \theta)$ and $\sigma(t, \theta)$ for $\theta \in C([-\tau, 0], \mathbb{R})$ can be taken of the form

$$(18)\qquad g\left(t, \max_{u\in[\tau_{i-1},\tau_i]} \theta(u);\; i = 1, \cdots, r\right)$$

where $-\tau = \tau_0 < \tau_1 < \cdots < \tau_r = 0$, or

$$g\left(t, \int_{-\tau}^{0} \psi_i(u, \theta(u))\gamma_i(du);\ i = 1, \cdots, r\right), \tag{19}$$

where γ_i are finite measures on $[-\tau, 0]$, and g and ψ_i are continuous functions.
By taking in (19) $\psi_i(u, x) = x$ for all i, and $\gamma_i(ds)$ in the set $\{\delta_{-\tau}(ds),\ \delta_0(ds)\}$, we recover the fixed time delay model

$$dX(t) = g_\mu\big(X(t),\ X(t-\tau)\big)dt + g_\sigma\big(X(t),\ X(t-\tau)\big)dW(t). \tag{20}$$

For the delay equation (17) we assume the following standing hypotheses:

(A1) η is a $\mathcal{F}_0$-measurable $C([-\tau, 0], \mathbb{R})$ valued random variable, with

$$E\left(\|\Pi_0 X\|^{2k}\right) = E\Big(\sup_{s\in[-\tau,0]} |\eta(s)|^{2k}\Big) < \infty, \qquad k = 1, 2$$

(A2) The functionals $\mu(t, \theta)$ and $\sigma(t, \theta)$ on $[0, T] \times C([-\tau, 0], \mathbb{R})$ are jointly globally continuous, Hölder in time and Lipschitz in space, i.e. for $\alpha \in (0, 1]$

$$|\mu(t, \theta) - \mu(t', \bar{\theta})|^2 + |\sigma(t, \theta) - \sigma(t', \bar{\theta})|^2 \le K\left(|t - t'|^{2\alpha} + \|\theta - \bar{\theta}\|^2\right). \tag{21}$$

and satisfy the growth condition

$$|\mu(t, \theta)|^2 + |\sigma(t, \theta)|^2 \le K\left(1 + \|\theta\|^2\right), \tag{22}$$

for some constant $K > 0$.

Conditions **(A1)** and **(A2)**, with $t' = t$ in (21), assure the existence and the uniqueness of the solution of equation (17) together with

$$E\Big[\sup_{u\in[0,T]} \|\Pi_u X\|^{2k}\Big] < \infty, \qquad k = 1, 2, \tag{23}$$

(see Mohammed (1984), Theorem II.2.1 and Lemma III.1.2 and Mohammed (1996), Theorem I.2). Note that, under condition **(A2)**, with $t' = t$ in (21), the existence and the uniqueness of the solution of equation (17) follow without condition **(A1)** (see Kallianpur and Mandal (2002)). The latter condition is used to obtain (23), which together with the sublinearity of μ and σ implies that conditions (H1) and (H2) of Lemma 2 are satisfied with $p = 2$ and $\varepsilon = 1$. As a consequence we can state that there exists a constant $\tilde{C}$ such that, for all h sufficiently small

$$E\left[\mathbf{w}_X^2(h; [0, T])\right] \le \tilde{C}\, h\, \ln(\tfrac{T}{h}) \tag{24}$$

The above upper bound is the key point in order to prove our rate of convergence result (see Proposition 3) for the approximation process, given by the piecewise linear

Euler-Maryuama scheme (see (25) and Remark 1, for the peculiarities due to the delay). Together with this scheme we also consider a continuous (diffusive) Euler-Maryuama scheme (see (27)).

First we consider the sequence of processes $\boldsymbol{X^n}=\big(X^n(t)\big)_{t\in[-\tau,T]}$ of the state process $\boldsymbol{X}=\big(X(t)\big)_{t\in[-\tau,T]}$ we consider is the piecewise linear Euler-Maruyama scheme, that is the linear interpolation of the Euler discretization scheme with step $\delta=\delta_n=T/n$, with $\tau=m\delta$ (for the sake of simplicity, we assume that T/τ is rational):

$$\begin{cases} X^n(\ell\delta)=\eta(\ell\delta), & -m\leq\ell\leq 0,\\ \\ X^n((\ell+1)\delta)=X^n(\ell\delta)+\mu(\ell\delta,\Pi_{\ell\delta}X^n)\delta & \\ \qquad\qquad +\sigma(\ell\delta,\Pi_{\ell\delta}X^n)\big[W((\ell+1)\delta)-W(\ell\delta)\big], & 0\leq\ell\leq n-1. \end{cases} \tag{25}$$

With this approximation for the process $\boldsymbol{X}$ we can consider the piecewise-constant $C([-\tau,0],\mathbb{R})$-valued process $\big(\Pi_{\lfloor t/\delta\rfloor\cdot\delta}X^n\big)_{t\in[0,T]}$ as an approximation of the $C([-\tau,0],\mathbb{R})$-valued process $\big(\Pi_t X\big)_{t\in[0,T]}$.

Note that the process $\boldsymbol{X^n}$ is not adapted nor Markov, while the process $\big(\Pi_{\lfloor t/\delta\rfloor\cdot\delta}X^n\big)_{t\in[0,T]}$ is an adapted process.

Remark 1. Note that, unlike in the finite dimensional Euler scheme, the interpolation has to be performed at every step in order to evaluate $\Pi_{\ell\delta}X^n$. Nevertheless it is clear that

$$\big\{\big(X^n(\ell\delta),X^n((\ell-1)\delta),\cdots,X^n((\ell-m)\delta)\big)\big\}_{0\leq\ell\leq n} \tag{26}$$

is an $(m+1)$-dimensional Markov chain, and for $t\in[\ell\delta,(\ell+1)\delta]$, $0\leq\ell\leq n-1$

$$\begin{aligned} X^n(t)=X^n(\ell\delta)&+\mu(\ell\delta,\Pi_{\ell\delta}X^n)(t-\ell\delta)\\ &+\sigma(\ell\delta,\Pi_{\ell\delta}X^n)\big[W((\ell+1)\delta)-W(\ell\delta)\big](t-\ell\delta)/\delta, \end{aligned} \tag{27}$$

with $X^n(0)=\eta(0)$.

When the process is given by fixed time delay model (20), the linear interpolation, in the above discrete Euler-Maruyama scheme, is not needed in order to compute the sequence $\{X^n(\ell\delta)\}_{0\leq\ell\leq n}$. Indeed in this case

$$\begin{aligned} X^n((\ell+1)\delta)=X^n(\ell\delta)&+g_a\big(\ell\delta,X^n(\ell\delta),X^n((\ell-m)\delta)\big)\delta\\ &+g_b\big(\ell\delta,X^n(\ell\delta),X^n((\ell-m)\delta)\big)\big[W((\ell+1)\delta)-W(\ell\delta)\big], \end{aligned}$$

with $X^n(0)=\eta(0)$, and therefore the computation of the discrete Markov chain (26) is much simpler.

We consider also the continuous Euler-Maruyama scheme, i.e. the diffusion processes $\boldsymbol{Z^n} = \big(Z^n(t)\big)_{t\in[0,T]}$ where, setting as above $\delta = \delta_n = T/n$,

$$
\begin{cases}
Z^n(t) := \eta(t) & -\tau \le t \le 0, \\
Z^n(t) := \eta(0) + \displaystyle\int_0^t \mu(\delta \cdot \lfloor s/\delta \rfloor, \Pi_{\delta\cdot\lfloor s/\delta\rfloor} X^n) ds & \\
\qquad\qquad + \displaystyle\int_0^t \sigma(\delta \cdot \lfloor s/\delta \rfloor, \Pi_{\delta\cdot\lfloor s/\delta\rfloor} X^n) dW(s), & 0 \le t \le T,
\end{cases}
\tag{28}
$$

which can be considered as intermediate approximation processes for the state $\boldsymbol{X}$.

The aim of this section is to compute an upper bound for the rate of convergence of our scheme under the further hypotheses that $1/2 \le \alpha \le 1$ in (21), i.e. to prove the following Proposition.

Proposition 3. *Assume that conditions* **(A1)***, for* $k = 1, 2$*, and* **(A2)***, with* $1/2 \le \alpha \le 1$*, are satisfied, and furthermore that the initial condition* η *satisfies*

$$
E\big[\mathbf{w}^2_\eta(\delta;[-\tau,0])\big] \le C_\eta\, \delta\, \log(\tfrac{1}{\delta}). \tag{29}
$$

Then there exist constants C_X *and* C_Z *such that*

$$
E\left[\sup_{t\in[0,T]} \big\|\Pi_{\delta\cdot\lfloor t/\delta\rfloor} X^n - \Pi_t X\big\|^2\right] \le C_X\, \frac{\log n}{n}, \tag{30}
$$

$$
E\left[\sup_{u\in[0,T]} \|\Pi_u Z^n - \Pi_u X\|^2\right] \le C_Z\, \frac{\log n}{n} \tag{31}
$$

The above result generalizes Proposition 4.2 in Calzolari et al. (2007), since it does not require the boundedness of the coefficients μ and σ (there denote by a and b, respectively). We point out that Proposition 3 requires that condition **(A1)**, holds also for $k = 2$, while when the diffusion coefficient $|\sigma(t,\theta)|$ is bounded above by a constant, then $k = 1$ is sufficient.

By (24), the above result implies that a similar upper bound holds also for the expectation of $\sup_{u\in[-\tau,T]} \|X^n(t) - X(t)\|^2$. In this respect Proposition 3 can be considered also as a generalization to SDDE the old result, due to Faure (1992), for piecewise linear approximations of solutions of ordinary stochastic differential equations (see Bouleau and Lépingle (1994)).

The proof of Proposition 3 can be achieved exactly as the proof of the above quoted Proposition 4.2 (see Section 5 of Calzolari et al. (2007)). Note that the boundedness condition on the coefficients was used only to get the upper bound (24). For the ease of the reader, though unnecessary, we now sketch briefly the proof.

The processes $\boldsymbol{Z^n}$ have the property that

$$Z^n(\ell\delta) = X^n(\ell\delta), \qquad \text{for } \ell \geq -m, \tag{32}$$

as can be easily seen. The latter property implies that the piecewise linear interpolation of $\boldsymbol{Z^n}$ coincides with $\boldsymbol{X^n}$. In other words

$$P^\delta Z^n(s) = X^n(s), \qquad \text{for } s \in [-\tau, T], \tag{33}$$

where P^δ denotes the operator which gives the linear interpolation of a function $\big(f(s)\big)_{s\in[-\tau,T]}$, with step δ, i.e.

$$P^\delta f(v) = \lambda(v)\, f(\delta \cdot \lfloor v/\delta \rfloor + \delta) + (1-\lambda(v))\, f(\delta \cdot \lfloor v/\delta \rfloor)$$

with $\lambda(v) = v/\delta - \lfloor v/\delta \rfloor$. By rewriting $f(v) = \lambda(v)\, f(v) + (1-\lambda(v))\, f(v)$, we have

$$\begin{aligned} \left|P^\delta f(v) - f(v)\right| &= \left|\lambda(v)\ [f(\delta \cdot \lfloor v/\delta \rfloor + \delta) - f(v)] + (1-\lambda(v))\ [f(\delta \cdot \lfloor v/\delta \rfloor) - f(v)]\,\right| \\ &\leq \lambda(v)\, \mathbf{w}_f(\delta) + (1-\lambda(v))\mathbf{w}_f(\delta) = \mathbf{w}_f(\delta). \end{aligned}$$

Furthermore, taking into account (33), we have

$$\begin{aligned} &\sup_{t\in[0,T]} \|\Pi_t X^n - \Pi_t P^\delta X\| = \sup_{k:k\delta\in[-\tau,T]} |X^n(k\delta) - X(k\delta)| \\ &= \sup_{k:k\delta\in[-\tau,T]} |Z^n(k\delta) - X(k\delta)| \leq \sup_{t\in[-\tau,T]} |Z^n(t) - X(t)| = \sup_{t\in[0,T]} \|\Pi_t Z^n - \Pi_t X\|. \end{aligned}$$

Therefore one can prove immediately that

$$\begin{aligned} &\left\|\Pi_{\delta\cdot\lfloor t/\delta\rfloor} X^n - \Pi_t X\right\|^2 \\ &\leq 2\left\|\Pi_{\delta\cdot\lfloor t/\delta\rfloor} X^n - \Pi_{\delta\cdot\lfloor t/\delta\rfloor} P^\delta X\right\|^2 + 2\left\|\Pi_{\delta\cdot\lfloor t/\delta\rfloor} P^\delta X - \Pi_t X\right\|^2 \\ &\leq 2 \sup_{t\in[0,T]} \|\Pi_t Z^n - \Pi_t X\|^2 + 2\,\mathbf{w}_X^2(\delta, [-\tau, T]). \end{aligned} \tag{34}$$

Then the result follows by the inequality

$$E\left[\sup_{u\in[0,T]} \|\Pi_u Z^n - \Pi_u X\|^2\right] \leq C_1(T)\left(E\left[\mathbf{w}_X^2(\delta; [-\tau, T])\right] + \delta^{2\alpha}\right), \tag{35}$$

(see (5.12) in the proof of Lemma 5.3 in Calzolari et al. (2007)), by (24), and by assumption on the moments of the modulus of continuity of η.

We end this section by observing that, on the one hand inequality (31) is far from being optimal and the constant in (30) could be improved, while on the other hand the rate of convergence in (30) cannot be improved. Indeed consider the toy model with $\mu = 0$ and $\sigma = 1$, with $\eta(s) = 0$ for all $s \in [-\tau, 0]$, i.e. the case $X(t) = W(t)$, for all $t \in [0, T]$. In this case $Z^n(t) = X(t)$, for all $t \in [-\tau, T]$ so that the continuous Euler approximation is useless, while, $X^n(t) = P^\delta X(t)$, for all $t \in [-\tau, T]$, and therefore, on the one hand

$$\sup_{t\in[-\tau,T]} |X^n(t) - X(t)| = \sup_{t\in[0,T]} |P^\delta W(t) - W(t)| \leq \mathbf{w}_W^E(\delta; [0,T]),$$

while on the other hand

$$\begin{aligned}
&\sup_{t\in[-\tau,T]} |X^n(t) - X(t)| = \sup_{t\in[0,T]} |P^\delta W(t) - W(t)| \\
&\geq \max_{1\leq i\leq n; i \text{ even}} \left| \tfrac{1}{2} \left[W(i\delta + \delta) - W(\tfrac{\delta}{2} + i\delta)\right] + \tfrac{1}{2} \left[W(i\delta) - W(\tfrac{\delta}{2} + i\delta)\right] \right| \\
&= \max_{1\leq 2\ell\leq n} |Y_\ell|,
\end{aligned}$$

where

$$Y_\ell := \tfrac{1}{2} \left[W(2\ell\delta + \delta) - W(\tfrac{\delta}{2} + 2\ell\delta)\right] + \tfrac{1}{2} \left[W(2\ell\delta) - W(\tfrac{\delta}{2} + 2\ell\delta)\right]$$

are independent Gaussian random variables with mean zero and variance $\frac{1}{4}\frac{\delta}{2} + \frac{1}{4}\frac{\delta}{2} = \frac{\delta}{4}$.

With the same extreme value technique used in the previous section one can prove that $\max_{1\leq 2\ell\leq n} |Y_\ell|/\sqrt{2 \ln n}$ converge in probability to 1, and that the same happens to the moments. Therefore we can get a result of the following kind

$$\mathbf{E}\left(\|P^\delta W - W\|^p\right) = O\big((\delta \ln \tfrac{T}{\delta})^{p/2}\big).$$

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