The conditional law of a continuous time random walk with respect to its local time: explicit form and approximations*

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Abstract

A filtering problem is considered in the case when the state process is a continuous time random walk X_t and the observation process is its local time L_t. An explicit construction of the filter is given. This construction is then applied to a suitable approximation of a Brownian motion and to an asymptotically symmetric M/M/1 queueing model. In both these cases it has been possible to give a convergence result for the respective filters.

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1 Introduction

The kind of problems we are interested in arises from the following situation. Suppose that in a queue we can observe, up to time t, whether the queue is busy or idle, but we cannot observe the size of the queue, so that the observation process is the total time the queue has spent in 0, i.e. the so called *idle time* (see Prabhu [14]). Then the problem is to evaluate the size of the queue at time t, given this information, i.e. to compute the conditional law (or the filter) of the queue given the observation process up to time t. In the setup of heavy traffic limit, the rescaled queue converges to a reflected Brownian motion and the observation process converges to its local time. Then the limit model can be constructed as $(W_t + \Lambda_t, \Lambda_t)$, where W_t is a Brownian motion and $\Lambda_t = -\inf_{0 \le s \le t}(W_s)$ is the corresponding local time in the sense of Skorohod definition. In the limit model the problem is the computation of the conditional law of a reflected Brownian motion $W_t + \Lambda_t$ when the observation process is its local time Λ_t, i.e. the computation of the filter $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^{\Lambda}\big]$, for g in a sufficiently large class of functions.
A first problem is to find the exact expression for the filters, both for the filter of the limit model of reflected Brownian motion, and for the filter of the rescaled queue model. A second problem concerns the convergence of the latter to $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^{\Lambda}\big]$.
The filter of the limit Brownian motion model is derived in G. Nappo, B. Torti [13] (Sections 4 and 6), where it is obtained by means of a suitable sequence of processes Λ^n approximating the observation process Λ. Each process Λ^n is proportional to a counting process, and therefore the nonlinear filtering techniques for counting processes are used. The filter can also be derived by means of the Azéma martingale, and this derivation is shortly discussed in [13]. For sake of completeness we recall its explicit expression.

Theorem 1.1. *Let W_t be a Brownian motion with diffusion coefficient a^2 and drift $c \in \mathbb{R}$ and let Λ_t be its local time. Let g be a bounded measurable function.*

Denote by

$$\Pi(g)(s,l) = \int_0^{\infty} g\left(-l + y\sqrt{s}\right) y \exp\left(-\frac{y^2}{2}\right) dy \quad s \ge 0, \tag{1}$$

or, equivalently,

$$\Pi(g)(s,l) = E\Big[g(-l + W_s^*)/\Lambda_s^* = 0\Big], \tag{2}$$

where W^* is any standard Brownian motion and Λ^* is its local time.

Then

$$\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^{\Lambda}\big] = \left.\frac{\Pi\left(g(\cdot)\exp(\frac{c}{a^2}\cdot)\right)(a^2 s, l)}{\Pi\left(\exp(\frac{c}{a^2}\cdot)\right)(a^2 s, l)}\right|_{s=\zeta_t, l=\Lambda_t}, \tag{3}$$

and

$$\hat{\pi}_t(g) = E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^{\Lambda}\big] = \left.\frac{\Pi\left(g(\cdot)\exp(\frac{c}{a^2}\cdot)\right)(a^2 s, 0)}{\Pi\left(\exp(\frac{c}{a^2}\cdot)\right)(a^2 s, 0)}\right|_{s=\zeta_t}, \tag{4}$$

where

$\mathcal{F}_t^{\Lambda}$ *is the history generated by Λ_u up to time t;*

ζ_t *is the elapsed time from the last visit to 0 for the process $W_t + \Lambda_t$, i.e.*

$$\zeta_t = \gamma_t^0(W + \Lambda) = \gamma_t(\Lambda), \tag{5}$$

where

$$\gamma_t^0(x) = t - \sup\{s < t : x_s = 0\}. \tag{6}$$

$$\gamma_t(x) = t - \sup\{s < t : x_s < x_t\}. \tag{7}$$

In this paper we start by considering a somehow simplified version of the motivating problem: we consider a continuous time random walk Y_t and its conditional law w.r.t. its local time L_u up to time t. Indeed, up to a reflection, several queueing models and birth and death processes can be represented just by choosing the law of a continuous time random walk.

More precisely, we start by deriving an explicit expression for the filter under the assumption that the process Y_t can be decomposed as $Y_t = V_{Z_t}$, where Z_t is a renewal process, V_k is a discrete time random walk, with Z_t and V_k mutually independent. It turns out that the filter of Y_t w.r.t. L_u up to time t can be expressed as a deterministic function of L_t and $\zeta_t = \gamma_t(L)$, where γ_t is the deterministic functional of L_u up to time t defined in (7).

Then, we consider two situations when a sequence (X_t^n, L_t^n), of rescaled random walks, converges to a Brownian motion and its local time (W_t, Λ_t), and we deal with the problem whether the corresponding filter converges to the filter of the limit. As discussed at the end of Section 3, the convergence of the previous systems is equivalent to the convergence of the system with the reflected random walk $(X_t^n + L_t^n, L_t^n)$ to the system with the reflected Brownian motion $(W_t + \Lambda_t, \Lambda_t)$, and the convergence of the filters for the rescaled random walks is equivalent to the convergence of the filters for the reflected random walks.

In particular, for the models we propose, we study two different kind of problems. The first is a theoretical one, and concerns the problem of checking if the weak limit of the filter is the the filter of the limit model. In turn, this question give rise to another problem, more interesting from a computational point of view, regarding to find a good approximation of the filter of the discrete system. If the filter, as functional of the observation process, satisfies some regularity properties both in the discrete case and in the limit case, we can think to use the limit functional, computed on the true observation, in order to give a manageable approximation of the filter. This situation will be analyzed in Section 6.
These problems have been studied in more general situations by many authors, among which we recall in particular Bhatt et al. in [1] and Goggin ([9], [10]). Most of these results concern diffusive models and do not apply to our case. Moreover, usually the applications concern the problem to approximate a given signal/observation process with a suitably chosen sequence of signal/observation processes so that the corresponding sequence of filters converges to the the filter of the original process. We start from a different point of view: the sequence of processes is given and the problem is to show the convergence of the filters in the sense specified above.
The above problem of convergence of the sequence of filters is not a trivial problem as even strong convergence of random variables does not imply convergence of the conditional laws. This is clearly explained by the following simple and illuminating example (see E. Goggin [9]).
Let ξ be a real random variable, and $(\xi_n, \eta_n) = (\xi, \xi/n)$. Then (ξ_n, η_n) converges strongly to (ξ, η), with $\eta = 0$. Nevertheless, for any measurable function g, $E(g(\xi_n)/\eta_n) = g(\xi)$, so that the conditional law of ξ_n given η_n is the measure concentrated in $\xi(\omega)$, while $E(g(\xi)/\eta) = E(g(\xi))$, so that the conditional law of ξ given η coincides with the (deterministic) law $P \circ \xi^{-1}$ of ξ.

In the example above, although the sequence of the conditional laws of ξ_n given η_n does not converge to the conditional law of the limit, it is a constant sequence and therefore is a converging sequence. This is not surprising indeed in the light of the next general result, which is a slight generalization of a result of E. Goggin [9] (proof of Theorem 2.1, Step 1): one has only to replace the sequence of $\sigma-$algebras used in [9] with a general sequence.

Lemma 1.2. *Let R_n be a sequence of random variables with values in a Polish space, let $\mathcal{H}^n$ be a sequence of $\sigma-$algebras, let α^n be a regular version of the conditional distribution of R_n given $\mathcal{H}^n$. If $\{R_n, n \in \mathbb{N}\}$ is tight, then $\{\alpha^n, n \in \mathbb{N}\}$ is tight.*

Then, as far as weak convergence is concerned, for the systems converging to a Brownian motion, the main problem is to check whether the limit points of the sequence of filters are all equal to the filter of a standard Brownian motion w.r.t. its local time.

The first model we consider is a non-markovian queueing model arising when a Brownian motion W is approximated by a sequence of continuous time random walks W^n, obtained with a suitable interpolation procedure. The approximation scheme we propose for W follows some of the ideas used in [10] to study a filter approximation problem in diffusive models, and is related with the approximation scheme used in [13]. In addition the processes W^n can be identified with our rescaled model. In this particular case we also get a strong convergence result for the approximating filter.

The second model we consider is the case when the renewal process Z_t is a Poisson process, so that the reflected random walk is an M/M/1 queue. We consider the case when the drift of the limit Brownian motion is zero, and we find, under suitable conditions, the weak limit of the corresponding filter by a limiting procedure. It is important to note that the observation process is the local time of the random walk, which is not the cumulative time spent in 0, as in the original motivating problem. The last Section outlines how this problem can be dealt with.

2 The model

Fix a probability space $(\Omega, \mathcal{F}, P)$ and consider on it two sequences of random variables $\{T_j, j \geq 1\}$, $\{U_j, j \geq 1\}$ satisfying the assumptions

H1 the sequences $\{T_j, j \geq 1\}$ $\{U_j, j \geq 1\}$ are mutually independent;

H2 the random variables $\{T_j, j \geq 1\}$ are non-negative, mutually independent, with identical distribution function F;

H3 $\{U_j, j \geq 1\}$ is a sequence of i.i.d. random variables such that

$$P(U_j = 1) = p \quad P(U_j = -1) = 1 - p = q, \quad p \in (0, 1).$$

Put

$$\tau_0 = 0, \ \tau_k = \sum_{j=1}^{k} T_j \quad \text{for } k \geq 1,$$

consider the renewal process

$$Z_t = \sum_{j=1}^{\infty} \mathbb{I}\left(\tau_j \leq t\right)$$

and the random walk $\{V_j, j \geq 0\}$ defined by

$$V_0 = 0, \ V_j = V_{j-1} + U_j, j \geq 1.$$

Finally, consider the marked point process

$$Y_t = V_{Z_t} = \sum_{j=1}^{\infty} U_j \mathbb{I}\left(\tau_j \le t\right) = \sum_{j \le Z_t} U_j. \tag{8}$$

The solution of the Skorohod problem [1] for the process Y_t is given by the pair $(Y_t + L_t, L_t)$ where L_t is the *local time at level 0* for the process Y_t, that is

$$L_t = \ell_t(Y),$$

where

$$\ell : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty) \quad \text{s.t.} \quad \ell_t(x) = -\inf_{s \le t} x(s) \wedge 0. \tag{9}$$

It is easy to see that, for our model, L_t admits the representation

$$L_t = \sum_{j=1}^{\infty} \mathbb{I}\left(\sigma_j \le t\right), \tag{10}$$

where $\{\sigma_j, j \ge 0\}$ is the sequence of its jump times. We can see that $\{\sigma_j, j \ge 0\}$ is the subsequence of $\{\tau_j, j \ge 0\}$ defined by $\sigma_0 = 0$ and

$$\sigma_j = \inf\left\{\tau_k \quad \text{s.t.} \quad Y_{\tau_k} \le -j\right\} = \inf\left\{t > 0 \quad \text{s.t.} \quad Y_t \le -j\right\} \quad \text{for } j \ge 1.$$

Set $\mathcal{G}_t = \mathcal{F}_t^L = \sigma\left\{L_s, s \le t\right\}$ and $\mathcal{F}_t = \mathcal{F}_t^Y = \sigma\left\{Y_s, s \le t\right\}$. Obviously $\{\sigma_j, j \ge 0\}$ are stopping times w.r.t. both the histories $\mathcal{G}_t$ and $\mathcal{F}_t$.

A first representation for the filter of Y_t given $\mathcal{G}_t$ is stated in the following Proposition, where we use a condition, which is implied by the above conditions **H1**, **H2**, **H3**.

H0 The $\mathbb{R}^+ \times \{+1, -1\}-$valued random variables (T_j, U_j), for $j \ge 1$, form a sequence of identically distributed and mutually independent random variables.

Proposition 2.1. *Assume* **H0**. *Then the conditional law of Y_t given $\mathcal{G}_t$ admits the following representation P-a.s.*

$$E\left[g(Y_t)/\mathcal{G}_t\right] = \sum_{j=0}^{\infty} \frac{E\left[g(-j + Y_{s+\sigma_j} - Y_{\sigma_j})\mathbb{I}\left(S_{j+1} > s\right)\right]_{s=t-\sigma_j}}{E\left[\mathbb{I}(S_{j+1} > s)\right]_{s=t-\sigma_j}} \mathbb{I}\left\{\sigma_j \le t < \sigma_{j+1}\right\}, \tag{11}$$

where $S_{j+1} = \sigma_{j+1} - \sigma_j, j \ge 0$.

Proof. The thesis can be proved by a slight modification of the argument in [13], Proposition 3.1. □

Remark 2.2. *The main ingredient in the proof of the previous Proposition is that condition* **H0** *implies that the process $Y_{s+\sigma_j} - Y_{\sigma_j}$ is independent of $\mathcal{G}_{\sigma_j}$ and is equal in law to the process Y_s. In its turn the last property implies that (11) admits the representation*

[1] For any $x \in D_{\mathbb{R}}[0,\infty)$, with $x(0) \ge 0$ the pair $(z, v) = (x + \ell(x), \ell(x))$ is the unique solution of the Skorohod problem, i.e. is the unique pair of functions satisfying $z(t) = x(t) + v(t)$, and such that $z(t) \ge 0$, for all $t \ge 0$, $v(0) = 0$, v is nondecreasing and increases only when $z(t) = 0$.

$$E\big[g(Y_t)/\mathcal{G}_t\big] = \sum_{j=0}^{\infty} \frac{E\Big[g(-j+Y^*_s)\mathbb{I}\,(\sigma^*_1 > s)\Big]_{s=t-\sigma_j}}{E\Big[\mathbb{I}(\sigma^*_1 > s)\Big]_{s=t-\sigma_j}} \mathbb{I}\big\{\sigma_j \le t < \sigma_{j+1}\big\}, \tag{12}$$

where Y^*_s is another birth and death process, with the same law as Y_t, defined by the rule (8) starting from a sequence $\{(T^*_j, U^*_j), \ge 1\}$ satisfying condition **H0***, and σ^*_1 is its first visit time to the state -1.*
Then the problem reduces to the computation of

$$\frac{E\Big[g(-j+Y^*_s)\mathbb{I}\,(\sigma^*_1 > s)\Big]}{E\Big[\mathbb{I}(\sigma^*_1 > s)\Big]} = \frac{E\Big[g(-j+Y_s)\mathbb{I}\,(\sigma_1 > s)\Big]}{E\Big[\mathbb{I}(\sigma_1 > s)\Big]},$$

for any $s \ge 0$.

The previous Remark implies also that the process L_t is a renewal process, with inter-arrival times $\{S_h, h \ge 1\}$. It is possible to represent them in terms of the sequences $\{T_i, i \ge 1\}$ and $\{M_i, i \ge 0\}$, where the sequence $\{M_i, i \ge 0\}$ is recursively defined by the rule

$$\begin{cases} M_0 = 0 \\ M_i = \inf\big\{k \ge 0 : V_{M_0+\ldots+M_{i-1}+k} - V_{M_0+\ldots+M_{i-1}} = -1\big\}. \end{cases} \tag{13}$$

Then

$$\sigma_0 = 0, \quad \sigma_h = \sum_{i=1}^{M_1+\ldots+M_h} T_i = \tau_{M_1+\ldots+M_h}, \qquad h \ge 1,$$

and

$$S_h = \sigma_h - \sigma_{h-1} = \sum_{i=M_1+\ldots+M_{h-1}+1}^{M_1+\ldots+M_h} T_i, \qquad h \ge 1.$$

Under Condition **H0** (and therefore under Conditions **H2** and **H3** for a suitable F and a suitable value of p) the sequence $\{M_i, i \ge 1\}$ is a sequence of i.i.d. random variables. Moreover, under Condition **H1**, it is obvious that $\{M_i, i \ge 1\}$ and $\{T_i, i \ge 1\}$ are mutually independent. Next result provides an explicit expression for the terms of the sum in (11).

Proposition 2.3. *Assume conditions* **H1**, **H2**, **H3**. *Then the filter $E\big[g(Y_t)/\mathcal{G}_t\big]$ admits the representation*

$$E\big[g(Y_t)/\mathcal{G}_t\big] = $$
$$= \sum_{j=0}^{\infty} \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\,(M_1 \ge k)\, g(-j+V_{k-1})\Big]\,(F_{k-1}(t-\sigma_j) - F_k(t-\sigma_j))}{\sum_{m=1}^{\infty} P\big(M_1 = m\big)\,(1 - F_m(t-\sigma_j))} \mathbb{I}\big\{\sigma_j \le t < \sigma_{j+1}\big\} \tag{14}$$

where F_k is the distribution function of τ_k i e. $F_k = F^{*k}$, the k-fold convolution of F.

Proof. As explained in Remark 2.2 we need only to compute $E\Big[\mathbb{I}(\sigma_1 > s)\Big]$, and $E\Big[g(-j+Y_s)\mathbb{I}\,(\sigma_1 > s)\Big]$, for all $j \ge 0$.
Taking into account the equality $\sigma_1 = \tau_{M_1}$, and the independence of the sequences $\{T_i, i \ge 1\}$ and $\{M_i, i \ge 0\}$

$$E\big[\mathbb{I}\,(\sigma_1 > s)\,\big] = E\big[\mathbb{I}\,(\tau_{M_1} > s)\,\big] = E\left[\sum_{m=1}^{\infty} \mathbb{I}\,(\tau_m > s)\,\mathbb{I}\,(M_1 = m)\right] = \sum_{m=j}^{\infty} P\big(M_1 = m\big)\big(1 - F_m(s)\big).$$

Finally, taking into account also (8),

$$\begin{aligned}
E\Big[g(-j+Y_s)\mathbb{I}\,(\sigma_1 > s)\,\Big] =& E\left[g\left(-j+V_{Z_s}\right)\mathbb{I}\,(\tau_{M_1} > s)\right] = E\left[\sum_{m=1}^{\infty} g\left(-j+V_{Z_s}\right)\mathbb{I}\,(\tau_m > s)\,\mathbb{I}\,(M_1 = m)\right] = \\
=& E\left[\sum_{m=1}^{\infty}\sum_{i=1}^{\infty} g\left(-j+V_{i-1}\right)\mathbb{I}\,(\tau_{i-1} \le s < \tau_i)\,\mathbb{I}\,(\tau_m > s)\,\mathbb{I}\,(M_1 = m)\right] = \\
=& \sum_{m=1}^{\infty} E\Big[\mathbb{I}\,(M_1 = m)\sum_{i=1}^{m} g(-j+V_{i-1})\mathbb{I}\,(\tau_{i-1} \le s < \tau_i)\,\Big] = \\
=& \sum_{m=1}^{\infty}\sum_{i=1}^{m} E\Big[\mathbb{I}\,(M_1 = m)\, g(-j+V_{i-1})\Big] P\left(\tau_{i-1} \le s < \tau_i\right) = \\
=& \sum_{i=1}^{\infty}\sum_{m=i}^{\infty} E\Big[\mathbb{I}\,(M_1 = m)\, g(-j+V_{i-1})\Big]\Big(F_{i-1}(s) - F_i(s)\Big) = \\
&= \sum_{i=1}^{\infty} E\Big[\mathbb{I}\,(M_1 \ge i)\, g(-j+V_{i-1})\Big]\Big(F_{i-1}(s) - F_i(s)\Big).
\end{aligned}$$

□

Now the state space of the process Y_t is discrete and then the filter $E\big[g(Y_t)/\mathcal{G}_t\big]$ is determined by its discrete density $\nu_t(x),\ x \in \mathbb{Z}$, i.e. $\nu_t(x) = E\big[g(Y_t)/\mathcal{G}_t\big]$ with $g(z) = \mathbb{I}_{\{x\}}(z),\ z \in \mathbb{Z}$.
In the next Proposition we give the expression of the discrete density $\nu_t(x)$.

Proposition 2.4. *The filter of Y_t given $\mathcal{G}_t$ has discrete density*

$$\nu_t(x) = \sum_{j=0}^{\infty} \frac{\sum_{k=1}^{\infty} P\left(V_{k-1} = x+j, M_1 \ge k\right)\left(F_{k-1}(t-\sigma_j) - F_k(t-\sigma_j)\right)}{\sum_{m=1}^{\infty} P\big(M_1 = m\big)\big(1 - F_m(t-\sigma_j)\big)}\mathbb{I}\big\{\sigma_j \le t < \sigma_{j+1}\big\},$$

where

$$P\left(V_{k-1} = a, M_1 \ge k\right) =$$

$$= \begin{cases} \dfrac{a+1}{k}\dbinom{k}{\frac{k+a+1}{2}} p^{\frac{k+a-1}{2}} q^{\frac{k-a-1}{2}} & \textit{if} \quad k+a-1 \quad \textit{is even and} \quad |a| \le k-1 \\ \\ 0 & \textit{otherwise.} \end{cases} \tag{15}$$

Proof. By using (14) for the functions $g(z) = \mathbb{I}_{\{x\}}(z),\ z \in \mathbb{Z}$ we achieve the result.

The joint law of $(V_{k-1}, \max_{j \le k-1} V_j)$ is well-known (see e.g. Theorem 3.10.10 in [11]), and in particular

$$P\left(V_{k-1}=-a,\max_{j\le k-1}V_j<1\right)=$$

$$=\begin{cases}\dfrac{a+1}{k}\dbinom{k}{\frac{k+a+1}{2}}q^{\frac{k+a-1}{2}}p^{\frac{k-a-1}{2}} & \text{if}\quad k+a-1\quad\text{is even and}\quad |a|\le k-1\\[2ex] 0 & \text{otherwise.}\end{cases}\tag{16}$$

Observe that $\{V_{k-1}=a, M_1\ge k\}=\{V_{k-1}=a, \min_{j\le k-1}V_j>-1\}$.
Then by noting that

$$P\left(V_{k-1}=-a,\max_{j\le k-1}V_j<1\right)=P\left(-V_{k-1}=a,\min_{j\le k-1}(-V_j)>-1\right)$$

we obtain the thesis interchanging the role of p and q in (16). □

3 Scaling

Let $\{\tilde X^n_t,\ n\in\mathbb{N}\}$ be a sequence of continuous time random walks defined in $(\Omega^n,\mathcal{F}^n,P^n)$. We assume that $\tilde X^n_t=\tilde V^n_{\tilde Z^n_t}$, where $\tilde V^n_k$ and $\tilde Z^n_t$ are defined as in Section 1 starting from a sequence $\{(\tilde T^n_j,\tilde U^n_j);j\ge 1\}$. For each n, we assume that the sequence $\{(\tilde T^n_j,\tilde U^n_j);j\ge 1\}$ satisfies the assumption **H0** stated in Section 2.
Consider the deterministic linear time-space scaling

$$X^n_t=b_n\tilde X^n_{a_nt},$$

where $\{a_n,\ n\in\mathbb{N}\}$ and $\{b_n,\ n\in\mathbb{N}\}$ are suitable sequences of real numbers. The local time $L^n_t=\ell_t(X^n)$ of the rescaled process X^n_t can be obtained just applying the same scaling to the process $\tilde L^n_t=\ell_t(\tilde X^n)$, i.e.

$$L^n_t=b_n\tilde L^n_{a_nt}.$$

We are interested in the conditional law of X^n_t w.r.t. $\mathcal{F}^{L^n}_t=\mathcal{F}^{\tilde L^n}_{a_nt}$, i.e. the filter

$$\pi^n_t(g)=E^n\big[g(X^n_t)/\mathcal{F}^{L^n}_t\big]=E^n\big[g(b_n\tilde X^n_{a_nt})/\mathcal{F}^{\tilde L^n}_{a_nt}\big],$$

where E^n denotes the expectation w.r.t. P^n. For the sake of notational convenience we will denote $\mathcal{F}^{L^n}_t$ as $\mathcal{G}^n_t$ and, when unnecessary, drop the symbol n in the expectation, so that the filter becomes

$$\pi^n_t(g)=E\big[g(X^n_t)/\mathcal{G}^n_t\big].$$

Following the same lines as in the previous Section we get the representation

$$\pi^n_t(g)=E\big[g(X^n_t)/\mathcal{G}^n_t\big]=\sum_{j=0}^{\infty}\frac{E\Big[g(-b_nj+X^{n*}_s)\mathbb{I}\left(\sigma^{n*}_1>s\right)\Big]_{s=t-\sigma^n_j}}{E\Big[\mathbb{I}(\sigma^{n*}_1>s)\Big]_{s=t-\sigma^n_j}}\mathbb{I}\left(\sigma^n_j\le t<\sigma^n_{j+1}\right),\tag{17}$$

where

- $\{\sigma_j^n, j \geq 0\}$ is defined by
$$\sigma_j^n = \inf\{t > 0 \;\text{ s. t. }\; X_t^n \leq -b_n j\}$$
and represents a renewal process that coincides with the local time L_t^n;

- X^{n*} is a process with the same law as X^n and σ_1^{n*} is its first exit time from the set $(-b_n, \infty)$.

Note that, when $t \in [\sigma_j^n,\ \sigma_{j+1}^n)$, the stopping time σ_j^n can be written as

$$\sigma_j^n = \sup\{u \leq t \quad \text{s. t.} \quad L_u^n < L_t^n\}.$$

This observation allows us to write (17) as follows

$$\pi_t^n(g) = E\big[g(X_t^n)/\mathcal{G}_t^n\big] = E\Big[g(-l + X_s^{n*})/\{L_s^{n*} < b_n\}\Big]\Big|_{l=L_t^n,\ s=\xi_t^n} = \\ = E\Big[g(-l + X_s^{n*})/\{L_s^{n*} = 0\}\Big]\Big|_{l=L_t^n,\ s=\xi_t^n}, \tag{18}$$

where

$$\xi_t^n = t - \sup\{u \leq t \quad \text{s. t.} \quad L_u^n < L_t^n\} = \gamma_t(L^n), \tag{19}$$

with $\gamma_t(x)$ the functional defined by (7).
Then, denoting by $\Sigma^n(g)(s,l)$

$$\Sigma^n(g)(s,l) = E\Big[g(-l + X_s^{n*})/\{L_s^{n*} < b_n\}\Big] = \frac{E\Big[g(X_s^{n*} - l)\mathbb{I}\,(\sigma_1^{n*} > s)\Big]}{E\Big[\mathbb{I}(\sigma_1^{n*} > s)\Big]} \tag{20}$$

(17) can be shortly written as

$$\pi_t^n(g) = E\big[g(X_t^n)/\mathcal{G}_t^n\big] = \Sigma^n(g)(\xi_t^n, L_t^n). \tag{21}$$

Remark 3.1. *Note that $\Sigma^n(g)(s,l)$ depends also on the probability measure P^n, i.e. $\Sigma^n(g)(s,l) = \Sigma^n_{P^n}(g)(s,l)$. This dependence will be emphasized when necessary.*

By taking into account that

$$\hat{\pi}_t^n(g) = E\big[g(X_t^n + L_t^n)/\mathcal{G}_t^n\big] = E\big[g(X_t^n + m)/\mathcal{G}_t^n\big]\big|_{m=L_t^n}$$

the filter can be written as

$$\hat{\pi}_t^n(g) = E\big[g(X_t^n + L_t^n)/\mathcal{G}_t^n\big] = \hat{\Sigma}^n(g)(\xi_t^n), \tag{22}$$

where

$$\hat{\Sigma}^n(g)(s) = \Sigma^n(g)(s,0) = E\left[g(X_s^{n*})/\{L_s^{n*} < b_n\}\right] = \frac{E\Big[g(X_s^{n*})\mathbb{I}\,(\sigma_1^{n*} > s)\Big]}{E\Big[\mathbb{I}(\sigma_1^{n*} > s)\Big]}. \tag{23}$$

We end this Section by noting that, when the process $\tilde{X}^n = V^n_{Z_t^n}$ satisfies the assumptions **H1**, **H2** and **H3** stated in Section 2, then Proposition 2.3 easily provides the explicit expressions of $\Sigma^n(g)(s,l)$ and $\hat{\Sigma}^n(g)(s)$

$$\Sigma^n(g)(s,l) = \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\,(M_1^n \geq k)\, g(b_n V_{k-1}^n - l)\Big]\big(F_{k-1}^n(s) - F_k^n(s)\big)}{\sum_{m=1}^{\infty} P\big(M_1^n = m\big)\big(1 - F_m^n(s)\big)} \tag{24}$$

$$\hat{\Sigma}^n(g)(s) = \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\left(M_1^n \geq k\right) g(b_n V_{k-1}^n)\Big] \big(F_{k-1}^n(s) - F_k^n(s)\big)}{\sum_{m=1}^{\infty} P\big(M_1^n = m\big)\big(1 - F_m^n(s)\big)}, \tag{25}$$

where $M_1^n,\ F_k^n$ have a similar meaning as $M_1,\ F_k$ in (14). In particular it is interesting to note that if $\tilde{F}_1^n$ denotes the distribution function of

$$\tilde{\sigma}_1^n = \inf\Big\{t > 0 \ \text{ s. t. } \ \tilde{X}_t^n \leq -1\Big\},$$

then $\sigma_1^n = \tilde{\sigma}_1^n / a_n$, and therefore $F_1^n(s) = \tilde{F}_1^n(a_n s)$, and $F_k^n(s) = \tilde{F}_k^n(a_n s)$, where $\tilde{F}_k^n$ is the $k-$fold convolution of $\tilde{F}_1^n$.

The remainder of this paper is devoted to the problem of finding the limit of the filter (21) (or of the filter (22)), or some good approximation for it, in the case when the rescaled sequence X_t^n converges to a process X_t.
When X_t^n converges to a continuous process X_t, then the local time $L_t^n = \ell_t(X^n)$ of X_t^n also converges to the local time $\ell_t(X)$ of X_t. In fact the functional $\ell_t(x)$ defined by (9) is continuous w.r.t. the topology of the uniform convergence on compact sets. In this case it is also natural to investigate whether the limit of the filter is the corresponding filter of the limits. Finally we observe that, if the limit of (22) exists, then the same holds for (21), and these limits can be obtained one from each other, as we can obtain (22) by (21) and vice-versa.

4 The interpolating Brownian motion model: strong convergence for the filter

In this Section we study the case of a continuous time random walk arising when a Brownian motion W is approximated with a sequence of processes W^n. The processes W^n are defined on the probability space of W, and are obtained pathwise by an interpolation procedure. For this model we are able to get a strong convergence result for the filter. When the process W has drift zero, then the approximating model falls into the frame of the previous sections, in particular the common distribution function of T_j is F^0 (see (39) below) and the random walk is symmetric.
The asymmetric case can also be managed, and we will briefly explain how at the end of this Section.

We start by introducing the approximating model, then we show the convergence result.

The basic idea is to approximate the state W by the stepwise interpolation of the random points where W hits a uniform grid and consider as approximating observation the local time of the approximating state. This procedure is a deterministic one and therefore we describe it in the deterministic case.

Let $z \in D_{\mathbb{R}}[0, +\infty)$ and let $h \in \mathbb{R}^+$ be a fixed threshold.

Consider the sequence $\{\hat{\tau}_k^h(z), k \geq 0\}$

$$\begin{cases} \hat{\tau}_0^h(z) = 0 \\ \hat{\tau}_k^h(z) = \inf\big\{t > \hat{\tau}_{k-1}^h(z) : |z(t) - z(\hat{\tau}_{k-1}^h(z))| \geq h\big\} \quad k \geq 1. \end{cases} \tag{26}$$

Put $z^h_k = z(\widehat{\tau}^h_k(z))$ and consider the function $z^h \in D_{\mathbb{R}}[0,+\infty)$

$$z^h(t) = \sum_{k=0}^{\infty} \mathbb{I}_{\left[\widehat{\tau}^h_k(z),\widehat{\tau}^h_{k+1}(z)\right)}(t) z^h_k. \tag{27}$$

We will need the following results

Lemma 4.1. *Let $z \in C_{\mathbb{R}}[0,+\infty)$. Then, for each $t \in \mathbb{R}^+$*

$$|z^h(t) - z(t)| \leq h, \tag{28}$$

and

$$z^h \underset{h\to 0}{\to} z \quad \text{and} \quad \ell(z^h) \underset{h\to 0}{\to} \ell(z)$$

w.r.t. the topology of the uniform convergence, where the functional ℓ is defined by (9).

Proof. The bound (28) is due to the continuity of z, and the uniform convergence of z^h to z follows. The continuity of the functional ℓ implies the other limit. □

Remark 4.2. *The process $\ell(z^h)$ admits the representation*

$$\ell_t(z^h) = 0 \vee (-z_0) - \sum_{j=0}^{\infty} z_{\sigma^h_j(z)} \mathbb{I}_{[\sigma^h_j(z),\sigma^h_{j+1}(z))}(t),$$

where

$$\sigma^h_j(z) = \inf\left\{t \quad s.t. \quad z(t) - z_0 \leq -jh\right\} = \inf\left\{\widehat{\tau}^h_k(z) \quad s.t. \quad z(\widehat{\tau}^h_k(z)) - z_0 \leq -jh\right\}. \tag{29}$$

When z is a continuous function and $z_0 = 0$ then $z_{\sigma^h_j(z)} = -jh$ and

$$\ell_t(z^h) = \sum_{j=0}^{\infty} jh\mathbb{I}(\sigma^h_j(z) \leq t < \sigma^h_{j+1}(z)) = \sum_{j=0}^{\infty} h\mathbb{I}(\sigma^h_j(z) \leq t) \tag{30}$$

We now apply this approximating procedure to the Brownian motion W_t.

The local time Λ_t of the process W_t is

$$\Lambda_t = \ell_t(W),$$

where ℓ is the deterministic functional defined by (9).

Observe that $W_0 = 0$ implies

$$\Lambda_t = -\inf_{0\leq s\leq t}\big(W_s\big). \tag{31}$$

Fix now the sequence of thresholds $h_n = \frac{1}{2^n}$, and consider the stopping times $\tau^n_k := \widehat{\tau}^h_k(W)$, when using $h = h_n = \frac{1}{2^n}$ in (26). Then the approximating signal/observation process $(W^n, \ell(W^n))$ is a $D_{\mathbb{R}^2}[0,+\infty)$-valued process, where

$$W^n_t = \sum_{k=0}^{\infty} W^n_k \, \mathbb{I}_{\left[\tau^n_k,\tau^n_{k+1}\right)}(t), \quad \text{with } W^n_k = W(\tau^n_k). \tag{32}$$

Note that Lemma 4.1 provides the following convergence result.

Lemma 4.3. *Let W^n be defined as in (32). Then, for each $t \in \mathbb{R}^+$*

$$|W_t^n - W_t| \leq \frac{1}{2^n}, \tag{33}$$

and

$$(W^n, \ell(W^n)) \underset{n\to\infty}{\rightarrow} (W, \Lambda) \quad a.s.,$$

w.r.t. the topology of the uniform convergence.

Remark 4.2 provides

$$\ell_t(W^n) = \sum_{j=0}^{\infty} \frac{1}{2^n} \mathbb{I}(\sigma_j^n \leq t), \tag{34}$$

where

$$\sigma_j^n = \inf\{t \quad \text{s.t.} \quad W_t \leq -\frac{j}{2^n}\} = \inf\{t \quad \text{s.t.} \quad \Lambda_t \geq \frac{j}{2^n}\}. \tag{35}$$

With the above choice of the threshold, the n-th grid is generated by considering the dyadic intervals of rank n. Then in the passage from the n-th grid to the $(n+1)$-th grid each threshold is split into two parts, and therefore $\sigma_{2j}^{n+1} = \sigma_j^n$. This property is decisive since it guarantees that $\{\mathcal{G}_t^n = \sigma\big(\ell_s(W^n), s \leq t\big), n \in \mathbb{N}\}$ is an increasing family of σ-algebras, and allows us to show the claimed strong convergence result.

Theorem 4.4. *Let $g \in C_b(\mathbb{R})$ uniformly continuous. Then*

$$E\big[g(W_t^n)/\mathcal{G}_t^n\big] \to E\big[g(W_t)/\mathcal{F}_t^\Lambda\big], \quad a.s.\ and\ in\ L^1, \tag{36}$$

where $E\big[g(W_t)/\mathcal{F}_t^\Lambda\big]$ is given in (3).

Proof. Observe that

$$E\Big|E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] - E\big[g(W_t^n)/\mathcal{G}_t^n\big]\Big| \leq E\Big|E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] - E\big[g(W_t)/\mathcal{G}_t^n\big]\Big| + \\ + E\Big|E\big[g(W_t)/\mathcal{G}_t^n\big] - E\big[g(W_t^n)/\mathcal{G}_t^n\big]\Big|.$$

In [13] the r.h.s. of (34) defines the process Λ_t^n, and then $\ell_t(W^n) = \Lambda_t^n$, and $\mathcal{G}_t^n = \sigma\big(\Lambda_s^n, s \leq t\big)$. Moreover $\mathcal{G}_t^n \uparrow \mathcal{F}_t^\Lambda$ (see Lemma 2.3 of [13]) and then we get (see Theorem 2.4 of [13])

$$E\big[g(W_t)/\mathcal{G}_t^n\big] \to E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] \qquad \text{a.s. and in } L^1, \tag{37}$$

and therefore the first term converges to zero. As far as the second term is concerned, we have

$$E\Big|E\big[g(W_t)/\mathcal{G}_t^n\big] - E\big[g(W_t^n)/\mathcal{G}_t^n\big]\Big| \leq E\Big|E\big[\big|g(W_t) - g(W_t^n)\big|/\mathcal{G}_t^n\big]\Big| \leq \omega_g\,(1/2^n) \to 0,$$

where $\omega_g\,(\delta) = \sup_{|x-y|\leq\delta} \big|g(x) - g(y)\big|$ is the modulus of continuity of g. □

If we consider the space of probability measures on $\mathbb{R}$ endowed with the topology of convergence in distribution, then (36) states that the conditional laws defined by $E\big[g(W_t^n)/\mathcal{G}_t^n\big]$ converge a.s. to the conditional law defined by (3).

By Lemma 4.3, we are in the situation discussed at the end of Section 3 with $X^n = W^n$ and $X = W$. Then one can get the analogous convergence results for the corresponding queueing model generated by reflecting W^n. In particular the conditional laws $E\big[g(W_t^n + \ell_t(W^n))/\mathcal{G}_t^n\big]$ converge a.s. to $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]$, (see (4)).

We now start to show that the approximating model falls into the frame of the previous sections. When the drift is zero, we can assume without loss of generality that W is a standard Brownian motion (when the diffusion coefficient is a^2 we can use the deterministic time change $\frac{t}{a^2}$ instead of t). In this case the process W_t^n can be represented on the space $(\Omega, \mathcal{F}, P)$ as follows

$$W_t^n = \frac{1}{2^n} V^n_{Z_t^n}, \tag{38}$$

where

- Z_t^n is a renewal process defined by the sequence of i.i.d. interarrival times $\{T_k^n = \tau_k^n - \tau_{k-1}^n,\ k \in \mathbb{N}\}$ with common law (see for example [7] page 342)

$$F^n(t) = P\big(T_k^n \leq t\big) = 4\sum_{j=0}^{+\infty}(-1)^j \frac{1}{\sqrt{2\pi}} \int_{\frac{(2j+1)}{2^n\sqrt{t}}}^{+\infty} \exp\left(-\frac{x^2}{2}\right) dx; \tag{39}$$

- $V_k^n = 2^n W_{\tau_k^n}$ is a symmetric random walk;
- Z_t^n and V_k^n are mutually independent.

Let V be the symmetric random walk V^0, and Z be the renewal process Z^0, with interarrival time distribution $F^0(t)$ (see (39) with $n = 0$). Define $\tilde{X}_t = V_{Z_t}$, and the rescaled process $X_t^n = \frac{1}{2^n} V_{Z_{2^{2n}t}}$. Then $\tilde{X} = W^0$, and it is easy to verify that

$$X^n = W^n \ \text{ in law.}$$

Consequently $(X^n, \ell(X^n)) = (W^n, \ell(W^n))$ in law, where ℓ is the functional defined by (9). Therefore, recalling (21), $E\big[g(W_t^n)/\mathcal{G}_t^n\big] = \Sigma^n(g)(\gamma_t(W^n), \ell(W^n))$ where $\Sigma^n(g)$ is defined in (24), with F^n as in (39), and γ_t as in (7). A similar result holds also for $E\big[g(W_t^n + \ell_t(W^n))/\mathcal{G}_t^n\big]$.

When the drift coefficient is $c \neq 0$, and the diffusion coefficient is a^2, the approximating model does not fall into the frame of the previous sections, since Z_t^n and V_k^n, defined as above, are not mutually independent. Nevertheless condition **H0** holds , and one could represent the approximating filter as explained in Remark 2.2. Alternatively by Kallianpur Striebel formula and Girsanov Theorem

$$E\big[g(W_t^n)/\mathcal{G}_t^n\big] = \frac{E^{P_0}\big[g(W_t^n)\exp(\frac{c}{a^2}W_t)/\mathcal{G}_t^n\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W_t)/\mathcal{G}_t^n\big]}$$

where P_0 is equivalent to P, and under P_0 the process W has drift coefficient zero. From this formula one can compute exactly $E\big[g(W_t^n)/\mathcal{G}_t^n\big]$. Moreover, for g a uniformly continuous function with modulus of continuity ω_g, the filter $E\big[g(W_t^n)/\mathcal{G}_t^n\big]$ can be expressed as

$$\frac{E^{P_0}\big[g(W_t)\exp(\frac{c}{a^2}W_t)/\mathcal{G}_t^n\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W_t)/\mathcal{G}_t^n\big]} + R(n,t),$$

where, taking into account that $|W_t - W_t^n| \leq 1/2^n$ (see Lemma 4.3),

$$|R(n,t)| \leq \frac{E^{P_0}\big[|g(W_t) - g(W_t^n)| \exp(\frac{c}{a^2} W_t)/\mathcal{G}_t^n\big]}{E^{P_0}\big[\exp(\frac{c}{a^2} W_t)/\mathcal{G}_t^n\big]} \leq \omega_g(1/2^n).$$

Therefore the filter converges a.s. to

$$\frac{E^{P_0}\big[g(W_t) \exp(\frac{c}{a^2} W_t)/\mathcal{F}_t^\Lambda\big]}{E^{P_0}\big[\exp(\frac{c}{a^2} W_t)/\mathcal{F}_t^\Lambda\big]} = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big].$$

5 The M/M/1 queueing model: weak convergence for the filter

In this Section we consider a random walk with exponential interarrival times, and the M/M/1 queue generated by reflecting the random walk. We use the techniques introduced in Section 2 to derive the filter of the M/M/1 queue (and therefore of the random walk) with respect to the local time of the random walk. Moreover, under a suitable set of conditions, we will get also the weak limit of the filter of the rescaled system.

In order to do so, in Subsection 5.1 we show the claimed convergence in the case when the arrival intensity λ_n and the service potential μ_n of the M/M/1 queue are equal; then, in Subsection 5.2, by using a suitable change of measure, we will extend this result to the sequences of M/M/1 queues satisfying conditions **C1**, **C2**, **C3** below.

Let $\{\lambda_n, n \in \mathbb{N}\}$ and $\{\mu_n, n \in \mathbb{N}\}$ be two sequences of real numbers satisfying the conditions

C1 $0 < \lambda_n \leq \mu_n$

C2 $(\lambda_n, \mu_n) \underset{n\to+\infty}{\to} (\lambda, \lambda)$

C3 $\sqrt{n}(\lambda_n - \lambda) \underset{n\to+\infty}{\to} 0$, and $\sqrt{n}(\mu_n - \lambda) \underset{n\to+\infty}{\to} 0$

As a consequence of condition **C3** we have $\sqrt{n}(\mu_n - \lambda_n) \underset{n\to+\infty}{\to} 0$.
The sequence of random walks we consider is defined by means of the same rule (8), namely for each $n \in \mathbb{N}$

$$\tilde{X}_t^n = \tilde{V}_{\tilde{Z}_t^n} = \sum_{j=1}^{\tilde{Z}_t^n} \tilde{U}_j^n,$$

where

A1 $\tilde{Z}_t^n$ is a Poisson process with intensity $(\lambda_n + \mu_n)$

A2 $\tilde{V}_j^n$ is defined by

$$\tilde{V}_j^n = \tilde{V}_{j-1}^n + \tilde{U}_j^n$$

$\{\tilde{U}_j^n, j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with

$$\begin{cases} P^n(\tilde{U}_k^n = +1) = \dfrac{\lambda_n}{\lambda_n + \mu_n} \\[2ex] P^n(\tilde{U}_k^n = -1) = \dfrac{\mu_n}{\lambda_n + \mu_n} \end{cases}$$

A3 $\{\tilde{U}_k^n, k \in \mathbb{N}\}$ and $\tilde{Z}_t^n$ are mutually independent.

So, in this case the interarrival times $\tilde{T}_k^n$ of the renewal process $\tilde{Z}_t^n$ are exponential random variables with expectation $1/(\lambda_n + \mu_n)$.
Observe that the processes $\tilde{X}_t^n$ and $\tilde{Z}_t^n$ can be represented as

$$\tilde{X}_t^n = \tilde{A}_t^n - \tilde{N}_t^n, \quad \tilde{Z}_t^n = \tilde{A}_t^n + \tilde{N}_t^n$$

where

$$\tilde{A}_t^n = \sum_{k=0}^{\infty} \mathbb{I}(U_k^n = 1)\mathbb{I}(\tau_k^n \leq t) \tag{40}$$

$$\tilde{N}_t^n = \sum_{k=0}^{\infty} \mathbb{I}(U_k^n = -1)\mathbb{I}(\tau_k^n \leq t). \tag{41}$$

Lemma 5.1. *The processes $\tilde{A}^n$ and $\tilde{N}^n$, defined by (40) and (41), are mutually independent Poisson processes with intensities λ_n and μ_n respectively.*

Proof. We just note that $\tilde{A}^n$ and $\tilde{N}^n$ do not have common jumps and that they are Poisson processes with parameters λ_n, μ_n respectively. Then Watanabe's Theorem (see, for instance, [3]) yields the thesis. □

The solution of the Skorohod problem for the process $\tilde{X}_t^n$ is the pair $(\tilde{Q}_t^n, \tilde{L}_t^n)$, where

$$\tilde{Q}_t^n = \tilde{X}_t^n + \tilde{L}_t^n$$

is a M/M/1 queue and $\tilde{L}_t^n$ is the local time for the process $\tilde{X}_t^n$ (see, for instance, [3]).
Under the assumptions stated in this Section, a continuous map argument applies to show that

$$(X_t^n, Q_t^n, L_t^n) = \left(\frac{\tilde{X}_{nt}^n}{\sqrt{n}}, \frac{\tilde{Q}_{nt}^n}{\sqrt{n}}, \frac{\tilde{L}_{nt}^n}{\sqrt{n}}\right) \Rightarrow (W_t, W_t + \Lambda_t, \Lambda_t), \tag{42}$$

where W_t is a Brownian motion with diffusion coefficient 2λ and Λ_t is the local time of W_t.
Therefore we are in the situation discussed at the end of Section 3, with $a_n = n$ and $b_n = \frac{1}{\sqrt{n}}$.
In the sequel we provide the weak limit of the filter of X_t^n and Q_t^n given the filtration $\mathcal{G}_t^n$, where, as in Section 3, $\mathcal{G}_t^n$ denotes the filtration generated by L_t^n.
The main result is stated in the following Theorem which is proved in Subsection 5.2.

Theorem 5.2. *Assume conditions* **C1**, **C2**, **C3** *and* **A1**, **A2**, **A3**, *and let g be a bounded continuous function. Then*

$$\pi_t^n(g) = E\big[g(X_t^n)/\mathcal{G}_t^n\big] \Rightarrow \pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] \tag{43}$$

and

$$\hat{\pi}_t^n(g) = E\big[g(Q_t^n)/\mathcal{G}_t^n\big] \Rightarrow \hat{\pi}_t(g) = E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big], \tag{44}$$

moreover π_t^n converge weakly to π_t, and $\hat{\pi}_t^n$ converge weakly to $\hat{\pi}_t$, as random variables with values in the space of probability measures endowed with the topology of weak convergence.

The proof of Theorem 5.2 is based on the representations (21) and (22) for π_t^n and $\hat{\pi}_t^n$, respectively, and therefore we need a preliminary result concerning the weak convergence of $\xi_t^n = \gamma_t(L^n)$ to $\zeta_t = \gamma_t^0(W + \Lambda) = \gamma_t(\Lambda)$, with γ_t^0 and γ_t defined in (6) and (7) respectively.
We show a slightly stronger result concerning the weak convergence of $\gamma_t^0(X^n + L^n) = \gamma_t^0(Q^n)$ to ζ_t. This result is used later in Section 7.

Proposition 5.3. *For each $t > 0$*

$$(\gamma_t^0(Q^n), \gamma_t(L^n), L_t^n) \Rightarrow (\zeta_t, \zeta_t, \Lambda_t).$$

Proof. Define

$$\begin{aligned} \eta_t^n &= \sup\{s < t : L_s^n < L_t^n\}, \quad \eta_t = \sup\{s < t : \Lambda_s < \Lambda_t\}, \\ \beta_t^n &= \sup\{s < t : \ \ Q_s^n = 0\}, \quad \beta_t = \sup\{s < t : W_s + \Lambda_s = 0\}, \end{aligned}$$

with $\eta_t^n = t$, $\eta_t = t$, $\beta_t^n = t$ and $\beta_t = t$ if the corresponding set is empty.
Note that

$$\begin{aligned} \eta_t^n &= \sup\{s < t : X_t^n - X_s^n < Q_t^n - Q_s^n\}, \\ \eta_t \ &= \sup\{s < t : W_t - W_s < W_t + \Lambda_t - W_s - \Lambda_s\}, \end{aligned}$$

Applying the Skorohod representation theorem, we can assume

$$\sup_{s \le t}(|X_s^n - W_s| + |Q_s^n - W_s - \Lambda_s|) \to 0 \quad \text{a.s.} \tag{45}$$

This implies that $L^n \to \Lambda$ uniformly in $[0, t]$ a.s. and

$$\liminf_{n\to\infty}\ \eta_t^n \ge \eta_t.$$

Then, since

$$\gamma_t^0(Q^n) = t - \beta_t^n, \qquad\qquad \gamma_t(L^n) = t - \eta_t^n,$$

the result is achieved once we prove that the sequence (β_t^n, η_t^n) converges a.s. to (η_t, η_t) and $\zeta_t = t - \eta_t$.
Let $\beta_t^\infty = \limsup_{n\to\infty} \beta_t^n$ and note that $Q^n(\beta_t^n)$ assumes only the values 0 or $\frac{1}{\sqrt{n}}$.
Then, by (45), $W_{\beta_t^\infty} + \Lambda_{\beta_t^\infty} = 0$. It follows that

$$\limsup_{n\to\infty} \beta_t^n \le \beta_t.$$

Moreover, if $\eta_t^n < t$, then $Q^n(\eta_t^n) = 0$, if $\eta_t^n = t$, then $\beta_t^n = t$, and it follows that

$$\eta_t^n \le \beta_t^n \qquad\qquad \text{for all } t, \text{ a.s.}$$

and then

$$\eta_t \le \liminf_{n\to\infty}\ \eta_t^n \le \limsup_{n\to\infty} \beta_t^n \le \beta_t, \qquad\qquad \text{for all } t, \text{ a.s.}$$

The proof is achieved since

$$P(\eta_t = \beta_t) = 1$$

and then $\zeta_t = \gamma_t^0(W + \Lambda) = t - \beta_t = t - \eta_t = t - \gamma_t(\Lambda)$. For sake of completeness we prove the previous statement. Note that $W_s + \Lambda_s = 0$ if and only if $W_s = \inf_{r<s} W_r \le 0$. If $\eta_t < \beta_t$, then for $\eta_t \le s \le \beta_t$,

$$W_t - W_s = W_t + \Lambda_t - W_s - \Lambda_s \le W_t + \Lambda_t,$$

and for any rational r satisfying $\eta_t < r < \beta_t$, we must have

$$\inf_{s\le r} W_s = \inf_{r\le s\le t} W_s.$$

But $P\{\inf_{s\le r} W_s = \inf_{r\le s\le t} W_s\} = 0$, and hence

$$P\{\eta_t < \beta_t\} \le \sum_{r\in\mathbb{Q}} P\{\inf_{s\le r} W_s = \inf_{r\le s\le t} W_s\} = 0.$$

□

Remark 5.4. *In the Skorohod space used in the proof of Proposition 5.3, choose a jointly measurable version of* $\xi^n_t = \gamma_t(L^n) = t - \eta^n_t$*. Then* $M = \big\{(\omega,t) \in \Omega \times [0,T] \ \ s.t.\ \xi^n_t(\omega) \not\to \zeta_t(\omega)\big\}$ *is a zero* $dP \times dt$*-measure set. Moreover a similar result holds for* $\gamma^0_t(Q^n) = t - \beta^n_t$*, namely* $M_0 = \big\{(\omega,t) \in \Omega \times [0,T] \ \ s.t.\ \gamma^0_t(Q^n)(\omega) \not\to \zeta_t(\omega)\big\}$ *is a zero* $dP \times dt$*-measure set.*

The following remarks summarize the results we need to find the weak limit of $E\big[g(Q^n_t)/\mathcal{G}^n_t\big]$.

- Both the filter $E\big[g(Q^n_t)/\mathcal{G}^n_t\big]$ and the filter $E\big[g(W_t + \Lambda_t)/\mathcal{F}^\Lambda_t\big]$ for the limit model are given as the compositions of deterministic functionals with processes measurable w.r.t. the respective observed histories. More specifically, the representation (22) for the filter $E\big[g(Q^n_t)/\mathcal{G}^n_t\big]$ is the deterministic functional $\hat\Sigma^n(g)(s)$ (23) evaluated in $s = \xi^n_t$ and $E\big[g(W_t + \Lambda_t)/\mathcal{F}^\Lambda_t\big]$ is the functional $\Pi(g)(s,0)$ evaluated in $s = \zeta_t$ (see Theorem 1.1).

- The sequence Q^n converges weakly to a reflected Brownian motion, hence by Lemma 1.2, the sequence $E\big[g(Q^n_t)/\mathcal{G}^n_t\big]$ is tight, and moreover, by Proposition 5.3, $\xi^n_t \Rightarrow \zeta_t$.

The previous remarks induce to conjecture that the limit of $E\big[g(Q^n_t)/\mathcal{G}^n_t\big]$ is $E\big[g(W_t + \Lambda_t)/\mathcal{F}^\Lambda_t\big]$. We prove this intuitive idea by developing the following steps.

- In Proposition 5.5, under the assumption $\lambda_n = \mu_n = \lambda$, we compute the limit of the sequence of measures defined by $\hat\Sigma^n(g)(s)$ in (23), and we note that it turns out to be the probability measure defined by the functional $\Pi(g)(s,0)$ (see (2) in Theorem 1.1).

- Then, by using Proposition 5.3 we show that the weak limit of $E\big[g(Q^n_t)/\mathcal{G}^n_t\big] = \hat\Sigma^n(g)(\xi^n_t)$ is $\Pi(g)(s,0)$ evaluated in $s = \zeta_t$, that is the filter of a reflected Brownian motion w.r.t. its local time (see Theorem 1.1).

- Finally we extend previous result to the more general case defined by conditions **C1**, **C2**, **C3** by making a suitable change of measure. Then Theorem 5.2 is completely shown.

5.1 Weak convergence of the filter: the symmetric case

On a suitable space $(\Omega, \mathcal{F}, P)$ let Z_t be a Poisson process with intensity $2\,\lambda$ and let V_k be a symmetric random walk. Let $\tilde X_t$ be the process defined by the rule (8), that is $\tilde X_t = V_{Z_t}$. In this particular situation the process $X^n_t = \frac{\tilde X_{nt}}{\sqrt{n}}$ depends on the index n just by the time-space scaling.

These properties for the model are equivalent to assume

$$\lambda_n = \mu_n = \lambda \tag{46}$$

and, consequently, $c = 0$. In this case the process W_t defined by (42) is a driftless Brownian motion with diffusion coefficient 2λ and Λ_t is its local time.
Without loss of generality we can assume $\lambda = \frac{1}{2}$. Otherwise we can use the deterministic change of time $\frac{t}{2\lambda}$ instead of t.

Denote by $\hat\Pi(s)$ the probability measure defined by

$$\hat\Pi(g)(s) = \begin{cases} \Pi(g)(s,0) = \int_0^\infty g(y\sqrt{s})y \exp\left(-\frac{y^2}{2}\right) dy & \text{if} \ \ s > 0, \\ g(0) \ \ \text{if} \ \ s = 0, \end{cases} \tag{47}$$

for each $g : \mathbb{R}^+ \to \mathbb{R}$ bounded and continuous, where $\Pi(g)(s,l)$ is the function defined by (1) and (2) in Theorem 1.1.

Proposition 5.5. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a bounded continuous function. Under the assumption (46), for each s,*

$$\hat{\Sigma}^n(g)(s) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(g)(s). \tag{48}$$

Moreover for any sequence s_n converging to $s > 0$

$$\hat{\Sigma}^n(g)(s_n) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(g)(s). \tag{49}$$

In order to prove Proposition 5.5 we need a preliminary result.

Lemma 5.6. *For each $s > 0$ and $n \in \mathbb{N}$, let F_s^n be the distribution function of the random variable X_s^n. Then*

$$F_s^n(x) - \Phi\left(\frac{x}{\sqrt{s}}\right) = o\left(\frac{1}{\sqrt{ns}}\right), \quad \textit{uniformly in } x \in \Gamma_n, \tag{50}$$

where

$$\Gamma_n = \{\frac{1}{\sqrt{n}}\left(z + \frac{1}{2}\right), z \in \mathbb{Z}\},$$

and $\Phi(x)$ is the distribution function of a standard normal random variable.

Proof. The random variable X_s^n can be written as follows

$$X_s^n = \frac{1}{\sqrt{n}}\sum_{k=0}^{n} \Delta_s^k,$$

where $\{\Delta_s^k = \tilde{X}_{ks} - \tilde{X}_{(k-1)s},\ k \in \mathbb{N}\}$. Under the assumption stated at the beginning of this Subsection, the sequence $\{\Delta_s^k,\ k \in \mathbb{N}\}$ is a sequence of i.i.d. symmetric random variables, with characteristic function $\exp\{s(\cos(u) - 1)\}$, with first and third moments equal to zero and $Var\left(\Delta_s^1\right) = s$. Moreover their common distribution function F_s is concentrated on the lattice Γ_n, as well as the distribution function F_s^n. Then, since the third moments of Δ_s^k are zero, by Theorem 2, in [7], Chapter XVI.4 we get

$$\hat{F}_s^n(x) - \Phi(x) = o\left(\frac{1}{\sqrt{n}}\right), \quad \text{uniformly in } x \text{ s.t.} \frac{x}{\sqrt{s}} \in \Gamma_n. \tag{51}$$

where $\hat{F}_s^n(x) = F_s^n(x\sqrt{s})$ is the distribution function of $\frac{X_s^n}{\sqrt{s}}$.
The expansion (51) does not imply immediately the thesis, nevertheless (50) can be proved by repeating and adapting the proof to the case when the limit distribution function is $\Phi\left(\frac{x}{\sqrt{s}}\right)$ instead of $\Phi(x)$. □

Remark 5.7. *Let $s_n \to s$ and $s > 0$, so that there exists $\bar{n}$ such that $s_n \geq \frac{1}{2}s$, for any $n > \bar{n}$. Then, under the assumptions of Lemma 5.6*

$$F_{s_n}^n(x) - \Phi\left(\frac{x}{\sqrt{s_n}}\right) = o\left(\frac{1}{\sqrt{n}}\right), \quad \textit{uniformly in } x \in \Gamma_n. \tag{52}$$

Proof of Proposition 5.5. We prove in detail only (48). The proof of (49) for the case of a sequence s_n converging to $s > 0$, is treated in a similar way, by using Lemma 5.6 and Remark 5.7.

When $s = 0$ (48) is trivial. When $s > 0$ we note that $\hat{\Sigma}^n(s)$ is the conditional law of X_s^{n*} given the event $\{L_s^{n*} < \frac{1}{\sqrt{n}}\}$. Then

$$\hat{\Sigma}^n(g)(s) = \frac{\Theta^n(g)(s)}{\Theta^n(1)(s)},$$

where 1 denotes the constant function $1(x) = 1$, and where $\Theta^n(s)$ is the measure defined by

$$\Theta^n(g)(s) = E\left[g(X_s^{n*})\mathbb{I}\{L_s^{n*} < \frac{1}{\sqrt{n}}\}\right].$$

We can restrict to the functions $g \in C^1(\mathbb{R})$ with $g'(0) = 0$ since this class is convergence determining.
Without loss of generality we can also assume that $g(0) = 0$. In fact

$$\hat{\Sigma}^n(g)(s) = \hat{\Sigma}^n(g - g(0))(s) + g(0).$$

By an explicit computation we get

$$\Theta^n(g)(s) = \sum_{k=0}^{\infty} g\left(\frac{k}{\sqrt{n}}\right) P\left(X_s^{n*} = \frac{k}{\sqrt{n}}, \min_{0\le u\le s} X_s^{n*} > \frac{-1}{\sqrt{n}}\right).$$

Moreover, the reflection principle yields

$$\begin{aligned} P\left(X_s^{n*} = \frac{k}{\sqrt{n}}, \min_{0\le u\le s} X_s^{n*} > \frac{-1}{\sqrt{n}}\right) =& P\left(X_s^{n*} = \frac{k}{\sqrt{n}}\right) - P\left(X_s^{n*} = -\frac{k+2}{\sqrt{n}}\right) = \\ =& P\left(X_s^{n*} = \frac{k}{\sqrt{n}}\right) - P\left(X_s^{n*} = \frac{k+2}{\sqrt{n}}\right), \end{aligned}$$

and so

$$\begin{aligned} \Theta^n(g)(s) =& \sum_{k=0}^{\infty} g\left(\frac{k}{\sqrt{n}}\right)\left[P\left(X_s^{n*} = \frac{k}{\sqrt{n}}\right) - P\left(X_s^{n*} = \frac{k+2}{\sqrt{n}}\right)\right] = \\ =& E\left[g(X_s^{n*})\mathbb{I}\{X_s^{n*} \ge 0\} - g(X_s^{n*} - \frac{2}{\sqrt{n}})\mathbb{I}\{X_s^{n*} - \frac{2}{\sqrt{n}} \ge 0\}\right]. \end{aligned} \tag{53}$$

Set $\tilde{g} : \mathbb{R} \to \mathbb{R}$, $x \to \tilde{g}(x) = g(x)\mathbb{I}(x \ge 0)$. Then

$$\Theta^n(g)(s) = E\left[\tilde{g}(X_s^{n*}) - \tilde{g}(X_s^{n*} - \frac{2}{\sqrt{n}})\right] = \frac{2}{\sqrt{n}} E\left[\tilde{g}'(X_s^{n*} - \theta\frac{2}{\sqrt{n}})\right],$$

where θ is a $(0, 1)$-valued random variable.
Then the sequence of measures $\hat{\Sigma}^n(s)$ admits the following representation

$$\hat{\Sigma}^n(g)(s) = E\left[\tilde{g}'(X_s^{n*} - \theta\frac{2}{\sqrt{n}})\right] \frac{2}{\sqrt{n}}\ \frac{1}{\Theta^n(1)(s)}, \tag{54}$$

for each $g \in C^1(\mathbb{R})$ with $g'(0) = 0$.
On the other hand

$$\lim_{n\to\infty} E\left[\tilde{g}'(X_s^{n*} - \theta\frac{2}{\sqrt{n}})\right] = E\left[\tilde{g}'(W_s)\right], \tag{55}$$

and an explicit computation yields

$$E\left[\tilde{g}'(W_s)\right] = \frac{1}{\sqrt{2\pi s}} \int_0^{\infty} g'(y)\exp\left(-\frac{y^2}{2s}\right) dy = \frac{1}{\sqrt{2\pi s}}\left(-g(0) + \int_0^{\infty} g(y)\frac{y}{s}\exp\left(-\frac{y^2}{2s}\right) dy\right).$$

By the change of variable $x = \frac{y}{\sqrt{s}}$ in the integral, and recalling that $g(0) = 0$ we get

$$E\left[\tilde{g}'(W_s)\right] = \frac{1}{\sqrt{2\pi s}}\left(\int_0^\infty g(\sqrt{s}x)x\exp\left(-\frac{x^2}{2}\right)dx\right) = \frac{1}{\sqrt{2\pi s}}\hat{\Pi}(g)(s).$$

Then, by (54), the proof of (48) is completely achieved by showing that

$$\lim_{n\to\infty}\frac{\sqrt{n}\Theta^n(1)(s)}{2} = \frac{1}{\sqrt{2\pi s}}\,. \tag{56}$$

By (53) we get

$$\Theta^n(1)(s) = P\left(X_s^{n*} \in \left[0, \frac{1}{\sqrt{n}}\right]\right) = F_s^n\left(\frac{1}{\sqrt{n}}\right) - F_s^n\left(-\frac{1}{\sqrt{n}}\right), \tag{57}$$

and, by using the expedient to write the r.h.s. of the above formula as

$$F_s^n\left(\frac{3}{2\sqrt{n}}\right) - F_s^n\left(-\frac{1}{2\sqrt{n}}\right),$$

we can use the expansion (50), so that

$$\Theta^n(1)(s) = \Phi\left(\frac{3}{2}\frac{1}{\sqrt{ns}}\right) - \Phi\left(-\frac{1}{2}\frac{1}{\sqrt{ns}}\right) + o\left(\frac{1}{\sqrt{ns}}\right) \simeq \frac{2}{\sqrt{ns}}\Phi'(\gamma_n^s) + o\left(\frac{1}{\sqrt{ns}}\right),$$

where $\gamma_n^s \in \left(-\frac{1}{2}\frac{1}{\sqrt{ns}}, \frac{3}{2}\frac{1}{\sqrt{ns}}\right)$. Then

$$\lim_{n\to\infty}\frac{\sqrt{n}\Theta^n(1)(s)}{2} = \lim_{n\to\infty}\frac{1}{\sqrt{s}}\Phi'(\gamma_n^s) + o(1) = \frac{1}{\sqrt{2\pi s}}.$$

□

Theorem 5.8. *Under the same assumptions of Proposition 5.5 for each* $t > 0$

$$\hat{\Sigma}^n(g)(\xi_t^n) \Rightarrow \hat{\Pi}(g)(\zeta_t). \tag{58}$$

for each $g : \mathbb{R}^+ \to \mathbb{R}$ *bounded and continuous. Moreover* $\hat{\pi}_t^n$ *converge weakly to* $\hat{\pi}_t$.

Proof. Proposition 5.5 guarantees that, for any sequence $s_n \to s$, $s > 0$,

$$\hat{\Sigma}^n(g)(s_n) \to \int_0^\infty g(\sqrt{s}x)x\exp\left(-\frac{x^2}{2}\right)dx = \hat{\Pi}(g)(s).$$

Then the thesis follows since we can use the Skorohod representation probability space as in Proposition 5.3, and in this space, for each $t > 0$, $\xi_t^n = \gamma_t(L^n) \to \zeta_t$, a.s., and on the other hand $P\{\omega :\ \zeta_t(\omega) = 0\} = 0$. Indeed therefore

$$\hat{\Sigma}^n(g)(\xi_t^n) \to \hat{\Pi}(g)(\zeta_t),\ \text{for each } g : \mathbb{R}^+ \to \mathbb{R} \text{ bounded and continuous} \quad \text{a.s.} \tag{59}$$

□

Remark 5.9. *Proposition 5.5 applies also to show that, for any sequence $s_n \to s$, $s > 0$ and for each $g : \mathbb{R}^+ \to \mathbb{R}$ bounded continuous,*

$$\Sigma^n(g)(s_n, l) \underset{n\to\infty}{\longrightarrow} \Pi(g)(s, l) = \int_0^\infty g(-l + \sqrt{s}x)x \exp\left(-\frac{x^2}{2}\right) dx. \tag{60}$$

Then, following the same procedure as in Theorem 5.8 we can state that

$$\Sigma^n(g)(\xi_t^n, L_t^n) \Rightarrow \Pi(g)(\zeta_t, \Lambda_t), \tag{61}$$

for each $g : \mathbb{R}^+ \to \mathbb{R}$ bounded and continuous, and that π_t^n converge weakly to π_t, so that Theorem 5.2 is completely achieved in the symmetric case.

5.2 Weak convergence of the filter: the asymptotically symmetric case

In this subsection our aim is to prove Theorem 5.2 in the asymptotically symmetric case defined by conditions **C1**, **C2**, **C3** and **A1**, **A2**, **A3**, with the same space-time scaling. The symmetric case investigated in previous subsection is obtained by defining on a suitable space $(\Omega, \mathcal{F}, P)$ a Poisson process Z_t of intensity 2λ and a symmetric random walk $V_k = \sum_{j=1}^k U_j$.
Starting from these processes, and in analogy with (40) and (41), we can define the process $\tilde{A}_t$ as the process counting the positive jumps of $\tilde{X}_t$, and the process $\tilde{N}_t$ as the process counting the negative jumps of $\tilde{X}_t$.
On $(\Omega, \mathcal{F})$ we consider the filtration $\{\mathcal{F}_t^n,\ t \in [0, T]\}$ generated by the rescaled processes $(\hat{A}_t^n, \hat{N}_t^n) = (\tilde{A}_{nt}, \tilde{N}_{nt})$, and the probability measure P^n, absolutely continuous with respect to P, such that

$$\frac{dP^n}{dP}\big|_{\mathcal{F}_t^n} = \mathcal{L}_t^n = \mathcal{L}_t^{A,n} \times \mathcal{L}_t^{N,n} \tag{62}$$

where

$$\mathcal{L}_t^{A,n} := \exp\left\{\int_0^t \log\frac{\lambda_n}{\lambda} d\hat{A}_s^n - \int_0^t n\left(\lambda_n - \lambda\right) ds\right\} = \exp\left\{\log\left(\frac{\lambda_n}{\lambda}\right)\hat{A}_t^n - n\left(\lambda_n - \lambda\right)t\right\} \tag{63}$$

$$\mathcal{L}_t^{N,n} := \exp\left\{\int_0^t \log\frac{\mu_n}{\lambda} d\hat{N}_s^n - \int_0^t n\left(\mu_n - \lambda\right) ds\right\} = \exp\left\{\log\left(\frac{\lambda_n}{\lambda}\right)\hat{N}_t^n - n\left(\lambda_n - \lambda\right)t\right\}. \tag{64}$$

Under the measure P, the processes $\hat{A}_t^n$, $\hat{N}_t^n$ are mutually independent Poisson processes with intensities $n\lambda$, $n\lambda$ while (see [3], Chapter VIII) under P^n the processes $\hat{A}_t^n$, $\hat{N}_t^n$ are mutually independent Poisson processes with intensities $n\lambda_n$, $n\mu_n$. Moreover, under P^n, the conditions **A1**, **A2**, **A3** are satisfied with $Z^n = Z$, $U_j^n = U_j$, namely:

1 $Z_t^n = \hat{A}_t^n + \hat{N}_t^n = Z_{nt}$ is a Poisson process with intensity $n(\lambda_n + \mu_n)$;

2 $Z_{nt}^n = Z_{nt}$ is a Poisson process with intensity $n(\lambda_n + \mu_n)$;

3 $\{U_k, k \in \mathbb{N}\}$ is a sequence of i.i.d. random variable with law

$$\begin{cases} P^n(U_k = \ \ \ 1) = & \frac{\lambda_n}{\lambda_n+\mu_n} \\ P^n(U_k = -1) = & \frac{\mu_n}{\lambda_n+\mu_n} \end{cases}$$

4 $\{U_k, k \in \mathbb{N}\}$ and Z_t^n are mutually independent.

We can now state the main result of this subsection.

Theorem 5.10. *Let $\mathcal{G}_t^n$ be the filtration generated by L_t^n, and g be a bounded and continuous function. Then*

$$E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big] \Rightarrow \hat{\Pi}(g)(\zeta_t), \tag{65}$$

where $\hat{\Pi}(s)$ is defined by (47).

Proof. The thesis is achieved if we prove that for any bounded and Lipschitz continuous function φ

$$E^n\big[\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big] \to E\big[\varphi(\hat{\Pi}(g)(\zeta_t))\big]$$

$$\begin{aligned}
&\Big|E^n\big[\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big] - E\big[\varphi(\hat{\Pi}(g)(\zeta_t))\big]\Big| \\
&\qquad = \Big|E\big[\mathcal{L}_t^n\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big] - E\big[\varphi(\hat{\Pi}(g)(\zeta_t))\big]\Big| \\
&\qquad \leq \quad \Big|E\big[\mathcal{L}_t^n\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big] - E\big[\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big]\Big| \\
&\qquad\quad + \Big|E\big[\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big] - E\big[\varphi(E\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big]\Big| \\
&\qquad\quad + \Big|E\big[\varphi(E\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big] - E\big[\varphi(\hat{\Pi}(g)(\zeta_t))\big]\Big|
\end{aligned}$$

The last addend in the above inequality converges to zero by Theorem 5.8 for the symmetric case.
The first one is bounded above by

$$\Big|E\big[\mathcal{L}_t^n\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big] - E\big[\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big]\Big| \leq \| \varphi \|_\infty E\big[\,|\mathcal{L}_t^n - 1|\,\big].$$

The second addend is bounded above by

$$\begin{aligned}
&E\Big[\,\big|\varphi(E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big]) - \varphi(E\big[g(Q_t^n)/\mathcal{G}_t^n\big])\big|\,\Big] \\
&\quad \leq L_\varphi E\Big[\,\big|E^n\big[g(Q_t^n)/\mathcal{G}_t^n\big] - E\big[g(Q_t^n)/\mathcal{G}_t^n\big]\big|\,\Big] \leq L_\varphi 2 \| g \|_\infty E\big[\,|\mathcal{L}_t^n - 1|\,\big],
\end{aligned}$$

where L_φ is the Lipschitz constant of φ, and the last inequality follows by Lemma 5.12 below. The thesis is achieved since $E\big[\,|\mathcal{L}_t^n - 1|\,\big]$ converges to zero, as is proven il Lemma 5.11 below. □

Lemma 5.11. *Let $\mathcal{L}_t^n := \frac{dP^n}{dP}\big|_{\mathcal{F}_t^n}$ be defined as in (62), then*

$$E\left[(\mathcal{L}_t^n)^2\right] = \exp\left(\frac{t}{\lambda}\left[(\sqrt{n}(\lambda_n - \lambda))^2 + (\sqrt{n}(\mu_n - \lambda))^2\right]\right) \tag{66}$$

Moreover, under the conditions **C1**, **C2**, **C3**, *on the space $(\Omega, \mathcal{F}, P)$*

$$E\big[\,|\mathcal{L}_t^n - 1|\,\big] \to 0, \tag{67}$$

and

$$\sup_n E\left[(\mathcal{L}_t^n)^2\right] < \infty, \tag{68}$$

Proof. We only need to prove (66), since

$$E^2\big[\,|\mathcal{L}_t^n-1|\,\big]\le E\big[\,|\mathcal{L}_t^n-1|^2\,\big]$$
$$=E\big[(\mathcal{L}_t^n)^2\big]-2E\big[\mathcal{L}_t^n\big]+1=E\big[(\mathcal{L}_t^n)^2\big]-1$$
$$=\exp\left(\frac{t}{\lambda}\left[(\sqrt{n}(\lambda_n-\lambda))^2+(\sqrt{n}(\mu_n-\lambda))^2\right]\right)-1\to 0,$$

and then, by condition **C3**, we get (67) and (68).

To this end we observe that

$$\mathcal{L}_t^n=\left(\frac{\lambda_n}{\lambda}\right)^{\hat{A}_t^n}\exp\left(-n\left(\lambda_n-\lambda\right)t\right)\times\left(\frac{\mu_n}{\lambda}\right)^{\hat{N}_t^n}\exp\left(-n\left(\mu_n-\lambda\right)t\right),$$

and so

$$E\left[(\mathcal{L}_t^n)^2\right]=\sum_{k=0}^{\infty}\left(\frac{\lambda_n}{\lambda}\right)^{2k}\frac{(n\lambda t)^k}{k!}\exp\left(-n\lambda t\right)\exp\left(-2n\left(\lambda_n-\lambda\right)t\right)\times$$
$$\times\sum_{k=0}^{\infty}\left(\frac{\mu_n}{\lambda}\right)^{2k}\frac{(n\lambda t)^k}{k!}\exp\left(-n\lambda t\right)\exp\left(-2n\left(\mu_n-\lambda\right)t\right)=$$
$$=\exp\left(\frac{t}{\lambda}\left[(\sqrt{n}(\lambda_n-\lambda))^2+(\sqrt{n}(\mu_n-\lambda))^2\right]\right).$$

□

Lemma 5.12. *Let $(\Omega,\mathcal{F},\mathbb{P})$ and $(\Omega,\mathcal{F},\mathbb{P}^0)$ be probability spaces with $\mathbb{P}$ and $\mathbb{P}^0$ equivalent probability measures, $\mathcal{G}\subset\mathcal{F}$, $P=\mathbb{P}|_{\mathcal{G}}$ and $P^0=\mathbb{P}^0|_{\mathcal{G}}$, then*

$$\mathbb{E}^0\left[\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]-\mathbb{E}\left[\psi/\mathcal{G}\right]\right|\right]\le 2\alpha\mathbb{E}^0\left[\left|1-(d\mathbb{P}/d\mathbb{P}^0)\right|\right]=2\alpha\mathbb{E}\left[\left|(d\mathbb{P}^0/d\mathbb{P})-1\right|\right],$$

for any random variable ψ bounded above by $\alpha=\|\psi\|_\infty$.
The same bound holds for $\mathbb{E}\left[\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]-\mathbb{E}\left[\psi/\mathcal{G}\right]\right|\right]$.

Proof. The proof is based on the Kallianpur-Striebel formula and is a particular case of Lemma 4.2 in [4].
For the sake of notational convenience we denote $\mathbb{L}=(d\mathbb{P}/d\mathbb{P}^0)$, $L=\mathbb{E}^0\left[\mathbb{L}/\mathcal{G}\right]=(dP/dP^0)$, $\mathbb{L}^0=(d\mathbb{P}^0/d\mathbb{P})$, and $L^0=\mathbb{E}\left[\mathbb{L}^0/\mathcal{G}\right]=(dP^0/dP)$

$$\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]-\mathbb{E}\left[\psi/\mathcal{G}\right]\right|=\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]-\frac{\mathbb{E}^0\left[\mathbb{L}\psi/\mathcal{G}\right]}{\mathbb{E}^0\left[\mathbb{L}/\mathcal{G}\right]}\right|$$
$$=\frac{1}{\mathbb{E}^0\left[\mathbb{L}/\mathcal{G}\right]}\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]\mathbb{E}^0\left[\mathbb{L}/\mathcal{G}\right]-\mathbb{E}^0\left[\mathbb{L}\psi/\mathcal{G}\right]\right|$$
$$=\frac{1}{L}\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]L-\mathbb{E}^0\left[\mathbb{L}\psi/\mathcal{G}\right]\right|$$
$$\le\frac{1}{L}\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]L-\mathbb{E}^0\left[\psi/\mathcal{G}\right]\right|+\left|\mathbb{E}^0\left[\psi/\mathcal{G}\right]-\mathbb{E}^0\left[\mathbb{L}\psi/\mathcal{G}\right]\right|.$$

Moreover

$$\Big|\mathbb{E}^0\big[\psi/\mathcal{G}\big]L-\mathbb{E}^0\big[\psi/\mathcal{G}\big]\Big|\le\|\psi\|_\infty\big|L-1\big|,$$

and

$$\Big|\mathbb{E}^0\big[\psi/\mathcal{G}\big]-\mathbb{E}^0\big[\mathbb{L}\psi/\mathcal{G}\big]\Big|\le\mathbb{E}^0\big[\psi\big|1-\mathbb{L}\big|/\mathcal{G}\big]\le\|\psi\|_\infty\mathbb{E}^0\big[\big|1-\mathbb{L}\big|/\mathcal{G}\big].$$

Therefore

$$\begin{aligned}&\big|\mathbb{E}^0\big[\psi/\mathcal{G}\big]-\mathbb{E}\big[\psi/\mathcal{G}\big]\big|\le\|\psi\|_\infty\frac{1}{L}\left\{\big|L-1\big|+\mathbb{E}^0\big[\big|1-\mathbb{L}\big|/\mathcal{G}\big]\right\}\\&\qquad\qquad(\text{ taking into account that } 1/L=1/(dP/dP^0)=(dP^0/dP)=L^0)\\&=\|\psi\|_\infty L^0\left\{\big|L-1\big|+\mathbb{E}^0\big[\big|1-\mathbb{L}\big|/\mathcal{G}\big]\right\},\end{aligned}$$

and so

$$\mathbb{E}^0\Big[\Big|\mathbb{E}^0\big[\psi/\mathcal{G}\big]-\mathbb{E}\big[\psi/\mathcal{G}\big]\Big|\Big]\le\|\psi\|_\infty\mathbb{E}^0\left[\big|L-1\big|+\mathbb{E}^0\big[\big|1-\mathbb{L}\big|/\mathcal{G}\big]\right].$$

Then the proof is achieved by noting that, by Jensen inequality,

$$\mathbb{E}^0\left[\big|L-1\big|\right]=\mathbb{E}^0\left[\Big|(dP/dP^0)-1\Big|\right]=\mathbb{E}^0\left[\Big|\mathbb{E}^0\big[(d\mathbb{P}/d\mathbb{P}^0)-1/\mathcal{G}\big]\Big|\right]\le\mathbb{E}^0\left[\Big|(d\mathbb{P}^0/d\mathbb{P})-1\Big|\right].$$

□

6 The M/M/1 queueing model: an approximation for the filter

As in Section 5, we suppose that under the measure P the process $\tilde{Q}_t^n$ is an M/M/1 queue with parameters λ,λ (symmetric case), while, under the measure P^n the process $\tilde{Q}_t^n$ is an M/M/1 queue with parameters λ_n,μ_n (general case).

The explicit form (22) for the filter of the discrete system is $\hat{\Sigma}^n(g)(s)$ (see (23)), evaluated in $s=\xi_t^n$, and it is evident from (25) that, from a computational point of view, it can be very hard to use it. Then the problem arises of finding a good approximation for this filter which, at the same time, is simpler to handle, and depends on the really observed trajectory, so that it can be actually used in applications.

A natural candidate is $\hat{\Pi}(g)(\xi_t^n)$, where $\hat{\Pi}(g)(s)$ is defined by (47). We will show in Theorem 6.1 that $\hat{\Pi}(g)(\xi_t^n)$, approximates the discrete filter in the $L_1\,(\Omega\times[0,T])$-norm, for each $T>0$. The statement holds both for the symmetric case and for the general one.

Before stating formally the result we recall that $\hat{\Sigma}^n(g)(s)$ depends on the law of $\tilde{Q}_t^n$, or equivalently on the probability measure we use (see Remark 3.1). Then, to emphasize this dependence we write, in analogy with (23)

$$\hat{\Sigma}^n_{P^n}(g)(s)=E^n\left[g(X_s^{n*})/\{L_s^{n*}<b_n\}\right],\tag{69}$$

and

$$\hat{\Sigma}^n_{P}(g)(s)=E\left[g(X_s^{n*})/\{L_s^{n*}<b_n\}\right].\tag{70}$$

Theorem 6.1. *For all g bounded and continuous and for each $T > 0$*

$$\int_0^T E\Big|\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\Big|dt \underset{n\to\infty}{\longrightarrow} 0.$$

$$\int_0^T E^n\Big|\hat{\Sigma}^n_{P^n}(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\Big|dt \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. Consider first the symmetric case.

The limit we are looking for depends only on the distribution of ξ^n_t, therefore, as observed in Remark 5.4, using the Skorohod representation theorem as in the proof of Proposition 5.3, we can assume that

$$\xi^n_t(\omega) \to \zeta_t(\omega) \quad dP \times dt - \text{a.e.}$$

Proposition 5.5 implies that

$$\hat{\Sigma}^n(g)(s_n) \equiv \hat{\Sigma}^n_P(g)(s_n) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(g)(s), \tag{71}$$

whenever $s_n \to s$, $s > 0$, moreover it is clear that

$$\hat{\Pi}(g)(s_n) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(g)(s),$$

whenever $s_n \to s$, $s > 0$.
Combining these results we get

$$\hat{\Sigma}^n_P(g)(\xi^n_t) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(g)(\zeta_t) \qquad \text{and} \qquad \hat{\Pi}(g)(\xi^n_t) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(g)(\zeta_t) \tag{72}$$

for each (ω, t) such that $\zeta_t(\omega) > 0$.

The observation that $\{(\omega, t) \in \Omega \times [0, T]$ such that $\zeta_t(\omega) = 0\}$ is a zero measure set with respect to $dP \times dt$, and an easy application of the dominated convergence theorem imply that

$$\int_0^T E\left[\Big|\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\zeta_t)\Big|^p\right] \to 0, \qquad \text{for any } p > 0$$

and

$$\int_0^T E\left[\Big|\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\Big|^p\right] \to 0, \qquad \text{for any } p > 0. \tag{73}$$

Consider now the asymmetric case.

Note that

$$\begin{aligned}&\int_0^T E^n\left[\Big|\hat{\Sigma}^n_{P^n}(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\Big|\right]dt \\ &\le \int_0^T E^n\left[\Big|\hat{\Sigma}^n_{P^n}(g)(\xi^n_t) - \hat{\Sigma}^n_P(g)(\xi^n_t)\Big|\right]dt + \int_0^T E^n\left[\Big|\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\Big|\right]dt\end{aligned}$$

Therefore by Lemma 5.12

$$E^n\left[\left|\hat{\Sigma}^n_{P^n}(g)(\xi^n_t) - \hat{\Sigma}^n_P(g)(\xi^n_t)\right|\right] = E^n\big[\,\big|E^n\big[g(Q^n_t)/\mathcal{G}^n_t\big] - E\big[g(Q^n_t)/\mathcal{G}^n_t\big]\big|\,\big]$$
$$\leq 2\parallel g\parallel_\infty E^n\Big[\Big|1-(dP/dP^n)|_{\mathcal{F}^n_t}\Big|\Big] = 2\parallel g\parallel_\infty E\big[\,|\mathcal{L}^n_t - 1|\,\big].$$

Moreover, by Cauchy-Schwartz inequality

$$E^n\left[\left|\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\right|\right] = E\left[\mathcal{L}^n_t\left|\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\right|\right]$$
$$\leq E\left[(\mathcal{L}^n_t)^2\right] E\left[\left(\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\right)^2\right].$$

Summarizing

$$\int_0^T E^n\left[\left|\hat{\Sigma}^n_{P^n}(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\right|\right] dt$$
$$\leq 2\parallel g\parallel_\infty \int_0^T E\big[\,|\mathcal{L}^n_t - 1|\,\big]dt + \int_0^T E\left[(\mathcal{L}^n_t)^2\right] E\left[\left(\hat{\Sigma}^n_P(g)(\xi^n_t) - \hat{\Pi}(g)(\xi^n_t)\right)^2\right] dt,$$

and the thesis follows by the bounds (67) and (68) of Lemma 5.11 and by (73) of the symmetric case. □

Remark 6.2. *Note that*

$$E^n\left[\left|\hat{\Sigma}^n_{P^n}(g)(\xi^n_t) - \hat{\Sigma}^n_P(g)(\xi^n_t)\right|\right] = E^n\big[\,\big|E^n\big[g(Q^n_t)/\mathcal{G}^n_t\big] - E\big[g(Q^n_t)/\mathcal{G}^n_t\big]\big|\,\big] \tag{74}$$

*is a particular case of the approximation problems considered in [4], where the observation process is a counting process: $\mathcal{G}^n_t$ coincides with the σ-algebra generated by the jump times of the observation process. Nevertheless in order to apply Theorem 2.1 in [4]) we need to assume the further and more restrictive assumptions that $n(\mu_n - \lambda) \to 0$ and $n(\lambda - \lambda_n) \to 0$, which, under condition **C3**, may even converge to infinity.*

7 The M/M/1 queueing model: observing the idle time process

In this section we are interested in the conditional law of the M/M/1 queue $\tilde{Q}^n_s$, when the observation process is the *idle time* process, i.e.

$$\tilde{C}^n_t = \int_0^t \mathbb{I}\left(\tilde{Q}^n_s = 0\right) ds,$$

the cumulative time the queue has spent in 0, up to t.

Equivalently one can consider as observation process the bivariate point process $(\tilde{I}^n_t, \tilde{B}^n_t)$, where $\tilde{I}^n_t$ is the process that counts the times when the system starts an idle period and $\tilde{B}^n_t$ is the process that counts the times when the system starts a busy period, that is

$$\tilde{I}^n_t = \int_0^t \mathbb{I}(\tilde{Q}^n_{s-} = 1) d\tilde{N}^n_s \tag{75}$$

$$\tilde{B}^n_t = \int_0^t \mathbb{I}(\tilde{Q}^n_{s-} = 0) d\tilde{A}^n_s. \tag{76}$$

Indeed the filtration generated by the idle time process $\tilde{C}_t^n$ and the filtration generated by the observation process $(\tilde{I}_t^n, \tilde{B}_t^n)$ coincide, or more precisely $\mathcal{F}_{t^+}^{\tilde{C}^n} = \mathcal{F}_t^{\tilde{I}^n, \tilde{B}^n}$.

Our first aim is to study the conditional law

$$E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big], \tag{77}$$

where for the notational convenience we denote

$$\tilde{\mathcal{H}}_t^n = \mathcal{F}_t^{\tilde{I}^n, \tilde{B}^n} = \mathcal{F}_{t^+}^{\tilde{C}^n},$$

and the explicit expression for the filter (77) in terms of $\gamma_t^0(\tilde{Q}^n)$, is given in (84).
Then we consider the rescaled processes

$$Q_t^n := \frac{\tilde{Q}_{nt}^n}{\sqrt{n}}, \qquad I_t^n := \frac{\tilde{I}_{nt}^n}{\sqrt{n}}, \qquad B_t^n := \frac{\tilde{B}_{nt}^n}{\sqrt{n}}, \qquad C_t^n := \sqrt{n}\mu_n \tilde{C}_{nt}^n,$$

and the conditional law of the rescaled queue

$$E\big[g(Q_t^n)/\mathcal{H}_t^n\big], \tag{78}$$

where $\mathcal{H}_t^n$ is the filtration generated by the rescaled observation

$$\mathcal{H}_t^n = \tilde{\mathcal{H}}_{nt}^n = \mathcal{F}_t^{I^n, B^n} = \mathcal{F}_{t^+}^{C^n}.$$

We are interested in the limit behaviour of the filter (78) under the same assumption **C1**, **C2**, **C3** and **A1**, **A2**, **A3** of Section 5. Under these assumptions we already know that Q_t^n converge weakly to a driftless Brownian motion W_t with diffusion coefficient 2λ.
Moreover if one defines $\bar{X}_t^n := Q_t^n - C_t^n$, then clearly $Q_t^n = \bar{X}_t^n + C_t^n$, and therefore, since by definition C_t^n increases only when $Q_t^n = 0$, the pair (Q_t^n, C_t^n) is the solution of the Skorohod problem corresponding to $\bar{X}_t^n$. Consequently

$$\big(\bar{X}_t^n, Q_t^n, C_t^n\big) \Rightarrow (W_t, W_t + \Lambda_t, \Lambda_t),$$

where as usual Λ_t is the local time of W_t (for a deeper investigation of these results, we refer to Kurtz [12]) [2] .
It is therefore natural to expect that $E\big[g(Q_t^n)/\mathcal{H}_t^n\big]$ converges weakly to $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^{\Lambda}\big] = \hat{\Pi}(g)(\zeta_t)$. This result is proven in Theorem 7.4. Moreover $\hat{\Pi}(g)(\gamma_t^0(Q^n))$ is a good approximation of the filter for the rescaled model (see Theorem 7.6).
Before giving the technical details we observe that the results of this section rely on the results of Sections 5 and 6. In these sections we have used the measures P in the symmetric case, and P^n otherwise. However in this section, as in Sections 2 and 3, we do not distinguish between

[2] By the above considerations

$$C_t^n \Rightarrow \Lambda_t \qquad \text{in } D_{\mathbb{R}}([0,T])$$

and

$$C_t^n := \sqrt{n}\mu_n \tilde{C}_{nt}^n = \sqrt{n}\mu_n \int_0^t \mathbb{I}\left(\tilde{Q}_{ns}^n = 0\right) ds = \sqrt{n}\mu_n \int_0^t \mathbb{I}\left(Q_s^n = 0\right) ds.$$

In general weak convergence does not imply immediately the convergence of the expected values, i.e. that

$$\lim_{n\to\infty} E^n\left[\mu_n \int_0^t \sqrt{n}\mathbb{I}(Q_s^n = 0) ds\right] = E\left[\Lambda_t\right].$$

Nevertheless, after some explicit calculations, it is possible to show that this is the case (see (95) in Appendix B).

the two cases for notational convenience, unless necessary.

Let $\{\sigma_k^{Bn},\ k\in\mathbb{N}\}$ and $\{\sigma_k^{In},\ k\in\mathbb{N}\}$ be the jump times of the process $\tilde I_t^n$ and the process $\tilde B_t^n$, respectively. Under the assumption $\tilde Q_0^n=0$, it easy to verify that

$$\sigma_k^{Bn}<\sigma_k^{In}<\sigma_{k+1}^{Bn}<\sigma_{k+1}^{In}, \qquad \text{for each } k\geq 1,$$

and that $Q_t^n=0$ when $\sigma_k^{In}\leq t<\sigma_{k+1}^{Bn}$, for $k\geq 0$, while $Q_t^n>0$ otherwise.
We start by observing some regenerative properties of the above jump times, which are fundamental in the sequel.

Lemma 7.1. *For each $k\in\mathbb{N}$ the processes $\tilde Q_{k,t}^{In}=\tilde Q_{t+\sigma_k^{In}}^n-\tilde Q_{\sigma_k^{In}}^n$ and $\tilde Q_{k,t}^{Bn}=\tilde Q_{t+\sigma_k^{Bn}}^n-\tilde Q_{\sigma_k^{Bn}}^n$ are independent of $\mathcal{F}_{\sigma_k^{In}}^{\tilde Q^n}$ and $\mathcal{F}_{\sigma_k^{Bn}}^{\tilde Q^n}$ respectively. Moreover, the process $\tilde Q_{k,t}^{In}$ has the same law as the process $\tilde Q_t^n$.*

Proof. Clearly $\tilde Q_{k,0}^{In}=\tilde Q_0^n=0$, moreover it is easy to see that $\tilde Q_{k,t}^{In}$ solves the same martingale problem as $\tilde Q_t^n$, hence these processes share the same law. The independence property follows by the strong Markov property of $\tilde Q_t^n$, since $\tilde Q_{\sigma_k^{In}}^n=0$:

$$\begin{aligned}
&E\left[\exp\left(iu\left(\tilde Q_{t+\sigma_k^{In}}^n-\tilde Q_{\sigma_k^{In}}^n\right)\right)/\mathcal{F}_{\sigma_k^{In}}^{\tilde Q^n}\right]=E\left[\exp\left(iu\left(\tilde Q_{t+\sigma_k^{In}}^n-\tilde Q_{\sigma_k^{In}}^n\right)\right)/\tilde Q_{\sigma_k^{In}}^n\right]=\\
=&E\left[\exp\left(iu\left(\tilde Q_{t+\sigma_k^{In}}^n-\tilde Q_{\sigma_k^{In}}^n\right)\right)\right]=E\left[\exp\left(iu\left(\tilde Q_{t+\sigma_k^{In}}^n-\tilde Q_{\sigma_k^{In}}^n\right)\right)\right]\\
=&E\left[\exp\left(iu\left(\tilde Q_t^n\right)\right)\right].
\end{aligned}$$

Similar arguments apply to show that the process $\tilde Q_{k,t}^{Bn}$ is independent of $\mathcal{F}_{\sigma_k^{Bn}}^{\tilde Q^n}$. □

The process $\tilde I_t^n$ is a renewal process, and $\tilde B_t^n$ is a delayed renewal process, i.e. (see, for example, [7] VI.7.3 page 187) the sequence $\{\sigma_{k+1}^{Bn}-\sigma_k^{Bn}\}_{k\geq 0}$ is a sequence of mutually independent random variables and $\{\sigma_{k+1}^{Bn}-\sigma_k^{Bn}\}_{k\geq 1}$ are also identically distributed. Also $\{\sigma_k^{In}-\sigma_k^{Bn}\}_{k\geq 1}$ is a sequence of mutually independent random variables.
In the setting of this section, the above considerations and Lemma 7.1 guarantee that the filter of $\tilde Q_t^n$ given $\tilde{\mathcal{H}}_t^n$ admits a representation similar to that given in Proposition 2.1. More precisely

Proposition 7.2. *The conditional law of $\tilde Q_t^n$ given $\tilde{\mathcal{H}}_t^n$ admits the following representation*

$$\begin{aligned}
E\big[g(\tilde Q_t^n)/\tilde{\mathcal{H}}_t^n\big]=&\,\mathbb{I}\left(\tilde Q_t^n=0\right)g(0)+ \qquad (79)\\
&+\mathbb{I}\left(\tilde Q_t^n>0\right)\sum_{j=1}^{\infty}\frac{E\Big[g(\tilde Q_{s+\sigma_j^{Bn}}^n-\tilde Q_{\sigma_j^{Bn}}^n+1)\mathbb{I}\left(\sigma_j^{In}-\sigma_j^{Bn}>s\right)\Big]_{s=t-\sigma_j^{Bn}}}{E\Big[\mathbb{I}(\sigma_j^{In}-\sigma_j^{Bn}>s)\Big]_{s=t-\sigma_j^{Bn}}}\mathbb{I}\{\sigma_j^{Bn}\leq t<\sigma_j^{In}\}.
\end{aligned}$$

Even if this Proposition is quite similar to Proposition 2.1, we sketch the proof because the assumptions on the processes involved are a little different.
Moreover we point out that since the processes involved are all Markovian, the proof of (79) could be given by the techniques used in [5].

Proof. We begin by noting that $\mathbb{I}\left(\tilde Q_t^n>0\right)$ is $\tilde{\mathcal{H}}_t^n$-adapted. Moreover we can state (see [3] Chapter. III T5)

$$\tilde{\mathcal{H}}_t^n\cap\{\sigma_j^{Bn}\leq t<\sigma_j^{In}\}\cap\{\tilde Q_t^n>0\}=\tilde{\mathcal{H}}_{\sigma_j^{Bn}}^n\cap\{\sigma_j^{Bn}\leq t<\sigma_j^{In}\}\cap\{\tilde Q_t^n>0\}.$$

By using, as in Proposition 2.1 the arguments of Proposition 3.1 in [13], we obtain

$$\mathbb{I}\left(\tilde{Q}^n_t > 0\right) E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big]\mathbb{I}\{\sigma^{Bn}_j \le t < \sigma^{In}_j\} =$$

$$\mathbb{I}\left(\tilde{Q}^n_t > 0\right) \frac{E\Big[g(\tilde{Q}^n_t)\mathbb{I}\left(\sigma^{In}_j - \sigma^{Bn}_j > t - \sigma^{Bn}_j\right)/\tilde{\mathcal{H}}^n_{\sigma^{Bn}_j}\Big]}{P\Big(\sigma^{In}_j - \sigma^{Bn}_j > t - \sigma^{Bn}_j/\tilde{\mathcal{H}}^n_{\sigma^{Bn}_j}\Big)}\mathbb{I}\{\sigma^{Bn}_j \le t < \sigma^{In}_j\}.$$

Note that

$$\{\sigma^{In}_j - \sigma^{Bn}_j > s\} = \left\{\inf_{0\le u\le s} \tilde{Q}^{Bn}_{j,u} > 0\right\}$$

and is independent of $\mathcal{F}^{\tilde{Q}^n}_{\sigma^{Bn}_j}$ by Lemma 7.1.
Then, denoting by $s = t - \sigma^{Bn}_j$, so that

$$\tilde{Q}^n_t = \tilde{Q}^n_{\sigma^{Bn}_j + s} = \tilde{Q}^n_{\sigma^{Bn}_j + s} - \tilde{Q}^n_{\sigma^{Bn}_j} + 1 = \tilde{Q}^{Bn}_{j,s} + 1$$

$$\frac{E\Big[g(\tilde{Q}^n_{\sigma^{Bn}_j + s})\mathbb{I}\left(\sigma^{In}_j - \sigma^{Bn}_j > s\right)\Big|_{s=t-\sigma^{Bn}_j}/\tilde{\mathcal{H}}^n_{\sigma^{Bn}_j}\Big]}{P\Big(\left(\sigma^{In}_j - \sigma^{Bn}_j > s\right)\Big|_{s=t-\sigma^{Bn}_j}/\tilde{\mathcal{H}}^n_{\sigma^{Bn}_j}\Big)}$$

$$= \frac{E\Big[g\left(\tilde{Q}^{Bn}_{j,s} + 1\right)\mathbb{I}\left(\sigma^{In}_j - \sigma^{Bn}_j > s\right)\Big]_{s=t-\sigma^{Bn}_j}}{E\Big[\mathbb{I}\left(\sigma^{In}_j - \sigma^{Bn}_j > s\right)\Big]_{s=t-\sigma^{Bn}_j}}, \tag{80}$$

Then the thesis follows by writing $g(\tilde{Q}^n_t)$ as

$$g(\tilde{Q}^n_t) = \mathbb{I}\left(\tilde{Q}^n_t = 0\right) g(0) + \mathbb{I}\left(\tilde{Q}^n_t > 0\right)\sum_{j=1}^{\infty} g(\tilde{Q}^n_t)\mathbb{I}\{\sigma^{Bn}_j \le t < \sigma^{In}_j\},$$

and, therefore, $E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big]$ as

$$E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big] = \mathbb{I}\left(\tilde{Q}^n_t = 0\right) E\big[g(0)/\tilde{\mathcal{H}}^n_t\big] + \mathbb{I}\left(\tilde{Q}^n_t > 0\right)\sum_{j=1}^{\infty} E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big]\mathbb{I}\{\sigma^{Bn}_j \le t < \sigma^{In}_j\}.$$

□

It is important to note that

$$\sigma^{In}_j - \sigma^{Bn}_j = \inf\left\{u \ge 0 :\ \tilde{Q}^{Bn*}_{j,u} + 1 = 0\right\}, \tag{81}$$

and that the process $\tilde{Q}^{Bn}_{j,s} + 1$ for $s <$
$sigma^{In}_j - \sigma^{Bn}_j$ behaves like the continuous time random walk $\tilde{X}^n_s + 1$ for $s < \tilde{\sigma}^n_1 = \inf\left\{u \ge 0 :\ \tilde{X}^n_u = -1\right\}$,
and hence

$$\frac{E\Big[g\left(\tilde{Q}^{Bn}_{j,s} + 1\right)\mathbb{I}\left(\sigma^{In}_j - \sigma^{Bn}_j > s\right)\Big]}{E\Big[\mathbb{I}\left(\sigma^{In}_j - \sigma^{Bn}_j > s\right)\Big]} = \frac{E\Big[g\left(\tilde{X}^n_s + 1\right)\mathbb{I}\left(\tilde{\sigma}^n_1 > s\right)\Big]}{E\Big[\mathbb{I}\left(\tilde{\sigma}^n_1 > s\right)\Big]}. \tag{82}$$

As a consequence

$$E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big] = \mathbb{I}\left(\tilde{Q}^n_t = 0\right) g(0) + \tag{83}$$
$$+\mathbb{I}\left(\tilde{Q}^n_t > 0\right) \sum_{j=1}^{\infty} \frac{E\Big[g\left(\tilde{X}^n_s + 1\right)\mathbb{I}\left(\tilde{\sigma}^n_1 > s\right)\Big]_{s=t-\sigma^{Bn}_j}}{E\Big[\mathbb{I}\left(\tilde{\sigma}^n_1 > s\right)\Big]_{s=t-\sigma^{Bn}_j}} \mathbb{I}\{\sigma^{Bn}_j \le t < \sigma^{In}_j\}.$$

Observing that, by definition (6),

$$\gamma^0_t(\tilde{Q}^n) = t - \sup\{s < t \text{ such that } \tilde{Q}^n_s = 0\} = \sum_{j=1}^{\infty}(t - \sigma^{Bn}_j)\mathbb{I}\{\sigma^{Bn}_j \le t < \sigma^{In}_j\}$$

we can rewrite

$$E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big] = \mathbb{I}\left(\tilde{Q}^n_t = 0\right) g(0) + \mathbb{I}\left(\tilde{Q}^n_t > 0\right) \left.\frac{E\Big[g\left(\tilde{X}^n_s + 1\right)\mathbb{I}\left(\tilde{\sigma}^n_1 > s\right)\Big]}{E\Big[\mathbb{I}\left(\tilde{\sigma}^n_1 > s\right)\Big]}\right|_{s=\gamma^0_t(\tilde{Q}^n)}. \tag{84}$$

The above considerations leads us to state the following result

Theorem 7.3. *Consider the rescaled process $Q^n_t = \frac{\tilde{Q}^n_{nt}}{\sqrt{n}}$, the rescaled observation processes $I^n_t = \frac{\tilde{I}^n_{nt}}{\sqrt{n}}$ and $B^n_t = \frac{\tilde{B}^n_{nt}}{\sqrt{n}}$, and denote by $\mathcal{H}^n_t$ the history generated by (I^n_u, B^n_u) for $u \le t$, i.e.*

$$\mathcal{H}^n_t = \mathcal{F}^{I^n,B^n}_t = \tilde{\mathcal{H}}^n_{nt}.$$

Then

$$E\big[g(Q^n_t)/\mathcal{H}^n_t\big] = \mathbb{I}\left(Q^n_t = 0\right) g(0) + \mathbb{I}\left(Q^n_t > 0\right) \hat{\Sigma}^n(\overline{g}_n)(\gamma^0_t(Q^n)), \tag{85}$$

where $\hat{\Sigma}^n$ is the functional defined in (23), and $\overline{g}_n(x) = g(x + \frac{1}{\sqrt{n}})$.

Proof. Equality (84) implies

$$E\big[g(Q^n_t)/\mathcal{H}^n_t\big] = \mathbb{I}\left(Q^n_t = 0\right) g(0) + \mathbb{I}\left(Q^n_t > 0\right) \left.\frac{E\Big[g\left(X^n_s + \frac{1}{\sqrt{n}}\right)\mathbb{I}\left(\sigma^n_1 > s\right)\Big]}{E\Big[\mathbb{I}\left(\sigma^n_1 > s\right)\Big]}\right|_{s=\gamma^0_t(Q^n)}$$

and clearly

$$\frac{E\Big[g\left(X^n_s + \frac{1}{\sqrt{n}}\right)\mathbb{I}\left(\sigma^n_1 > s\right)\Big]}{E\Big[\mathbb{I}\left(\sigma^n_1 > s\right)\Big]} = \hat{\Sigma}^n(\overline{g}_n)(s).$$

□

As a consequence of the above Theorem, for any g uniformly continuous

$$\begin{aligned} E\big[g(Q^n_t)/\mathcal{H}^n_t\big] &= \mathbb{I}\left(Q^n_t = 0\right) g(0) + \mathbb{I}\left(Q^n_t > 0\right) \hat{\Sigma}^n(\overline{g}_n)(\gamma^0_t(Q^n)) \\ &= \mathbb{I}\left(Q^n_t = 0\right) g(0) + \mathbb{I}\left(Q^n_t > 0\right) \hat{\Sigma}^n(g)(\gamma^0_t(Q^n)) + \varepsilon(n, g) \end{aligned} \tag{86}$$

with

$$|\varepsilon(n,g)| \le \omega_g(\frac{1}{\sqrt{n}}) = \sup_{|x-y|\le \frac{1}{\sqrt{n}}} \big|g(x)-g(y)\big|. \tag{87}$$

We can now state the main result of this section.

Theorem 7.4. *For any g uniformly continuous*

$$E\big[g(Q_t^n)/\mathcal{H}_t^n\big] \Rightarrow E\big[g(W_t)/\mathcal{F}_t^{\Lambda}\big] = \hat{\Pi}(g)(\zeta_t), \qquad \textit{for any } t\ge 0.$$

Proof. It is sufficient to prove that

$$\mathbb{I}\,(Q_t^n = 0) \xrightarrow{Prob} 0, \qquad \text{for any } t \ge 0 \tag{88}$$

and that

$$\hat{\Sigma}^n(g)(\gamma_t^0(Q^n)) \Rightarrow \hat{\Pi}(g)(\zeta_t), \qquad \text{for any } t\ge 0, \tag{89}$$

so that

$$\mathbb{I}\,(Q_t^n = 0)\, g(0) + \varepsilon(n,g) \xrightarrow{Prob} 0$$

and

$$\mathbb{I}\,(Q_t^n > 0)\, \hat{\Sigma}^n(g)(\gamma_t^0(Q^n)) \Rightarrow \hat{\Pi}(g)(\zeta_t),$$

since (88) is equivalent to $\mathbb{I}\,(Q_t^n > 0) \xrightarrow{Prob} 1$.

As recalled in (42), the sequence Q_t^n converges weakly to a reflected Brownian motion $W_t + \Lambda_t$. Then, the limit (88) can be obtained by noting that

- the function $\mathbb{I}\,(x=0)$ has a discontinuity point at $x=0$,
- $P(W_t+\Lambda_t = 0) = 0$,
- the continuous mapping theorem implies that $\mathbb{I}\,(Q_s^n = 0) \Rightarrow 0$.

To prove (89) we use the weak convergence of $\gamma_t^0(Q^n)$ to ζ_t (see Proposition 5.3), and same techniques of the previous sections, in particular Remark 5.4, Proposition 5.5.

□

Remark 7.5. *As already observed at the beginning of this section,*

$$\big(\bar{X}_t^n, Q_t^n, C_t^n\big) \Rightarrow (W_t, W_t+\Lambda_t, \Lambda_t)$$

where $\bar{X}_t^n := Q_t^n - C_t^n$. Thanks to the continuity of the limit processes, the convergence can be considered in the space $D_{\mathbb{R}^3[0,\infty)}$ endowed with the topology of the uniform convergence on compact sets. Moreover it is interesting to note that $\gamma_t^0(Q^n) = \gamma_t(C^n) \Rightarrow \gamma_t(\Lambda) = \zeta_t$,
Then, similarly to Proposition 5.3, it is possible to prove that

$$\big(\gamma_t^0(Q^n), \gamma_t(C^n), C_t^n\big) \Rightarrow \big(\gamma_t^0(W_t+\Lambda_t), \gamma_t^0(\Lambda_t), \Lambda_t\big) = (\zeta_t, \zeta_t, \Lambda_t)\,,$$

and therefore an alternative proof of the previous Theorem can be achieved, by using these properties.

We end this section by noting that, even in this new situation, it is possible to give the same approximation for the filter as in Theorem 6.1, namely for $E\big[g(Q_t^n)/\mathcal{H}_t^n\big]$ in the symmetric case and for $E^n\big[g(Q_t^n)/\mathcal{H}_t^n\big]$ in the asymptotically symmetric case. More precisely the following result holds

Theorem 7.6. *For all g bounded and continuous and for each $T > 0$*

$$\int_0^T E\Big|E\big[g(Q_t^n)/\mathcal{H}_t^n\big] - \hat{\Pi}(g)(\gamma_t^0(Q^n))\Big|dt \underset{n\to\infty}{\longrightarrow} 0.$$

$$\int_0^T E^n\Big|E^n\big[g(Q_t^n)/\mathcal{H}_t^n\big] - \hat{\Pi}(g)(\gamma_t^0(Q^n))\Big|dt \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. For the symmetric case, note that

$$\begin{aligned}
&\Big|E\big[g(Q_t^n)/\mathcal{H}_t^n\big] - \hat{\Pi}(g)(\gamma_t^0(Q^n))\Big|^p = \\
&\quad = \Big|\mathbb{I}\left(Q_t^n = 0\right) g(0) + \mathbb{I}\left(Q_t^n > 0\right) \hat{\Sigma}^n(g)(\gamma_t^0(Q^n)) + \varepsilon(n,g) - \hat{\Pi}(g)(\gamma_t^0(Q^n))\Big|^p \\
&\qquad \leq C(p)\Big|\mathbb{I}\left(Q_t^n = 0\right)\, \Big(g(0) + \hat{\Sigma}^n(g)(\gamma_t^0(Q^n))\Big) + \varepsilon(n,g)\Big|^p + C(p)\Big|\hat{\Sigma}^n(g)(\gamma_t^0(Q^n)) - \hat{\Pi}(g)(\gamma_t^0(Q^n))\Big|^p,
\end{aligned}$$

where $C(p)$ is a suitable constant, and $\varepsilon(n,g)$ is defined in (87).
Then

$$\begin{aligned}
&E\left[\Big|E\big[g(Q_t^n)/\mathcal{H}_t^n\big] - \hat{\Pi}(g)(\gamma_t^0(Q^n))\Big|^p\right] \leq \\
&\leq C(p)E\left[\Big|\mathbb{I}\left(Q_t^n = 0\right) 2 \parallel g \parallel_\infty + \varepsilon(n,g)\Big|^p\right] + C(p)E\left[\Big|\hat{\Sigma}^n(g)(\gamma_t^0(Q^n)) - \hat{\Pi}(g)(\gamma_t^0(Q^n))\Big|^p\right].
\end{aligned}$$

The thesis follows since both the addends at the right hand side of the previous inequality converge to zero. The first addend at converges to zero by the bounded convergence theorem. To prove that the the second addend converges to zero one has just to substitute ξ_t^n with $\gamma_t^0(Q^n)$ in the proof of Theorem 6.1.
The proof in the asymptotically symmetric case is analogous. □

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A Appendix

A.1 Identification of the interpolating Brownian motion model

W_t^n defined by (32) admits the following representation

$$W_t^n = \frac{1}{2^n} V^n_{Z_t^n}, \tag{90}$$

where

- Z_t^n is the point process defined by the sequence τ_k^n;

- V_k^n is the discrete-time process satisfying the relation

  $$V_k^n = V_{k-1}^n + U_k^n,$$

  where $\{U_k, k \in \mathbb{N}\}$ is the sequence of $\{-1, 1\}$-valued random variables defined by the rule $U_k^n = 2^n \left(W_{\tau_k^n} - W_{\tau_{k-1}^n} \right)$.

Let us consider the interarrival time sequence $\{T_k^n = \tau_k^n - \tau_{k-1}^n,\ k \geq 1\}$ of the point process Z_t^n. The following two lemmas are an easy consequence of the strong Markov property of W and of the fact that W and $-W$ have the same law.

Lemma A.1. *Z_t^n is a renewal process, that is $\{T_k^n, k \geq 1\}$ ia a sequence of i.i.d. random variables.*

Lemma A.2. *V_k^n is a symmetric random walk, i.e.*

$$V_k^n = V_{k-1}^n + U_k^n,$$

$\{U_k^n, k \geq 1\}$ is a sequence of i.i.d. random variables s.t. $P(U_k^n = 1) = P(U_k^n = -1) = \frac{1}{2}$

Moreover the following result can be found, for example, in [6] page 480 or [7] page 342.

Lemma A.3. *The distribution function F^n of the random variables T_k^n is*

$$F^n(t) = P\big(T_k^n \leq t\big) = 4 \sum_{j=0}^{+\infty} (-1)^j \frac{1}{\sqrt{2\pi}} \int_{\frac{(2j+1)}{2^n \sqrt{t}}}^{+\infty} \exp\left(-\frac{x^2}{2} \right) dx \tag{91}$$

and its generating function is

$$E\left[\exp\left(-\alpha T_k^n\right)\right] = \frac{1}{\cosh\left(\frac{\sqrt{2\alpha}}{2^n}\right)}.$$

Remark A.4. *Note that the distribution function (91) can be written in the equivalent way*

$$P\big(T_k^n \leq t\big) = 4 \sum_{j=0}^{+\infty} (-1)^j P\left(W_t \geq \frac{(2j+1)}{2^n} \right).$$

Lemma A.5. *The processes* $\{Z^n_t, t \in \mathbb{R}^+\}$ *and* $\{V^n_k, k \in \mathbb{N}\}$ *are mutually independent.*

Proof. Note that, for each $k \in \mathbb{N}$, U^n_k and $(T^n_1, ...T^n_{k-1})$ are independent. In fact U^n_k and $\mathcal{F}_{\tau^n_{k-1}}$ are independent and $(T^n_1, ...T^n_{k-1})$ are $\mathcal{F}_{\tau^n_{k-1}}$-measurable.
Moreover U^n_k is $\mathcal{F}_{\tau^n_k}$-measurable, so it is independent of $(T^n_{k+1}, T^n_{k+2}, ...)$.

So we just have to prove the independence between U^n_k and T^n_k. We consider the case $k = 1$ taking into account that this procedure easily extends to the general case.

To this end it will suffices to verify

$$E\left[\mathbb{I}\left(U_1 = 1\right)\exp(-\alpha\tau^n_1)\right] = E\left[\mathbb{I}\left(U_1 = 1\right)\right]E\left[\exp(-\alpha\tau^n_1)\right]. \tag{92}$$

Setting

$$\begin{aligned} y_{-1}(\alpha) =& E\left[\mathbb{I}\left(U_1 = -1\right)\exp(-\alpha\tau^n_1)\right] \\ y_1(\alpha) =& E\left[\mathbb{I}\left(U_1 = 1\right)\exp(-\alpha\tau^n_1)\right] \end{aligned}$$

by a direct computation we get

$$y_1(\alpha) = y_{-1}(\alpha) = \frac{1}{2\cosh\left(\frac{\sqrt{2\alpha}}{2^n}\right)}$$

and then

$$y_1(\alpha) = y_1(0)\left[y_1(\alpha) + y_{-1}(\alpha)\right] \tag{93}$$

which is equivalent to (92). □

$$\Pi^{a,c}(g)(s,l) = \frac{\Pi\left(g \cdot \exp(\frac{c}{a^2}\cdot)\right)(a^2 s, l)}{\Pi\left(\exp(\frac{c}{a^2}\cdot)\right)(a^2 s, l)};$$

B A limit result and an upper bound for the rescaled queue

In this section we prove that, under conditions **C1**, **C2**, and **C3**

$$E[\Lambda_t] = \lim_{n\to\infty} E^n\left[\mu_n \int_0^t \sqrt{n}\mathbb{I}(Q^n_{s^-} = 0)ds\right] = \lim_{n\to\infty} \mu_n \int_0^t \sqrt{n}P^n(Q^n_s = 0/Q^n_0 = 0)ds, \qquad (94)$$

where Λ_t is the local time of a Wiener process with diffusion coefficient 2λ and drift coefficient 0, and moreover we prove that for any $1 < \alpha < 2$, there exists n_α such that for any $n \geq n_\alpha$

$$\int_0^t \sqrt{n}P^n(Q^n_s = 0/Q^n_0 = 0)ds \leq n^{(1-\alpha)/2} + \sqrt{n}\big(1 - \frac{\lambda_n}{\mu_n}\big)\, t + K\sqrt{t}, \qquad (95)$$

for a computable constant K.
As a consequence, since $\sqrt{n}\left(1 - \frac{\lambda_n}{\mu_n}\right) = \frac{1}{\mu_n}\sqrt{n}\,(\mu_n - \lambda_n) \to 0$, under assumption **C3**,

$$\limsup_{n\to\infty} \int_0^t \sqrt{n}P^n(Q^n_s = 0/Q^n_0 = 0)ds \leq K\sqrt{t}.$$

To this end we start by recalling a few properties that we need, then we sketch the proof of the upper bound and of the limit, and finally we present the technical details.

B.1 Preliminary results

1) Modified Bessel functions

The modified Bessel functions of order k, for $k \geq 0$ are defined as

$$I_k(x) = \sum_{i=0}^{\infty} \frac{1}{(k+i)!i!}\left(\frac{x}{2}\right)^{k+2i},$$

while, for for $k \leq 0$

$$I_k(x) = I_{-k}(x).$$

2) The asymptotic behaviour of modified Bessel functions

It is well known that, for any k,

$$I_k(x) \sim \frac{e^x}{\sqrt{2\pi\, x}} \qquad \text{when } x \to \infty, \qquad (96)$$

and therefore, for any k, there exist an $\overline{x}_k > 0$ and an L_k such that

$$\sup_{x \geq \overline{x}_k} \frac{I_k(x)}{\frac{e^x}{\sqrt{2\pi\, x}}} \leq L_k \qquad (97)$$

3) The one-dimensional distribution of continuous time random walk

Let Z_t be a standard Poisson process and $Y_t = V_{Z_t}$, where as usual $\{V_k, k \in \mathbb{N}\}$ is a discrete time random walk, such that $V_0 = 0$, and $V_k = V_{k-1} + U_k$, where $\{U_k, k \in \mathbb{N}\}$ is a sequence of independent $\{-1, 1\}$-valued random variables with $P(U_k = 1) = p$ and $P(U_k = -1) = q$. Then, for any $u > 0$ (see, for instance, [7] II.7 page 58)

$$P(Y_u = k) = \exp\{-u\}\left(\frac{p}{q}\right)^{\frac{k}{2}} I_k(2\sqrt{pq}u).$$

As a consequence, for any $u > 0$ and any positive numbers $p, q \in (0, 1)$, with $p + q = 1$

$$\sum_{k=-\infty}^{\infty} \exp\{-u\} \left(\frac{p}{q}\right)^{\frac{k}{2}} I_k(2\sqrt{pq}u) = 1,$$

and therefore

$$\sum_{k=2}^{\infty} \exp\{-u\} \left(\frac{p}{q}\right)^{\frac{k}{2}} I_k(2\sqrt{pq}u) \leq 1,$$

and, taking into account that $I_k(x) = I_{-k}(x)$ (or equivalently interchanging the role of p and q)

$$\sum_{k=2}^{\infty} \exp\{-u\} \left(\frac{q}{p}\right)^{\frac{k}{2}} I_k(2\sqrt{pq}u) \leq 1, \tag{98}$$

4) Transition probabilities $p_{k,m}(t)$ of a birth and death process with reflection in l

Assume that Q_t is a birth and death process, assume that the upward jump rate is ν and the downward jump rate is ρ, and that the process lives in $[l, \infty) \cap \mathbb{N}$, with l a reflecting state. Then the transition probabilities (see e.g. [8]) are

$$\begin{aligned} p_{k,m}(t) = P(Q_t = m/Q_0 = k) \quad &= \quad \left(\frac{\nu}{\rho}\right)^{\frac{k-m}{2}} \exp\{-t(\nu+\rho)\} \Big[I_{k-m}(2\sqrt{\rho\nu}\ t) \\ &+ \quad \big(\frac{\rho}{\nu}\big)^{\frac{1}{2}} I_{m+k+1-2l}(2\sqrt{\rho\nu}\ t) \\ &+ \quad \big(1-\frac{\nu}{\rho}\big) \sum_{j=2}^{\infty} \big(\frac{\rho}{\nu}\big)^{\frac{j}{2}} I_{m+k-2l+j}(2\sqrt{\rho\nu}\ t) \Big] \end{aligned}$$

5) The behaviour of $p_{0,0}(t)$ with reflection in 0 When $k = m = l = 0$, then

$$\begin{aligned} p_{0,0}(t) = P(Q_t = 0/Q_0 = 0) = \exp\{-t(\nu+\rho)\} \Big[I_0(2\sqrt{\rho\nu}\ t) + \big(\frac{\rho}{\nu}\big)^{\frac{1}{2}} I_1(2\sqrt{\rho\nu}\ t) \\ + \big(1-\frac{\nu}{\rho}\big) \sum_{j=2}^{\infty} \big(\frac{\rho}{\nu}\big)^{\frac{j}{2}} I_j(2\sqrt{\rho\nu}\ t). \Big] \end{aligned}$$

$$\begin{aligned} p_{0,0}(t) - \exp\{-t(\nu+\rho)\} \Big[I_0(2\sqrt{\rho\nu}\ t) + \big(\frac{\rho}{\nu}\big)^{\frac{1}{2}} I_1(2\sqrt{\rho\nu}\ t) \Big] \\ = \big(1-\frac{\nu}{\rho}\big) \Big[\exp\{-t(\nu+\rho)\} \sum_{j=2}^{\infty} \big(\frac{\rho}{\nu}\big)^{\frac{j}{2}} I_j(2\sqrt{\rho\nu}\ t). \Big] \end{aligned}$$

Note that if we set $t = \nu+\rho$, $p = \frac{\nu}{\nu+\rho}$, and $q = \frac{\rho}{\nu+\rho}$, then the expression in square brackets in the previous expression is identical to (98) with $u = (\nu+\rho)t$, and therefore it is bounded by 1. As a consequence we have that

$$\Big| p_{0,0}(t) - \exp\{-t(\nu+\rho)\} \Big[I_0(2\sqrt{\rho\nu}\ t) + \big(\frac{\rho}{\nu}\big)^{\frac{1}{2}} I_1(2\sqrt{\rho\nu}\ t) \Big] \Big| \leq \big(1-\frac{\nu}{\rho}\big) \tag{99}$$

and in particular

$$p_{0,0}(t) \leq \exp\{-t(\nu+\rho)\} \Big[I_0(2\sqrt{\rho\nu}\ t) + \big(\frac{\rho}{\nu}\big)^{\frac{1}{2}} I_1(2\sqrt{\rho\nu}\ t) \Big] + \big(1-\frac{\nu}{\rho}\big). \tag{100}$$

B.2 Sketch of the proof

In our case we are interested to give an upper bound for

$$E^n\left[\int_0^t \sqrt{n}\mathbb{I}(Q_s^n=0)ds\right] = \int_0^t \sqrt{n}P^n(Q_s^n=0/Q_0^n=0)ds.$$

We can use (100) in the case when $\nu = n\lambda_n$, and $\rho = n\mu_n$ and get

$$\begin{aligned}&\sqrt{n}P^n(Q_s^n=0/Q_0^n=0)\\ &\quad\le \sqrt{n}\exp\{-sn(\lambda_n+\mu_n)\}\Big[I_0(x_n(s)) + (\frac{\mu_n}{\lambda_n})^{\frac{1}{2}} I_1(x_n(s))\Big] + \sqrt{n}(1-\frac{\lambda_n}{\mu_n}),\end{aligned}$$

where

$$x_n(s) = 2n\sqrt{\mu_n\lambda_n}\ s.$$

For any real number $\alpha > 1$

$$\begin{aligned}\int_0^t \sqrt{n}P^n(Q_s^n=0/Q_0^n=0)ds &= \int_0^{n^{-\alpha/2}} \sqrt{n}P^n(Q_s^n=0/Q_0^n=0)ds + \int_{n^{-\alpha/2}}^t \sqrt{n}P^n(Q_s^n=0/Q_0^n=0)ds\\ &\le n^{(1-\alpha)/2} + \int_{n^{-\alpha/2}}^t \sqrt{n}\exp\{-sn(\lambda_n+\mu_n)\}\Big[I_0(x_n(s)) + (\frac{\mu_n}{\lambda_n})^{\frac{1}{2}} I_1(x_n(s))\Big]ds\\ &\quad + \sqrt{n}(1-\frac{\lambda_n}{\mu_n})\ t.\end{aligned}$$

Then, using (97), it is possible to show (see the technical details at the end of this section) that

$$\text{for any } 1<\alpha<2 \quad \text{ and for any } n^{1-\alpha/2} \ge \frac{\max(\overline{x}_0,\overline{x}_1)}{2\sqrt{\mu_n\lambda_n}}$$

the following bound holds

$$\begin{aligned}\int_0^t \sqrt{n}P^n(Q_s^n=0/Q_0^n=0)ds &\le n^{(1-\alpha)/2} + \sqrt{n}(1-\frac{\lambda_n}{\mu_n})\ t\\ &\quad + \int_{n^{-\alpha/2}}^t \frac{1}{2\sqrt{\pi\sqrt{\mu_n\lambda_n}s}}\left[L_0 + (\frac{\mu_n}{\lambda_n})^{\frac{1}{2}} L_1\right]ds.\end{aligned}$$

Starting from the bound (99) instead of (100), and using (96), the same technique can be used to show that

$$\lim_{n\to\infty} E^n\left[\int_0^t \sqrt{n}\mu_n\mathbb{I}(Q_{s^-}^n=0)ds\right] = \lim_{n\to\infty}\mu_n\int_{n^{-\alpha/2}}^t \frac{1}{2\sqrt{\pi\sqrt{\mu_n\lambda_n}s}}\left[1+(\frac{\mu_n}{\lambda_n})^{\frac{1}{2}}\right]ds = E[\Lambda_t]$$

where Λ_t is the local time of a Wiener process with diffusion coefficient 2λ and drift coefficient 0. Indeed, on the one hand

$$\begin{aligned}&\lim_{n\to\infty}\mu_n\int_{n^{-\alpha/2}}^t \frac{1}{2\sqrt{\pi\sqrt{\mu_n\lambda_n}s}}\left[1+(\frac{\mu_n}{\lambda_n})^{\frac{1}{2}}\right]ds = \lambda\int_0^t \frac{1}{2\sqrt{\lambda\pi s}}\left[1+(\frac{\lambda}{\lambda})^{\frac{1}{2}}\right]ds\\ &= \int_0^t \frac{\sqrt{\lambda}}{\sqrt{\pi s}}ds = \frac{2\sqrt{\lambda t}}{\sqrt{\pi}},\end{aligned}$$

On the other hand, by the reflection principle, the density of the random variable Λ_t is

$$f_{\Lambda_t}(u) = \frac{e^{-\frac{u^2}{4\lambda t}}}{\sqrt{\pi\lambda t}}\mathbb{I}(u>0),$$

and then

$$E[\Lambda_t] = \frac{1}{\sqrt{\pi\lambda t}}\int_0^\infty ue^{-\frac{u^2}{4\lambda t}}du = \frac{2\sqrt{\lambda t}}{\sqrt{\pi}}.$$

B.3 Technical details

Taking into account (97), we get, for $s \in [n^{-\alpha/2}, t]$

$$\begin{aligned}
&\sqrt{n} P^n (Q_s^n = 0 / Q_0^n = 0) \\
&\leq \sqrt{n} \exp\{-sn(\lambda_n + \mu_n)\} \frac{\exp\{x_n(s)\}}{\sqrt{2\pi x_n(s)}} \left[\sup_{u \in [n^{-\alpha/2}, t]} \frac{I_0(x_n(u))}{\frac{\exp\{x_n(u)\}}{\sqrt{2\pi x_n(u)}}} + \left(\frac{\mu_n}{\lambda_n}\right)^{\frac{1}{2}} \sup_{u \in [n^{-\alpha/2}, t]} \frac{I_1(x_n(u))}{\frac{\exp\{x_n(u)\}}{\sqrt{2\pi x_n(u)}}} \right]
\end{aligned}$$

Now

$$\begin{aligned}
&\sqrt{n} \exp\{-sn(\lambda_n + \mu_n)\} \frac{\exp\{x_n(s)\}}{\sqrt{2\pi x_n(s)}} = \sqrt{n} \exp\{-sn[(\lambda_n + \mu_n) - 2\sqrt{\mu_n \lambda_n}\ ]\} \frac{1}{\sqrt{4\pi n \sqrt{\mu_n \lambda_n} s}} \\
&= \exp\{-sn[(\lambda_n + \mu_n) - 2\sqrt{\mu_n \lambda_n}\ ]\} \frac{1}{2\sqrt{\pi \sqrt{\mu_n \lambda_n} s}} \leq \frac{1}{2\sqrt{\pi \sqrt{\mu_n \lambda_n} s}}
\end{aligned}$$

since $(\lambda_n + \mu_n) - 2\sqrt{\mu_n \lambda_n} \geq 0$.

Moreover, for any n, and for $k = 0, 1$

$$\sup_{u \in [n^{-\alpha/2}, t]} \frac{I_k(x_n(u))}{\frac{\exp\{x_n(u)\}}{\sqrt{2\pi x_n(u)}}} = \sup_{x \in [x_n(n^{-\alpha/2}, x_n(t)]} \frac{I_k(x)}{\frac{\exp\{x\}}{\sqrt{2\pi x}}} \leq \sup_{x \geq 2n\sqrt{\mu_n \lambda_n} n^{-\alpha/2}} \frac{I_0(x)}{\frac{\exp\{x\}}{\sqrt{2\pi x}}}$$

and therefore, with the same notations of (97), for $\alpha < 2$, for n sufficiently large so that $2n^{1-\alpha/2}\sqrt{\mu_n \lambda_n} \geq \overline{x}_k$, we get

$$\sup_{u \in [n^{-\alpha/2}, t]} \frac{I_k(x_n(u))}{\frac{\exp\{x_n(u)\}}{\sqrt{2\pi x_n(u)}}} \leq L_k$$

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