A CRITERION OF CONVERGENCE IN THE AUGMENTED TEICHMÜLLER SPACE

GABRIELE MONDELLO

ABSTRACT. We prove a criterion of convergence in the augmented Teichmüller space that can be phrased in terms of convergence of the hyperbolic metrics or of quasiconformal convergence away from the nodes.

1. INTRODUCTION

The purpose of this note is to prove a criterion of convergence in the augmented Teichmüller space $\overline{\mathcal{T}}(S)$ of a hyperbolic surface S (Theorem 6.1). Basically, $[f_n : S \to \Sigma_n]$ converges to $[f : S \to \Sigma] \in \overline{\mathcal{T}}(S)$ if and only if we can find "good" representatives $\tilde{f}_n \in [f_n]$ such that $F_n := f \circ \tilde{f}_n^{-1} : \Sigma_n \to \Sigma$ displays a "standard" behavior near the nodes of Σ (in our terminology, it restricts to a standard map of hyperbolic annuli around each node of Σ). Moreover, if we call $N := f^{-1}(\text{nodes of } \Sigma) \subset S$, we say that the representatives $\tilde{f}_n$ are "good" for f if either of the following properties holds:

(i) $f_n^*(g_n) \to f^*(g)$ uniformly away from N, where g_n is the hyperbolic metric of Σ_n and g is the hyperbolic metric of Σ

(ii) on every compact $C \subset S \setminus N$, the distortion $K(\tilde{f}_n^* J_n, f^* J) \to 1$ uniformly, J_n being the complex structure on Σ_n and J that on Σ.

To show that $[f_n] \to [f]$ implies condition (i) and (ii), we construct F_n by gluing some "standard" maps of hyperbolic pair of pants, which depend only on the Fenchel-Nielsen coordinates of Σ_n and Σ. Incidentally, we remark that the idea of constructing good quasiconformal representatives for points in $\mathcal{T}(S)$ using some "standard" map of pair of pants is also exploited in [Bis02].

In order to prove that (ii) implies that $[f_n] \to [f]$, we prove the following result of quasiconformal "modifications" that might be interesting in its own right (see Section 2).

Theorem. *Let $\Delta \subset \mathbb{C}$ be the unit disc, $F \subset \varepsilon\overline{\Delta}$ a compact Jordan domain (for some $\varepsilon < 1$) and $g : \overline{\Delta} \setminus F \to \overline{\Delta}$ a K-quasiconformal map which is a homeomorphism onto its image.*

(a) *There exists a $\tilde{K}$-quasiconformal homeomorphism $\tilde{g} : \overline{\Delta} \to \overline{\Delta}$ that coincides with g on $\partial\Delta$. Moreover, $\tilde{K}$ depends only on K and ε, and $\tilde{K} \to 1$ as $(K, \varepsilon) \to (1, 0)$.*

(b) *If $0 \in g(F)$, then there exists a $\hat{K}$-quasiconformal homeomorphism $\hat{g} : \overline{\Delta} \to \overline{\Delta}$ that coincides with g on $\partial\Delta$ and such that $\hat{g}(0) = 0$. Moreover, $\hat{K}$ depends only on K and ε, and $\hat{K} \to 1$ as $(K, \varepsilon) \to (1, 0)$.*

In Section 3 we recall the collar lemma and some geometry of hyperbolic hexagons, which are used in Section 4 to construct the standard maps between annuli and between pair of pants. Basic facts and notations about the Teichmüller space and its augmentation are recalled in Section 5, together with Fenchel-Nielsen coordinates. Finally, in Section 6 the convergence criterion is proven.

We will use this criterion in [Mon] to prove (roughly speaking) that the modulus of a Riemann surface, which is constructed by gluing flat and hyperbolic tiles along a graph, varies continuously with respect to the length parameters of the graph.

We want just to stress that the statement of Theorem 6.1 looks quite intuitive and is probably folkloristically accepted, even though we were not able to find a precise reference.

Similar considerations hold for Theorem 2.7, even though we have not seen a similar result stated in the literature.

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2. Quasiconformal modifications

Definition 2.1. *Given a bounded Jordan domain $S \subset \mathbb{C}$ and distinct points $z_1, z_2, z_3, z_4 \in \partial S$, call $S(z_1, z_2, z_3, z_4)$ the quadrilateral with vertices $z_1, \dots, z_4$ and whose interior is S. The portions of ∂S running from z_1 to z_2 and from z_3 to z_4 are called a-sides; the portions from z_2 to z_3 and from z_4 to z_1 are called b-sides. The modulus of the quadrilateral is*

$$\mathrm{mod}(S(z_1, z_2, z_3, z_4)) = \inf_{\rho \in \mathcal{R}} \frac{\mathrm{Area}_\rho(S)}{\inf_{\gamma \in \mathcal{F}} \ell^2_\rho(\gamma)}$$

where $\mathcal{F}$ is the set of smooth curves in S that join the two a-sides of $S(z_1, z_2, z_3, z_4)$ and $\mathcal{R}$ is the set of smooth metrics on S in the given conformal class.

Given $z_1, z_2, z_3, z_4 \in \mathbb{C}$ distinct points, we denote by $R(z_1, z_2, z_3, z_4)$ the quadrilateral with straight edges that has them as vertices. One can show that the modulus of the rectangle $R(0, a, a + ib, ib)$ is a/b (Grötzsch).

Definition 2.2. *A quasiconformal map of quadrilaterals $f : S(z_1, \dots, z_4) \to S'(w_1, \dots, w_4)$ is a quasiconformal map $f : S \to S'$, whose continuous extension $\overline{f} : \overline{S} \to \overline{S}'$ (which always exists) takes z_i to w_i for $i = 1, \dots, 4$.*

The modulus of a quadrilateral is clearly invariant under biholomorphisms.

Definition 2.3. *An annular domain is a Riemann surface A homeomorphic to an open annulus. Its modulus is defined as*

$$\mathrm{mod}(A) = \inf_{\rho \in \mathcal{R}} \frac{\mathrm{Area}_\rho(A)}{\inf_{\gamma \in \mathcal{F}} \ell^2_\rho(\gamma)}$$

where $\mathcal{F}$ is the set of smooth curves in S that generate $\pi_1(A)$ and $\mathcal{R}$ is the set of smooth metrics on A in the given conformal class.

It follows from Grötzsch's result on quadrilaterals that, for $0 < r < 1$, the standard annulus $A(r) = \{z \in \mathbb{C} \,|\, r < |z| < 1\}$ (with $0 < r < 1$) has modulus $-(2\pi)^{-1} \log(r)$.

Lemma 2.4 ([McM94])**.** *Let $D, R \subset \mathbb{C}$ be Jordan domains with finite area, such that D is contained in R° and is compact. Then,*

$$\frac{\mathrm{Area}(D)}{\mathrm{Area}(R)} \leq \frac{1}{1 + 4\pi \,\mathrm{mod}(R \setminus D)}$$

where areas are taken with respect to the Euclidean measure.

Proof. Let $\mathcal{F}$ be the set of simple closed curves in the annulus $R \setminus D$ which are homotopically nontrivial. The isoperimetric inequality in the plane says that the Euclidean length of $\gamma \in \mathcal{F}$ satisfies $\ell^2(\gamma) \geq 4\pi \mathrm{Area}(D)$. By definition,

$$\mathrm{mod}(R \setminus D) \leq \frac{\mathrm{Area}(R \setminus D)}{\ell^2(\gamma)}$$

and the result follows. □

The following is similar to Theorem I.8.3 of [LV73].

Proposition 2.5. *Let $\Delta \subset \mathbb{C}$ be the unit disc and $F \subset \Delta$ a compact Jordan domain contained in the smaller disc $\varepsilon\overline{\Delta}$ with $0 < \varepsilon < 1$. Consider a homeomorphism $g : \overline{\Delta} \to \overline{\Delta}$ whose restriction to $\overline{\Delta} \setminus F$ is K-quasiconformal.*

Then, for every distinct $z_1, \dots, z_4 \in \partial\Delta$, we have

$$\frac{M}{K_\varepsilon} \leq M' \leq K_\varepsilon M \quad \text{and } K_\varepsilon \to K \text{ as } \varepsilon \to 0$$

where $M = \mathrm{mod}(Q)$ and $Q = \Delta(z_1, z_2, z_3, z_4)$ (similarly, $M' = \mathrm{mod}(Q')$ and $Q' = g(Q)$). In particular, one can take

$$K_\varepsilon = K\left(1 + \frac{\beta_0 - 1 + \beta_1(1 - \sqrt{\varepsilon})^{-2}}{1 + \log(1/\varepsilon)}\right)$$

where $\beta_0 > 1$ and $\beta_1 > 0$ are universal constants.

Notation. *For $0 < r < 1$, call $\mu(r)$ the modulus of Grötzsch extremal domain $\Delta \setminus [0, r]$ and let $\lambda(K) = (\mu^{-1}(\pi K/2))^{-2} - 1$.*

As μ is continuous and strictly decreasing from $+\infty$ to zero, the function λ is continuous and strictly increasing from zero to $+\infty$. It can be shown that $\lambda(1/K) = 1/\lambda(K)$, so that $\lambda(1) = 1$ (see [LV73]).

Lemma 2.6 (Grötzsch)**.** *Let $0 \leq r < 1$ and let $A \subset \Delta$ be an open annular domain that separates $\{0, r\}$ from $\partial\Delta$. Then, $\mathrm{mod}(A) \leq \mu(r)$ and equality is attained if and only if $A = \Delta \setminus [0, r]$.*

Theorem 2.7. *Let $F \subset \varepsilon\overline{\Delta} \subset \Delta$ be a compact Jordan domain and $g : \overline{\Delta} \setminus F \to \overline{\Delta}$ a K-quasiconformal map which is a homeomorphism onto its image.*

(a) *There exists a $\tilde{K}$-quasiconformal homeomorphism $\tilde{g} : \overline{\Delta} \to \overline{\Delta}$ that coincides with g on $\partial\Delta$. Moreover, $\tilde{K}$ depends only on K and ε, and $\tilde{K} \to 1$ as $(K, \varepsilon) \to (1, 0)$.*

(b) *If $0 \in g(F)$, then there exists a $\hat{K}$-quasiconformal homeomorphism $\hat{g} : \overline{\Delta} \to \overline{\Delta}$ that coincides with g on $\partial\Delta$ and such that $\hat{g}(0) = 0$. Moreover, $\hat{K}$ depends only on K and ε, and with $\hat{K} \to 1$ as $(K, \varepsilon) \to (1, 0)$.*

Clearly, by shrinking Δ, one can even produce similar *quasiconformal modifications* $\tilde{g}$ and $\hat{g}$ that agree with g on a neighbourhood of $\partial\Delta$ and such that the above theorem holds (with slightly worse $\tilde{K}$ and $\hat{K}$).

Corollary 2.8. *If $F \subset \Delta$ is a compact Jordan domain with* $\mathrm{mod}(\Delta \setminus F) \geq M$ *and g is a K-quasiconformal map as above, then there exist a modification $\tilde{g}$ and $\tilde{K}$ as in Theorem 2.7, with $\tilde{K}$ depending on K and M only. Moreover, $\tilde{K} \to 1$ as $(K, M) \to (1, \infty)$. The analogous conclusion holds for $\hat{g}$ and $\hat{K}$, if $0 \in g(F)$.*

This is an easy consequence. In fact, for every $z \in F$, Grötzsch's lemma gives $\mu(|z|) \geq \mathrm{mod}(\Delta \setminus F)$ and so $|z| \leq \mu^{-1}(M)$. Thus, $F \subset \varepsilon\overline{\Delta}$ with $\varepsilon = \mu^{-1}(M)$ and Theorem 2.7 applies.

Lemma 2.9. *Let $h : \overline{\Delta} \to \overline{\Delta}$ be a K-quasiconformal homeomorphism and let $d = |h(0)|$. Then, there exists a $K(1+d)/(1-d)$-quasiconformal homeomorphism $\hat{h} : \overline{\Delta} \to \overline{\Delta}$ that agrees with h on $\partial\Delta$ and such that $\hat{h}(0) = 0$.*

Proof. Up to rotations, we can assume that $h(0) = d$. The conformal map $T : \overline{\Delta} \to \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ given by $z \mapsto i(1-z)/(1+z)$ sends $T(0) = i$ and $T(d) = i(1-d)/(1+d)$. The homeomorphism $(u+iv) = T \circ h \circ T^{-1} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ sends $(u+iv)(i) = i(1-d)/(1+d)$. The map $(u + iv(1+d)/(1-d)) : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ is a $(1+d)/(1-d)$-quasiconformal homeomorphism, it agrees with $(u+iv)$ on $\partial\mathbb{H}$ and it sends i to i. Thus, $\hat{h} := T^{-1} \circ (u + iv(1+d)/(1-d)) \circ T$ has the required properties. □

The lemma below is a reformulation of Theorem II.6.4 in [LV73] (see also [Väi62]).

Lemma 2.10. *There exist constants $\beta_0 \geq 1$ and $\beta_1 > 0$ such that, given a K-quasiconformal homeomorphism $h : A(r) \to \Delta$ onto its image, there exists another KC-quasiconformal homeomorphism $\tilde{h} : \overline{\Delta} \to \overline{\Delta}$ that coincides with h on $\partial\Delta$, with $C = \beta_0 + \beta_1(1-r)^{-2} > 1$.*

Proof of Theorem 2.7. Using Riemann's conformal map theorem, we can assume that $\partial\Delta$ is contained in the image of g.

Applying Lemma 2.10 to the restriction of g to $\sqrt{\varepsilon}\Delta$, we can assume that g extends to a quasiconformal homeomorphism $\overline{\Delta} \to \overline{\Delta}$, whose restriction to $A(\sqrt{\varepsilon})$ is K-quasiconformal.

Claim (a) immediately follows considering Ahlfors-Beurling's extension $\tilde{g}$ of $g\Big|_{\partial\Delta}$ [BA56] and using the estimate obtained in Proposition 2.5. We obtain $\tilde{K} = \lambda(K_\varepsilon)^2$.

For (b), we proceed as follows. As before, using Lemma 2.10, we modify g so that it becomes a homeomorphism of the unit disc to itself, which is K-quasiconformal on $A(\sqrt{\varepsilon})$ and KC-quasiconformal on $\sqrt{\varepsilon}\Delta$, with $C = \beta_0 + \beta_1(1-\sqrt{\varepsilon})^{-2}$.

Now, we apply part (a) to the restriction of g to $\sqrt[4]{\varepsilon}\overline{\Delta}$ and so we obtain a $\tilde{g}$ which agrees with g on $A(\sqrt[4]{\varepsilon})$ (where is K-quasiconformal) and which is $\tilde{K}'$-quasiconformal on $\sqrt[4]{\varepsilon}\Delta$, where $\tilde{K}' = \lambda(K'_\varepsilon)^2$ and

$$K'_\varepsilon = K\left(1 + \frac{\beta_0 - 1 + \beta_1(1-\sqrt[8]{\varepsilon})^{-2}}{1 + \frac{1}{8}\log(1/\varepsilon)}\right)$$

Consider a path $\gamma \subset g(\sqrt[4]{\varepsilon}\Delta)$ that joins 0 with $\tilde{g}(0)$. By Grötzsch's theorem,

$$\begin{aligned}\mu(d) &\geq \mathrm{mod}(\Delta \setminus \gamma) \geq \mathrm{mod}(\Delta \setminus g(\sqrt[4]{\varepsilon}\Delta)) \geq \\ &\geq \frac{1}{K}\mathrm{mod}(A(\sqrt[4]{\varepsilon})) = \frac{1}{8\pi K}\log(1/\varepsilon)\end{aligned}$$

where $d = |\tilde{g}(0)|$. The inequality

$$d \le \mu^{-1}\left(\frac{1}{8\pi K}\log(1/\varepsilon)\right)$$

implies that $d \to 0$ if $\varepsilon \to 0$ while K stays bounded.

Applying Lemma 2.9 to $\tilde{g}$, we obtain a $\hat{K}$-quasiconformal homeomorphism $\hat{g} : \overline{\Delta} \to \overline{\Delta}$ that fixes 0 and which agrees with g on $\partial\Delta$, where

$$\hat{K} = \tilde{K}'\left(\frac{1+d}{1-d}\right) \le \lambda\left[K\left(1+\frac{\beta_0 - 1 + \beta_1(1-\sqrt[8]{\varepsilon})^{-2}}{1+\frac{1}{8}\log(1/\varepsilon)}\right)\right]^2\left(\frac{1+\mu^{-1}\left(\frac{1}{8\pi K}\log(1/\varepsilon)\right)}{1-\mu^{-1}\left(\frac{1}{8\pi K}\log(1/\varepsilon)\right)}\right)$$

Notice that $\hat{K}$ depends only on K and ε and that, if $K \to 1$ and $\varepsilon \to 0$, then $\hat{K} \to 1$. $\square$

Proof of Proposition 2.5. Clearly, it is sufficient to find K_ε such that $M' \le K_\varepsilon M$ holds. We can assume that $F = \varepsilon\overline{\Delta}$ and call $D = \sqrt{\varepsilon}\Delta$, so that $\mathrm{mod}(\Delta \setminus D) = \frac{1}{4\pi}\log(1/\varepsilon)$. Applying Lemma 2.10 to the restriction of $g\Big|_D$, we can assume that g is KC-quasiconformal on D, with $C = \beta_0 + \beta_1(1-\sqrt{\varepsilon})^{-2}$.

Consider the canonical biholomorphisms $f : Q \to R = R(0, M, M+i, i)$ and $f : Q' \to R' = R(0, M', M'+i, i)$ and call $\tilde{g} : R \to R'$ the composition $\tilde{g} := f' \circ g \circ f^{-1}$.

Notice that, as $\partial f(D)$ is a smooth real-analytic arc, it can be subdivided into a finite number of arcs, each intersecting any horizontal line of R at most once.

Thus, chosen a small $\delta > 0$, we can divide R into rectangles $R = \bigcup_{h,k=1}^m R_{h,k}$, where $R_{h,k} = R(x_{k-1} + iy_{h-1}, x_k + iy_{h-1}, x_k + iy_h, x_{k-1} + iy_h)$ with $0 = y_0 < y_1 < \cdots < y_m = 1$ and $0 = x_0 < x_1 < \cdots < x_m = M$ such that

(1) $R_h \cap \partial f(D)$ is a finite union of arcs joining the horizontal sides of R_h, where $R_h = \bigcup_{k=1}^m R_{h,k}$
(2) $\mathrm{Area}(U) - \delta \le \mathrm{Area}(D) \le \mathrm{Area}(U)$, where $U = U_1 \cup \cdots \cup U_m$, $U_h = \bigcup_{k \in I_h} R_{h,k}$ and $I_h = \{k \mid R_{h,k} \cap f(D) \ne \emptyset\}$.
(3) For every $h = 1, \ldots, m$ the sum of the diameters of the connected components of $\tilde{g}(R_h \cap f(\partial D))$ is smaller than δ.

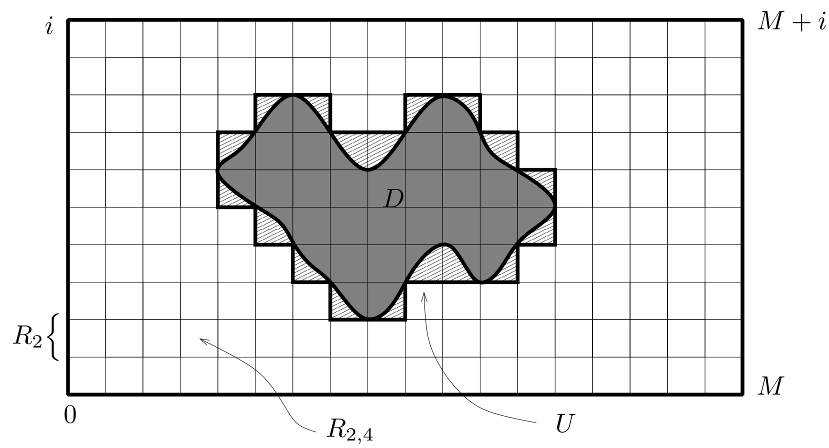

FIGURE 1. Subdivision of R.

Let $R'_{h,k} = \tilde{g}(R_{h,k})$ and $R'_h = \tilde{g}(R_h)$ and call $s_{h,k}$ the distance between the b-sides of $R'_{h,k}$. By Rengel's inequality (see Section I.4.3 of [LV73])

$$\mathrm{mod}(R'_{h,k}) \geq \frac{s_{h,k}^2}{\mathrm{Area}(R'_{h,k})}$$

Thus, we obtain

$$K\mathrm{mod}(R_h) + K(C-1)\sum_{k\in I_h}\mathrm{mod}(R_{h,k}) = \sum_{k\notin I_h} K\mathrm{mod}(R_h) + \sum_{k\in I_h} KC\mathrm{mod}(R_{h,k}) \geq$$
$$\geq \sum_{k=1}^{m}\mathrm{mod}(R'_{h,k}) \geq \sum_{k=1}^{m}\frac{s_{h,k}^2}{\mathrm{Area}(R'_{h,k})}$$

By Schwarz's inequality

$$\sum_{k=1}^{m}\frac{s_{h,k}^2}{\mathrm{Area}(R'_{h,k})} \geq \frac{\left(\sum_{k=1}^m s_{h,k}\right)^2}{\sum_{k=1}^m \mathrm{Area}(R'_{h,k})} \geq \frac{(M'-\delta)^2}{\mathrm{Area}(R'_h)}$$

As $R_{h,k}$ are rectangles, $\mathrm{mod}(R_{h,k}) = \dfrac{\mathrm{Area}(R_{h,k})}{(y_h - y_{h-1})^2}$, and so

$$\frac{K\mathrm{Area}(R_h) + K(C-1)\mathrm{Area}(U_h)}{(y_h-y_{h-1})^2} \geq \frac{(M'-\delta)^2}{\mathrm{Area}(R'_h)}$$

The left-hand side can be rewritten as

$$\frac{KM}{(y_h-y_{h-1})}\left(1+\frac{C-1}{M(y_h-y_{h-1})}\mathrm{Area}(U_h)\right)$$

Taking inverses

$$\frac{(y_h-y_{h-1})}{KM}\left(1+\frac{C-1}{M(y_h-y_{h-1})}\mathrm{Area}(U_h)\right)^{-1} \leq \frac{\mathrm{Area}(R'_h)}{(M'-\delta)^2}$$

Summing over $h = 1,\dots,m$ and using the convexity of the function $x \mapsto 1/x$ (for $x > 0$), we get

$$\frac{1}{KM}\left(1+\frac{C-1}{M}(\mathrm{Area}(D)+\delta)\right)^{-1} \leq \frac{M'}{(M'-\delta)^2}$$

As $\delta > 0$ was arbitrary, we conclude

$$M' \leq MK\left(1+(C-1)\frac{\mathrm{Area}(D)}{\mathrm{Area}(R)}\right) \leq MK\left(1+\frac{C-1}{1+\log(1/\varepsilon)}\right)$$

where the last inequality is obtained applying Lemma 2.4 to $f(D) \subset R$. □

3. The collar lemma

Let γ be a simple closed geodesic in a hyperbolic surface Σ and let ℓ be its length. Call ℓ' the length of an embedded hypercycle γ', whose distance from ℓ is d, and let R the annulus enclosed by γ and γ'. The following is a simple computation.

Lemma 3.1. *$\ell' = \ell\cosh(d)$ and $\mathrm{Area}(R) = \ell\sinh(d)$. So $\ell' = \sqrt{\ell^2 + \mathrm{Area}(R)^2}$.*

For every simple closed geodesic γ in a hyperbolic surface Σ there is a correspondence between sides of γ (i.e. connected components $\overrightarrow{N}$ of the complement of the zero section in the normal bundle $N(\gamma/\Sigma)$) and orientations $\overrightarrow{\gamma}$ of γ in such a way that $(\overrightarrow{\gamma}, \overrightarrow{N})$ determines a positive frame along γ.

Lemma 3.2 (Collar lemma, [Kee74]-[Mat76]). *For every oriented simple closed geodesic $\overrightarrow{\gamma} \subset \Sigma$ in a hyperbolic surface, there exists an embedded hypercycle γ' parallel to γ (on the corresponding side) such that the area of the annulus $A_1(\gamma)$ enclosed by γ and γ' is $\ell/2\sinh(\ell/2)$. For $\ell = 0$, the geodesic γ must be intended to be a cusp and γ' a horocycle of length 1. Furthermore, all annuli corresponding to distinct oriented simple closed geodesics are disjoint.*

One can easily show that the length ℓ' of the hypercycle γ' provided by the collar lemma always satisfies $\ell' \geq 1$.

Motivated by the collar lemma, we call $A_t(\ell)$ to be a closed hyperbolic annulus of area $t\ell/2\sinh(\ell/2)$, bounded by a geodesic of length ℓ and a hypercycle. As usual, by $A_t(0)$ we will mean a closed horoball of area t (which does include the cusp).

Concretely, given an oriented simple closed geodesic $\overrightarrow{\gamma}$ in a hyperbolic surface Σ, we will denote by $A_t(\overrightarrow{\gamma}) \subset \Sigma$ the annulus isometric to $A_t(\ell_\gamma)$ bounded by γ on the corresponding side. If γ is not a boundary geodesic, we will denote by $dA_t(\gamma)$ the collar $A_t(\overrightarrow{\gamma}) \cup A_t(\overleftarrow{\gamma})$.

Consider now a closed hyperbolic hexagon $H = H(h_1, h_2, h_3)$ with six right angles and (cyclically ordered) sides $(\alpha_1, \beta_3, \alpha_2, \beta_1, \alpha_3, \beta_2)$, where α_i has length h_i. Notice that H is compact, that is it contains possible ideal points.

The double dH of H along the β-sides is a pair of pants with boundary lengths $\ell_{d\alpha_i} = 2h_i$. The collar $A_1(\alpha_i) \subset H$ is then the restriction of $A_1(d\alpha_i) \subset dH$ to H. Notice that we need not specify an orientation of α_i as only one side is possible.

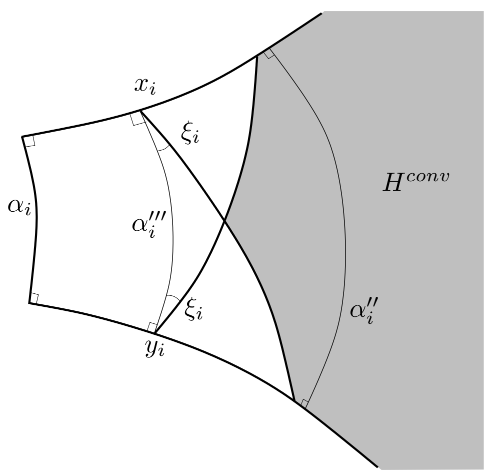

FIGURE 2. α_i'' and α_i''' bound $A_{\frac{1}{2}}(\alpha_i)$ and $A_t(\alpha_i)$ respectively.

Given $\xi > 0$ and $0 < t < 1$, denote by $\hat{A}_{t,\xi}(\alpha_i) \subset A_1(\alpha_i)$ the union of $A_t(\alpha_i)$ and all geodesics in H that hit $\partial A_t(\alpha_i)$ with an angle smaller than ξ (or, equivalently, bigger than $\pi - \xi$). As usual, we denote by $\hat{A}_{t,\xi}(d\alpha_i) \subset dH$ the subset of $A_1(d\alpha_i)$ isometric to $d\hat{A}_{t,\xi}(2h_i)$. Notice that $\hat{A}_{t,\xi}(\alpha_i)$ is delimited by the two geodesics that hit x_i and y_i (that is, the extremal points of the hypercycle α_i''' that bounds $A_t(\alpha_i)$) with an angle ξ (see Figure 2).

Clearly, there exist smooth decreasing functions $t, \xi : [0, \infty) \to \mathbb{R}_+$ such that $\hat{A}_{t(h_i),\xi(h_i)}(\alpha_i) \subset A_{\frac{1}{2}}(\alpha_i)$ for all $h_i \in [0, \infty)$. We can also assume that $t(0) = 1/4$ and $\xi(0) = \pi/4$.

Hence, $H^\circ := H \setminus \bigcup_i A_{t(h_i)}(\alpha_i)$ contains the nonempty convex subset $H^{conv} := H \setminus \bigcup_i \hat{A}_{t(h_i),\xi(h_i)}(\alpha_i)$. Notice that the diameter of H° is bounded above in terms of $h_1 + h_2 + h_3$. As a consequence, every geodesic that joins the *baricenter* B of H^{conv} to ∂H° forms an angle which is bounded from below (and this angle may go to zero only if $h_1 + h_2 + h_3$ diverges).

4. Standard maps

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative function with compact support inside $(0, 1/2)$ and with unit integral and let $\Phi(s) = \int_0^s \varphi(x)dx$.

Let $\ell, \tilde{\ell} \geq 0$ and assume that $\ell = 0$ only if $\tilde{\ell} = 0$. Then, for every $0 < t \leq 1$ and $\vartheta \in \mathbb{R}$, the *standard map of annuli* $\sigma_a(\vartheta) : A_t(\ell) \to A_t(\tilde{\ell})$ is defined as follows.

The longest hypercycle of $A_t(\ell)$ has length $\ell''' = \ell\sqrt{1 + \dfrac{t^2}{4\sinh(\ell)^2}}$ and sits at distance d''' from the geodesic (similarly, for $A_t(\tilde{\ell})$).

For every $0 \leq s \leq 1$, $\sigma_a(\vartheta)$ maps the hypercycle of length $s\ell + (1-s)\ell'''$ to the hypercycle of length $s\tilde{\ell} + (1-s)\tilde{\ell}'''$ by a homothety (the scaling factor will clearly depend on s). Moreover, the hypercycle that is mapped to the one of length $\tilde{\ell}\cosh(\tilde{d})$ (which is at distance $\tilde{d}$ from the closed geodesic of $A_t(\tilde{\ell})$) is twisted[1] by an angle $\vartheta\Phi(\tilde{d}/\tilde{d}''')$.

Clearly, if $\tilde{\ell} = 0$, then the cusp of $A_t(\tilde{\ell})$ is at infinite distance and so the actual value of ϑ is ineffective.

Consider hyperbolic hexagons $H = H(h_1, h_2, h_3)$ and $\tilde{H} = H(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$ as defined in the previous section and call $B \in H$ (resp. $\tilde{B} \in \tilde{H}$) the baricenter of H^{conv} (resp. of $\tilde{H}^{conv}$).

The standard map $\sigma^\circ : H^\circ \to \tilde{H}^\circ$ is defined as follows.

We prescribe $\sigma^\circ(B) := \tilde{B}$ and we ask σ° to map the vertices of H° to the corresponding vertices of $\tilde{H}^\circ$, α_i'' to $\tilde{\alpha}_i''$ and $\beta_j \cap H^\circ$ to $\tilde{\beta}_j \cap \tilde{H}^\circ$ by homotheties. Finally, for every $p \in \partial H^\circ$, we require σ° to map the geodesic segment $\widehat{Bp}$ homothetically onto $\widehat{\tilde{B}\sigma^\circ(p)}$.

Clearly, σ° is a homeomorphism. Moreover, σ° is quasiconformal and Lipschitz and it is differentiable everywhere except (possibly) along the six geodesic segments that join B to the vertices of H°.

Now, assume that $h_i = 0$ implies $\tilde{h}_i = 0$. Given pair of pants dH and $d\tilde{H}$, obtained by doubling hexagons H and $\tilde{H}$ as above, and given $\Theta = (\vartheta_1, \vartheta_2, \vartheta_3) \in \mathbb{R}^3$, we define the *standard map of pair of pants* $\sigma(\Theta) : dH \to d\tilde{H}$ as the double of σ° on dH° and as $\sigma_a(\vartheta_i)$ on each $A_{t(h_i)}(d\alpha_i)$.

Notation. *If g and g' are two symmetric bilinear forms, we will write $g' < \varepsilon g$ to mean that $\varepsilon g - g'$ is positive-definite and $|g'| < \varepsilon g$ to mean that both $\varepsilon g - g'$ and $\varepsilon g + g'$ are positive-definite. If g is positive-definite, we will say that $g_n \to g$ if, for every $\varepsilon > 0$, there exists N such that $|g_n - g| < \varepsilon g$ for all $n \geq N$.*

[1] In our conventions, the twist is positive if a pedestrian that walks across the crack is pushed to his *right*.

Lemma 4.1. (a) *The standard map of annuli* $\sigma_a(\vartheta_i) : A_{t(h_i)}(d\alpha_i) \to A_{t(\tilde{h}_i)}(d\tilde{\alpha}_i)$ *satisfies* $|\sigma_a^{-1}(\vartheta_i)^*(g) - \tilde{g}| < c(h_i, \tilde{h}_i, \vartheta_i)\tilde{g}$*, where* g *and* $\tilde{g}$ *are the hyperbolic metrics and* $c(h_i, \tilde{h}_i, \vartheta)$ *depends continuously on* $h_i, \tilde{h}_i > 0$ *and* $\vartheta_i \in \mathbb{R}$*, and is zero if* $h_i = \tilde{h}_i$ *and* $\vartheta_i = 0$*. Moreover, if* $h_i, \tilde{h}_i \geq 0$*, then the restriction of* $\sigma_a(\vartheta_i)$ *to* $A_{t(h_i)}(d\alpha_i) \setminus A_{t(h_i)/n}(d\alpha_i)$ *satisfies* $|\sigma_a^{-1}(\vartheta_i)^*(g) - \tilde{g}| < c(h_i, \tilde{h}_i, \vartheta_i, n)\tilde{g}$*, where* $c(h_i, \tilde{h}_i, \vartheta_i, n)$ *depends continuously on* $h_i, \tilde{h}_i \geq 0$ *and* $\vartheta_i \in \mathbb{R}$*, and is zero if* $h_i = \tilde{h}_i = 0$*.*

(b) *The standard map* $\sigma^\circ : H^\circ \to \tilde{H}^\circ$ *satisfies* $|(\sigma^\circ)_*(g) - \tilde{g}| < c(h, \tilde{h})\tilde{g}$*, where* $c(h, \tilde{h})$ *depends continuously on the* h_i*'s and the* $\tilde{h}_j$*'s and is equal to zero if* $h = \tilde{h}$*.*

The proof is by direct estimate. Similar computations are also in [Bis02]. The key point in (b) is that the angle of incidence of a geodesic ray that joins B (resp. $\tilde{B}$) to ∂H° (resp. $\partial \tilde{H}^\circ$) is bounded below by $\xi(h)$ (resp. $\xi(\tilde{h})$).

Corollary 4.2. (a) *Let* (ℓ_k, ϑ_k) *be a sequence of pairs in* $\mathbb{R}_{\geq 0} \times \mathbb{R}$ *such that: either* $\ell_k \to 0$ *or* $(\ell_k, \vartheta_k) \to (\tilde{\ell}, \tilde{\vartheta})$*. Let* $\sigma_{a,(k)} : A_{t(\ell_k)}(\ell_k) \to A_{t(\ell)}(\tilde{\ell})$ *be the standard map that twists by an angle* ϑ_k*. Then,* $(\sigma_{a,(k)}^{-1})^*(g) \to \tilde{g}$ *(and so* $K(\sigma_{(k)}^{-1}) \to 1$*) uniformly on every bounded subset of* $A_{t(\tilde{\ell})}(\tilde{\ell})$*.*

(b) *Let* $\{H_{(k)}\}$ *be a sequence of closed hexagons* $H_{(k)} = H(h_{1,(k)}, h_{2,(k)}, h_{3,(k)})$ *such that* $h_{i,(k)} \to h_i$ *and let* $\tilde{H} = H(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$*. Moreover, consider a sequence* $\Theta_{(k)} = (\vartheta_{1,(k)}, \vartheta_{2,(k)}, \vartheta_{3,(k)})$ *such that* $\vartheta_{i,(k)} \to \vartheta_i$ *whenever* $\tilde{h}_i > 0$*. If* $\sigma_{(k)} : dH_{(k)} \to d\tilde{H}$ *is the standard map between pair of pants with twist data* $\Theta_{(k)}$*, then* $(\sigma_{(k)}^{-1})^*(g) \to \tilde{g}$ *(and so* $K(\sigma_{(k)}^{-1}) \to 1$*) uniformly on the bounded subsets of* $d\tilde{H}$*.*

In fact, straightforward computations show that, if a map $f : (S, g) \to (\tilde{S}, \tilde{g})$ between surfaces satisfies $|f^*(\tilde{g}) - g| < \varepsilon g$, then $|K(f) - 1| < 2\varepsilon + o(\varepsilon)$.

5. The Teichmüller space

Let R be a compact oriented surface and assume that $\chi(R) < 0$.

An *R-marked Riemann surface* is an isotopy class of oriented diffeomorphisms $[f : R \to R']$, where R' is a Riemann surface. Two R-marked Riemann surfaces $[f' : R \to R']$ and $[f'' : R \to R'']$ are *equivalent* if there exists a biholomorphism $h : R' \to R''$ such that $h \circ f \simeq f'$. The *Teichmüller space* of R is the set $\mathcal{T}(R)$ of equivalence classes of R-marked Riemann surfaces.

Clearly, the uniformization theorem endows each Riemann surface R' with $\chi(R') < 0$ with a unique hyperbolic metric. Thus, we can consider $\mathcal{T}(R)$ as the set of equivalence classes of R-marked hyperbolic surfaces.

Here, we recall just two ways to topologize $\mathcal{T}(R)$.

The *Teichmüller distance* between $[f_1 : R \to R_1]$ and $[f_2 : R \to R_2]$ is

$$d_T([f_1], [f_2]) = \frac{1}{2} \log \inf_{g_i \simeq f_i} K(g_2 \circ g_1^{-1})$$

where clearly $g_i : R \to R_i$ ranges among quasiconformal homeomorphisms.

However, in order to give a topology to the "augmented" Teichmüller space it is easier to use Fenchel-Nielsen coordinates.

A *pair of pants decomposition* of R is the choice of a maximal set of disjoint simple closed curves $\{\gamma_1, \dots, \gamma_N\}$ of R, which are not pairwise isotopic nor homotopically trivial. The *length function* $\ell_{\gamma_i} : \mathcal{T}(R) \to \mathbb{R}_+$ sends $[f : R \to R']$ to the hyperbolic length of the unique geodesic representative (which we will denote by $f_*(\gamma_i)$) in the free homotopy class of $f(\gamma_i) \subset R'$.

Knowing $\ell_{\gamma_1}, \dots, \ell_{\gamma_N}$, we are able to uniquely determine a hyperbolic structure on $R' \setminus \bigcup_i f_*(\gamma_i)$, which consists of a disjoint union of pair of pants.

The recipe to glue the pair of pants and obtain R' and $f : R \to R'$ is encoded in the twist parameters $\tau_1, \dots, \tau_N \in \mathbb{R}$, which are uniquely determined after fixing some conventions. The important fact is that, if $[f : R \to R_1]$ corresponds to parameters $(\ell_{\gamma_1}, \tau_1, \dots, \ell_{\gamma_N}, \tau_N)$, then an R-marked surface with coordinates $(\ell_{\gamma_1}, \tau_1 + x, \dots, \ell_{\gamma_N}, \tau_N)$ is obtained by performing a right twist on R_1 along $f_*(\gamma_1)$, with traslation distance x.

The *Fenchel-Nielsen coordinates* associated to $\{\gamma_1, \dots, \gamma_N\}$ are the functions $(\ell_{\gamma_i}, \tau_i)$.

A hyperbolic surface with cusps is a compact oriented surface R'', endowed with a hyperbolic metric of finite volume that can blow-up at a finite number of points (the cusps). A *nodal hyperbolic surface* R' is a topological space obtained from a hyperbolic surface R'' by identifying some cusps of R'' in pairs (thus determining the *nodes* $\nu_1, \dots, \nu_k$ of R'). Denote by $R'_{sm} := R' \setminus \{\nu_1, \dots, \nu_k\}$ the smooth locus of R'.

An *R-marking* of R' is an isotopy class of maps $[f : R \to R']$ such that $f^{-1}(\nu_i)$ is a smooth circle for $i = 1, \dots, k$ and f is a diffeomorphism elsewhere. Equivalence of R-marked hyperbolic surfaces is defined in the natural way.

The *augmented Teichmüller space* of R is the set $\overline{\mathcal{T}}(R)$ of equivalence classes of R-marked (possibly nodal) hyperbolic surfaces (see also [Ber74]).

$\overline{\mathcal{T}}(R)$ clearly contains $\mathcal{T}(R)$. To describe the topology around some nodal $[f : R \to R']$, let $\gamma_i = f^{-1}(\nu_i)$ be disjoint (homotopically nontrivial) simple closed curves on R that are shrunk to nodes $\nu_1, \dots, \nu_k$ of R'. Complete $\gamma_1, \dots, \gamma_k$ to a pair of pants decomposition $\{\gamma_1, \dots, \gamma_N\}$ of R and consider the associated Fenchel-Nielsen coordinates $(\ell_{\gamma_i}, \tau_i)$. Clearly, $\ell_{\gamma_1}, \dots, \ell_{\gamma_k}$ continuously extend to zero on $[f]$, whereas $\tau_1, \dots, \tau_k$ are no longer well-defined at $[f]$. Thus, declare that a sequence $[f_n : R \to R_n]$ converges to $[f]$ if and only if

(a) $\exists n_0$ such that $f_n^{-1}(\nu)$ is homotopic to some γ_i for every node $\nu \in R_n$ and every $n \geq n_0$
(b) as $n \to \infty$, we have $\ell_{\gamma_i}(f_n) \to \ell_{\gamma_i}(f)$ for $i = 1, \dots, N$ and $\tau_j(f_n) \to \tau_j(f)$ for $j = k+1, \dots, N$.

Remark 1. *It turns out that $\overline{\mathcal{T}}(S)$ is the completion of $\mathcal{T}(S)$ with respect to the Weil-Petersson metric* [Mas76].

If S is a compact oriented surface with boundary and $\chi(S) < 0$, then we can consider its double dS, obtained by gluing two copies of S (with opposite orientations) along their boundary. The compact surface (without boundary) dS comes equipped with a natural orientation-reversing involution σ, so that $dS/\sigma \cong S$.

We can define a *Riemann surface with boundary* (resp. *hyperbolic surface with geodesic boundary*) to be a surface Σ together with a conformal (resp. hyperbolic) structure on its double such that σ is anti-holomorphic (resp. an isometry). Clearly, this is the same as giving a hyperbolic metric on Σ with *geodesic boundary* or a complex structure on Σ which is *real* on $\partial\Sigma$.

Thus, we can define the Teichmüller space of S to be $\mathcal{T}(S) := \mathcal{T}(dS)^\sigma$ and, similarly, $\overline{\mathcal{T}}(S) := \overline{\mathcal{T}}(dS)^\sigma$.

We will represent a point $[f : dS \to d\Sigma] \in \overline{\mathcal{T}}(S)^\sigma$ by the class of maps $[g : S \to \Sigma]$ whose doubles are isotopic to f. Clearly, such g's can shrink a boundary component of S to a cusp of Σ. Again, denote by $\Sigma_{sm} := (d\Sigma)_{sm} \cap \Sigma$ the smooth locus of Σ.

6. A convergence criterion

Let $[f : S \to \Sigma] \in \overline{\mathcal{T}}^{WP}(S)$ and let $\gamma \subset S$ be a (homotopically nontrivial) simple closed curve. We define $R_\gamma(f)$ in the following way.

- if $f(\gamma)$ is homotopic to a boundary cusp or to a boundary geodesic, then $R_\gamma(f) := A_{t(\ell_\gamma(f))}(f(\gamma))$;
- otherwise, $f(\gamma)$ is homotopic to a node or to a closed geodesic in the interior of Σ, and we let $R_\gamma(f) := dA_{t(\ell_\gamma(f))}(f(\gamma))$.

Theorem 6.1. *Let $[f : S \to (\Sigma, g)] \in \overline{\mathcal{T}}(S)$ and call $\gamma_1, \dots, \gamma_k$ the simple closed curves of S that are contracted to a point by f, $R_i = R_{\gamma_i}(f)$ and $\Sigma^\circ := \Sigma \setminus (R_1 \cup \dots \cup R_k) \subset \Sigma_{sm}$. For every sequence $\{[f_n : S \to (\Sigma_n, g_n)]\}$ of points in $\overline{\mathcal{T}}(S)$, the following are equivalent:*

1. *$[f_n] \to [f]$ in $\overline{\mathcal{T}}(S)$*
2. *$\ell_{\gamma_i}(f_n) \to 0$ and there exist representatives $\tilde{f}_n \in [f_n]$ such that $(f \circ \tilde{f}_n^{-1})\Big|_{R_{\gamma_i}(f_n)}$ is standard and $(\tilde{f}_n \circ f^{-1})^*(g_n) \to g$ uniformly on Σ°*
3. *$\exists \tilde{f}_n \in [f_n]$ such that the metrics $(\tilde{f}_n \circ f^{-1})^*(g_n) \to g$ uniformly on the compact subsets of Σ_{sm}*
4. *$\ell_{\gamma_i}(f_n) \to 0$ and $\exists \tilde{f}_n \in [f_n]$ such that $(f \circ \tilde{f}_n^{-1})\Big|_{R_{\gamma_i}(f_n)}$ is standard and $(\tilde{f}_n \circ f^{-1})\Big|_{\Sigma^\circ}$ is K_n-quasiconformal with $K_n \to 1$*
5. *$\exists \tilde{f}_n \in [f_n]$ such that, for every compact subset $F \subset \Sigma_{sm}$, the homeomorphism $(\tilde{f}_n \circ f^{-1})\Big|_F$ is $K_{n,F}$-quasiconformal and $K_{n,F} \to 1$.*

Proof. For (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (4), complete $\{\gamma_1, \dots, \gamma_k\}$ to a maximal system of curves $\{\gamma_1, \dots, \gamma_N\}$ of S. Fenchel-Nielsen coordinates tell us that $\ell_{\gamma_i}(f_n) \to \ell_{\gamma_i}(f)$ as $n \to \infty$ for every $i = 1, \dots, N$; moreover, if $\ell_{\gamma_i}(f) > 0$, then $\tau_{\gamma_i}(f_n) \to \tau_{\gamma_i}(f)$.

Let $\Sigma \setminus \cup_i f_*(\gamma_i) = P_1 \cup \dots \cup P_M$ to be the pair of pants decomposition associated to $\{\gamma_1, \dots, \gamma_N\}$, in such a way that P_i is bounded by $f_*(\gamma_{i_1}), f_*(\gamma_{i_2}), f_*(\gamma_{i_3})$. Call $P_1^{(n)}, \dots, P_M^{(n)}$ be the analogous decomposition of Σ_n. Define $\tilde{f}_n$ so that the restriction of $f \circ \tilde{f}_n^{-1}$ to $P_i^{(n)}$ is a standard map of pair of pants $P_i^{(n)} \to P_i$ with twist data $\frac{1}{2}(\Theta_i(f) - \Theta_i(f_n))$, where $\theta_\gamma := \tau_\gamma / \ell_\gamma$ and $\Theta_i = \big(\vartheta_{\gamma_{i_1}}, \vartheta_{\gamma_{i_2}}, \vartheta_{\gamma_{i_3}}\big)$ be the collection of twist parameters along ∂P_i. The result follows from Corollary 4.2.

(2) $\Longrightarrow$ (3) and (4) $\Longrightarrow$ (5) are also a consequence of Corollary 4.2.

(3) $\Longrightarrow$ (1) relies on the following remark. Let $\rho_{i,l}$ be a horocycle of g-length l in R_i. For every $\varepsilon > 0$, there exists n_0 such that the $(\tilde{f}_n \circ f^{-1})^*(g_n)$-length of $\rho_{i,l}$ is less than $(1+\varepsilon)l$ for all $\varepsilon \le l \le 1$ and $n \ge n_0$. Thus, $\ell_{\gamma_i}(f_n) \to 0$ for all i. Moreover, for every simple closed curve $\alpha \subset S$ disjoint from γ_i, we have $f_*(\alpha) \cap R_i = \tilde{f}_{n,*}(\alpha) \cap R_{\gamma_i}(\tilde{f}_n) = \emptyset$ for $n \ge n_\alpha$. Hence, $\ell_\beta(f_n) \to \ell_\beta(f)$ for all simple closed curves $\beta \subset S$, which implies that $[f_n] \to [f]$.

(5) $\Longrightarrow$ (1) is more elaborate. For every $M > 0$, consider a compact annulus $A_i(M) \subset R_i \cap \Sigma_{sm}$ with modulus M. Then, $\exists n_0$ such that $(f_n \circ f^{-1})(A_i(M))$ has modulus at least $M/2$ and so $\ell_{\gamma_i}(f_n) \leq 2\pi/M$ for all $n \geq n_0$. Thus, $\ell_{\gamma_i}(f_n) \to 0$.

Let S' be the surface with boundary obtained from S by cutting along the γ_i's and then compactifying so that $S \setminus \bigcup_i \gamma_i \simeq S'$. Similarly, let Σ_n' be the compact hyperbolic surface obtained from Σ_n by cutting along the geodesics (or the cusps) $\tilde{f}_{n,*}(\gamma_1), \dots, \tilde{f}_{n,*}(\gamma_k)$ (and then compactifying) and call $\tilde{f}_n' : S' \to (\Sigma_n', g_n')$ the associated map. With the same procedure, we also get an $f' : S' \to (\Sigma', g')$.

To prove (1), it is sufficient to show that $[\tilde{f}_n'] \to [f']$ in $\mathcal{T}(S')$. We split it into two steps.

(a) We perform an infinite grafting at the boundary of Σ_n' by gluing infinite flat cylinders at $\partial\Sigma_n'$ in order to obtain a new surface $(\mathrm{gr}_\infty(\Sigma_n'), \mathrm{gr}_\infty(g_n'))$ with an inclusion $j_n : \Sigma_n' \hookrightarrow \mathrm{gr}_\infty(\Sigma_n')$, and we show that the hyperbolic lengths satisfy

$$1/C < \ell_\alpha(\Sigma_n')/\ell_\alpha(\mathrm{gr}_\infty(\Sigma_n')) < C$$

for every simple closed curve $\alpha \subset S'$ not homotopic to a boundary component, where $C = C(\ell_{\boldsymbol{\gamma}})$ and $C \to 1$ as $\ell_{\boldsymbol{\gamma}} = \ell_{\gamma_1} + \cdots + \ell_{\gamma_k} \to 0$. Thus, the (Weil-Petersson) distance between $[\tilde{f}_n' : S' \to \Sigma_n']$ and $[j_n \circ \tilde{f}_n' : S' \to \mathrm{gr}_\infty(\Sigma_n')]$ goes to zero.

(b) Modifying $j_n \circ \tilde{f}_n'$, we produce a $\hat{f}_n' : S' \to \mathrm{gr}_\infty(\Sigma_n')$ such that $\hat{f}_n' \circ f^{-1} : \Sigma' \to \mathrm{gr}_\infty(\Sigma_n')$ is a K_n-quasiconformal homeomorphism, with $K_n \to 1$.

As for (a), let $\alpha_n \subset (\Sigma_n', g_n')$ and $\hat{\alpha}_n \subset (\mathrm{gr}_\infty(\Sigma_n'), g_n'')$ be simple closed geodesics for the hyperbolic metric in the same homotopy class (here g_n'' is hyperbolic and conformally equivalent to $\mathrm{gr}_\infty(g_n')$), that is $j_n(\alpha_n) \simeq \hat{\alpha}_n$. Schwarz's lemma implies that the process of grafting and uniformizing contracts lengths, thus $\exists n_0$ such that the (g_n'')-distance between $j_n(\partial\Sigma_n')$ and a unit horocycle is at least $c_n = -\log(\ell_{\boldsymbol{\gamma}}(g_n'))/2$ for all $n \geq n_0$.

Again by Schwarz's lemma (applied to the lift of j_n to the universal covers of Σ_n' and $\mathrm{gr}_\infty(\Sigma_n')$), we have $|\nabla j(p)| \geq \tanh d_{g''}(j(p), j(\partial\Sigma_n'))$ for every $p \in \Sigma_n'$.

As no simple closed geodesic of $(\mathrm{gr}_\infty(\Sigma_n'), g_n'')$ enters unit horoballs, we obtain $\ell_{\alpha_n}(g_n') \geq \ell_{j_n(\alpha_n)}(g_n'') \geq \ell_{\hat{\alpha}_n}(g_n'') \geq \ell_{j^{-1}(\hat{\alpha}_n)}(g_n')\tanh(c_n) \geq \ell_{\alpha_n}(g_n')\tanh(c_n)$. As $C := \tanh(c_n) \to 1$, we obtain (a).

For part (b), let $F_m := \Sigma' \setminus \bigcup_i A_{\frac{t(0)}{m}}(\gamma_i')$, where $\{\gamma_i'\}$ is the set of cusps of Σ' for $m \geq 2$. The restriction of $j_n \circ \tilde{f}_n' \circ (f')^{-1}$ defines $g_{i,n}$ through the following commutative diagram.

$$\begin{tikzcd}
\overline{A}_{\frac{t(0)}{m}}(\gamma_i') \arrow[r, hook] & \overline{A}_{t(0)}(\gamma_i') \arrow[r, hook, "j_n\circ\tilde{f}_n'\circ(f')^{-1}"] & \mathrm{gr}_\infty(\Sigma_n') \\
\overline{A}(\exp\{2\pi[1-m/t(0)]\}) \arrow[r, hook] \arrow[u, "\cong"] & \overline{A}(\exp\{2\pi[1-1/t(0)]\}) \arrow[r, hook, "g_{i,n}"] \arrow[u, "\cong"] & \overline{\Delta} \arrow[u, hook]
\end{tikzcd}$$

We can also assume that $\partial\Delta$ is contained in the image of $g_{i,n}$ and that the last vertical arrow sends the origin to the cusp.

As $\tilde{f}_n' \circ (f')^{-1}$ is $K_{n,m}$-quasiconformal on F_m, so is $g_{i,n}$ outside $\overline{A}(\exp\{2\pi[1 - m/t(0)]\})$ for every i. It follows from Theorem 2.7(b) that there exists $\hat{g}_{i,n} : \overline{\Delta} \to \overline{\Delta}$ which is a $\hat{K}_{n,m}$-quasiconformal homeomorphism that fixes the origin and which coincides with $g_{i,n}$ near $\partial\Delta$. Thus, replacing $\coprod_i g_{i,n}$ by $\coprod_i \hat{g}_{i,n}$ we obtain a modification of $\tilde{f}'_n$, which we call *m-th modification of* $\tilde{f}'_n$.

Define $\hat{f}'_n$ to be the m-th modification of $\tilde{f}'_n$ for all $n_{m-1} \leq n < n_m$ and all $m \geq 2$. Adjusting the sequence $n_2 = 1 < n_3 < n_4 < \dots$, one obtains the wished result. $\square$

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Imperial College of London, Department of Mathematics, South Kensington Campus, London SW7 2AZ

E-mail address: `g.mondello@imperial.ac.uk`