Riemann surfaces, arc systems and Weil-Petersson form

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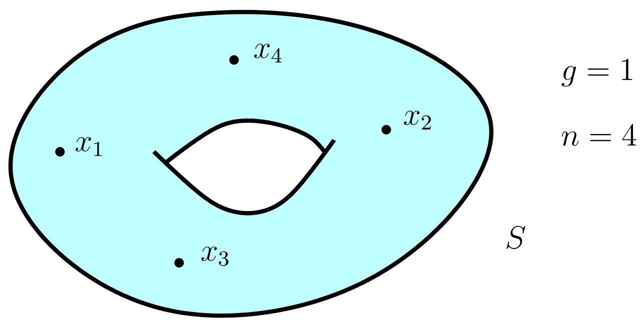
1. – Introduction

Let S be a Riemann surface of genus g together with n distinct points $x = \{x_1, \dots, x_n\} \subset S$ and let $\dot{S} = S \setminus x$ be the corresponding pointed surface.

Assume throughout the paper that $\chi(\dot{S}) = 2 - 2g - n < 0$.



The **moduli space** of (S, x) is the orbifold $\mathcal{M}(S, x)$ that classifies x-marked Riemann surfaces diffeomorphic to (S, x) up to isomorphism.

Its universal cover (in the orbifold category) is the **Teichmüller space** $\mathcal{T}(S, x)$ that parametrizes complex structures on S up to isotopy relative to x.

In other words, $\mathcal{T}(S, x)$ is the space of equivalence classes of oriented diffeomorphisms $f : S \to \Sigma$ of Riemann surfaces, where two oriented diffeomorphisms $f : S \to \Sigma$ and $g : S \to \Sigma'$ of Riemann surfaces are equivalent if there exists a biholomorphic map $h : \Sigma \to \Sigma'$ such that $h \circ f \simeq g$ are isotopic relative to x.

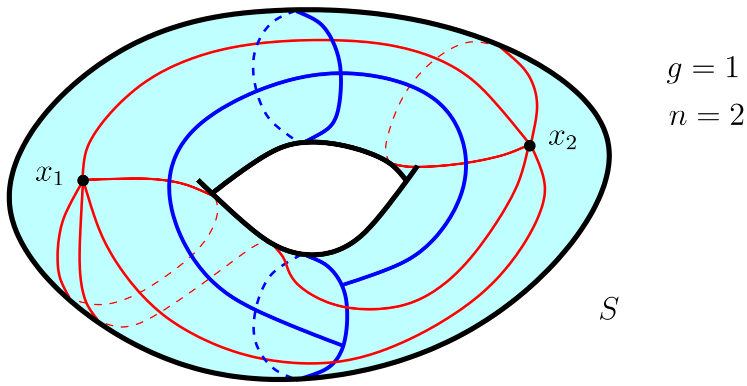
In this context, the cover $\mathcal{T}(S, x) \to \mathcal{M}(S, x)$ is simply the forgetful map $[f : S \to \Sigma] \mapsto [(\Sigma, f(x))]$. It is also clear that the **mapping class group** $\Gamma(S, x) = \pi_0 \mathrm{Diff}_+(S, x)$ of isotopy classes of oriented diffeomorphisms of (S, x) acts on $\mathcal{T}(S, x)$ by right composition and that $\mathcal{M}(S, x)$ identifies to the quotient $\mathcal{T}(S, x)/\Gamma(S, x)$. The group $\Gamma(S, x)$ is infinite but it acts properly and discontinuously on $\mathcal{T}(S, x)$; in particular, the stabilizer of $[f : S \to \Sigma]$ is naturally isomorphic to the automorphism group $\mathrm{Aut}(\Sigma, f(x))$, which is known to be finite (because $\chi(\dot{S}) < 0$).

A classical result of Teichmüller states that $\mathcal{T}(S, x)$ is differentiably isomorphic to a Euclidean space (of dimension $6g - 6 + 2n$). Thus, $\mathcal{M}(S, x)$ locally looks like $\mathbb{C}^{3g-3+n}/G$ around $[(\Sigma, f(x))]$, where $G = \mathrm{Aut}(\Sigma, x)$.

From the orbifold point of view, which we will privilege, $\mathcal{T}(S,x) \to \mathcal{M}(S,x)$ is a $\Gamma(S,x)$-principal bundle; whereas it is certainly not so at the level of underlying topological spaces, whenever some Riemann surface in $\mathcal{M}(S,x)$ has nontrivial automorphisms.

This way, to give a $\Gamma(S,x)$-equivariant cellular model for $\mathcal{T}(S,x)$ is the same as giving an orbi-cellular structure to $\mathcal{M}(S,x)$: even though we will not speak of orbi-cells, this is our basic purpose.

Two constructions are known in literature: starting from a Riemann surface (S,x) and a collection of numbers $p_1,\dots,p_n > 0$, both of them produce (in different ways) a metric graph G together with an isotopy class of homotopy equivalences $G \hookrightarrow \dot{S}$, where "metric" means that each side of G is associated a positive weight.



Hence, both constructions induce a map

$$\mathcal{T}(S,x) \times \Delta^{n-1} \times \mathbb{R}_+ \longrightarrow \begin{pmatrix} \text{metric graphs } G \subset \dot{S} \\ \text{up to isotopy} \end{pmatrix}$$

where $\Delta^{n-1} \times \mathbb{R}_+$ must be understood as $\mathbb{R}^n_{\geq 0} \setminus \{0\}$.

Our aim is to relate these two maps.

2. – The arc complex

Let S be a Riemann surface with marked points $x = \{x_i\}$ and/or boundary.

An **arc** on (S,x) is a nontrivial isotopy class (relative to $x \cup \partial S$) of simple paths that intersect $x \cup \partial S$ at the extremal points and nowhere else.

A **k-system of arcs** is a set $\underline{\boldsymbol{\alpha}} = \{\alpha_1,\dots,\alpha_k\}$ of k distinct arcs that admit disjoint representatives, up to isotopies of systems of arcs.

The **arc complex** $\mathfrak{A}(S,x)$ (see [3]) is the simplicial complex whose m-simplices are $(m+1)$-systems of arcs in (S,x).

We say that $\underline{\boldsymbol{\alpha}} = \{\alpha_0,\dots,\alpha_k\}$ is **proper** if

$$\dot{S} \setminus \underline{\boldsymbol{\alpha}} := \dot{S} \setminus \cup_{i=0}^k \alpha_i$$

is a disjoint union of discs and pointed discs. The subset of $\mathfrak{A}(S,x)$ consisting of proper simplices is denoted by $\mathfrak{A}^\circ(S,x)$.

Lefschetz duality on (S,x) will induce the correspondences shown below, which will allow us to identify the space of embedded graphs $G \subset \dot{S}$ to the topological realization of $\mathfrak{A}^\circ(S,x)$.

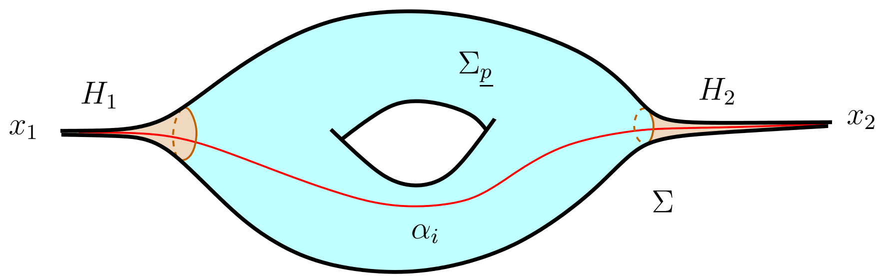
$$\begin{array}{rcl}
e_i \text{ edge of } G \subset \dot{S} & \longleftrightarrow & \text{dual arc } \alpha_i \subset S \\
\text{weight of } e_i & \longleftrightarrow & \text{weight } w_i \text{ of } \alpha_i \\
G \subset \dot{S} \text{ homotopy equiv.} & \longleftrightarrow & \text{proper system of arcs } \underline{\boldsymbol{\alpha}} \\
\text{metric on } G & \longleftrightarrow & \text{system of weights } \{w_i\} \\
\left\{ \begin{array}{c} \text{metric graphs } G \subset \dot{S} \\ \text{homotopy equiv.} \\ \text{up to isotopy} \end{array} \right\} & \longleftrightarrow & \begin{array}{c} \text{topological realization} \\ |\mathfrak{A}^\circ(S,x)|_{\mathbb{R}} := |\mathfrak{A}^\circ(S,x)| \times \mathbb{R}_+ \end{array}
\end{array}$$

See also [10] for a more detailed treatment of arc systems and ribbon graphs.

3. – First construction (Penner, Bowditch-Epstein)

Given $[f : S \to \Sigma] \in \mathcal{T}(S,x)$, let us endow $\dot{\Sigma}$ with the Poincaré metric (namely, the hyperbolic metric obtained from a uniformization $\mathbb{H} \to \dot{\Sigma}$) and consider the horoball H_i around the cusp x_i of circumference p_i.

Call $(\Sigma, H_1, \dots, H_n)$ a **decorated surface** (see [13]) and $\Sigma_{\underline{p}} = \Sigma \setminus \cup H_i^\circ$ the corresponding truncated surface.



Just for semplicity (as it is not necessary), assume that $0 < p_i < 1$, in such a way that the H_i's are pairwise disjoint.

For every arc $\alpha_i \subset S$, identify $f(\alpha_i) \subset \Sigma$ with its unique geodesic representative and call its *reduced length* $\tilde{a}_i := \ell(\alpha_i \cap \Sigma_{\underline{p}})$. The following is due to Penner.

Proposition 1 (Penner [13]) *If $\underline{\boldsymbol{\alpha}} = \{\alpha_i\} \in \mathfrak{A}^\circ(S,x)$ is an ideal triangulation (i.e. a maximal system of arcs) of (S,x), then*

$$\begin{aligned}
\tilde{F}_{\underline{\boldsymbol{\alpha}}} : \mathcal{T}(S,x) \times \mathbb{R}^n_+ &\longrightarrow \mathbb{R}^{6g-6+3n}_+ \\
([f : S \to \Sigma], \underline{p}) &\longmapsto (\tilde{a}_i)
\end{aligned}$$

is a real-analytic diffeomorphism.

The problem with the coordinates $\{\tilde{a}_i\}$ is that they are not $\Gamma(S,x)$-invariant, unless one is able to canonically associate to every point in $\mathcal{T}(S,x) \times \mathbb{R}^n_+$ a system of arcs.

In order to do that, define the **valence** of a point $q \in \Sigma$ as

$$\mathbf{val}(q) := \# \left\{ \begin{array}{c} \text{paths of minimal length} \\ \text{from } q \text{ to } \partial\Sigma_{\underline{p}} \end{array} \right\}$$

and the **spine** of $\Sigma_{\underline{p}}$ (see [2]) to be

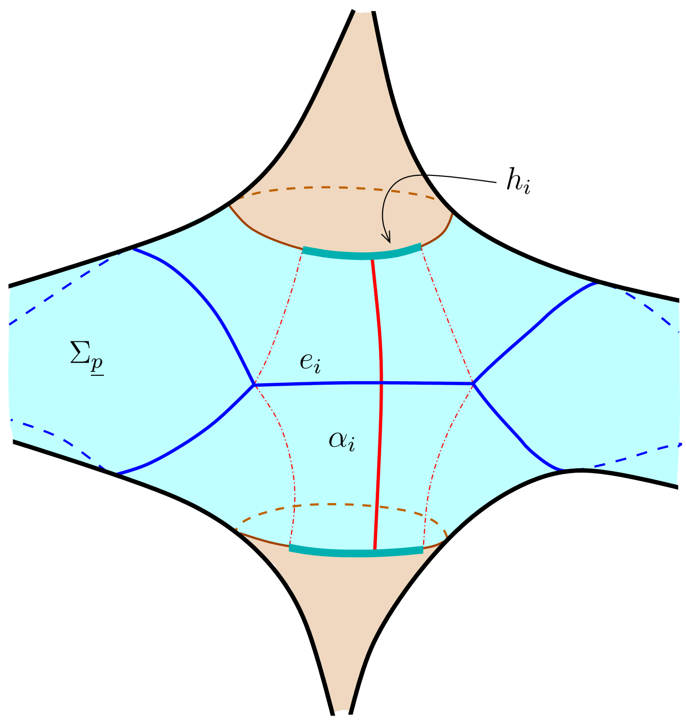
$$G = \mathrm{Sp}(\Sigma_{\underline{p}}) := \{ q \in \Sigma_{\underline{p}} \,|\, \mathrm{val}(q) \geq 2 \}$$

The components of $\mathrm{val}^{-1}(2)$ are geodesic **edges** of G and $\mathrm{val}^{-1}([3,\infty))$ are isolated **vertices**, so that G is a 1-dimensional CW complex embedded in $\dot{\Sigma}$ as a deformation retract. In fact, for every $q \in \dot{\Sigma} \setminus G$, we can consider the unique shortest geodesic from $\partial\Sigma_{\underline{p}}$ to q, prolong it until it meets G and flow q along this prolongation.

Now, for every edge e_i of $\mathrm{Sp}(\Sigma_{\underline{p}})$, we can define a **dual arc** α_i by joining the two shortest geodesics that contribute $\mathrm{val}(q)$, where q is any point internal to e_i.

One can notice that the system of arcs $\underline{\boldsymbol{\alpha}}_{sp}$ dual to the spine $\mathrm{Sp}(\Sigma_{\underline{p}})$ is proper. More generally, every proper system of arcs is dual to a graph embedded in $\dot{\Sigma}$ through a homotopy equivalence.

It is not hard to see that, for a general Σ, the spine $\mathrm{Sp}(\Sigma_{\underline{p}})$ has trivalent vertices and so $\underline{\boldsymbol{\alpha}}_{sp}$ is an ideal triangulation of Σ.



Define the **weight** of e_i (and dually of α_i) to be equal to the length h_i of the portion of boundary horocycle obtained by "projecting" e_i onto $\partial\Sigma_{\underline{p}}$.

The result below provides the wished equivariant cellularization of the Teichmüller space (times a contractible $\Delta^{n-1} \times \mathbb{R}_+$).

THEOREM **1 (Penner [13], Bowditch-Epstein [2])** *The following map*

$$\begin{aligned} \Phi_{BPE}: \ \mathcal{T}(S,x) \times \Delta^{n-1} \times \mathbb{R}_+ &\longrightarrow |\mathfrak{A}^\circ(S,x)|_{\mathbb{R}} \\ ([f : S \to \Sigma], \underline{p}) &\longmapsto (\underline{\boldsymbol{\alpha}}_{sp}, \{h_i\}) \end{aligned}$$

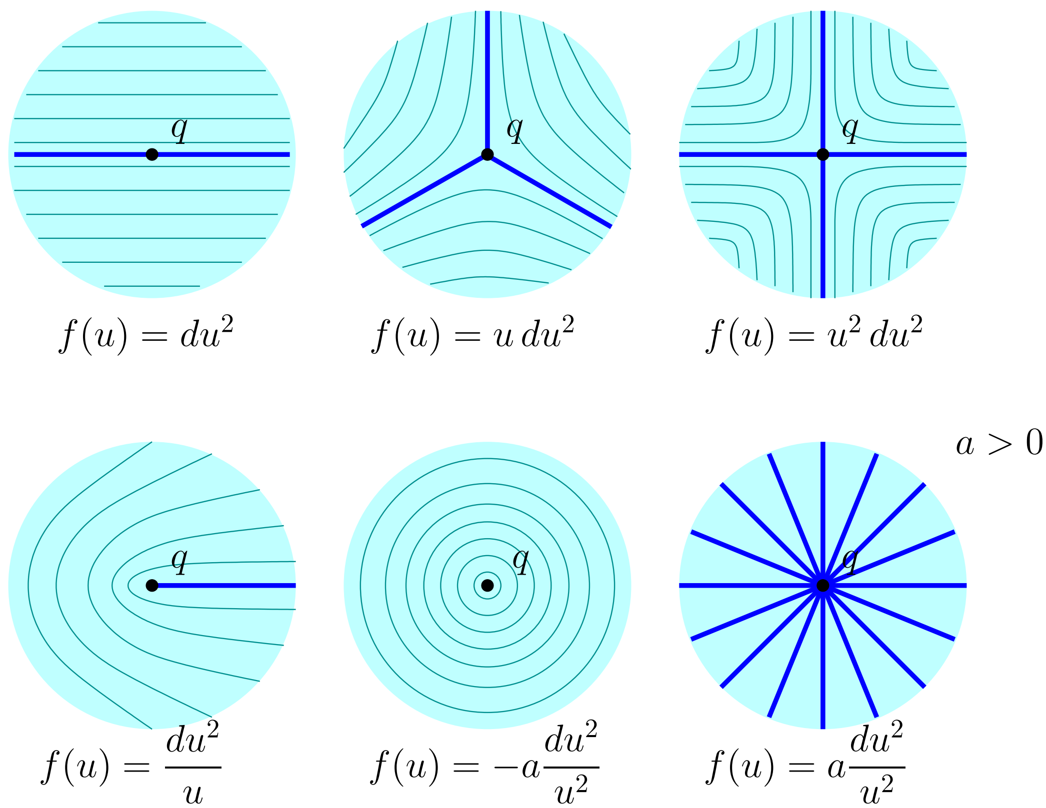
is a $\Gamma(S,x)$-equivariant homeomorphism. Moreover, its restriction to every open cell of $|\mathfrak{A}^\circ(S,x)|_{\mathbb{R}}$ is a real-analytic diffeomorphism.

4. – Second construction (Harer-Mumford-Thurston)

Consider a holomorphic quadratic differential φ on $\dot{\Sigma}$, which locally looks like $\varphi = f(z)dz^2$. Its absolute value $|\varphi| = |f(z)|dz\,d\bar{z}$ induces a flat metric, which has conical singularities at the **critical points** (that is, where the metric degenerates).

In particular, given $q \in \dot{\Sigma}$ with $f(q) \neq 0$, there exists a local holomorphic coordinate u around q such that $u(q) = 0$ and $\varphi(u) = du^2$. As u is uniquely determined up to sign, the **(horizontal) trajectories** $\mathrm{Im}(u) = \mathrm{cost}$ are well-defined. We call **critical** those trajectories that meet a critical point.

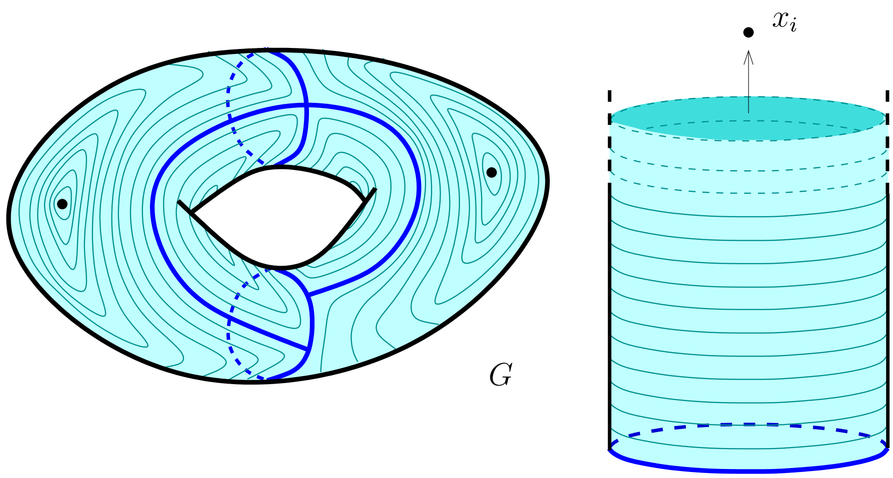
The structure of trajectories of φ is completely classified. In particular, periodic trajectories sweep out **annular domains** (see [16]).



The result below attaches to every surface Σ with weights at its marked points a quadratic differential whose trajectories are a simple as possible.

Theorem **2 (Strebel [16])** *Given $p_1, \ldots, p_n \geq 0$ (not all zero), $\exists! \varphi$ holomorphic quadratic differential on $\dot{\Sigma}$ such that*

- *$\exists$ a local holomorphic coordinate z_i at $x_i \in \Sigma$ such that $\varphi(z_i) = -\left(\frac{p_i}{2\pi}\right)^2 \frac{dz_i^2}{z_i^2}$*
- *the horizontal trajectories of φ are closed*
- *the complement of the critical trajectories is a disjoint union of annular domains, each one isometric to a semi-infinite cylinder whose parallels wind around x_i and have length p_i.*



Let G be the critical graph of φ and let $\underline{\boldsymbol{\alpha}}_{JS}$ be an arc system on S such that $f(\underline{\boldsymbol{\alpha}}_{JS})$ is dual to G. An associated system of weights is simply determined using the $|\varphi|$-lengths of the edges of G, namely $\tilde{w}_i = \ell_{|\varphi|}(e_i)$.

The following is due to Harer-Mumford-Thurston.

Theorem **3** (**[3]**) *The map*

$$\begin{aligned} \Phi_{JS} : \mathcal{T}(S,x) \times \Delta^{n-1} \times \mathbb{R}_+ &\longrightarrow |\mathfrak{A}^\circ(S,x)|_{\mathbb{R}} \\ ([f:S\to\Sigma],\underline{p}) &\longmapsto (\underline{\boldsymbol{\alpha}}_{JS},\{\tilde{w}_i\}) \end{aligned}$$

is a $\Gamma(S,x)$*-equivariant homeomorphism.*

Hubbard-Masur [4] showed that the restriction of Φ_{JS} to each open simplex is a real-analytic diffeomorphism.

Notice that one can act by $\mathbb{R}_+$ rescaling both sides $\mathcal{T}(S,x)\times\Delta^{n-1}\times\mathbb{R}_+$ and $|\mathfrak{A}^\circ(S,x)|_{\mathbb{R}}$. It is immediate to see that both constructions are $\mathbb{R}_+$-equivariant.

5. – Symplectic structure

The space $\mathcal{T}(S,x)$ is can be made into a complex variety of (complex) dimension $3g-3+n$. In fact, its holomorphic cotangent space at $[f:S\to\Sigma]$ can be identified to $H^0(\Sigma,K_\Sigma^{\otimes 2}(x))$ and its holomorphic tangent space to $H^{0,1}(\Sigma,T_\Sigma(-x))$. The natural pairing between them is

$$\begin{aligned} H^0(\Sigma,K_\Sigma^{\otimes 2}(x))\times H^{0,1}(\Sigma,T_\Sigma(-x)) &\longrightarrow \mathbb{C} \\ (\varphi,\mu) &\longmapsto \int_\Sigma \varphi\,\mu \end{aligned}$$

The **Weil-Petersson Kähler** metric is defined on the cotangent space as

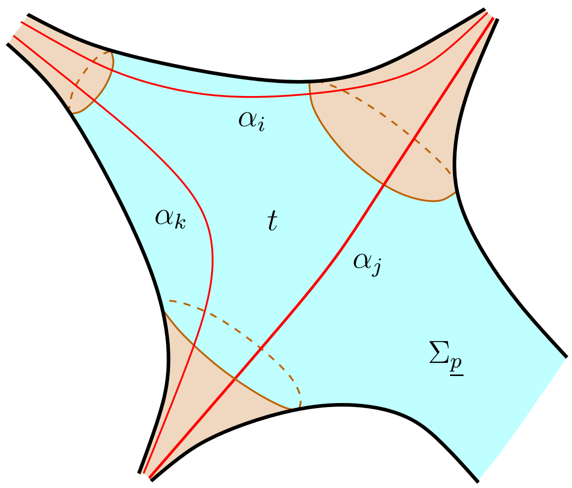
$$h^*(\varphi,\psi)=\int_\Sigma \frac{\varphi\,\overline{\psi}}{\lambda}$$

where $\varphi, \psi \in T^*_{[f]}\mathcal{T}(S,x)$ and λ is the hyperbolic metric; the corresponding h on the tangent space is obtained using the pairing above. If $h = g + i\omega$, then ω is called **Weil-Petersson sympletic form**.

The following result expresses the Weil-Petersson form in the α-coordinates.

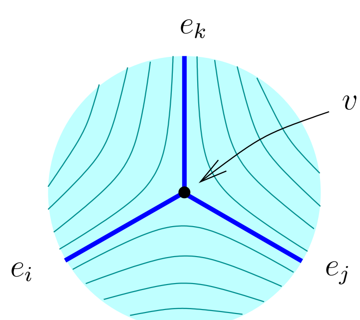
THEOREM **4 (Penner [14])** *Let $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}^\circ(S,x)$ be an ideal triangulation. Then*

$$\tilde{F}^*_{\underline{\boldsymbol{\alpha}}}\omega = -\frac{1}{2}\sum_t (da_i \wedge da_j + da_j \wedge da_k + da_k \wedge da_i)$$



On the other hand, Kontsevich [5] defined a symplectic piecewise-linear form Ω on $|\mathfrak{A}^\circ(S,x)|_{\mathbb{R}}$, which is an incarnation (up to some coefficient) of the cohomology class $\sum_k p_k^2 \psi_k$ on $\mathcal{M}(S,x)$. The dual β of Ω can be written as

$$\beta = \sum_v \left(\frac{\partial}{\partial e_i} \wedge \frac{\partial}{\partial e_j} + \frac{\partial}{\partial e_j} \wedge \frac{\partial}{\partial e_k} + \frac{\partial}{\partial e_k} \wedge \frac{\partial}{\partial e_i}\right)$$



The aim of the following sections is to relate ω and Ω.

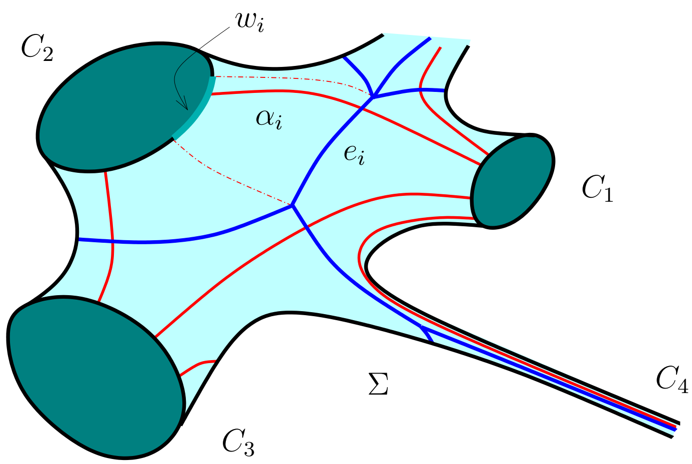
6. – Surfaces with boundary

The key idea (see [11]) is to interpolate between the two constructions using hyperbolic surfaces with geodesic boundary.

This way, we will recover BPE's construction as $\underline{p} \to 0$ and HTM's as $\underline{p} \to \infty$.

Let S be a hyperbolic surface with boundary $\partial S = C_1 \cup \cdots \cup C_n$ consisting of n geodesic circles and let $\mathcal{T}(S)$ be the Teichmüller space of S. Its cotangent space $T^*_{[f:S\to\Sigma]}\mathcal{T}(S)$ can be naturally identified to the *real* vector space of holomorphic quadratic differentials on Σ, whose restriction to $\partial\Sigma$ is real.

Thus, $\mathcal{T}(S)$ is a *real-analytic* manifold of dimension $6g - 6 + 3n$.



The geometry of hyperbolic hexagons shows that, if $\underline{\boldsymbol{\alpha}} = \{\alpha_i\}$ is a maximal system of arcs on S and $a_i = \ell_{\alpha_i}$, then

$$\begin{aligned} F_{\underline{\boldsymbol{\alpha}}} :\quad \mathcal{T}(S) &\longrightarrow \mathbb{R}_+^{6g-6+3n} \\ [f : S \to \Sigma] &\longmapsto (a_i) \end{aligned}$$

is a real-analytic diffeomorphism.

Similarly to the BPE construction, the spine $\mathrm{Sp}(\Sigma)$ induces a system of arcs $\underline{\boldsymbol{\alpha}}_{sp} = \{\alpha_i\} \in \mathfrak{A}(S)$. The weight w_i of α_i is the length of the projection of e_i (i.e. the edge of $\mathrm{Sp}(\Sigma)$ transverse to α_i) to the boundary of Σ. Extend the Teichmüller space $\mathcal{T}(S) \subset \tilde{\mathcal{T}}(S)$ in order to contain those $[f : S \to \Sigma]$, which collapse some C_i to a cusp of Σ.

Then, the boundary length function extends to $\mathcal{L} : \tilde{\mathcal{T}}(S) \longrightarrow \mathbb{R}^n_{\geq 0}$ and $\tilde{\mathcal{T}}(S)(0)$ identifies to the Teichmüller space of a pointed surface.

Theorem **5 (Luo [6])** *The map*

$$\begin{aligned} \Phi :\ \tilde{\mathcal{T}}(S) \setminus \tilde{\mathcal{T}}(S)(0) &\longrightarrow |\mathfrak{A}^\circ(S)|_{\mathbb{R}} \\ [f : S \to \Sigma] &\longmapsto (\underline{\boldsymbol{\alpha}}_{sp}, \{w_i\}) \end{aligned}$$

is a $\Gamma(S)$-equivariant homeomorphism. Moreover, the restriction of Φ to each open simplex is a real-analytic diffeomorphism.

A Weil-Petersson pairing can still be defined on $T^*\mathcal{T}(S)$ as

$$h^*(\varphi,\psi) = \int_S \frac{\varphi\,\overline{\psi}}{\lambda}$$

and we can write $h = g + i\omega$. However, g is not Kähler and ω is degenerate.

More precisely, $\eta = \mathrm{Im}(h^*)$ defines a Poisson structure and $\mathcal{T}(S)(\underline{p}) := \mathcal{L}^{-1}(\underline{p})$ is a symplectic leaf.

In this context, the analogue of Penner's result is the following.

Theorem **6 (Mondello [11])** *If $\underline{\boldsymbol{\alpha}} \in \mathfrak{A}^\circ(S)$ is a maximal system of arc on S, then*

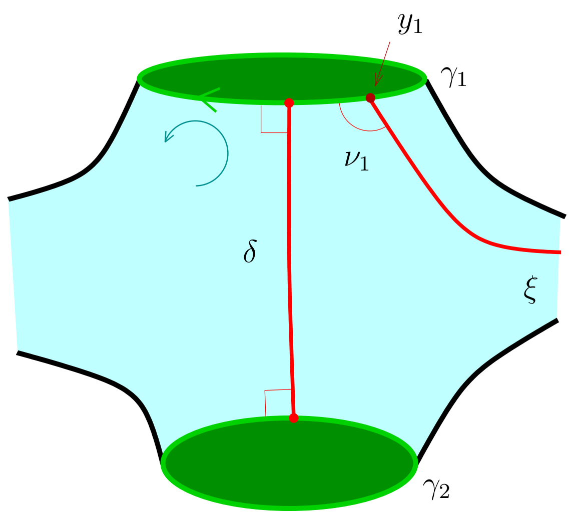
$$\eta = \frac{1}{4}\sum_C \sum_{\substack{y_i\in\alpha_i\cap C\\ y_j\in\alpha_j\cap C}} \frac{\sinh(p_C/2 - d_C(y_i,y_j))}{\sinh(p_C/2)} \frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial a_j}$$

where d_C is the distance (with sign) along C and p_C is the length of C.

The proof uses Wolpert's formula $\omega = \sum_i d\ell_i \wedge d\tau_i$ for a compact hyperbolic surface (see [17]) and the following.

Theorem **7 (Mondello [11])** *Let R be a closed hyperbolic surface and let $\gamma_1,\gamma_2,\xi \subset R$ be simple closed geodesics. Let δ be a geodesic between γ_1 and γ_2, that meets the γ_i's perpendicularly. For simplicity, suppose that no portion of ξ is homotopic to δ and that they do not intersect each other. Then*

$$\frac{\partial \ell(\delta)}{\partial \tau_\xi} = \frac{1}{2}\sum_{i=1}^{2}\sum_{y_i\in\xi\cap\gamma_i} \frac{\sinh(p_i/2 - d_{\gamma_i}(\delta,y_i))}{\sinh(p_i/2)}\sin(\nu_{y_i})$$



7. – Limits for long/short boundary components

Let $\mathbb{R}^n_+ \hookrightarrow \Delta^{n-1}\times[0,\infty]$ as $\underline{p}\mapsto(\underline{p}/\boldsymbol{p},\boldsymbol{p})$, where $\boldsymbol{p} = p_1+\cdots+p_n$.

We want to complete the family of Teichmüller spaces $\mathcal{L} : \tilde{\mathcal{T}}(S) \to \Delta^{n-1}\times[0,\infty]$ over $\Delta^{n-1}\times\{0\}$ and $\Delta^{n-1}\times\{\infty\}$.

In order to do that, we interpret boundaries of zero length as cusps, and so we identify the space of surfaces with projective decoration $\tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1}$ with $\mathcal{L}^{-1}(\Delta^{n-1} \times \{0\})$.

Thus, $\Phi : \tilde{\mathcal{T}}(S) \setminus \tilde{\mathcal{T}}(S)(0) \longrightarrow |\mathfrak{A}^\circ(S)| \times (0, \infty)$ extends over 0 by setting $\Phi_0 := \Phi_{BPE}$ and η limits to

$$\eta_0 = \frac{1}{4} \sum_H \sum_{\substack{y_i \in \alpha_i \cap H \\ y_j \in \alpha_j \cap H}} \left(1 - \frac{2 d_H(y_i, y_j)}{p_H}\right) \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

To understand the limit $\boldsymbol{p} \to \infty$, we introduce the space of measured laminations.

Let R be a closed hyperbolic surface and let $\mathcal{S}(R)$ be the set of (nontrivial isotopy classes of) simple closed curves on R.

The hyperbolic length function $\mathcal{T}(R) \times \mathcal{S}(R) \longrightarrow \mathbb{R}_+$ induces an embedding

$$\begin{array}{ccc} \mathcal{T}(R) & \longrightarrow & \mathbb{R}_+^{\mathcal{S}(R)} \\ & \searrow^{j} & \downarrow \\ & & \mathbb{P}(\mathbb{R}_+^{\mathcal{S}(R)}) \end{array}$$

of $\mathcal{T}(R)$ inside $\mathbb{P}(\mathbb{R}_+^{\mathcal{S}(R)})$.

Thurston's compactification of $\mathcal{T}(R)$ is the closure $\overline{\mathcal{T}}^{Th}(R) = \mathcal{T}(R) \cup \partial^{Th}\mathcal{T}(R)$ of $j(\mathcal{T}(R))$. Incidentally, we recall that $\overline{\mathcal{T}}^{Th}(R)$ is homeomorphic to a closed ball, whose boundary sphere identifies to $\partial^{Th}\mathcal{T}(R)$.

A **measured lamination** m on R is a closed $F \subset R$ which is foliated by complete geodesics together with a transverse measure. Though not evident, the space of measured lamination does not depend on the specific hyperbolic metric of R but only on its diffeomorphism type.

Notice that $\mathcal{ML}(R) \hookrightarrow \mathbb{R}_+^{\mathcal{S}(R)}$ as $m \mapsto \iota(m, \cdot)$, where $\iota(\cdot, \cdot)$ is the geometric intersection pairing, and so $\mathbb{P}\mathcal{ML}(R) \hookrightarrow \mathbb{P}(\mathbb{R}_+^{\mathcal{S}(R)})$. Moreover, the boundary $\partial^{Th}\mathcal{T}(R)$ can be identified to the space $\mathbb{P}\mathcal{ML}(R)$ of projective measured laminations.

An example of measured lamination is a **multi-curve**, namely a (nonzero) weighted sum of disjoint simple closed curves $m = w_1\gamma_1 + \cdots + w_n\gamma_n$.

When S is a hyperbolic surface with geodesic boundary and dS is its double (with real involution σ), we define $\mathcal{ML}(S) := \mathcal{ML}(dS)^\sigma$, so that $|\mathfrak{A}(S)| \hookrightarrow \mathbb{P}\mathcal{ML}(S)$.

We also define $\overline{\mathcal{T}}^{Th}(S) := \overline{\mathcal{T}}^{Th}(dS)^\sigma$, which is thus equal to $\mathcal{T}(S) \cup \mathbb{P}\mathcal{ML}(S)$. Then, we can compare $|\mathfrak{A}(S)| \hookrightarrow \mathbb{P}\mathcal{ML}(S) \hookrightarrow \overline{\mathcal{T}}^{Th}(S)$ with $\mathcal{L}^{-1}(\Delta^{n-1} \times \{\infty\})$.

Theorem **8 (Mondello [12])** *The map*

$$\Phi^{-1} : |\mathfrak{A}^\circ(S)| \times (0, \infty) \to \overline{\mathcal{T}}^{Th}(S)$$

extends over ∞ *by setting* Φ_∞^{-1} *to be equal to the natural embedding*

$$|\mathfrak{A}^\circ(S)| \times \{\infty\} \hookrightarrow \mathbb{P}\mathcal{ML}(S)$$

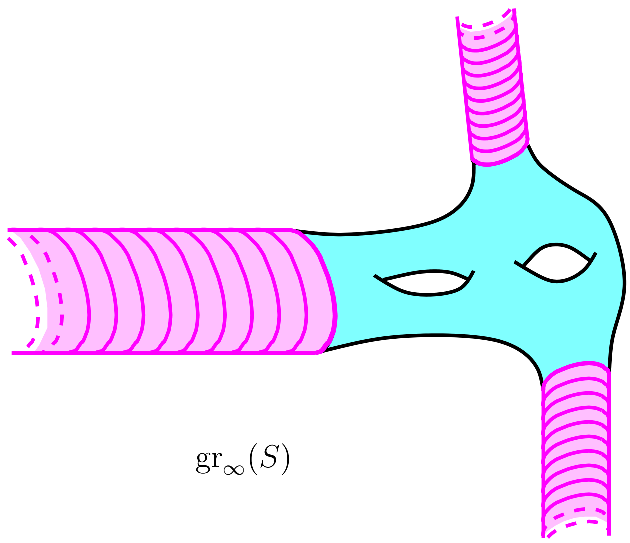
Moreover, $\tilde{\eta} = (\boldsymbol{p}/2)^2\eta \to \tilde{\beta}/2 = (\boldsymbol{p}/2)^2\beta/2$ *as* $\boldsymbol{p} \to \infty$.

Consider the S^1-bundle $\mathcal{C}_i \to \overline{\mathcal{M}}(S)$ associated to the i-th boundary component of S, and let $\psi_i = c_1(\mathcal{C}_i)$.

Mirzakhani [8] observed that Wolpert's $[\omega_0] = \pi^2\kappa_1$ (see [18]) and standard facts of symplectic geometry imply $[\omega_{\underline{p}}] = \pi^2\kappa_1 + \sum_i (p_i^2/2)\psi_i$. Hence, $(2/\boldsymbol{p})^2[\omega] \to 2\sum_i (p_i/\boldsymbol{p})^2\psi_i$, which proves that *intersection numbers of* ψ *classes* are *Weil-Petersson symplectic volumes* in the large boundary limit. Thus, for this purpose, a detailed analysis of $\partial\mathcal{M}(S)$ is not essential. (About the problems that can emerge when dealing with $\partial\mathcal{M}(S)$, see [1] and [9], for instance.)

8. – Grafting

Let S be a hyperbolic surface with geodesic boundary. The **grafted surface** $\mathrm{gr}_\infty(S)$ is obtained from S by glueing semi-infinite cylinders at ∂S. Notice that $\mathrm{gr}_\infty(S)$ is biholomorphic to a pointed surface.



Thus, grafting defines the following real-analytic map

$$\mathrm{gr}_\infty : \mathcal{T}(S) \longrightarrow \mathcal{T}(S)(0)$$

Theorem **9 (Mondello [12])** *For every* $\underline{p}$*, the map* $\mathrm{gr}_\infty : \mathcal{T}(S)(\underline{p}) \to \mathcal{T}(S)(0)$ *is a real-analytic diffeomorphism.*

We remark that continuity and properness of gr_∞ are not difficult. To prove that the differential $d\mathrm{gr}_\infty$ is injective, we mimick the proof of Scannell-Wolf [15] with suitable variations.

Intuitively, as $\boldsymbol{p} \to \infty$, the collar of ∂S becomes thinner and thinner. If we rescale the metric so that $\boldsymbol{p} = 1$ (and so the area goes to zero), the hyperbolic portion shrinks to a graph G.

Clearly, one can obtain the surface $\mathrm{gr}_\infty(S)$ simply glueing semi-infinite cylinders along G, which immediately defines a Strebel differential (up to multiples) on $\mathrm{gr}_\infty(S)$. We recall that the exceptional locus of the real-oriented blow-up $\mathrm{Bl}_0\widetilde{\mathcal{T}}(S)$ is made of projectively decorated surfaces and can be identified to $\widetilde{\mathcal{T}}(S)(0) \times \Delta^{n-1}$.

Hence, we can finally summarize our previous conclusions in the following commutative diagram of homeomorphisms

$$\begin{array}{ccc} \tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1} \times [0,\infty] & \xleftarrow{\ \mathrm{gr}_\infty\ } & \widehat{\mathcal{T}}(S) \\ & \searrow^{\Psi} & \downarrow{\scriptstyle \Phi} \\ & & |\mathfrak{A}^\circ(S)| \times [0,\infty] \end{array}$$

in which $\widehat{\mathcal{T}}(S) := \mathrm{Bl}_0\tilde{\mathcal{T}}(S) \cup |\mathfrak{A}^\circ(S)|$ and the two limits for short/long boundaries are given by

$$\Psi_0 = \Phi_{BPE} : \tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1} \to |\mathfrak{A}^\circ(S)|$$

and

$$\Psi_\infty = \Phi_{JS} : \tilde{\mathcal{T}}(S)(0) \times \Delta^{n-1} \to |\mathfrak{A}^\circ(S)|\,.$$

9. – A remark on the Riemannian volume

Regarding $\mathcal{T}(S)(\underline{p})$ as a family of moduli spaces with a Riemannian metric, obtained restricting the Weil-Petersson metric on $\mathcal{T}(S)$ to each leaf, the situation is more complicated, because $\mathcal{T}(S)(\underline{p})$ is not totally geodesic (except for $\underline{p} = 0$).

In any case, when the arc α_i is short, Masur [7] provided the expansion $\|\nabla a_i\|^2 \approx 2\pi a_i$. Moreover, Wolpert showed that, if α_i, α_j are two disjoint short arcs, then $\langle a_i, a_j\rangle = O(a_i^2 a_j^2)$.

As $w_i \approx 2\log(2/a_i)$ for a_i short, we get $\|\nabla w_i\| \approx 2\sqrt{\pi}\exp(w_i/4)$ in the former case, and $\langle \nabla w_i, \nabla w_j\rangle = O(\exp(-w_i/2 - w_j/2))$ in the latter.

Thus, given a maximal system of arcs $\underline{\boldsymbol{\alpha}} = \{\alpha_i\}$ for S, the coordinates $\{\nabla w_i\}$ become orthogonal as $\boldsymbol{p} \to \infty$.

In fact, the leading term of the associated Riemannian volume form looks like

$$\mathrm{vol} \approx (4\pi)^{-N/2}\exp(-\boldsymbol{p}/2) dw_1 \wedge \cdots \wedge dw_N$$

where $N = \dim_{\mathbb{R}}\mathcal{T}(S)$.

10. – Bordification of arcs

The Weil-Petersson completion $\overline{\mathcal{T}}(S)(\underline{p})$ of $\mathcal{T}(S)(\underline{p})$ is obtained by adding hyperbolic surfaces with nodes. Namely, a point of $\overline{\mathcal{T}}(S)$ is a class of maps $f : S \to \Sigma$ from S to a hyperbolic surface, such that f is a homeomorphism everywhere, except at some simple closed curves of S, which are shrunk to form nodes.

The family $(\overline{\mathcal{T}}(S)(\underline{p}), (2/\boldsymbol{p})^2\omega_{WP})$ of symplectic spaces can be completed by a "piecewise-linear symplectic space" $(|\mathfrak{A}(S)|, \tilde{\Omega}/2)$ over $\boldsymbol{p} = \infty$.

Quotienting by the action of $\Gamma(S)$, we obtain a family of moduli spaces $(\overline{\mathcal{M}}(S)(\underline{p}), (2/\boldsymbol{p})^2\omega_{WP})$ homeomorphic to the Deligne-Mumford compactification. This family can be compactified at ∞ by $(|\mathfrak{A}(S)|/\Gamma(S), \tilde{\Omega}/2)$.

In order to describe this compactification using quantities associated to arcs, we introduce the **transverse length** t_α of an arc $\alpha \subset S$, defined as $t_\alpha := T(\ell_\alpha)$, where

$$T(x) = 2\mathrm{arcsinh}\left(\frac{1}{\sinh(x/2)}\right)$$

The function $T : \mathbb{R} \to \mathbb{R}$ is smooth and strictly decreasing, and $T \circ T = id$. Moreover, $T(x) \approx 4e^{-x/2}$ as $x \to \infty$, and $T(x) \approx 2\log(4/x)$ as $x \to 0$.

The following map is an embedding

$$\begin{aligned} t_\bullet :\ \mathcal{T}(S) &\longrightarrow \mathbb{P}(\mathcal{A}(S)) \times [0,\infty] \\ [f] &\longmapsto (t_\bullet(f), \|t_\bullet(f)\|_\infty) \end{aligned}$$

where $\mathcal{A}(S) = \mathfrak{A}_0(S)$ is the set of arcs of S and $\mathbb{P}(\mathcal{A}(S))$ is the projectivization of $L^\infty(\mathcal{A}(S))$.

In analogy to Thurston's compactification, we call **bordification of arcs** $\overline{\mathcal{T}}^a(S)$ the closure of $t_\bullet(\mathcal{T}(S))$ inside $\mathbb{P}(\mathcal{A}(S)) \times [0,\infty]$ (see [12]). We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_0\overline{\mathcal{T}}(S) \cup |\mathfrak{A}(S)| & \xrightarrow{\ \pi\ } & \overline{\mathcal{T}}^a(S) \\ \downarrow & \nearrow_{\mathrm{tr}} & \\ |\mathfrak{A}(S)| \times [0,\infty] & & \end{array}$$

in which π "forgets" the hyperbolic metric on the components without boundary (and without positively weighted cusps).

THEOREM **10 (Mondello [12])** *The map* $\mathrm{tr} : |\mathfrak{A}(S)| \times [0,\infty] \longrightarrow \overline{\mathcal{T}}^a(S)$ *is a* $\Gamma(S)$-*equivariant homeomorphism.*

We have thus recovered a different interpretation of Kontsevich's compactification $\mathcal{M}(S) \subset |\mathfrak{A}(S)|/\Gamma(S) \times [0,\infty]$ (see [5]).

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