RIEMANN SURFACES WITH BOUNDARY AND NATURAL TRIANGULATIONS OF THE TEICHMÜLLER SPACE

GABRIELE MONDELLO

Abstract. We compare some natural triangulations of the Teichmüller space of hyperbolic surfaces with geodesic boundary and of some bordifications. We adapt Scannell-Wolf's proof to show that grafting semi-infinite cylinders at the ends of hyperbolic surfaces with fixed boundary lengths is a homeomorphism. This way, we construct a family of equivariant triangulations of the Teichmüller space of punctured surfaces that interpolates between Penner-Bowditch-Epstein's (using the spine construction) and Harer-Mumford-Thurston's (using Strebel's differentials). Finally, we show (adapting arguments of Dumas) that on a fixed punctured surface, when the triangulation approaches HMT's, the associated Strebel differential is well-approximated by the Schwarzian of the associated projective structure and by the Hopf differential of the collapsing map.

1. Introduction

1.1. **Overview.** The aim of this paper is to compare two different ways of triangulating the Teichmüller space $\mathcal{T}(R,x)$ of conformal structures on a compact oriented surface R with distinct ordered marked points $x=(x_1,\dots,x_n)$. Starting with $[f:R\to R']\in\mathcal{T}(R,x)$ and a collection of weights $\underline{p}=(p_1,\dots,p_n)\in\Delta^{n-1}$, both constructions produce a ribbon graph G embedded inside the punctured surface $\dot{R}=R\setminus x$ as a deformation retract, together with a positive weight for each edge. The space of such weighted graphs can be identified to the topological realization of the arc complex $\mathfrak{A}(R,x)$ (via Poincaré-Lefschetz duality on (R,x), see for instance [Mon09b]), which is the simplicial complex of (isotopy classes of) systems of (homotopically nontrivial, pairwise nonhomotopic) arcs that join couples of marked points and that admit representatives with disjoint interior ([Har86], [BE88], [Loo95]).

Thus, both constructions provide a $\Gamma(R,x)$-equivariant homeomorphism $\mathcal{T}(R,x)\times\Delta^{n-1}\to|\mathfrak{A}^\circ(R,x)|$, where $\Gamma(R,x)=\pi_0\mathrm{Diff}_+(R,x)$ is the mapping class group of (R,x) and $\mathfrak{A}^\circ(R,x)\subset\mathfrak{A}(R,x)$ consists of proper systems of arcs $\boldsymbol{A}=\{\alpha_0,\dots,\alpha_k\}$, namely such that $\dot{R}\setminus(\alpha_0\cup\dots\cup\alpha_k)$ is a disjoint union of discs and pointed discs. In fact, properness of $\boldsymbol{A}$ is exactly equivalent to its dual ribbon graph being a deformation retract of $\dot{R}$.

The HMT construction (due to Harer, Mumford and Thurston) appears in [Har86]. It uses Strebel's result [Str67] on existence and uniqueness of meromorphic quadratic differential φ on a Riemann surface R with prescribed residues $\underline{p}$ at x to decompose $\dot{R}$ into a disjoint union of semi-infinite $|\varphi|$-flat cylinders (one for each puncture x_i with $p_i>0$), that are identified along a critical graph G which inherits this way a metric. The length of each edge of G will be its weight.

The PBE construction (due Penner [Pen87] and Bowditch-Epstein [BE88]) uses the unique hyperbolic metric on the punctured Riemann surface $\dot{R}$. Given a *(projectively) decorated surface*, that is a hyperbolic surface $\dot{R}$ with cusps plus a weight $\underline{p}\in\Delta^{n-1}$, there are disjoint embedded horoballs of circumference $p_1,\dots,p_n$ at the n cusps of $\dot{R}$. Removing the horoballs, we obtain a truncated surface R^{tr} with boundary, on which the function "distance from the boundary" is well-defined. The critical locus of this function is a spine G embedded in $R^{tr}\subset\dot{R}$ as a deformation retract and with geodesic edges, whose horocyclic lengths provide the associated weights.

Both constructions share similar properties of homogeneity and real-analiticity (see [HM79] and [Pen87]) and they also enjoy some good compatibility with the Weil-Petersson symplectic structure on $\mathcal{T}(R,x)$, as explained later.

In this paper, we will interpolate these two constructions using the Teichmüller space $\mathcal{T}(S)$ of *hyperbolic surfaces with geodesic boundary* (see also [Luo07]), where S is a surface with boundary endowed with a homotopy equivalence $S \hookrightarrow \dot{R}$. The spine construction works perfectly on such surfaces, even when nodal (which is the content of Theorem 3.16) and it can be easily seen to reduce to the PBE case as the boundary lengths $(p_1, \dots, p_n) \in \mathbb{R}_+^n \hookrightarrow \Delta^{n-1} \times (0, \infty)$ become infinitesimal (see also [Mon09c]). Also, the Weil-Petersson Poisson structure can be explicitly determined, thus obtaining a generalization of Penner's formula [Pen92].

Thus, the limit $\boldsymbol{p} := p_1 + \dots + p_n \to 0$ is completely understood and it behaves as the Weil-Petersson completion (or Bers's augmentation [Ber74]) of the Teichmüller space.

Instead, the limit $\boldsymbol{p} \to \infty$ behaves more like Thurston's compactification [FLP79] of the Teichmüller space; in fact, the arc complex $|\mathfrak{A}(S)|$ naturally embeds inside the space of projective measured laminations. From a symplectic point of view, the Weil-Petersson structure admits a precise limit as $\boldsymbol{p} \to \infty$, after a suitable normalization, which agrees with Kontsevich's piecewise-linear symplectic form on $|\mathfrak{A}(S)|$ defined in [Kon92] (see [Mon09c]).

To give a more geometric framework to these limiting considerations, we produce a few different bordifications of the Teichmüller space $\mathcal{T}(S)$ of a surface S with boundary, whose quotients by the mapping class group $\Gamma(S)$ give different compactifications of the moduli space. A convenient bordification from the point of view of the Weil-Petersson Poisson structure is the *extended Teichmüller space* $\widetilde{\mathcal{T}}(S)$; whereas the most suitable one for triangulations and spine constructions is the *bordification of arcs* $\overline{\mathcal{T}}^a(S)$ (described in Theorem 3.16), whose definition looks a bit like Thurston's but with some relevant differences (for instance, we use t-lengths related to hyperbolic collars instead of hyperbolic lengths). It is reasonable to believe that careful iterated blow-ups of $\overline{\mathcal{T}}^a(S)$ along its singular locus would produce finer bordifications of $\mathcal{T}(S)$ in the spirit of [Loo95] (see also [MP07]).

In order to explicitly link the HMT and PBE constructions, we construct a family of isotopic triangulations of $\mathcal{T}(R, x) \times \Delta^{n-1}$, parametrized by $t \in [0, \infty]$, that coincides with PBE for $t = 0$ and with HMT for $t = \infty$ (and this is the content of Corollary 5.6). In particular, we prove that, for every complex structure on $\dot{R}$ and every $(\underline{p}, t) \in \Delta^{n-1} \times [0, \infty]$, there exists a unique projective structure $\mathcal{P}(\dot{R}, t\underline{p})$ on $\dot{R}$, whose associated Thurston metric has flat cylindrical ends (with circumferences $t\underline{p}$) and a hyperbolic core. Rescaling the lengths by a factor $1/t$, we recognize that at $t = \infty$ the hyperbolic core shrinks to a graph G and the metric is of the type $|\varphi|$, where φ is a Strebel differential. This result (Theorem 5.4) can be restated in term of infinite grafting at the ends of a hyperbolic surface with geodesic boundary and the proof adapts arguments of Scannell-Wolf [SW02].

Finally, we show that, for large t, two results of Dumas [Dum06] [Dum07b] for compact surfaces still hold (Theorem 5.13). The first one says that, for t large, the Strebel differential φ is well-approximated in $L^1_{loc}(\dot{R})$ by the Hopf differential of the collapsing map associated to $\mathcal{P}(\dot{R}, t\underline{p})$, that is the quadratic differential which writes dz^2 on the flat cylinders $S^1 \times [0, \infty)$ and is zero on the hyperbolic part. The second result says that φ is also well-approximated by the Schwarzian derivative of the projective structure $\mathcal{P}(\dot{R}, t\underline{p})$.

1.2. **Detailed content of the paper.** In Section 2, we recall basic concepts like Teichmüller space $\mathcal{T}(R)$, measured laminations $\mathcal{ML}(R)$ and Thurston's compactification $\overline{\mathcal{T}}^{Th}(R) = \mathcal{T}(R) \cup \mathcal{ML}(R)$, when R is an oriented compact surface with $\chi(R) < 0$.

We also extend these concepts to the case of an oriented surface S with boundary and $\chi(S) < 0$, using the doubling construction $S \rightsquigarrow dS$. We also remark that the arc complex $|\mathfrak{A}(S)|$ embeds in $\mathcal{ML}(S) = \mathcal{ML}(dS)^\sigma$ (where σ is the natural anti-holomorphic involution of dS) and, even though its image is neither open nor closed, the subspace topology coincides with the metric topology.

Next, we introduce the Weil-Petersson pairing on a closed surface and on a surface with boundary, we describe the augmentation $\overline{\mathcal{T}}^{WP}$ of the Teichmüller space and we restate Wolpert's formula [Wol82], which expresses the WP symplectic structure in Fenchel-Nielsen coordinates. Finally, we recall the definition of the mapping class group $\Gamma(S) = \pi_0 \mathrm{Diff}_+(S)$, the moduli space $\mathcal{M}(S) = \mathcal{T}(S)/\Gamma(S)$ and the Deligne-Mumford compactification.

We begin Section 3 by defining the geometrical quantities that are associated to an arc in a hyperbolic surface S with boundary: the hyperbolic length $a_i = \ell_{\alpha_i}$ of (the geodesic representative of) $\alpha_i \in \mathcal{A}(S) = \{\text{isotopy classes of arcs in } S\}$, its associated s-length $s_{\alpha_i} =$

$\cosh(a_i/2)$ and t-length $t_{\alpha_i} = T(\ell_{\alpha_i})$, where T is defined by $\sinh(T(x)/2)\sinh(x/2) = 1$. The t-lengths give a continuous embedding

$$\begin{aligned} j: \quad \mathcal{T}(S) &\hookrightarrow \mathbb{P}L^\infty(\mathcal{A}(S)) \times [0,\infty] \\ [f : S \to \Sigma] &\longmapsto ([t_\bullet(f)], \|t_\bullet(f)\|_\infty) \end{aligned}$$

and we call *bordification of arcs* the closure of its image $\overline{\mathcal{T}}^a(S)$.

Then we define the *spine* $\mathrm{Sp}(\Sigma)$ (of a hyperbolic surface Σ) as the critical locus of the function "distance from the boundary" and we produce its dual *spinal arc system* $\boldsymbol{A}_{sp} \in \mathfrak{A}(\Sigma)$ and a system of weights (the *widths*) w_{sp} so that $w_{sp}(\alpha)$ is the length of either of the two projections of the edge α^* of the spine (dual to α) to the boundary. We also define the width of an arc α (and of an oriented arc $\overrightarrow{\alpha}$) associated to a maximal system of arcs $\boldsymbol{A}$ and we show that the two concepts agree [Ush99] (see also [Mon09c]).

We recall the PBE and Luo's result on the cellularization of $\mathcal{T}(S)$ using the spine construction.

Theorem 1.1 (Penner, Bowditch-Epstein, Luo)**.** *Let S be a compact oriented surface with $n \geq 1$ boundary components and $\chi(S) < 0$ and let (R, x) be a pointed surface such that $S \hookrightarrow \dot{R}$ is a homotopy equivalence.*

(a) *If $\mathcal{T}(R, x) \times \Delta^{n-1}$ is the Teichmüller space of (projectively) decorated surfaces, then the spine construction*

$$\begin{aligned} \boldsymbol{W}_{PBE}: \ \mathcal{T}(R,x) \times \Delta^{n-1} &\longrightarrow |\mathfrak{A}^\circ(R,x)| \\ ([f : R \to R'], \underline{p}) &\longmapsto f^*\tilde{w}_{sp,R',\underline{p}} \end{aligned}$$

induces a $\Gamma(R, x)$-equivariant homeomorphism ([Pen87], [BE88]*).*

(b) *The spine construction applied to hyperbolic surfaces with geodesic boundary*

$$\begin{aligned} \boldsymbol{W}: \quad \mathcal{T}(S) &\longrightarrow |\mathfrak{A}^\circ(S)| \times \mathbb{R}_+ \\ [f : S \to \Sigma] &\longmapsto f^* w_{sp,\Sigma} \end{aligned}$$

gives a $\Gamma(S)$-equivariant homeomorphism ([Luo07]*).*

To deal with stable surfaces, we first define $\widehat{\mathcal{T}}(S)$ as the real blow-up of $\overline{\mathcal{T}}^{WP}(S) := \bigcup_{\underline{p} \in \Delta^{n-1} \times [0,\infty)} \overline{\mathcal{T}}^{WP}(S)(\underline{p})$ along the locus $\overline{\mathcal{T}}^{WP}(S)(0)$ of surfaces with n boundary cusps and we identify the exceptional locus $\widehat{\mathcal{T}}(S)(0)$ with the space of projectively decorated surfaces (that is, of surfaces with n boundary cusps and weights $(p_1, \ldots, p_n) \in \Delta^{n-1}$). Then, we call *visible* the subsurface $\Sigma_+ \subset \Sigma$ consisting of the components of Σ which have positive boundary length (or some positively weighted cusp) and we declare that $[f_1 : S \to \Sigma_1]$ and $[f_2 : S \to \Sigma_2]$ in $\widehat{\mathcal{T}}(S)$ are *visibly equivalent* if there exists a third $[f : S \to \Sigma]$ and maps $h_i : \Sigma \to \Sigma_i$ for $i = 1, 2$ that are isomorphisms on the visible components and such that $h_i \circ f \simeq f_i$ for $i = 1, 2$.

Our first result is the following.

Theorem 3.16. *Let S be a compact oriented surface with $n \geq 1$ boundary components and $\chi(S) < 0$. The spine construction gives a $\Gamma(S)$-equivariant homeomorphism*

$$\begin{aligned} \widehat{\boldsymbol{W}}: \quad \widehat{\mathcal{T}}^{vis}(S) &\longrightarrow |\mathfrak{A}(S)| \times [0,\infty) \\ [f : S \to \Sigma] &\longmapsto f^* w_{sp,\Sigma} \end{aligned}$$

where $\widehat{\mathcal{T}}^{vis}(S)$ is obtained from $\widehat{\mathcal{T}}(S)$ by identifying visibly equivalent surfaces. Moreover, $\widehat{\boldsymbol{W}}$ extends Penner's and Luo's constructions.

Our second main result is the following description of the bordification of arcs.

Theorem 3.23. *The map $\Phi : |\mathfrak{A}(S)| \times [0,\infty] \longrightarrow \overline{\mathcal{T}}^a(S)$ defined as*

$$\Phi(w, \boldsymbol{p}) = \begin{cases} ([\lambda_\bullet^{-1}(\widehat{\boldsymbol{W}}^{-1}(w,0))], 0) & \text{if } \boldsymbol{p} = 0 \\ j(\widehat{\boldsymbol{W}}^{-1}(w,\boldsymbol{p})) & \text{if } 0 < \boldsymbol{p} < \infty \\ ([w], \infty) & \text{if } \boldsymbol{p} = \infty \end{cases}$$

is a $\Gamma(S)$-equivariant homeomorphism, where λ_α is Penner's λ-length of α.

The situation is illustrated in the following $\Gamma(S)$-equivariant commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{T}}(S) \arrow[r, two heads] \arrow[d, hook] & \widehat{\mathcal{T}}^{vis}(S) \arrow[r, "\widehat{\boldsymbol{W}}", "\sim"'] \arrow[d, hook] & |\mathfrak{A}(S)| \times [0,\infty) \arrow[d, hook] \\
\widetilde{\mathcal{T}}(S) \arrow[r, two heads] & \overline{\mathcal{T}}^{a}(S) & |\mathfrak{A}(S)| \times [0,\infty] \arrow[l, "\Phi"', "\sim"]
\end{tikzcd}$$

in which $\overline{\mathcal{T}}^a(S)$ is exhibited as a quotient of the *extended Teichmüller space* $\widetilde{\mathcal{T}}(S) := \widehat{\mathcal{T}}(S) \cup |\mathfrak{A}(S)|_\infty$ (endowed with a suitable topology, where $|\mathfrak{A}(S)|_\infty$ is just a copy of $|\mathfrak{A}(S)|$) by visible equivalence.

Section 4 describes how to extend the previous triangulations to the case of a surface with boundary S and a marked point v_i on each boundary component C_i (with $i = 1, \dots, n$), so that we obtain a commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{T}}^{vis}(S,v) \arrow[r, "\widehat{\boldsymbol{W}}_v"] \arrow[d] & |\mathfrak{A}(S,v)| \times [0,\infty) \arrow[d] \\
\widehat{\mathcal{T}}^{vis}(S) \arrow[r, "\widehat{\boldsymbol{W}}"] & |\mathfrak{A}(S)| \times [0,\infty)
\end{tikzcd}$$

in which the horizontal arrows are $\Gamma(S,v)$-equivariant homeomorphisms and the vertical arrows are $\mathbb{R}^n$-fibrations on the smooth locus (with some possible degenerations on the stable surfaces). After passing to the associated moduli spaces, the vertical arrows become $(S^1)^n$-bundles, which are actually products of the circle bundles $L_1, \dots, L_n$ associated to the respective boundary components $C_1, \dots, C_n$. This $(S^1)^n$-action is Hamiltonian for the Weil-Petersson structure with moment map $\mu = (p_1^2/2, \dots, p_n^2/2)$ and this shows that $[\omega_{\underline{p}}] = [\omega_0] + \sum_i p_i^2/2[c_1(L_i)]$ in cohomology ([Mir07]), where $\omega_{\underline{p}}$ is the restriction of the Weil-Petersson form to the symplectic leaf $\widehat{\mathcal{M}}(S)(\underline{p})$, that is the moduli space of surfaces with boundary lengths $\underline{p}$. Pointwise, the Poisson structure η on $\widehat{\mathcal{M}}(S)$ can be described as follows.

Theorem 4.3 ([Mon09c]). *Let $\boldsymbol{A}$ be a maximal system of arcs on S. Then*

$$\eta = \frac{1}{4} \sum_{k=1}^{n} \sum_{\substack{y_i \in \alpha_i \cap C_k \\ y_j \in \alpha_j \cap C_k}} \frac{\sinh(p_k/2 - d_{C_k}(y_i, y_j))}{\sinh(p_k/2)} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

on $\mathcal{T}(S)$, where $d_{C_k}(y_i, y_j)$ is the length of the geodesic running from y_i to y_j along C_k in the positive direction. Moreover, if we normalize $\tilde{w}_i = (\boldsymbol{p}/2)^{-1} w_i$ and $\tilde{\eta} = (1 + \boldsymbol{p}/2)^2 \eta$, then $\tilde{\eta}$ extends to $\widetilde{\mathcal{T}}(S)$ and

$$\tilde{\eta}_\infty = \frac{1}{2} \sum_r \left(\frac{\partial}{\partial \tilde{w}_{r_1}} \wedge \frac{\partial}{\partial \tilde{w}_{r_2}} + \frac{\partial}{\partial \tilde{w}_{r_2}} \wedge \frac{\partial}{\partial \tilde{w}_{r_3}} + \frac{\partial}{\partial \tilde{w}_{r_3}} \wedge \frac{\partial}{\partial \tilde{w}_{r_1}} \right)$$

where r ranges over all (trivalent) vertices of the ribbon graph representing a point in $|\mathfrak{A}^\circ(S)|$ and (r_1, r_2, r_3) is the (cyclically) ordered triple of edges incident at r.

The result can be seen to reduce to Penner's formula [Pen92] as $\boldsymbol{p} = 0$.

Finally, Section 5 relates hyperbolic surfaces with boundary homeomorphic to S to punctured surfaces homeomorphic to $R \setminus x = \dot{R} \simeq S$. We describe first Strebel's result and the HMT construction and its extension to $\widehat{\mathcal{T}}^{vis}(R,x)$ (see [Mon09b]), which provides a $\Gamma(R,x)$-equivariant homeomorphism

$$\boldsymbol{W}_{HMT} : \widehat{\mathcal{T}}^{vis}(R,x) \longrightarrow |\mathfrak{A}(R,x)|$$

Then, we define $\mathrm{gr}_\infty(\Sigma) \in \overline{\mathcal{T}}^{WP}(R,x) \times \Delta^{n-1} \times [0,\infty)$ to be the Riemann surface obtained from the hyperbolic surface $\Sigma \in \widehat{\mathcal{T}}(S)$ with geodesic boundary by *grafting semi-infinite flat cylinders at its ends*. Moreover, for every $w \in |\mathfrak{A}(S)|_\infty \cong |\mathfrak{A}(R,x)|$, we let $\mathrm{gr}_\infty(w) := \boldsymbol{W}_{HMT}^{-1}(w)$.

The key result is the following.

Theorem 5.4. *The map* $(\mathrm{gr}_\infty,\mathcal{L}) : \widetilde{\mathcal{T}}(S) \to \overline{\mathcal{T}}^{WP}(R,x)\times\Delta^{n-1}\times[0,\infty]/\sim$ *is a homeomorphism, where* $\mathcal{L}$ *is the boundary length map and* $\sim$ *identifies* $([f_1 : R\to R_1],\underline{p},\infty)$ *and* $([f_2 : R\to R_2],\underline{p},\infty)$ *if and only if* $([f_1],\underline{p})$ *and* $([f_2],\underline{p})$ *are visibly equivalent.*

The continuity at infinity requires some explicit computations, whereas the proof of the injectivity simply adapts arguments of Scannell-Wolf [SW02] to our situation.

We can summarize our results in the following commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{T}}^{vis}(R,x)\times[0,\infty] \arrow[dr, "\Psi"'] & \overline{\mathcal{T}}^{a}(S) \arrow[l, "(\mathrm{gr}_\infty{,}\mathcal{L})"'] \\
& |\mathfrak{A}(S)|\times[0,\infty] \arrow[u, "\Phi"']
\end{tikzcd}$$

in which $\Psi = \Phi^{-1}\circ(\mathrm{gr}_\infty,\mathcal{L})^{-1}$. Then we can condense our main results in the following statement.

Corollary 5.6. *The maps* $\Psi_t : \widehat{\mathcal{T}}^{vis}(R,x)\to|\mathfrak{A}(R,x)|$ *obtained by restricting* Ψ *to* $\widehat{\mathcal{T}}^{vis}(R,x)\times\{t\}$ *form a continuous family of* $\Gamma(S)$*-equivariant triangulations, which specializes to PBE for* $t=0$ *and to HMT to* $t=\infty$*.*

It would be interesting to investigate whether the techniques of Section 5.5 could be employed to attack the following problem (which I believe was raised by Dennis Sullivan).

Problem. *Does there exist a constant* $K\geq 1$ *(maybe dependent on* g *and* n*) and* $\underline{p}\in\Delta^{n-1}$ *such that the Teichmüller distance between* $[f : R\to R']$ *and* $\boldsymbol{W}_{HMT}^{-1}\circ\boldsymbol{W}_{PBE}([f],\underline{p})$ *is bounded by* K *uniformly in* $[f]\in\mathcal{T}(R,x)$*.*

Finally, we investigate the degeneration of the projective structure $\mathrm{Gr}_\infty(\Sigma)$ on the Riemann surface $\mathrm{gr}_\infty(\Sigma)$. It adapts arguments of Dumas [Dum06] [Dum07b] to our case.

Theorem 5.13. *Let* $\{f_m : S\to\Sigma_m\}\subset\mathcal{T}(S)$ *be a sequence such that* $(\mathrm{gr}_\infty,\mathcal{L})(f_m) = ([f : R\to R'],\underline{p}_m)\in\mathcal{T}(R,x)\times\mathbb{R}^n_+$*. The following are equivalent:*

1. $\underline{p}_m\to(\underline{p},\infty)$ *in* $\Delta^{n-1}\times(0,\infty]$
2. $[f_m]\to[w]$ *in* $\overline{\mathcal{T}}^{a}(S)$*, where* $[w]$ *is the projective multi-arc associated to the vertical foliation of the Jenkins-Strebel differential* φ_{JS} *on* R' *with weights* $\underline{p}$ *at* $x'=f(x)$*.*

When this happens, we also have

(a) $4\boldsymbol{p_m}^{-2}\boldsymbol{H}(\kappa_m)\to\varphi_{JS}$ *in* $L^1_{loc}(R',K(x')^{\otimes 2})$*, where* $\boldsymbol{H}(\kappa_m)$ *is the Hopf differential of the collapsing map* $\kappa_m : R'\to\Sigma_m$

(b) $2\boldsymbol{p_m}^{-2}\boldsymbol{S}(\mathrm{Gr}_\infty([f_m]))\to-\varphi_{JS}$ *in* $H^0(R',K(x')^{\otimes 2})$*, where* $\boldsymbol{S}$ *is the Schwarzian derivative with respect to the Poincaré projective structure.*

1.3. **Acknowledgments.** I would like to thank Enrico Arbarello, Curtis McMullen, Tomasz Mrowka, Mike Wolf and Scott Wolpert for very fruitful and helpful discussions.

2. Preliminaries

2.1. **Double of a surface with boundary.** By a *surface* with boundary and/or marked points we will always mean a compact oriented surface S possibly with boundary and/or distinct ordered marked points $x=(x_1,\dots,x_n)$, with $x_i\in S^\circ$. By a *nodal surface* with boundary and marked points we mean a compact, Hausdorff topological space S with countable basis in which every $q\in S$ has an open neighbourhood U_q such that (U_q,q) is homeomorphic to: either $(\mathbb{C},0)$ and q is called *smooth* point; or $(\{z\in\mathbb{C}\,|\,\mathrm{Im}(z)\geq 0\},0)$ and q is called *boundary* point; or $(\{(z,w)\in\mathbb{C}^2\,|\,zw=0\},0)$ and q is called *node*.

We will say that a (nodal) surface S is *closed* if it has no boundary and no marked points.

A *hyperbolic metric* on S is a complete metric g of finite volume on the smooth locus $\dot{S}_{sm}$ of the *punctured surface* $\dot{S} := S\setminus x$ of constant curvature -1, such that ∂S is geodesic. Clearly, such a g acquires cusps at the marked points and at the nodes.

Given a (possibly nodal) surface S with boundary and/or marked points, we can construct its double dS in the following way. Let S' be another copy of S, with opposite orientation, and call $q'\in S'$ the point corresponding to $q\in S$. Define dS to be $S\coprod S'/\sim$, where $\sim$ is the equivalence relation generated by $q \sim q'$ for every $q \in \partial S$ and every x_i. Clearly, dS is closed and it is smooth whenever S has no nodes and no marked points.

dS can be oriented so that the natural embedding $\iota : S \hookrightarrow dS$ is orientation-preserving. Moreover, dS comes naturally equipped with an orientation-reversing involution σ that fixes the boundary and the cusps of $\iota(S)$ and such that $dS/\sigma \cong S$. If S is hyperbolic, then dS can be given a hyperbolic metric such that ι and σ are isometries.

Clearly, on dS there is a correspondence between complex structures and hyperbolic metrics and, in fact, σ-invariant hyperbolic metrics correspond to complex structures such that σ is anti-holomorphic. Thus, the datum of a hyperbolic metric with geodesic boundary on S is equivalent to that of a complex structure on S, such that ∂S is totally real.

2.2. **Teichmüller space.** Let S be a hyperbolic surface with $n \geq 0$ boundary components $C_1, \dots, C_n$ and no cusps.

Definition 2.1. *An S-marked hyperbolic surface is an orientation preserving map $f : S \longrightarrow \Sigma$ of (smooth) hyperbolic surfaces that may shrink boundary components of S to cusps of Σ and that is a diffeomorphism everywhere else.*

Two S-marked surfaces $f_1 : S \longrightarrow \Sigma_1$ and $f_2 : S \longrightarrow \Sigma_2$ are *equivalent* if there exists an isometry $h : \Sigma_1 \longrightarrow \Sigma_2$ such that $h \circ f_1$ is homotopic to f_2.

Definition 2.2. *Call $\check{\mathcal{T}}(S)$ the space of equivalence classes of S-marked hyperbolic surfaces. The Teichmüller space $\mathcal{T}(S) \subset \check{\mathcal{T}}(S)$ is the locus of surfaces Σ with no cusps.*

The space $\mathfrak{Met}(S)$ of smooth metrics on $\dot{S}$ has the structure of an open convex subset of a Fréchet space. Consider the map $\mathfrak{Met}(S) \to \check{\mathcal{T}}(S)$ that associates to $g \in \mathfrak{Met}(S)$ the unique hyperbolic metric with geodesic boundary in the conformal class of g. Endow $\check{\mathcal{T}}(S)$ with the quotient topology.

Let $\boldsymbol{\gamma} = \{C_1, \dots, C_n, \gamma_1, \dots, \gamma_{3g-3+n}\}$ be a maximal system of disjoint simple closed curves of S such that no γ_i is contractible and no couple $\{\gamma_i, \gamma_j\}$ or $\{\gamma_i, C_j\}$ bounds a cylinder. The system $\boldsymbol{\gamma}$ induces a *pair of pants decomposition* of S, that is $S^\circ \setminus \bigcup_i \gamma_i = P_1 \cup \dots \cup P_{2g-2+n}$, and each P_i is a pair of pants (i.e. a surface homeomorphic to $\mathbb{C} \setminus \{0, 1\}$).

Given $[f : S \to \Sigma] \in \mathcal{T}(S)$, we can define $\ell_i(f)$ to be the length of the unique geodesic curve isotopic to $f(\gamma_i)$. Let $\tau_i(f)$ be the associated twist parameter (whose definition depends on some choices).

The *Fenchel-Nielsen coordinates* (p_j, ℓ_i, τ_i) exhibit a homeomorphism $\check{\mathcal{T}}(S) \xrightarrow{\sim} \mathbb{R}^n_{\geq 0} \times (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}$. In particular, the *boundary length map* $\mathcal{L} : \check{\mathcal{T}}(S) \longrightarrow \mathbb{R}^n_{\geq 0}$ is defined as $\mathcal{L}([f]) = (p_1, \dots, p_n)$ and we write $\mathcal{T}(S)(\underline{p}) := \mathcal{L}^{-1}(\underline{p})$ for $\underline{p} \in \mathbb{R}^n_{\geq 0}$. Thus, $\mathcal{T}(S) = \check{\mathcal{T}}(S)(\mathbb{R}^n_+)$.

2.3. **Measured laminations.** A *geodesic lamination* on a closed smooth hyperbolic surface (R, g) is a closed subset $\lambda \subset R$ which is foliated in complete simple geodesics. A *transverse measure* μ for λ is a function $\mu : \Lambda(\lambda) \longrightarrow \mathbb{R}_{\geq 0}$, where $\Lambda(\lambda)$ is the collection of compact smooth arcs imbedded in R with endpoints in $R \setminus \lambda$, such that

(1) $\mu(\alpha) = \mu(\beta)$ if α is isotopic to β through elements of $\Lambda(\lambda)$ (*homotopy invariance*)
(2) $\mu(\alpha) = \displaystyle\sum_{i \in I} \mu(\alpha_i)$ if $\alpha = \bigcup_{i \in I} \alpha_i$, if $\alpha_i \in \Lambda(\lambda)$ for all i in a countable set I and distinct α_i, α_j meet at most at their endpoints (σ-*additivity*)
(3) for every $\alpha \in \Lambda(\lambda)$, $\mu(\alpha) > 0$ if and only if $\alpha \cap \lambda \neq \emptyset$ (*the support of μ is λ*).

In this case, the couple (λ, μ) is called a *measured geodesic lamination* on (R, g) (often denoted just by μ).

Lemma-Definition 2.3. *If g and g' are hyperbolic metrics on R, then there is a canonical identification between measured g-geodesic laminations and measured g'-geodesic laminations. Thus, we call the set $\mathcal{ML}(R)$ of such (λ, μ)'s just the space of measured laminations on R (see* [FLP79] *and* [PH92] *for more details).*

Given a measured lamination (λ, μ) and a simple closed curve γ on R, one can decompose γ as a union of geodesic arcs $\gamma = \gamma_1 \cup \dots \cup \gamma_k$ with $\gamma_i \in \Lambda(\lambda)$. The *intersection* $\iota(\mu, \gamma)$ is defined to be $\mu(\gamma_1) + \dots + \mu(\gamma_k)$. Clearly, if $\gamma \simeq \gamma'$, then $\iota(\mu, \gamma) = \iota(\mu, \gamma')$.

Call $\mathcal{C}(R)$ the set of nontrivial isotopy classes of simple closed curves γ contained in R.

Remark 2.4. *There is a map $\mathcal{ML}(R) \times \mathcal{C}(R) \longrightarrow \mathbb{R}_{\geq 0}$ given by $(\mu, \gamma) \mapsto \iota(\mu, \gamma)$. The induced $\mathcal{ML}(R) \longrightarrow (\mathbb{R}_{\geq 0})^{\mathcal{C}(R)}$ is injective: identifying $\mathcal{ML}(R)$ with its image, we can induce a topology on $\mathcal{ML}(R)$ which is independent of the hyperbolic structure on R (see* [FLP79]*).*

A *k-system of curves* $\boldsymbol{\gamma} = \{\gamma_1, \dots, \gamma_k\} \subset \mathcal{C}(R)$ is a subset of curves of R, which admit disjoint representatives.

Definition 2.5. *The complex of curves $\mathfrak{C}(R)$ is the simplicial complex whose k-simplices are $(k+1)$-systems of curves on R (see* [Har81]*).*

Notation. *Given a simplicial complex $\mathfrak{X}$, denote by $|\mathfrak{X}|$ its geometric realization. It comes endowed with two natural topologies. The coherent topology is the finest topology that makes the realization of all simplicial maps continuous. The metric topology is induced by the path metric, for which every k-simplex is isometric to the standard $\Delta^k \subset \mathbb{R}^{k+1}$. Where $|\mathfrak{X}|$ is not locally finite, the metric topology is coarser than the coherent one. Now on, we will endow all realizations with the metric topology, unless differently specified.*

Clearly, there are continuous injective maps $|\mathfrak{C}(R)| \longrightarrow \mathbb{P}\mathcal{ML}(R)$ and $|\mathfrak{C}(R)| \times \mathbb{R}_+ \longrightarrow \mathcal{ML}(R)$. Points in the image of the latter map are called *multi-curves*.

Definition 2.6. *Let S is a compact hyperbolic surface with boundary. A geodesic lamination λ on S is a closed subset of S foliated in geodesics that can meet ∂S only perpendicularly; equivalently, a σ-invariant geodesic lamination on its double dS. A measured lamination on S is a σ-invariant measured lamination λ on dS.*

If S has at least a boundary component or a marked point, call $\mathcal{A}(S)$ the set of all nontrivial isotopy classes of simple arcs $\alpha \subset S$ with $\alpha^\circ \subset S^\circ$ and endpoints at ∂S or at the marked points of S. A *k-system of arcs* $\boldsymbol{A} = \{\alpha_1, \dots, \alpha_k\} \subset \mathcal{A}(S)$ is a subset of arcs of S, that admit representatives which can intersect only at the marked points. The system $\boldsymbol{A}$ *fills* (resp. *quasi-fills*) S if $S \setminus \boldsymbol{A} := S \setminus \bigcup_{\alpha_i \in \boldsymbol{A}} \alpha_i$ is a disjoint union of discs (resp. discs, pointed discs and annuli homotopic to boundary components); $\boldsymbol{A}$ is also called *proper* if it quasi-fills S ([Loo95]).

Definition 2.7. *The complex of arcs $\mathfrak{A}(S)$ of a surface S with boundary and/or cusps is the simplicial complex whose k-simplices are $(k+1)$-systems of arcs on S (see* [Har86]*).*

We will denote by $\mathfrak{A}^\circ(S) \subset \mathfrak{A}(S)$ the subset of proper systems of arcs, which is the complement of a lower-dimensional simplicial subcomplex, and by $|\mathfrak{A}^\circ(S)| \subset |\mathfrak{A}(S)|$ the locus of weighted proper systems, which is open and dense.

Notation. *If $\boldsymbol{A} = \{\alpha_1, \dots, \alpha_k\} \in \mathfrak{A}(S)$, then a point $w \in |\boldsymbol{A}| \subset |\mathfrak{A}(S)|$ is a formal sum $w = \sum_i w_i \alpha_i$ such that $w_i \geq 0$ and $\sum_i w_i = 1$, which can be also seen as a function $w : \mathcal{A}(S) \to \mathbb{R}$ supported on $\boldsymbol{A}$.*

We recall the following simple result.

Lemma 2.8. $|\mathfrak{A}(S)|/\Gamma(S)$ *is compact.*

Proof. It is sufficient to notice the following facts:

- $\Gamma(S)$ acts on $\mathfrak{A}(S)$
- the above action may not be simplicial, but it is on the second baricentric subdivision $\mathfrak{A}(S)''$
- $\mathfrak{A}(S)/\Gamma(S)$ is a finite set and so is $\mathfrak{A}(S)''/\Gamma(S)$.

∎

Clearly, for a hyperbolic surface S with nonempty boundary and no marked points, there are continuous injective maps $|\mathfrak{A}(S)| \times \mathbb{R}_+ \hookrightarrow |\mathfrak{C}(dS)|^\sigma \times \mathbb{R}_+ \hookrightarrow \mathcal{ML}(S)$ and $|\mathfrak{A}(S)| \hookrightarrow |\mathfrak{C}(dS)|^\sigma \hookrightarrow \mathbb{P}\mathcal{ML}(S)$.

Notice that, if R is a smooth compact surface without boundary, then the multi-curves are dense in $\mathcal{ML}(R)$ and so the metric topology on $|\mathfrak{C}(R)| \times \mathbb{R}_+$ is stricly finer than the one coming from $\mathcal{ML}(R)$. The situation for multi-arcs is different.

Lemma 2.9. *If S is a smooth surface with boundary, the metric topology on $|\mathfrak{A}(S)|$ agrees with the subspace topology induced by the inclusion $|\mathfrak{A}(S)| \hookrightarrow \mathbb{P}\mathcal{ML}(S)$. The image of $|\mathfrak{A}^\circ(S)|$ is open but the image of $|\mathfrak{A}(S)|$ is neither open nor close.*

Proof. The image of $|\mathfrak{A}^\circ(S)|$ is open by invariance of domain, because $|\mathfrak{A}^\circ(S)|$ is a topological manifold (for instance, see [HM79]). But if we consider an arc α and a simple closed curve β disjoint from α, then the lamination $(1-t)[\alpha]+t[\beta]\to[\alpha]$ in $\mathbb{P}\mathcal{ML}(S)$ as $t\to 0$, which shows that the image of $|\mathfrak{A}(S)|$ is not open. To show that it is not even closed, consider two disjoint arcs $\{\alpha,\beta\}\in\mathfrak{A}(S)$ and a simple closed curve γ (possibly, a boundary component of S) such that $\alpha\cap\gamma=\emptyset$ and $i(\beta,\gamma)=1$. Let $U\subset\mathcal{ML}(S)$ be an $\mathcal{ML}(S)$-neighbourhood of $[\alpha]$ that contains the $|\mathfrak{A}(S)|$-ball of radius 2ε centered at α. Consider the weighted arc systems $w^{(k)}=(1-\varepsilon)\alpha+\varepsilon\mathrm{tw}_{k\gamma}(\beta)$ in $|\mathfrak{A}(S)|$, where $\mathrm{tw}_{k\gamma}$ is the k-uple Dehn twist along γ. Then, the sequence $\{w^{(k)}\}$ is contained in U; moreover, it diverges in $|\mathfrak{A}(S)|$, but it converges to $[\gamma]$ in $\mathbb{P}\mathcal{ML}(S)$.

To compare the topologies, pick $w=\sum w_i\alpha_i\in|\mathfrak{A}(S)|$ and $w^{(k)}=\sum v_j^{(k)}\beta_j^{(k)}\in|\mathfrak{A}(S)|$ such that $w^{(k)}\to w$ in $\mathbb{P}\mathcal{ML}(S)$. Complete $\boldsymbol{A}=\{\alpha_i\}$ to a maximal system of arcs $\boldsymbol{A}'=\{\alpha_i\}\cup\{\alpha'_h\}$ and define $w'=w+\delta\sum_h\alpha'_h$, where $\delta=\min_i w_i$.

For every k, write $w^{(k)}$ as a sum $\tilde{w}^{(k)}+\hat{w}^{(k)}$ of two nonnegative multi-arcs in such a way that all arcs in the support of $\hat{w}^{(k)}$ cross $\boldsymbol{A}'$ and that $i(\tilde{w}^{(k)},w')=0$.

Let t_k be the sum of the weights on $\hat{w}^{(k)}$, so that $d(w,w^{(k)})\le d(w,\tilde{w}^{(k)})+t_k$, where d is the path metric on $|\mathfrak{A}(S)|$. Because

$$t_k\delta\le i(\hat{w}^{(k)},w')=i(w^{(k)},w')\to i(w,w')=0$$

it follows that $t_k\to 0$. Moreover, $\tilde{w}^{(k)}$ has support contained in $\boldsymbol{A}'$ and the result follows. □

2.4. **Thurston's compactification.** Let R be a closed hyperbolic surface and let $\mathcal{C}(R)$ be the set of nontrivial isotopy classes of simple closed curves of R. The map $\ell:\mathcal{T}(R)\times\mathcal{C}(R)\longrightarrow\mathbb{R}_+$, that assigns to $([f:R\to R'],\gamma)$ the length of the geodesic representative for $f(\gamma)$ in the hyperbolic metric of R', induces an embedding $I:\mathcal{T}(R)\hookrightarrow(\mathbb{R}_+)^{\mathcal{C}(R)}$, where $(\mathbb{R}_+)^{\mathcal{C}(R)}$ is given the product topology (which is the same as the weak* topology on $L^\infty(\mathcal{C}(R))$).

Theorem 2.10 (Thurston [FLP79])**.** *The composition $[I]:\mathcal{T}(R)\hookrightarrow\mathbb{R}_+^{\mathcal{C}(R)}\to\mathbb{P}(\mathbb{R}_+^{\mathcal{C}(R)})$ is an embedding with relatively compact image and the boundary of $\mathcal{T}(R)$ inside $\mathbb{P}(\mathbb{R}_+^{\mathcal{C}(R)})$ is exactly $\mathbb{P}\mathcal{ML}(R)$.*

Let S be a hyperbolic surface with boundary and no cusps. The doubling map $\mathcal{C}(S)\cup\mathcal{A}(S)\hookrightarrow\mathcal{C}(dS)$ identifies $\mathcal{C}(S)\cup\mathcal{A}(S)$ with $\mathcal{C}(dS)^\sigma$.

Corollary 2.11. *$\mathcal{T}(S)$ embeds in $\mathbb{P}(\mathbb{R}_+^{\mathcal{C}(S)\cup\mathcal{A}(S)})$ and its boundary is $\mathbb{P}\mathcal{ML}(S)$.*

For S a hyperbolic surface with no cusps, $\overline{\mathcal{T}}^{Th}(S):=\mathcal{T}(S)\cup\mathbb{P}\mathcal{ML}(S)$ is called *Thurston's compactification* of $\mathcal{T}(S)$. Notice that the doubling map $S\hookrightarrow dS$ induces a closed embedding $D:\overline{\mathcal{T}}^{Th}(S)\hookrightarrow\overline{\mathcal{T}}^{Th}(dS)$.

2.5. **Weil-Petersson metric.** Let S be a hyperbolic surface with (possibly empty) geodesic boundary $\partial S=C_1\cup\dots\cup C_n$, and let $[f:S\to\Sigma]$ a point of $\mathcal{T}(S)$.

Define $\mathcal{Q}_\Sigma$ to be the *real* vector space of holomorphic quadratic differentials $q(z)dz^2$ whose restriction to $\partial\Sigma$ is real. Similarly, define the real vector space of harmonic Beltrami differentials as $\mathcal{B}_\Sigma:=\{\mu=\mu(z)d\bar{z}/dz=\bar{\varphi}\,ds^{-2}\,|\,\varphi\in\mathcal{Q}_\Sigma\}$, where ds^2 is the hyperbolic metric on Σ.

It is well-known that $T_{[f]}\mathcal{T}(S)$ can be identified to $\mathcal{B}_\Sigma$ and, similarly, $T^*_{[f]}\mathcal{T}(S)\cong\mathcal{Q}_\Sigma$. The natural coupling is given by

$$\begin{array}{ccc}\mathcal{B}_\Sigma\times\mathcal{Q}_\Sigma & \longrightarrow & \mathbb{C}\\ (\mu,\varphi) & \longmapsto & \displaystyle\int_\Sigma\mu\varphi\end{array}$$

Definition 2.12. *The Weil-Petersson pairing on $T_{[f]}\mathcal{T}(S)$ is defined as*

$$h(\mu,\nu):=\int_\Sigma\mu\bar{\nu}\,ds^2\qquad\textit{with }\mu,\nu\in\mathcal{B}_\Sigma$$

Writing $h=g+i\omega$, we call g the Weil-Petersson Riemannian metric and ω the Weil-Petersson form. For $T^*_{[f]}\mathcal{T}(S)$, we similarly have $h^\vee(\varphi,\psi):=\int_\Sigma\varphi\bar{\psi}\,ds^{-2}$ with $\varphi,\psi\in\mathcal{Q}_\Sigma$. The Weil-Petersson Poisson structure is $\eta:=\mathrm{Im}(h^\vee)$.

It follows from the definition that the doubling map $D : \mathcal{T}(S) \longrightarrow \mathcal{T}(dS)$ is a homothety of factor 2 onto a real Lagrangian submanifold of $\mathcal{T}(dS)$.

From Wolpert's work [Wol83], we learnt that $\omega = \sum_{i=1}^{N} d\ell_i \wedge d\tau_i$, where $(p_1, \dots, p_n, \ell_1, \tau_1, \dots, \ell_N, \tau_N)$ are Fenchel-Nielsen coordinates, and so ω is degenerate whenever S has boundary. In this case, the symplectic leaves (which we will also denote by $\mathcal{T}(S)(\underline{p})$) are exactly the fibers $\mathcal{L}^{-1}(\underline{p})$ of $\mathcal{L}$, which are not totally geodesic subspaces for g (unless $p_1 = \dots = p_n = 0$ and the boundary components degenerate to cusps).

Using the Weil-Petersson metric, the cotangent space to $\mathcal{T}(S)(\underline{p})$ at $[f : S \to \Sigma]$ can be identified with $(dp_1 \oplus \dots \oplus dp_n)^\perp \subset T^*_{[f]}\mathcal{T}(S)$. It follows from [Wol89] that the elements of $(dp_1 \oplus \dots \oplus dp_n)^\perp$ are those $\varphi \in \mathcal{Q}_\Sigma$ such that $\int_{C_i} \varphi |ds|^{-1} = 0$ for all $i = 1, \dots, n$. Similarly, the tangent space $T_{[f]}\mathcal{T}(S)(\underline{p})$ is the subspace of those $\mu \in \mathcal{B}_\Sigma$ such that $\int_{C_i} \mu|\lambda| = 0$ for $i = 1, \dots, n$.

Remark 2.13. *$\mathcal{T}(S)$ is naturally a complex manifold if S is closed. In this case, ω and η are nondegenerate and the Weil-Petersson metric is Kähler (see* [Ahl61]*).*

2.6. **Augmented Teichmüller space.** Let S be a hyperbolic surface with geodesic boundary $\partial S = C_1 \cup \dots \cup C_n$ and no cusps. An *S-marked stable surface* Σ is a hyperbolic surface possibly with geodesic boundary components, cusps and nodes plus an isotopy class of maps $f : S \to \Sigma$ that may shrink some boundary components of S to cusps of Σ, some loops of S to the nodes of Σ and is an oriented diffeomorphism elsewhere.

We say that $f_1 : S \longrightarrow \Sigma_1$ and $f_2 : S \longrightarrow \Sigma_2$ are equivalent if there exists an isometry $h : \Sigma_1 \longrightarrow \Sigma_2$ such that $h \circ f_1$ is homotopic to f_2. The *augmented Teichmüller space* $\overline{\mathcal{T}}^{WP}(S)$ is the set of stable S-marked surfaces up to equivalence (see [Ber74]). Clearly, $\mathcal{T}(S) \subset \check{\mathcal{T}}(S) \subset \overline{\mathcal{T}}^{WP}(S)$.

To describe the topology of $\overline{\mathcal{T}}^{WP}(S)$ around a stable surface $[f : S \to \Sigma]$ with k cusps and d nodes, choose a system of curves $\{C_1, \dots, C_n, \gamma_1, \dots, \gamma_N\}$ on S (with $N = 3g - 3 + n$) *adapted to* f, i.e. such that $f^{-1}(\nu_j) = \gamma_j$ for each of the nodes $\nu_1, \dots, \nu_d$ of Σ. Clearly, the Fenchel-Nielsen coordinates $(p_1, \dots, p_n, \ell_1, \tau_1, \dots, \ell_N, \tau_N)$ extend over $[f]$, with the exception of $\tau_1(f), \dots, \tau_d(f)$, which are not defined (see [Abi80] for more details on the Fenchel-Nielsen coordinates).

We declare that the sequence $\{f_m : S \to \Sigma_m\} \subset \overline{\mathcal{T}}^{WP}(S)$ converges to $[f]$ if $p_i(f_m) \to p_i(f)$ for $1 \le i \le n$, $\ell_j(f_m) \to \ell_j(f)$ for $1 \le j \le N$ and $\tau_j(f_m) \to \tau_j(f)$ for $d + 1 \le j \le N$. By definition, the boundary length map extends with continuity to $\mathcal{L} : \overline{\mathcal{T}}^{WP}(S) \longrightarrow \mathbb{R}^n_{\ge 0}$ and we call $\overline{\mathcal{T}}^{WP}(S)(\underline{p})$ the fiber $\mathcal{L}^{-1}(\underline{p})$. We will write $\boldsymbol{p}$ for $p_1 + \dots + p_n$ and $\boldsymbol{\mathcal{L}}(f)$ for the L^1 norm of $\mathcal{L}(f)$.

A logarithmic version of the cotangent cone at a singular $[f : S \to \Sigma]$ can be related to the space of holomorphic quadratic differentials on Σ that are real at $\partial\Sigma$ and that have (at worst) double poles at the cusps with negative quadratic residues and (at worst) double poles at the nodes with the same quadratic residues on both branches. Details can be found in [Ber74], [Mas76] and [Wol03].

Notice that the Weil-Petersson metric diverges in directions transverse to $\partial\mathcal{T}(S)$. However, the divergence is so mild that $\partial\mathcal{T}(S)$ is at finite distance (see [Mas76]). In fact, for every $\underline{p} \in \mathbb{R}^n_{\ge 0}$ the augmented $\overline{\mathcal{T}}^{WP}(S)(\underline{p})$ is the completion of $\mathcal{T}(S)(\underline{p})$ with respect to the Weil-Petersson metric (it follows from the $\Gamma(S)$-invariance of the metric, its compatibility with the doubling map D and the compactness of the Deligne-Mumford moduli space [DM69]).

Remark 2.14. *According to our definition, if S has nonempty boundary, then $\overline{\mathcal{T}}^{WP}(S)$ is not WP-complete and in fact the image of $\overline{\mathcal{T}}^{WP}(S)$ inside $\overline{\mathcal{T}}^{WP}(dS)$ through the doubling map is not close because it misses thoses surfaces with boundaries of infinite length.*

We recall here a criterion of convergence in $\overline{\mathcal{T}}^{WP}(S)$ that will be useful later.

Proposition 2.15 ([Mon09a]). *Let $[f : S \to (\Sigma, g)] \in \overline{\mathcal{T}}^{WP}(S)$ and call $\gamma_1, \dots, \gamma_d$ the simple closed curves of S that are contracted to a point by f, and let $\{f_m : S \to (\Sigma_m, g_m)\}$ be a sequence of points in $\overline{\mathcal{T}}^{WP}(S)$. Denote by $V_{\gamma_i}(f_m)$ a standard collar (of fixed width) of the hyperbolic geodesic homotopic to $f_m(\gamma_i) \subset \Sigma_m$ and set $V_i = V_{\gamma_i}(f)$ and $\Sigma^\circ := \Sigma \setminus (V_1 \cup \dots \cup V_k) \subset \Sigma_{sm}$. The following are equivalent:*

(1) $[f_m] \to [f]$ *in* $\overline{\mathcal{T}}^{WP}(S)$
(2) $\ell_{\gamma_i}(f_m) \to 0$ *and there exist representatives* $\tilde{f}_m \in [f_m]$ *such that* $(f \circ \tilde{f}_m^{-1})\Big|_{V_{\gamma_i}(f_m)}$ *is standard and* $(\tilde{f}_m \circ f^{-1})^*(g_m) \to g$ *uniformly on* Σ°
(3) $\exists \tilde{f}_m \in [f_m]$ *such that the metrics* $(\tilde{f}_m \circ f^{-1})^*(g_m) \to g$ *uniformly on the compact subsets of* Σ_{sm}
(4) $\ell_{\gamma_i}(f_m) \to 0$ *and* $\exists \tilde{f}_m \in [f_m]$ *such that* $(f \circ \tilde{f}_m^{-1})\Big|_{V_{\gamma_i}(f_m)}$ *is standard and* $(\tilde{f}_m \circ f^{-1})\Big|_{\Sigma^\circ}$ *is* K_m*-quasiconformal with* $K_m \to 1$
(5) $\exists \tilde{f}_m \in [f_m]$ *such that, for every compact subset* $F \subset \Sigma_{sm}$*, the homeomorphism* $(\tilde{f}_m \circ f^{-1})\Big|_F$ *is* $K_{m,F}$*-quasiconformal and* $K_{m,F} \to 1$.

We denoted by g_m the hyperbolic metric on Σ_m and by Σ_{sm} the locus of Σ on which g is smooth (namely, Σ with cusps and nodes removed). By "standard collar" of width t of a boundary component γ (resp. an internal curve γ), we meant an annulus of the form $A_t(\gamma)$ (resp. the union of the two annuli isometric to $A_t(\gamma)$ that bound γ), as provided by the following celebrated result.

Lemma 2.16 (Collar lemma, [Kee74]- [Mat76]). *For every simple closed geodesic $\gamma \subset \Sigma$ in a hyperbolic surface and for every "side" of γ, and for every $0 < t \leq 1$, there exists an embedded hypercycle γ' parallel to γ (on the prescribed side) such that the area of the annulus $A_t(\gamma)$ enclosed by γ and γ' is $t\ell/2\sinh(\ell/2)$. For $\ell = 0$, the geodesic γ must be intended to be a cusp and γ' a horocycle of length t. Furthermore, all such annuli (corresponding to distinct geodesics and sides) are disjoint.*

Standard maps between annuli or between pair of pants are defined in [Mon09a].

2.7. **The moduli space.** Let S be a hyperbolic surface of genus g with geodesic boundary components $C_1, \dots, C_n$ and no cusps. The augmented Teichmüller space $\overline{\mathcal{T}}^{WP}(S)$ (as well as Thurston's compactification $\overline{\mathcal{T}}^{Th}(S)$) carries a natural right action of the group $\mathrm{Diff}_+(S)$ of orientation-preserving diffeomorphisms of S that send C_i to C_i for every $i = 1, \dots, n$.

$$\begin{array}{ccc} \overline{\mathcal{T}}^{WP}(S) \times \mathrm{Diff}_+(S) & \longrightarrow & \overline{\mathcal{T}}^{WP}(S) \\ ([f : S \to \Sigma], h) & \longmapsto & [f \circ h : S \to \Sigma] \end{array}$$

Clearly, the action is trivial on the connected component $\mathrm{Diff}_0(S)$ of the identity.

Definition 2.17. *The mapping class group of S is the quotient*

$$\Gamma(S) := \mathrm{Diff}_+(S)/\mathrm{Diff}_0(S) = \pi_0 \mathrm{Diff}_+(S)$$

The quotient $\overline{\mathcal{M}}(S) := \overline{\mathcal{T}}^{WP}(S)/\Gamma(S)$ is the moduli space of stable hyperbolic surfaces of genus g with n (ordered) boundary components.

The quotient map $\pi : \overline{\mathcal{T}}^{WP}(S) \longrightarrow \overline{\mathcal{M}}(S)$ can be identified with the forgetful map $[f : S \to \Sigma] \mapsto [\Sigma]$. Moreover, we can identify the stabilizer $\mathrm{Stab}_{[\Sigma]}(\Gamma(S))$ with the group $\mathrm{Iso}_+(\Sigma)$ of orientation-preserving isometries of Σ, which is finite.

$\overline{\mathcal{M}}(S)$ can be given a natural structure of orbifold (with corners), called *Fenchel-Nielsen smooth structure*. Let $[f : S \to \Sigma]$ be a point of $\overline{\mathcal{T}}^{WP}(S)$ and let $(p_1, \dots, p_n, \ell_1, \tau_1, \dots, \ell_N, \tau_N)$ be Fenchel-Nielsen coordinates adapted to f. A local chart for (the Fenchel-Nielsen smooth structure of) $\overline{\mathcal{M}}(S)$ around $[\Sigma]$ is given by

$$\begin{array}{ccc} \mathbb{R}^n_{\geq 0} \times \mathbb{C}^{3g-3} & \longrightarrow & \overline{\mathcal{M}}(S) \\ (p, z) & \longmapsto & \Sigma(p, z) \end{array}$$

where $[f' : S \to \Sigma(p,z)]$ is the point of $\overline{\mathcal{T}}^{WP}(S)$ with coordinates $(p_1,\dots,p_n,\ell_1,\tau_1,\dots,\ell_N,\tau_N)$ with $\ell_j = |z_j|$ and $\tau_j = |z_j|\arg(z_j)/2\pi$.

Remark 2.18. *As shown by Wolpert* [Wol85]*, the smooth structure at $\partial\mathcal{M}(S) = \overline{\mathcal{M}}(S) \setminus \mathcal{M}(S)$ coming from Fenchel-Nielsen coordinates and the one coming from algebraic geometry (for instance, see* [DM69] *or* [AC09]*) are not the same.*

We can identify the (co)tangent space to $\mathcal{T}(S)$ (with the analytic structure) at $[f : S \to \Sigma]$ with the (co)tangent space to $\mathcal{M}(S)$ at $[\Sigma]$. It follows by its very definition that the Weil-Petersson metric and the boundary lengths map descends to $\mathcal{M}(S)$ and that $\overline{\mathcal{M}}(S)(\underline{p})$ is the metric completion of $\mathcal{M}(S)(\underline{p})$ for every $\underline{p} \in \mathbb{R}^n_{\geq 0}$.

3. Triangulations

3.1. **Systems of arcs and widths.** Let S be a hyperbolic surface of genus g with boundary components $C_1,\dots,C_n$ and no cusps, and let $\boldsymbol{A} = \{\alpha_1,\dots,\alpha_N\} \in \mathfrak{A}^\circ(S)$ a maximal system of arcs on S (so that $N = 6g-6+3n$).

Fix a point $[f : S \to \Sigma]$ in $\mathcal{T}(S)$. For every $i = 1,\dots,N$ there exists a unique geodesic arc on Σ in the isotopy class of $f(\alpha_i)$ that meets $\partial\Sigma$ perpendicularly and which we will still denote by $f(\alpha_i)$: call $a_i = \ell_{\alpha_i}(f)$ its length and let $s_i = \cosh(a_i/2)$. Notice that $\{f(\alpha_i)\}$ decomposes Σ into a disjoint union of right-angles hexagons $\{H_1,\dots,H_{4g-4+2n}\}$, so that the following is immediate (see also [Ush99], [Mon09c]).

Lemma 3.1. *The maps $a_{\boldsymbol{A}} : \mathcal{T}(S) \longrightarrow \mathbb{R}^{\boldsymbol{A}}_+$ and $s_{\boldsymbol{A}} : \mathcal{T}(S) \longrightarrow \mathbb{R}^{\boldsymbol{A}}_+$ given by $a_{\boldsymbol{A}} = (a_1,\dots,a_N)$ and $s_{\boldsymbol{A}} = (s_1,\dots,s_N)$ are real-analytic diffeomorphisms.*

Let H be such a right-angled hexagon and let $(\overrightarrow{\alpha_i},\overrightarrow{\alpha_j},\overrightarrow{\alpha_k})$ the cyclic set of oriented arcs that bound H, so that $\partial H = \overrightarrow{\alpha_i} * \overrightarrow{\alpha_j} * \overrightarrow{\alpha_k}$. If $\overrightarrow{\alpha_x},\overrightarrow{\alpha_y}$ are oriented arcs with endpoint on the same boundary component C, denote by $d(\overrightarrow{\alpha_x},\overrightarrow{\alpha_y})$ the length of the portion of C running from the endpoint of $\overrightarrow{\alpha_x}$ to the endpoint of $\overrightarrow{\alpha_y}$ along the positive direction of C.

Define $w_{\boldsymbol{A}}(\overrightarrow{\alpha_i}) = \dfrac{1}{2}[d(\overrightarrow{\alpha_i},\overleftarrow{\alpha_j}) + d(\overrightarrow{\alpha_k},\overleftarrow{\alpha_i}) - d(\overrightarrow{\alpha_j},\overleftarrow{\alpha_k})]$, where $\overleftarrow{\alpha_x}$ the oriented arc obtained from $\overrightarrow{\alpha_x}$ by switching its orientation.

Definition 3.2. *The $\boldsymbol{A}$-width of α_i is $w_{\boldsymbol{A}}(\alpha_i) = w_{\boldsymbol{A}}(\overrightarrow{\alpha_i}) + w_{\boldsymbol{A}}(\overleftarrow{\alpha_i})$.*

In [Luo07]), Luo calls the width “E-invariant”.

3.2. **The t-coordinates.** Let S be a surface as in the previous section.

Definition 3.3. *The t(ransverse)-length of an arc α at $[f]$ is $t_\alpha(f) := T(\ell_\alpha(f))$, where $T(x) := 2\mathrm{arcsinh}\left(\dfrac{1}{\sinh(x/2)}\right)$.*

Notice that $T(x) : [0,+\infty] \to [0,+\infty]$ is decreasing function of x (similar to the width of the collar of a closed curve of length x provided by Lemma 2.16). Moreover, T is involutive, $T(x) \approx 4e^{-x/2}$ as $x \to \infty$ and $T(x) \approx 2\log(4/x)$ as $x \to 0$.

Back to the t-length, the following lemma reduces to a statement about hyperbolic hexagons with right angles.

Lemma 3.4. *For every maximal system of arcs $\boldsymbol{A}$*

$$\begin{aligned} t_{\boldsymbol{A}} :\ \check{\mathcal{T}}(S) &\longrightarrow \mathbb{R}^{\boldsymbol{A}}_{\geq 0} \\ [f] &\longmapsto (t_{\alpha_1}(f),\dots,t_{\alpha_N}(f)) \end{aligned}$$

is a continuous map that restricts to a real-analytic diffeomorphism $\mathcal{T}(S) \to \mathbb{R}^{\boldsymbol{A}}_+$. Moreover, for every $[f] \in \check{\mathcal{T}}(S)$ with $\mathcal{L}(f) \neq 0$, there exists an $\boldsymbol{A}$ such that $t_{\boldsymbol{A}}$ is a system of coordinates around $[f]$.

Consequently, the t-lengths map $\check{\mathcal{T}}(S) \times \mathcal{A}(S) \longrightarrow \mathbb{R}_{\geq 0}$ defined as $(f,\alpha) \mapsto t_\alpha(f)$ gives an injection

$$\begin{aligned} j :\ \mathcal{T}(S) &\hookrightarrow \mathbb{P}(\mathcal{A}(S)) \times [0,\infty] \\ [f] &\longmapsto ([t_\bullet(f)], \|t_\bullet(f)\|_\infty) \end{aligned}$$

where $L^\infty(\mathcal{A}(S))$ is the $\mathbb{R}_+$-cone of the bounded maps $t : \mathcal{A}(S) \to \mathbb{R}_{\geq 0}$ and $\mathbb{P}(\mathcal{A}(S))$ is its projectivization.

Notice that $\mathbb{P}(\mathcal{A}(S))$ has a metric induced by the unit sphere of $L^\infty(\mathcal{A}(S))$ and that $\Gamma(S)$ acts on $\mathbb{P}(\mathcal{A}(S))$ permuting some coordinates. Thus, $\mathbb{P}(\mathcal{A}(S)) \times [0,\infty]$ has a $\Gamma(S)$-invariant metric.

Lemma 3.5. *j is continuous.*

We proof is included in that of Proposition 3.24.

Definition 3.6. *Call* **bordification of arcs** *the closure $\overline{\mathcal{T}}^a(S)$ of $\mathcal{T}(S)$ inside $\mathbb{P}(\mathcal{A}(S))\times[0,\infty]$. By "finite part" of $\overline{\mathcal{T}}^a(S)$ we will mean $\overline{\mathcal{T}}^a(S) \cap \mathbb{P}(\mathcal{A}(S)) \times [0,\infty)$. Call* **compactification of arcs** *the quotient $\overline{\mathcal{M}}^a(S) := \overline{\mathcal{T}}^a(S)/\Gamma(S)$.*

We will give an explicit description of the boundary points in $\overline{\mathcal{T}}^a(S)$ and we will show that $\overline{\mathcal{M}}^a(S)$ is Hausdorff and compact.

3.3. **The spine construction.** Let S be a hyperbolic surface with geodesic boundary $\partial S = C_1 \cup \cdots \cup C_n$ and no cusps and let $[f : S \to \Sigma]$ be a point in $\mathcal{T}(S)$.

The *valence* $\mathrm{val}(p)$ of a point $p \in \Sigma$ is the number of paths from p to $\partial\Sigma$ of minimal length.

Definition 3.7. *The spine of Σ is the locus $\mathrm{Sp}(\Sigma)$ of points of Σ of valence at least 2.*

One can easily show that $\mathrm{Sp}(\Sigma) = V \cup E$ is a one-dimensional CW-complex embedded in Σ, where $V = \mathrm{val}^{-1}([3,\infty))$ is a finite set of points, called *vertices*, and $E = \mathrm{val}^{-1}(2)$ is a disjoint union of finitely many (open) geodesic arcs, called *edges*.

For every edge $E_i \subset E$ of $\mathrm{Sp}(\Sigma)$, we can define a *dual arc* α_i in the following way. Pick $p \in E_i$ and call γ_1 and γ_2 the two paths that join p to $\partial\Sigma$. Then α_i is the shortest arc in the homotopy class (with endpoints on $\partial\Sigma$) of $\gamma_1^{-1} * \gamma_2$. Let the *spinal arc system* $\boldsymbol{A}_{sp}(\Sigma)$ be the system of arcs dual to the edges of $\mathrm{Sp}(\Sigma)$, which is proper because Σ retracts by deformation onto $\mathrm{Sp}(\Sigma)$ just flowing away from the boundary.

Even if the spinal arc system is not maximal, widths w_{sp} can be associated to $\boldsymbol{A}_{sp}(\Sigma)$ in the following way. For every oriented arc $\overrightarrow{\alpha_i} \in \boldsymbol{A}_{sp}(\Sigma)$ ending at $y_i \in C_m$, orient the dual edge E_i in such a way that $(\overrightarrow{E_i}, \overrightarrow{\alpha_i})$ is positively oriented and call v the starting point of $\overrightarrow{E_i}$. Every point of E_i has exactly two projections, that is two closest points in $\partial\Sigma$: the endpoint of $\overrightarrow{\alpha_i}$ selects only one of these, which belongs to C_m. Call $v' \in C_m$ the projection of v determined by $\overrightarrow{\alpha_i}$. Define $w_{sp}(\overrightarrow{\alpha_i})$ to be the distance with sign $d_{C_m}(y_i, v')$ along C_m, which is certainly positive if α_i and E_i intersect. Clearly, the sum $w_{sp}(\alpha_i) = w_{sp}(\overrightarrow{\alpha_i}) + w_{sp}(\overleftarrow{\alpha_i})$ is always positive, being the length of either of the two projections of E_i.

Example 3.8. *In Figure 1, we have $\overrightarrow{\alpha_i} = \overrightarrow{z_i y_i}$, $v' = f_k$ and $w_{sp}(\overrightarrow{\alpha_i}) > 0$.*

Theorem 3.9 (Ushijima [Ush99])**.** *Given a hyperbolic surface with nonempty boundary Σ, let $\mathfrak{A}(\Sigma)_+$ be the set of all maximal systems of arcs $\boldsymbol{A}$ such that $w_{\boldsymbol{A}}(\alpha_i) \geq 0$ for all $\alpha_i \in \boldsymbol{A}$. Then $\mathfrak{A}(\Sigma)_+$ is nonempty and the intersection of all systems in $\mathfrak{A}(\Sigma)_+$ is exactly $\boldsymbol{A}_{sp}(\Sigma)$. Moreover, $w_{sp}(\alpha) = w_{\boldsymbol{A}}(\alpha) > 0$ for all $\alpha \in \boldsymbol{A}_{sp}(\Sigma)$ and all $\boldsymbol{A} \in \mathfrak{A}(\Sigma)_+$.*

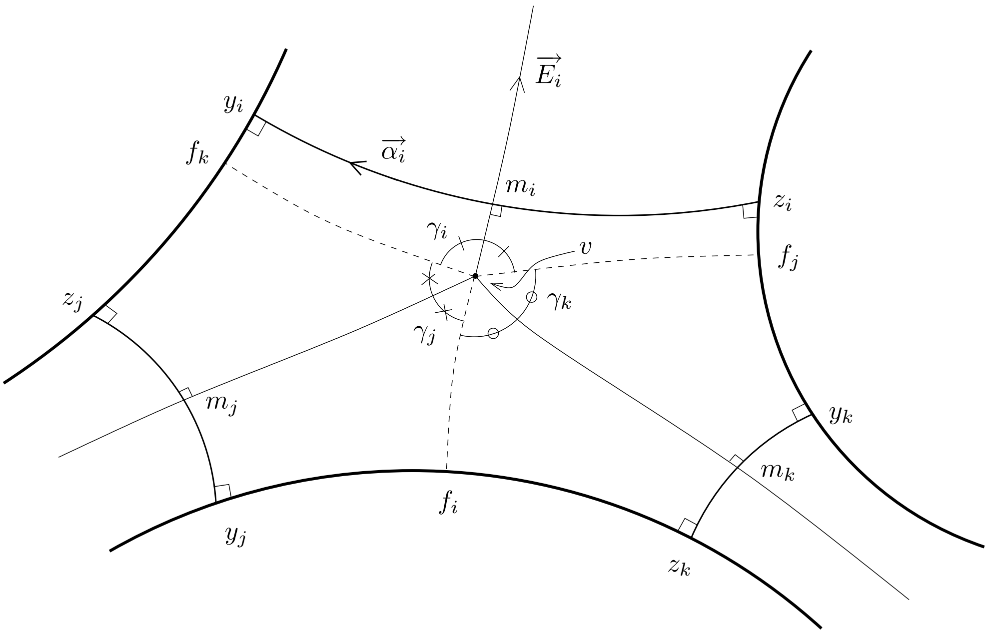

Figure 1: Geometry of the spine close to a trivalent vertex

Remark 3.10. *Let $H \subset \Sigma$ be a right-angled hexagon, bounded by $(\overrightarrow{\alpha_i}, \overrightarrow{\alpha_j}, \overrightarrow{\alpha_k})$, and call $\gamma_i = \gamma(\overrightarrow{\alpha_i})$ as in Figure 1. An easy computation* [Mon09c] *shows that*

$$\sinh(w_{sp}(\overrightarrow{\alpha_i}))\sinh(a_i/2) = \cos(\gamma_i) = \frac{s_j^2 + s_k^2 - s_i^2}{2 s_j s_k} \tag{1}$$

and so $\sinh(w_{sp}(\overrightarrow{\alpha_i})) = \dfrac{s_j^2 + s_k^2 - s_i^2}{2 s_j s_k \sqrt{s_i^2 - 1}}$ *and* $w_{sp}(\overrightarrow{\alpha_i}) \leq \dfrac{1}{2} t_{\alpha_i}$.

Theorem 3.11 (Luo [Luo07])**.** *Given a hyperbolic surface with boundary S and no cusps, the map*

$$\begin{array}{rccc} \boldsymbol{W} : & \mathcal{T}(S) & \longrightarrow & |\mathfrak{A}^\circ(S)| \times \mathbb{R}_+ \\ & [f : S \to \Sigma] & \longmapsto & f^* w_{sp} \end{array}$$

is a $\Gamma(S)$-equivariant homeomorphism.

Notice that the construction extends to $\check{\mathcal{T}}(S) \setminus \check{\mathcal{T}}(S)(0)$ but the locus $\check{\mathcal{T}}(S)(0)$ of surfaces with n cusps is problematic, because the function "distance from the boundary $\partial\Sigma$" diverges everywhere on Σ. This can be easily fixed by considering the real blow-up $\mathrm{Bl}_0\check{\mathcal{T}}(S)$ of $\check{\mathcal{T}}(S)$ along $\check{\mathcal{T}}(0)$. The exceptional locus can be identified to the space of *projectively decorated surfaces* [Pen87], that is of couples $([f : S \to \Sigma], \underline{p})$, where $[f]$ is an S-marked hyperbolic surface with n cusps and $\underline{p} \in \Delta^{n-1} \cong \mathbb{P}(\mathbb{R}^n_{\geq 0})$ is a ray of weights (the *decoration*).

We have the following two simple facts.

Lemma 3.12 ([Mon09c])**.** *For every maximal system of arcs $\boldsymbol{A}$ (of cardinality $N = 6g-6+3n$), the associated t-lengths extend to real-analytic map*

$$t_{\boldsymbol{A}} : \mathrm{Bl}_0\check{\mathcal{T}}(S) \longrightarrow \Delta^{N-1} \times [0, \infty)$$

On the exceptional locus, the projectivized t-lengths are inverses to the projectivized λ-lengths (defined by Penner in [Pen87]*). Thus, $t_{\boldsymbol{A}}$ gives a system of coordinates on $[\check{\mathcal{T}}(S)(0) \times (\Delta^{n-1})^\circ] \cup \mathcal{T}(S)$. Moreover, for every $(f, \underline{p}) \in \check{\mathcal{T}}(S)(0) \times \partial\Delta^{n-1}$ there exists an $\boldsymbol{A}$ such that $t_{\boldsymbol{A}}$ gives a chart around $(f, \underline{p})$.*

Theorem 3.13 ([Mon09c])**.** *The map $\boldsymbol{W}$ extends to a $\Gamma(S)$-equivariant homeomorphism*

$$\check{\boldsymbol{W}} : \mathrm{Bl}_0\check{\mathcal{T}}(S) \longrightarrow |\mathfrak{A}^\circ(S)| \times [0, \infty)$$

On the exceptional locus, the projectivized widths coincide with the projectived simplicial coordinates (defined by Penner in [Pen87]*) and so there $\check{\boldsymbol{W}}$ coincides with Penner's homeomorphism.*

3.4. **Spines of stable surfaces.** Notice that the spine construction extends to stable hyperbolic surfaces Σ (blowing up the locus of surfaces with n cusps), discarding the components of Σ where the distance from $\partial\Sigma$ is infinite. However, the weighted arc system we can produce does not allow to reconstruct the full surface, but just a visible portion of it.

Definition 3.14. *Let Σ a stable hyperbolic surface Σ with boundary (and possibly cusps) or let $(\Sigma, \underline{p})$ be a stable (projectively) decorated surface. A component of Σ is called visible if it contains a boundary circle or a positively weighted cusp; otherwise, it is called invisible. Denote by Σ_+ the visible subsurface of Σ, that is the union of the smooth points of all visible components and by Σ_- the invisible subsurface. Two points $[f_1 : S \to \Sigma_1]$ and $[f_2 : S \to \Sigma_2]$ of $\widehat{\mathcal{T}}(S) := \mathrm{Bl}_0\overline{\mathcal{T}}^{WP}(S)$ are visibly equivalent $[f_1] \sim_{vis} [f_2]$ if there exists a third point $[f : S \to \Sigma]$ and maps $h_i : \Sigma \to \Sigma_i$ for $i = 1, 2$ such that h_i restricts to an isometry $\Sigma_+ \to \Sigma_{i,+}$ and $h_i \circ f \simeq f_i$ for $i = 1, 2$.*

The spine $\mathrm{Sp}(\Sigma)$ of a stable hyperbolic surface Σ with geodesic boundary (or with weighted cusps) can only be defined inside $\overline{\Sigma}_+$, so that its dual system of arcs $\boldsymbol{A}_{sp}(\Sigma)$ will be contained in $\overline{\Sigma}_+$ too. Given a marking $[f : S \to \Sigma]$, we will write $S_+ = f^{-1}(\Sigma_+)$ and $S_- = f^{-1}(\Sigma_-)$, so that $\overline{S}_+$ will be the maximal subsurface of S (unique up to isotopy), quasi-filled by $f^*\boldsymbol{A}_{sp}(\Sigma)$, which carries positive weights f^*w_{sp}.

Conversely, given a system of arcs $\boldsymbol{A} \in \mathfrak{A}(S)$, the *visible subsurface* S_+ associated to $\boldsymbol{A}$ is the isotopy class of maximal open subsurfaces embedded (with their closures) in S° such that $\boldsymbol{A}$ is contined in $\overline{S}_+$ as a proper system of arcs. More concretely, $\overline{S}_+$ is the union of a closed tubular neighbourhood of $\boldsymbol{A}$ and all components of $S \setminus \boldsymbol{A}$ which are discs or annuli that retract onto some boundary component. If Σ is obtained from S by collapsing the boundary components of $\overline{S}_+$ and the possible resulting two-noded spheres to nodes of Σ, then we obtain an isotopy class of maps $f : S \to \Sigma$, which depends only on $\boldsymbol{A}$. We will refer to this map (or just to Σ, when we work in the moduli space) as the *topological type* of $\boldsymbol{A}$.

Given weights $w \in |\boldsymbol{A}|^\circ \times [0, \infty)$, the components of $\overline{\Sigma}_+ = f(\overline{S}_+)$ are quasi-filled by the arc system $f(\boldsymbol{A})$: because of Theorem 3.13, they can be given a hyperbolic metric such that $f(\boldsymbol{A})$ is its spinal arc system with weights $f_*(w)$. When no confusion is possible, we will still denote by $[f : S \to \Sigma]$ the class of visibly equivalent S-marked stable surfaces determined by f.

This construction defines a $\Gamma(S)$-equivariant extension of the previous $\boldsymbol{W}^{-1}$

$$\widehat{\boldsymbol{W}}^{-1} \ : \ |\mathfrak{A}(S)| \times [0, \infty) \longrightarrow \widehat{\mathcal{T}}^{vis}(S)$$

where $\widehat{\mathcal{T}}^{vis}(S) = \widehat{\mathcal{T}}(S)/\sim_{vis}$.

The argument above shows that $\widehat{\boldsymbol{W}}^{-1}$ is bijective. As already noticed in [BE88] and [Loo95], the map $\widehat{\boldsymbol{W}}$ is not continuous if $|\mathfrak{A}(S)|$ is endowed with the coherent topology.

Remark 3.15. *$|\mathfrak{A}(S)|$ is locally finite at $w \iff \boldsymbol{A} = \mathrm{supp}(w)$ is a proper system of arcs $\iff w$ has a countable fundamental system of coherent neighbourhoods. Moreover, a sequence converges (for the coherent topology) if and only if it is definitely in a fixed closed simplex and there it converges in the Euclidean topology.*

The discontinuity of $\widehat{\boldsymbol{W}}$ at $\partial\mathcal{T}(S)$ with respect to the coherent topology can be seen as follows. Consider a marked surface $[f : S \to \Sigma]$ with a node $f(\gamma) = q \in \Sigma$ such that not all the boundary components of Σ are cusps and call $\boldsymbol{A}$ a maximal system of arcs of S such that $\widehat{\boldsymbol{W}}(f) \in |\boldsymbol{A}| \times \mathbb{R}_+$. Choose a sequence $[f_m : S \to \Sigma_m]$ with $\widehat{\boldsymbol{W}}(f_m)$ contained in $|\boldsymbol{A}|^\circ \times \mathbb{R}_+$ and such that $[f_m] \to [f]$. If τ_γ is the right Dehn twist along γ and $f'_m = f_m \circ \tau_\gamma^m$, then $[f'_m]$ still converges to $[f]$. On the other hand, the $\widehat{\boldsymbol{W}}(f'_m)$'s all belong to the interior of distinct maximal simplices of $|\mathfrak{A}(S)|$ and so the sequence $\widehat{\boldsymbol{W}}(f'_m)$ is divergent for the coherent topology.

The correct solution (see [BE88]), anticipated in Section 2.3 and which we will adopt without further notice, is to equip $|\mathfrak{A}(S)|$ with the metric topology, whose importance will be also clear in the proof of Lemma 3.21.

Theorem 3.16. *The $\Gamma(S)$-equivariant natural extension*

$$\widehat{\boldsymbol{W}} \ : \ \widehat{\mathcal{T}}^{vis}(S) \longrightarrow |\mathfrak{A}(S)| \times [0, \infty)$$

is a homeomorphism.

The following proof shares some ideas with [ACGH] (to which we refer for a more detailed discussion of the case with n cusps). The bijectivity of $\widehat{\boldsymbol{W}}$ is a direct consequences of the work of Penner/Bowditch-Epstein and Luo. We begin with some preparatory observations.

Definition 3.17. *Let $([f : S \to \Sigma], \underline{p}) \in \widehat{\mathcal{T}}(S)(0)$ be a projectively decorated surface and let $B_i \subset \Sigma$ be the embedded horoball at $x_i = f(C_i)$ with radius p_i. The associated truncated surface is $\Sigma^{tr} := \Sigma \setminus (B_1 \cup \dots \cup B_n)$ and the reduced length of an arc $\alpha \in \mathcal{A}(S)$ at f is $\tilde{\ell}_\alpha(f) := \ell(\Sigma^{tr} \cap f(\alpha))$.*

Lemma 3.18. *Let $\{f_m : S \to \Sigma_m\} \subset \mathcal{T}(S)$ be a sequence that converges to $[f_\infty : S \to \Sigma_\infty] \in \widehat{\mathcal{T}}^{vis}(S)$.*

(a) *Assume $\boldsymbol{\mathcal{L}}(f_\infty) > 0$ and let $\mathcal{A}(S) = \mathcal{A}_{fin} \sqcup \mathcal{A}_\infty$, where $\mathcal{A}_\infty$ is the subset of arcs β such that $\ell_\beta(f_\infty) = \infty$. Then $\ell_\alpha(f_m)/\ell_\alpha(f_\infty) \to 1$ uniformly for all $\alpha \in \mathcal{A}_{fin}$. Moreover, if $\mathcal{A}_\infty \neq \emptyset$, then there exists a diverging sequence $\{L_m\} \subset \mathbb{R}_+$ such that $\ell_\beta(f_m) \geq L_m$ for all $\beta \in \mathcal{A}_\infty$. Hence, $t_\bullet(f_m) \to t_\bullet(f_\infty)$ uniformly.*

(b) *Assume $\mathcal{L}(f_\infty) = (\underline{\tilde{p}}, 0) \in \Delta^{n-1} \times \{0\}$ and let $\mathcal{A}(S) = \mathcal{A}_{fin} \sqcup \mathcal{A}_\infty$, where $\mathcal{A}_\infty$ is the subset of arcs β such that $\tilde{\ell}_\beta(f_\infty) = \infty$. Then $\tilde{\ell}_\alpha(f_m)/\tilde{\ell}_\alpha(f_\infty) \to 1$ uniformly for all $\alpha \in \mathcal{A}_{fin}$. Moreover, if $\mathcal{A}_\infty \neq \emptyset$, then there exists a diverging sequence $\{\tilde{L}_m\} \subset \mathbb{R}_+$ such that $\tilde{\ell}_\beta(f_m) \geq \tilde{L}_m$ for all $\beta \in \mathcal{A}_\infty$. Hence, $[t_\bullet(f_m)] \to [t_\bullet(f_\infty)]$.*

Remark 3.19. *A simple computation shows that a hypercycle at distance d from a closed geodesic of length ℓ has length $\ell \cosh(d)$. In case (b), we can assume that $\boldsymbol{\mathcal{L}}(f_m) \leq 1$ and so we can define $\vartheta_m \in [0, \pi/2]$ by $\sin(\vartheta_m) := \boldsymbol{\mathcal{L}}(f_m)$. For each boundary circle $C_{i,m}$ of Σ_m, let $B_{i,m} \subset \Sigma_m$ be the hypercycle parallel to $C_{i,m}$ at distance $d_m = -\log\tan(\vartheta_m/2)$ (i.e. $\cosh(d_m) = 1/\sin(\vartheta_m)$), which has length $\tilde{p}_i(f_m) := p_i(f_m)/\sin(\vartheta_m) \leq 1$ and so is embedded in Σ_m. Notice that the spine of Σ_m coincides with the spine of its subsurface Σ^{tr}_m obtained by removing the hyperballs bounded by the $B_{i,m}$'s: in fact, every geodesic that meets $C_{i,m}$ orthogonally also intersects $B_{i,m}$ orthogonally. For every arc α, define the reduced length of α at f_m to be $\tilde{\ell}_\alpha(f_m) := \ell_\alpha(f_m) - 2d_m$, namely the length of $f_m(\alpha) \cap \Sigma^{tr}_m$. Extending a definition of Penner's* [Pen87] *(and modifying it by a factor $\sqrt{2}$), we put $\lambda_\alpha(f_m) := \exp(\tilde{\ell}_\alpha/2)$. Because $B_{i,m}$ limits to a horoball of circumference $\tilde{p}_i = \tilde{p}_i(f_\infty)$ as $m \to \infty$, the length $\lambda_\alpha(f_m) \to \lambda_\alpha(f_\infty, \underline{p})$.*

Notation. *In the following proof, we will denote by $S_{\infty,+}$ the open subsurface $f_\infty^{-1}(\Sigma_{\infty,+})$ and by $S^{tr}_{\infty,+}$ the preimage through f_∞ of the analogous truncated surface. Similar notation for $S_{\infty,-}$.*

Proof of Lemma 3.18. About (a), if $\mathcal{A}_\infty \neq \emptyset$, then $\exists \gamma_1, \dots, \gamma_l \in \mathcal{C}(S)$ disjoint such that $c_m = \max_h \ell_{\gamma_h}(f_m) \to 0$ as $m \to \infty$. Clearly, $\beta \in \mathcal{A}_\infty \iff i(\beta, \gamma_1 + \dots + \gamma_l) > 0$.

By the collar lemma, $\ell_\beta(f_m) > L_m := T(c_m)/2$ and so $t_\beta(f_m) < T(L_m) \to 0$ for all $\beta \in \mathcal{A}_\infty$. On the other hand, by Proposition 2.15, we can assume that $f_m^*(g_m) \to f_\infty^*(g_\infty)$ in $L^\infty_{loc}(S_{\infty,+})$. Thus, $\dfrac{|\ell_\alpha(f_m) - \ell_\alpha(f_\infty)|}{\ell_\alpha(f_\infty)} \to 0$ uniformly for all $\alpha \in \mathcal{A}_{fin}$.

Fix $\varepsilon > 0$ and let $\alpha_1, \dots, \alpha_k \in \mathcal{A}_{fin}$ be the arcs such that $\ell_{\alpha_i}(f_\infty) \leq T(\varepsilon)/(1-\varepsilon)$ for $i = 1, \dots, k$. Clearly, $t_{\alpha_i}(f_m) \to t_{\alpha_i}(f_\infty)$. If $\alpha \in \mathcal{A}_{fin}$ and $\alpha \notin \{\alpha_1, \dots, \alpha_k\}$, then $\ell_\alpha(f_m) \geq \ell_\alpha(f_\infty)(1-\varepsilon) > T(\varepsilon)$ and so $t_\alpha(f_m) < \varepsilon$ for m large. Hence, $|t_\alpha(f_m) - t_\alpha(f_\infty)| < \varepsilon$ for m large and $\alpha \in \mathcal{A}_{fin} \setminus \{\alpha_1, \dots, \alpha_k\}$.

The proof of (b) is similar. Call $\gamma_1, \dots, \gamma_l$ the curves in the interior of S that are shrunk to nodes of Σ_∞ and let $J = \{j \,|\, \tilde{p}_j = 0\}$. We can assume that $p_i(f_m) < \tilde{p}_i(f_\infty)$.

Let $c_m = \max\{\ell_{\gamma_h}(f_m)\}$ and $c'_m = \max\{p_j \,|\, j \in J\}$. Clearly, if $\beta \in \mathcal{A}_\infty$ intersects some γ_h, then $\tilde{\ell}_\beta(f_m) \geq T(c_m)/2 \to \infty$. If $\beta \in \mathcal{A}_\infty$ does not intersect any γ_j, then it starts at some C_j with $j \in J$. Because of the collar lemma, there is a hypercycle embedded in Σ_m at distance $\delta_{i,m}$ from $f_m(C_i)$, with $p_i(f_m)\cosh(\delta_{i,m}) = 1$. As $p_i(f_m)\cosh(d_m) = p_i(f_m)/\sin(\vartheta_m)$, we get $\cosh(\delta_{i,m})/\cosh(d_m) = \sin(\vartheta_m)/p_i(f_m)$ and so $\delta_{j,m} - d_m \approx \log(\sin(\vartheta_m)/p_j(f_m)) \geq \log(\sin(\vartheta_m)/c'_m) \to \infty$ for $j \in J$. Hence, $\tilde{\ell}_\beta(f_m) \geq \tilde{L}_m := \min\{T(c_m)/2, \log(\sin(\vartheta_m)/c'_m)\} \to \infty$.

As before, we can assume that $f_m^*(g_m)$ converges uniformly over the compact subsets of $S^{tr}_{\infty,+}$, so that $\dfrac{|\tilde{\ell}_\alpha(f_m)-\tilde{\ell}_\alpha(f_\infty)|}{\tilde{\ell}_\alpha(f_\infty)} \to 0$ uniformly for all $\alpha \in \mathcal{A}_{fin}$.

Call $\alpha_0 \in \mathcal{A}_{fin}$ the arc with smallest $\tilde{\ell}_{\alpha_0}(f_\infty)$. Fix $\varepsilon > 0$ and let $\alpha_1, \dots, \alpha_k \in \mathcal{A}_{fin}$ be the arcs such that $\tilde{\ell}_{\alpha_i}(f_\infty) \leq \tilde{\ell}_{\alpha_0}(f_\infty) - 2\log(\varepsilon)$ for $i = 1, \dots, k$. Clearly, $\dfrac{t_{\alpha_i}(f_m)}{t_{\alpha_0}(f_m)} \to \dfrac{t_{\alpha_i}(f_\infty)}{t_{\alpha_0}(f_\infty)} = \dfrac{\lambda_{\alpha_0}(f_\infty)}{\lambda_{\alpha_i}(f_\infty)}$ for $i = 1, \dots, k$.

If $\alpha \in \mathcal{A}_{fin}$ and $\alpha \notin \{\alpha_1, \dots, \alpha_k\}$, then $\dfrac{t_{\alpha_i}(f_m)}{t_{\alpha_0}(f_m)} \leq \varepsilon + \sqrt{\exp[\tilde{\ell}_{\alpha_0}(f_m) - \tilde{\ell}_{\alpha_i}(f_m)]} < 2\varepsilon$ for m large. Hence, $\left|\dfrac{t_{\alpha_i}(f_m)}{t_{\alpha_0}(f_\infty)}\right| < 2\varepsilon$ for m large and $\alpha \in \mathcal{A}_{fin} \setminus \{\alpha_1, \dots, \alpha_k\}$. □

Proof of Theorem 3.16. The continuity of $\widehat{\boldsymbol{W}}$ is dealt with in Lemma 3.20 below. In order to prove that $\widehat{\boldsymbol{W}}$ is a homeomorphism, it is sufficient to show that the induced map below is.

$$\widehat{\boldsymbol{W}}' : \widehat{\mathcal{M}}^{vis}(S) \longrightarrow (|\mathfrak{A}(S)|/\Gamma(S)) \times [0, \infty)$$

where $\widehat{\mathcal{M}}^{vis}(S) = \widehat{\mathcal{T}}^{vis}(S)/\Gamma(S)$.

In fact, we first endow $\overline{\mathcal{T}}^{WP}(S)$ with a $\Gamma(S)$-equivariant metric, for example pulling it back from $\overline{\mathcal{T}}^{WP}(S) \to \overline{\mathcal{M}}(S)$. This way, we induce a metric on the quotient $\overline{\mathcal{T}}^{WP}(S) \times \Delta^{n-1}/\!\sim_{vis}$, where Δ^{n-1} has the Euclidean metric. Finally, we embed $\widehat{\mathcal{T}}^{vis}(S)$ inside $\overline{\mathcal{T}}^{WP}(S) \times \Delta^{n-1}/\!\sim_{vis}$ (where the second component of the map is given by the normalized boundary lengths), thus obtaining a $\Gamma(S)$-equivariant metric on $\widehat{\mathcal{T}}^{vis}(S)$.

Then, $\widehat{\mathcal{T}}^{vis}(S)$ and $|\mathfrak{A}(S)|$ are metric spaces and $\Gamma(S)$ acts on both by isometries. Moreover, the action on $|\mathfrak{A}(S)|$ is simplicial on the second baricentric subdivision, and so its orbits are discrete.

On the other hand, the map $\widehat{\boldsymbol{W}}'$ is clearly proper, because $\widehat{\mathcal{T}}^{vis}(S)(\underline{p})/\Gamma(S)$ is compact for every $\underline{p} \in \Delta^{n-1} \times [0, \infty)$. As the image of $\widehat{\boldsymbol{W}}'$ contains the dense open subset $(|\mathfrak{A}^\circ(S)|/\Gamma(S)) \times (0, \infty)$, we have that $\widehat{\boldsymbol{W}}'$ is a homeomorphism. By Lemma 3.21(b), $\widehat{\boldsymbol{W}}$ is a homeomorphism too. □

Lemma 3.20. $\widehat{\boldsymbol{W}}$ *is continuous.*

Proof. Notice that $\widehat{\mathcal{T}}^{vis}(S)$ and $|\mathfrak{A}(S)| \times [0, \infty)$ have countable systems of neighbourhoods at each point. As $\mathcal{T}(S)$ is dense in $\widehat{\mathcal{T}}^{vis}(S)$, in order to test the continuity of $\widehat{\boldsymbol{W}}$, we can consider a sequence $[f_m : S \to \Sigma_m] \subset \mathcal{T}(S)$ converging to a point $[f_\infty : S \to \Sigma_\infty] \in \widehat{\mathcal{T}}^{vis}(S) \setminus \widehat{\mathcal{T}}^{vis}(S)(0)$ (the case of $[f_\infty] \in \widehat{\mathcal{T}}^{vis}(S)(0)$ will be treated later).

Step 1. Because of Proposition 2.15, there are representatives $f_1, f_2, \dots, f_\infty$ such that the hyperbolic metric $f_m^*(g_m) \to f_\infty^*(g_\infty)$ in $L^\infty_{loc}(S_{\infty,+})$. Also, the distance function $d_{f_m}(-, \partial S_{\infty,+}) : S_{\infty,+} \to \mathbb{R}_+$ with respect to the metric $f_m^* g_m$ converges in $L^\infty_{loc}(S_{\infty,+})$.

Step 2. Let $\mathcal{E}$ be the set of edges of $\mathrm{Sp}(\Sigma_\infty)$ and let m_i be the midpoint of the edge $E_i \in \mathcal{E}$ in Σ_∞. Call $\gamma_{i,1}$ and $\gamma_{i,2}$ the shortest geodesics that join m_i to $\partial\Sigma_\infty$ and $\alpha_i := f_\infty^{-1}(\gamma_{i,1}^{-1} * \gamma_{i,2})$ the associated arc. Let $d(m_i, \partial\Sigma_\infty) = \ell(\gamma_{E,1}) = \ell(\gamma_{E,2})$ and call $d'(m_i, \partial\Sigma_\infty)$ to be the minimum length of a geodesics that join m_i to $\partial\Sigma_\infty$ and is not homotopic to $\gamma_{i,1}$ or $\gamma_{i,2}$. Finally, set $\varepsilon = \min\{d'(m_i, \partial\Sigma_\infty) - d(m_i, \partial\Sigma_\infty) \,|\, E_i \in \mathcal{E}\} > 0$.

Because $d_{f_m}(f_\infty^{-1}(m_i), \partial S_{\infty,+})$ and $d'_{f_m}(f_\infty^{-1}(m_i), \partial S_{\infty,+})$ also converge as $m \to \infty$, their difference is eventually positive and so the arc α_i (up to isotopy) is still dual to some edge of the spines of $(S, f_m^*(g_m))$ (which is equal to $f_m^{-1}(\mathrm{Sp}(\Sigma_m))$) for m large.

Thus, up to discarding finitely many terms of the sequence, we can assume that $f_\infty^* \boldsymbol{A}_{sp}(\Sigma_\infty) \subseteq f_m^* \boldsymbol{A}_{sp}(\Sigma_m)$.

Step 3. Let $\boldsymbol{A}_\infty$ be the system of arcs $f_\infty^* \boldsymbol{A}_{sp}(\Sigma_\infty)$ on S. Consider the subset $\widetilde{\mathfrak{St}}(\boldsymbol{A}_\infty) \subset \mathfrak{A}(S)$ of systems $\boldsymbol{A}$ that contain $\boldsymbol{A}_\infty$ and such that $f(\boldsymbol{A})$ can be represented inside $\Sigma_{\infty,+}$. Let $\boldsymbol{A}_1, \dots, \boldsymbol{A}_k$ the maximal elements of $\widetilde{\mathfrak{St}}(\boldsymbol{A}_\infty)$ and let $\widetilde{\mathfrak{St}}_i = \{\boldsymbol{A}_{i,r} \,|\, r \in R_i\}$ the set of maximal systems of arcs $\boldsymbol{A}_{i,r} \supseteq \boldsymbol{A}_i$ for $i = 1, \dots, k$.

Step 4. Clearly $\exists i_m \in \{1,\dots,k\}$ and $\exists r_m \in R_{i_m}$ such that $f_m^*\boldsymbol{A}_{sp}(\Sigma_m) \subseteq \boldsymbol{A}_{i_m,r_m}$ (and there are finitely many options for each m). We need to show that

$$\max\{|w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) - w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_\infty)| : \alpha \in \boldsymbol{A}_{i_m,r_m} \supseteq f_m^*\boldsymbol{A}_{sp}(\Sigma_m)\} \to 0$$

as $m \to \infty$.

Step 5. By Lemma 3.18(a), $\ell_\beta(f_m) \geq L_m$ equidiverge and $w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\beta}, f_m) \leq t_\beta(f_m)/2$ uniformly converge to zero, for all $\beta \in \mathcal{A}_\infty$.

Step 6. $\boldsymbol{A}_1 \cup \dots \cup \boldsymbol{A}_k$ is finite and the lengths $\ell_\alpha(f_m) \to \ell_\alpha(f_\infty) < \infty$ for every α in some $\boldsymbol{A}_i$. Define $M(\alpha) = \{m \,|\, \alpha \in f_m^*\boldsymbol{A}_{sp}(\Sigma_m)\}$. It is sufficient to prove that, if $M(\alpha)$ is infinite, then $|w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) - w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_\infty)| \to 0$ as $m \in M(\alpha)$ diverges. There are three cases.

Case 6(a). Let $H_m \subset S$ be a right-angled hexagon bounded by $(\overrightarrow{\alpha}, \overrightarrow{\alpha'_m}, \overrightarrow{\alpha''_m})$, with $\alpha'_m, \alpha''_m \in \boldsymbol{A}_{i_m}$ for suitable i_m. Then

$$\sinh\left(w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m)\right) = \frac{s_{\alpha'_m}(f_m)^2 + s_{\alpha''_m}(f_m)^2 - s_\alpha(f_m)^2}{2s_{\alpha'_m}(f_m)s_{\alpha''_m}(f_m)\sqrt{s_\alpha(f_m)^2 - 1}}$$

and so $|w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) - w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_\infty)| \to 0$.

Case 6(b). Suppose that there are hexagons $H_m \subset S$ with $\partial H_m = \overrightarrow{\alpha} * \overrightarrow{\alpha'_m} * \overrightarrow{\beta_m}$, where $\alpha'_m \in \boldsymbol{A}_{i_m}$ and $\beta_m \in \boldsymbol{A}_{i_m,r_m} \setminus \boldsymbol{A}_{i_m}$. We can extract a subsequence so that α'_m is a fixed arc α'. The divergence of b_m and the formula

$$\cosh(d(\overrightarrow{\alpha}, \overleftarrow{\alpha'})) = \frac{\cosh(a_m)\cosh(a'_m) + \cosh(b_m)}{\sinh(a_m)\sinh(a'_m)}$$

(where a_m, a'_m, b_m are the lengths of α, α', β_m at $[f_m]$) imply that $d(\overrightarrow{\alpha}, \overleftarrow{\alpha'})$ diverges, which contradicts the fact that the boundary lengths are bounded.

Case 6(c). Let $H_m \subset S$ be a right-angled hexagon bounded by $(\overrightarrow{\alpha}, \overrightarrow{\beta_m}, \overrightarrow{\beta'_m})$, with $\beta_m, \beta'_m \in \boldsymbol{A}_{i_m,r_m} \setminus \boldsymbol{A}_{i_m}$. Call $x_{\alpha,m}, x_{\beta,m}, x_{\beta',m}$ the length of the edges of H_m opposed to $\alpha, \beta_m, \beta'_m$ and let a_m, b_m, b'_m be the lengths of $\alpha, \beta_m, \beta'_m$ at $[f_m]$. Remember that $w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\beta_m}, f_m)$ and $w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\beta'_m}, f_m)$ converge to zero, whereas $w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m)$ is bounded (and so are $x_{\beta,m}$ and $x_{\beta',m}$). Notice that $x_{\beta,m} - w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m)$ converges to zero and so do the differences $\cosh(x_{\beta,m}) - \cosh(w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m))$ and $\cosh(x_{\beta,m}) - \cosh(x_{\beta',m})$. But

$$\cosh(x_{\beta,m}) = \frac{\cosh(b'_m)\cosh(a_m) + \cosh(b_m)}{\sinh(b'_m)\sinh(a_m)} = \frac{1}{\tanh(a_m)\tanh(b'_m)} + \frac{\cosh(b_m)}{\sinh(a_m)\sinh(b'_m)}$$

and similarly for $x_{\beta'_m}$, so that we obtain that

$$\cosh(x_{\beta,m}) - \cosh(x_{\beta',m}) \approx \frac{e^{b_m - b'_m} - e^{b'_m - b_m}}{\sinh(a_m)} = \frac{2\sinh(b_m - b'_m)}{\sinh(a_m)} \to 0$$

which implies that $|b_m - b'_m| \to 0$, because $a_m \to a_\infty \in (0, \infty)$.

Consequently, $\cosh(w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m)) \to \dfrac{1}{\tanh(a_\infty)} + \dfrac{1}{\sinh(a_\infty)} = \dfrac{1}{\tanh(a_\infty/2)}$, which gives $w_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) \to t_\alpha(f_\infty)/2 = w_{\boldsymbol{A}_\infty}(\overrightarrow{\alpha}, f_\infty)$.

Case of decorated surfaces.

Suppose now that $[f_\infty, \underline{p}] \in \widehat{\mathcal{T}}^{vis}(S)(0)$. We use the notation in Remark 3.19. Notice that

$$(*) \qquad \lambda_\alpha(f_m) = e^{\ell_\alpha/2 - d_m} = e^{-d_m}\left(s_\alpha(f_m) + \sqrt{s_\alpha(f_m)^2 - 1}\right) =$$
$$= \tan(\vartheta_m/2)\left(s_\alpha(f_m) + \sqrt{s_\alpha(f_m)^2 - 1}\right)$$

The normalized widths $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}} = 2w_{\boldsymbol{A}_{i_m,r_m}}/\sin(\vartheta_m)$ limit to Penner's simplicial coordinates (see below the modifications to step (6)). So the map $\widehat{\boldsymbol{W}}$ reduces to Penner's map for cusped surfaces, in which case we will still use the term "normalized widths" (instead of "simplicial coordinates") for brevity.

We follow the same path as before, with some modifications.

Step 1. As $\tilde{\boldsymbol{p}} = 1$, we can assume that $p_i(f_m) < \tilde{p}_i(f_m)$. Let Σ_∞^{tr} be the truncated surface as in the proof of Lemma 3.18(b). By Proposition 2.15, we can assume that $f_m(S^{tr}) = \Sigma_m^{tr}$ and that the metrics $f_m^*(g_m)$ converge in $L^\infty_{loc}(S^{tr}_{\infty,+})$.

Step 2. Essentially the same, up to replacing the distance from $\partial\Sigma_m$ by the distance from $\partial\Sigma_m^{tr}$.

Step 3. Identical.

Step 4. Now on, we have to replace the widths by the normalized widths $\tilde{w}$. Notice that $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) \leq t_\alpha(f_m)/\sin(\vartheta_m) \approx 2\exp(-\tilde{\ell}_\alpha/2)$ for all $\alpha \in \boldsymbol{A}_{i_m,r_m}$.

Step 5. Similar: by Lemma 3.18(b), $\tilde{\ell}_\beta(f_m) \geq \tilde{L}_m$ equidiverge and $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\beta}, f_m) \to 0$ uniformly, for all $\beta \in \mathcal{A}_\infty$.

Step 6. It follows from (1) that, as $m \to \infty$, for all $\alpha \in \boldsymbol{A}_1 \cup \dots \cup \boldsymbol{A}_k$ we have $\lambda_\alpha(f_\infty) < \infty$ and $|\lambda_\alpha(f_m) - \lambda_\alpha(f_\infty)| \to 0$.

It follows from $(*)$ that $\lambda_\alpha(f_m) \sim \vartheta_m s_\alpha(f_m) + O(\vartheta_m^3 s_\alpha(f_m))$. Hence, as $m \to \infty$, for all these α, we also have $|\lambda_\alpha(f_m) - \vartheta_m s_\alpha(f_m)| \to 0$. On the other hand, for all β belonging to some $\boldsymbol{A}_{i_m,r_m} \setminus \boldsymbol{A}_{i_m}$, we have $\lambda_\beta(f_\infty) = \infty$ and $\lambda_\beta(f_m) \sim \vartheta_m s_\beta(f_m)$ equidiverge.

Case 6(a). $|\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) - \tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_\infty)| \to 0$ and $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) \to X_{\boldsymbol{A}_\infty}(\overrightarrow{\alpha}, f_m)$, where $X_{\boldsymbol{A}_\infty}(\overrightarrow{\alpha}, f_\infty) = \dfrac{\lambda_{\alpha_i}(f_\infty)^2 + \lambda_{\alpha_j}(f_\infty)^2 - \lambda_\alpha(f_\infty)^2}{\lambda_{\alpha_i}(f_\infty)\lambda_{\alpha_j}(f_\infty)\lambda_\alpha(f_\infty)}$ is Penner's simplicial coordinate of $\overrightarrow{\alpha}$.

Case 6(b). β_m cannot cross a simple closed (nonboundary) curve of S that is contracted to a node by f_∞, because so it would either α or α': this would contradict the boundedness of $\tilde{\ell}_\alpha(f_m)$ and $\tilde{\ell}_{\alpha'}(f_m)$.

Case 6(c). Because $\cosh(x_{\alpha,m}) \approx 1 + x_{\alpha,m}^2/2$ and

$$\cosh(x_{\alpha,m}) = \frac{\cosh(b_m)\cosh(b'_m) + \cosh(a_m)}{\sinh(b_m)\sinh(b'_m)} \approx 1 + 2\exp(a_m - b_m - b'_m) + \\ + 2\exp(-2b_m) + 2\exp(-2b'_m)$$

we obtain that $x_{\alpha,m}^2/\vartheta_m^2 \approx \exp(\tilde{a}_m - \tilde{b}_m - \tilde{b}'_m) + O(\vartheta_m^2) \to 0$ as $m \to \infty$.

Remember that $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\beta}_m, f_m)$ and $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\beta}'_m, f_m)$ converge to zero uniformly and that $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m)$ is bounded (and so are $x_{\beta,m}/\sin(\vartheta_m)$ and $x_{\beta',m}/\sin(\vartheta_m)$).

On the other hand, $\cosh(x_{\beta',m}) \approx 1 + 2\exp(b'_m - b_m - a_m) + 2\exp(-2a_m) + 2\exp(-2b_m)$ and so $x_{\beta',m}^2 \approx 4\exp(b'_m - b_m - a_m) + 4\exp(-2b_m) + 4\exp(-2a_m)$ and $\dfrac{x_{\beta',m}^2}{\sin^2(\vartheta_m)} \approx \exp(\tilde{b}'_m - \tilde{b}_m - \tilde{a}_m) + O(\vartheta_m^2)$.

On the other hand, $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) \approx \dfrac{2x_{\beta,m}}{\sin(\vartheta_m)} \approx 2\exp\left(\dfrac{\tilde{b}'_m - \tilde{b}_m - \tilde{a}_m}{2}\right) + O(\vartheta_m)$ and an analogous estimate holds switching the roles of $\tilde{b}_m$ and $\tilde{b}'_m$. This implies that $|\tilde{b}'_m - \tilde{b}_m| \to 0$.

As a consequence, $\tilde{w}_{\boldsymbol{A}_{i_m,r_m}}(\overrightarrow{\alpha}, f_m) \approx 2\exp(-\tilde{a}_m/2) \to 2\exp(-\tilde{a}_\infty/2) = 2/\lambda_\alpha(f_\infty) = X_{\boldsymbol{A}_\infty}(\overrightarrow{\alpha}, f_\infty)$. □

Lemma 3.21. *Let $f : X \to Y$ be a G-equivariant continuous map between metric spaces on which the discrete group G acts by isometries. Assume that the G-orbits on Y are discrete.*

(a) *If X/G is compact and* $\mathrm{stab}(x) \subseteq \mathrm{stab}(f(x))$ *has finite index for all $x \in X$, then f is proper.*

(b) *If f is injective and the induced map $\overline{f} : X/G \to Y/G$ is a homeomorphism, then f is a homeomorphism.*

Proof. For (a) we argue by contradiction: let $\{x_n\} \subset X$ be a diverging subsequence such that $\{f(x_n)\} \subset Y$ is not diverging. Up to extracting a subsequence, we can assume that $f(x_n) \to y \in Y$ and that $[x_n] \to [x] \in X/G$. Thus $\exists g_n \in G$ such that $x_n \cdot g_n \to x$, that is $d_X(x_n, x \cdot g_n^{-1}) \to 0$. As $\{x_n\}$ is divergent, the sequence $\{[g_n]\} \subset G/\mathrm{stab}(x)$ is divergent too, and so is it in $G/\mathrm{stab}(f(x))$. Because $f(x_n \cdot g_n) \to f(x)$, we have $d_Y(f(x_n), f(x) \cdot g_n^{-1}) \to 0$ and so $\{f(x_n)\}$ is divergent, because $f(x) \cdot G$ is discrete.

For (b), let's show first that f is surjective. Because $\overline{f}$ is bijective, for every $y \in Y$ there exists a unique $[x] \in X/G$ such that $\overline{f}([x]) = [y]$. Hence, $f(x) = y \cdot g$ for some $g \in G$ and so $f(x \cdot g^{-1}) = y$.

The injectivity of f also implies that $\mathrm{stab}(x) = \mathrm{stab}(f(x))$ for all $x \in X$.

To prove that f^{-1} is continuous, let $(x_m) \subset X$ be a sequence such that $f(x_m) \to f(x)$ as $m \to \infty$ for some $x \in X$. Clearly, $[f(x_m)] \to [f(x)]$ in Y/G and so $[x_m] \to [x]$ in X/G, because $\overline{f}$ is a homeomorphism.

Consider the balls $U_k = B_X(x, 1/k)$ for $k > 0$ and set $U_0 = X$, so that $[U_k]$ is an open neighbourhood of $[x] \in X/G$. There exists an increasing sequence $\{m_k\}$ such that $[x_m] \in [U_k]$, that is $x_m \in \bigcup_{g \in G} U_k \cdot g$ for all $m \geq m_k$. Consequently, there is a $g_m \in G$ such that $x_m \in U_k \cdot g_m$ for every $m_k \leq m < m_{k+1}$. Thus, $z_m := x_m \cdot g_m^{-1} \to x$. By continuity of f, we have $f(z_m) \longrightarrow f(x)$ and by hypothesis $f(z_m) \cdot g_m \to f(x)$. Moreover, $d_Y(f(x) \cdot g_m, f(x)) \leq d_Y(f(x) \cdot g_m, f(x_m)) + d_Y(f(x_m), f(x)) = d_Y(f(x), f(z_m)) + d_Y(f(x_m), f(x)) \to 0$ and so $f(x) \cdot g_m \to f(x)$. Hence, $g_m \in \mathrm{stab}(f(x)) = \mathrm{stab}(x)$ for large m, because G acts with discrete orbits on Y. As a consequence, for m large enough $d_X(x_m, x) = d_X(z_m, x) \to 0$ and so $x_m \to x$ and f^{-1} is continuous at $f(x)$. □

Remark 3.22. *In order to check that the G-orbits on Y are discrete, it is sufficient to show the following:*

(∗) *whenever $y \cdot g_m \to y$ for a certain $y \in Y$ and $\{g_m\} \subset G$, the sequence $\{g_m\}$ definitely belongs to $\mathrm{stab}_G(y)$.*

Assuming (∗), there is an $\varepsilon > 0$ and a ball $B = B(z, \varepsilon)$ such that $B \cap z \cdot G = \{z\}$. Given a sequence $\{g_m\} \subset G$ such that $y \cdot g_m \to z \in Y$, then $d(z \cdot g_j^{-1} g_i, z) \leq d(z \cdot g_j^{-1}, y) + d(y, z \cdot g_i^{-1}) = d(z, y \cdot g_j) + d(y \cdot g_i, z) < \varepsilon$ for $i, j \geq N_\varepsilon$. Thus, $g_j^{-1} g_i \in \mathrm{stab}_G(z)$ and $d(y \cdot g_j, z) = d(y \cdot g_j g_j^{-1} g_i, z) = d(y \cdot g_i, z)$ for all $i, j \geq N_\varepsilon$. Hence, $y \cdot g_i = z$ for all $i \geq N_\varepsilon$ and so the orbit is discrete.

3.5. **The bordification of arcs.**

Define a map

$$\Phi : |\mathfrak{A}(S)| \times [0, \infty] \longrightarrow \overline{\mathcal{T}}^a(S)$$

in the following way:

$$\Phi(w, \boldsymbol{p}) = \begin{cases} ([\lambda_\bullet^{-1}(\widehat{\boldsymbol{W}}^{-1}(w, 0))], 0) & \text{if } \boldsymbol{p} = 0 \\ j(\widehat{\boldsymbol{W}}^{-1}(w, \boldsymbol{p})) & \text{if } 0 < \boldsymbol{p} < \infty \\ ([w], \infty) & \text{if } \boldsymbol{p} = \infty. \end{cases}$$

The situation is thus as in the following diagram.

$$\begin{tikzcd} |\mathfrak{A}(S)| \times [0, \infty) \arrow[d, hook] & \widehat{\mathcal{T}}^{vis}(S) \arrow[l, "\widehat{\boldsymbol{W}}"', "\cong"] \arrow[d, hook, "\hat{j}"] \\ |\mathfrak{A}(S)| \times [0, \infty] \arrow[r, "\Phi"] & \overline{\mathcal{T}}^a(S) \end{tikzcd}$$

Theorem 3.23. *Φ is a $\Gamma(S)$-equivariant homeomorphism. Thus, $\overline{\mathcal{M}}^a(S) = \overline{\mathcal{T}}^a(S)/\Gamma(S)$ is compact.*

For homogeneity of notation, we will call $\overline{\boldsymbol{W}}^a := \Phi^{-1} : \overline{\mathcal{T}}^a(S) \to |\mathfrak{A}(S)| \times [0, \infty]$.

In order to prove Theorem 3.23, we need a few preliminary results.

Proposition 3.24. *The map $\mathcal{T}(S) \hookrightarrow \overline{\mathcal{T}}^a(S)$ extends to a continuous $\hat{j} : \widehat{\mathcal{T}}^{vis}(S) \hookrightarrow \overline{\mathcal{T}}^a(S)$.*

Proof. The continuity of $\hat{j}$ follows from Lemma 3.18. Moreover, Lemma 3.4 and Lemma 3.12 assure that the t-lengths separate the points of $\widehat{\mathcal{T}}^{vis}(S)$ and so $\hat{j}$ is injective. □

Lemma 3.25. *Let $[f_m : S \to \Sigma_m]$ be a sequence in $\mathcal{T}(S)$.*

(a) $\|t_\bullet(f_m)\|_\infty \to 0$ *if and only if* $\boldsymbol{\mathcal{L}}(f_m) \to 0$
(b) $\|t_\bullet(f_m)\|_\infty \to \infty$ *if and only if* $\boldsymbol{\mathcal{L}}(f_m) \to \infty$
(c) $\exists M > 0$ *such that* $1/M \leq \|t_\bullet(f_m)\|_\infty \leq M$ *if and only if* $\exists M' > 0$ *such that* $1/M' \leq \boldsymbol{\mathcal{L}}(f_m) \leq M'$.

Proof. Because $w_{sp}(\alpha, f_m) \leq t_\alpha(f_m)$ for $\alpha \in \boldsymbol{A}_{sp}(f_m)$, we conclude

$$(6g-6+3n)\|t_\bullet(f_m)\|_\infty \geq 2\boldsymbol{\mathcal{L}}(f_m)$$

By the collar lemma, $\ell_\alpha(f_m) \geq T(\boldsymbol{\mathcal{L}}(f_m))/2$ for all $\alpha \in \mathcal{A}(S)$ and so $\|t_\bullet(f_m)\| \leq T(T(\boldsymbol{\mathcal{L}}(f_m))/2)$. □

Lemma 3.26. *The map Φ is continuous and injective.*

Proof. Injectivity of Φ is immediate.

As we already know that $\hat{j}$ is continuous, consider a sequence $\{f_m : S \to \Sigma_m\} \subset \mathcal{T}(S)$ such that $\boldsymbol{W}(f_m) \to w \in |\mathfrak{A}(S)| \times \{\infty\}$, where $\boldsymbol{A} := \mathrm{supp}(w) = \{\alpha_0, \dots, \alpha_k\}$, and assume that $w(\alpha_0) \geq w(\alpha_i)$ for every $1 \leq i \leq k$. Thus,

$$\sup_{\beta \in \boldsymbol{A}_{sp}(f_m)\setminus \boldsymbol{A}} \frac{w_{sp}(\beta, f_m)}{w_{sp}(\alpha_0, f_m)} \to 0$$

We want to show that $j(f_m) \to ([w], \infty)$; equivalently, that for every subsequence of (f_m) (which we will still denote by (f_m)) we can extract a further subsequence that converges to $([w], \infty)$.

Because of Equation 1 (applied to any maximal system of arcs $\boldsymbol{A}'$ that contains $\boldsymbol{A}$), $\ell_{\alpha_i}(f_m) \to 0$ for all $\alpha_i \in \boldsymbol{A}$.

The collar lemma ensures that $\exists \delta > 0$ such that two simple closed geodesics of length $\leq \delta$ in a closed hyperbolic surface cannot intersect each other. Thus, $\boldsymbol{A} \subseteq \boldsymbol{A}_{sp}(f_m)$ for m large.

Claim: for all $\beta \notin \boldsymbol{A}$, the ratio $t_\beta(f_m)/t_{\alpha_0}(f_m) \to 0$ uniformly.

By contradiction, suppose $\exists \eta > 0$ and $\{\beta_m\} \subset \mathcal{A}(S) \setminus \boldsymbol{A}$ such that $t_{\beta_m}(f_m)/t_{\alpha_0}(f_m) \geq \eta$. Thus, $\ell_{\beta_m}(f_m) \to 0$ and $\beta_m \in \boldsymbol{A}_{sp}(f_m)$. By Equation 1,

$$\sinh(w_{sp}(\overrightarrow{\beta_m}, f_m)) = \frac{s_x^2 + s_y^2 - s_{\beta_m}^2}{2 s_x s_y \sqrt{s_{\beta_m}^2 - 1}} \approx \frac{s_x^2 + s_y^2 - 1}{s_x s_y \ell_{\beta_m}} \geq \frac{1}{2\ell_{\beta_m}}$$

Thus, asymptotically $w_{sp}(\beta_m, f_m) \geq 2\log(1/\ell_{\beta_m}(f_m)) \approx t_{\beta_m}(f_m)$. As $w_{sp}(\alpha_0, f_m) \approx t_{\alpha_0}(f_m)$, we conclude that $w_{sp}(\beta_m, f_m)/w_{sp}(\alpha_0, f_m) \geq \eta/2$ for m large. This contradiction proves the claim.

Given a small $\varepsilon > 0$, we pick a $\delta > 0$ as above such that $\cosh(\delta/2)^2 < 1 + 2\varepsilon$. If $\liminf \ell_\beta(f_m) < \delta$ for $\beta \notin \boldsymbol{A}$, then we can extract a subsequence of (f_m) such that $\ell_\beta(f_m) < \delta$ for large m. Again, we can assume that β belongs to $\boldsymbol{A}_{sp}(f_m)$ for large m and so also to $\boldsymbol{A}$. Clearly, we can only add a finite number of β's to $\boldsymbol{A}$ and so we extract a subsequence only a finite number of times.

Again up to subsequences, we can assume that for every $\alpha \in \boldsymbol{A}$ either: $\ell_\alpha(f_m) \in (\delta', \delta)$ or $\ell_\alpha(f_m) \to 0$. If $\ell_\alpha(f_m) \in (\delta', \delta)$, then clearly $t_\alpha(f_m)/t_{\alpha_0}(f_m) \to 0$. Instead, if $\lim \ell_\alpha(f_m) \to 0$, then Equation 1 gives

$$\cos(\gamma(\overrightarrow{\alpha}, f_m)) \geq 1 - s_\alpha^2/2 \geq 1/2 - \varepsilon$$

for large m. This implies that

$$w_{sp}(\alpha, f_m) \approx \log(16\cos(\gamma(\overrightarrow{\alpha}, f_m))\cos(\gamma(\overleftarrow{\alpha}, f_m))) - 2\log(\ell_\alpha(f_m))$$

and so $\dfrac{t_\alpha(f_m)}{t_{\alpha_0}(f_m)} \approx \dfrac{\log(\ell_\alpha(f_m))}{\log(\ell_{\alpha_0}(f_m))} \approx \dfrac{w_{sp}(\alpha, f_m)}{w_{sp}(\alpha_0, f_m)} \to \dfrac{w(\alpha)}{w(\alpha_0)}$. □

Proposition 3.27. $\Gamma(S)$ *acts on $\overline{\mathcal{T}}^a(S)$ by isometries and with discrete orbits. Hence, $\overline{\mathcal{M}}^a(S) = \overline{\mathcal{T}}^a(S)/\Gamma(S)$ is Hausdorff.*

Proof. Suppose $t \cdot g_m \to t$, with $t \in \overline{\mathcal{T}}^a(S)$ and $g_m \in \Gamma(S)$. Consider a sequence $[f_m : S \to \Sigma_m]$ such that $j(f_m) \to t$ in $\overline{\mathcal{T}}^a(S)$.

Case 1: $\|t\|_\infty < \infty$. Passing to a subsequence, $[f_m] \cdot h_m \to [f_\infty : S \to \Sigma_\infty] \in \widehat{\mathcal{T}}^{vis}(S)$ for suitable $h_m \in \Gamma(S)$. Thus, $\hat{j}(f_\infty) \cdot h_m^{-1} \to t$.

Assume first that $\|t\|_\infty > 0$ and so Σ_∞ does not have n cusps.

Let $\boldsymbol{A} = \boldsymbol{A}_{sp}(f_\infty)$, so that it is supported on $f_\infty^{-1}(\Sigma_{\infty,+})$ and $\ell_\alpha(f_\infty) < \infty$ for all $\alpha \in \boldsymbol{A}$. Because the length spectrum of finite arcs in Σ_∞ is discrete (with finite multiplicities) and $\hat{j}(f_\infty) \cdot h_m^{-1}$ is a Cauchy sequence, there exists an integer M such that (up to subsequences) h_m^{-1} fixes $\boldsymbol{A}$ for all $m \geq M$. Thus, h_m is the composition of a diffeomorphism of $f_\infty^{-1}(\Sigma_{\infty,-})$ and an isometry of $f_\infty^{-1}(\Sigma_{\infty,+})$ (with the pull-back metric) for $m \geq M$. Hence, $t = \hat{j}(f_\infty) \cdot h_m^{-1}$ for $m \geq M$ and so $t = \hat{j}(\hat{f}_\infty)$ for some $\hat{f}_\infty : S \to \Sigma_\infty$. Similarly, $\hat{j}(\hat{f}_\infty) \cdot g_m \to \hat{j}(\hat{f}_\infty)$ and so g_m is the composition of a diffeomorphism of $\hat{f}_\infty^{-1}(\Sigma_{\infty,-})$ and an isometry of $\hat{f}_\infty^{-1}(\Sigma_{\infty,+})$ for large m. Hence, $t \cdot g_m$ cannot accumulate at t.

Assume now that $\|t\|_\infty = 0$, so that Σ_∞ has n cusps.

It follows from the classical case that the spectrum of the finite reduced lengths (and so of the finite λ-lengths) of $(\Sigma_\infty, \underline{p})$ is discrete and with finite multiplicities. Because $[t_\bullet(f_\infty)] = [\lambda_\bullet(f_\infty)]$, we can conclude as in the previous case.

Case 2: $\|t\|_\infty = \infty$**.**
Let $w^m = \boldsymbol{W}(f_m)$. Up to subsequences, $w^m \cdot h_m \to w^\infty$ in $|\mathfrak{A}(S)| \times [0, \infty]$ for suitable $h_m \in \Gamma(S)$ and $w^\infty \in |\mathfrak{A}(S)| \times \{\infty\}$. As before, $\Phi(w^\infty) \cdot h_m^{-1} \to t$ in $\overline{\mathcal{T}}^a(S)$. Because w^∞ has finite support, $t = \Phi(w^\infty) \cdot h_m^{-1}$ for $m \geq M$ and so $t = \Phi(\hat{w}^\infty)$, for some $\hat{w}^\infty \in |\mathfrak{A}(S)| \times \{\infty\}$. Thus, $\Phi(\hat{w}^\infty) \cdot g_m \to \Phi(\hat{w}^\infty)$ and g_m is the composition of a diffeomorphism of S_- and an isometry of S_+, where S_+ (resp. S_-) is the $\hat{w}^\infty$-visible (resp. invisible) subsurface of S. Hence, $t \cdot g_m$ cannot accumulate at t. □

Proof of Theorem 3.23. In order to apply Lemma 3.21(b), we only need to prove that $\Phi' : |\mathfrak{A}(S)|/\Gamma(S) \times [0, \infty] \to \overline{\mathcal{M}}^a(S)$ is a homeomorphism.

We already know that Φ' is continuous, injective. Moreover, its image contains $\mathcal{M}(S)$ which is dense in $\overline{\mathcal{M}}^a(S)$. As $|\mathfrak{A}(S)|/\Gamma(S)$ is compact and $\overline{\mathcal{M}}^a(S)$ is Hausdorff, the map Φ' is closed and so it is also surjective. Hence, Φ' is a homeomorphism. □

Corollary 3.28. *$\hat{j}$ is a homeomorphism onto the finite part of $\overline{\mathcal{T}}^a(S)$.*

3.6. **The extended Teichmüller space.** We define the *extended Teichmüller space* $\widetilde{\mathcal{T}}(S)$ to be

$$\widetilde{\mathcal{T}}(S) := \overline{\mathcal{T}}^{WP}(S) \cup |\mathfrak{A}(S)|_\infty$$

where $|\mathfrak{A}(S)|_\infty$ is just a copy of $|\mathfrak{A}(S)|$.

Clearly, there is map $\mathrm{Bl}_0\widetilde{\mathcal{T}}(S) \to \overline{\mathcal{T}}^a(S)$, which identifies visibly equivalent surfaces of $\widehat{\mathcal{T}}(S) \subset \mathrm{Bl}_0\widetilde{\mathcal{T}}(S)$.

We define a topology on $\widetilde{\mathcal{T}}(S)$ by requiring that $\overline{\mathcal{T}}^{WP}(S) \hookrightarrow \widetilde{\mathcal{T}}(S)$ and $|\mathfrak{A}(S)|_\infty \hookrightarrow \widetilde{\mathcal{T}}(S)$ are homeomorphisms onto their images, that $\overline{\mathcal{T}}^{WP}(S) \subset \widetilde{\mathcal{T}}(S)$ is open and we declare that a sequence $\{f_m\} \subset \overline{\mathcal{T}}^{WP}(S)$ is converging to $w \in |\mathfrak{A}(S)|_\infty$ if and only if $\boldsymbol{W}(f_m) \to (w, \infty)$ in $|\mathfrak{A}(S)| \times (0, \infty]$.

Notice that $\widetilde{\mathcal{M}}(S) := \widetilde{\mathcal{T}}(S)/\Gamma(S)$ is an orbifold with corners, which acquires some singularities at infinity. In fact, $\widetilde{\mathcal{M}}(S)$ is homeomorphic to $\overline{\mathcal{M}}^{WP}(R, x) \times \Delta^{n-1} \times [0, \infty]/\sim$, where (R, x) is a closed x-marked surface such that $S \simeq R \setminus x$ and $(R', \underline{p}', t') \sim (R'', \underline{p}'', t'') \iff t' = t'' = \infty$ and $(R', \underline{p}')$ is visibly equivalent to $(R'', \underline{p}'')$.

4. Weil-Petersson form and circle actions

4.1. **Circle actions on moduli spaces.** Let S be a compact surface of genus g with boundary components $C_1, \ldots, C_n$ (assume as usual that $2g - 2 + n > 0$). Let v_i be a point of C_i and set $v = (v_1, \ldots, v_n)$.

We denote by $\mathrm{Diff}_+(S, v)$ the group of orientation-preserving diffeomorphisms of S that fix v pointwise and by $\mathrm{Diff}_0(S, v)$ its connected component of the identity.

The Teichmüller space $\mathcal{T}(S, v)$ is the space of hyperbolic metrics on S up to action of $\mathrm{Diff}_0(S, v)$ and the mapping class group of (S, v) is $\Gamma(S, v) = \mathrm{Diff}_+(S, v)/\mathrm{Diff}_0(S, v)$. Thus, $\mathcal{M}(S, v) = \mathcal{T}(S, v)/\Gamma(S, v)$ is the resulting moduli space.

Clearly, $\mathbb{R}^n$ acts on $\mathcal{T}(S, v)$ by Fenchel-Nielsen twist (with unit angular speed) around the boundary components and $\mathcal{T}(S, v)/\mathbb{R}^n = \mathcal{T}(S)$. Similarly, the torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ acts on $\mathcal{M}(S, v)$ and the quotient is $\mathcal{M}(S, v)/\mathbb{T}^n = \mathcal{M}(S)$.

Mimicking what done for $\overline{\mathcal{T}}^{WP}(S)$, we can define an augmented Teichmüller space $\overline{\mathcal{T}}^{WP}(S, v)$ and an action of $\mathbb{R}^n$ on it. However, we want to be a little more careful and require that a marking $[f : S \to \Sigma] \in \overline{\mathcal{T}}^{WP}(S, v)$ that shrinks C_i to a cusp $y_i \in \Sigma$ is smooth with $\mathrm{rk}(df) = 1$ at C_i, so that f identifies C_i to the sphere tangent bundle to ST_{Σ, y_i} and v_i to a point in ST_{Σ, y_i}.

Thus, $\overline{\mathcal{T}}^{WP}(S,v)\to\overline{\mathcal{T}}^{WP}(S)$ is an $\mathbb{R}^n$-bundle and $\overline{\mathcal{M}}(S,v)\to\overline{\mathcal{M}}(S)$ is a $\mathbb{T}^n$-bundle, which is a product $L_1\times\cdots\times L_n$ of circle bundles L_i associated to $v_i\in C_i$.

If one wish, one can certainly lift the action to $\widehat{\mathcal{T}}(S,v)=\mathrm{Bl}_0\overline{\mathcal{T}}^{WP}(S,v)$.

As for the definition of $\widehat{\mathcal{T}}^{vis}(S,x)=\widehat{\mathcal{T}}(S,x)/\!\sim_{vis}$, we declare $[f_1:S\to\Sigma_1],[f_2:S\to\Sigma_2]\in\widehat{\mathcal{T}}(S,v)$ to be visibly equivalent if $\exists\,[f:S\to\Sigma]\in\widehat{\mathcal{T}}(S,v)$ and maps $h_i:\Sigma\to\Sigma_i$ for $i=1,2$ such that h_i restricts to an isometry $\Sigma_+\to\Sigma_{i,+}$ and $h_i\circ f\simeq f_i$ (for $i=1,2$) through homotopies that fix $f^{-1}(\overline{\Sigma}_+)\cap v$ (but not necessarily $f^{-1}(\overline{\Sigma}_-)\cap v$).

This means that, if $[f:S\to\Sigma]\in\widehat{\mathcal{T}}^{vis}(S,v)$ has $f(C_i)\subset\Sigma_-$, then $[f]$ does not record the exact position of the point $v_i\in C_i$. In other words, the i-th component of $\mathbb{R}^n$ acts trivially on $[f]$.

4.2. **The arc complex of** (S,v)**.** Let $\mathcal{A}(S,v)$ to be the set of nontrivial isotopy classes of simple arcs in S that start and end at $\partial S\setminus v$ and let β_i be a (fixed) arc from C_i to C_i that separates v_i from the rest of the surface.

A subset $\boldsymbol{A}=\{\beta_1,\dots,\beta_n,\alpha_1,\dots,\alpha_k\}\subset\mathcal{A}(S,v)$ is a k-system of arcs on (S,v) if $\beta_1,\dots,\beta_n,\alpha_1,\dots,\alpha_k$ admit disjoint representatives. The arc complex $\mathfrak{A}(S,v)$ is the set of systems of arcs on (S,v). A point in $\mathfrak{A}(S,v)$ can be represented as a sum $\sum_j w_j\alpha_j$, provided we remember the β_i's (that is, as $\sum_j w_j\alpha_j+\sum_i 0\beta_i$) or as a function $w:\mathcal{A}(S,v)\to\mathbb{R}$.

We can define $\mathfrak{A}^\circ(S,v)\subset\mathfrak{A}(S,v)$ to be the subset of simplices representing systems of arcs that cut S into a disjoint union of discs and annuli homotopic to some boundary component.

Remark that there is a natural map $\mathfrak{A}(S,v)\to\mathfrak{A}(S)$, induced by the inclusion $S\setminus v\hookrightarrow S$ and that forgets the β_i's, and so a simplicial map $|\mathfrak{A}(S,v)|\to|\mathfrak{A}(S)|$.

We can also define a suitable map $\widehat{\boldsymbol{W}}_v$ for the pointed surface (S,v) in such a way that the following diagram commutes.

$$\begin{tikzcd}
\widehat{\mathcal{T}}^{vis}(S,v) \arrow[r, "\widehat{\boldsymbol{W}}_v"] \arrow[d] & {|\mathfrak{A}(S,v)|\times[0,\infty)} \arrow[d] \\
\widehat{\mathcal{T}}^{vis}(S) \arrow[r, "\widehat{\boldsymbol{W}}"] & {|\mathfrak{A}(S)|\times[0,\infty)}
\end{tikzcd}$$

Let $[f:S\to\Sigma]\in\widehat{\mathcal{T}}^{vis}(S,v)$. If we consider it as a point of $\widehat{\mathcal{T}}^{vis}(S)$, then $\widehat{\boldsymbol{W}}(f)$ is a system of arcs in S.

For every $i=1,\dots,n$ such that $f(C_i)\in\Sigma_+$, consider the geodesic $\rho_i\subset\Sigma$ coming out from $f(v_i)$ and perpendicular to $f(C_i)$ (if $f(C_i)$ is a cusp, let ρ_i be the geodesic originating at $f(C_i)$ in direction $f(v_i)$). Call z_i the point where ρ_i first meets the spine of Σ and e_i an infinitesimal portion of ρ_i starting at z_i and going towards $f(C_i)$.

Define $\mathrm{Sp}(\Sigma,f(v))$ to be the one-dimensional CW-complex obtained from $\mathrm{Sp}(\Sigma)$ by adding the vertices z_i (in case z_i was not already a vertex) and the infinitesimal edges e_i. Consequently, we have a well-defined system of arcs $\boldsymbol{A}_{sp}(\Sigma,f(v))$ dual to $\mathrm{Sp}(\Sigma,f(v))$ and widths $w_{sp,f(v)}$, in which the arc dual to e_i plays the role of $f(\beta_i)$ (which thus has zero weight).

We set $\widehat{\boldsymbol{W}}_v(f)=f^*w_{sp,f(v)}$.

The following is an immediare consequence of Theorem 3.16.

Proposition 4.1. *The map $\widehat{\boldsymbol{W}}_v$ is a $\Gamma(S,v)$-equivariant homeomorphism.*

We can make $\mathbb{R}^n$ act on $|\mathfrak{A}(S,v)|$ via $\widehat{\boldsymbol{W}}_v$ and so on $|\mathfrak{A}(S,v)|\times[0,\infty]$. Thus, the action also prolongs to the extended Teichmüller space $\widetilde{\mathcal{T}}(S,v):=\overline{\mathcal{T}}^{WP}(S,v)\cup|\mathfrak{A}(S,v)|_\infty$.

4.3. **Weil-Petersson form.** Chosen a maximal set of simple closed curves $\boldsymbol{\gamma}=\{\gamma_1,\dots,\gamma_{6g-6+2n},C_1,\dots,C_n\}$ on S, we can define a symplectic form ω_v on $\mathcal{T}(S,v)$ by setting

$$\omega_v=\sum_{i=1}^{6g-6+2n}d\ell_i\wedge d\tau_i+\sum_{j=1}^n dp_j\wedge dt_j$$

where $t_j=p_j\vartheta_j/2\pi$ is the twist parameter at C_j. As usual, ω_v does not depend on the choice of $\boldsymbol{\gamma}$ and it descends to $\mathcal{M}(S,v)$. Its independence of the particular Fenchel-Nielsen coordinates permits to extend ω_v to a symplectic form on $\overline{\mathcal{M}}(S,v)$.

Moreover, as $\omega_v(dp_j, -) = \partial/\partial t_j$, the twist flow on $\overline{\mathcal{M}}(S,v)$ is Hamiltonian and the associated moment map is exactly $\mu = (p_1^2/2, \dots, p_n^2/2)$. Thus, the leaves $(\overline{\mathcal{M}}(S)(\underline{p}), \omega_{\underline{p}})$ are exactly the symplectic reductions of $(\overline{\mathcal{M}}(S,v), \omega_v)$ with respect to the $\mathbb{T}^n$-action.

As remarked by Mirzakhani [Mir07], it follows by standards results of symplectic geometry that there is a symplectomorphism $\overline{\mathcal{M}}(S)(\underline{p}) \to \overline{\mathcal{M}}(S)(0)$ which pulls $[\omega_0] + \sum_{i=1}^n \frac{p_i^2}{2} c_1(L_i)$ back to $[\omega_{\underline{p}}]$.

Penner has provided a beautiful formula for ω_0 in term of the $\tilde{a}$-coordinates.

Theorem 4.2 ([Pen92]). *Let $\boldsymbol{A}$ be a maximal system of arcs on S. If $\pi : \mathcal{T}(S)(0) \times \mathbb{R}_+^n \to \mathcal{T}(S)(0)$ is the projection onto the first factor, then*

$$\pi^*\omega_0 = -\frac{1}{2}\sum_{t \in T} (d\tilde{a}_{t_1} \wedge d\tilde{a}_{t_2} + d\tilde{a}_{t_2} \wedge d\tilde{a}_{t_3} + d\tilde{a}_{t_3} \wedge d\tilde{a}_{t_1})$$

where T is the set of ideal triangles in $S \setminus \boldsymbol{A}$, the triangle $t \in T$ is bounded by the (cyclically ordered) arcs $(\alpha_{t_1}, \alpha_{t_2}, \alpha_{t_3})$.

The whole $\mathcal{T}(S)$ is naturally a Poisson manifold with the Weil-Petersson pairing η on the cotangent bundle, whose symplectic leaves are the $\mathcal{T}(S)(\underline{p})$. A general formula expressing η in term of lengths of arcs and widths is given by the following.

Theorem 4.3 ([Mon09c]). *Let $\boldsymbol{A}$ be a maximal system of arcs on S. Then*

$$\eta = \frac{1}{4}\sum_{k=1}^{n} \sum_{\substack{y_i \in \alpha_i \cap C_k \\ y_j \in \alpha_j \cap C_k}} \frac{\sinh(p_k/2 - d_{C_k}(y_i, y_j))}{\sinh(p_k/2)} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_j}$$

where $d_{C_k}(y_i, y_j)$ is the length of the geodesic running from y_i to y_j along C_k in the positive direction.

In order to understand the limit for large $\underline{p}$, it makes sense to rescale the main quantities as $\tilde{w}_i = (\boldsymbol{\mathcal{L}}/2)^{-1} w_i$, $\tilde{\omega} = (1 + \boldsymbol{\mathcal{L}}/2)^{-2}\omega$ and $\tilde{\eta} = (1 + \boldsymbol{\mathcal{L}}/2)^2\, \eta$.

Lemma 4.4 ([Kon92]). *The class $[\tilde{\omega}_\infty] \in H^2_{\Gamma(S)}(|\mathfrak{A}(S)|)$ is represented by a piecewise linear 2-form on $|\mathfrak{A}(S)|$ whose dual can be written (on the maximal simplices) as*

$$\tilde{H} = \frac{1}{2}\sum_r \left(\frac{\partial}{\partial \tilde{w}_{r_1}} \wedge \frac{\partial}{\partial \tilde{w}_{r_2}} + \frac{\partial}{\partial \tilde{w}_{r_2}} \wedge \frac{\partial}{\partial \tilde{w}_{r_3}} + \frac{\partial}{\partial \tilde{w}_{r_3}} \wedge \frac{\partial}{\partial \tilde{w}_{r_1}} \right)$$

where r ranges over all (trivalent) vertices of the ribbon graph represented by a point in $|\mathfrak{A}^\circ(S)|$ and (r_1, r_2, r_3) is the (cyclically) ordered triple of edges incident at r.

The above result admits a pointwise sharpening as follows.

Theorem 4.5 ([Mon09c]). *The bivector field $\tilde{\eta}$ extends over $\widetilde{\mathcal{T}}(S)$ and, on the maximal simplices of $|\mathfrak{A}(S)|_\infty$, we have*

$$\tilde{\eta}_\infty = \tilde{H}$$

pointwise.

Thus, we have a description of the degeneration of η when the boundary lengths of the hyperbolic surface become very large.

5. From surfaces with boundary to pointed surfaces

5.1. **Ribbon graphs.** Let S be a compact oriented surface of genus g with boundary components $C_1, \dots, C_n$ and assume that $\chi(S) = 2 - 2g - n < 0$. Let $\boldsymbol{A} = \{\alpha_0, \dots, \alpha_k\} \in \mathfrak{A}(S)$ be a system of arcs in S and S_+ the corresponding visible subsurface of S.

If $\vec{\alpha}$ is an oriented arc supported on α, then we will refer to the symbol $\vec{\alpha}^*$ as to the *oriented edge dual to $\vec{\alpha}$*.

Remark 5.1. *If S carries a hyperbolic metric and $\boldsymbol{A}$ is its spinal system of arcs, then $\vec{\alpha}^*$ must be considered the edge of the spine dual to α and oriented in such a way that, at the point $\vec{\alpha}^* \cap \vec{\alpha}$ (unique, up to prolonging $\vec{\alpha}^*$) the tangent vectors $\langle v_{\vec{\alpha}^*}, v_{\vec{\alpha}} \rangle$ form a positive basis of T_pS.*

Let $E(\boldsymbol{A}) := \{\overrightarrow{\alpha}^*, \overleftarrow{\alpha}^* \mid \alpha \in \boldsymbol{A}\}$ and define the following operators $\sigma_0, \sigma_1, \sigma_\infty$ on $E(\boldsymbol{A})$:

(1) σ_1 reverses the orientation of each arc (i.e. $\sigma_1(\overrightarrow{\alpha}^*) = \overleftarrow{\alpha}^*$)

(∞) if $\overrightarrow{\alpha}$ ends at $x_\alpha \in C_i$, then $\sigma_\infty(\overrightarrow{\alpha}^*)$ is dual to the oriented arc $\overrightarrow{\beta}$ that ends at $x_\beta \in C_i$, where x_β comes just *before* x_α according to the orientation induced on C_i by S

(0) σ_0 is defined by $\sigma_0 = \sigma_1 \sigma_\infty^{-1}$.

If we call $E_i(\boldsymbol{A})$ the orbits of $E(\boldsymbol{A})$ under the action of σ_i, then

(1) $E_1(\boldsymbol{A})$ can be identified with $\boldsymbol{A}$

(∞) $E_\infty(\boldsymbol{A})$ can be identified with the subset of the boundary components of S that belong to S_+

(0) $E_0(\boldsymbol{A})$ can be identified to the set of connected components of $S_+ \setminus \boldsymbol{A}$.

5.2. **Flat tiles and Jenkins-Strebel differentials.** Keeping the notation as before, let $f : S \to \hat{S}$ be the topological type of $\boldsymbol{A}$ (see Section 3.4).

For every system of weights w supported on $\boldsymbol{A}$, the surface $\hat{S}_+$ can be endowed with a flat metric (with conical singularities) in the following way.

Every component $\hat{S}_{i,+}$ of $\hat{S}_+$ is quasi-filled by the arc system $f(\boldsymbol{A}) \cap \hat{S}_{i,+}$. As we can carry on the construction componentwise, we can assume that $\boldsymbol{A}$ quasi-fills S.

In this case, we consider the flat *tile* $T = [0,1] \times [0,\infty] \Big/ [0,1] \times \{\infty\}$ and we call *point at infinity* the class $[0,1] \times \{\infty\}$. Moreover, we define $\Sigma := \bigcup_{\overrightarrow{e} \in E(\boldsymbol{A})} T_{\overrightarrow{e}}/\sim$, where $T_{\overrightarrow{e}} := T \times \{\overrightarrow{e}\}$ and

- $(u, 0, \overrightarrow{e}) \sim (1-u, 0, \overleftarrow{e})$ for all $\overrightarrow{e} \in E(\boldsymbol{A})$ and $u \in [0,1]$
- $(1, v, \overrightarrow{e}) \sim (0, v, \sigma_\infty(\overrightarrow{e}))$ for all $\overrightarrow{e} \in E(\boldsymbol{A})$ and $v \in [0,\infty]$.

We can also define an embedded graph $G \subset \Sigma$ by gluing the segments $[0,1] \times \{0\} \subset T$ contained in each tile. Thus, we can identify α^* with an edge of G for every $\alpha \in \boldsymbol{A}$.

It is easy to check that there is a homeomorphism $\hat{S} \to \Sigma$, well-defined up to isotopy, that takes boundary components to points at infinity or to vertices.

Moreover, for every $\overrightarrow{\alpha}^* \in E(\boldsymbol{A})$, we can endow $T_{\overrightarrow{\alpha}^*}$ with the quadratic differential dz^2, where $z = w(\alpha)u + iv$. These quadratic differentials glue to give a global φ (and so a conformal structure on the whole Σ), which has double poles with negative quadratic residue at the points at infinity and is holomorphic elsewhere, with zeroes of order $k-2$ at the k-valent vertices of G. Furthermore, α^* has length $w(\alpha)$ with respect to the induced flat metric $|\varphi|$.

Finally, the *horizontal trajectories* of φ (that is, the curves along which φ is positive-definite) are: either closed circles that wind around some point at infinity, or edges of G.

Thus, φ is a *Jenkins-Strebel quadratic differential* and G is its *critical graph*, i.e. the union of all horizontal trajectories that hit some zero or some pole of φ.

If $\boldsymbol{A}$ does not quasi-fill S, then we will define the Jenkins-Strebel differential componentwise, by setting it to zero on the invisible components.

See [Har86], [Kon92], [Loo95] and [Mon09b] for more details.

5.3. **HMT construction.** We begin by recalling the following result of Strebel.

Theorem 5.2 ([Str67]). *Let R' be a compact Riemann surface of genus g with $x' = (x'_1, \dots, x'_n)$ distinct points such that $n \geq 1$ and $2g - 2 + n > 0$. For every $(p_1, \dots, p_n) \in \mathbb{R}^n_{\geq 0}$ (but not all zero), there exists a unique (nonzero) quadratic differential φ on R' such that*

- *φ is holomorphic on $R' \setminus x'$*
- *horizontal trajectories of φ are either circles that wind around some x'_i or closed arcs between critical points*
- *the critical graph G of φ cuts R' into semi-infinite flat cylinders (according to the metric $|\varphi|$), whose circumferences are closed trajectories*
- *if $p_i = 0$, then x'_i belongs to the critical graph*
- *if $p_i > 0$, then the cylinder around x'_i has circumference length p_i.*

Notice that the graph G plays a role analogous to the spine of a hyperbolic surface. In fact, given a point $[f : R \to R'] \in \mathcal{T}(R, x)$ and $(p_1, \dots, p_n) \in \Delta^{n-1}$, we can consider the unique φ given by the theorem above and the system of arcs $\boldsymbol{A} \in \mathfrak{A}^\circ(R, x)$ such that $f(\boldsymbol{A})$ is dual to the critical graph G of φ, and we can define the width $w(\alpha)$ to be the $|\varphi|$-length of the edge α^* of G dual to $\alpha \in \boldsymbol{A}$.

Theorem 5.3 (Harer-Mumford-Thurston [Har86])**.** *The map $\mathcal{T}(R,x)\times\Delta^{n-1}\longrightarrow|\mathfrak{A}^\circ(R,x)|$ just constructed is a $\Gamma(R,x)$-equivariant homeomorphism.*

Clearly, if R' is a stable Riemann surface, then the theorem can be applied on every visible component of R' (i.e. on every component that contains some x'_i with $p_i>0$) and φ can be extended by zero on the remaining part of R'. Hence, we can extend the previous map to

$$\boldsymbol{W}_{HMT}:\overline{\mathcal{T}}^{vis}(R,x)\times\Delta^{n-1}\longrightarrow|\mathfrak{A}(R,x)|$$

which is also a $\Gamma(R,x)$-equivariant homeomorphism (see, for instance, [Loo95] and [Mon09b]).

The purpose of the following sections is to relate this $\boldsymbol{W}_{HMT}$ to the spine construction.

5.4. **The grafting map.** Given a hyperbolic surface Σ with boundary components $C_1,\dots,C_n$, we can *graft a semi-infinite flat cylinder at each C_i* of circumference $p_i=\ell(C_i)$. The result is a surface $\mathrm{gr}_\infty(\Sigma)$ with a $C^{1,1}$-metric, called the *Thurston metric* (see [SW02] for the case of a general lamination, or [KP94] a higher dimensional analogues). If Σ has cusps, we do not glue any cylinder at the cusps of Σ. Notice that $\mathrm{gr}_\infty(\Sigma)$ has the conformal type of a punctured Riemann surface and it will be sometimes regarded as a closed Riemann surface with marked points.

Notation. *Choose a closed surface R with distinct marked points $x=(x_1,\dots,x_n)\subset R$ and an identification $R\setminus x\cong\mathrm{gr}_\infty(S)$ such that x_i corresponds to C_i. Clearly, we can identify $\mathfrak{A}(S)\cong\mathfrak{A}(R,x)$ and $\Gamma(S)\cong\Gamma(R,x)$.*

We use the grafting construction to define a map

$$(\mathrm{gr}_\infty,\mathcal{L}):\widetilde{\mathcal{T}}(S)\longrightarrow\overline{\mathcal{T}}^{WP}(R,x)\times\Delta^{n-1}\times[0,\infty]/\sim$$

where $\sim$ identifies $([f_1],\underline{p},\infty)$ and $([f_2],\underline{p},\infty)$ if $([f_1],\underline{p})$ and $([f_2],\underline{p})$ are visibly equivalent.

We set $\mathrm{gr}_\infty(f:S\to\Sigma):=[\mathrm{gr}_\infty(f):R\to\mathrm{gr}_\infty(\Sigma)]$, on the bounded part $\overline{\mathcal{T}}^{WP}(S)\subset\widetilde{\mathcal{T}}(S)$. On the other hand, if $\tilde{w}\in|\mathfrak{A}(S)|_\infty$ represents a point at infinity of $\widetilde{\mathcal{T}}(S)$, then we define $(\mathrm{gr}_\infty,\mathcal{L})(\tilde{w}):=(\boldsymbol{W}_{HMT}^{-1}(\tilde{w}),\infty)$.

Theorem 5.4. *The map $(\mathrm{gr}_\infty,\mathcal{L})$ is a $\Gamma(S)$-equivariant homeomorphism that preserves the topological types and whose restriction to each topological stratum of the finite part and to each simplex of $|\mathfrak{A}(S)|_\infty$ is a real-analytic diffeomorphism.*

Corollary 5.5. *(a) The induced $\widetilde{\mathcal{M}}(S)\longrightarrow\overline{\mathcal{M}}(R,x)\times\Delta^{n-1}\times[0,\infty]/\sim$ is a homeomorphism, which is real-analytic on $\widehat{\mathcal{M}}(S)$ and piecewise real-analytic on $|\mathfrak{A}(S)|_\infty/\Gamma(S)$.*

(b) Let $\widehat{\mathcal{T}}^{vis}(R,x)$ (resp. $\widehat{\mathcal{M}}^{vis}(R,x)$) be obtained from $\overline{\mathcal{T}}^{WP}(R,x)\times\Delta^{n-1}$ (resp. $\overline{\mathcal{M}}^{WP}(R,x)\times\Delta^{n-1}$) by identifying visibly equivalent surfaces. Then, the induced $\overline{\mathcal{T}}^a(S)\longrightarrow\widehat{\mathcal{T}}^{vis}(R,x)\times[0,\infty]$ and $\overline{\mathcal{M}}^a(S)\longrightarrow\widehat{\mathcal{M}}^{vis}(R,x)\times[0,\infty]$ are homeomorphisms.

We can summarize our results in the following commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{T}}^{vis}(R,x)\times[0,\infty] \arrow[rd, "\Psi"'] & \overline{\mathcal{T}}^a(S) \arrow[l, "(\mathrm{gr}_\infty,\mathcal{L})"'] \arrow[d, "\overline{\boldsymbol{W}}^a"] \\
 & |\mathfrak{A}(R,x)|\times[0,\infty]
\end{tikzcd}$$

in which $\Psi=\overline{\boldsymbol{W}}^a\circ(\mathrm{gr}_\infty,\mathcal{L})^{-1}$ and all maps are $\Gamma(R,x)$-equivariant homeomorphisms. For every $t\in[0,\infty]$, call $\Psi_t:\widehat{\mathcal{T}}^{vis}(R,x)\to|\mathfrak{A}(R,x)|$ the restriction of Ψ to $\widehat{\mathcal{T}}^{vis}(R,x)\times\{t\}$ followed by the projection onto $|\mathfrak{A}(R,x)|$.

Corollary 5.6. *Ψ_t is a continuous family of $\Gamma(R,x)$-equivariant triangulations of $\widehat{\mathcal{T}}^{vis}(R,x)$, whose extremal cases are Penner/Bowditch-Epstein's for $t=0$ and Harer-Mumford-Thurston's for $t=\infty$.*

In order to prove Theorem 5.4, we will show first that $(\mathrm{gr}_\infty,\mathcal{L})$ is continuous. Lemma 3.21(a) will ensure that it is proper. Finally, we will prove that the restriction of $(\mathrm{gr}_\infty,\mathcal{L})$ to each stratum is bijective onto its image, and so that $(\mathrm{gr}_\infty,\mathcal{L})$ is bijective.

5.5. **Continuity of** $(\mathrm{gr}_\infty, \mathcal{L})$**.** To test the continuity of $(\mathrm{gr}_\infty, \mathcal{L})$ at $q \in \widetilde{\mathcal{T}}(S)$, we split the problem into two distinct cases:

(1) $\mathcal{L}(q)$ bounded and so $q = [f : S \to \Sigma]$
(2) $\mathcal{L}(q)$ not bounded and so $q = \tilde{w} \in |\mathfrak{A}(S)|_\infty$.

5.5.1. $\mathcal{L}(q)$ *bounded.* Let $\{f_m : S \to \Sigma_m\} \subset \mathcal{T}(S)$ be a sequence that converges to $[f]$, so that $\mathcal{L}(f_m) \to \mathcal{L}(f)$.

Condition (2) of Proposition 2.15 ensures that there are maps $\tilde{f}_m : S \to \Sigma_m$ which have a standard behavior on a neighbourhood of the thin part of Σ_m and such that the metric $\tilde{f}_m^*(g_m)$ converges to $f^*(g)$ uniformly on the complement. Fixed some $\mathrm{gr}_\infty(f) : R \to \mathrm{gr}_\infty(\Sigma)$, define $\hat{f}_m : R \to \mathrm{gr}_\infty(\Sigma_m)$ in such a way that $F_m = \mathrm{gr}_\infty(f) \circ \hat{f}_m^{-1} : \mathrm{gr}_\infty(\Sigma_m) \to \mathrm{gr}_\infty(\Sigma)$ has the following properties.

If $\ell_{C_i}(f) > 0$, then let $\phi_m^i : \partial^i\Sigma_m \to \partial^i\Sigma$ be restriction of $f \circ \tilde{f}_m^{-1}$ to the i-th boundary component. Moreover, we can give orthonormal coordinates (x, y) (with $y \geq 0$ and $x \in [0, \ell_{C_i}))$ on the i-th cylinder in such a way that $C_i = \{y = 0\}$ and S induced on C_i the orientation along which x *decreases*.

For every i such that $\ell_{C_i}(f) > 0$, we define F_m to be $(x, y) \mapsto (\phi_m^i(x), y)$ on the i-th cylinder.

For every i such that $\ell_{C_i}(f) = 0$ and $\ell_{C_i}(f_m) > 0$, we can assume that $\ell_{C_i}(f_m) < 1/2$ and we can consider a hypercycle $H_i \subset \Sigma_m \subset \mathrm{gr}_\infty(\Sigma_m)$ parallel to the i-th boundary component and of length $2\ell_{C_i}(f_m)$.

We define F_m to agree with $f \circ \tilde{f}_m^{-1}$ on the portion of $\mathrm{gr}_\infty(\Sigma_m)$ which is hyperbolic and bounded by the possible hypercycles H_i's.

Finally, we extend F_m outside the possible hypercycles H_i's by a diffeomorphism.

Clearly, condition (5) of Proposition 2.15 for the sequence $\{\hat{f}_m\}$ and $\mathrm{gr}_\infty(f)$ is verified and so $[f_m] = [\hat{f}_m] \to [\mathrm{gr}_\infty(f)]$ in $\widehat{\mathcal{T}}(R, x)$.

5.5.2. $\mathcal{L}(q)$ *not bounded.* Let $S_+ \subset S$ be the visible subsurface determined (up to isotopy) by $\boldsymbol{A} = \mathrm{supp}\tilde{w}$ and let $\{\boldsymbol{A}_i \,|\, i \in I\}$ be the set of all maximal system of arcs of S that contain $\boldsymbol{A}$.

Consider a sequence $[f_m : S \to \Sigma_m] \in \mathcal{T}(S)$ that converges to $q = \tilde{w} \in |\mathfrak{A}(S)|_\infty \subset \widetilde{\mathcal{T}}(S)$ and such that $\widehat{\boldsymbol{W}}(f_m) \in |\boldsymbol{A}_{i_m}|^\circ \times \mathbb{R}_+$, with $i_m \in I$.

We must show that $\mathrm{gr}_\infty(f_m) \to \mathrm{gr}_\infty(q) = [f : R \to \Sigma]$ in $\overline{\mathcal{T}}(R, x) \times \Delta^{n-1} \times [0, \infty]/\sim$, where f and Σ are constructed as in Section 5.2.

It will be convenient to denote by $\tilde{w}(\alpha, f)$ the weight $\tilde{w}(\alpha)$ for every $\alpha \in \boldsymbol{A}$. Moreover, we will use the notation $\tilde{w}(-, f_m)$ to denote the normalized quantity $2w_{sp}(-, f_m)/\boldsymbol{\mathcal{L}}(f_m)$ and $\tilde{w}_m(\overrightarrow{\alpha}, f)$ to denote $\dfrac{\tilde{w}(\alpha, f)}{\tilde{w}(\alpha, f_m)}\tilde{w}(\overrightarrow{\alpha}, f_m)$.

Remark 5.7. *Using Proposition 2.15, it is sufficient to show that condition (5) for the sequence* $\{\mathrm{gr}_\infty(f_m)\}$ *and* $\mathrm{gr}_\infty(q)$ *is satisfied on the positive components of* Σ*. As usual, we will define a sequence of homeomorphisms* $\hat{f}_m : R \to \mathrm{gr}_\infty(\Sigma_m)$ *(that satisfies condition (5)) by describing* $F_m = f \circ \hat{f}_m^{-1} : \mathrm{gr}_\infty(\Sigma_m) \to \Sigma$.

For every m and every small $\varepsilon > 0$, define the following regions of $\mathrm{gr}_\infty(\Sigma_m)$ and of Σ.

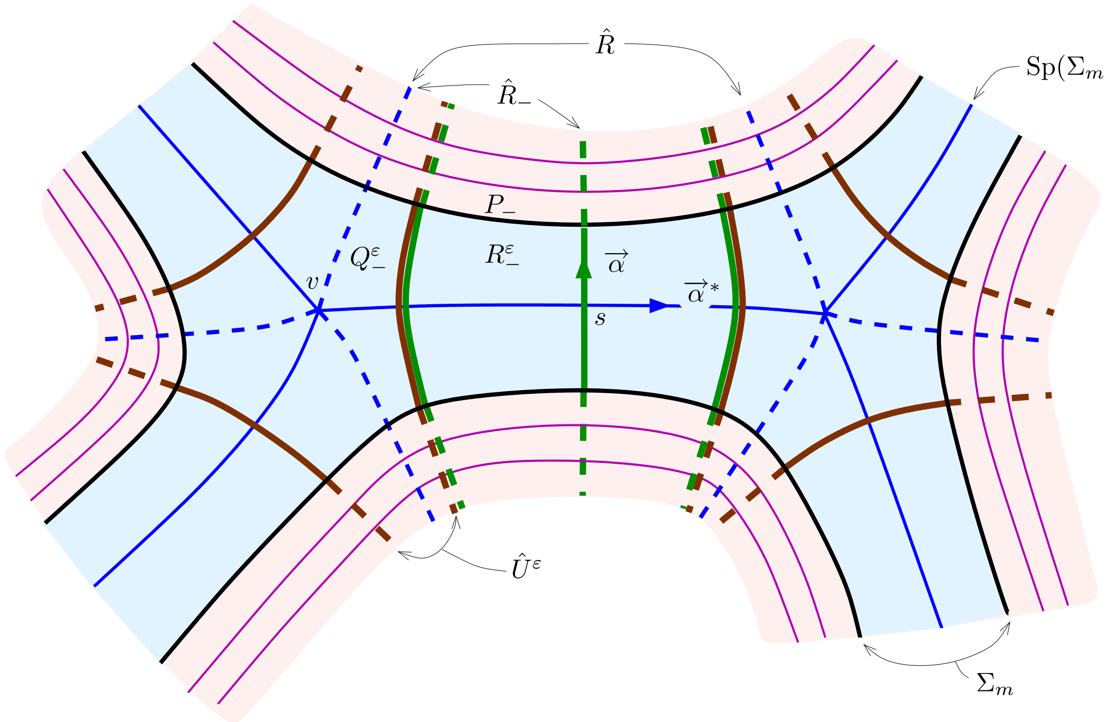

Figure 2: Regions of $\mathrm{gr}_\infty(\Sigma_m)$: $\hat{U}^\varepsilon$ refers to v and P, Q, R, U to $\overrightarrow{\alpha}^*$.

- Let $\overrightarrow{\alpha}$ be an oriented arc on S with support $\alpha \in \boldsymbol{A}_{i_m}$. Call $P(\overrightarrow{\alpha}^*, \Sigma_m)$ the projection of the geodesic edge $f_m(\alpha)^*$ of $\mathrm{Sp}(\Sigma_m)$ to the boundary component of Σ_m pointed by $f(\overrightarrow{\alpha})$ and orient $P(\overrightarrow{\alpha}^*, \Sigma_m)$ coherently with $\overrightarrow{\alpha}^*$ (and so reversing the orientation induced by Σ_m).

  For every $b \in P(\overrightarrow{\alpha}^*, \Sigma_m)$, call g_b the geodesic arc that leaves $P(\overrightarrow{\alpha}^*, \Sigma_m)$ perpendicularly at b and ends at α^* and define the quadrilateral
$$R(\overrightarrow{\alpha}^*, \Sigma_m) := \bigcup_{b \in P(\overrightarrow{\alpha}^*, \Sigma_m)} g_b$$
  and let $\hat{R}(\overrightarrow{\alpha}^*, \Sigma_m)$ be the union of $R(\overrightarrow{\alpha}^*, \Sigma_m)$ and the flat rectangle of $\mathrm{gr}_\infty(\Sigma_m)$ of infinite height with basis $P(\overrightarrow{\alpha}^*, \Sigma_m)$.
- Assume now $\alpha \in \boldsymbol{A} \subset \boldsymbol{A}_{i_m}$ and let $(\overrightarrow{\alpha}, \overrightarrow{\beta_1}, \overrightarrow{\beta_2})$ bound a hexagon of $S \setminus \boldsymbol{A}_{i_m}$. The formula
$$\sinh(a/2)\sinh(w(\overrightarrow{\alpha})) = \frac{s_{\beta_1}^2 + s_{\beta_2}^2 - s_\alpha^2}{2 s_{\beta_1} s_{\beta_2}}$$
  shows that $w_{sp}(\overrightarrow{\alpha}, f_m) > 0$ for m large enough, because $a(f_m) = \ell_\alpha(f_m) \to 0$. Thus, we can assume that $w_{sp}(\overrightarrow{\alpha}, f_m), w_{sp}(\overleftarrow{\alpha}, f_m) > 0$ for all $\alpha \in \boldsymbol{A}$ and all m.

  Call x the arc-length coordinate on $P(\overrightarrow{\alpha}^*, \Sigma_m)$ that is zero at the projection of $s := \alpha^* \cap \alpha$ and let $P_- = P \cap \{x \leq 0\}$. Define
$$R^\varepsilon(\overrightarrow{\alpha}^*, \Sigma_m) := \{r_x \,|\, x \in [-(1-\varepsilon)w_{sp}(\overrightarrow{\alpha}, f_m), (1-\varepsilon)w_{sp}(\overleftarrow{\alpha}, f_m)]\}$$
  where r_x is the hypercyclic arc parallel to $f(\alpha)$ that joins $x \in P(\overrightarrow{\alpha}^*, \Sigma_m)$ and $f(\alpha)^*$, and let $\hat{R}^\varepsilon(\overrightarrow{\alpha}^*, \Sigma_m)$ be the union of $R^\varepsilon(\overrightarrow{\alpha}^*, \Sigma_m)$ and the flat rectangle of $\mathrm{gr}_\infty(\Sigma_m)$ that leans on it.

  We can clearly put coordinates (x, y) on $R^\varepsilon(\overrightarrow{\alpha}^*, \Sigma_m) \cup (\hat{R}(\overrightarrow{\alpha}^*, \Sigma_m) \setminus \Sigma_m)$ such that
  - x extends the arc-length coordinate of P
  - (x, y) are orthonormal on the flat part $\hat{R}(\overrightarrow{\alpha}^*, \Sigma_m) \setminus \Sigma_m$, which corresponds to $[-w(\overrightarrow{\alpha}, f_m), w(\overleftarrow{\alpha}, f_m)] \times [0, \infty)$
  - (x, y) are orthogonal on the hyperbolic part $R^\varepsilon(\overrightarrow{\alpha}^*, \Sigma_m)$, which corresponds to $[-(1-\varepsilon)w(\overrightarrow{\alpha}, f_m), (1-\varepsilon)w(\overleftarrow{\alpha}, f_m)] \times [-a(f_m)/2, 0]$; moreover, $\{x = const\}$ is a hypercycle parallel to $f(\alpha)$ and $\{y = const\}$ is a geodesic that crosses $f(\alpha)$ perpendicularly.

Finally, we set $R^\varepsilon_- := R^\varepsilon \cap \{x \leq 0\}$ and $\hat{R}^\varepsilon_- := \hat{R}^\varepsilon \cap \{x \leq 0\}$, and we let $\hat{Q}^\varepsilon_- := \hat{R}_- \setminus \hat{R}^\varepsilon_-$.

Define analogously the regions for Σ, some of which will depend on m. First, we call $\hat{R}(\overrightarrow{\alpha}^*, \Sigma, m) := T_{\overrightarrow{\alpha}^*} \subset \Sigma$ and we put coordinates $\tilde{x} = -\tilde{w}_m(\overrightarrow{\alpha}, f) + \tilde{w}(\alpha, f)u$ (which depends on m) and $\tilde{y} = \tilde{w}(\alpha, f)v$ on it, so that the Jenkins-Strebel differential φ on Σ restricts to $(d\tilde{x} + id\tilde{y})^2$ on $\hat{R}(\overrightarrow{\alpha}^*, \Sigma, m)$. Then, we define $\hat{R}_-(\overrightarrow{\alpha}^*, \Sigma, m) := \hat{R}(\overrightarrow{\alpha}, \Sigma, m) \cap \{\tilde{x} \leq 0\}$ and $\hat{R}^\varepsilon_-(\overrightarrow{\alpha}^*, \Sigma, m) := \hat{R}(\overrightarrow{\alpha}, \Sigma, m) \cap \{-(1-\varepsilon)\tilde{w}_m(\overrightarrow{\alpha}^*, f) \leq \tilde{x} \leq 0\}$ and finally $\hat{Q}^\varepsilon_- := \hat{R}_- \setminus \hat{R}^\varepsilon_-$. Define similarly the regions with $\tilde{x} \geq 0$.

- If v is a vertex of $G \subset \Sigma$, then let $f(\overrightarrow{\beta_1})^*, \dots, f(\overrightarrow{\beta_j})^*$ be the (cyclically ordered) set of edges of G outgoing from v, where $\beta_h \in \boldsymbol{A}$ (the indices of the β's are taken in $\mathbb{Z}/j\mathbb{Z}$). For every m and h there is an $l_h \geq 1$ such that $\overrightarrow{\beta_h}, \sigma_\infty^{-1}(\overrightarrow{\beta_h}), \sigma_\infty^{-2}(\overrightarrow{\beta_h}), \dots, \sigma_\infty^{-l_h}(\overrightarrow{\beta_h}) = \overleftarrow{\beta_{h+1}}$ are distinct. Call
$$\hat{U}^\varepsilon(\overrightarrow{\beta_h}, \Sigma_m) := \hat{Q}^\varepsilon_-(\overrightarrow{\beta_h}, \Sigma_m) \cup \hat{Q}^\varepsilon_+(\overrightarrow{\beta_{h+1}}, \Sigma_m) \cup \bigcup_{i=1}^{l_h-1} \hat{R}(\sigma_\infty^{-i}(\overrightarrow{\beta_h}), \Sigma_m)$$
and let $\tilde{w}(v, f_m) = \sum_h \sum_{i=1}^{l_h-1} \tilde{w}(\sigma_\infty^{-i}(\overrightarrow{\beta_h}), \Sigma_m)$ be the total (normalized) weight of the edges $\{\eta_k\}$ of $G_m \subset \Sigma_m$ *that shrink to* v, that is the edges supporting $\sigma_\infty^{-i}\overrightarrow{\beta_h}$ with $h = 1, \dots, j$ and $i = 1, \dots, l_h - 1$.

  Set $U^\varepsilon := \hat{U}^\varepsilon \cap \Sigma_m$ and, similarly, $\hat{U}^\varepsilon(\overrightarrow{\beta_h}^*, \Sigma, m) := \hat{Q}^\varepsilon_-(\overrightarrow{\beta_h}^*, \Sigma, m) \cup \hat{Q}^\varepsilon_+(\overrightarrow{\beta_{h+1}}^*, \Sigma, m)$ and $\hat{U}^\varepsilon(v, \Sigma, m) := \bigcup_{h=1}^j \hat{U}^\varepsilon(\overrightarrow{\beta_h}^*, \Sigma, m)$.
- If v is a nonmarked (smooth or singular) vertex of $G \subset \Sigma$, then we simply set $\hat{U}^\varepsilon(v, \Sigma_m) := \bigcup_{h=1}^j \hat{U}^\varepsilon(\overrightarrow{\beta_h}^*, \Sigma_m)$.
- If v is a smooth vertex of Σ marked by x_i, then we set $\hat{U}^\varepsilon(v, \Sigma_m) := \{x_i\} \cup \tilde{C}_i \cup \bigcup_{h=1}^j \hat{U}^\varepsilon(\overrightarrow{\beta_h}^*, \Sigma_m)$, where $\tilde{C}_i$ is the flat cylinder corresponding to x_i.

We choose $\varepsilon_m = \max\{1/\boldsymbol{\mathcal{L}}(f_m), 1 - \sum_{\alpha \in \boldsymbol{A}} \tilde{w}(\alpha, f_m)\}^{1/2}$, so that $\varepsilon_m \to 0$, $\varepsilon_m \boldsymbol{\mathcal{L}}(f_m) \to \infty$ and $(1 - \sum_{\alpha \in \boldsymbol{A}} \tilde{w}(\alpha, f_m))/\varepsilon_m \to 0$. Moreover, we set $\delta_m = \exp(-\varepsilon_m w(\alpha_0, f_m)/4) \to 0$, where $\alpha_0 \in \boldsymbol{A}$ and $\tilde{w}(\alpha_0, f) = \min\{\tilde{w}(\alpha, f) > 0 \,|\, \alpha \in \boldsymbol{A}\}$, so that $a_i(f_m) < \delta_m$ for m large.

Define $F_m : \mathrm{gr}_\infty(\Sigma_m) \to \Sigma$ according to the following prescriptions.

Edges. For every $\alpha \in \boldsymbol{A}$ and every orientation $\overrightarrow{\alpha}$, F_m continuously maps $\hat{R}^{\varepsilon_m}_+(\overrightarrow{\alpha}^*, \Sigma_m)$ onto $\hat{R}^{\varepsilon_m}_+(\overrightarrow{\alpha}^*, \Sigma, m)$ in such a way that $F_m(x, y) = \dfrac{2}{\boldsymbol{\mathcal{L}}(f_m)} \left(\dfrac{\tilde{w}_m(\overrightarrow{\alpha}, f)}{\tilde{w}(\overrightarrow{\alpha}, f_m)} x, y \right)$ for $y \geq \delta_m$ and the vertical arcs $\{x\} \times [-a/2, \delta_m]$ (whose length is $\delta_m + a\cosh(x)/2$) are homothetically mapped to vertical trajectories $\{\tilde{x}'\} \times [0, 2\delta_m/\boldsymbol{\mathcal{L}}(f_m)]$. Thus, the differential of F_m (from the xy-coordinates on Σ_m to the $\tilde{x}\tilde{y}$-coordinates on Σ) is

$$dF_m = \begin{cases} \dfrac{2}{\boldsymbol{\mathcal{L}}(f_m)} \begin{pmatrix} \dfrac{\tilde{w}_m(\overrightarrow{\alpha}, f)}{\tilde{w}(\overrightarrow{\alpha}, f_m)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } y \geq \delta_m \\[2ex] \dfrac{2}{\boldsymbol{\mathcal{L}}(f_m)} \begin{pmatrix} \dfrac{\tilde{w}_m(\overrightarrow{\alpha}, f)}{\tilde{w}(\overrightarrow{\alpha}, f_m)} & 0 \\ -\dfrac{ya\sinh(x)}{2\delta_m \left(1 + \frac{a}{2\delta_m}\cosh(x)\right)^2} & \left(1 + \dfrac{a}{2\delta_m}\cosh(x)\right)^{-1} \end{pmatrix} & \text{if } 0 \leq y \leq \delta_m \\[2ex] \dfrac{2}{\boldsymbol{\mathcal{L}}(f_m)} \begin{pmatrix} \dfrac{\tilde{w}_m(\overrightarrow{\alpha}, f)}{\tilde{w}(\overrightarrow{\alpha}, f_m)} & 0 \\ \dfrac{y\sinh(x)}{\left(1 + \frac{a}{2\delta_m}\cosh(x)\right)^2} & \dfrac{\cosh(x)}{1 + \frac{a}{2\delta_m}\cosh(x)} \end{pmatrix} & \text{if } y \leq 0 \end{cases}$$

Because the metric g_m on $\hat{R}_+^{\varepsilon_m}(\overrightarrow{\alpha}^*, \Sigma_m)$ in the xy-coordinates is

$$g_m = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } y \geq 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & \cosh(x)^2 \end{pmatrix} & \text{if } y \leq 0 \end{cases}$$

we obtain $(F_m^{-1})^*(g_m) = M^t\, M$ (with respect to the $\tilde{x}\tilde{y}$-coordinates), where

$$M = \sqrt{g_m} dF_m^{-1} = \begin{cases} \frac{\boldsymbol{\mathcal{L}}(f_m)}{2} \begin{pmatrix} \frac{\tilde{w}(\overrightarrow{\alpha}, f_m)}{\tilde{w}_m(\overrightarrow{\alpha}, f)} & 0 \\ 0 & 1 \end{pmatrix} \\ \frac{\boldsymbol{\mathcal{L}}(f_m)}{2} \begin{pmatrix} \frac{\tilde{w}(\overrightarrow{\alpha}, f_m)}{\tilde{w}_m(\overrightarrow{\alpha}, f)} & 0 \\ \frac{ya\sinh(x)\tilde{w}(\overrightarrow{\alpha}, f_m)}{2\delta_m \tilde{w}_m(\overrightarrow{\alpha}, f)(1+\frac{a}{2\delta_m}\cosh(x))} & 1+\frac{a}{2\delta_m}\cosh(x) \end{pmatrix} \\ \frac{\boldsymbol{\mathcal{L}}(f_m)}{2} \begin{pmatrix} \frac{\tilde{w}(\overrightarrow{\alpha}, f_m)}{\tilde{w}_m(\overrightarrow{\alpha}, f)} & 0 \\ -\frac{y\sinh(x)\tilde{w}(\overrightarrow{\alpha}, f_m)}{\tilde{w}_m(\overrightarrow{\alpha}, f)(1+\frac{a}{2\delta_m}\cosh(x))} & 1+\frac{a}{2\delta_m}\cosh(x) \end{pmatrix} \end{cases}$$

in the three different regions.

If $w(\overrightarrow{\alpha}, f_m) \geq w(\alpha, f_m)/2$, then $\frac{a}{2}\sinh(w(\overrightarrow{\alpha}, f_m)) \leq 1$ implies $\frac{a}{2}\sinh(x) \leq \frac{a}{2}\sinh[(1-\varepsilon_m)w(\overrightarrow{\alpha}, f_m)] \lessapprox \exp(-\varepsilon_m w(\overrightarrow{\alpha}, f_m))$ and so $\frac{ya\sinh(x)}{2\delta_m} \lessapprox \exp(-\varepsilon_m w(\alpha_0, f_m)/2)$ and $\frac{a\cosh(x)}{2\delta_m} \lessapprox \exp(-\varepsilon_m w(\alpha_0, f_m)/4)$. Hence, on the region where we have defined F_m, the distortion is bounded by

$$\max\left\{ \frac{\tilde{w}_m(\overrightarrow{\alpha}, f)}{\tilde{w}(\overrightarrow{\alpha}, f_m)}, \frac{\tilde{w}(\overrightarrow{\alpha}, f_m)}{\tilde{w}_m(\overrightarrow{\alpha}, f)} \,\Big|\, \alpha \in \boldsymbol{A} \right\} \cdot (1 + \exp[-\varepsilon_m w(\alpha_0, f_m)/4]) \to 1\,.$$

Around the vertices. For every vertex $v \in G \subset \Sigma$ with outgoing edges $\{\overrightarrow{\beta_h}^*\}$, we require F_m to map $\hat{U}^{\varepsilon_m}(v, \Sigma_m) \cap \{y \geq \delta_m\}$ onto $\hat{U}^{\varepsilon_m}(v, \Sigma, m) \cap \{\tilde{y} \geq 2\delta_m/\boldsymbol{\mathcal{L}}(f_m)\}$ with differential (from (x, y) to $(\tilde{x}, \tilde{y})$) constantly equal to

$$\frac{2}{\boldsymbol{\mathcal{L}}(f_m)} \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \text{ where } c = \frac{\varepsilon_m \sum_h \tilde{w}_m(\overrightarrow{\beta_h}, f)}{\varepsilon_m \sum_h \tilde{w}(\overrightarrow{\beta_h}, f_m) + \tilde{w}(v, f_m)}$$

Notice that

$$c - 1 = \frac{\sum_h \left(\tilde{w}_m(\overrightarrow{\beta_h}, f) - \tilde{w}(\overrightarrow{\beta_h}, f_m)\right) - \varepsilon_m^{-1}\tilde{w}(v, f_m)}{\sum_h \tilde{w}(\overrightarrow{\beta_h}, f_m) + \varepsilon_m^{-1}\tilde{w}(v, f_m)} \to 0$$

because $\varepsilon_m^{-1}\tilde{w}(v, f_m) \leq \varepsilon_m^{-1}(1 - \sum_{\alpha\in\boldsymbol{A}} \tilde{w}(\alpha, f_m)) \to 0$. Hence, the distortion of F_m goes to 1.

Neighbourhoods of the vertices. If $v \in G \subset \Sigma$ is smooth, then define F_m to be a diffeomorphism between $\hat{U}^{\varepsilon_m}(v, \Sigma_m) \setminus \{y \geq \delta_m\}$ and $\hat{U}^{\varepsilon_m}(v, \Sigma, m) \setminus \{\tilde{y} \geq 2\delta_m/\boldsymbol{\mathcal{L}}(f_m)\}$. If v is also marked, then we can require F_m to preserve the marking.

If v is a node between two visible components, then F_m maps $\hat{U}^{\varepsilon_m}(v, \Sigma_m) \setminus \{y \geq \delta_m\}$ onto $\hat{U}^{\varepsilon_m}(v, \Sigma, m) \setminus \{\tilde{y} \geq 2\delta_m/\boldsymbol{\mathcal{L}}(f_m)\}$ shrinking the edges $\{\eta_i^*\}$ to v and as a diffeomorphism elsewhere.

If $\Sigma' \subset \Sigma$ is an invisible component and $v_1, \ldots, v_l$ are vertices of $G \subset \Sigma_+$ and nodes of Σ', then let $\{\eta_i\}$ be the sub-arc-system $\boldsymbol{A}_{i_m} \cap f^{-1}(\Sigma')$. We require F_m to map

$$\left(\bigcup_i \left(\hat{R}^{\varepsilon_m}(\overrightarrow{\eta_i}^*, \Sigma_m) \cup \hat{R}^{\varepsilon_m}(\overleftarrow{\eta_i}^*, \Sigma_m) \right) \cup \bigcup_h \hat{U}^{\varepsilon_m}(v_h, \Sigma_m) \right) \setminus \{y \geq \delta_m\}$$

onto $\Sigma' \cup \left(\bigcup_h \hat{U}^{\varepsilon_m}(v_h, \Sigma, m) \setminus \{\tilde{y} \geq 2\delta_m/\boldsymbol{\mathcal{L}}(f_m)\} \right)$ by shrinking $\hat{U}^{\varepsilon_m}(v_h, \Sigma_m) \cap \{y = \delta_m/2\}$ to v_h and as a diffeomorphism elsewhere.

5.6. **Bijectivity of** $(\mathrm{gr}_\infty, \mathcal{L})$**.** The bijectivity at infinity (namely, for $\boldsymbol{\mathcal{L}} = \infty$) follows from Theorem 5.2. Thus, let's select a (possibly empty) system of curves $\boldsymbol{\gamma} = \{\gamma_1, \dots, \gamma_k\}$ on S and let's consider the stratum $\mathcal{S}(\boldsymbol{\gamma}) \subset \widehat{\mathcal{T}}(S)$ in which $\boldsymbol{\mathcal{L}} < \infty$ and $\ell_{\gamma_i} = 0$.

To show that $(\mathrm{gr}_\infty, \mathcal{L})$ gives a bijection of $\mathcal{S}(\boldsymbol{\gamma})$ onto its image, it is sufficient to work separately on each component of $S \setminus \boldsymbol{\gamma}$. Thus, we can reduce to the case in which $\gamma_i = C_i \subset \partial S$ and gr_∞ glues a cylinder at the boundary components $C_{k+1}, \dots, C_n$. Hence, we are reduced to show that the grafting map

$$\mathrm{gr}'_\infty : \mathcal{T}(S)(\underline{p}) \longrightarrow \mathcal{T}(S)(0)$$

is bijective for every $p_{k+1}, \dots, p_n \in \mathbb{R}_+$, where $p_1 = \dots = p_k = 0$. We already know that gr'_∞ is continuous and proper: we will show that it is a local homeomorphism by adapting the argument of [SW02]. Here we describe what considerations are needed to make their proof work in our case.

Remark 5.8. *Here we are using the notation $\mathcal{T}(S)(0)$ instead of $\mathcal{T}(R, x)$ because we want to stress that we are regarding $\mathrm{gr}_\infty(\Sigma)$ as a hyperbolic surface, with the metric coming from the uniformization.*

The grafted metrics are $C^{1,1}$ but the map gr'_∞ is real-analytic. In fact, given a real-analytic arc $[f_t : S \to \Sigma_t]$ in $\mathcal{T}(S)(\underline{p})$ and chosen representatives f_t so that $f_0 \circ f_t^{-1} : \Sigma_t \to \Sigma_0$ is an isometry on the boundary components $C_{k+1,t}, \dots, C_{n,t}$ and harmonic in the interior with respect to the hyperbolic metrics (so that the hyperbolic metrics pull back to a real-analytic family σ_t on S), we can choose the grafted maps $\mathrm{gr}'_\infty(f_t) : S \to \Sigma_t$ so that $\mathrm{gr}'_\infty(f_0) \circ \mathrm{gr}'_\infty(f_t)^{-1}$ extend $f_0 \circ f_t^{-1}$ as isometries on the cylinders $\tilde{C}_{i,t} := C_{i,t} \times [0, \infty)$. Hence, the family of metrics $\mathrm{gr}'_\infty(\sigma_t)$ on S, obtained by pulling the Thurston metric back thourgh $\mathrm{gr}'_\infty(f_t)$, is real-analytic in t and so the arc $[\mathrm{gr}'_\infty(f_t)]$ in $\mathcal{T}(S)(0)$ is real-analytic.

Thus, it is sufficient to show that the differential $d\mathrm{gr}'_\infty$ is injective at every point of $\mathcal{T}(S)(\underline{p})$.

Given a real-analytic one-parameter family $f_t : S \to \Sigma_t$ corresponding to a tangent vector $v \in T_{[f_0]}\mathcal{T}(S)(\underline{p})$, assume that the grafted family $[\mathrm{gr}'_\infty(f_t) : S \to \mathrm{gr}'_\infty(\Sigma_t)]$ defined above determines the zero tangent vector in $T_{[\mathrm{gr}'_\infty(f_0)]}\mathcal{T}(S)(0)$.

Call $\mathrm{gr}'_\infty(\sigma_t)$ the pull-back via f_t of the hyperbolic metric of Σ_t and construct the harmonic representative $F_t : (S, \mathrm{gr}'_\infty(\sigma_t)) \to (S, \mathrm{gr}'_\infty(\sigma_0))$ in the class of the identity as follows.

Give orthonormal coordinates (x, y) to the cylinder $\tilde{C}_{i,t} \cong C_{i,t} \times [0, \infty)$ that is glued at the boundary component $C_{i,t} \subset \Sigma_t$ for $i = k+1, \dots, n$, in such a way that x is the arc-length parameter of the circumferences and $y \in [0, \infty)$.

Remark 5.9. *The (x, y) coordinates can be extended to an orthogonal system in a small hyperbolic collar of $C_{i,t}$ in such a way that y is the arc-length parameter along the geodesics $\{x = const\}$. Thus, for $y \in (-\varepsilon, 0)$, the metric looks like* $\cosh(y)^2 dx^2 + dy^2 = dx^2 + dy^2 + O(\varepsilon^2)$.

Call *M-ends* of $\mathrm{gr}'_\infty(\Sigma_t)$ the subcylinders $\tilde{C}_{i,t} \times [M, \infty)$ for $i = k+1, \dots, n$ and use the same terminology for their images in S via $\mathrm{gr}'_\infty(f_t)^{-1}$.

For every t, let $\mathfrak{F}_t := \bigcup_{M \geq 0} \mathfrak{F}_t(M)$ where $\mathfrak{F}_t(M)$ is the set of $C^{1,1}$ diffeomorphisms $g_t : (S, \mathrm{gr}_\infty(\sigma_t)) \to (S, \mathrm{gr}_\infty(\sigma_0))$ homotopic to the identity, such that g_t isometrically preserves the M-ends. Clearly, $\mathfrak{F}_t(M) \subseteq \mathfrak{F}_t(M')$, if $M \leq M'$.

Let $e(g_t) = \frac{1}{2}\|\nabla g_t\|^2$ be the energy density of g_t, $\mathcal{H}(g_t) = \|dg_t(\partial_z)\|^2 \dfrac{dz\, d\bar{z}}{\mathrm{gr}'_\infty(\sigma_t)}$, where z is a local conformal coordinate on $(S, \mathrm{gr}'_\infty(\sigma_t))$, and $\mathcal{J}(g_t)$ the Jacobian determinant of g_t, so that $e(g_t) = 2\mathcal{H}(g_t) - \mathcal{J}(g_t)$. Notice that, if g_t is an oriented diffeomorphism, then $0 < \mathcal{J}(g_t) \leq \mathcal{H}(g_t) \leq e(g_t)$ at each point.

Define also the reduced quantities $\tilde{e}(g_t) = e(g_t) - 1$, $\tilde{\mathcal{H}}(g_t) = \mathcal{H}(g_t) - 1$ and $\tilde{\mathcal{J}}(g_t) = \mathcal{J}(g_t) - 1$, so that the reduced energy

$$\tilde{E}(g_t) := \int_S \tilde{e}(g_t)\mathrm{gr}'_\infty(\sigma_t)$$

is well-defined for every $g_t \in \mathfrak{F}_t$. For instance, the identity map on S belongs to $\mathfrak{F}_t(0)$ and its reduced energy is $E(f_0 \circ f_t^{-1}) - 2\pi\chi(S)$.

As $\mathrm{gr}'_\infty(\sigma_0)$ is nonpositively curved, the map $F_{t,M}$ of least energy in $\mathfrak{F}_t(M)$ is harmonic away from the M-ends and so is an oriented diffeomorphism. Thus,

$$0 = \int_S \tilde{\mathcal{J}}(F_{t,M})\mathrm{gr}'_\infty(\sigma_t) \leq \int_S \tilde{\mathcal{H}}(F_{t,M})\mathrm{gr}'_\infty(\sigma_t) \leq \tilde{E}(F_{t,M})$$

Thus, the map F_t of least (reduced) energy in $\mathfrak{F}_t$ can be obtained as a limit of the $F_{t,M}$'s and it is clearly unique. Call $\tilde{\mathcal{H}}_t := \tilde{\mathcal{H}}(F_t)$ and similarly $\tilde{e}_t = \tilde{e}(F_t)$.

Following Scannell-Wolf (but noticing that the roles of x and y here are exchanged compared to their paper), one can show that

- the family $\{F_t\}$ is real-analytic in t
- for every small t, the map F_t is (locally) $C^{2,\alpha}$ on S; so is the vector field $\dot{F} := \dot{F}_0$ (hence, the analyticity of F_t implies that $\tilde{\mathcal{H}}_t$ and $\tilde{e}_t$ are real-analytic in t too)
- the function $\dot{\tilde{\mathcal{H}}} := \dot{\tilde{\mathcal{H}}}_0$ is locally Lipschitz and it is harmonic on the flat cylinders
- along every $C_{k+1}, \dots, C_n$, we have
$$V = -\frac{1}{2}\left((\partial_y \dot{\tilde{\mathcal{H}}})_+ - (\partial_y \dot{\tilde{\mathcal{H}}})_-\right)$$
where $V(x,y)$ is a harmonic function defined on the cylinders $\tilde{C}_{i,0}$ (and on first-order thickenings of $C_{i,0}$) that can be identified to the y-component of $\dot{F}$ and $w(x,0)_+$ simply means $\lim_{y\to 0^+} w(x,y)$.
- $V_y = \dfrac{1}{2}\dot{\mathcal{H}} + c_i$ on each $\tilde{C}_i$, where c_i is a constant that may depend on the cylinder.

Now on, let all line integrals be with respect to the arc-length parameter dx and all surface integrals with respect to $\mathrm{gr}'_\infty(\sigma_0)$. Notice that

$$\int_S \dot{\mathcal{H}} = \int_S \dot{\tilde{\mathcal{H}}} = \lim_{t\to 0}\frac{1}{t}\int_S \tilde{\mathcal{H}}_t$$

because $\tilde{\mathcal{H}}_0 = 0$. As the integral on the right is a real-analytic function of t which vanishes at $t = 0$, we conclude that $\dot{\tilde{\mathcal{H}}}$ is integrable. By the same argument, so is $\dot{\tilde{e}}$.

On the other hand, $\dot{\tilde{e}} = \dfrac{1}{2}\dfrac{\partial}{\partial t}\|\nabla F_t\|^2\Big|_{t=0} \geq |V_y|$ and so V_y is integrable too and all constants $c_i = 0$. Thus, V and $\dot{\tilde{\mathcal{H}}}$ decay at least as $\exp(-2\pi y/p_i)$ on $\tilde{C}_{i,0}$ and we can write

$$0 = \int_{\tilde{C}_{i,0}} V\Delta V = -\int_{\tilde{C}_{i,0}} \|\nabla V\|^2 + \int_{C_{i,0}} V\partial_n V$$

Moreover,

$$0 \leq \int_{\tilde{C}_{i,0}} \|\nabla V\|^2 = \int_{C_{i,0}} V_y V = \frac{1}{2}\int_{C_{i,0}} \dot{\tilde{\mathcal{H}}} V \tag{2}$$

On the other hand, multiplying by $\dot{\tilde{\mathcal{H}}} = \dot{\mathcal{H}}$ and integrating by parts the linearized equation

$$(\Delta_{\mathrm{gr}'_\infty(\sigma_0)} + 2K_0)\dot{\mathcal{H}} = 0$$

where K_0 is the curvature of $\mathrm{gr}'_\infty(\sigma_0)$, we obtain

$$\begin{cases} 0 \leq \displaystyle\int_{\tilde{C}_{i,0}} \|\nabla\dot{\mathcal{H}}\|^2 = \int_{C_{i,0}} \dot{\mathcal{H}}(\partial_n \dot{\mathcal{H}})_+ \\ 0 \leq \displaystyle\int_{S_{hyp}} \|\nabla\dot{\mathcal{H}}\|^2 + 2|\dot{\mathcal{H}}|^2 = -\sum_{i=k+1}^{n}\int_{-C_{i,0}} \dot{\mathcal{H}}(\partial_n\dot{\mathcal{H}})_- \end{cases}$$

where S_{hyp} is the $\mathrm{gr}'_\infty(\sigma_0)$-hyperbolic part of S.

From $0 \leq \displaystyle\sum_{i=k+1}^{n}\int_{C_{i,0}} \dot{\mathcal{H}}\left((\partial_y\dot{\mathcal{H}})_- - (\partial_y\dot{\mathcal{H}})_+\right)$, we finally get

$$0 \leq 2\int_S \|\nabla\dot{\mathcal{H}}\|^2 - 2K\|\dot{\mathcal{H}}\|^2 = -\sum_{i=k+1}^{n}\int_{C_{i,0}} \dot{\mathcal{H}}V \tag{3}$$

Combining Equation 2 and Equation 3, we obtain

$$\int_{C_{i,0}} \dot{\mathcal{H}} V = 0 \qquad \forall\ i = k+1, \dots, n$$

and so $\dot{\mathcal{H}} = 0$ on S.

Hence, F_t is a $(1+o(t))$-isometry between $\mathrm{gr}'_\infty(\sigma_t)$ and $\mathrm{gr}'_\infty(\sigma_0)$ and one can easily conclude that σ_t and σ_0 are $(1+o(t))$-isometric too.

5.7. **More on infinitely grafted structures.**

5.7.1. *Projective structures.* Consider a compact Riemann surface R without boundary and of genus at least 2. A *projective structure* on a marked surface $[f : R \to R']$ is an equivalence class of holomorphic atlases $\mathfrak{U} = \{f_i : U_i \to \mathbb{CP}^1 \,|\, R' \supset U_i \text{ open}\}$ for R' such that the transition functions belong to $\mathrm{Aut}(\mathbb{CP}^1) \cong \mathrm{PSL}(2,\mathbb{C})$, that is $f_i\Big|_{U_i \cap U_j}$ and $f_j\Big|_{U_i \cap U_j}$ are *projectively equivalent.*

Given two projective structures, represented by maximal atlases $\mathfrak{U}$ and $\mathfrak{V}$, on the same $[f : R \to R'] \in \mathcal{T}(R)$ and a point $p \in R'$, we want to measure how charts of $\mathfrak{U}$ are not projectively equivalent to charts in $\mathfrak{V}$ around p. So, let $f : U \to \mathbb{CP}^1$ be a chart in $\mathfrak{U}$ and $g : U \to \mathbb{CP}^1$ a chart in $\mathfrak{V}$, with $U \subset R'$. There exists a unique $\sigma \in \mathrm{PSL}(2,\mathbb{C})$ such that f and $\sigma \circ g$ agree up to second order at p. Then, $(f - \sigma \circ g)''' : T_pU \to T_{f(p)}\mathbb{CP}^1$ is a homogeneous cubic map and $f'(p)^{-1} \circ (f - \sigma \circ g)'''$ is a homogeneous cubic endomorphism of T_pU, and so an element $\boldsymbol{S}(f,g)(p)$ of $(T_p^*U)^{\otimes 2}$. The holomorphic quadratic differential $\boldsymbol{S}(\mathfrak{U},\mathfrak{V})$ on R' is called *Schwarzian derivative.* It is known that, given a $\mathfrak{U}$ and a holomorphic quadratic differential $\varphi \in \mathcal{Q}_{R'}$, there exists a unique projective structure $\mathfrak{V}$ on R' such that $\boldsymbol{S}(\mathfrak{U},\mathfrak{V}) = \varphi$.

Thus, the natural projection $\pi : \mathcal{P}(R) \to \mathcal{T}(R)$ from the set $\mathcal{P}(R)$ of projective structures on R (up to isotopy) to the Teichmüller space of R is a principal $\mathcal{Q}$-bundle, where $\mathcal{Q} \to \mathcal{T}(R)$ is the bundle of holomorphic quadratic differentials.

On the other hand, the grafting map $\mathrm{gr} : \mathcal{T}(R) \times \mathcal{ML}(R) \to \mathcal{T}(R)$ admits a lifting

$$\mathrm{Gr} : \mathcal{T}(R) \times \mathcal{ML}(R) \xrightarrow{\sim} \mathcal{P}(R)$$

which is a homeomorphism (Thurston) and such that $\mathrm{Gr}(-,0)$ corresponds to the Poincaré structure. We recall that a surface with projective structure comes endowed with a Thurston $C^{1,1}$ metric: in particular, if $\lambda = c_1\gamma_1 + \cdots + c_n\gamma_n$ is a multi-curve on R, then $\mathrm{Gr}(R',\lambda)$ is made of a hyperbolic piece, isometric to $R' \setminus \mathrm{supp}(\lambda)$ and n flat cylinders $F_1, \dots, F_n$, with F_i homotopic to γ_i and of height c_i.

It is a general fact that $\mathrm{Gr}(-,\lambda)$ is a real-analytic section of π for all $\lambda \in \mathcal{ML}$.

5.7.2. *A compactification of* $\mathcal{P}(R)$. The homeomorphism $\mathcal{T}(R) \times \mathcal{ML}(R) \cong \mathcal{P}(R)$ shows that sequences $([f_m : R \to R'_m], \lambda_m)$ in $\mathcal{P}(R)$ can diverge in two "directions".

Dumas [Dum06] provides a *grafting compactification* of $\mathcal{P}(R)$ by separately compactifying $\mathcal{T}(R)$ and $\mathcal{ML}(R)$. In particular, he defines $\overline{\mathcal{P}}(R) := \overline{\mathcal{T}}^{Th}(R) \times \overline{\mathcal{ML}}(R)$, where $\overline{\mathcal{T}}^{Th}(R) = \mathcal{T}(R) \cup \mathbb{P}\mathcal{ML}(R)$ is Thurston's compactification and $\overline{\mathcal{ML}}(R) = \mathcal{ML}(R) \cup \mathbb{P}\mathcal{ML}(R)$ is the natural projective compactification of $\mathcal{ML}(R)$. In particular, the locus $\overline{\mathcal{T}}^{Th}(R) \times \mathbb{P}\mathcal{ML}(R)$ corresponds to "infinitely grafted surfaces".

In order to describe the asymptotic properties of $\overline{\mathcal{P}}(R)$, we recall the following well-known result.

Theorem 5.10 ([HM79])**.** *The map*

$$\Lambda : \mathcal{Q} \to \mathcal{T}(R) \times \mathcal{ML}(R)$$

defined as $([f : R \to R'], \varphi) \mapsto ([f : R \to R'], f^*\Lambda_{R'}(\varphi))$ *is a homeomorphism, where* $\Lambda_{R'}(\varphi)$ *is the measured lamination on* R' *obtained by straightening the (measured) horizontal foliation of* φ*.*

The *antipodal map* is the homeomorphism $\iota : \mathcal{T}(R) \times \mathcal{ML}(R) \to \mathcal{T}(R) \times \mathcal{ML}(R)$ given by $\iota([f : R \to R'], \lambda) = \Lambda(-\Lambda^{-1}([f], \lambda))$. The following result shows that the restriction $\iota_f : \mathcal{ML}(R) \to \mathcal{ML}(R)$ of i to a certain point $[f] \in \mathcal{T}(R)$ controls the asymptotic behavior of $\pi^{-1}(f)$.

Theorem 5.11 ([Dum06], [Dum07b]). *Let $\{([f_m : R \to R_m], \lambda_m)\} \subset \mathcal{T}(R) \times \mathcal{ML}(R)$ be a diverging sequence such that $\pi \circ \mathrm{Gr}_{\lambda_m}(f_m) = [f : R \to R']$. The following are equivalent:*

(1) $\lambda_m \to [\lambda]$ *in* $\overline{\mathcal{ML}}(R)$, *where* $[\lambda] \in \mathbb{P}\mathcal{ML}(R)$
(2) $[f_m] \to [\iota_f(\lambda)]$ *in* $\overline{\mathcal{T}}^{Th}(R)$
(3) $\Lambda_f(-\boldsymbol{S}(\mathrm{Gr}_{\lambda_m}(f_m))) \to [\lambda]$ *in* $\overline{\mathcal{ML}}(R)$
(4) $\Lambda_f(\boldsymbol{S}(\mathrm{Gr}_{\lambda_m}(f_m))) \to [i_f(\lambda)]$ *in* $\overline{\mathcal{ML}}(R)$, *where the Schwarzian derivative is considered with respect to the Poincaré structure.*

When this happens, we also have $[\boldsymbol{H}(\kappa_m)] \to [\Lambda^{-1}(R', \lambda)]$ in $\mathbb{P}L^1(R', K^{\otimes 2})$, where $\boldsymbol{H}(\kappa_m)$ is the Hopf differential of the collapsing map $\kappa_m : R' \to R_m$.

We recall that, if λ_m is a multi-curve $c_1\gamma_1 + \cdots + c_n\gamma_n$, then κ_m collapses the n grafted cylinders onto the respective geodesics and is the identity elsewhere. Thus, if the j-th flat cylinder is isometric to $[0, \ell_j] \times [0, c_j]/(0, y) \sim (\ell_j, y)$, then $\boldsymbol{H}(\kappa_m)$ restricts to dz^2 on the grafted cylinders and vanishes on the remaining hyperbolic portion of R'.

Remark 5.12. *The theorem implies that the boundary of $\pi^{-1}(f) \subset \overline{\mathcal{P}}(R)$ is exactly the graph of the projectivization of i_f.*

5.7.3. *Surfaces with infinitely grafted ends.* We can adapt Theorem 5.4 to our situation, when we restrict our attention to smooth hyperbolic surfaces with large boundary.

Let S be a compact oriented surface of genus g with boundary components $C_1, \dots, C_n$ (and $\chi(S) = 2 - 2g - n < 0$) and let dS be its double.

Theorem 5.13. *Let $\{f_m : S \to \Sigma_m\} \subset \mathcal{T}(S)$ be a sequence such that $(\mathrm{gr}_\infty, \mathcal{L})(f_m) = ([f : (R, x) \to (R', x')], \underline{p}_m) \in \mathcal{T}(R, x) \times \mathbb{R}^n_+$. The following are equivalent:*

(1) $\underline{p}_m \to (\underline{p}, \infty)$ *in* $\Delta^{n-1} \times (0, \infty]$
(2) $[f_m] \to \tilde{w}$ *in* $\overline{\mathcal{T}}^a(S)$, *where $\tilde{w}$ is the projective multi-arc associated to the vertical foliation of the Jenkins-Strebel differential φ_{JS} on (R', x') with weights $\underline{p}$ (see Theorem 5.2).*

When this happens, we also have

(a) $4\boldsymbol{p_m}^{-2}\boldsymbol{H}(\kappa_m) \to \varphi_{JS}$ *in* $L^1_{loc}(R', K(x')^{\otimes 2})$, *where $\boldsymbol{H}(\kappa_m)$ is the Hopf differential of the collapsing map $\kappa_m : R' \to \Sigma_m$*
(b) *with respect to the Poincaré projective structure,* $2\boldsymbol{p_m}^{-2}\boldsymbol{S}(\mathrm{Gr}_\infty(f_m)) \to -\varphi_{JS}$ *in* $H^0(R', K(x')^{\otimes 2})$.

Remark 5.14. *We have denoted by $\mathrm{Gr}_\infty(f_m : S \to \Sigma_m)$ the (S-marked) surface with projective structure obtained from Σ_m by grafting cylinders of infinite length at its ends. This is a somewhat "very exotic" projective structure, whose developing map wraps infinitely many times around $\mathbb{CP}^1$. Its Schwarzian with respect to the Poincaré structure has double poles at the cusps.*

We have already shown that (1) and (2) are equivalent to each other.

Lemma 5.15. *Given an increasing sequence $\{m_k\} \subset \mathbb{N}$ and a diverging sequence $\{t_k\} \subset \mathbb{R}_+$, we have $\bigcup_k R'_k = \dot{R}'$, where we consider $R'_k := \mathrm{gr}_{t_k \boldsymbol{p_{m_k}} \partial\Sigma_{m_k}}(\Sigma_{m_k})$ as embedded inside $\dot{R}'$.*

Proof. Let $z \in R' \setminus \bigcup_k R'_k$ and notice that, for each k, $R' \setminus R'_k$ is a disjoint union of n discs. Up to extracting a subsequence, we can assume that z belongs to the j-th disc, together with x'_j. The image of $C_j \subset S$ inside R' separates $\{x_j, z\}$ from the rest of the surface. Because $t_k \to \infty$, the extremal length $Ext_{C_j}(R'_k) \to 0$ as $k \to \infty$. This implies $z = x_j$. □

The proof of (a) follows [Dum06] (see also [Dum07a]) with minor modifications:

- because of the previous lemma, for every compact $K \subset \dot{R}'$, there exist $t_0 > 0$ such that $K \subset K^t_m := \mathrm{gr}_{t\boldsymbol{p_m}\partial\Sigma_m}(\Sigma_m) \subset \dot{R}'$ for every $t \geq t_0$
- let $h_m : \dot{R}' \to \Sigma_m$ be the harmonic map homotopic to κ_m, that is the limit as $s \to \infty$ of the harmonic maps $h^s_m : \mathrm{gr}_{s\partial\Sigma_m}(\Sigma_m) \to \Sigma_m$ that restrict to isometries at the boundary: we clearly have
$$\|\boldsymbol{H}(h_m) - \boldsymbol{H}(\kappa_m)\|_{L^1(K)} \leq \|\boldsymbol{H}(h_m) - \boldsymbol{H}(\kappa_m)\|_{L^1(K^t_m)}$$
and $E_{K^t_m}(h_m) < E_{K^t_m}(\kappa_m) = 2\pi|\chi(S)| + t\boldsymbol{p_m}^2/2 \leq E_{K^t_m}(h_m) + 2\pi|\chi(S)|$, where $E_{K^t_m}$ is the integral of the energy density on K^t_m

- the statement that $[\boldsymbol{H}(h_m)] \to [\varphi]$ as $m \to \infty$ is basically proven by Wolf in [Wol89]; in fact, the considerations involved in his argument do not require the integrability of $\boldsymbol{H}(h_m)$ or φ over the whole $\dot{R}'$: rescaling the Hopf differential in order to have the right boundary lengths, one obtains
$$4\boldsymbol{p_m}^{-2}\boldsymbol{H}(h_m) \to \varphi \qquad \text{in } L^1_{loc}(\dot{R}')$$
- the local estimate
$$\|\boldsymbol{H}(h_m) - \boldsymbol{H}(\kappa_m)\|_{L^1(K)} \leq \sqrt{2(E_K(h_m) - E_K(\kappa_m))}\left(\sqrt{E_K(h_m)} + \sqrt{E_K(\kappa_m)}\right)$$
is obtained in the proof of Proposition 2.6.3 of [KS93]
- one easily concludes, because $\|\boldsymbol{H}(h_m) - \boldsymbol{H}(\kappa_m)\|_{L^1(K^t_m)} = O(\boldsymbol{p_m}\sqrt{t})$ and $\|\boldsymbol{H}(h_m)\|_{L^1(K^t_m)} = O(\boldsymbol{p_m}^2 t)$.

Assertion (b) is also basically proven in [Dum07b] up to minor considerations.

- Call ρ the hyperbolic metric on $\dot{R}'$ and ρ_m the Thurston metric on $\mathrm{Gr}_\infty(\Sigma_m) \cong \dot{R}'$; moreover, let β_m be the Schwarzian tensor $\beta(\rho, \rho_m) = [\mathrm{Hess}_\rho(\sigma_m) - d\sigma_m \otimes d\sigma_m]^{2,0}$, where $\sigma_m = \sigma(\rho, \rho_m) = \log(\rho_m/\rho)$.
- The decomposition ([Dum07b], Theorem 7.1)
$$\boldsymbol{S}(\mathrm{Gr}_\infty(\Sigma_m)) = 2\beta_m - 2\boldsymbol{H}(\kappa_m)$$
(where $\boldsymbol{S}$ is with respect to the Poincaré structure on $\dot{R}'$) still holds, because it relies on local considerations.
- Let K be the compact subsurface of $\dot{R}'$ obtained by removing all n horoballs of circumference $1/4$ at x'. Moreover, let $\rho_{\underline{p}}$ be the Thurston metric on Σ obtained by grafting infinite flat cylinders at the boundary of $(\mathrm{gr}_\infty, \mathcal{L})^{-1}([f], \underline{p})$ and call $\hat{\rho}_{\underline{p}} := (1 + \boldsymbol{p}^2)\rho_{\underline{p}}$ the normalized metrics. The set $\mathcal{N} = \{\hat{\rho}_{\underline{p}} \,|\, \underline{p} \in \Delta^{n-1} \times [0, \infty]\}$ is compact in $L^\infty(K)$. Thus, $\|\hat{\rho}_{\underline{p}}/\rho\|_{L^\infty(K)} < c$ and all restrictions to K of metrics in $\mathcal{N}$ are pairwise Hölder equivalent with factor and exponent dependent on $\dot{R}'$ only (same proof as in Theorem 9.2 of [Dum07b]).
- The same estimates of [Dum07b] give (Theorem 11.4)
$$\|\beta_m\|_{L^1(D_{\delta/4}, \rho)} \leq c$$
where c depends on $\dot{R}'$ and δ.
- All norms are equivalent on $H^0(R', K(x')^{\otimes 2})$, so we consider the L^1 norm on $K \subset \dot{R}'$ and we observe that $\|\psi\|_{L^1(D_{\delta/4}, \rho)} \leq c'\|\psi\|_{L^1(K)}$ for any ρ-ball of radius $\delta/4$ embedded in K.
- There exists t_0 (dependent only on $\dot{R}'$) such that $K \subset K^t_m$ for all m. Thus,
$$\begin{aligned} &\|2\boldsymbol{S}(\mathrm{Gr}_\infty(\Sigma_m)) + \varphi_{JS}\|_{L^1(K)} \leq \\ &\leq c_1\|2\boldsymbol{S}(\mathrm{Gr}_\infty(\Sigma_m)) + 4\boldsymbol{H}(\kappa_m)\|_{L^1(D_{\delta/4}, \rho)} + \|4\boldsymbol{H}(\kappa_m) - \boldsymbol{p_m}^2\varphi_{JS}\|_{L^1(K^{t_0}_m)} \leq \\ &\leq 4c_1\|\beta_m\|_{L^1(D_{\delta/4}, \rho)} + c_2(1 + \boldsymbol{p_m}\sqrt{t_0}) \leq c_3(1 + \boldsymbol{p_m}\sqrt{t_0}) \end{aligned}$$
where c_3 depends on $\dot{R}'$ only. We conclude as in (a).

References

[Abi80] William Abikoff, *The real analytic theory of Teichmüller space*, Lecture Notes in Mathematics, vol. 820, Springer-Verlag, New York, 1980.

[AC09] Enrico Arbarello and Maurizio Cornalba, *Teichmüller space via Kuranishi families*, Ann. Sc. Norm. Super. Pisa C. Sci. (5) **8** (2009), no. 1, 89–116.

[ACGH] Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths, and Joe Harris, *Geometry of Algebraic Curves II*, book in preparation.

[Ahl61] Lars Valerian Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces*, Ann. of Math. **74** (1961), 171–191.

[BE88] B. H. Bowditch and D. B. A. Epstein, *Natural triangulations associated to a surface*, Topology **27** (1988), no. 1, 91–117. MR MR935529 (89e:57004)

[Ber74] Lipman Bers, *Spaces of degenerating Riemann surfaces*, Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Princeton Univ. Press, Princeton, N.J., 1974, pp. 43–55. Ann. of Math. Studies, No. 79. MR MR0361051 (50 #13497)

[DM69] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109. MR MR0262240 (41 #6850)

[Dum06] David Dumas, *Grafting, pruning, and the antipodal map on measured laminations*, J. Differential Geom. **74** (2006), no. 1, 93–118. MR MR2260929 (2007g:30065)

[Dum07a] ———, *Erratum to: "Grafting, pruning, and the antipodal map on measured laminations" [J. Differential Geom.* **74** *(2006), no. 1, 93–118]*, J. Differential Geom. **77** (2007), no. 1, 175–176. MR MR2344358

[Dum07b] ———, *The Schwarzian derivative and measured laminations on Riemann surfaces*, Duke Math. J. **140** (2007), no. 2, 203–243. MR MR2359819

[FLP79] *Travaux de Thurston sur les surfaces*, Astérisque, vol. 66, Société Mathématique de France, Paris, 1979, Séminaire Orsay, With an English summary. MR MR568308 (82m:57003)

[Har81] W. J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978) (Princeton, N.J.), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, pp. 245–251.

[Har86] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), no. 1, 157–176. MR MR830043 (87c:32030)

[HM79] John Hubbard and Howard Masur, *Quadratic differentials and foliations*, Acta Math. **142** (1979), no. 3-4, 221–274. MR MR523212 (80h:30047)

[Kee74] Linda Keen, *Collars on Riemann surfaces*, Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Princeton Univ. Press, Princeton, N.J., 1974, pp. 263–268. Ann. of Math. Studies, No. 79.

[Kon92] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23. MR MR1171758 (93e:32027)

[KP94] Ravi S. Kulkarni and Ulrich Pinkall, *A canonical metric for Möbius structures and its applications*, Math. Z. **216** (1994), no. 1, 89–129. MR MR1273468 (95b:53017)

[KS93] Nicholas J. Korevaar and Richard M. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. **1** (1993), 561–659.

[Loo95] Eduard Looijenga, *Cellular decompositions of compactified moduli spaces of pointed curves*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 369–400. MR MR1363063 (96m:14031)

[Luo07] Feng Luo, *On Teichmüller spaces of surfaces with boundary*, Duke Math. J. **139** (2007), no. 3, 463–482. MR MR2350850

[Mas76] Howard Masur, *Extension of the Weil-Petersson metric to the boundary of Teichmuller space*, Duke Math. J. **43** (1976), no. 3, 623–635. MR MR0417456 (54 #5506)

[Mat76] J. Peter Matelski, *A compactness theorem for Fuchsian groups of the second kind*, Duke Math. J. **43** (1976), no. 4, 829–840.

[Mir07] Maryam Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. **20** (2007), no. 1, 1–23 (electronic). MR MR2257394 (2007g:14029)

[Mon09a] Gabriele Mondello, *A criterion of convergence in the augmented Teichmüller space*, Bull. Lond. Math. Soc. **41** (2009), no. 4, 733–746. MR MR2521369

[Mon09b] ———, *Riemann surfaces, ribbon graphs and combinatorial classes*, Handbook of Teichmüller theory. Vol. II, IRMA Lect. Math. Theor. Phys., vol. 13, Eur. Math. Soc., Zürich, 2009, pp. 151–215. MR MR2497787

[Mon09c] ———, *Triangulated Riemann surfaces with boundary and the Weil-Petersson Poisson structure*, J. Differential Geom. **81** (2009), no. 2, 391–436. MR MR2472178 (2010a:32027)

[MP07] Greg McShane and Robert C. Penner, *Stable curves and screens on fatgraphs*, e-print, `arXiv:0707.1468`, 2007.

[Pen87] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. **113** (1987), no. 2, 299–339. MR MR919235 (89h:32044)

[Pen92] ———, *Weil-Petersson volumes*, J. Differential Geom. **35** (1992), no. 3, 559–608. MR MR1163449 (93d:32029)

[PH92] Robert C. Penner and John Harer, *Combinatorics of train tracks*, Annals of Mathematics Studies, vol. 125, Princeton University Press, Princeton, NJ, 1992.

[Str67] Kurt Strebel, *On quadratic differentials with closed trajectories and second order poles*, J. Analyse Math. **19** (1967), 373–382. MR MR0224808 (37 #407)

[SW02] Kevin P. Scannell and Michael Wolf, *The grafting map of Teichmüller space*, J. Amer. Math. Soc. **15** (2002), no. 4, 893–927 (electronic). MR MR1915822 (2003d:32011)

[Ush99] Akira Ushijima, *A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary*, Comm. Math. Phys. **201** (1999), no. 2, 305–326. MR MR1682230 (2000a:32029)

[Wol82] Scott Wolpert, *The Fenchel-Nielsen deformation*, Ann. of Math. (2) **115** (1982), no. 3, 501–528. MR MR657237 (83g:32024)

[Wol83] ———, *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math. (2) **117** (1983), no. 2, 207–234. MR MR690844 (85e:32028)

[Wol85] ———, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985), no. 4, 969–997. MR MR796909 (87b:32040)

[Wol89] Michael Wolf, *The Teichmüller theory of harmonic maps*, J. Differential Geom. **29** (1989), no. 2, 449–479. MR MR982185 (90h:58023)

[Wol03] Scott Wolpert, *Geometry of the Weil-Petersson completion of Teichmüller space*, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003, pp. 357–393.

Università di Roma "La Sapienza", Dipartimento di Matematica, Piazzale Aldo Moro, 2, 00185 Roma - Italy

E-mail address: `mondello@mat.uniroma1.it`