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Combinatorial classes on the moduli space of curves are tautological

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Introduction

Let g and n be nonnegative integers such that $2g - 2 + n > 0$ and set $P = \{p_1, \ldots, p_n\}$. Let $\mathcal{M}_{g,P}$ denote the moduli orbifold of smooth compact Riemann surfaces S of genus g with an injection $P \hookrightarrow S$. Mumford first noticed that for $n \geq 1$ Strebel's results on quadratic differentials [Str84] make it possible to give a combinatorial description of $\mathcal{M}_{g,P}$ in terms of metrized ribbon graphs, in which each orbicell corresponds to an isomorphism class of ribbon graphs of genus g with n holes marked by P. Then Harer [Har86] used this model to establish the virtual cohomological dimension of the modular group $\Gamma_{g,P} = \pi_1^{orb}(\mathcal{M}_{g,P})$ (remember that $\mathcal{M}_{g,P}$ is a $K(\Gamma_{g,P}, 1)$ as an orbifold) and to compute the orbifold Euler characteristic of $\mathcal{M}_{g,P}$ in a joint work with Zagier [HZ86]. We refer to Harer's survey [Har88] for a more detailed bibliography.

The same model (which we denote by $\mathcal{M}_{g,P}^{comb}$) was the starting point of Kontsevich's work [Kon92] and allowed Witten and Kontsevich to guess that the tautological classes κ are related to the W cycles, where W_{2i+3} is supported on the subcomplex of ribbon graphs with a vertex of valency at least $2i + 3$. In fact W_{2i+3} determines a homology class with noncompact support on $\mathcal{M}_{g,P}$, so we naturally obtain a cohomology class with coefficients in $\mathbb{Q}$ by Poincaré duality. More precisely Kontsevich [Kon92] conjectured that W_{2i+3} is a polynomial in the kappa classes.

First results in this direction were obtained by Wolpert [Wol83] and Penner [Pen92] [Pen93], who dealt (with some minor mistakes) with the simplest case $W_5 = 12\kappa_1$. The approach of Arbarello and Cornalba [AC96] passes through Di Francesco-Itzykson-Zuber's theorem [DFIZ93] and Kontsevich's compactification $\overline{\mathcal{M}}^{comb}_{g,P}$ and led to stronger results.

In fact let $m_* = (m_{-1}, m_0, m_1, \dots)$ be a sequence of nonnegative integers such that $\sum_{i\geq -1}(2i+3)m_i = 2(6g-6+3n)$ and let $\mathcal{M}^{comb}_{m_*,P}$ be the orbicellular complex of ribbon graphs whose top-dimensional orbicells are parametrized by ribbon graphs with m_i vertices of valency $2i+3$. Notice, by the way, that $\mathcal{M}^{comb}_{m_*,P} \cong \mathcal{M}^{comb}_{g,P}$ if $m_* = (0, 4g-4+n, 0, 0, \dots)$. For every $l = (l_{p_1}, \dots, l_{p_n}) \in \mathbb{R}^P_+$ denote by $\mathcal{M}^{comb}_{m_*,P}(l)$ the subset of graphs in $\mathcal{M}^{comb}_{m_*,P}$ such that the p_i-th hole has perimeter $2l_{p_i}$. Remark that $\mathcal{M}^{comb}_{m_*,P}$ is (not canonically) isomorphic to $\mathcal{M}^{comb}_{m_*,P}(l) \times \mathbb{R}^P_+$ for any $l \in \mathbb{R}^P_+$.

Kontsevich [Kon92] proved that for every $l \in \mathbb{R}^P_+$ the orbicomplex $\mathcal{M}^{comb}_{m_*,P}(l)$ has an orientation and the classifying map $\mathcal{M}^{comb}_{m_*,P}(l) \to \mathcal{M}_{g,P}$ defines a homology class with noncompact support $W_{m_*,P}$ on $\mathcal{M}_{g,P}$ that does not depend on the choice of $l \in \mathbb{R}^P_+$ and which will be called *combinatorial class*. Moreover he introduced combinatorial realtive compactifications $\overline{\mathcal{M}}^{comb}_{m_*,P}$ which still have orientations and (in the case $m_{-1} = 0$) embed as subcomplexes into $\overline{\mathcal{M}}^{comb}_{g,P}$; so that they define cycles $\overline{W}_{m_*,P}(l)$ in $H^{nc}_*(\overline{\mathcal{M}}^{comb}_{g,P}(l); \mathbb{Q})$. Even if $\overline{\mathcal{M}}^{comb}_{g,P}(l)$ is homeomorphic to a quotient $\overline{\mathcal{M}}'_{g,P}$ of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,P}$ for all $l \in \mathbb{R}^P_+$ and the class $\overline{W}_{m_*,P}(l)$ on $\overline{\mathcal{M}}'_{g,P}$ does not depend on l, however $\overline{\mathcal{M}}'_{g,P}$ has ugly singularities, so we cannot use Poincaré duality to lift the cohomology class $\overline{W}_{m_*,P}$ via the projection

$$\overline{\mathcal{M}}_{g,P} \longrightarrow \overline{\mathcal{M}}'_{g,P} \cong \overline{\mathcal{M}}^{comb}_{g,P}(l)$$

to $\overline{\mathcal{M}}_{g,P}$.

Back to Arbarello and Cornalba's work, they found a way to compute in principle all the $W_{m_*,P}$ in terms of the kappa classes and reported their results in lower codimensions, giving a strong evidence to Witten-Kontsevich's conjecture. For example they discovered that on $\mathcal{M}_{g,P}$ the cycle $W_{(0,m_0,3,0,\dots),P}$ is dual (in a sense to be made precise) to $288\kappa_1^3 - 4176\kappa_1\kappa_2 + 20736\kappa_3$. Looking at a number of results such as the previous one, they refined the conjecture as follows.

Conjecture ([AC96]). *Consider the algebra of polynomials* $\mathbb{Q}[t_*] := \mathbb{Q}[t_1, t_2, \dots]$ *where each* t_i *has degree* 1. *Then for every* m_*

such that $m_{-1} = 0$ *there exists a polynomial* $f_{m_*} \in \mathbb{Q}[t]$ *of degree* $\sum_{i\geq 1} m_i$ *such that* $W_{m_*,P} = f_{m_*}(\kappa)$*. Moreover* f_{m_*} *looks like*

$$f_{m_*}(t) = \prod_{i\geq 1} \frac{(2^{i+1}(2i+1)!!)^{m_i}}{m_i!} t_i^{m_i} + \textit{(terms of lower degree).}$$

In any event, the meaning of the other coefficients of f_{m_*} was still obscure.

Really they compared the combinatorial classes and the kappa classes as functionals on the algebra generated by the psi classes, which are defined both on $\overline{\mathcal{M}}_{g,P}$ and on $\overline{\mathcal{M}}^{comb}_{g,P}(l)$. In this way they were able to compute the difference $W_{m_*,P} - f_{m_*}(\kappa)$ in some concrete cases up to some minor uncertainty.

In this thesis we give an affirmative answer to the previous conjecture and we exhibit a formula that permits to compute all the polynomials f_{m_*} inductively on their degree.

Quite recently K. Igusa [Igu02] [Igu03] and K. Igusa-M. Kleber [IK03] have proven very similar results by different methods.

The proof proceeds in the following way. Given the projection map $\overline{\mathcal{M}}_{g,P} \to \overline{\mathcal{M}}^{comb}_{g,P}(l)$ for some l, we canonically lift the cycles $\overline{W}(l)$ in $\overline{\mathcal{M}}^{comb}_{g,P}(l)$ to cycles $\widehat{W}$ on $\overline{\mathcal{M}}_{g,P}$ using, in an essential way, a modification $\widehat{A}(S_g, P)$ of the arc complex (where (S_g, P) is a P-pointed compact orientable surface of genus g) introduced by Looijenga (see [Loo95]). This modification comes equipped with a map

$$\widehat{\mathcal{M}}^{comb}_{g,P} := \widehat{A}(S_g, P)/\Gamma_{g,P} \longrightarrow \overline{\mathcal{M}}_{g,P} \times \Delta_P$$

which is generically $1-1$.

Then we remark that the $\widehat{W}_{m_*,P}$ classes are push-forward via the forgetful map $\pi_Q : \overline{\mathcal{M}}_{g,P\cup Q} \to \overline{\mathcal{M}}_{g,P}$ of some generalized combinatorial classes $\widehat{W}_{m_*,\rho,P}$ associated to some $\rho : Q \to \mathbb{Z}_{\geq -1}$ defined prescribing that every $q \in Q$ marks a vertex of valency $2\rho(q) + 3$. The simplest case is the class

W^q_{2r+3} supported on the subcomplex of $P \cup \{q\}$-marked ribbon graphs in which q marks a vertex of valency $2r+3$.

Notice that the kappa classes are obtained as push-forward of psi classes via the forgetful morphisms and in particular that $(\pi_q)_*(\psi_q^{r+1}) = \kappa_r$. So for example in order to prove that $c_r \widehat{W}_{2r+3} + \widehat{B}_{2r+3} = \kappa_r$ in $H^{2r}(\overline{\mathcal{M}}_{g,P};\mathbb{Q})$ where $r \geq 1$, $c_r \in \mathbb{Q}$ and $\widehat{B}_{2r+3}$ is a boundary class, it is sufficient to prove that $c_r \widehat{W}^q_{2r+3} + \widehat{B}^q_{2r+3} = \psi_q^{r+1}$ in $H^{2r+2}(\overline{\mathcal{M}}_{g,P\cup\{q\}};\mathbb{Q})$ and that $(\pi_q)_*(\widehat{B}^q_{2r+3}) = \widehat{B}_{2r+3}$, since $(\pi_q)_*(\widehat{W}^q_{2r+3}) = \widehat{W}_{2r+3}$, if $r \geq 1$.

Hence the problem translates to showing that

$$\int_{\overline{\mathcal{M}}_{g,P\cup\{q\}}} \psi_q^{r+1} \smile \eta = c_r \int_{\widehat{W}^q_{2r+3}} \eta + \int_{\widehat{B}^q_{2r+3}} \eta$$

for all $\eta \in H^*(\overline{\mathcal{M}}_{g,P\cup\{q\}};\mathbb{Q})$.

As Kontsevich found a nice PL differential form $\overline{\omega}_q$ on $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$ that pulls back to ψ_q on $\overline{\mathcal{M}}_{g,P\cup\{q\}}$, the key ideas are:

1. Translate the calculation on the combinatorial spaces to exploit the explicit differential forms $\overline{\omega}_q$: this is not difficult but involves some technicalities and a little uncertainty in the description of the boundary component $\widehat{B}^q_{2r+3}$.

2. Find a deformation retraction $\mathcal{H}_0$ that shrinks the q-th hole and makes it possible to recover the combinatorial class as "push-forward" of $\overline{\omega}_q^{r+1}$ via $\mathcal{H}_0$. To do so we must restrict our attention to η's living on $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$ which are pull-back via $\mathcal{H}_0$. However this will be sufficient for our purposes.

Once we have our retraction $\mathcal{H}_0$, we can look at $\mathcal{H}_0^*\eta$ and discover that $\overline{\omega}_q^{r+1} \wedge \mathcal{H}_0^*\eta$ is supported on the smallest subcomplex $\overline{Y}_{2r+3} \subset \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$ which contains all the cells parametrized by ordinary ribbon graphs whose q-th hole is bordered by $2r+3$ edges. Then we dissect $\overline{Y}_{2r+3}$ into subcomplexes $\overline{Y}^i_{2r+3}$ according to the topology of the q-th hole. In this way the restriction of $\mathcal{H}_0$ to each $\overline{Y}^i_{2r+3}$ is generically a fibration whose fibers F^i are simplicial complexes of dimension $2r+2$. Hence

$$\int_{\widehat{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(l)} \psi_q^{r+1} \smile \mathcal{H}_0^*\eta = \sum_i \int_{\mathcal{H}_0(\overline{Y}^i_{2r+3}(l))} \eta \int_{F^i} \overline{\omega}_q^{r+1}$$

and we get the result analyzing $\mathcal{H}_0(\overline{Y}^i_{2r+3})$ and computing the integral on the fibers. For example, the class $\widehat{W}^q$ arises as image via $\mathcal{H}_0$ of top-dimensional simplices when the hole q is contractible, i.e. no edge borders the hole q from both sides. In this case the fiber is just one simplex and the integral on the fiber is exactly $\frac{(r+1)!}{(2r+2)!}$.

Theorem A. *For any g and $n \geq 1$ the equality*

$$\widehat{W}^q_{2r+3} = \frac{(2r+2)!}{(r+1)!}\psi_q^{r+1}$$

holds in $H^{2r+2}(\overline{\mathcal{M}}_{g,P\cup\{q\}})$ up to terms in the kernel of $H_{6g-6+2n-2r}(\overline{\mathcal{M}}_{g,P\cup\{q\}}) \to H_{6g-6+2n-2r}(\overline{\mathcal{M}}_{g,P}, \partial\mathcal{M}_{g,P})$. As a consequence

$$\widehat{W}_{2r+3} = \begin{cases} 0 & \text{if } r=-1 \\ [\overline{\mathcal{M}}_{g,P}] & \text{if } r=0 \\ 2^{r+1}(2r+1)!!\,\kappa_r & \text{if } r \geq 1 \end{cases}$$

holds in $H^{2r}(\overline{\mathcal{M}}_{g,P})$ up to boundary terms.

In fact our proof shows more as it determines quite precisely the boundary terms $\widehat{B}^q_{2r+3}$ and $\widehat{B}_{2r+3}$ up to some uncertainty. As an example we have the following corollary which was already proven by Arbarello and Cornalba in a very different manner [AC96].

Corollary A.1. *For every g and $n \geq 1$ such that $2g-2+n > 0$ the following equality*

$$\widehat{W}_5 = 12\kappa_1 - \delta_{irr} - \sum_{g',I\neq\emptyset,P} \delta_{g',I}$$

holds in $H^2(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ up to Poincaré duals of elements in the kernel of $H_{6g-8+2n}(\overline{\mathcal{M}}_{g,P};\mathbb{Q}) \longrightarrow H_{6g-8+2n}(\overline{\mathcal{M}}'_{g,P};\mathbb{Q})$, where δ_{irr} is the divisor of irreducible surfaces with one node and $\delta_{g',I}$ is the divisor of surfaces with two components of type (g', I) and $(g-g', P\setminus I)$ intersecting in a node.

Next we pass to a general combinatorial class $\widehat{W}_{m_*,P}$. As explained before, we recover them as push-forward of some $\widehat{W}_{m_*,\rho,P}$ on $\widehat{\mathcal{M}}^{comb}_{g,P\cup Q}$ via π_Q. However the notations and the results about the classes $\widehat{W}_{m_*,\rho,P}$ are quite heavy to state, so here we content ourselves to state the theorem in the simpler case of $\widehat{W}_{m_*,P}$ and we refer to Chapter 5 for more complete results.

The techniques are analogous as in the proof of Theorem A but here new combinatorial problems arise. However the only new idea is to think of the retraction $\mathcal{H}_0 : \overline{\mathcal{M}}^{comb}_{g,P\cup Q} \to \overline{\mathcal{M}}^{comb}_{g,P\cup Q}$ as a sequence of shrinkings $\mathcal{H}^i_0$ of the holes q_i in a fixed order and then to reduce the problem to the shrinking of one hole only, which we have already dealt with before. Now we just state the main result.

For every $m_* = (0, m_0, m_1, m_2, \dots)$ choose Q' such that $|Q'| = \sum_{i\geq 1} m_i$ and a $\tilde{\rho} : Q' \to \mathbb{N}$ such that $|\tilde{\rho}^{-1}(j)| = m_j$ for all $j \geq 1$. Let $\mathfrak{P}_{Q'}$ be the set of partitions of Q' and for all $\mu \subset Q'$ define $\tilde{\rho}_\mu = \sum_{i\in\mu} \tilde{\rho}(i)$.

Theorem B (simplified version). *For any g and $n \geq 1$ the following relation holds in $H^*(\overline{\mathcal{M}}_{g,P})$ up to boundary terms:*

$$\left(2^{\sum_{q\in Q'}(\tilde{\rho}(q)+1)} \prod_{q\in Q'} (2\tilde{\rho}(q)+1)!!\right) \sum_{\sigma\in\mathfrak{S}_{Q'}} \kappa_{r(\sigma)} = \\ = \sum_{M\in\mathfrak{P}_{Q'}} \frac{m_*(M)!}{m_0!} \tilde{c}_M \widehat{W}_{m_*(M),P}$$

where $m_i(M) = |\{\mu \in M | \tilde{\rho}_\mu = i\}| + \delta_{i,0} m_0$ and

$$\tilde{c}_M := \prod_{\mu\in M} \tilde{c}_\mu \qquad \tilde{c}_\mu := \frac{(2\tilde{\rho}_\mu + 2|\mu| - 1)!!}{(2\tilde{\rho}_\mu + 1)!!}$$

and moreover $\kappa_{r(\sigma)}$ is a monomial in the kappa classes (see Section 1.3).

The theorem gives an inductive recipe on $|Q'|$ to calculate all the coefficients of f_{m_*}. As an example we have the following.

Corollary B.2. *For every nonnegative g and positive n such that $2g-2+n > 0$ and for every $a, b \geq 1$ the following identity*

$$2^{\delta_{a,b}} \widehat{W}_{2a+1,2b+1} = 2^{a+b+2}(2a+1)!!(2b+1)!!(\kappa_a\kappa_b + \kappa_{a+b}) \\ - 2^{a+b+1}(2a+2b+3)!!\kappa_{a+b}$$

holds in $H^{2a+2b}(\overline{\mathcal{M}}_{g,P}; \mathbb{Q})$ up to boundary terms.

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Chapter 1

The geometric point of view

In this chapter we recall some basic definitions and some elementary facts from the theory of moduli spaces of curves. We follow Teichmüller's point of view but we introduce also the Deligne-Mumford compactification and its stratification by topological type. Then we define the tautological ring and its cohomological analogue and we recall some properties of the psi and kappa classes that will be useful later. In the last section we construct a slight modification of Kontsevich's compactification of $\mathcal{M}_{g,P}$ which is a quotient of Deligne-Mumford's one. We explain why the contraction map cannot be a morphism of schemes but just a continuous surjection.

1.1 The Teichmüller functor and the moduli space of Riemann surfaces

Let S be a compact connected oriented surface of genus g and let $P \hookrightarrow S$ be an injection of n points such that $2g-2+n>0$.

Definition 1.1.1. A family of P-pointed surfaces is a couple (π, s) where $\pi : \mathcal{C} \to B$ is a proper differentiable submersion whose fibers are oriented connected surfaces and $\{s_p : B \to \mathcal{C} \,|\, p \in P\}$ is a collection of disjoint sections. An (S,P)-*marking* is an equivalence class of oriented diffeomorphisms

$$f : S \times B \xrightarrow{\sim} \mathcal{C}$$

that commute with the projections onto B and such that $f(p,b) = s_p(b)$ for every $p \in P$. Two markings $f \sim \tilde{f}$ are *equivalent* if and only if

$$\tilde{f}^{-1} \circ f : (S,P) \times B \longrightarrow (S,P) \times B$$

is vertically (i.e. over B) isotopic to the identity relatively to P.

A *conformal structure* on $S \backslash P$ is an atlas such that the changes of coordinates are differentiable and preserve the angles. There is an obvious bijection between conformal structures and complex structures (via isothermal coordinates) and between conformal structures and Riemannian metrics up to multiplication by a positive function. Remark that every complex structure on $S \setminus P$ can be extended to the whole S in a unique way.

Definition 1.1.2. Let (π, s) be a family of P-pointed surfaces. A *conformal structure* on (π, s) is a differentiable atlas of $\mathcal{C}$ which endows $\mathcal{C}_b \setminus \cup s_p(b)$ with a conformal structure for all $b \in B$; equivalently, it is a family of metrics h_b on $\mathcal{C}_b \setminus \cup s_p(b)$ smoothly depending on b up to multiplication by a positive function on $\mathcal{C} \setminus \cup s_p(B)$.

We say that two marked families $(\mathcal{C}, f)$ and $(\mathcal{C}', f')$ of P-pointed surfaces with conformal structure are *isomorphic* if there is a diffeomorphism $t : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ such that $t \circ f' = f$ and the restriction to each fiber $t_b : \mathcal{C}_b \xrightarrow{\sim} \mathcal{C}'_b$ is conformal outside the sections.

The *Teichmüller functor*

$$\mathfrak{T}_{S,P} : (\text{Top. Spaces}) \longrightarrow (\text{Sets})$$

associates to every manifold B the set of isomorphism classes of (S,P)-marked families of P-pointed surfaces over B with conformal structure. It is represented by a complex smooth manifold $\mathcal{T}_{S,P}$ analytically isomorphic to a ball of complex dimension $3g - 3 + n$. Except in the case $(g,n) = (0,3)$ it is never compact.

The *modular group* $\Gamma_{S,P} := \mathrm{Diff}_+(S,P)/\mathrm{Diff}_0(S,P)$ of connected components of the space of oriented diffeomorphisms of (S,P) acts on the (S,P)-markings and so on $\mathfrak{T}_{S,P}$. Its quotient is denoted by $\mathfrak{M}_{g,P}$ and classifies smooth families of compact P-pointed Riemann surfaces of genus g. For $g = 0$ this functor is represented by a smooth affine variety of complex dimension $n - 3$. On the contrary $\mathfrak{M}_{g,P}$ is not represented by a space for $g \geq 1$ (due to the existence of Riemann surfaces with nontrivial automorphisms); hence the topological quotient $M_{g,P} := \mathcal{T}_{S,P}/\Gamma_{S,P}$ which is a normal quasi-projective irreducible variety of (complex) dimension $3g-3+n$ is only a *coarse moduli space*.

The functor $\mathfrak{M}_{g,P}$ admits a natural extension $\overline{\mathfrak{M}}_{g,P}$ that classifies flat families of stable P-pointed complex curves of (arithmetic) genus g, where *stable* means that the singularities look like $\{xy = 0\} \subset \mathbb{C}^2$ in local analytic coordinates and that each connected component of the smooth locus has negative topological Euler characteristic. The functor $\overline{\mathfrak{M}}_{g,P}$ hass a coarse moduli space $\overline{M}_{g,P}$ which is a normal irreducible projective variety with quotient singularities and which contains $M_{g,P}$ as a Zariski-dense open subset. It can be seen that $\overline{\mathfrak{M}}_{g,P}$ is in fact represented by a smooth Deligne-Mumford stack $\overline{\mathcal{M}}_{g,P}$ (or an orbifold) which is proper and connected. As before, the stack $\overline{\mathcal{M}}_{0,P}$ is in fact a smooth projective variety.

1.2 The system of moduli spaces of curves

Many facts suggest that one should not look at the moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$ each one separately, but one must consider the whole system $(\overline{\mathcal{M}}_{g,n})_{2g-2+n>0}$. An evidence is given by the existence of three families of maps that connect different moduli spaces.

1. The *forgetful map* is a projective flat morphism

$$\pi_q : \overline{\mathcal{M}}_{g,P\cup\{q\}} \longrightarrow \overline{\mathcal{M}}_{g,P}$$

that forgets the point q and stabilizes the curve (i.e. contracts a possible two-pointed sphere). This map can be identified to the universal family and so is endowed with natural sections

$$\vartheta_{0,\{p_i,q\}} : \overline{\mathcal{M}}_{g,P} \to \overline{\mathcal{M}}_{g,P\cup\{q\}}$$

for all $p_i \in P$.

2. The *boundary map* corresponding to irreducible curves is the finite map
$$\vartheta_{irr} : \overline{\mathcal{M}}_{g-1,P\cup\{p',p''\}} \longrightarrow \overline{\mathcal{M}}_{g,P}$$
(defined for $g > 0$) that glues p' and p'' together. It is generically $2-1$ and its image sits in the boundary of $\overline{\mathcal{M}}_{g,P}$.

3. The *boundary maps* corresponding to reducible curves are the finite maps
$$\vartheta_{g',I} : \overline{\mathcal{M}}_{g',I\cup\{p'\}} \times \overline{\mathcal{M}}_{g-g',I^c\cup\{p''\}} \longrightarrow \overline{\mathcal{M}}_{g,P}$$
(defined for every $0 \leq g' \leq g$ and $I \subseteq P$ such that the spaces involved are nonempty) that take two curves and glues them together identifying p' and p''. They are generically $1-1$ (except in the case $g = 2g'$ and $P = \emptyset$ when the map is generically $2-1$) and their images sit in the boundary of $\overline{\mathcal{M}}_{g,P}$ too.

The boundary maps naturally define Cartier divisors in $\overline{\mathcal{M}}_{g,P}$ corresponding to their images. We will denote by $\delta_{irr} \subset \overline{\mathcal{M}}_{g,P}$ and $\delta_{g',I} \subset \overline{\mathcal{M}}_{g,P}$ the cycles supported on the image of ϑ_{irr} and $\vartheta_{g',I}$ respectively.

We observe that $\overline{\mathcal{M}}_{g,P}$ has a natural *stratification* by topological type of the complex curve. In fact we can attach to every stable curve S its *dual graph* γ_S whose vertices correspond to irreducibile components and whose edges correspond to nodes of S. Moreover every vertex is labelled by a couple (g_v, P_v), where g_v is the geometric genus of the component S_v associated to v and $P_v \subset P$ is the set of marked points lying on S_v. Moreover we call Q_v the singular points of S_v.

For every such labelled graph γ we can construct a boundary map
$$\vartheta_\gamma : \prod_v \overline{\mathcal{M}}_{g_v,P_v\cup Q_v} \longrightarrow \overline{\mathcal{M}}_{g,P}$$
which is a finite morphism. We call its image δ_γ.

When there is no risk of confusion, we will denote by the same symbol the cycles and the associated classes in the Chow ring (or in cohomology). Remember by the way that the moduli spaces $\overline{\mathcal{M}}_{g,P}$ of complex projective stable curves have also the structure of smooth proper Deligne-Mumford stacks over $\mathbb{C}$ (or of compact analytic orbifolds). So it is possible to define the Chow intersection ring with rational coefficients $CH^*(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}}$ (in fact it is also possible to define integral Chow rings such that $CH^*(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}} = CH^*(\overline{\mathcal{M}}_{g,P}) \otimes \mathbb{Q}$).

1.3 Tautological classes

All the maps we have defined are in some sense tautological as they are very naturally constructed and they reflect intrinsic relations among the various moduli spaces. It is apparent that one can look at them as classifying maps in the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,P}$, which obviously descend to maps between coarse moduli spaces. Hence we can consider all the cycles obtained by push-forward or pull-back via these map as tautologically defined. However there is an ingredient we have not considered yet: it is the relative dualizing sheaf of the universal curve π_q. One expects that it carries many informations and that it can produce many classes of interest.

Denoted by ω_{π_q} the relative dualizing sheaf, define the *Miller classes* as

$$\psi_{p_i} := c_1(\mathcal{L}_i) \in CH^1(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}}$$

where $\mathcal{L}_i := \vartheta^*_{0,\{p_i,q\}}\omega_{\pi_q}$ and the modified (by Arbarello-Cornalba) *Mumford-Morita classes* as

$$\kappa_j := (\pi_q)_*(c_1(\omega_{\pi_q}(\sum_i \delta_{0,\{p_i,q\}}))^{j+1}) \in CH^j(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}}.$$

One could moreover define the l-th *Hodge bundle* as $\mathbb{E}_l := (\pi_q)_*(\omega_{\pi_q}^{\otimes l})$ and consider the Chern classes of these bundles (for example, the lambda classes $\lambda_i := c_i(\mathbb{E}_1)$). However, using Grothendieck-Riemann-Roch, Mumford [Mum83] and Bini [Bin02] proved that $c_i(\mathbb{E}_j)$ can be expressed as a linear combination of Mumford-Morita classes up to elements in the boundary, so that they do not introduce anything really new.

When there is no risk of ambiguity, we will denote in the same way the classes ψ and κ belonging to different $\overline{\mathcal{M}}_{g,P}$'s as it is now traditional.

Because of the natural definition of κ and ψ classes, as explained before, the subring $R^*(\mathcal{M}_{g,P})$ of $CH^*(\mathcal{M}_{g,P})_{\mathbb{Q}}$ they generate is called the *tautological ring* of $\mathcal{M}_{g,P}$. Its image $RH^*(\mathcal{M}_{g,P})$ through the cycle class map is called cohomology tautological ring.

The *system of tautological rings* $(R^*(\overline{\mathcal{M}}_{g,P}) \subset CH^*(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}})_{2g-2+n>0}$ is the minimal system of subrings that contain the classes κ and ψ which is closed under the push-forward maps π_*, $(\vartheta_{irr})_*$ and $(\vartheta_{g',I})_*$. The definition is the same for the rational cohomology.

As it is evident from the definition, the classes psi and kappa are very strictly related. In fact

$$(\pi_q)_*(\psi_{p_1}^{r_1}\cdots\psi_{p_n}^{r_n}) = \sum_{\{i|r_i>0\}} \psi_{p_1}^{r_1}\cdots\psi_{p_i}^{r_i-1}\cdots\psi_{p_n}^{r_n}$$

$$(\pi_q)_*(\psi_{p_1}^{r_1}\cdots\psi_{p_n}^{r_n}\psi_q^{b+1}) = \psi_{p_1}^{r_1}\cdots\psi_{p_n}^{r_n}\kappa_b$$

where the first one is the so-called *string equation* and the second one for $b=0$ is the *dilaton equation*. They have been generalized by Faber for maps that forget more than one point in a formula which can be proven using the second equation before and the relation

$$\pi_q^*(\kappa_j) = \kappa_j - \psi_q^j.$$

Let $Q := \{q_1,\ldots,q_m\}$ and let $\pi_Q : \overline{\mathcal{M}}_{g,P\cup Q} \to \overline{\mathcal{M}}_{g,P}$ be the forgetful map. Then

$$(\pi_Q)_*(\psi_{p_1}^{r_1}\cdots\psi_{p_n}^{r_n}\psi_{q_1}^{b_1+1}\cdots\psi_{q_m}^{b_m+1}) = \psi_{q_1}^{r_1}\cdots\psi_{q_n}^{r_n}K_{b_1\cdots b_m}$$

where $K_{b_1\cdots b_m} = \sum_{\sigma\in\mathfrak{S}_m}\kappa_{b(\sigma)}$ and $\kappa_{b(\sigma)}$ is defined in the following way. If $\gamma = (c_1,\ldots,c_l)$ is a cycle, then set $b(\gamma) := \sum_{j=1}^{l} b_{c_j}$. If $\sigma = \gamma_1\cdots\gamma_\nu$ is the decomposition in disjoint cycles (including 1-cycles), then we pose $k_{b(\sigma)} := \prod_{i=1}^{\nu}\kappa_{b(\gamma_i)}$. We refer to [KMZ96] for more details on Faber's formula, to [AC96] and [AC98] for more properties of tautological classes and to [Fab99] for a conjectural description of the tautological rings.

1.4 Kontsevich's compactification

It has been observed by Witten [Wit91] that the intersection theory of kappa and psi classes can be reduced to that of psi classes only by using the push-pull formula with respect to the forgetful morphisms. Moreover recall that

$$\psi_p = c_1(\omega_{\pi_p}(D_p))$$

on $\overline{\mathcal{M}}_{g,P}$, where $D_p = \sum_{p' \in P} \delta_{0,\{p,p'\}}$. So, in order to find a "minimal" projective compactification of $\mathcal{M}_{g,P}$ where to compute the intersection numbers of the psi classes, it is natural to look at the maps induced by the linear system $\mathbb{L} := \sum_{p \in P} \omega_{\pi_p}(D_p)$. It is well-known that $\mathbb{L}$ is nef (Arakelov) and big, so that the problem is to decide whether $\mathbb{L}$ is semi-ample and to determine its exceptional locus $Ex(\mathbb{L}^{\otimes l})$ for $l >> 0$.

It is easy to see that $\mathbb{L}^{\otimes l}$ pulls back to the trivial line bundle via the boundary map $\overline{\mathcal{M}}_{g',\{p'\}} \times \{C\} \longrightarrow \overline{\mathcal{M}}_{g,P}$, where C is a fixed curve of genus $g - g'$ with a $P \cup \{p''\}$-marking and the map glues p' with p''. Hence the map induced by the linear system $\mathbb{L}^{\otimes l}$ (if base-point-free) should restrict to the projection $\overline{\mathcal{M}}_{g,\{p'\}} \times \overline{\mathcal{M}}_{g-g',P\cup\{p''\}} \longrightarrow \overline{\mathcal{M}}_{g-g',P\cup\{p''\}}$ on these boundary components.

While $\mathbb{L}$ is semi-ample in characteristic p, it is not in characteristic 0 (Keel [Kee99]). However one can still topologically contract the exceptional (with respect to $\mathbb{L}$) curves to obtain Kontsevich's map

$$\xi' : \overline{\mathcal{M}}_{g,P} \longrightarrow \overline{\mathcal{M}}'_{g,P}$$

which is a proper continuous surjection of orbispaces. A consequence of Keel's result is that the coarse $\overline{M}'_{g,P}$ cannot be given a scheme structure such that the contraction map is a morphism. It is in some sense unexpected because the morphism behaves as if it were algebraic, in particular the fiber product $\overline{M}_{g,P} \times_{\overline{M}'_{g,P}} \overline{M}'_{g,P}$ is projective.

So now we leave the realm of algebraic geometry and proceed topologically to construct and describe this different compactification. In fact we introduce a slight modification of Kontsevich's construction (see [Kon92]) which will be very useful in the future. We realize it as a quotient of

$\overline{\mathcal{M}}_{g,P} \times \Delta_P$ by an equivalence relation, where Δ_P is the standard simplex $\Delta_P := \{l \in \mathbb{R}^P_{\geq 0} | \sum_{p \in P} l_p = 1\}$.

If (S, l) is an element of $\overline{\mathcal{M}}_{g,P} \times \Delta_P$, then we say that an irreducible component of S is *positive* (with respect to l) if it contains a point $p \in P$ such that $l_p > 0$. Similarly we say that a vertex v of the dual graph γ_S is positive if the associated irreducible component S_v is.

Next we declare that (S, l) is equivalent to (S', l') if $l = l'$ and there is a homeomorphism of pointed surfaces $S \xrightarrow{\sim} S'$ which is analytic on the positive components of S. As this relation would not give back a Hausdorff space we consider its closure, which can be described as follows.

Given (S, l) as before, consider the following two moves on the dual graph γ_S:

1. if two nonpositive vertices v and v' are joined by an edge e, then we can build a new graph discarding e, melting v and v' together and obtaining a new vertex w which we label with $(g_w, P_w) := (g_v + g_{v'}, P_v \cup P_{v'})$

2. if a nonpositive vertex v has a loop e, we can make a new graph discarding e and labelling v with $(g_v + 1, P_v)$.

Applying these moves to γ_S iteratively until the process ends, we are given back a *reduced dual graph* $\gamma^{red}_{S,l}$. Call $V_0(S, l)$ the subset of vertices v of $\gamma^{red}_{S,l}$ such that $l_p = 0$ for every $p \in P_v$ and $V_+(S, l)$ the subset of positive vertices of $\gamma^{red}_{S,l}$.

For every couple (S, l) denote by $\bar{S}$ the quotient of S obtained collapsing every nonpositive component to a point. Given (S, l) and (S', l') it is clear that a homeomophism $\bar{S} \to \bar{S}'$ of P-pointed spaces induces an isomorphism of graphs $\gamma^{red}_{S,l} \to \gamma^{red}_{S',l'}$ which does not necessarily preserve the labels g_v.

We say that (S, l) and (S', l') are equivalent if $l = l'$ and there exists a homeomorphism $\bar{f} : \bar{S} \xrightarrow{\sim} \bar{S}'$ whose restriction to each component is analytic and which induces an isomorphism $f^{red} : \gamma^{red}_{S,l} \xrightarrow{\sim} \gamma^{red}_{S',l'}$ of reduced dual graphs. Finally call

$$\xi : \overline{\mathcal{M}}_{g,P} \times \Delta_P \longrightarrow \overline{\mathcal{M}}^{\triangle}_{g,P} := \overline{\mathcal{M}}_{g,P} \times \Delta_P / \sim$$

the quotient map and remark that $\overline{\mathcal{M}}^{\triangle}_{g,P}$ is compact and that ξ commutes with the projection onto Δ_P.

For every l in Δ_P we will denote by $\overline{\mathcal{M}}^{\triangle}_{g,P}(l)$ the subset of points of the type $[S,l]$ and we will write $\overline{\mathcal{M}}^{\triangle}_{g,P}(L)$ for $\cup_{l\in L}\overline{\mathcal{M}}^{\triangle}_{g,P}(l)$ where $L\subset\Delta_P$. Then it is easy to see that $\overline{\mathcal{M}}^{\triangle}_{g,P}(\Delta_P^{\circ})$ is in fact homeomorphic to a product $\overline{\mathcal{M}}^{\triangle}_{g,P}(l)\times\Delta_P^{\circ}$ for any given $l\in\Delta_P^{\circ}$. Observe that $\overline{\mathcal{M}}^{\triangle}_{g,P}(l)$ is isomorphic to $\overline{\mathcal{M}}'_{g,P}$ for all $l\in\Delta_P^{\circ}$ in such a way that

$$\xi_l:\overline{\mathcal{M}}_{g,P}\cong\overline{\mathcal{M}}_{g,P}\times\{l\}\longrightarrow\overline{\mathcal{M}}^{\triangle}_{g,P}(l)$$

identifies to ξ'.

Notice by the way that the fibers of ξ are isomorphic to moduli spaces. More precisely consider a point $[S,l]$ of $\overline{\mathcal{M}}^{\triangle}_{g,P}$. For every $v\in V_0(S,l)$ call Q_v the subset of edges of $\gamma^{red}_{S,l}$ outgoing from v. Then we have the natural isomorphism

$$\xi^{-1}([S,l])\cong\prod_{v\in V_0(S,l)}\overline{\mathcal{M}}_{g_v,P_v\cup Q_v}$$

according to the fact that $\overline{M}_{g,P}\times_{\overline{M}'_{g,P}}\overline{M}_{g,P}$ is projective.

Chapter 2

The combinatorial point of view

Now we want to introduce a different approach to the moduli space of Riemann surfaces, namely we want to give simplicial or cellular structure to the objects we have met so far. First we define the arc complex (see [Har88]) and we describe how an open subset of it triangulates $\mathcal{M}_{g,P}$ when P is nonempty. In this description, simplices are parametrized by systems of disjoint arcs between couples of punctures and ribbon graphs appear in some sense as a dual notion. However they become the central object when we want to deal with stable surfaces. We follow Looijenga's treatment (see [Loo95]) and look at stable surfaces as degeneration of smooth surfaces obtained by iterated collapses. In this way we can define a modified arc complex using stable ribbon graphs that nearly cellularizes the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,P}$. In the last section, we cellularizes Kontsevich's compactification (see [Kon92]) by means of ordinary ribbon graphs and we illustrate how these different complexes are related to one another.

2.1 The arc complex

Fix a compact connected oriented surface S of genus g and an injection $P := \{p_1, \dots, p_n\} \hookrightarrow S$ with $n > 0$.

Let $\mathcal{A}$ be the set of isotopy classes relative to P of embedded unoriented loops or arcs $\alpha \subset S$ that intersect P exactly in the extremal point(s). The *arc complex* is the abstract simplicial complex $A(S,P)$ whose k-simplices are subsets $\alpha = \{\alpha_0, \ldots, \alpha_k\}$ of $\mathcal{A}$ that are representable by a system of $k+1$ arcs and loops intersecting only in P. We will denote its geometric realization by $|A|$.

A simplex $\alpha = \{\alpha_0, \ldots, \alpha_k\}$ of A is called *proper* if its star is finite, or equivalently if $S \setminus \cup_{i=0}^{k} \alpha_i$ is a disjoint union of open disks, each one containing at most one point of P. The subset $A_\infty \subset A$ of improper simplices is a subcomplex; we denote $A^\circ := A \setminus A_\infty$ the subset of proper ones and by $|A^\circ|$ its "geometric realization" $|A| \setminus |A_\infty|$.

We will associate a marked ribbon graph G_α to every proper simplex α in a natural way and a metric on G_α to every internal point of $|\alpha|$. Let us fix some notation first.

Definition 2.1.1. An *(ordinary) ribbon graph* G is a triple $(X(G), \sigma_0, \sigma_1)$ such that $X(G)$ is a nonempty finite set, σ_0 is a permutation of $X(G)$ and σ_1 is a fixed-point-free involution of $X(G)$. Let denote by $X_i(G)$ the set of orbits in $X(G)$ with respect to the action of σ_i for $i = 0, 1$.

Observe that this definition is equivalent to the intuitive one given in terms of a graph plus a cyclic ordering of the half-edges outcoming from each vertex (see Fig. 2.1). In fact we should look at $X(G)$ as the set of oriented edges of G, at $X_0(G)$ as the set of vertices and at $X_1(G)$ as the set of unoriented edges. So we can identify σ_0 with the operator that sends an edge outcoming from a vertex v to the following edge outcoming from v with respect to a given cyclic order, and σ_1 with the operator that simply reverses the orientation of the given (oriented) edge.

Given an oriented edge $\vec{e}$ in $X(G)$ we will denote by $e = [\vec{e}]_1$, $[\vec{e}]_0$ and $[\vec{e}]_\infty$ its classes in $X_1(G)$, $X_0(G)$ and $X_\infty(G)$ respectively.

Observe that there is a natural bijection between connected components of the ribbon graph G and orbits in $X(G)$ under the action of the subgroup $\langle \sigma_0, \sigma_1 \rangle \subset \mathfrak{S}(X(G))$. Finally we can define σ_∞ requiring that $\sigma_\infty \sigma_1 \sigma_0 = 1$, so that $X_\infty(G)$ naturally corresponds to the set of holes of G and σ_∞ rotates

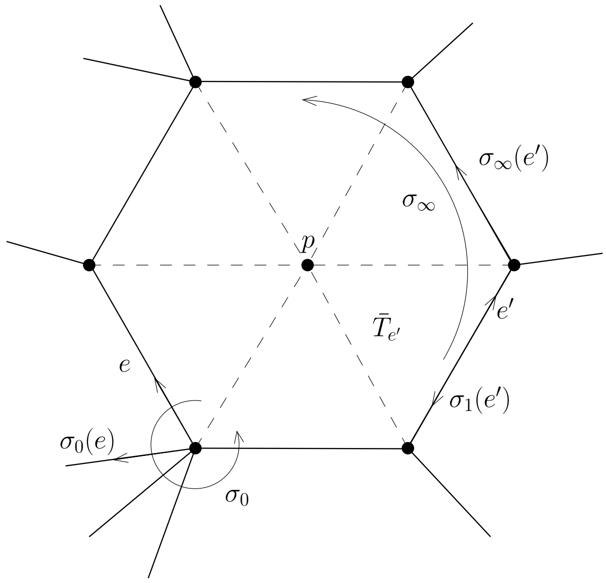


Figure 2.1: Geometric representation of a ribbon graph

the edges that border each hole. To each ribbon graph $(X(G), \sigma_0, \sigma_1)$ we can associate a dual one $G^* := (X(G), \sigma_\infty^{-1}, \sigma_1)$ such that $(G^*)^* = G$.

Definition 2.1.2. A *P-marking* of G is an injection

$$x : P \hookrightarrow X_0(G) \cup X_\infty(G)$$

such that $X_\infty(G)$ is in the image of x. A *metrized* ribbon graph is a couple (G, l) where G is a ribbon graph and l is a *unital metric* on G, i.e. a point of $\Delta^\circ_{X_1(G)}$.

We call (G, x) *reduced* if every unmarked vertex has valency greater than two. One can associate a reduced marked ribbon graph $(\overline{G}, \overline{x})$ to any (G, x) (provided it is not a one-pointed or two-pointed sphere) simply "forgetting" bivalent vertices and contracting unmarked tails (i.e. edges with a univalent unmarked extremal point), so that a metric on (G, x) descends to its reduction.

To each proper simplex $\alpha = \{\alpha_0, \dots, \alpha_k\}$ we can associate a connected ribbon graph G_α^* simply taking as $X(G_\alpha^*)$ the set of oriented versions of the

α_i's, as σ_1 the sense-reversing operator and making σ_0 rotate edges outcoming from a point p counterclockwise with respect to the given orientation of S. It is easy to see that $G_\alpha := (G^*_\alpha)^*$ inherits a P-marking: we call it the "dual" ribbon graph associated to α. By the way, notice that G_α can be "concretely" realized as embedded in S.

Actually, it is clear that a point a of $|\alpha|^\circ \subset |A^\circ|$ correspond to a unital metric on G^*_α and so on G_α. Moreover, if $\lambda : |A| \to \Delta_P$ is the simplicial map that sends a vertex $\{\alpha\}$ of $|A|$ to the barycenter of the extremal points of the arc α (or to the extremal point if α is a loop), then the restriction of λ to a proper simplex is the *circumference function* of the associated ribbon graph, that is it sends a metrized ribbon graph (G, a) to the point whose p-th coordinate is half the perimeter of the p-marked hole (it is zero in the case in which p marks a vertex).

To each metrized ribbon graph (G, a) we can canonically associate a Riemann surface

$$S(G, a) := \left(\coprod_{\vec{e} \in X(G)} T_{\vec{e}} \right) \Big/ \sim$$

where $T_{\vec{e}} = [0, e(a)] \times [0, \infty]/[0, e(a)] \times \{\infty\}$ and $\sim$ is the equivalence relation generated by $T_{\vec{e}} \ni (t, 0) \sim (e(a) - t, 0) \in T_{\sigma_1(\vec{e})}$ and $T_{\vec{e}} \ni (e(a), s) \sim (0, s) \in T_{\sigma_\infty(\vec{e})}$. Call $\bar{T}_{\vec{e}}$ the image of $T_{\vec{e}}$ under the above identification and (if G is P-marked) $\bar{T}_p$ the union of the $\bar{T}_{\vec{e}}$'s for all $\vec{e} \in x(p)$ and notice that the conformal structures on $\bar{T}_{\vec{e}} \setminus (\{\infty\} \cup \{0\} \times \{0\} \cup \{e(a)\} \times \{0\}) \subset \mathbb{R}^2 \cong \mathbb{C}$ glue to give a conformal structure on $S(G)$ minus a finite set. So we get a well-defined unique complex structure on $S(G)$ and it is clear that a P-marking descends to $x' : P \hookrightarrow S(G, a)$, thus determining a well-defined isotopy class of homeomorphisms $(S, P) \longrightarrow (S(G, a), x'(P))$ and a continuous classifying map

$$\Psi : |A^\circ(S, P)| \longrightarrow \mathcal{T}_{S,P}$$

to the Teichmüller space.

2.2 Strebel's theorem

Consider the continuous application

$$(\Psi, \lambda) : |A^{\circ}(S, P)| \longrightarrow \mathcal{T}_{S,P} \times \Delta_P$$

which is clearly $\Gamma_{S,P}$-equivariant. We want to say that it is an homeomorphism, so we need to prove that it is bijective and open.

Remember how we constructed the metrized surface $S(G, a)$: every $T_{\vec{e}} \subset \mathbb{R}^2 \sim \mathbb{C}$ has the flat metric $dz \otimes d\bar{z}$. Then notice that the holomorphic quadratic differentials dz^2 on each $T_{\vec{e}}$ glue to give a global meromorphic differential β on $S(G, a)$. It is regular outside $x(P)$ and has quadratic residues $-\frac{1}{4\pi^2}(2\lambda(p))^2$ at $x(p)$. Moreover its horizontal trajectories (i.e. the trajectories defined by $\mathrm{Arg}(\beta) = 0$) are either closed or critical (i.e. they begin and end in a zero or a simple pole of β). In particular the k-th order zeroes of β correspond to $(k + 2)$-valent vertices of G and the critical graph of β (i.e. the union of all critical horizontal trajectories) corresponds to the union of the edges of G.

Definition 2.2.1. A meromorphic quadratic differential β on a Riemann surface S' is called *Strebel differential* if its closed horizontal trajectories cover the surface S' up to a subset of measure zero.

It can be proved that nonclosed horizontal trajectories of a Strebel differential are necessarily critical and in fact there are only finitely many of them.

Summarizing, given a metrized ribbon graph G with a P-marking we can construct a P-marked Riemann surface S' plus a Strebel differential β on S' whose critical graph corresponds to G. Conversely, given a P-marked Riemann surface S' plus a Strebel differential β we can define a P-marked metrized ribbon graph G from the critical graph of β.

Now we are ready to understand the full strength of the following result.

Theorem 2.2.2 (Strebel, [Str67]). *Let S' be a compact Riemann surface and $P' \subset S'$ a nonempty subset (such that P' contains at least two points* *if S' is a sphere). Then for every function $h : P' \to \mathbb{R}_{\geq 0}$ there exists a unique Strebel differential $\beta(S', P', h)$ on S' that is holomorphic on $S' \setminus P'$ and which has a double pole on every $p' \in P' \setminus h^{-1}(0)$ of quadratic residue $-\frac{1}{4\pi^2}(2h(p'))^2$ and at most a simple pole on every $p' \in h^{-1}(0)$.*

This assures that the map (Ψ, λ) is bijective. In fact the previous theorem provides a set-theoretic inverse of (Ψ, λ).

We are left to prove that the map is open. The quickiest way to do that is to notice that $|A^\circ(S, P)|$ can be given a structure of differentiable manifold compatible with the piecewise linear one (see [HM79]), hence Ψ is an open map by invariance of domain.

As a consequence, we get the desired isomorphism

$$\Phi : |A^\circ(S, P)|/\Gamma_{S,P} \xrightarrow{\sim} \mathcal{M}_{g,P} \times \Delta_P$$

of orbifolds.

Remark. We may notice that one can construct a tautological family of Riemann surfaces $\mathcal{C} \longrightarrow |A^\circ(S, P)|$ whose restriction over a simplex α is real-analytic. So Ψ is continuous by the universal property of the Teichmüller space and $\Psi|_\alpha$ is real-analytic for every α.

2.3 The modified arc complex

Let $Z \subset X_1(G)$ be a nonempty subset of edges of an ordinary connected ribbon graph G. We can construct two new ribbon graphs. The *subgraph* $G_Z = (X(G_Z), \sigma_0', \sigma_1')$ has $X(G_Z)$ equal to the set of orientations of edges in Z, its σ_1' is the natural restriction of σ_1 and its σ_0' sends an edge to the next one belonging to $X(G_Z)$ with respect to the cyclic order induced by σ_0. If Z does not coincide with $X_1(G)$, then G_Z has some new *exceptional holes* corresponding to orbits in $X(G_Z) \subset X(G)$ under σ_∞' which are not orbits under the action of σ_∞.

Consider now a proper subset Z of $X_1(G)$. Then the *quotient graph* G/G_Z has $X(G/G_Z)$ equal to $X(G) \setminus X(G_Z)$, its σ_1' is the restriction and its σ_∞' sends an edge to the next one of $X(G/G_Z)$ with respect to the cyclic order induced by σ_∞. If Z is nonempty, then G/G_Z has *exceptional vertices* corresponding to orbits in $X(G_Z) \subset X(G)$ under σ_0' that are not orbits under the action of σ_0.

Notice that there is a canonical correspondence between exceptional vertices of G/G_Z and exceptional holes of G_Z (see Fig. 2.2). In fact consider an exceptional hole H of G_Z. For every (oriented) edge $\vec{e} \in H$ call $b_{\vec{e}} > 0$ the minimum integer such that $\sigma_1\sigma_0^{b_{\vec{e}}}(\vec{e})$ belongs to H. Then the subset $\{\sigma_0^i(\vec{e}) | \, \vec{e} \in H \,\text{and}\, 0 < i < b_{\vec{e}}\}$ is the corresponding exceptional vertex in G/G_Z. Conversely, given an exceptional vertex V of G/G_Z and an $\vec{e} \in V$ call $b_{\vec{e}} > 0$ the minimum integer such that $\sigma_\infty^{b_{\vec{e}}}\sigma_1(\vec{e})$ belongs to V. Then $\{\sigma_1\sigma_\infty^i\sigma_1(\vec{e}) | \, \vec{e} \in H \,\text{and}\, 0 < i < b_{\vec{e}}\}$ is the corresponding exceptional hole in G_Z.

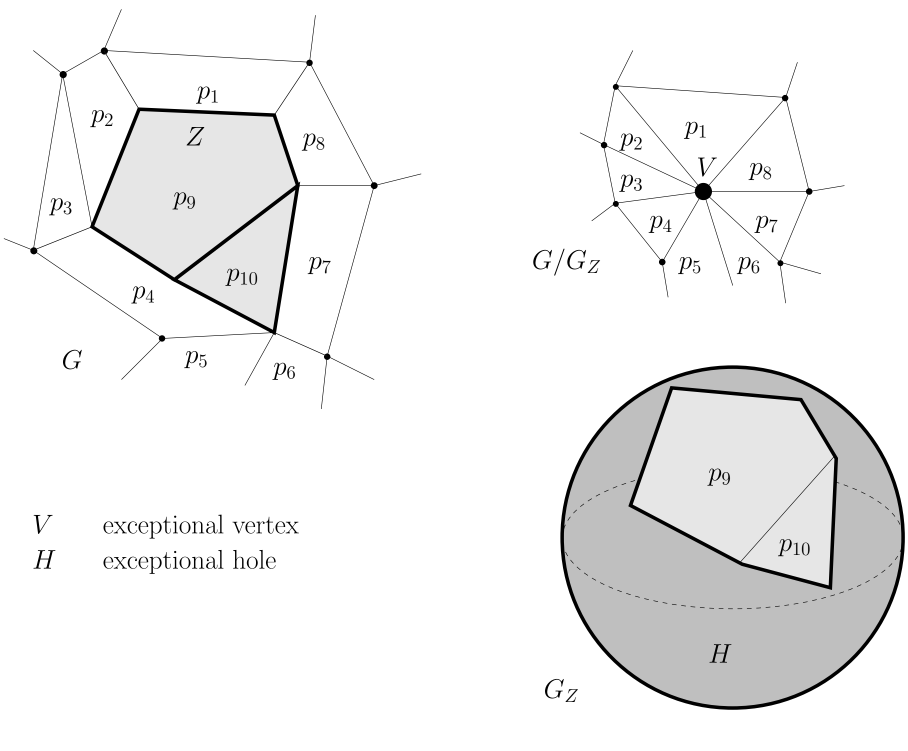


Figure 2.2: Example of correspondence between exceptional holes and exceptional vertices

To introduce the definition of stable P-marked ribbon graph think at how an ordinary metrized P-marked ribbon graph G can degenerate: it happens when the lengths of a subset Z of edges go to zero. As we can work componentwise, we suppose Z connected. Then various cases can occur:

1. Z is a tree and contains at most one marked point, so it is *contractible*: then we can collapse it to a vertex and put if necessary the marking on it, that is we simply obtain G/G_Z (see Fig. 2.3)

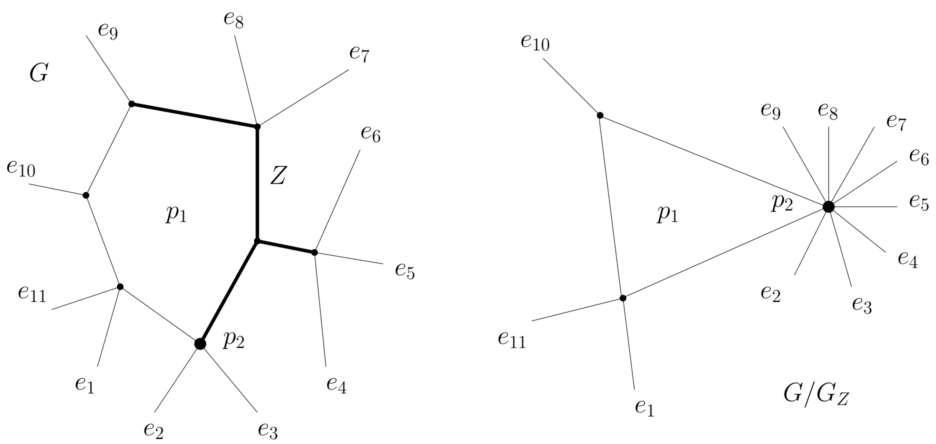


Figure 2.3: Example of contractible subset Z of edges

2. Z is homotopic to a circle and has no marked points, so we call it *semistable*. If Z surrounds a single hole, then it shrinks to a vertex which inherits the marking in G/G_Z; otherwise G/G_Z contains two exceptional vertices (see Fig. 2.4)

3. if Z is neither contractible nor semistable, then its collapsing gives rise to a new irreducible component. If Z contains no unmarked tails, then we call Z a *stable* subset. Notice that every Z contains a maximal stable subset Z^{st} (see Fig. 2.5).

Now we want to produce a stable version of ribbon graphs successively collapsing semistable or stable subsets of edges.

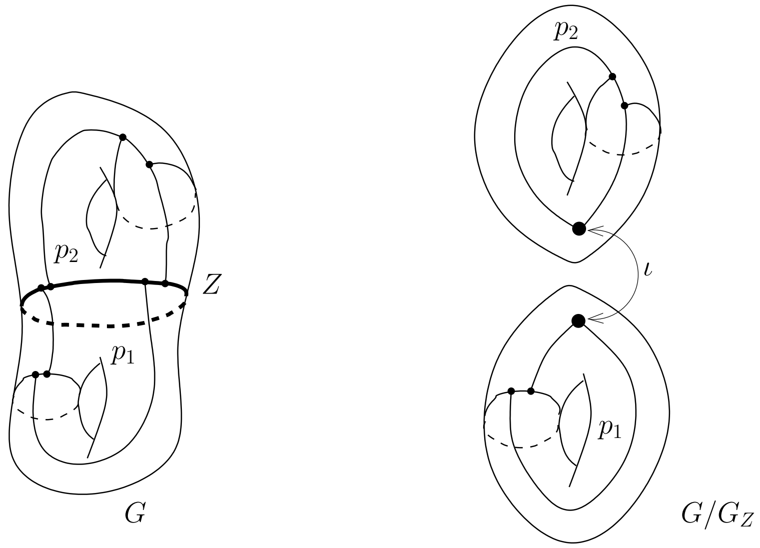


Figure 2.4: Example of semistable subset Z of edges

Given an ordinary connected P-marked ribbon graph (G, x), we call $Z_\bullet = (Z_0, Z_1, \ldots, Z_k)$ a *permissible sequence* for (G, x) if $Z_0 = X_1(G)$ and $Z_{j+1} \subset \overline{Z_j^{st}}$ is a nonempty subset not containing a whole component of $\overline{Z_j^{st}}$ for every $j = 0, \ldots, k-1$. Given such a $Z_\bullet$ we can produce a (quasi)stable P-marked ribbon graph taking the triple $(G(Z_\bullet), \bar{x}, \iota)$ where

$$G(Z_\bullet) := (G_{Z_0}/G_{Z_1}) \sqcup (G_{\overline{Z_1^{st}}}/G_{Z_2}) \sqcup \cdots \sqcup (G_{\overline{Z_{k-1}^{st}}}/G_{Z_k}) \sqcup G_{\overline{Z_k^{st}}},$$

$\bar{x} : P \hookrightarrow X_\infty(G(Z_\bullet)) \cup X_0(G(Z_\bullet))$ is induced by x and ι is a fixed-point-free involution that exchanges every exceptional hole with its corresponding exceptional vertex. The "stabilized" P-marked ribbon graph is simply obtained discarding possible unstable components, namely unmarked spheres with two exceptional holes, and making ι exchange the two corresponding exceptional vertices. In any case, ι never exchanges two holes. We say that the (stable) components of $G_{\overline{Z_i^{st}}}/G_{Z_{i+1}}$ have *order* i and we define H_i as the set of holes belonging to components of order i and V_i as the set of marked or exceptional vertices belonging to components of order i. Finally we say that $\Sigma := \cup_i(H_i \cup V_i)$ is the set of *special points*.

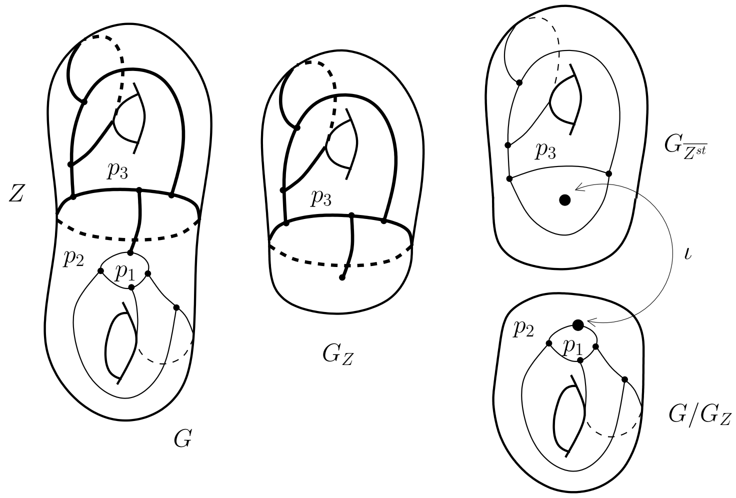


Figure 2.5: Example of stable subset Z^{st} of edges

Definition 2.3.1. A *stable metric* with respect to $Z_\bullet$ is a sequence of metrics $(a_i)_{i=0}^k$ where $a_0 \in \Delta^\circ_{Z_0 \setminus Z_1}$ and a_i is a metric on $\overline{Z_i^{st}} \setminus Z_{i+1}$ which is unital on every irreducible component.

So given a stable metric for $Z_\bullet$ we can build a stable marked Riemann surface $S(G, Z_\bullet, a_\bullet)$. In fact we first consider the disjoint union of the surfaces $S(G_{\overline{Z_i^{st}}}/G_{Z_{i+1}}, a_i)$ for $i = 0, \dots, k$ and then we identify some pairs of points according to ι. Remark that there is an extended circumference function

$$\widehat{\Lambda} : \{\text{unital metrics on } S(G, Z_\bullet)\} \longrightarrow \prod_{i=0}^{k} \Delta_{H_i}$$

that restricts to a map $\hat{\lambda} := \widehat{\Lambda}_0 : \{\text{unital metrics on } S(G, Z_\bullet)\} \longrightarrow \Delta_P$.

Remark. Let C be a stable P-marked Riemann surface and let

$$\nu : \tilde{C} = \sqcup_j C_j \to C$$

be its normalization. Let $P_j := \nu^{-1}(P) \cap C_j$ and $Q_j := \nu^{-1}(\text{nodes}) \cap C_j$. Consider the set $Cir(C)$ of all circumference functions $\widehat{\Lambda}$ associated to P-marked stable ribbon graphs $G(Z_\bullet)$ such that $S(G, Z_\bullet)$ is a stable P-marked

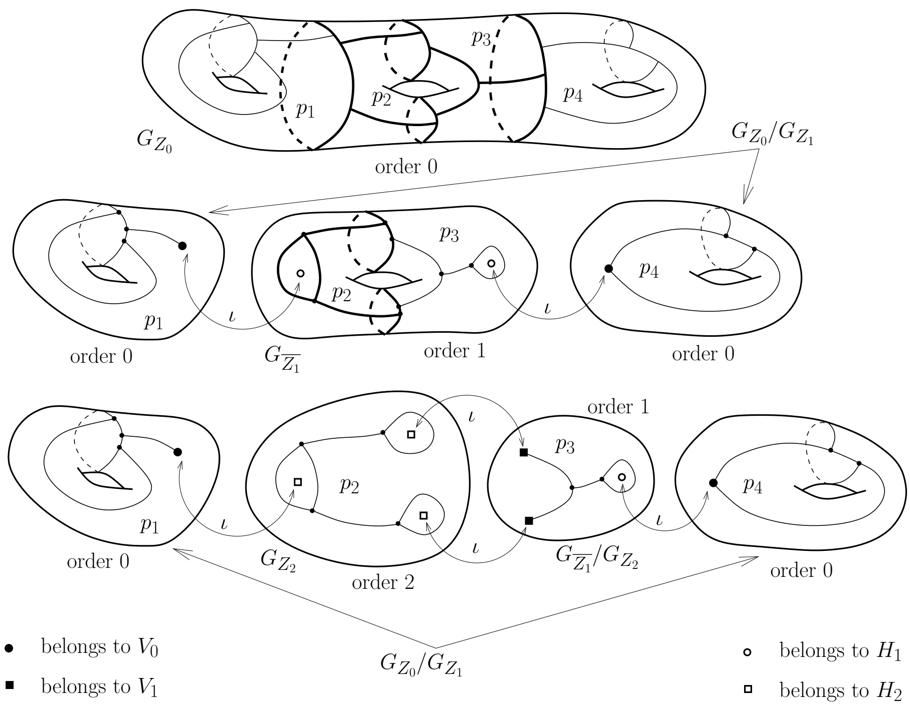


Figure 2.6: Example of iterated collapses

surface homeomorphic to C. Then the elements of $Cir(C)$ take values in $\prod_j \Delta_{P_j \cup Q_j}$, so that we can define $\mathrm{Im}(Cir(C)) \subset \prod_j \Delta_{P_j \cup Q_j}$ as the union of the images of all $\widehat{\Lambda}$'s in $Cir(C)$. Notice that for any $l \in \Delta_P$ we can define the subset $Cir(C, l) \subset Cir(C)$ of circumference functions $\widehat{\Lambda}$ such that their restrictions $\hat{\lambda}$ is constantly l. It is easy to see that both $Cir(C)$ and $Cir(C, l)$ are simplicial complexes.

Now we can give the formal definition of stable P-marked ribbon graph.

Definition 2.3.2. Consider a metrized (possibly disconnected) ribbon graph G with an injection $x : P \hookrightarrow \Sigma$ in a subset of "special points" $\Sigma \subset X_0(G) \sqcup X_\infty(G)$ containing $X_\infty(G)$ plus a fixed-point-free involution ι acting on the set of "exceptional points" $\Sigma \setminus x(P)$. We say that an order function that assigns a natural number to each connected component of G is *admissible* if

- components of order 0 contain at least a P-marked hole
- if ι exchanges each point of the component G_j with a point in a component of order $\leq k$ then G_j has order $\leq k+1$
- every $h \in X_\infty(G) \setminus x(P)$ belongs to a component of order $k > 0$ and the point $\iota(p)$ sits in a component of order at most $k-1$ (and so is a vertex).

We call (G, x, ι) a *P-marked stable ribbon graph* if there exists an admissible order function on G and we say that (G, x, ι) is *reduced* if (G, x) is. A *stable metric* on (G, x, ι) is the datum of a unital metric a_j for every connected component G_j of G.

Now let $\alpha(G)$ be a proper simplex of A whose associated marked ribbon graph is $G = G_\alpha$. Consider the set $\mathcal{Z}(G)$ of connected stable subsets of $X_1(G)$ and for every $a \in |\alpha(G)|^\circ$ and every $Z \in \mathcal{Z}(G)$ let $|\alpha(G)|^\circ \to |\alpha(G_{\overline{Z}})|$ be the projection to a face. Define $\hat{\alpha}(G)$ to be the closure of the graph of the map

$$|\alpha(G)|^\circ \hookrightarrow |\alpha(G)| \times \prod_{Z \in \mathcal{Z}(G)} |\alpha(G_{\overline{Z}})|$$

in $|\alpha(G)| \times \prod |\alpha(G_{\overline{Z}})|$.

It can be proven that $\hat{\alpha}(G)$ parametrizes all stable degenerations of the ribbon graph G_α. Moreover all the $\hat{\alpha}$'s can be glued to obtain a *modified arc complex* $\widehat{A}$. Remark that $\widehat{A}(S, P)$ comes with an obvious cellularization indexed by permissible sequences: for every $Z_\bullet = (Z_0, \ldots, Z_k)$ there is a (closed) cell isomorphic to $|\alpha_0| \times \cdots \times |\alpha_k|$ that parametrizes stable metrics on $G(Z_\bullet)$. The projections $|\alpha_0| \times \cdots \times |\alpha_k| \to |\alpha_0|$ glue to give a continuous surjection $\widehat{A}(S, P) \to |A(S, P)|$ which is actually a quotient (i.e. $|A(S, P)|$ has the quotient topology).

Theorem 2.3.3 ([Loo95]). *The modular group $\Gamma_{g,P}$ naturally acts on $\widehat{A}(S, P)$ respecting the cellularization. The product of the classifying map*

$$\widehat{A}(S, P)/\Gamma_{S,P} \longrightarrow \overline{\mathcal{M}}_{g,P}$$

with $\hat{\lambda}$ *is a continuous surjection*

$$\widehat{\Phi} : \widehat{A}(S,P)/\Gamma_{S,P} \longrightarrow \overline{\mathcal{M}}_{g,P} \times \Delta_P$$

that extends Φ*, so it is one-to-one when restricted to the dense open subset* $|A^\circ(S,P)|/\Gamma_{S,P}$*. More precisely the fiber of* $\widehat{\Phi}$ *over* (C,l) *naturally identifies to* $\mathrm{Im}(Cir(C,l))$.

In what follows we will always identify $\Delta_P \times \mathbb{R}_+$ with $\mathbb{R}^P_{\geq 0} \setminus \{0\}$ and we will still denote by $\widehat{\Phi}$ the map

$$\widehat{\Phi} : \widehat{\mathcal{M}}^{comb}_{g,P} := (\widehat{A}(S,P)/\Gamma_{S,P}) \times \mathbb{R}_+ \longrightarrow \overline{\mathcal{M}}_{g,P} \times (\mathbb{R}^P_{\geq 0} \setminus \{0\}).$$

2.4 The ribbon graph complex

Here we introduce the last complex we are interested in, which is due to Kontsevich (see [Kon92]). The point of view is reversed: the central object is the ribbon graph and no longer the arc system.

Form the category $\mathcal{RG}_{g,P}$ of P-marked ribbon graphs of genus g as follows. Its objects are the ribbon graphs G_α with α in $A^\circ(S,P)$, and its morphisms are compositions of isomorphisms of pointed ribbon graphs and contractions of one edge. Denote by $\mathcal{M}$ (resp. $\overline{\mathcal{M}}$) the functor $\mathcal{RG}_{g,P} \longrightarrow$ (Top. spaces) that associates $(|\alpha| \cap |A^\circ|) \times \mathbb{R}_+$ (resp. $|\alpha| \times \mathbb{R}_+$) to every G_α and by $\mathcal{M}^{comb}_{g,P}$ (resp. $\overline{\mathcal{M}}^{comb}_{g,P}$) its limit functor. Remark that both functors are represented by orbicellular complexes and that $\mathcal{M}^{comb}_{g,P} \subset \overline{\mathcal{M}}^{comb}_{g,P}$ embeds as a dense open subspace. Moreover we can define a circumference function $\bar{\lambda} : \overline{\mathcal{M}}^{comb}_{g,P} \to \mathbb{R}^P_{\geq 0} \setminus \{0\}$ as in the case of the arc complex.

Remark. Notice that our definition of $\mathcal{M}^{comb}_{g,P}$ and $\overline{\mathcal{M}}^{comb}_{g,P}$ slightly differs from Kontsevich's one. In fact we allow some holes to have perimeter zero (i.e. we admit marked vertices) while Kontsevich does not. Briefly Kontsevich's spaces are obtained from ours by intersecting $\mathcal{M}^{comb}_{g,P}$ and $\overline{\mathcal{M}}^{comb}_{g,P}$ with $\bar{\lambda}^{-1}(\mathbb{R}^P_+)$.

We observe that the points of $\overline{\mathcal{M}}^{comb}_{g,P}$ correspond to the following data:

- a connected graph γ (the "dual graph of the pointed surface") with vertices labelled by pairs (g_v, P_v) such that $\sqcup_v P_v = P$ and $h^1(\gamma) + \sum_v g_v = g$
- a subset V_+ of vertices of γ (the "positive vertices of the dual graph")
- for every vertex $v \in V_+$ an ordinary $P_v \cup Q_v$-marked ribbon graph (G_v, x_v) of genus g with (nonunital) metric such that Q_v marks only vertices of G_v, where Q_v bijectively correspond to the set of half-edges of γ outgoing from v.

We require moreover that no edge of γ joins two nonpositive vertices (the dual graph γ is "reduced").

It is clear that an isomorphism $G_\alpha \xrightarrow{\sim} G_{\alpha'}$ of ribbon graphs with $\alpha, \alpha' \in A^\circ(S,P)$ lifts to an isotopy class of oriented diffeomorphisms $(S,P) \xrightarrow{\sim} (S,P)$. So the natural map $(|A^\circ(S,P)|/\Gamma_{S,P}) \times \mathbb{R}_+ \xrightarrow{\sim} \mathcal{M}^{comb}_{g,P}$ is a homeomorphism and commutes with λ and $\bar{\lambda}$, hence

$$\mathcal{M}_{g,P} \times (\mathbb{R}^P_{\geq 0} \setminus \{0\}) \xrightarrow{\sim} \mathcal{M}^{comb}_{g,P}$$

is a homeomorphism too. At the same time the continuous surjection

$$\overline{\Phi} : (|A(S,P)|/\Gamma_{S,P}) \times \mathbb{R}_+ \longrightarrow \overline{\mathcal{M}}^{comb}_{g,P}$$

(naturally induced by the definition of $\overline{\mathcal{M}}^{comb}_{g,P}$) is a quotient and the preimage of a point can be described as follows. Pick a point $(\gamma, V_+, \{G_v\})$ in $\overline{\mathcal{M}}^{comb}_{g,P}$ and consider the disjoint union of the marked surfaces $X := \sqcup_{v \in V_+} (S(G_v, x_v) \setminus Q_v)$, so that each point of Q_v corresponds to an ideal boundary component of $X_v := S(G_v, x_v) \setminus Q_v$. An orientation-preserving embedding of $f : X \hookrightarrow S$ is *admissible* if

- for every positive v the restriction of f to $S(G_v, x_v) \setminus Q_v$ preserves the P_v-marking
- every edge joining two positive vertices v and v' (which determines points $q \in Q_v$ and $q' \in Q_{v'}$) corresponds to a cylinder in $S \setminus f(X)$

that connects the ideal boundary components of $f(X_v)$ and $f(X_{v'})$ corresponding to q and q' respectively

- every nonpositive w corresponds to a connected component C_w of $S \setminus f(X)$ of genus g_w which contains P_w and every edge of γ joining w with a positive v (which determines a point q in Q_v) corresponds to the ideal boundary component of $f(X_v)$ labelled by q coinciding with a boundary component of C_w.

Then $\overline{\Phi}^{-1}(G)$ can be identified to $\mathrm{AdmEmb}(G^*, S)/\mathrm{Diff}_+(S, P)$.

Finally it is easy to see that $\widehat{\Phi}$ set-theoretically descends to a well-defined $R : \overline{\mathcal{M}}^{comb}_{g,P} \xrightarrow{\sim} \overline{\mathcal{M}}^{\triangle}_{g,P} \times \mathbb{R}_+$. We want to show that R is a homeomorphism. To see that R is continuous it is sufficient to prove that $\widehat{\mathcal{M}}^{comb}_{g,P} \to \overline{\mathcal{M}}^{\triangle}_{g,P} \times \mathbb{R}_+$ is. This is obvious as this map is exactly $\xi\widehat{\Phi}$. Bijectivity relies again on Strebel's result applied componentwise; moreover R is proper, hence closed.

We summarize the preceding observations in the following commutative diagram

$$
\begin{tikzcd}
 & & (|A^{\circ}(S,P)|/\Gamma_{S,P}) \times \mathbb{R}_+ \arrow[d, equal] & & \\
(|\widehat{A}(S,P)|/\Gamma_{S,P}) \times \mathbb{R}_+ \arrow[d, equal] & & \mathcal{M}^{comb}_{g,P} \arrow[dll, hook'] \arrow[drr, hook] \arrow[dd, "\cong"', "\Phi"] & & \\
\widehat{\mathcal{M}}^{comb}_{g,P} \arrow[rrrr, two heads] \arrow[dd, "\widehat{\Phi}"] & & & & (|A(S,P)|/\Gamma_{S,P}) \times \mathbb{R}_+ \arrow[d, two heads, "\overline{\Phi}"] \\
 & & \mathcal{M}_{g,P} \times (\mathbb{R}^P_{\geq 0} \setminus \{0\}) \arrow[rr, hook] \arrow[dll, hook'] & & \overline{\mathcal{M}}^{comb}_{g,P} \arrow[d, "\cong"', "R"] \\
\overline{\mathcal{M}}_{g,P} \times (\mathbb{R}^P_{\geq 0} \setminus \{0\}) \arrow[rrrr, two heads, "\xi"] & & & & \overline{\mathcal{M}}^{\triangle}_{g,P} \times \mathbb{R}_+
\end{tikzcd}
$$

and we recall that ξ is the map that collapses nonpositive components so that its fibers are isomorphic to products of smaller moduli spaces; while $\widehat{\Phi}$

is the classifying map of Looijenga's modification of the arc complex and its fibers are the simplicial complexes $Cir(C, l)$ of circumference values. The map $\widehat{\mathcal{M}}^{comb}_{g,P} \twoheadrightarrow (|A(S,P)|/\Gamma_{S,P}) \times \mathbb{R}_+$ is the natural projection, so it is a sort of simplicial "blow-up".

In what follows we will always identify $\overline{\mathcal{M}}^{comb}_{g,P}$ and $\overline{\mathcal{M}}^{\triangle}_{g,P} \times \mathbb{R}_+$ via R, so that ξ will be a map from $\overline{\mathcal{M}}_{g,P} \times (\mathbb{R}_{\geq 0} \setminus \{0\})$ to $\overline{\mathcal{M}}^{comb}_{g,P}$.

Remark. The (orbi)spaces $\overline{\mathcal{M}}^{comb}_{g,P}$ and $\widehat{\mathcal{M}}^{comb}_{g,P}$ have an (orbi)piecewise-linear structure so de Rham theorem holds giving an isomorphism between rational singular cohomology and rational PL de Rham cohomology. Now all cohomology groups will be considered with rational coefficients even though tautological and combinatorial classes are defined over $\mathbb{Z}$, so that all results still hold in integral cohomology modulo torsion.

Chapter 3

Combinatorial classes

Now we introduce some remarkable subcomplexes of the combinatorial moduli spaces which define interesting cycles in simplicial (cellular/singular) homology. This subcomplexes are simply defined prescribing that some vertices have assigned odd valencies. It can be easily shown that, if we assign even valency to some vertex, the subcomplex we obtain is not a cycle (even with $\mathbb{Z}/2\mathbb{Z}$ coefficients!).

We follow Kontsevich ([Kon92]) for the orientation of the combinatorial cycles. In the last section we define a slight generalization of the combinatorial classes by allowing some vertices to be marked, which will turn very useful in the following chapter.

3.1 Combinatorial complexes

Fix S a compact Riemann surface of genus g and $P = \{p_1, \dots, p_n\} \subset S$ a subset of n points such that $2g - 2 + n > 0$. Let $m_* = (m_{-1}, m_0, m_1, \dots)$ be a sequence of nonnegative integers such that

$$\sum_{i \geq -1} (2i+1) m_i = 4g - 4 + 2n$$

and define $(m_*)! := \prod_{i \geq -1} m_i!$ and $r := \sum_{i \geq -1} i\, m_i$. We assume that $m_{-1} = 0$.

Reasoning as in Section 2.4, it is possible to construct an orbispace $\overline{\mathcal{M}}^{comb}_{m_*,P}$ whose cells of maximal dimension are indexed by isomorphism classes of ordinary ribbon graphs that have exactly m_i vertices of valency $2i+3$. Analogously it is possible to define an arc complex $A(S,P)_{m_*}$ (resp. a modified arc complex $\widehat{A}(S,P)_{m_*}$) as the smallest subcomplex of $A(S,P)$ (resp. of $\widehat{A}(S,P)$) that contains all simplices α such that G_α is an ordinary ribbon graph with exactly m_i vertices of valency $2i+3$. Notice that both these complexes are acted on by $\Gamma_{S,P}$ and so is $A^\circ(S,P)_{m_*} := A(S,P)_{m_*} \cap A^\circ(S,P)$. Hence we can set $\widehat{\mathcal{M}}^{comb}_{m_*,P} := (\widehat{A}(S,P)_{m_*}/\Gamma_{S,P}) \times \mathbb{R}_+$.

Remark. In the case $m_{-1} > 0$ it is still possible to define the complexes $A(S,P)_{m_*}$, $A^\circ(S,P)_{m_*}$ and $\widehat{A}(S,P)_{m_*}$ from an (extended) arc complex $\widetilde{A}(S,P)$, obtained adding to $\mathcal{A}$ contractible loops (i.e. unmarked tails in the corresponding ribbon graph picture). However $A(S,P)_{m_*}$ is no longer a subcomplex of $A(S,P)$, so we only have the classifying maps $\widehat{\mathcal{M}}^{comb}_{m_*,P} \to \overline{\mathcal{M}}_{g,P} \times (\mathbb{R}^P_{\geq 0} \setminus \{0\})$ and $\mathcal{M}^{comb}_{m_*,P} \to \mathcal{M}_{g,P} \times (\mathbb{R}^P_{\geq 0} \setminus \{0\})$, which are quite mysterious if we consider them as maps of cellular complexes.

All the spaces we have introduced fit in the following commutative diagram

$$\begin{tikzcd}
(|A^\circ(S,P)_{m_*}|/\Gamma_{S,P}) \times \mathbb{R}_+ \arrow[r, "\cong"] \arrow[d, hook] & \mathcal{M}^{comb}_{m_*,P} \arrow[r, hook] \arrow[d, hook] & \mathcal{M}^{comb}_{g,P} \arrow[d, hook] \\
(|A(S,P)_{m_*}|/\Gamma_{S,P}) \times \mathbb{R}_+ \arrow[r, two heads] & \overline{\mathcal{M}}^{comb}_{m_*,P} \arrow[r, hook] & \overline{\mathcal{M}}^{comb}_{g,P} \\
\widehat{\mathcal{M}}^{comb}_{m_*,P} \arrow[u, two heads] \arrow[rr, hook] & & \widehat{\mathcal{M}}^{comb}_{g,P} \arrow[u, two heads, "\xi\,\widehat{\Phi}"]
\end{tikzcd}$$

For every $l \in \mathbb{R}^P_{\geq 0}\setminus\{0\}$ call $\overline{\mathcal{M}}_{g,P}(l)$ the slice $\overline{\mathcal{M}}_{g,P}\times\{l\} \subset \overline{\mathcal{M}}_{g,P}\times(\mathbb{R}^P_{\geq 0}\setminus\{0\})$ and similarly $\overline{\mathcal{M}}^{comb}_{g,P}(l) := \bar{\lambda}^{-1}(l) \subset \overline{\mathcal{M}}^{comb}_{g,P}$ and $\widehat{\mathcal{M}}^{comb}_{g,P}(l) := \hat{\lambda}^{-1}(l)$. In the same way we can define $\widehat{\mathcal{M}}^{comb}_{m_*,P}(l)$ and $\overline{\mathcal{M}}^{comb}_{m_*,P}(l)$ and the restrictions ξ_l and $\widehat{\Phi}_l$ of ξ and $\widehat{\Phi}$ respectively. Notice that the dimensions of the slices are the expected ones because in every cell they are described by n linear equations.

3.2 Orientation

Define L_p as the space of couples (G, v), where G is a P-marked metrized ribbon graph in $\overline{\mathcal{M}}^{comb}_{g,P}(\{l_p > 0\})$ and v is a point of $S(G)$ belonging to an edge that borders the p-th hole. It will be given the topology induced by the natural piecewise-linear structure.

Clearly $L_p \longrightarrow \overline{\mathcal{M}}^{comb}_{g,P}(\{l_p > 0\})$ is a combinatorial bundle with fiber homeomorphic to S^1. It is easy to see that, for a fixed $l \in \Delta_P$ such that $l_p > 0$, the pull-back of L_p via

$$\xi_l : \overline{\mathcal{M}}_{g,P} \longrightarrow \overline{\mathcal{M}}^{comb}_{g,P}(l)$$

is isomorphic (as a topological bundle) to the sphere bundle associated to $\mathcal{L}^*_p$.

Lemma 3.2.1 ([Kon92]). *Fix p in P and $l \in \mathbb{R}^P_{\geq 0}$ such that $l_p > 0$. Then on every simplex $\alpha \in \overline{\mathcal{M}}^{comb}_{g,P}(l)$ define*

$$\overline{\omega}_p|_\alpha := \sum_{1 \leq s < t \leq k-1} d\tilde{e}_s \wedge d\tilde{e}_t$$

where $\tilde{e}_j = \frac{e_j(a)}{2l_p}$ and $x(p)$ is a hole with cyclically ordered sides $(e_1, \ldots, e_k)$. These 2-forms glue to give a piecewise-linear 2-form $\overline{\omega}_p$ on $\overline{\mathcal{M}}^{comb}_{g,P}(l)$ which represents $-c_1(L_p)$. Hence the pull-back class $\xi^*_l[\overline{\omega}_p]$ is exactly $\psi_p = c_1(\mathcal{L}_p)$ in $H^2(\overline{\mathcal{M}}_{g,P})$.

Proof. We will define a differentiable 1-form β on L_p such that its integral on each fiber is 1 and such that $d\beta$ is the pull-back of $-\overline{\omega}_p$. This will prove that $\overline{\omega}_p$ represents $-c_1(L_p)$.

Remember that a fiber of L_p is a k-uple of cyclically ordered distinct points $\bar{\phi}_1, \ldots, \bar{\phi}_k$ of the circle $\mathbb{R}/l_p\mathbb{Z}$. For all $i = 1, \ldots, k$ consider representative $\phi_i \in \mathbb{R}$ of $\bar{\phi}_i = \phi_i + l_p\mathbb{Z}$ such that $\phi_i \in [0, l_p)$.

Then the length of the i-th edge is

$$e_i = \begin{cases} \phi_{i+1} - \phi_i & \text{if } i = 1, \ldots, k-1 \\ \phi_1 - \phi_k + l_p & \text{if } i = k \end{cases}$$

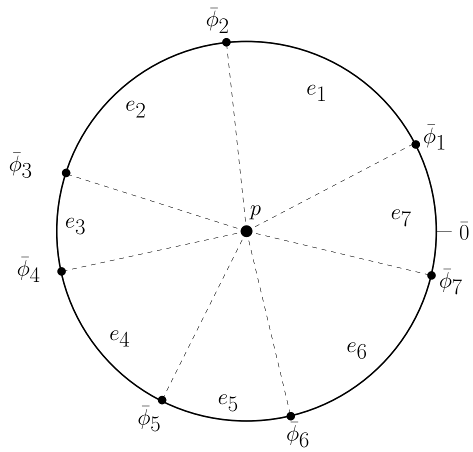


Figure 3.1: A fiber of L_p

so that we can define

$$\beta := \sum_{i=1}^{k} \left(\frac{e_i}{l_p}\right) d\left(\frac{\phi_i}{l_p}\right).$$

Then for every fiber of L_p we obtain

$$\int_{\text{fiber of } L_p} \beta = \sum_{i=1}^{k} \frac{e_i}{l_p} \int_0^{l_p} \frac{d\phi_k}{l_p} = 1$$

and $d\beta$ is exactly

$$d\beta = -\sum_{1\le i<j\le k-1} d\left(\frac{e_i}{l_p}\right) \wedge d\left(\frac{e_j}{l_p}\right).$$

□

Lemma 3.2.2 ([Kon92]). *For every $l \in \mathbb{R}_+^P$ the restriction of*

$$\overline{\Omega} := \sum_{p\in P} l_p^2 \overline{\omega}_p$$

to $\overline{\mathcal{M}}_{g,P}^{comb}(l)$ is a nondegenerate symplectic form, so $\overline{\Omega}^r$ defines an orientation on $\overline{\mathcal{M}}_{m_*,P}^{comb}(l)$. Hence $\overline{\Omega}^r \wedge \bar\lambda^* dvol_{\mathbb{R}_+^P}$ is an orientation on $\overline{\mathcal{M}}_{m_*,P}^{comb}$.

Proof. Let α be a cell of $\overline{\mathcal{M}}^{comb}_{g,P}(l)$ whose associated ribbon graph G_α has only vertices of odd valency. Then on α the differentials de_i span the cotangent space. As the perimeters l_p are fixed, we have the relation $dl_p = 0$ for all $p \in P$. Hence

$$T^*\overline{\mathcal{M}}^{comb}_{g,P}(l)|_\alpha \cong \alpha \times \bigoplus_{e \in X_1(G_\alpha)} \mathbb{R} \cdot de \Big/ \Big(\sum_{\vec{e} \in x(p)} de \,|\, p \in P \Big).$$

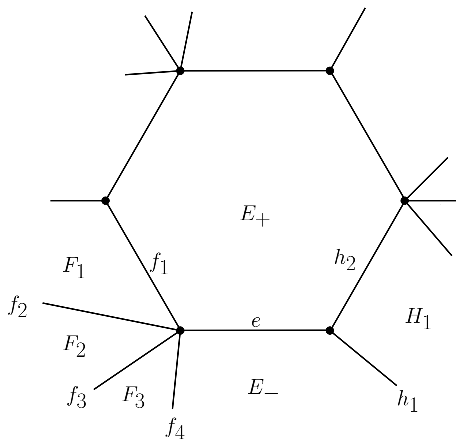
On the other hand the tangent bundle is

$$T\overline{\mathcal{M}}^{comb}_{g,P}(l)|_\alpha \cong \alpha \times \left\{ \sum_{e \in X_1(G_\alpha)} b_e \frac{\partial}{\partial e} \,\Big|\, \sum_{\vec{e} \in x(p)} b_e = 0 \quad \text{for all } p \in P \right\}.$$

In order to prove that $\overline{\Omega}|_\alpha : T\alpha \longrightarrow T^*\alpha$ is nondegenerate, we construct its right-inverse. Define $B : T^*\alpha \longrightarrow T\alpha$ as

$$B(de) = \sum_{i=1}^{2s} (-1)^i \frac{\partial}{\partial[\sigma_0^i(\vec{e})]_1} + \sum_{j=1}^{2t} (-1)^j \frac{\partial}{\partial[\sigma_0^j \sigma_1(\vec{e})]_1}$$

where $\vec{e}$ is any orientation of e, while $2s+1$ and $2t+1$ are the cardinalities of $[\vec{e}]_0$ and $[\sigma_1(\vec{e})]_0$ respectively. We want to prove that $\overline{\Omega}B(de) = de$ for every $e \in X_1(G)$. To shorten the notation, set $f_i := [\sigma_0^i(\vec{e})]_1$ and $h_j := [\sigma_0^j \sigma_1(\vec{e})]_1$



and call $F_i := [\sigma_0^i(\vec{e})]_\infty$ for $i = 1, \dots, 2s-1$ and $H_j := [\sigma_0^j \sigma_q(\vec{e})]_\infty$ for $j = 1, \dots, 2t-1$ the holes bordered respectively by $\{f_i, f_{i+1}\}$ and $\{h_j, h_{j+1}\}$.

Finally call E_+ and E_- the holes adjacent to e as in the previous figure. Consequently denote by $l_{_{F_i}}$ and $l_{_{H_j}}$ the lengths of the half-perimeters of the holes F_i and H_j respectively. Remark that neither the edges f and h nor the holes F and H are necessarily distinct. This however has no importance in the following computation.

First of all we have

$$B(de) = -\sum_{i=1}^{2s}(-1)^i \frac{\partial}{\partial f_i} - \sum_{j=1}^{2t}(-1)^j \frac{\partial}{\partial h_j}.$$

Then it is easy to see (using that the perimeters are constant) that

$$l^2_{_{F_i}} \overline{\omega}_{_{F_i}}(\frac{\partial}{\partial f_i} - \frac{\partial}{\partial f_{i+1}}) = -\frac{1}{4}(df_i + df_{i+1})$$

and analogously for the h's. Moreover

$$l^2_{_{E_+}} \overline{\omega}_{_{E_+}}(\frac{\partial}{\partial h_{2s}} - \frac{\partial}{\partial f_1}) = \frac{1}{4}dh_{2s} + \frac{1}{4}df_1 + \frac{1}{2}de$$

and similarly for E_-. At last we obtain $\overline{\Omega}B(de) = de$. □

Lemma 3.2.3 ([Kon92]). *With the given orientation $\overline{\mathcal{M}}^{comb}_{m_*,P}(l)$ is a cycle for all $l \in \mathbb{R}^P_+$ and $\overline{\mathcal{M}}^{comb}_{m_*,P}(\mathbb{R}^P_+)$ is a cycle with noncompact support.*

Proof. Given a top-dimensional cell α in $\overline{\mathcal{M}}^{comb}_{m_*,P}(l)$, each face in the boundary $\partial\alpha$ is obtained shrinking one edge of G_α. This contraction may merge two vertices as in Fig. 3.2. Otherwise the shrinking produces a node (obtained identifying two vertices) as in Fig. 3.3. Let $\alpha' \in \partial\alpha$ be the face of α obtained by shrinking the edge L. Then $\Lambda^{6g-7+2n-2r}T\alpha' = \Lambda^{6g-6+2n-2r}T\alpha \otimes N^*_{\alpha'/\alpha}$ and so the dual of the orientation form induced by α on α' is $\iota_{dL}(B_\alpha^{6g-6+2n-2r}) = (6g-6+2n-2r)\iota_{dL}(B_\alpha) \wedge B_\alpha^{6g-8+2n-2r}$, where B_α is the 2-vector field on α defined in Lemma 3.2.2.

Consider the graph $G_{\alpha'}$ that occurs in the boundary of a top-dimensional cell of $\overline{\mathcal{M}}^{comb}_{m_*,P}(l)$. Suppose it is obtained merging two vertices of valencies $2t_1+3$ and $2t_2+3$ in a vertex v of valency $2(t_1+t_2)+4$. Then a' is in the boundary of exactly $2(t_1+t_2)+4$ cells of $\overline{\mathcal{M}}^{comb}_{m_*,P}(l)$ or t_1+t_2+2 ones in the case $t_1 = t_2$. In any case the number of cells α' is border of are even: we

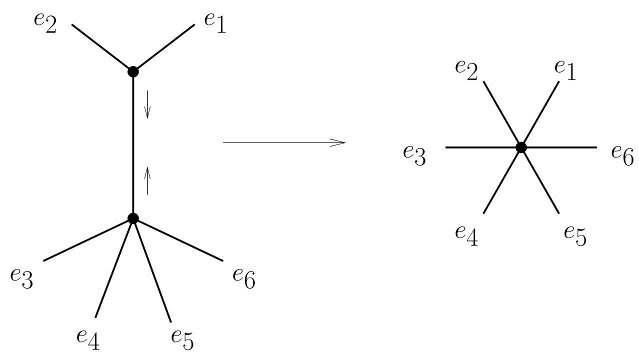


Figure 3.2: A contraction that merges a 3-valent and a 5-valent vertex

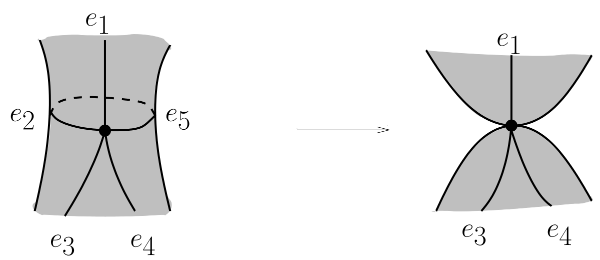


Figure 3.3: A contraction that produces a node

need to prove that half of them induces on α' an orientation and the other half induces the opposite one. If α' is obtained from some α contracting an edge L, then we just have to compute the vector field $\iota_{dL}(B_\alpha)$, which turns to be

$$\iota_{dL}(B_\alpha) = \pm \sum_{i=1}^{2(t_1+t_2)+4} (-1)^i \frac{\partial}{\partial f_i}$$

where $f_1, \ldots, f_{2(t_1+t_2)+4}$ are the edges of $G_{\alpha'}$ outgoing from v. It is a straightforward computation to check that one obtains in half the cases a plus and in half the cases a minus.

When $G_{\alpha'}$ has a node with $2t_1+2$ edges on one side (which we will denote by $f_1, \ldots, f_{2t_1+2}$) and $2t_2+3$ edges on the other side, the computation is similar. The cell occurs as boundary of exactly $(2t_1+2)(2t_2+3)$ top-dimensional cells and, if α' is obtained by α contracting the edge L, then

$$\iota_{dL}(B_\alpha) = \pm \sum_{i=1}^{2t_1+2} (-1)^i \frac{\partial}{\partial f_i}.$$

A quick check ensures that the signs cancel. Hence $\overline{\mathcal{M}}^{comb}_{m_*,P}(l)$ is a cycle and as a consequence $\overline{\mathcal{M}}^{comb}_{m_*,P}$ is a cycle with noncompact support. □

Set $\widehat{\omega}_p := \widehat{\Phi}^*\xi^*\overline{\omega}_p$. With some modifications we have the following analogous result for combinatorial cycles on $\widehat{\mathcal{M}}^{comb}_{g,P}$. Really one does not need it to establish Theorem A and Theorem B on the locus of smooth surfaces, but only if one wants to deal with boundary terms.

Lemma 3.2.4. *The symplectic form $\widehat{\Omega} := \sum_{p\in P} l_p^2 \widehat{\omega}_p$ (resp. $\widehat{\Omega}^r \wedge \bar{\lambda}^* dvol_{\mathbb{R}^P_+}$) gives an orientation to the complex $\widehat{\mathcal{M}}^{comb}_{m_*,P}(l)$ for any $l \in \mathbb{R}^P_+$ (resp. $\widehat{\mathcal{M}}^{comb}_{m_*,P}(\mathbb{R}^P_+)$) so that $\widehat{\mathcal{M}}^{comb}_{m_*,P}(l)$ (resp. $\widehat{\mathcal{M}}^{comb}_{m_*,P}(\mathbb{R}^P_+)$) is a cycle (resp. a cycle with noncompact support).*

The spaces $\widehat{\mathcal{M}}^{comb}_{m_*,P}$ and $\overline{\mathcal{M}}^{comb}_{m_*,P}$ reduce to $\mathcal{M}^{comb}_{m_*,P}$ when restricted to the locus of ordinary ribbon graphs and they coincide with the closure of $\mathcal{M}^{comb}_{m_*,P}$ in $\widehat{\mathcal{M}}^{comb}_{g,P}$ and $\overline{\mathcal{M}}^{comb}_{g,P}$ respectively. Hence $\xi_*\widehat{\Phi}_*[\widehat{\mathcal{M}}^{comb}_{m_*,P}] = [\overline{\mathcal{M}}^{comb}_{m_*,P}]$.

Define the *combinatorial classes* $\widehat{W}_{m_*,P}(l) := \widehat{\Phi}_*[\widehat{\mathcal{M}}^{comb}_{m_*,P}(l)]$ and $\overline{W}_{m_*,P}(l) := [\overline{\mathcal{M}}_{m_*,P}(l)]$ and observe that

$$\begin{array}{ccc} H^{nc}_{6g-6+3n-2r}(\overline{\mathcal{M}}_{g,P} \times \mathbb{R}^P_+) & \xrightarrow{\sim} & H_{6g-6+2n-2r}(\overline{\mathcal{M}}_{g,P}(l)) \\ \widehat{W}_{m_*,P}(\mathbb{R}^P_+) & \longmapsto & \widehat{W}_{m_*,P}(l) \end{array}$$

and

$$\begin{array}{ccc} H^{nc}_{6g-6+3n-2r}(\overline{\mathcal{M}}^{comb}_{g,P}(\mathbb{R}^P_+)) & \xrightarrow{\sim} & H_{6g-6+2n-2r}(\overline{\mathcal{M}}^{comb}_{g,P}(l)) \\ \overline{W}_{m_*,P}(\mathbb{R}^P_+) & \longmapsto & \overline{W}_{m_*,P}(l) \end{array}$$

for every $l \in \mathbb{R}^P_+$ naturally with respect to ξ. So, from now on we will write $\overline{W}_{m_*,P}$ and $\widehat{W}_{m_*,P}$ instead of $\overline{W}_{m_*,P}(l)$ and $\widehat{W}_{m_*,P}(l)$ for the homology classes they define in $\overline{\mathcal{M}}'_{g,P} \cong \overline{\mathcal{M}}^{comb}_{g,P}(l)$ and $\overline{\mathcal{M}}_{g,P}$ respectively for any $l \in \mathbb{R}^P_+$. Moreover we will identify $\widehat{W}_{m_*,P}$ with its Poincaré dual in $H^{2r}(\overline{\mathcal{M}}_{g,P})$.

3.3 Generalized combinatorial classes

It is possible to define a slight generalization of the previous classes, prescribing that some markings hit vertices with assigned valency.

Given a finite set $Q := \{q_1, \dots, q_h\}$ and a map $\rho : Q \to \mathbb{Z}_{\geq -1}$ we define $m_*^\rho = (m_{-1}^\rho, m_0^\rho, \dots)$ as $m_i^\rho := |\rho^{-1}(i)|$. Consider now an m_* and a ρ such that $m_{-1}^\rho = m_{-1}$, $m_*^\rho \leq m_*$ and $\sum_{i\geq -1}(2i+1)m_i = 4g-4+2|P|$ and call $\overline{\mathcal{M}}^{comb}_{m_*,\rho,P}$ the subcomplex of $\overline{\mathcal{M}}^{comb}_{m_*,P\cup Q}$ whose simplices of maximal dimension are ordinary ribbon graphs in which q_j marks a vertex of valency $2\rho(q_j)+3$ for every $j = 1, \dots, h$ and denote by $\overline{W}_{m_*,\rho,P}$ its cohomology class in $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(\{l_q = 0 \,|\, q \in Q\})$ (as before the orientation is determined by $\sum_{p\in P} l_{p_i}^2 \overline{\omega}_p$). Define analogously $\widehat{\mathcal{M}}^{comb}_{m_*,\rho,P}$ and let $\widehat{W}_{m_*,\rho,P}$ be its cohomology class in in $\overline{\mathcal{M}}_{g,P\cup Q}$. Notice that these classes live in codimension $2\sum_{i\geq -1} i\, m_i + 2|Q| = 2r + 2|Q|$. The following statement is straightforward.

Lemma 3.3.1. *Let $\pi_Q : \overline{\mathcal{M}}_{g,P\cup Q} \to \overline{\mathcal{M}}_{g,P}$ be the forgetful morphism. Then one has*

$$(\pi_Q)_*(\widehat{W}_{m_*,\rho,P}) = \frac{(m_*)!}{(m_* - m_*^\rho)!}\widehat{W}_{m_*,P}$$

where $(\pi_Q)_* : H^{2r+2|Q|}(\overline{\mathcal{M}}_{g,P\cup Q}) \to H^{2r}(\overline{\mathcal{M}}_{g,P})$ is the induced push-forward map.

Chapter 4

Classes with one special vertex

We deal with the simplest combinatorial class, namely the class W_{2r+3} of graphs with only one vertex of valency $2r+3$.

At a first reading the proof may look quite involved, because of some technicalities. However the basic ideas are quite simple. We want to describe them in some detail before going to the formal proof.

The first observation is that $\mathcal{M}^{comb}_{g,P\cup\{q\}}(l)$ is homeomorphic to $\mathcal{M}_{g,P\cup\{q\}}$ for every $l \in R^{P\cup\{q\}}_{\geq 0} \setminus \{0\}$. The second remark is that the differential form $\overline{\omega}_q$ lives on the slices $\mathcal{M}^{comb}_{g,P\cup\{q\}}(l)$ such that $l_q > 0$, while the (generalized) combinatorial class (which we briefly denote by W^q_{2r+3}), defined prescribing that q marks a vertex of valency (at least) $2r+3$, lives on the slices that have $l_q = 0$.

So a deformation retraction $\mathcal{H}_0$ of $\mathcal{M}^{comb}_{g,P\cup\{q\}}$ onto the slice defined by $l_q = 0$ would help us to compare $\overline{\omega}^{r+1}_q$ and the combinatorial class W^q_{2r+3} as functionals on the cohomology of $\mathcal{M}^{comb}_{g,P\cup\{q\}}(l)$.

The deformation retraction $\mathcal{H}_0$ we will construct however does not preserve the locus of the smooth curves, but it retracts the whole $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$ onto the slice $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(l_q = 0)$. In fact $\mathcal{H}_0$ is defined sending all the edges bordering the q-th hole to zero (it is defined only when l_q is "small", because we must avoid the situation in which $\mathcal{H}_0$ would squeeze another hole beside q). So it shrinks a circular q-th hole (i.e. such that $\bar{T}_q$ is a disk) to a q-marked vertex, while it produces a "singular" graph if the topology of the q-th hole is more complicated. But, if we subdivide the complex $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$ into subcomplexes $\overline{Y}^{\bullet}_{*}$ according to the topology of the q-th hole, then the restriction of $\mathcal{H}_0$ to each subcomplex is a simplicial fibration.

Then we consider a differential form η on $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(l_q = 0)$ and we compare the integral of η on $\overline{W}^q_{2r+3}(l)$ (the closure of $W^q_{2r+3}(l)$) for an l such that $l_q = 0$ with the integral of $\varpi_q^{r+1} \wedge \mathcal{H}_0^*\eta$ on $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(l')$ for an l' such that $l'_q > 0$. Here we notice that the form $\varpi_q^{r+1} \wedge \mathcal{H}_0^*\eta$ has support on the top-dimensional simplices whose q-th hole has *exactly* $2r+3$ distinct edges. Then the integral of $\varpi_q^{r+1} \wedge \mathcal{H}_0^*\eta$ is performed by calculating for each $\overline{Y}^{\bullet}_{*}$ the integral of ϖ_q^{r+1} on the fibers of $\mathcal{H}_0$. In the case of a circular q-th hole with $2r+3$ edges we obtain the factor $2^{r+1}(2r+1)!!$.

The analogous result for the (ordinary) combinatorial class $\overline{W}_{2r+3}$ and κ_r is derived from the preceding one by simply noticing that the forgetful morphism π_q has a combinatorial analogue π_q^{comb} on the combinatorial spaces (another little technical problem is due to the fact that π_q^{comb} is not defined on the whole $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$ but what we get is enough to conclude). So that $(\pi_q)_*(\psi_q^{r+1}) = \kappa_r$ and $(\pi_q^{comb})_*$ sends $\overline{W}^q_{2r+3}$ to $\overline{W}_{2r+3}$. Hence we obtain our result for the kappa classes too.

4.1 The retraction $\mathcal{H}_0$ and π_q^{comb}

Fix $g \geq 0$ and $n > 0$ such that $2g-2+n > 0$ and define $P := \{p_1, \ldots, p_n\}$ and

$$C_{P,q} := \{l \in \mathbb{R}^{P\cup\{q\}}_{\geq 0} \,|\, l_q < l_p \quad \text{for all } p \in P\}.$$

Denote by $\pi_q : \overline{\mathcal{M}}_{g,P\cup\{q\}} \times C_{P,q} \to \overline{\mathcal{M}}_{g,P} \times \mathbb{R}^P_+$ the map that forgets q and the q-th coordinate. We can define π_q^{comb} forcing the commutativity of the following diagram

$$\begin{tikzcd}
(\overline{\mathcal{M}}_{g,P\cup\{q\}} \setminus D_q) \times C_{P,q} \arrow[r, "\tilde{\xi}"] \arrow[d, "\pi_q"'] & \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C_{P,q}) \setminus D_q^{comb} \arrow[d, "\pi_q^{comb}"] \\
\overline{\mathcal{M}}_{g,P} \times \mathbb{R}_+^P \arrow[r, "\xi"] & \overline{\mathcal{M}}^{comb}_{g,P}(\mathbb{R}_+^P)
\end{tikzcd} \qquad (\star)$$

where $D_q := \cup_{p\in P}\delta_{0,\{q,p\}}$ and $D_q^{comb} = \tilde{\xi}(D_q \times C_{P,q})$. We remark that $\xi\pi_q$ does not factorize through $\tilde{\xi} : \overline{\mathcal{M}}_{g,P\cup\{q\}} \times C_{P,q} \to \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C_{P,q})$. In fact

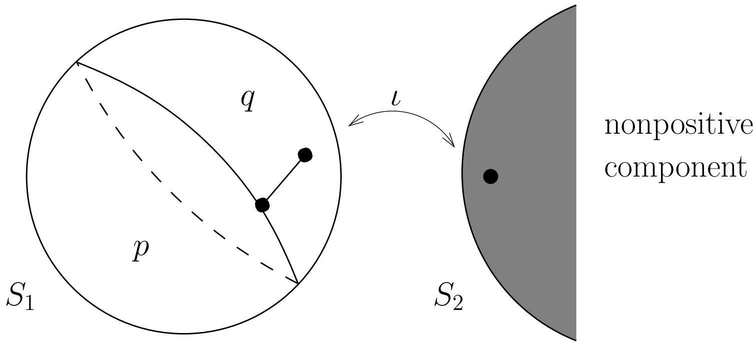


Figure 4.1: π_q^{comb} is not defined in this case

pick a point (S, l) in $\overline{\mathcal{M}}_{g,P\cup\{q\}} \times C_{P,q}$ such that q and p lie on a two-pointed component S_1 of S of genus zero which has only one singular point and suppose that the adjacent component S_2 is nonpositive (see Fig. 4.1). Then $\tilde{\xi}(S, l)$ does not "remember" the analytic type of S_2 but $\xi\pi_q(S, l)$ does (if $l_p > 0$) because the p-marking now hits S_2 after forgetting q and stabilizing. However this is the only case, so it sufficient to cut away D_q and D_q^{comb}.

Remark. The behaviour of the map π_q^{comb} is really misterious as we do not know in general how Strebel's differential changes when we delete the marked point q and consequently how the critical graph modifies. However we know that if q marks a vertex then the new critical graph is obtained simply forgetting the marking. On the other hand, when this happens the form $\overline{\omega}_q$ is not defined because $l_q = 0$. All the technical problems derive from this dichotomy. We will overcome this difficulty by keeping the perimeter l_q positive so that $\overline{\omega}_q$ makes sense and by taking the limit for $l_q \to 0$. Then we will show that, in this limit, π_q^{comb} is well approximated by the simplicial map that shrinks the hole q to a vertex and forgets the q-marking.

Notation. Call $\widehat{Y}_h \subset \widehat{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$ (resp. $\overline{Y}_h \subset \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}$) the closure of the locus of graphs where the hole $x(q)$ has positive perimeter and consists exactly of h distinct (unoriented) edges. Set $\widehat{Y}_{\geq h} := \cup_{i\geq h}\widehat{Y}_i$ (resp. $\overline{Y}_{\geq h} := \cup_{i\geq h}\overline{Y}_i$).

Clearly the topological boundary $\partial\widehat{Y}_{\geq h}$ (resp. $\partial\overline{Y}_{\geq h}$) is contained inside $\widehat{Y}_{\leq h-1} := \cup_{1\leq i\leq h-1}\widehat{Y}_i$ (resp. $\overline{Y}_{\leq h-1} := \cup_{1\leq i\leq h-1}\overline{Y}_i$). Moreover $\widehat{\Phi}(\widehat{Y}_{\geq 2}(C_{P,q}))$ is contained inside $(\overline{\mathcal{M}}_{g,P\cup\{q\}} \setminus D_q) \times C_{P,q}$ and similarly $\overline{Y}_{\geq 2}(C_{P,q})$ is contained inside $(\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}} \setminus D_q^{comb}) \times C_{P,q}$. In fact $\overline{Y}_1(C_{P,q})$ is a closed neighbourhood of $D_q^{comb}(C_{P,q})$ inside $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C_{P,q})$. Remark, by the way, that $\overline{\omega}_q|_{\overline{Y}_1} = \widehat{\omega}_q|_{\widehat{Y}_1} = 0$ because the hole q does not contain enough edges.

Proposition 4.1.1. *There is a deformation retraction*

$$\tilde{\mathcal{H}} : \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C_{P,q}) \times [0,1] \longrightarrow \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C_{P,q})$$

such that $\tilde{\mathcal{H}}_1$ is the identity and $\tilde{\mathcal{H}}_0$ is the piecewise-linear retraction onto $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(\mathbb{R}^P_+ \times \{0\})$ that "shrinks" the q-th hole. Moreover $\tilde{\mathcal{H}}_t(\overline{Y}_h) \subset \overline{Y}_h$ for all $t \in [0,1]$.

Proof. Consider a cell $\bar{\lambda}^{-1}(C_{P,q}) \cap |\alpha| \times \mathbb{R}_+$ inside $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C_{P,q})$. Denote by $e_1, \dots, e_h$ the coordinates of $|\alpha| \times \mathbb{R}_+$ corresponding to the unoriented edges of G_α that border the hole q and by $f_1, \dots, f_k$ the remaining ones. Then it is sufficient to define $\tilde{\mathcal{H}}_t$ as the map that sends $e_i \longmapsto t \cdot e_i$ and $f_j \longmapsto f_j$ and to observe that all these deformation retractions glue to give a global $\tilde{\mathcal{H}}$. By definition it is obvious that $\tilde{\mathcal{H}}_t(\overline{Y}_h) \subset \overline{Y}_h$. □

Call $\mathcal{H}_0 : \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C_{P,q}) \twoheadrightarrow \overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(\mathbb{R}^P_+ \times \{0\})$ the restriction of $\tilde{\mathcal{H}}_0$. Since we will work with classes of the form $\psi_q^{r+1} \smile \tilde{\xi}^*\mathcal{H}_0^*\eta$ and we would like to exploit the explicit representative $\overline{\omega}_q$ which is defined only where $l_q > 0$, then we let the perimeters vary in the subset $C^+_{P,q} := C_{P,q} \cap \{l_q > 0\}$ only.

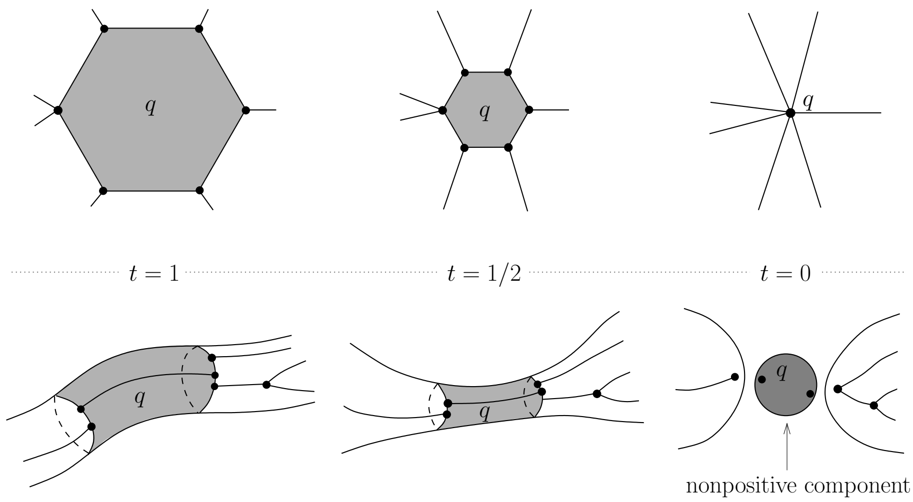


Figure 4.2: The deformation retraction $\tilde{\mathcal{H}}$

Proposition 4.1.2. *Let η be a piecewise-linear differential form on $\overline{Y}_{\geq 2}(\mathbb{R}_+^P \times \{0\})$. Then $\overline{\omega}_q \wedge \mathcal{H}_0^* \eta$ (which is defined only on $\overline{Y}_{\geq 2}(C_{P,q}^+)$) regularly extends by zero to the whole $\overline{\mathcal{M}}_{g,P\cup\{q\}}^{comb}(C_{P,q}^+)$. Moreover, if $[\eta]$ is the restriction of $(\pi_q^{comb})^*\varphi$ to $\overline{Y}_{\geq 2}(\mathbb{R}_+^P \times \{0\})$, then $\tilde{\xi}^*[\overline{\omega}_q \wedge \mathcal{H}_t^*\eta]$ is exactly $\psi_q \smile \pi_q^* \xi^* \varphi$.*

Proof. The first assertion is trivial because $\overline{\omega}_q$ vanishes on $\partial \overline{Y}_{\geq 2}(C_{P,q}^+) \subset \overline{Y}_1(C_{P,q}^+)$. For the second assertion, remember that de Rham isomorphism holds on $\overline{\mathcal{M}}_{g,P\cup\{q\}}^{comb}(C_{P,q}^+)$ and that $\overline{\omega}_q$ defines a cohomology class relative to $\overline{Y}_1(C_{P,q}^+)$. Hence the result follows from the commutativity of $(\star)$ and from the excision of D_q and D_q^{comb}. □

4.2 Proof of Theorem A

Now we can pass to analyze the simplest kind of combinatorial classes, namely those with just one special vertex.

Let $P := \{p_1, \ldots, p_n\}$ and for every integer $r \geq -1$ denote by $\widehat{W}_{2r+3}^q$ the combinatorial class of $\overline{\mathcal{M}}_{g,P\cup\{q\}}$ whose vertices are all trivalent except one which has valency $2r+3$ and is marked by q. Analogously call $\widehat{W}_{2r+3}$

the combinatorial class on $\overline{\mathcal{M}}_{g,P}$ whose vertices are all trivalent except one which has valency $2r+3$ (in the case $r=0$ all the vertices are trivalent).

Theorem A. *For any g and $n \geq 1$ the equality*

$$\widehat{W}^q_{2r+3} = \frac{(2r+2)!}{(r+1)!}\psi_q^{r+1}$$

holds in $H^{2r+2}(\overline{\mathcal{M}}_{g,P\cup\{q\}})$ up to terms in the kernel of

$$\zeta_* : H_{2s}(\overline{\mathcal{M}}_{g,P\cup\{q\}}) \to H_{2s}(\overline{\mathcal{M}}_{g,P}, \partial\mathcal{M}_{g,P})$$

where $s = 3g-3+n-r$. As a consequence

$$\widehat{W}_{2r+3} = \begin{cases} 0 & \text{if } r=-1 \\ [\overline{\mathcal{M}}_{g,P}] & \text{if } r=0 \\ 2^{r+1}(2r+1)!!\,\kappa_r & \text{if } r \geq 1 \end{cases}$$

holds in $H^{2r}(\overline{\mathcal{M}}_{g,P})$ up to boundary terms.

Strategy. The $(2r+2)$-form $\hat{\omega}_q^{r+1}$ determines a class in

$$H^{2r+2}(\widehat{Y}_{\geq 2}(C^+_{P,q}), \partial\widehat{Y}_{\geq 2}(C^+_{P,q}))$$

and so it couples with forms of $H^{2s}(\widehat{Y}_{\geq 2}(C^+_{P,q}))$ by Poincaré duality. Hence $[\hat{\omega}_q]^{r+1}$ may be viewed as an element of the dual space $H^{2s}(\widehat{Y}_{\geq 2}(C^+_{P,q}))^*$, which maps to $H^{2s}(\widehat{Y}_{\geq 2}(C_{P,q}))^*$.

We will determine a boundary class $\widehat{B}^q_{2r+3}$ in $\widehat{Y}_{\geq 2}(C_{P,q})$ such that the equality

$$\widehat{W}^q_{2r+3} = 2^{r+1}(2r+1)!![\widehat{\omega}_q]^{r+1} - \widehat{B}^q_{2r+3}$$

holds in $H^{2s}(\widehat{Y}_{\geq 2}(C_{P,q}))^*$ when coupled with cocycles in the image of

$$\widehat{\Phi}^*\tilde{\xi}^*\mathcal{H}_0^* : H^{2s}(\overline{Y}_{\geq 2}(\mathbb{R}^P_+ \times \{0\})) \to H^{2s}(\widehat{Y}_{\geq 2}(C_{P,q})).$$

Now by the commutativity of the following diagram

$$\begin{tikzcd}
H^{2s}(\widehat{Y}_{\geq 2}(C_{P,q}))^* \arrow[r, "(\mathcal{H}_0\tilde{\xi}\widehat{\Phi})_*"] \arrow[d] & H^{2s}(\overline{Y}_{\geq 2}(\mathbb{R}^P_+ \times \{0\}))^* \arrow[d, "(\pi_q)^{comb}_*"] \\
H_{2s}(\overline{\mathcal{M}}_{g,P\cup\{q\}}) \arrow[d, "\zeta_*"] & H^{2s}(\overline{\mathcal{M}}^{comb}_{g,P}(\mathbb{R}^P_+))^* \arrow[d] \\
H_{2s}(\overline{\mathcal{M}}_{g,P}, \partial\mathcal{M}_{g,P}) \arrow[r, "\sim"] & H^{2s}(\overline{\mathcal{M}}^{comb}_{g,P}(\mathbb{R}^P_+), \partial\overline{\mathcal{M}}^{comb}_{g,P}(\mathbb{R}^P_+))^*
\end{tikzcd}$$

the image of $\ker(\mathcal{H}_0\tilde{\xi}\widehat{\Phi})_*$ in $H_{2s}(\overline{\mathcal{M}}_{g,P\cup\{q\}})$ is contained inside $\ker\zeta_*$. This concludes the argument. Because of Proposition 4.1.2 and Lemma 3.3.1 we then immediately obtain the second claim.

Proof of Theorem A. Consider a closed PL differential form η on $\overline{Y}_{\geq 2}(\mathbb{R}^P_+ \times \{0\})$ of degree $2s$ which is the pull-back of a form via a projection onto $\overline{Y}_{\geq 2}(l, 0)$ for some $l \in \mathbb{R}^P_+$. By Proposition 4.1.2 the form $\mathcal{H}_0^*\eta \wedge \overline{\omega}_q^{r+1}$ extends by zero to $\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(C^+_{P,q})$ and its pull-back via $\tilde{\xi}$ is $\psi_q^{r+1} \smile \tilde{\xi}^*\mathcal{H}_0^*[\eta]$. Moreover $\mathcal{H}_0^*\eta \wedge \overline{\omega}_q^{r+1}$ has support inside $\overline{Y}_{2r+3}(C^+_{P,q})$. In fact $\overline{\omega}_q^{r+1}$ has support inside $\overline{Y}_{\geq 2r+3}(C^+_{P,q})$, while $\mathcal{H}_0^*\eta$ has support inside $\overline{Y}_{\leq 2r+3}(C^+_{P,q})$ because η has support inside $\overline{Y}_{\leq 2r+3}(\mathbb{R}^P_+ \times \{0\})$.

Now decompose $\overline{Y}_{2r+3}(C^+_{P,q})$ into three families of subsets:

1. the closure $\overline{Y}^{disk}_{2r+3}(C^+_{P,q})$ of the locus of graphs where the surface $\bar{T}_q$ is a disk; in this case $\mathcal{H}_0(\overline{Y}^{disk}_{2r+3}(C^+_{P,q}))$ is exactly the support of $\overline{W}^q_{2r+3}(\mathbb{R}^P_+)$ consisting of graphs with one vertex of valency $2r+3$ marked by q

2. the closure $\overline{Y}^{cyl}_{v_1,v_2}(C^+_{P,q})$ of the locus of graphs where $\bar{T}_q$ is a cylinder with exactly one internal edge e, which divides the other edges of $x(q)$ into two subsets of cardinality v_1+1 and $v_2+1 = 2r - v_1 + 1$; its image via $\mathcal{H}_0$ is the union of loci $\overline{N}^q_{v_1,v_2}(\mathbb{R}^P_+)$ of graphs with one node marked by q that is obtained identifying two vertices of valencies v_1 and v_2

3. the closure $\overline{Y}^{surf}_{h,\{v_1,\ldots,v_\nu\}}(C^+_{P,q})$ of the locus of graphs where $\bar{T}_q$ a surface of genus h with $\nu > 2-2h$ boundary components which touch

$v_1, \dots, v_\nu$ external edges (i.e. not in $\bar{T}_q$) respectively, where $6h-6+\sum_{j=1}^{\nu}(v_j+3) = 2r$; its image via $\mathcal{H}_0$ is the locus $\overline{Z}^q_{h,\{v_1,\dots,v_\nu\}}(\mathbb{R}_+^P)$ of graphs with one nonpositive component of genus h which has the q-marking and ν nodes corresponding to vertices of $v_1, \dots, v_\nu$ valencies.

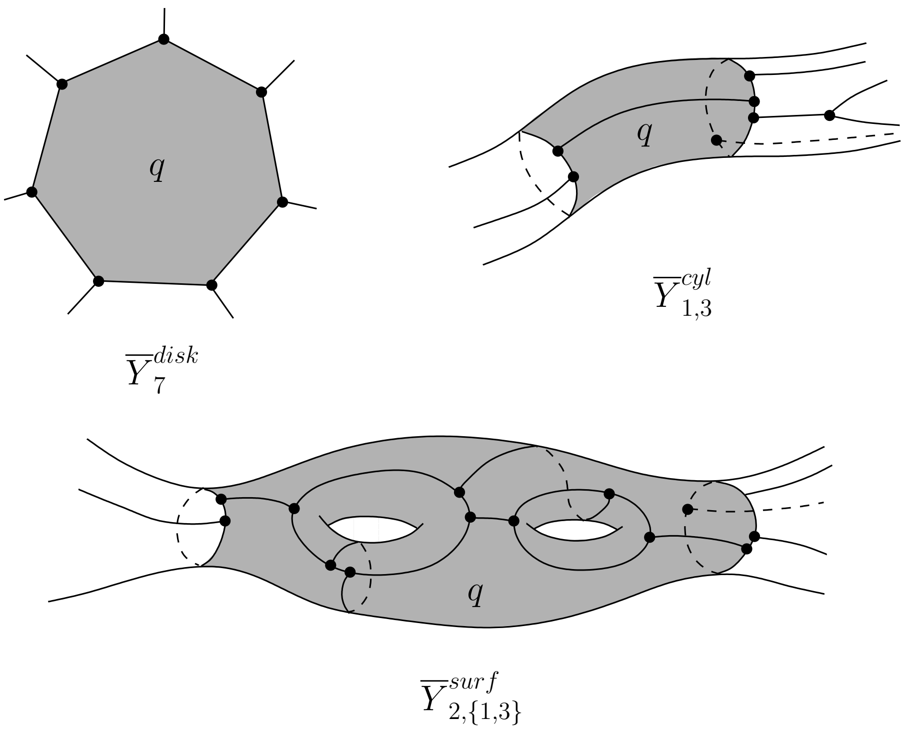


Figure 4.3: Three examples of loci $\overline{Y}$

Remark that $\bar{\lambda}_{p_i}(\mathcal{H}_0(\overline{Y}_{2r+3}(\tilde{l})))$ takes values between $\tilde{l}_{p_i} - \tilde{l}_q$ and $\tilde{l}_{p_i}$. So choose $0 < \varepsilon < L'' << L'$ and notice that $\mathcal{H}_0(\overline{Y}_{2r+3}([L'', L']^n, \varepsilon))$ contains

$$\left(\cup_{h,v_*} \overline{Z}^q_{h,v_*} \cup \mathrm{supp}(\overline{W}^q_{2r+3}) \cup_{v_1,v_2} \overline{N}^q_{v_1,v_2}\right)([L'', L'-\varepsilon]^n)$$

and is contained inside

$$\left(\cup_{h,v_*} \overline{Z}^q_{h,v_*} \cup \mathrm{supp}(\overline{W}^q_{2r+3}) \cup_{v_1,v_2} \overline{N}^q_{v_1,v_2}\right)([L''-\varepsilon, L']^n).$$

Since the volume of the difference $[L''-\varepsilon, L']^n \setminus [L'', L'-\varepsilon]^n$ goes to zero as ε decreases, we have

$$
\begin{aligned}
&\int_{[L'',L']^n} dl_{p_1}\wedge\cdots\wedge dl_{p_n}\int_{\overline{\mathcal{M}}_{g,P\cup\{q\}}}\psi_q^{r+1}\smile\tilde{\xi}^*\mathcal{H}_0^*[\eta]= \qquad (\bullet)\\
&=\lim_{\varepsilon\to 0}\int_{\overline{Y}_{2r+3}([L'',L']^n,\varepsilon)}\bar{\lambda}^*(dl_{p_1}\wedge\cdots\wedge dl_{p_n})\wedge\overline{\omega}_q^{r+1}\wedge\mathcal{H}_0^*\eta=\\
&=\lim_{\varepsilon\to 0}\Big(\int_{\overline{Y}^{disk}_{2r+3}([L'',L']^n,\varepsilon)}\bar{\lambda}^*(dl_{p_1}\wedge\ldots dl_{p_n})\wedge\overline{\omega}_q^{r+1}\wedge\mathcal{H}_0^*\eta+\\
&+\sum_{v_1+v_2=2r}\int_{\overline{Y}^{cyl}_{v_1,v_2}([L'',L']^n,\varepsilon)}\bar{\lambda}^*(dl_{p_1}\wedge\ldots dl_{p_n})\wedge\overline{\omega}_q^{r+1}\wedge\mathcal{H}_0^*\eta+\\
&+\sum_{h,v_*}\int_{\overline{Y}^{surf}_{h,v_*}([L'',L']^n,\varepsilon)}\bar{\lambda}^*(dl_{p_1}\wedge\ldots dl_{p_n})\wedge\overline{\omega}_q^{r+1}\wedge\mathcal{H}_0^*\eta\Big)= \qquad (\bullet\bullet)\\
&=\lim_{\varepsilon\to 0}\Big(\int_{\overline{W}^q_{2r+3}([L'',L'-\varepsilon]^n)}\bar{\lambda}^*(dl_{p_1}\wedge\ldots dl_{p_n})\wedge\eta\int_{F^{disk}(\varepsilon)}\overline{\omega}_q^{r+1}+\\
&+\sum_{v_1+v_2=2r}\int_{\overline{N}^q_{v_1,v_2}([L'',L'-\varepsilon]^n)}\bar{\lambda}^*(dl_{p_1}\wedge\ldots dl_{p_n})\wedge\eta\int_{F^{cyl}_{v_1,v_2}(\varepsilon)}\overline{\omega}_q^{r+1}+\\
&+\sum_{h,v_*}\int_{\overline{Z}^q_{h,v_*}([L'',L'-\varepsilon]^n)}\bar{\lambda}^*(dl_{p_1}\wedge\ldots dl_{p_n})\wedge\eta\int_{F^{surf}_{h,v_*}(\varepsilon)}\overline{\omega}_q^{r+1}\Big)=\\
&=\int_{[L'',L']^n}dl_{p_1}\wedge\cdots\wedge dl_{p_n}\Big(\int_{\overline{W}^q_{2r+3}(l)}\eta\int_{F^{disk}(\varepsilon)}\overline{\omega}_q^{r+1}+\\
&+\sum_{v_1+v_2=2r}\int_{\overline{N}^q_{v_1,v_2}(l)}\eta\int_{F^{cyl}_{v_1,v_2}(\varepsilon)}\overline{\omega}_q^{r+1}+\sum_{h,v_*}\int_{\overline{Z}^q_{h,v_*}(l)}\eta\int_{F^{surf}_{h,v_*}(\varepsilon)}\overline{\omega}_q^{r+1}\Big)
\end{aligned}
$$

where l belongs to $\mathbb{R}_+^P$ and $F^{disk}(\varepsilon)$ is the intersection of the generic fiber of $\mathcal{H}_0$ over $\mathrm{supp}(\overline{W}^q_{2r+3}([L'',L'-\varepsilon]^n))$ with $\overline{Y}_{2r+3}((\varepsilon,+\infty)^n,\varepsilon)$ and similarly for F^{cyl} and F^{surf}.

Remark. In $(\bullet)$ we used Proposition 4.1.2 and the push-forward through the map

$$\tilde{\xi}_\varepsilon:\overline{\mathcal{M}}_{g,P\cup\{q\}}\times\mathbb{R}^n_+\times\{\varepsilon\}\longrightarrow\overline{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(\mathbb{R}^n_+\times\{\varepsilon\}).$$

In $(\bullet\bullet)$ we used that $\mathcal{H}_0$ restricts to

$$
\begin{tikzcd}
\overline{Y}^{disk}_{2r+3}([L'',L']^n,\varepsilon) \arrow[r, "\mathcal{H}_0^{disk}"] & \mathrm{supp}(\overline{W}^q_{2r+3}([L''-\varepsilon,L']^n)) \\
(\mathcal{H}_0^{disk})^{-1}(\mathrm{supp}(\overline{W}^q_{2r+3}([L'',L'-\varepsilon]^n))) \arrow[u, hook] \arrow[r, two heads] & \mathrm{supp}(\overline{W}^q_{2r+3}([L'',L'-\varepsilon]^n)) \arrow[u, hook]
\end{tikzcd}
$$

where the lower map is a fibration with fiber $F^{disk}(\varepsilon)$ and the differences $\overline{Y}^{disk}_{2r+3}\setminus(\mathcal{H}^{disk}_0)^{-1}(\mathrm{supp}(\overline{W}^q_{2r+3}([L'',L'-\varepsilon]^n)))$ and $\mathrm{supp}(\overline{W}^q_{2r+3}([L''-\varepsilon,L']^n))\setminus\mathrm{supp}(\overline{W}^q_{2r+3}([L'',L'-\varepsilon]^n))$ tend to zero with ε.

It is easy to see that $F^{disk}(\varepsilon)$ is a simplex of dimension $2r+2$ with affine coordinates $e_0,\dots,e_{2r+2}$ where e_j are the (unoriented) edges of the hole $x(q)$, so that $\sum_{j=0}^{2r+2} e_j = 2\varepsilon$. It is also immediate to see that $\overline{\omega}_q^{r+1}$ is equal to $(r+1)!d(\frac{e_1}{2\varepsilon})\wedge\dots\wedge d(\frac{e_{2r+2}}{2\varepsilon})$ so that the integral of $\overline{\omega}_q^{r+1}$ on $F^{disk}(\varepsilon)$ is equal to $\frac{(r+1)!}{(2r+2)!}$.

A simple computation shows that $\overline{\omega}_q^{r+1}$ vanishes on $F^{cyl}_{v_1,v_2}$ if v_1 and v_2 are even; while $\overline{\omega}_q^{r+1}=(r+1)!d(\frac{e_1}{2\varepsilon})\wedge\dots\wedge d(\frac{e_{2r+2}}{2\varepsilon})$ if v_1 and v_2 are odd, where $2e_0+\sum_{j=1}^{2r+2}e_j=2\varepsilon$ and e_0 is the "separating" edge of the cylinder. We conclude that for v_1 and v_2 odd

$$\int_{F^{cyl}_{v_1,v_2}} \overline{\omega}_q^{r+1} = v_1 v_2 \frac{(r+1)!}{(2r+2)!}$$

because $F^{cyl}_{v_1,v_2}$ contains v_1v_2 top-dimensional simplices. On the other hand the integral of $\overline{\omega}_q^{r+1}$ on F^{surf}_{h,v_*} is nontrivial to compute.

However the cycles $\overline{N}^q_{v_1,v_2}$ and $\overline{Z}^q_{h,v_*}$ clearly lift to cycles $\widehat{N}^q_{v_1,v_2}$ and $\widehat{Z}^q_{h,v_*}$ on $\widehat{\mathcal{M}}^{comb}_{g,P\cup\{q\}}(\mathbb{R}^P_+\times\{0\})$ and so on $\overline{\mathcal{M}}_{g,P\cup\{q\}}$, then we can define

$$\widehat{B}^q_{2r+3} := \sum_{\substack{i,j\ge 0\\ i+j=r-1}} (2i+1)(2j+1)\frac{(r+1)!}{(2r+2)!}\widehat{N}^q_{2i+1,2j+1} + \sum_{h,v_*}\widehat{Z}^q_{h,v_*}\int_{F^{surf}_{h,v_*}}\overline{\omega}_q^{r+1}$$

so that the equation $\widehat{W}^q_{2r+3}=2^{r+1}(2r+1)!!\psi_q^{r+1}-\widehat{B}^q_{2r+3}$ in $H^{2r+2}(\overline{\mathcal{M}}_{g,P\cup\{q\}})$ is satisfied in the sense explained before.

For the second claim, we can use an $\eta=(\pi_q^{comb})^*\varphi$. Because of the commutativity of the diagram $(\star)$ and Proposition 4.1.2 it follows that $(\pi_q)_*(\widehat{W}^q_{2r+3}) = 2^{r+1}(2r+1)!!\kappa_r - \widehat{B}_{2r+3}$ (with $\widehat{B}_{2r+3}=(\pi_q)_*(\widehat{B}^q_{2r+3})$) holds in $H^{2r}(\overline{\mathcal{M}}_{g,P})$ up to terms in the kernel of $\xi'_*:H_{2s}(\overline{\mathcal{M}}_{g,P})\to H_{2s}(\overline{\mathcal{M}}'_{g,P})$. □

Remark. In fact we have proven more than what is stated in Theorem A as we have determined $\widehat{B}^q_{2r+3}$ and $\widehat{B}_{2r+3}$ up to minor uncertainties. Moreover the classes $\widehat{B}^q_{2r+3}$ and $\widehat{B}_{2r+3}$ are push-forward of combinatorial classes via some boundary maps. The problem now would be to compute the coefficients of $\widehat{Z}^q_{h,v_*}$ and to clarify what terms we must add to $\widehat{B}^q_{2r+3}$ and $\widehat{B}_{2r+3}$ to obtain the full equality.

Corollary A.1. *For every g and $|P| = n \geq 1$ such that $2g - 2 + n > 0$ the following equalities hold up to elements in the kernel of*

$$(\widehat{\Phi}\tilde{\xi}\mathcal{H}_0)_* : H_*(\widehat{Y}_{\geq 2}(C^+_{P,q})) \to H_*(\overline{Y}_{\geq 2}(\mathbb{R}^P_+ \times \{0\}))$$

and in $\ker(\xi'_*)$ *respectively*

$$\widehat{W}^q_5 = 12\psi^2_q - \delta^q_{irr} - \sum_{g', I \neq \emptyset, P} \delta^q_{g',I} \quad \text{in } H^4(\overline{\mathcal{M}}_{g,P\cup\{q\}})$$
$$\widehat{W}_5 = 12\kappa_1 - \delta_{irr} - \sum_{g', I \neq \emptyset, P} \delta_{g',I} \quad \text{in } H^2(\overline{\mathcal{M}}_{g,P})$$

where $\delta^q_{g',I}$ is the image of the morphism

$$\overline{\mathcal{M}}_{g',I\cup\{p'\}} \times \overline{\mathcal{M}}_{0,\{q,q',q''\}} \times \overline{\mathcal{M}}_{g-g',I^c\cup\{p''\}} \to \overline{\mathcal{M}}_{g,P\cup\{q\}}$$

that glues p' with q' and p'' with q'' (analogously for δ^q_{irr}).

The second equality of the previous corollary has been proven first by Arbarello and Cornalba [AC96] in a very different manner. Here it is a consequence of the proof of Theorem A, because for $r = 1$ the subset $\overline{Y}^{surf}$ does not contain simplices of top dimension while all simplices of top dimension in $\overline{Y}^{cyl}$ have $v_1 = v_2 = 1$.

Chapter 5

Classes with many special vertices

The case of a general combinatorial class $\widehat{W}_{M_*,\rho,P}$ is not much more complicated. The only real obstacle is the notation that becomes cumbersome, but the main ideas are already present in the previous chapter.

The only substantially new proof concerns Lemma 5.1.3 where we compute the number of admissible clusters by an inductive argument.

5.1 Proof ot Theorem B

We now want to examine the case of an arbitrary class $\widehat{W}_{m_*,\rho,P}$ on $\overline{\mathcal{M}}_{g,P\cup Q}$ for some $\rho : Q \to \mathbb{Z}_{\geq -1}$. So fix $P := \{p_1, \dots, p_n\}$ with $n \geq 1$ and $Q := \{q_1, \dots, q_u, q_{u+1}, \dots, q_s\}$ such that $\tilde{Q} := \rho^{-1}(-1) = \{q_{u+1}, \dots, q_s\}$ and let $r = \sum_{i\geq -1} i\, m_i$. Clearly one must have $4g - 4 + 2|P| = \sum_{i\geq -1}(2i+1)m_i$. We always assume $m_{-1} = m^{\rho}_{-1}$.

Notation. We denote by $\mathfrak{P}_Q$ the set of partitions of Q and by M_0 the discrete partition $\{\{q_1\}, \dots, \{q_s\}\}$. We denote by $\mathfrak{P}_{Q,Q'}$ the subset of $\mathfrak{P}_{Q\cup Q'}$ consisting of $M = \{\mu_1, \dots, \mu_k\}$ such that the restriction $M \cap Q := \{\mu_1 \cap Q, \dots, \mu_k \cap Q\}$ is the discrete partition of Q.

Definition 5.1.1. Given P and ρ as before and M a partition of Q, consider the boundary map

$$\vartheta_{M,\rho,P} : \overline{\mathcal{M}}_{g,P\cup M} \times \prod_{\mu\in M} \overline{\mathcal{M}}_{0,\mu\cup\{\iota_\mu\}} \to \overline{\mathcal{M}}_{g,P\cup Q}$$

that glues every point $\mu \in M$ with ι_μ. Then we call combinatorial class *with rational tails* $\widehat{W}^{rt}_{M,\rho,P}$ the image through $\vartheta_{M,\rho,P}$ of $\widehat{W}_{m_*(M),\rho|_M,P} \times \{pt\}^M$, where $\rho|_M : M \to \mathbb{Z}_{-1}$ sends μ to $\rho_\mu := \sum_{q\in\mu} \rho(q)$ and

$$m_i(M) := |\{\mu \in M | \rho_\mu = i\}| + \delta_{i,0}(m_0 - m_0^\rho).$$

Theorem B. *Suppose $m_i = m_i^\rho$ for $i \neq 0$ and $Q \neq \emptyset$. Then, up to elements in the kernel of*

$$H_{6g-6+2n+2s}(\overline{\mathcal{M}}_{g,P\cup Q}) \to H_{6g-6+2n-2s}(\overline{\mathcal{M}}_{g,P\cup\tilde{Q}}, \partial\mathcal{M}_{g,P\cup\tilde{Q}}),$$

the following equation holds in $H^{2r+2s}(\overline{\mathcal{M}}_{g,P\cup Q})$:

$$\left(2^{\sum_{q\in Q}(\rho(q)+1)} \prod_{q\in Q} (2\rho(q)+1)!!\right) \prod_{q\in Q} \psi_q^{\rho(q)+1} =$$

$$= \widehat{W}_{m_*,\rho,P} + \sum_{M_0\neq M\in\mathfrak{P}_Q} c_M \widehat{W}^{rt}_{M,\rho,P}$$

where

$$c_M := \prod_{\mu\in M} c_\mu \quad \textit{and} \quad c_\mu = \frac{(2\rho_\mu + 2|\mu| - 1)!!}{(2\rho_\mu + 1)!!}$$

For the general case, choose Q' such that $|Q'| = \sum_{i\geq 1}(m_i - m_i^\rho)$ and a $\tilde{\rho} : Q \cup Q' \to \mathbb{Z}_{\geq -1}$ such that $|\tilde{\rho}^{-1}(j)| = m_j$ for all $j \neq 0$ and $\tilde{\rho}|_Q = \rho$. For every $\mu \subset Q \cup Q'$ define $\tilde{\rho}_\mu := \sum_{q\in\mu} \tilde{\rho}(q)$.

Corollary B.1. *Suppose $m_{-1} = m_{-1}^\rho$ and $P, Q' \neq \emptyset$. Then the following relation holds in $H^{2r+2s}(\overline{\mathcal{M}}_{g,P\cup Q})$ up to boundary terms:*

$$\left(2^{\sum_{q\in Q\cup Q'}(\tilde{\rho}(q)+1)} \prod_{q\in Q\cup Q'} (2\tilde{\rho}(q)+1)!!\right) \prod_{q\in Q} \psi_q^{\rho(q)+1} \sum_{\sigma\in\mathfrak{S}_{Q'}} \kappa_{r(\sigma)} =$$

$$= \sum_{M\in\mathfrak{P}_{Q,Q'}} \frac{m_*(M)!}{(m_*(M) - m_*^{\tau_M}(M))!} \tilde{c}_M \widehat{W}_{m_*(M),\tau_M,P}$$

where $\tilde{c}_\mu := \frac{(2\tilde{\rho}_\mu+2|\mu|-1)!!}{(2\tilde{\rho}_\mu+1)!!}$ and $\tau_M : Q \to \mathbb{Z}_{\geq -1}$ sends q to $\tilde{\rho}_{\mu_q}$ where $\mu_q \ni q$.

We remark that Theorem A and Corollary B.1 give an inductive method to express all $\widehat{W}_{m_*,\rho,P}$ in terms of the tautological classes and vice versa. In fact it is sufficient to isolate the term on the right hand side which corresponds to the discrete partition to obtain the recursion or to isolate the term on the left hand side that corresponds to $\sigma = e$.

Proof of Corollary B.1. It follows immediately from Theorem B applying Faber's formula and Lemma 3.3.1. □

Proof of Theorem B. Define $Q_i := \{q_1, \ldots, q_i\}$ and and let ψ_i denote ψ_{q_i}. Analogously to the previous section, let $C_{P,k}$ for $k = 1, \ldots, u$ be the subset of $l \in \mathbb{R}^{P\cup Q}_{\geq 0}$ defined by

$$\begin{cases} l_{q_j} = 0 & \text{for all } j = k+1, s \\ \sum_{i=j+1}^{k} l_{q_i} < l_{q_j} & \text{for all } j = 1, \ldots, k-1 \\ \sum_{i=1}^{k} l_{q_i} < l_{p_j} & \text{for all } j = 1, \ldots, n \end{cases}$$

and set $C^+_{P,k} := C_{P,k} \cap \{l_{q_k} > 0\}$. Notice that $C_{P,0} = \mathbb{R}^P_+ \times \{0\}^Q$.

Call $\mathcal{H}_0 : \overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,u}) \twoheadrightarrow \overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,0})$ the composition $\mathcal{H}^1_0\mathcal{H}^2_0 \cdots \mathcal{H}^u_0$ of all the retractions

$$\mathcal{H}^i_0 := \mathcal{H}^{q_i}_0 : \overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,i}) \to \overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,i-1})$$

and remark the important fact that $(\mathcal{H}^i_0)^*\overline{\omega}_{i-1}$ is not the $\overline{\omega}_{i-1}$ on $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,i})$ but $\overline{\omega}_i \wedge (\mathcal{H}^i_0)^*\overline{\omega}_{i-1}$ and $\overline{\omega}_i \wedge \overline{\omega}_{i-1}$ are cohomologous. So $\tilde{\xi}^*[\overline{\omega}_i^{\rho(q_i)+1} \smile (\mathcal{H}^i_0)^*\overline{\omega}_{i-1}^{\rho(q_{i-1})+1}]$ is exactly $\psi_i^{\rho(q_i)+1} \smile \psi_{i-1}^{\rho(q_{i-1})+1}$ if $\rho(q_i) \geq 0$.

Now pick a closed PL differential form η of degree $6g-6+2n-2r$ on $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,0})$ which is the pull-back of a form via a projection $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,0}) \twoheadrightarrow \overline{\mathcal{M}}^{comb}_{g,P\cup Q}(l, \{0\}^Q)$ with $l \in \mathbb{R}^P_+$. To produce a more useful representative for the class of

$$\overline{\omega}_u^{\rho(q_u)+1}\overline{\omega}_{u-1}^{\rho(q_{u-1})+1} \cdots \overline{\omega}_1^{\rho(q_1)+1}\mathcal{H}^*_0\eta$$

on $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C^+_{P,u})$ we proceed in the following inductive way. We start with $\beta_0(\eta) := \eta$ on $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,0})$. The first step is to pull it back via

$$\mathcal{H}^1_0 : \overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,1}) \longrightarrow \overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,0})$$

and to cup it with the $\overline{\omega}_1^{\rho(q_1)+1}$ living on $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C^+_{P,1})$. Then we obtain a well-defined form $\beta_1(\eta)$ on $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C^+_{P,1})$. Now suppose we have already produced $\beta_{k-1}(\eta)$ for $k<u$. Then we can pull $\beta_{k-1}(\eta)$ back via $\mathcal{H}_0^k$ and cup it with $\overline{\omega}_k^{\rho(q_k)+1}$ to obtain a well-defined β_k on the whole $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C^+_{P,k})$.

Finally we get a form $\beta(\eta) := \beta_u(\eta)$ on $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C^+_{P,u})$ with the property that the pull-back of its class to $\overline{\mathcal{M}}_{g,P\cup Q}\times C^+_{P,u}$ is

$$\psi_u^{\rho(q_u)+1}\psi_{u-1}^{\rho(q_{u-1})+1}\cdots\psi_1^{\rho(q_1)+1}\tilde{\xi}^*\mathcal{H}_0^*[\eta].$$

Call $\overline{Y}_{t_1,\dots,t_u}(C^+_{P,u})\subset\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C^+_{P,u})$ the closure of the locus of graphs such that

- the hole $x(q_u)$ has t_u distinct (unoriented) edges
- for all $i=1,\dots,u-1$ the hole $x(q_i)$ has t_i distinct (unoriented) edges beside those which border any of the holes $x(q_u),\dots,x(q_{i+1})$.

As in the previous section, it is easy to see that $\beta(\eta)$ has support contained inside the locus $\overline{Y}_{2\rho(q_1)+3,\dots,2\rho(q_u)+3}(C^+_{P,u})$. Now we want to analyze its image through $\mathcal{H}_0$ which consists of several components.

Definition 5.1.2. Given an ordinary ribbon graph, we say that a subset μ of markings form a *cluster* if

- any vertex of $x(\mu)$ contains an edge that belongs to a hole in $x(\mu)$
- any two distinct holes in $x(\mu)$ are joined by a chain of adjacent holes belonging to $x(\mu)$.

Two clusters μ and μ' are *disjoint* if $\mu\cup\mu'$ is not a cluster (in particular μ and μ' are disjoint as sets).

We associate to every partition $M=\{\mu_1,\dots,\mu_k\}$ in $\mathfrak{P}_Q$ the closure $\overline{Y}_M(C^+_{P,u})$ of the locus of top-dimensional simplices of $\overline{Y}_{2\rho(q_1)+3,\dots,2\rho(q_u)+3}(C^+_{P,u})$ such that $\mu_1,\dots,\mu_k$ form disjoint clusters. It is obvious that $\{\overline{Y}_M(C^+_{P,u})\}$ is a dissection of $\overline{Y}_{2\rho(q_1)+3,\dots,2\rho(q_u)+3}(C^+_{P,u})$. Really they overlap on simplices of nonmaximal dimension, but it is not important for what follows. Strictly speaking, we would need a refinement of this dissection: for every tripartition $M^\bullet = \{M^{disk}, M^{cyl}, M^{surf}\}$ of M we denote by $\overline{Y}_{M^\bullet}(C^+_{P,u})$ the closure of the locus in $\overline{Y}_M(C^+_{P,u})$ where every cluster in M^{disk} (resp. in M^{cyl} or in M^{surf}) form a disk (resp. a cylinder or a surface with negative Euler characteristic).

Then $\mathcal{H}_0(\overline{Y}_{M^\bullet}(C^+_{P,u}))$ is the union of the simplices in $\overline{\mathcal{M}}^{comb}_{g,P\cup Q}(C_{P,0})$ indexed by ribbon graphs G such that:

1. every $\mu \in M^{disk}$ marks a vertex lying in the smooth locus of G of valency $2\rho_\mu+3$ (if $|\mu| > 1$ we should say: μ marks a nonpositive sphere that intersects only one positive component in a vertex of valency $2\rho_\mu + 3$), while all the other vertices in the smooth locus are trivalent
2. every $\mu \in M^{cyl}$ marks a node which is obtained identifying two vertices of valencies v_1 and v_2 with $v_1 + v_2 = 2\rho_\mu$ (i.e. μ marks a nonpositive sphere that intersects only two positive components in vertices of valencies v_1 and v_2)
3. every $\mu \in M^{surf}$ marks a nonpositive component of genus h and with ν nodes of valencies $v_1, \dots, v_\nu$ such that $6h - 6 + \sum_{j=1}^{\nu}(v_j + 3) = 2\rho_\mu$.

As in the simplest case, the length

$$\bar{\lambda}_{p_i}(\mathcal{H}_0(\overline{Y}_M(l_1, \dots, l_n, \varepsilon_1, \dots, \varepsilon_u, \{0\}^{\tilde{Q}})))$$

may vary between $l_i - \varepsilon$ and l_i, where $\varepsilon = \sum_{j=1}^{u} \varepsilon_j$. Reasoning in the same way as in the previous section, we obtain

$$\int_{[L'',L']^n} dl_{p_1} \wedge \cdots \wedge dl_{p_n} \int_{\overline{\mathcal{M}}_{g,P\cup Q}} \psi_1^{\rho(q_1)+1} \smile \cdots \smile \psi_u^{\rho(q_s)+1} \smile \tilde{\xi}^* \mathcal{H}_0^*[\eta] =$$
$$= \lim_{\varepsilon\to 0} \sum_{M^\bullet} \int_{\overline{Y}_{M^\bullet}([L'',L']^n, \varepsilon_1, \dots, \varepsilon_u, \{0\}^{\tilde{Q}})} dl_{p_1} \wedge \dots dl_{p_n} \wedge \beta(\eta) =$$
$$= \int_{[L'',L']^n} dl_{p_1} \wedge \cdots \wedge dl_{p_n} \sum_{M^\bullet} \int_{\mathcal{H}_0(\overline{Y}_{M^\bullet})(l)} \eta \int_{F_{M^\bullet}(\varepsilon_1,\dots,\varepsilon_u)} \beta(1)$$

where l belongs to $C_{P,0}$ and $F_{M^\bullet}(\varepsilon_1, \dots, \varepsilon_u)$ is the intersection of the generic fiber of $\mathcal{H}_0$ over $\mathcal{H}_0(\overline{Y}_{M^\bullet})([L'', L' - \varepsilon]^n, \{0\}^Q)$ with $\overline{Y}_{M^\bullet}((\varepsilon, +\infty)^n, \varepsilon_1, \dots, \varepsilon_u, \{0\}^{\tilde{Q}})$.

As we are now interested only in cycles that do not lie in the boundary after forgetting $Q\backslash\tilde{Q}$, we may restrict to the case in which $M^{cyl} = M^{surf} = \emptyset$ and $M^{disk} \cap \tilde{Q}$ is the discrete partition because of the wise choice of keeping the perimeters inside $C^+_{P,u}$. Hence $\mathcal{H}_0(\overline{Y}_{M,\emptyset,\emptyset}(C^+_{P,u}))$ is exactly the support of $\tilde{\xi}(\widehat{W}^{rt}_{M,\rho,P}(C^+_{P,0}))$.

Then we only need to compute

$$\frac{1}{2^{\sum_{i=1}^{u}(\rho(q_i)+1)}\prod_{i=1}^{u}(2\rho(q_i)+1)!!}\int_{F_{M,\emptyset,\emptyset}(\varepsilon_1,\dots,\varepsilon_u)}\beta(1)$$

which is in fact the number of isomorphism class of $(P \cup Q)$-marked ribbon graphs in $F_{M,\emptyset,\emptyset}(\varepsilon_1,\dots,\varepsilon_u)$ that parametrize simplices of top dimension.

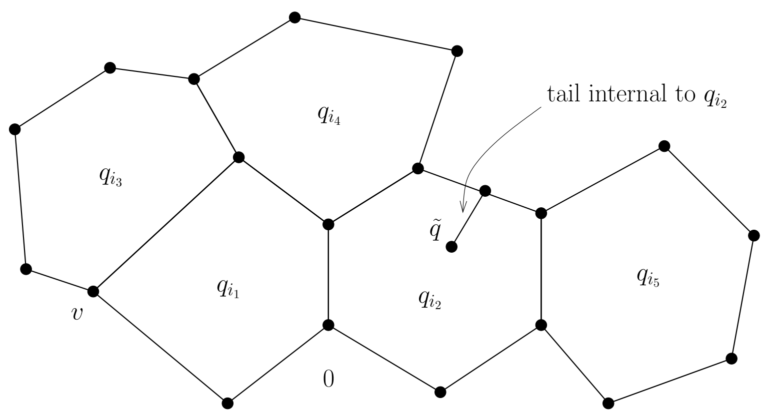


Figure 5.1: An example of admissible cluster

To complete the proof, we need to determine the number c_μ of all possible isomorphism types of *admissible* cluster associated to μ. To be precise, c_μ is exactly the number of isomorphism classes of ribbon graphs G such that:

- G is a connected ordinary ribbon graph marked by the set $\mu \cup \{0, v\}$
- $S(G)$ is a sphere and μ forms a cluster
- the vertices of G have valency at most three; the bivalent ones always border the hole 0 and one of them is marked by v

- if $\mu \cap \tilde{Q} = \{\tilde{q}\}$ is nonempty, then $\tilde{q}$ marks the only univalent vertex; otherwise there are no univalent vertices

- if $\mu = \{q_{i_1} < \cdots < q_{i_h}\}$ or $\{q_{i_1} < \cdots < q_{i_h}, \tilde{q}\}$ then the hole i_h has $2\rho(q_{i_h}) + 3$ sides and for all $j = 1, \ldots, h-1$ the hole i_j has $2\rho(q_{i_j}) + 3$ sides beside those which border the holes $i_h, \ldots, i_{j+1}$.

So we are left to prove the following lemma. □

Lemma 5.1.3. *Let* $\rho(\mu) = \sum_{q \in \mu} \rho(q)$. *Then*

$$c_\mu = \frac{(2\rho_\mu + 2|\mu| - 1)!!}{(2\rho_\mu + 1)!!}.$$

where we have conventionally set $(-1)!! = 1$.

Proof. Remark that the calculation has a clear geometrical meaning even if we allow some $\rho(q_{i_j})$ to assume the value -1. However in what follows we will bound ourselves to the case $\rho(q_{i_1}) \geq -1$ and $\rho(q_{i_j}) \geq 0$ for all $j > 1$.

We proceed by induction on $|\mu|$.

Clearly, if $|\mu| = h = 1$ then $c_\mu = 1$. If $h > 1$ then the cluster has no symmetries and so the possible v-markings are exactly $2\rho_\mu + 3$. In particular if $h = 2$ and $\mu \cap \tilde{Q}$ is empty, then the cluster consists of the holes q_{i_1} with $2\rho(q_{i_1}) + 4$ sides and q_{i_2} with $2\rho(q_{i_2}) + 3$ sides that have exactly one edge in common. If $h = 2$ and $\mu \cap \tilde{Q} = \{\tilde{q}\}$, then the cluster is made of a hole q_{i_1} with $2\rho(q_{i_1}) + 5$ distinct edges and an internal tail marked by $\tilde{q}$. In both cases $c_\mu = 2\rho_\mu + 3$.

Now we deal with the case $h > 2$. Remember that $\rho(q_{i_j}) \geq 0$ for $j = 2, 3, \ldots, h$.

If $\rho(q_{i_1}) = -1$ and so the hole q_{i_1} does not contain an internal tail, then we look at the situation just before shrinking q_{i_1}. We have a loop surrounding q_{i_1} and its vertex has valency $2\rho_\mu + 5$. So this vertex is obtained collapsing the subcluster $\mu' = \mu \setminus \{q_{i_1}\}$. By induction hypothesis, c_μ is $(2\rho_\mu + 3)\rho_{\mu'} = (2\rho_\mu + 3)(2(\rho_\mu + 1) + 2(|\mu| - 1) - 1) \cdots (2(\rho_\mu + 1) + 3) = (2\rho_\mu + 3)(2\rho_\mu + 2|\mu| - 1) \cdots (2\rho_\mu + 5)$.

If $\rho(q_{i_1}) \geq 0$, then we look at the situation before collapsing q_{i_2} and q_{i_1}. There are two possibilities: the holes may touch each other in one edge (case a) or in one vertex (case b). Moreover $\tilde{q}$ may appear as internal tail in q_{i_1} or q_{i_2}. We want to show that in both cases the number of configurations (which we denote respectively by c_μ^a and c_μ^b) depends only on $\rho(q_{i_1})+\rho(q_{i_2})$ and not on $\rho(q_{i_1})$ and $\rho(q_{i_2})$ separately. Hence c_μ depends only on $\rho(q_{i_1})+\rho_(q_{i_2})$ too as $c_\mu = c_\mu^a + c_\mu^b$. Hence we can apply the previous computation. Case (a)

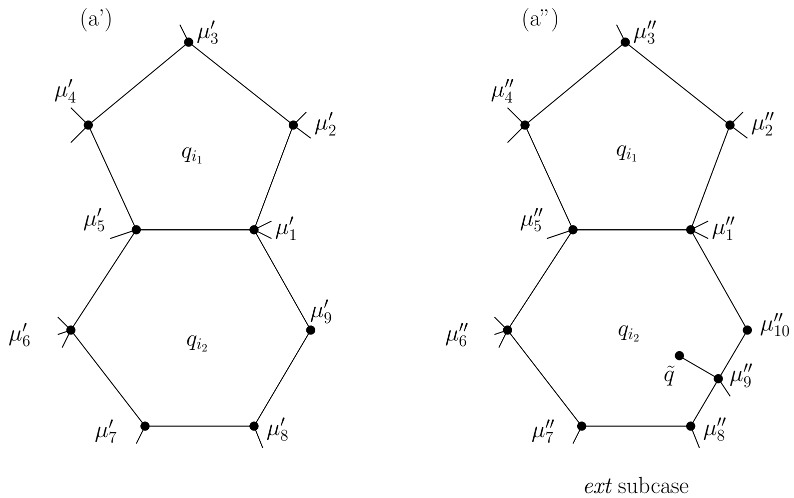


Figure 5.2: Examples of cases (a') and (a")

immediately split into two subcases, so that $c_\mu^a = c_\mu^{a'} + c_\mu^{a''}$.

(a') The two holes touch in an edge and there is not an internal tail inside q_{i_1} or q_{i_2}. So the holes $q_{i_3},\ldots,q_{i_h}$ (and possibly $\tilde{q}$) are distributed in $t = (2\rho(q_{i_1})+3)+(2\rho(q_{i_2})+3)-1 = 2(\rho(q_{i_1})+\rho(q_{i_2}))+5$ subclusters $\mu_1',\ldots,\mu_t'$. Then we obtain

$$c_\mu^{a'} = \sum_{j\in J}\prod_{k=1}^{t} c_{j^{-1}(k)}$$

where $J = \{j : \mu \setminus \{q_{i_1}, q_{i_2}\} \to \{1,\ldots,t\} | \rho_{j^{-1}(k)} \geq 0 \quad \forall k\}$.

(a") The two holes touch in an edge and there is an internal tail in q_{i_1} or in q_{i_2}. Then the holes $q_{i_3},\ldots,q_{i_h}$ are distributed in $t_{int} = 2(\rho(q_{i_1})+$

$\rho(q_{i_2})) + 3$ clusters if the tail hangs on the separating edge (*int*) and in $t_{ext} = t_{vert} = 2(\rho(q_{i_1}) + \rho(q_{i_2})) + 4$ clusters if it hangs on an extremal point of the separating edge (*vert*) or the external perimeter (*ext*). Hence

$$c_\mu^{a''} = 2 \sum_{j \in J_{int}} \prod_{k=1}^{t_{int}} c_{j^{-1}(k)} + 4 \sum_{j \in J_{vert}} \prod_{k=1}^{t_{vert}} c_{j^{-1}(k)} + \sum_{j \in J_{ext}} \prod_{k=1}^{t_{ext}} c_{j^{-1}(k)}$$

where

$$J_{sep} = \{j : \mu \setminus \{q_{i_1}, q_{i_2}, \tilde{q}\} \to \{1, \ldots, t_{int}\} | \rho_{j^{-1}(k)} \geq 0 \quad \forall k\}$$
$$J_{vert} = \{j : \mu \setminus \{q_{i_1}, q_{i_2}, \tilde{q}\} \to \{1, \ldots, t_{vert}\} | \rho_{j^{-1}(k)} \geq \delta_{1,k} \quad \forall k\}$$
$$J_{ext} = \{j : \mu \setminus \{q_{i_1}, q_{i_2}, \tilde{q}\} \to \{1, \ldots, t_{ext}\} | \rho_{j^{-1}(k)} \geq 0 \quad \forall k\}$$

so we are done again.

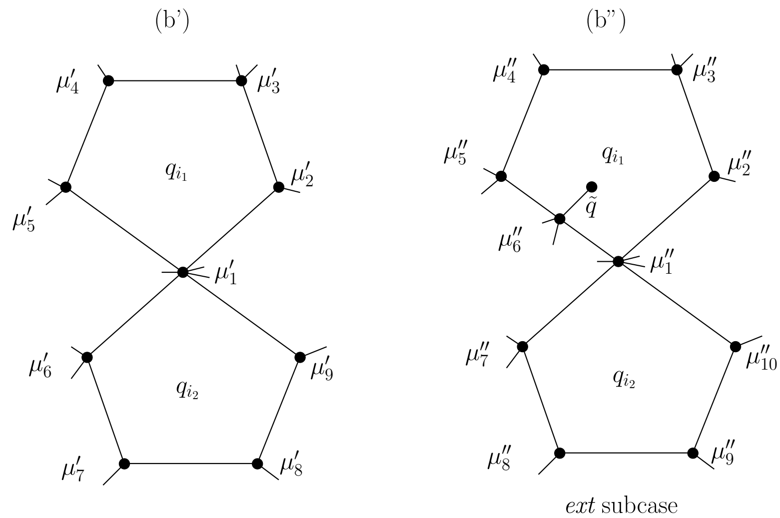


Figure 5.3: Examples of cases (b') and (b")

Also case (b) splits into two subcases and $c_\mu^b = c_\mu^{b'} + c_\mu^{b''}$.

(b') The two holes touch in a vertex and there is no internal tail in q_{i_1} or q_{i_2}. So we have $t = 2(\rho(q_{i_1}) + \rho(q_{i_2})) + 5$ subclusters but μ'_1 corresponding to the common vertex must be at least 4-valent. Moreover the two holes can touch the cluster μ_1' in $2\rho_{\mu_1'}(2\rho_{\mu_1'}+3)$ ways (or in $2\rho_{\mu_1'}$ ways if $|\mu_1'|=1$), hence we obtain

$$c_\mu^{b'} = (2\rho_\mu + 3)\sum_{j\in J} 2\rho_{\mu_1'} \prod_{k=1}^{t} c_{j^{-1}(k)}$$

where $J = \{j : \mu \setminus \{q_{i_1}, q_{i_2}\} \to \{1,\dots,t\} | \rho_{j^{-1}(k)} \geq \delta_{1,k} \quad \forall k\}$.

(b") The two holes touch in a vertex and there is an internal tail in q_{i_1} or q_{i_2}. Then the tail may hang on a vertex (*vert*) or on the external perimeter (*ext*). Anyway the holes $q_{i_3},\dots,q_{i_h}$ are distributed in $t = 2(\rho(q_{i_1}) + \rho(q_{i_2})) + 4$ clusters. Hence

$$c_\mu^{b''} = 2\sum_{j\in J}\prod_{k=1}^{t} c_{j^{-1}(k)} + (2\rho(q_{i_1}) + 2\rho(q_{i_2}) + 3)\sum_{j\in J}\prod_{k=1}^{t} c_{j^{-1}(k)}$$

where

$$J = \{j : \mu \setminus \{q_{i_1}, q_{i_2}, \tilde{q}\} \to \{1,\dots,t\} | \rho_{j^{-1}(k)} \geq \delta_{1,k} \quad \forall k\}$$

and finally we are done.

Remark that Figure 5.3 illustrates the exact situation, i.e. the two holes q_{i_1} and q_{i_2} cannot be nested. In fact the "internal" hole should be contracted before the other, and so in our case it should be q_{i_2}. But an "internal" hole should also have one side only, while we are assuming that $\rho(q_{i_2}) \geq 0$. This explains why we have assumed that only $\rho(q_{i_1})$ can take value -1, which, on the other hand, we need to perform the above explicit computation. $\square$

As an example we compute the class $W_{2a+1,2b+1}$ of graphs with two nontrivalent vertices.

Corollary B.2. *For every nonnegative g and positive n such that $2g-2+n>0$ and for every $a,b \geq 1$ the following identity holds*

$$
\begin{aligned}
&2^{\delta_{a,b}}\widehat{W}_{2a+1,2b+1} = \\
&\quad = 2^{a+b+2}(2a+1)!!(2b+1)!!(\kappa_a\kappa_b + \kappa_{a+b}) - (2a+2b+3)\widehat{W}_{2a+2b+1} = \\
&\quad = 2^{a+b+2}(2a+1)!!(2b+1)!!(\kappa_a\kappa_b + \kappa_{a+b}) - 2^{a+b+1}(2a+2b+3)!!\kappa_{a+b}
\end{aligned}
$$

in $H^{2a+2b}(\overline{\mathcal{M}}_{g,n})$ *up to boundary terms.*

List of Notation

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Bibliography

[AC96] Enrico Arbarello and Maurizio Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*, J. Algebraic Geom. **5** (1996), no. 4, 705–749.

[AC98] Enrico Arbarello and Maurizio Cornalba, *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 88, 97–127 (1999).

[Bin02] G. Bini, *A combinatorial algorithm related to the geometry of the moduli space of pointed curves*, J. Algebraic Combin. **15** (2002), no. 3, 211–221.

[DFIZ93] P. Di Francesco, C. Itzykson, and J.-B. Zuber, *Polynomial averages in the Kontsevich model*, Comm. Math. Phys. **151** (1993), no. 1, 193–219.

[Fab99] Carel Faber, *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and Abelian varieties, Aspects Math., E33, Vieweg, Braunschweig, 1999, pp. 109–129.

[Har86] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), no. 1, 157–176.

[Har88] John L. Harer, *The cohomology of the moduli space of curves*, Theory of moduli (Montecatini Terme, 1985), Springer, Berlin, 1988, pp. 138–221.

[HM79] John Hubbard and Howard Masur, *Quadratic differentials and foliations*, Acta Math. **142** (1979), no. 3-4, 221–274.

[HZ86] John Harer and Don Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), no. 3, 457–485.

[Igu02] Kiyoshi Igusa, *Combinatorial Miller-Morita-Mumford classes and Witten cycles*, 2002, e-print: `math.GT/0207042`.

[Igu03] Kiyoshi Igusa, *Graph cohomology and Kontsevich cycles*, 2003, e-print: `math.AT/0303157`.

[IK03] Kiyoshi Igusa and Michael Kleber, *Increasing trees and Kontsevich cycles*, 2003, e-print: `math.AT/0303353`.

[Kee99] Seán Keel, *Basepoint freeness for nef and big line bundles in positive characteristic*, Ann. of Math. (2) **149** (1999), no. 1, 253–286.

[KMZ96] R. Kaufmann, Yu. Manin, and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves*, Comm. Math. Phys. **181** (1996), no. 3, 763–787.

[Kon92] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[Loo95] Eduard Looijenga, *Cellular decompositions of compactified moduli spaces of pointed curves*, The moduli space of curves (Texel Island, 1994), Birkhäuser Boston, Boston, MA, 1995, pp. 369–400.

[Mum83] David Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328.

[Pen92] R. C. Penner, *Weil-Petersson volumes*, J. Differential Geom. **35** (1992), no. 3, 559–608.

[Pen93] Robert C. Penner, *The Poincaré dual of the Weil-Petersson Kähler two-form*, Comm. Anal. Geom. **1** (1993), no. 1, 43–69.

[Str67] Kurt Strebel, *On quadratic differentials with closed trajectories and second order poles*, J. Analyse Math. **19** (1967), 373–382.

[Str84] Kurt Strebel, *Quadratic differentials*, Springer-Verlag, Berlin, 1984.

[Wit91] Edward Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310.

[Wol83] Scott Wolpert, *On the homology of the moduli space of stable curves*, Ann. of Math. (2) **118** (1983), no. 3, 491–523.