A cyclic extension of the earthquake flow
(joint work with Francesco Bonsante and Jean-Marc Schlenker)

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geodesic flow — horocyclic flow — rotation flow

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Use **conformal/flat structure** - No hyperbolic geometry involved

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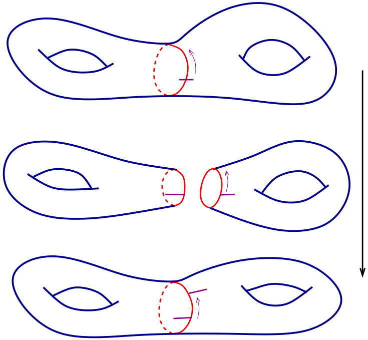

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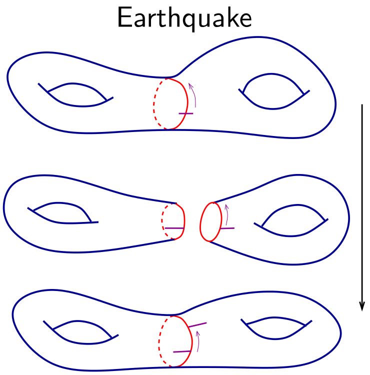

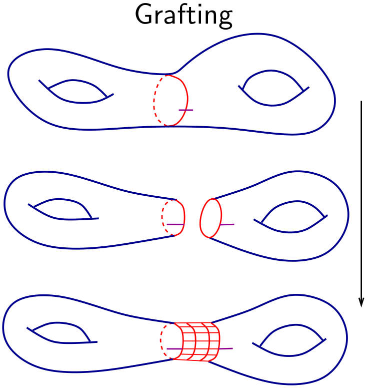

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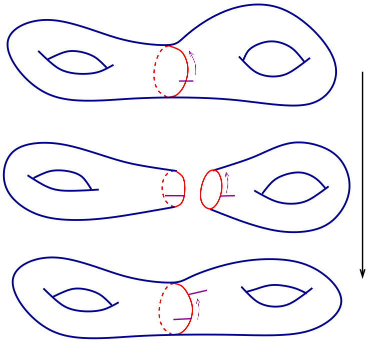

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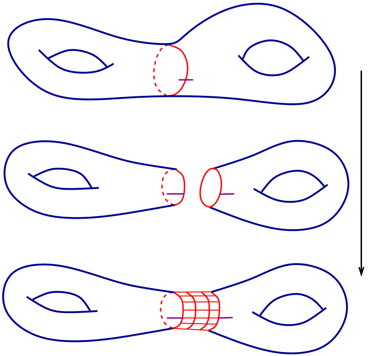

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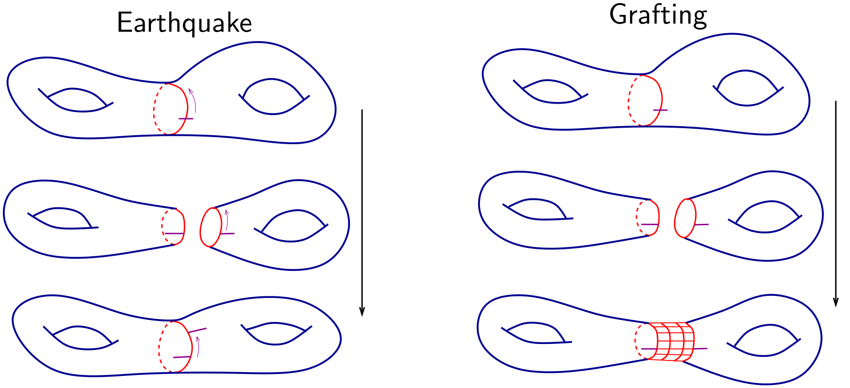

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The earthquake $E_t : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{T}(S) \times \mathcal{ML}(S)$ defined as $E_t(X, \lambda) := (E_{t\lambda}(X), \lambda)$ is an $\mathbb{R}$-flow.

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Grafting map $gr_t : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{T}(S) \times \mathcal{ML}(S)$ is **<u>not</u>** a flow!

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$G_\mu \subset \mathrm{PSL}_2(\mathbb{C})$ is **quasi-Fuchsian**, i.e. acts prop. discont. on $\mathbb{CP}^1 \setminus \Lambda$

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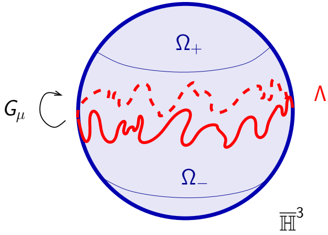
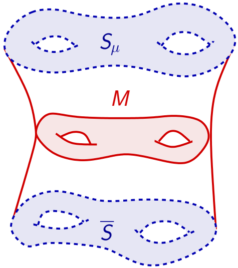
with boundary components at infinity $(\mathbb{CP}^1_\infty \setminus \Lambda)/G_\mu = (S, J_\mu) \sqcup (S, -J)$





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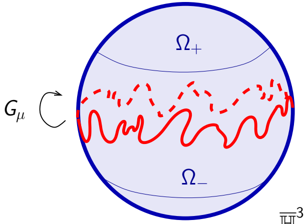

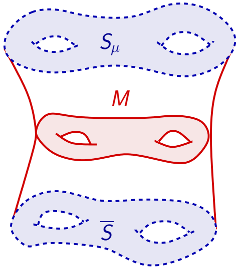

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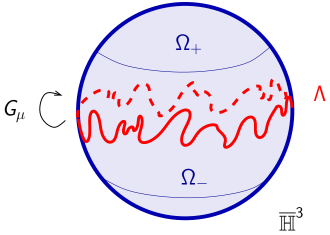

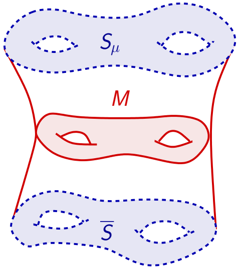

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Theorem (Bers, 1960)

$\mathcal{QF} = \{$*quasi-Fuchsian manifolds* $M\} \leftrightarrow \mathcal{T} \times \overline{\mathcal{T}}$ *is biholomorphic.*

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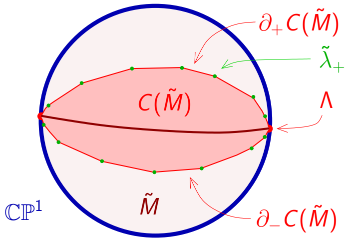

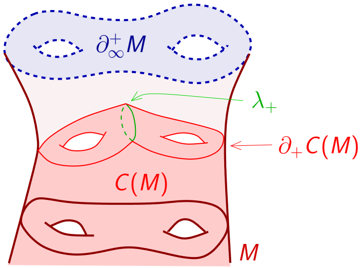

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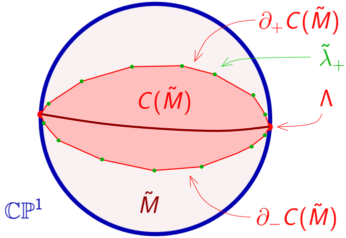
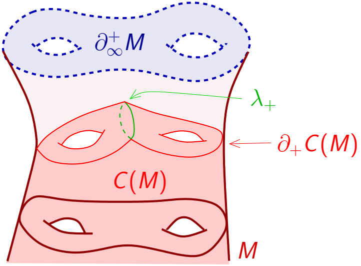
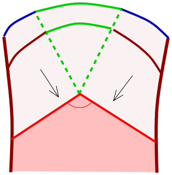


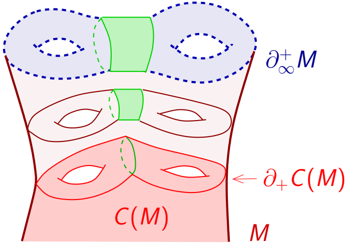


$\partial_+ C(M)$ is a **bent surface**, i.e. $I = h$ hyperbolic metric and $III = \lambda_+$

Quasi-Fuchsian manifolds

Closest point projection $\partial^+_\infty(M) \to \partial_+ C(M)$





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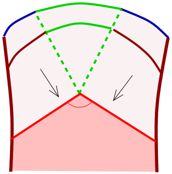

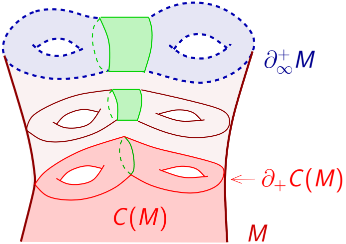

$\partial^+_\infty(M) \cong Gr_{\lambda_+}(\partial_+ C(M))$

Quasi-Fuchsian manifolds

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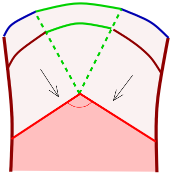

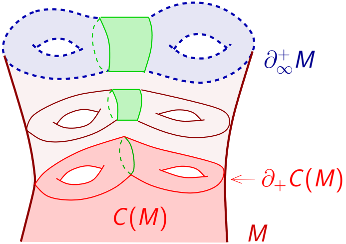

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Theorem (Thurston)

$Gr : \mathcal{T} \times \mathcal{ML} \to \mathcal{P} = \{\mathbb{CP}^1$-structures on S up to isotopy$\}$ is a homeomorphism.

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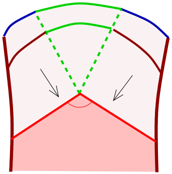

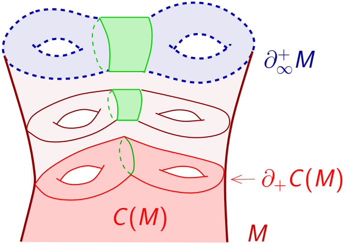

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Recipe for grafting: hyperbolic surface (S, h) and measured lamination λ

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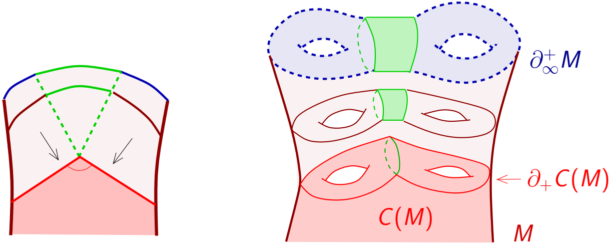

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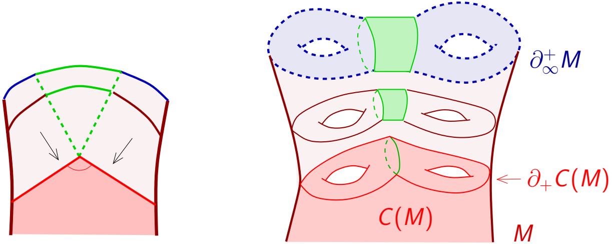

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λ small $\rightsquigarrow$ get M QF manifold, otherwise M hyperbolic end

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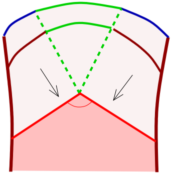

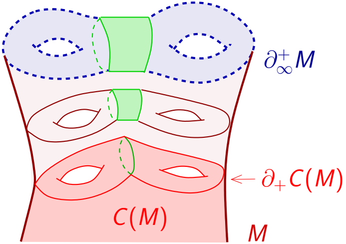

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From the $\sigma(S) \subset M$, look at the concave direction: see $Gr_\lambda(h)$

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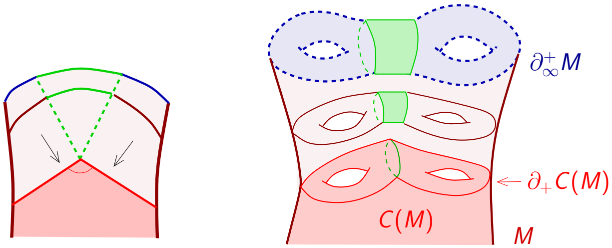

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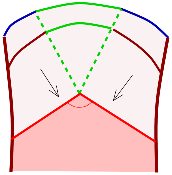

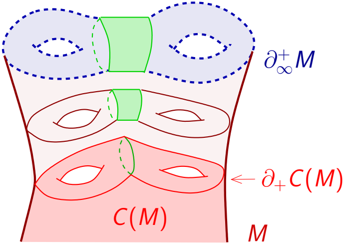

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$B = -\nabla\nu$ shape operator, normal flow away from $\sigma(S)$ $\rightsquigarrow$
$h_t = h(\cosh(t) + \sinh(t)B, \cosh(t) + \sinh(t)B)$

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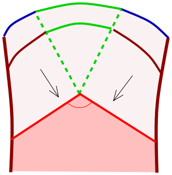

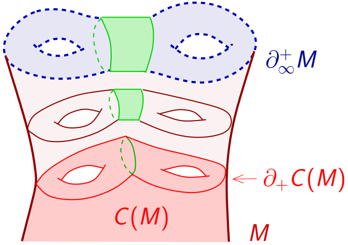

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$Gr_\lambda(h)$ is conformally equivalent to $[h_\infty] = [h(id + B, id + B)]$

Anti de Sitter 3-space

Similar description for the earthquake?

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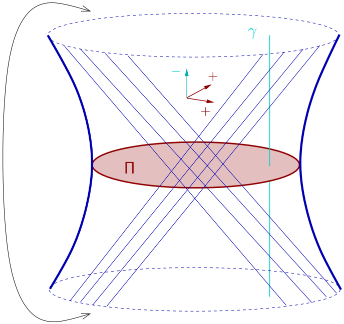

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$\mathbb{AdS}^3 = \{x \in \mathbb{R}^{2,2} \,|\, \langle x, x\rangle = -1\}$

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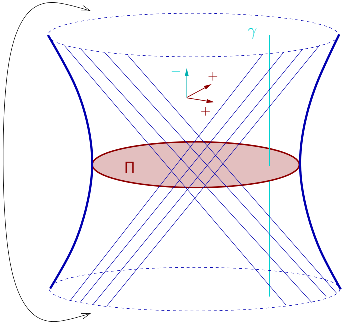

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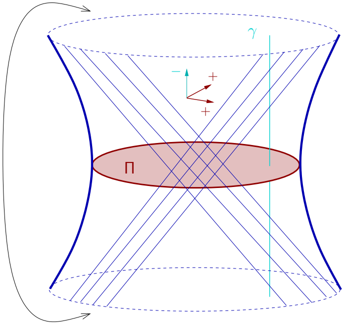

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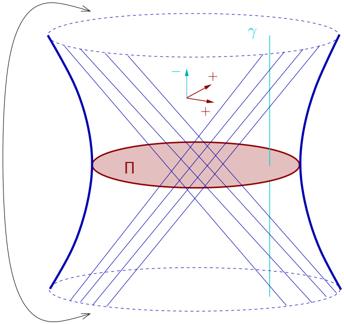

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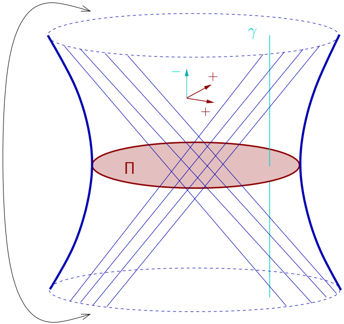

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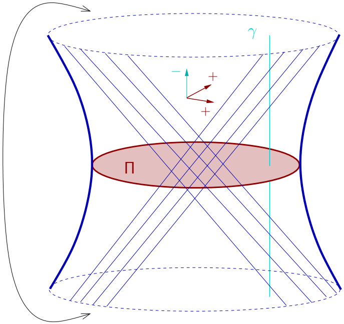

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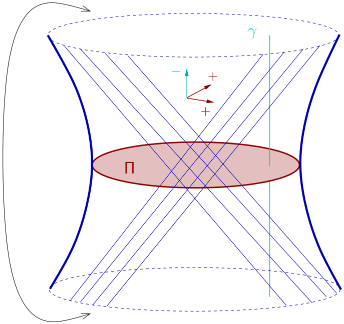

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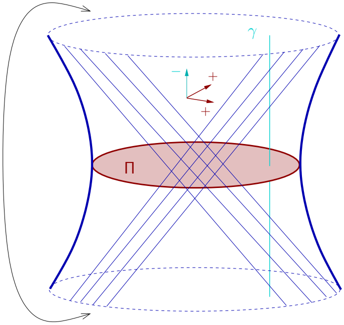

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- Time-like geodesics γ are closed of length 2π

MGH AdS 3-manifolds

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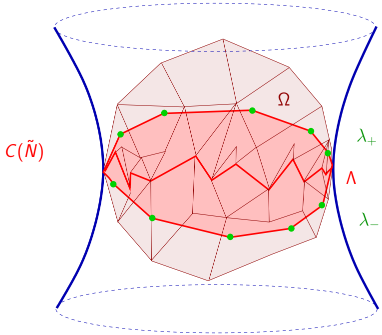
Theorem (Mess, 1990)

$\mathcal{MGH}(S) \longrightarrow \mathcal{T}(S) \times \mathcal{T}(S)$ *is a diffeomorphism.*

MGH AdS 3-manifolds



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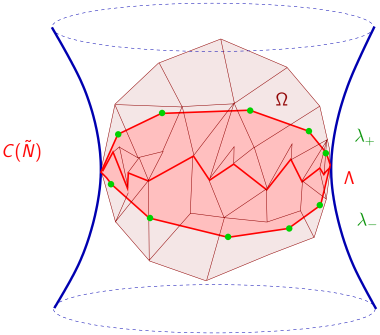

Λ limit set for $\rho(\pi_1(S))$ acting on Ω
Hölder Jordan curve on $\partial\Omega$

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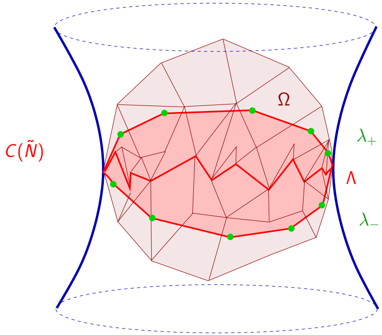

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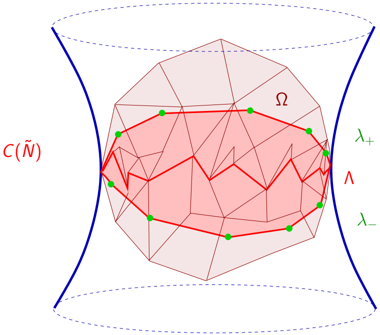

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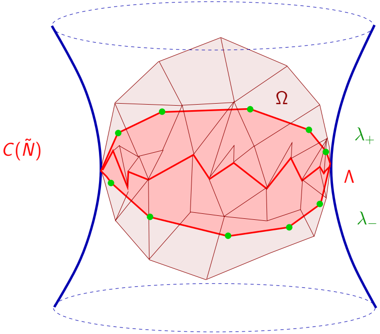

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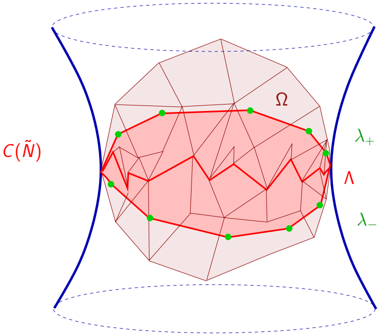

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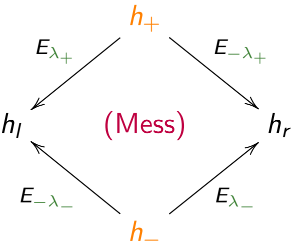

Landslide

Earthquake is a flow $E_t : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \times \mathcal{ML}$

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Theorem L (Bonsante-M-Schlenker, 2011)

There exists a real-analytic S^1-**landslide flow** $L_{e^{i\theta}} : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ s.t.

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- $L_{-1}(h, h^*) = (h^*, h)$

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Idea: embed S in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ using $I \propto h$ and $III \propto h^*$ with $h, h^* \in \mathcal{T}(S)$

Theorem L (Bonsante-M-Schlenker, 2011)

There exists a real-analytic S^1-**landslide flow** $L_{e^{i\theta}} : \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ s.t.

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We need to "fix the gauge" between h and h^*

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Definition

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Remark

If h fixed and $h^*_n \to [\lambda]$, *then* $\lim_n c_n$ *does not depend only on h and* $[\lambda]$

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(2) As a map $L_\bullet(h, h^*) : S^1 \to \mathcal{T}$, it extends over $z = 0$ as $L_0(h, h^*) := c$.

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  Given h^* and $s \in (0,1)$, is $Sgr_s(\cdot, h^*) : \mathcal{T} \to \mathcal{T}$ a diffeomorphism?
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- Description of other $\mathrm{SL}_2(\mathbb{R})$-flows in terms of surfaces in $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifolds?