Degenerations of Algebraic Surfaces and applications to Moduli problems.

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Introduction.

An important question concerning algebraic geometry and differential topology is the so called def=diff? problem:
Are two complex structures on a closed compact differentiable $2n$-manifold deformation of each other?
In case $n = 1$ it is a classical result (cf. [E-C] III.33) that the answer is yes, while in case $n = 2$ the above question (Friedman-Morgan conjecture) has a positive answer in some cases, but it is in general still unsolved. The reader can see the survey article [Do] for a discussion of recent results about this problem and [Li-W] for the higher dimensional case.
If we restrict to minimal algebraic surfaces of general type the above question can be interpreted in terms of properties of the moduli space of surfaces of general type. In fact for a given oriented smooth four-manifold X the (possibly empty) set $\mathcal{M}^{diff}(X)$ of minimal surfaces of general type orientedly diffeomorphic to X can be endowed with the structure of a quasiprojective variety in such a way that two surfaces $S_1, S_2 \in \mathcal{M}^{diff}(X)$ can be deformed the one in the other if and only if they belong to the same connected component of $\mathcal{M}^{diff}(X)$.
The main goal of this thesis is to study the general connectedness properties of moduli spaces of surfaces of general type and to give some general recipes to construct examples of pairs of "very similar" complex algebraic surfaces with the same underlying topological 4-manifold such that their complex structures cannot be continuously deformed the one in the other. It is important to say at this point that our methods belong to the realm of algebraic geometry, no computation of differentiable invariants is made and it is not clear to us if, in some cases, our examples may have the same differential structure.
Let S be a minimal surface of general type and let $\mathcal{M}^{top}(S)$ be the moduli space of surfaces of general type homeomorphic (by an orientation preserving homeomorphism) to S.
Let $k_S \in H^2(S, \mathbb{Z})$ be the first Chern class of the canonical bundle of S and let $r(S)$ its divisibility, i.e.

$$r(S) = \max\{r \in \mathbb{N} \,|\, k_S = rc \text{ for some } c \in H^2(S, \mathbb{Z})\}$$

If $S' \in \mathcal{M}^{top}(S)$ is in the same connected component of S then there exists an orientation preserving diffeomorphism $f\colon S' \to S$ such that $f^*(k_S) = k_{S'}$ and $r(S) = r(S')$.

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Catanese ([Ca4]) was the first to prove that "in general" $\mathcal{M}^{top}(S)$ is not connected, giving examples of homeomorphic simply connected surfaces with different divisibility r. His examples include the so called simple bihyperelliptic surfaces.

Denote by $\mathcal{O}(a,b)$ the line bundle on $\mathbb{P}^1\times\mathbb{P}^1$ whose sections are bihomogeneous polynomials of bidegree a,b. A minimal surface of general type is said to be simple bihyperelliptic of type $(a,b)(n,m)$ if its canonical model is defined in $\mathcal{O}(a,b)\oplus\mathcal{O}(n,m)$ by the equations

$$z^2 = f(x,y) \qquad w^2 = g(x,y) \tag{$*$}$$

where f,g are bihomogeneous polynomials of respective bidegree $(2a,2b)$, $(2n,2m)$.

If $a,b,c,d\geq 3$ then simple bihyperelliptic surfaces of type $(a,b),(c,d)$ are simply connected ([Ca1]).

Catanese also considered the subset $\hat{N}_{(a,b)(n,m)}$ of the moduli space of surfaces of general type $\mathcal{M}$ whose members are simple bihyperelliptic surfaces of type $(a,b)(n,m)$ and proved ([Ca3]) that if $a\geq \max(2n+1,b+2)$, $m\geq\max(2b+1,n+2)$ then $\hat{N}_{(a,b),(n,m)}$ is an irreducible component of the moduli space. In chapter II of this thesis we make the necessary computations in order to prove that under the above conditions about a,b,n,m the set $\hat{N}_{(a,b),(n,m)}$ is open in the moduli space and then it is a connected components.

This result enables us to prove (chapter V) the following

Theorem 1. *For every $k>0$ there exist simple bihyperelliptic surfaces $S_1,...,S_k$ orientedly homeomorphic to each other, such that $r(S_i)=r(S_j)$ and such that they belong to k distinct connected components of the moduli space.*

After Donaldson's work about polynomial invariants of smooth four manifolds it was clear that for a large class of simply connected minimal surfaces of general type the divisibility r is a differential invariant ([F-M-M]) and using this fact Friedman, Morgan and Moishezon were able to construct the first examples of homeomorphic but nondiffeomorphic surfaces of general type. Later Salvetti ([Sal]) proved that the number of surfaces of general type with the same underlying oriented topological 4-manifold but with nonequivalent underlying differential structures can be arbitrarily large.

Very recently, using a new differential invariant, Witten [Wi] proved in particular that r is a differential invariant for every simply connected minimal surface of general type.

Moreover, if Witten's speculations, based on supersymmetric quantum field theory, are correct, then homeomorphic surfaces with the same divisibility r have the same Donaldson's polynomials and therefore to decide whether they have the same differential structure will probably be one of the most challenging problems in four-dimensional differential topology.

If X is the surface defined in $(*)$, every deformation of X defined by the equations

$$z^2 = f'(x,y) + w\phi(x,y) \qquad w^2 = g'(x,y) + z\psi(x,y) \tag{$**$}$$

where f', g', ϕ, ψ are bihomogeneous polynomials or respective bidegree $(2a, 2b), (2n, 2m), (2a-n, 2b-m), (2n-a, 2m-b)$, is called a *natural deformation* of X. Assume now $a > 2n, m > 2b$, in this case every natural deformation is obtained by deforming the polynomials f, g of the equations $(*)$ in their linear systems since the polynomials ϕ, ψ of the above equations $(**)$ must be equal to 0. Therefore the canonical models of simple bihyperelliptic surfaces are stable under small natural deformations and then the openess of $\hat{N}_{(a,b),(n,m)}$, $a > 2n, m > 2b$, is equivalent to the surjectivity of the Kodaira-Spencer map of the family of natural deformations of X for every X as in $(*)$ with at most rational double points as singularities.

More generally it is possible to extend the notion of natural deformations to every smooth abelian covering of algebraic varieties ([Ca1],[Ca2],[Par],[F-P]) and this notion finds useful application in the explicit determination of complete families of deformations.

The generalization of the notion of natural deformations to normal abelian coverings presents in general some difficulty, for example in the case, considered in chapter II, of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ Galois covers $X \to Y$ with Y smooth and X normal, in order to prove some interesting results we need the assumption that every irreducible component of the ramification locus $R \subset X$ is a locally principal divisor (cf.II.4.2).

In this thesis the theory of natural deformations is also used in the explicit description of the connected components in the moduli space of some surfaces of general type different from the ones considered in theorem 1. The first cases we consider are the double coverings of the projective plane.

As before for every $h \geq 4$ we define $N(\mathbb{P}^2, \mathcal{O}(h)) \subset \mathcal{M}$ as the set of surfaces of general type whose canonical model is a double cover of $\mathbb{P}^2$ ramified over a plane curve of degree $2h$. In this case the natural deformations are obtained by deforming the branching divisor and are a complete family (VI.2.9), therefore $N(\mathbb{P}^2, \mathcal{O}(h))$ is an open irreducible subset of the moduli space. The following questions becomes natural:

i) Is $N(\mathbb{P}^2, \mathcal{O}(h))$ closed in $\mathcal{M}$?

ii) Is the closure in $\mathcal{M}$ of $N(\mathbb{P}^2, \mathcal{O}(h))$ a connected component?

A theorem of Horikawa ([B-P-V] VII.10.1) asserts that every minimal surface of general type with $K^2 = 2$ and $p_g = 3$ belongs to $N(\mathbb{P}^2, \mathcal{O}(4))$ and then for $h = 4$ the above questions have positive answer. In chapter VII we shall prove the following

Theorem 2. *The subset $N = N(\mathbb{P}^2, \mathcal{O}(h))$, $h \geq 4$ is closed in the moduli space if and only if h is even.*

For every $h \geq 4$ the closure of N in the moduli space is a connected component.

The main step in the proof of theorem 2 is the classification of degenerate double covers of the projective plane. By definition a degenerate double cover of $\mathbb{P}^2$ is a normal projective surface Y_0 with at most rational double points and ample canonical bundle such that there exists a proper flat map $f: Y \to \Delta$ with $f^{-1}(0) = Y_0$ and $f^{-1}(t) = Y_t$ a double cover of $\mathbb{P}^2$ for every $t \neq 0$.

In general the classification of degenerations is a very difficult problem, fortunately in our case, using the fact that for every finite group G the subset $\mathcal{M}^G \subset \mathcal{M}$ of minimal surfaces of general type admitting a faithful G-action is closed ([Ca2],[F-P]), we shall show that there exists a nontrivial involution τ on the degenerate double cover Y_0 and its quotient $X_0 = Y_0/\tau$ is a normal degeneration of $\mathbb{P}^2$ with at most quotient singularities.

We are therefore reduced, as a preliminary step, to consider the following problem of independent interest.

Classify the normal surfaces X_0 admitting a deformation $X \to \Delta = \{t \in \mathbb{C} | \, |t| < 1\}$ such that $X_t \simeq \mathbb{P}^2$ for every $t \neq 0$.

If X_0 is smooth then it is isomorphic to $\mathbb{P}^2$ and the family is locally trivial. Chapter IV is devoted to study the case where X_0 is a normal surface. We shall call this case a normal degeneration of $\mathbb{P}^2$.

For every projective arithmetically Cohen-Macaulay surface V it is possible to construct normal degenerations of V by taking the intersection of the projective cone of Y with the hyperplanes of a generic pencil (see chapter IV for details). Taking as V a Veronese embedding of $\mathbb{P}^2$ we are able to construct examples of normal nonsmooth degenerations of $\mathbb{P}^2$ which are cones over projectively normal curves. The natural question which arises (cf. [Ba1],[Ba2]) is whether these "classical" degenerations are the only ones and, if they aren't, what other normal surfaces can appear.

We observe that "classical" degenerations of $\mathbb{P}^2$ with at most quotient singularities are $\mathbb{P}^2$ and W_0 =cone over the rational smooth curve of degree 4 in $\mathbb{P}^4$.

A quite surprising result we find is the existence of infinitely many examples of normal degenerations of $\mathbb{P}^2$ with at most cyclic quotient singularities. These examples are constructed using the following theorem (IV.B)

Theorem 3.

1) Let X_0 be a normal degeneration of $\mathbb{P}^2$ with at most quotient singularities, then the following properties hold:

- *a) X_0 is projective algebraic.*
- *b) $q(X_0) = P_n(X_0) = 0 \ \forall n \geq 1$*
- *c) $\varrho(X_0) = 1$*
- *d) Every singularity of X_0 is cyclic of type $\dfrac{1}{n^2}(1, na-1)$ for some pair of positive integers a, n with $(a,n) = 1$ ((a,n) is the g.c.d. of a and n)*
- *e) If $p_1, p_2 \in X_0$ and the singularities (X_0, p_i) are cyclic of type $\dfrac{1}{n_i^2}(1, n_i a_i - 1)$ then the n_i's are not divisible by 3, moreover if $p_1 \neq p_2$ then $(n_1, n_2) = 1$*
- *f) X_0 has at most 3 singular points.*

2) Conversely if a normal surface X_0 satisfies a), b), c) and d) of 1) then X_0 is a degeneration of $\mathbb{P}^2$, in particular e) and f) hold too.

The part f) is a consequence of some more general results about normal projective surfaces with Picard number $\varrho = 1$ and $P_{-1} \geq 5$.
The study of these surfaces is made in chapter III where we prove the following

Theorem 4. *Let $\delta: S \to X$ be the minimal resolution of a normal projective surface X with $\varrho(X) = 1$, $P_{-1} \geq 5$ and at most rational singularities. Then S is a rational surface and there exists a birational morphism $\mu: S \to \mathbb{F}_d$, $d \geq 2$ such that the exceptional locus of δ is exactly the union of $\mu^{-1}(\sigma_\infty)$ and the irreducible curves with selfintersection ≤ -2 contained in the fibres of the composite morphism $S \xrightarrow{\mu} \mathbb{F}_d \xrightarrow{p} \mathbb{P}^1$.*

In the statement of theorem 4 $\mathbb{F}_d$ denotes the Segre-Hirzebruch surface and σ_∞ the section of p such that $\sigma_\infty^2 = -d$. Note in particular that the irregularity of X is 0.
A consequence of theorem 4 is that if the singularities of X are taut (e.g. quotient singularities) then X is uniquely determined by the combinatorial data of the sequence of blowings-up composing μ and by a combinatorial argument we shall show that if the singularities are cyclic then X has at most 3 singular points. With the additional information of theorem 3 we shall moreover prove that for every normal degeneration of $\mathbb{P}^2$ with at most quotient singularities X_0 then either X_0 is the cone W_0 over the rational curve of degree 4 in $\mathbb{P}^4$ or if S is the minimal resolution of X and $\mu: S \to \mathbb{F}_d$ is as in theorem 4 then $d = 7, 10$. The degenerations with $d = 7$ are infinitely many and completely classified (IV.4.3) while in case $d = 10$ the situation is more complicated.
As a consequence of our classification of normal degenerations of $\mathbb{P}^2$ we shall prove that every degenerate double cover Y_0 of the projective plane is either a double cover of $\mathbb{P}^2$ or it is a nonflat double cover of the cone W_0 with the vertex $w_0 \in W_0$ as an isolated branch point. This second possibility can appear only if $K_{Y_0}^2$ is divisible by 8 and therefore for h even, the subset $N(\mathbb{P}^2, \mathcal{O}(h))$ is closed since the surfaces belonging to $N(\mathbb{P}^2, \mathcal{O}(h))$ have invariants $K^2 = 2(h-3)^2, 2(\chi - 1) = (h-1)(h-2)$.
The last step of the proof of theorem 2 follows from the fact (VII.3.5) that every degenerate double cover of $\mathbb{P}^2$ has unobstructed deformations. The proof of VII.3.5 when Y_0 is a nonflat double cover of W_0 will require a quite long computation since in this case the "natural deformations" are not a complete family.
Much easier is deformation theory for flat double covers of normal surfaces. Let $X \xrightarrow{\pi} Y$ be a flat double cover of normal surfaces, then $\pi_* \mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module and there exists an eigensheaves decomposition $\pi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$ for a line bundle $L \to Y$, this implies that X can be embedded in the total space of L as the square root of a section of $2L$.
In chapter VI (VI.2.11 and its proof) we prove the following "expected" result

Theorem 5. *In the above notation assume:*

i) $H^1(\mathcal{O}_Y)=0$ *and* L *extends to every deformation of* Y. *(Note that since by the previous assumption* Y *is assumed to be regular, on every deformation of* Y *there exists at most one extension of* L*).*

ii) $\mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y,-L)=0$.

iii) The sections of $2L$ *extend to every deformation of* Y *(e.g. if* $H^1(Y,2L)=0$*).*

If Y *has unobstructed deformations then the same holds for* X *and every deformations of* X *is a flat double cover of a deformation of* Y.

Theorem 5 is the starting point for the construction of a large number of examples of connected components of moduli spaces of surfaces of general type.

For every minimal surface S of general type the set

$$\mathcal{M}_d(S)=\{S'\in\mathcal{M}^{top}(S)|r(S)=r(S')\}$$

is a quasiprojective variety and has a finite number of components, denote by $\delta(S)$ (resp.: $i(S)$) the number of connected (resp.: irreducible) components of $\mathcal{M}_d(S)$. Clearly $i(S)\geq\delta(S)$ and with a more accurate computation in theorem 1 it is possible to find simple bihyperelliptic surfaces S such that $i(S)\geq C_1K_S^2$, $\delta(S)\geq C_2\log\log(K_S^2)$ where C_1,C_2 are absolute positive constants.

These lower bounds are quite unsatisfactory since simple bihyperelliptic surfaces are very special surfaces and it is natural to expect much greater values of $\delta(S)$ and $i(S)$. In ([Ca5]) Catanese gives some effective upper bounds for the number $i(S)$ in terms of K_S^2, the best of which is $i(S)\leq Cy^{77y^2}$, $y=K_S^2$, C absolute constant, for every regular surface S. Catanese's bounds are not very satisfactory and it seems that improvements are possible, in any case $i(S)$ and $\delta(S)$ are in general quite big. In fact in chapter VI we prove:

Theorem 6. *For every real number* $4\leq\beta\leq 8$ *there exists a sequence* S_n *of simply connected surfaces of general type such that:*

a) $y_n=K^2_{S_n}, x_n=\chi(\mathcal{O}_{S_n})\to\infty$ *as* $n\to\infty$.

b) $\lim_{n\to\infty}\frac{y_n}{x_n}=\beta$.

c) $\delta(S_n)\geq y_n^{\frac{1}{5}\log y_n}$.

Theorem 6 is proved by using simple iterated double covers of $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$.

Definition 7. A finite map between normal algebraic surfaces $p\colon X\to Y$ is called a *simple iterated double cover* associated to a sequence of line bundles $L_1,...,L_n\in\mathrm{Pic}(Y)$ if the following conditions hold:

1) There exist $n+1$ normal surfaces $X=X_0,...,X_n=Y$ and n flat double covers $\pi_i\colon X_{i-1}\to X_i$ such that $p=\pi_n\circ....\circ\pi_1$.

2) If $p_i\colon X_i\to Y$ is the composition of π_j's $j>i$ then we have for every $i=1,...,n$ the eigensheaves decomposition $\pi_{i*}\mathcal{O}_{X_{i-1}}=\mathcal{O}_{X_i}\oplus p_i^*(-L_i)$.

For any sequence $L_1, ..., L_n \in \mathrm{Pic}(Y)$ define $N(Y, L_1, ..., L_n)$ as the image in the moduli space of the set of surfaces of general type whose canonical model is a simple iterated double cover of Y associated to $L_1, ..., L_n$.

In case $Y = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ we are able to find sufficient conditions on the sequence $L_1, ..., L_n$ in such a way that the set $N(Y, L_1, .., L_n)$ has "good" properties; the condition we find are summarized in the following definition:

Definition 8. A sequence $L_1, ..., L_n$, $L_i = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_i, b_i)$ $n \geq 2$ of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ is called a *good sequence* if satisfies the following conditions.

C1) $a_i, b_i \geq 3$ for every $i = 1, ..., n$.

C2) $\max_{j<i} \min(2a_i - a_j, 2b_i - b_j) < 0$.

C3) $a_n \geq b_n + 2$, $b_{n-1} \geq a_{n-1} + 2$.

C4) a_i, b_i are even for $i = 2, ..., n$.

C5) For every $i < n$ $2a_i - a_{i+1} \geq 2, 2b_i - b_{i+1} \geq 2$.

A sequence of line bundles $L_1, .., L_n \in \mathrm{Pic}(\mathbb{P}^2)$, $L_i = \mathcal{O}_{\mathbb{P}^2}(l_i)$, is called a *good sequence* if satisfies the following 3 conditions:

C6) $l_i \geq 4$ for every $i = 1, ..., n$.

C7) $l_i > 2l_{i+1}$ for every $i = 1, ..., n-1$.

C8) l_n is odd, l_i is even for $i = 1, ..., n-1$.

The main result we prove is:

Theorem 9. *For $Y = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ let $L_1, ..., L_n$ be a good sequence in sense of definition 8, then:*

a) $N(Y, L_1, ..., L_n)$ is open in the moduli space and its closure is a nonempty connected component.

b) $N(Y, L_1, ..., L_n)$ is reduced, irreducible and unirational.

c) The generic $[S] \in N(Y, L_1, ..., L_n)$ has $Aut(S) = \mathbb{Z}/2\mathbb{Z}$.

d) If $M_1, ..., M_m$ is another good sequence and $N(Y, L_1, ..., L_n) = N(Y, M_1, ..., M_m)$ then $n = m$ and $L_i = M_i$ for every $i = 1, ..., n$.

Moreover in case $Y = \mathbb{P}^1 \times \mathbb{P}^1$ the set $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ is closed in the moduli space.

The proof of the openess of $N(Y, L_1, ..., L_n)$ for $L_1, ..., L_n$ good sequence is an easy consequence of theorem 5. In the understanding of the closure the key results we use is the following (VI.3.1)

Theorem 10. *Let $f: X \to \Delta = \{t \in \mathbb{C} | \, |t| < 1\}$ be a proper flat family of normal projective surfaces and let $\tau: X \to X$ be an involution preserving f. Let $\pi: X \to Y = X/\tau$ be the projection to the quotient and assume that:*

i) X_t, Y_t are smooth surfaces for every $t \neq 0$.

ii) X_0 has at most rational double points (RDP) as singularities.

iii) The divisibility of the canonical class of Y_t is even for $t \neq 0$.

Then Y_0 has at most RDP's and the map $\pi: X \to Y$ is flat.

The example of degenerate double covers of $\mathbb{P}^2$ shows that the above theorem is false without the assumption $r(Y_t)$ even. The proof of theorem 10 is based on the idea (already used in [Ca3]) that we can get information about the number and type of singular points of Y_0 from the intersection product on $H_2(Y_t, \mathbb{Z})$, $t \neq 0$. On the same idea is based also the proof of item *e)* of theorem 3.

In the proof of theorem 10 this idea is used as follows. Let $y_0 \in Y_0$ be a singular point and let $F_t \subset Y_t$ be its Milnor fibre; since in a neighbourhood of y_0, Y is the quotient of a smoothing of a rational double point a classification theorem (VI.3.2) shows that either *a)* (Y_0, y_0) is a rational double point and π is flat at y_0 or *b)* the canonical class of F_t is not 2-divisible in $H^2(F_t, \mathbb{Z})$.

Since the canonical class of F_t is the image of the canonical class of Y_t under the natural restriction homomorphism $H^2(Y_t, \mathbb{Z}) \to H^2(F_t, \mathbb{Z})$ if $r(Y_t)$ is even then the situation *b)* above cannot appear.

Chapter I is almost completely expository and contains the definitions and the main properties of rational and quotient singularities.

The main theme of chapter II is the application of deformation theory to the computation of the Kuranishi family of simple bihyperelliptic surfaces and prove their stability under small holomorphic deformations (II.5.2).

Chapter III contains the proofs of some results used in chapters IV and VII. However we consider these results to be of independent interest (e.g. the above theorem 4) and, with the exception of section III.5, the method used are completely elementary.

Chapter IV is completely devoted to the study of normal degenerations of the projective plane and in section IV.2 are introduced the concepts of Milnor fibre of a smoothing and of a $\mathbb{Q}$-Gorenstein smoothing of a normal twodimensional singularity.

Chapter V is mainly an exposition of the definition and of the main properties of the moduli space of surfaces of general type and in the last section we join the results of Chapter II, [Ca1] and [Ca3] in order to prove the stability of simple bihyperelliptic surfaces (of suitable type) under arbitrary holomorphic deformations and the above theorem 1.

Finally in chapters VI and VII we develop the theory of simple iterated double covers.

The main results of the first six chapters are contained in the papers [Ma1], [Ma3], [Ma4] and [Ma6] while chapter VII contains the yet unpublished contributions of this thesis concerning coverings of $\mathbb{P}^2$.

With respect to the above papers some simplification and improvement in the presentation are made, moreover with the aim of making this thesis more readable and selfcontained, several known facts and related results are recalled.

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Notation.

Unless otherwise stated we shall use the following general notation.

μ_n is the cyclic multiplicative group of complex n-roots of 1.

Given a group G acting on the left on two sets X, Y, a map $f\colon X \to Y$ is said to be G-equivariant or a G-morphism if $f(gx) = gf(x)$ for every $g \in G, x \in X$. A subset $A \subset X$ is called G-stable if $GA \subset A$, it is called G-fixed if $ga = a \ \forall g \in G, a \in A$.

For every topological space X, $b_i(X)$ is its i-th Betti number and $e(X)$ its topological Euler-Poincaré characteristic.

For a complex algebraic variety X and a rational function f on X we shall write $div(f)$ for the principal divisor defined by f, $\mathrm{Pic}(X)$ for the Picard group of X and $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ for the connected component of 0.

The Picard number $\varrho(X)$ is by definition the rank of the Neron-Severi group

$$NS(X) = \mathrm{Pic}(X)/(algebraic \quad equivalence)$$

For every sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules on X the number $h^i(\mathcal{F})$ denotes the dimension of the complex vector space $H^i(X, \mathcal{F})$ and $\mathcal{F}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ is the dual sheaf of $\mathcal{F}$.

We shall denote respectively by Ω^1_X and $\theta_X = (\Omega^1_X)^\vee$ the sheaf of Kaehler differentials and tangent vector fields, note that if X is normal then θ_X is a reflexive sheaf.

For a normal surface X we shall use the following notations:

$q(X) = h^1(\mathcal{O}_X)$ is the irregularity of X.

$p_g(X) = h^2(\mathcal{O}_X)$ is the geometric genus of X.

For every Weil divisor D on X, $\mathcal{O}_X(D)$ is the sheaf of rational functions f such that $div(f) + D \geq 0$ (note that $\mathcal{O}_X(D)$ is reflexive and $\mathcal{O}_X(D)^\vee = \mathcal{O}_X(-D)$).

$\omega_X = (\wedge^2 \Omega^1_X)^{\vee\vee}$ is the canonical sheaf of X.

K_X is the canonical divisor i.e. the Weil divisor, unique up to rational equivalence, such that $\omega_X = \mathcal{O}_X(K_X)$.

For every integer n, $\omega_X^{(n)} = (\omega_X^{\otimes n})^{\vee\vee} = \mathcal{O}_X(nK_X)$ is the n-canonical sheaf and $P_n(X) = h^0(\omega_X^{(n)})$ the n-th plurigenus.

$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$ is the algebraic Poincaré characteristic.

A smooth irreducible complete curve E contained in a smooth surface S is called a *(-1)-curve* if it is rational and $E^2 = -1$, it is called a *nodal curve* if it is rational and $E^2 = -2$.

I. Quotient Singularities.

1. Generalities about minimal resolutions.

Let (X,p) be a normal two-dimensional singularities, a well known theorem (for a historical sketch [Lip1]), generalized to higher dimension by Hironaka, says that there exists a resolution $\delta\colon (S,E) \to (X,p)$ where S is a smooth complex surface, δ is a proper holomorphic map, $E=\delta^{-1}(p)$ is a reduced curve and δ is biholomorphic on $S-E$. (for proofs see also [La2], [B-P-V]).

Note that since (X,p) is normal $\delta_*\mathcal{O}_S = \mathcal{O}_X$, E is connected and if $E=\cup E_i$ is the irreducible decomposition then by Grauert-Mumford theorem ([Mu]) the intersection matrix $E_i\cdot E_j$ is negative definite.

A resolution $(S,E)\to(X,p)$ is *minimal* if E doesn't contains (-1)-curves. From Castelnuovo criterion of decomposition of bimeromorphic maps it follows easily that every normal two-dimensional singularity has a unique minimal resolution.

A resolution $(S,E)\to(X,p)$ is *good* or *global normal crossing* if E satisfies the following conditions:

1) All the irreducible components of E are smooth and intersect transversally.
2) Not more than 2 components pass through any given point.
3) 2 different components intersect at most once.

According to desingularization theorem of curves in surfaces good resolutions always exist although the minimal resolution is generally not good.

If $C_1, C_2 \subset E$ are (-1)-curves then $(C_1+C_2)^2<0$ and then $C_1\cdot C_2 \le 0$. From this we see easily that there exists a unique resolution (S,E), called the *minimal canonical resolution*, which is minimal in the class of resolutions satisfying the above conditions 1) and 2).

We now introduce some invariants of a normal two-dimensional singularity (X,p) with minimal canonical resolution $\delta\colon (S,E)\to(X,p)$. Let $E=\cup_{i=1}^n E_i$ be the irreducible decomposition.

The *Dynkin diagram* D_X is the weighted dual graph of its minimal canonical resolution. D_X is a graph whose vertices corresponds to irreducible components E_i with associated their

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selfintersection E_i^2 and their geometric genus $g(E_i)$; the number of edges connecting E_i to E_j is the intersection number $E_i \cdot E_j$.

The Dynkin diagram depends only on the topological type of the pair (S, E) minimal canonical resolution and is an invariant of the germ (X, p), if the singularity is uniquely determined by D_X then it is called *taut*.

The *genus* of the singularity is defined as $g(X, p) = h^0(R^1\delta_*\mathcal{O}_S)$. If X is Stein then by Leray spectral sequence it follows that $g(X, p) = h^1(\mathcal{O}_S)$, in particular since the irregularity is invariant under blow-up the definition of the genus is independent from the resolution.

It is not difficult to prove (cf. [Ar2] Prop.2) that for every $c = (c_1, ..., c_n) \in \mathbb{Z}^n$ there exists a unique minimal effective divisor $Z_c = \sum a_i E_i$ such that $Z_c \cdot E_i \leq c_i$ for every $i = 1, ..., n$. The divisor $Z = Z_0$, $0 \in \mathbb{Z}^n$ is called the *fundamental cycle*. Some important relations between the fundamental cycle and the genus are discussed in the next section.

2. Rational Singularities

If S is a smooth complex, possibly non compact, surface denote by $K_S \in \mathrm{Pic}(S)$ the canonical line bundle and by $k_S \in H^2(S, \mathbb{Z})$ its first Chern class.

If D is a divisor is S with compact support and $\mathcal{L} \in \mathrm{Pic}(S)$ the intersection product $D \cdot \mathcal{L}$ is well defined and depends only on the cohomology classes $[D] \in H_2(S, \mathbb{Z}) = H_c^2(S, \mathbb{Z}), c_1(\mathcal{L}) \in H^2(S, \mathbb{Z})$.

The arithmetic genus of D is by definition

$$p_a(D) = 1 + \frac{1}{2} D \cdot (D + K_S)$$

For D irreducible curve this definition is the same of the usual arithmetic genus $h^1(\mathcal{O}_D)$ while for general effective divisor we have $p_a(D) = 1 - \chi(\mathcal{O}_D)$ (this is clear if S is compact since $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-D))$ but can be proved without difficulties also for general S, cf. [B-P-V] II.11).

Proposition-Definition 2.1. *Let* $(S, E) \xrightarrow{\delta} (X, p)$ *be the minimal resolution of a normal surface singularity.* (X, p) *is called a* Rational singularity *if one of the following equivalent conditions holds:*

i) The genus of (X, p) *is* 0.

ii) For every effective divisor D *supported in* E, $h^1(\mathcal{O}_D) = 0$.

iii) For every effective divisor D *supported in* E, $p_a(D) \leq 0$.

iv) The arithmetic genus of the fundamental cycle is 0.

For a proof we refer to the original paper of Artin ([Ar2]) or to the books ([B-P-V] chapter III),([Ba3]).

Corollary 2.2. *The minimal resolution of a rational singularity is good, the irreducible components of the exceptional curve are smooth rational and the Dynkin diagram is a weighted tree.*

Proof. Every irreducible component has arithmetic genus 0 and then is smooth rational. The other results follows from *2.1.iii)* and formula $p_a(A+B) = p_a(A) + p_a(B) + A \cdot B - 1$ for every pair of compactly supported divisors A, B. □

Note that we can recognize if a singularity is rational from its Dynkin diagram D_X, in fact if all components of E are smooth rational then the fundamental cycle and its genus depend only on D_X.

A rational singularity with fundamental cycle Z is called a *rational n-point* if $-Z^2 = n$; this definition is motivated from the following

Theorem 2.3.(Artin [Ar2])*Let $(S,E) \xrightarrow{\delta} (X,p)$ be the minimal resolution of a rational singularity with fundamental cycle Z. Then:*

(i) For every $k > 0$ $\delta^*(\mathcal{M}^k) = \mathcal{O}_S(-kZ)$ where $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

(ii) The multiplicity of X at p is $-Z^2$.

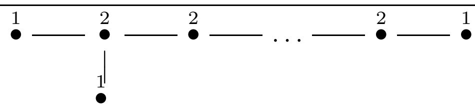
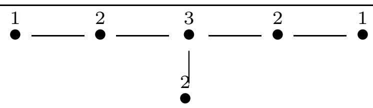
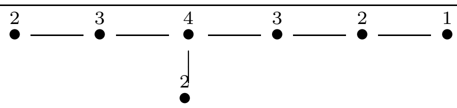
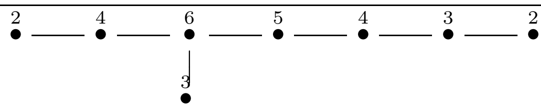
(iii) The embedding dimension of X at p is $-Z^2 + 1$.

Therefore a simple rational point is smooth and a rational double point (RDP from now on) is defined in $\mathbb{C}^3$ by a function of multiplicity 2.

If E is the exceptional curve of a RDP the every component of E has selfintersection -2. In fact by minimality $E_i \cdot K_S \geq 0$ for every component E_i. By definition of RDP $K_S \cdot Z = -2 - Z^2 + p_a(Z) = 0$ and then $K_S \cdot E_i = 0$, $E_i^2 = -2$.

Conversely is a trivial consequence of 2.1 that every normal singularity $(X,0)$ whose irreducible components of the exceptional curve of its minimal resolution are nodal curves then $(X,0)$ is a rational singularity.

In the next table is showed the complete classification (made first by Du Val) of rational double points.

Name	Canonical equation	Dynkin diagram and fundamental cycle
A_n $n \geq 1$	$z^2 = x^2 + y^{n+1}$	 n vertices
D_n $n \geq 4$	$z^2 = xy^2 + x^{n-1}$	 n vertices
E_6	$z^2 = x^3 + y^4$	
E_7	$z^2 = x^3 + xy^3$	
E_8	$z^2 = x^3 + y^5$	

Let $(S,E) \to (X,p)$ be the minimal resolution of a rational singularity and assume X Stein and contractible, then $H^1(\mathcal{O}_S) = H^2(\mathcal{O}_S) = 0$ and E is a deformation retract of S, in particular the exponential sequence on S gives an isomorphism $\mathrm{Pic}(S) = H^2(E,\mathbb{Z})$.

Therefore a line bundle $\mathcal{L}$ on S is trivial if and only if $\mathcal{L} \cdot E_i = 0$ for every irreducible component of E and for every divisor $D \subset S$ with $D \cdot E_i = 0$ for every i there exists, possibly shrinking S, a meromorphic function f such that $\mathrm{div}(f) = D$.

If (X,p) is a RDP then $K_S = \mathcal{O}_S$, $K_X = \mathcal{O}_X$. Conversely every rational Gorenstein singularity is a RDP, in fact there exists a cohomology (with integer coefficient) exact sequence in the minimal resolution

$$0 \longrightarrow H^2(S,\partial S) = H_2(E) \xrightarrow{q} H^2(S) = H^2(E) \xrightarrow{p} H^2(\partial S) \longrightarrow 0$$

where q is the map induced from the intersection form on E. Note that since q is nondegenerate the group $H_1(\partial X) = H^2(\partial S)$ is finite.

Our assertion is a consequence of the following:

Lemma 2.4. *If $p(k_S) = 0$ then $E_i^2 = -2$ for every irreducible component of E*

Proof. Assume $p(k_S) = 0$, then k_S is the Chern class of a divisor K supported on E. Let A, B be the minimal effective divisors such that $K = A - B$, since $0 \leq K \cdot A \leq A^2$, A must be 0 and $-K$ effective.

The arithmetic genus is $p_a(-K) = 1$ and since by assumption the singularity is rational K must be 0, proving the lemma. □

For later use we recall now some other important properties of rational singularities.

Let $\delta\colon S \to X$ be a bimeromorphic map between compact surfaces with S smooth and X normal, let $E =$ be the exceptional curve.

Assume X with only rational singularities and let $\mathcal{L}$ be a line bundle on S, then there exist a positive integer n and a divisor D supported on E such that $n\mathcal{L} + D$ is the trivial line bundle on a neighbourhood of E and then there exists a line bundle $\mathcal{L}'$ on X such that $n\mathcal{L} + D == \delta^*\mathcal{L}'$. Moreover the $\mathbb{Q}$-divisor $\frac{1}{n}D$ is uniquely determined by the intersection products $\mathcal{L}\cdot E_i$ and the map $\delta^*\colon NS(X,\mathbb{Q}) \to NS(S,\mathbb{Q})$ is injective. As a consequence we have:

Proposition 2.5. *There exists a natural isomorphism of* $\mathbb{Q}$*-vector spaces*

$$NS(S,\mathbb{Q}) = NS(X,\mathbb{Q}) \oplus (\bigoplus_{E_i} \mathbb{Q}E_i)$$

where the direct sum is taken over all irreducible components E_i *of* E*, in particular* $\varrho(S) = \varrho(X) + b_2(E)$.

If S is algebraic and $\mathcal{L}$ is ample then it is reasonable to expect that also $\mathcal{L}'$ is ample, in fact this is true ([Ar1]) and we have:

Theorem 2.6. (Artin contractibility criterion) *Let* $S \to X$ *be a bimeromorphic map with* S *projective algebraic and* X *normal with at most rational singularities, then* X *is algebraic.*

For a proof we refer to ([Ar1]). Note that the statement of theorem 2.6 is generally false without the assumption on the type of singular points (cf [Ha1] Example V.5.7.3).

3. Finite group actions on singularities

Let G be a finite group of automorphisms of a complex analytic space X. In [Car] Cartan proved that the orbit space has a natural structure of analytic space, the main ingredient of his proof was the following beautiful result nowadays known as "Cartan's Lemma".

Lemma 3.1. (Cartan) *Let* (X,x) *be a germ of complex space with Zariski tangent space* T *and let* G *a finite group of automorphisms of* (X,x).

Then there exists a G*-embedding* $(X,x) \to (T,0)$*, in particular the induced representation* $G \to GL(T)$ *is faithful.*

As application of this lemma we prove a result that we shall use in the next chapters.

Proposition 3.2. *Every finite group* G *of automorphisms of a RDP of type* E_7 *or* E_8 *is cyclic.*

Proof. The action of G lift to a faithful action on the minimal resolution (S, E) of the RDP, since the Dynkin diagram has no automorphism G leave fixed every irreducible component of E.

Let $E_0 \subset E$ be the central component, i.e. the component intersecting the others in three points, then G acts trivially on E_0 and for every $p \in E_0$, G acts on the tangent space T_pS. By Cartan lemma the action of G is faithful in T_pS, trivial on the hyperplane $T_pE_0 \subset T_pS$ and then G is cyclic. □

Actually holds a stronger statement, two finite subgroups of automorphisms of a RDP of type E_7 or E_8 with the same cardinality are conjugated ([Ca3]).

A *Quotient* two-dimensional singularity is a singularity isomorphic to $(\mathbb{C}^2, 0)/G$ for a finite group $G \subset Aut(\mathbb{C}^2, 0)$. According to Cartan lemma we can assume without loss of generality $G \subset GL(2, \mathbb{C})$ and after a linear change of coordinates $G \subset U(2)$. Every quotient singularity is rational ([Bri] Satz 1.7) and if $G \subset SU(2)$ then $X = \mathbb{C}^2/G$ is a rational double point. In fact $\omega = dz_1 \wedge dz_2$ is a G-invariant nowhere vanishing holomorphic two-form in $\mathbb{C}^2 - \{0\}$ and since G acts freely on $\mathbb{C}^2 - \{0\}$ $\omega \in H^0(X - \{0\}, K_X)$. Thus K_X is the trivial line bundle on $X - \{0\}$ and X is Gorenstein.

Conversely every RDP is the quotient of $\mathbb{C}^2$ by a finite subgroup of $SU(2)$ ([Lo2]). We refer also to ([E-C] Volume 1, Libro 2, II.10) for an explicit classification of finite subgroups of $SU(2)$ based on the homomorphism $SU(2) \to Aut(\mathbb{P}^1)$ and Hurwitz formula and to ([Bri]) for a complete classification to quotient two-dimensional singularities and their Dynkin diagrams.

Example 3.3. *Cyclic singularities.*
By a cyclic singularity of type $\frac{1}{n}(a, b)$ we mean the quotient of $\mathbb{C}^2$ by the action of a diagonal automorphism with eigenvalues $\exp(2\pi i \frac{a}{n})$, $\exp(2\pi i \frac{b}{n})$. Since the quotient of $\mathbb{C}^2$ by a complex reflection (i.e. a linear map of finite order leaving a hyperplane pointfixed) is again smooth it is easy to see that every cyclic singularity is isomorphic to a cyclic singularity of type $\frac{1}{n}(1, a)$ with $G.C.D.(a, n) = 1$.
The standard torus action on $\mathbb{C}^2$

$$(\lambda, \mu)(x, y) = (\lambda x, \mu y) \quad \lambda, \mu \in \mathbb{C}^* \quad x, y \in \mathbb{C}$$

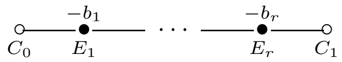
commute with every diagonal linear endomorphism of $\mathbb{C}^2$ and then induces a faithful action on the quotient $X = \mathbb{C}^2/H$ where

$$H = \{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix} \mid \epsilon^n = 1\} \quad G.C.D.(n, a) = 1$$

of the group $G = (\mathbb{C}^*)^2/H \simeq (\mathbb{C}^*)^2$.

In particular there exists a direct system of commuting faithful actions of the groups $\mu_h \times \mu_k$ on the minimal resolution $(S, E) \to (X, 0)$ for all pairs of positive integers h, k.

It is clear that every irreducible component of E is invariant and reasoning as in the proof of proposition 3.2 we see that every component of E must intersect the others at most twice. Then the Dynkin diagram D_X is a string and there exists, apart the components of E, exactly two closed irreducible invariant curves C_0, C_1 intersecting transversally E. The dual weighted graph of $E \cup C_0 \cup C_1$ is



Let $\pi\colon \mathbb{C}^2 \to X$ be the projection, the only invariant irreducible curves on X are the image of the coordinate axis and then, up to permutation of indices

$$C_0 = \text{ strict transform of } \pi(\{y=0\}) \qquad C_1 = \text{ strict transform of } \pi(\{x=0\})$$

According to the above description of the H-action on $\mathbb{C}^2$ we have $n = \min\{i > 0 | x^i \in \mathcal{O}_X\}$, $a = \min\{i > 0 | yx^{n-i} \in \mathcal{O}_X\}$ and by general properties of rational singularities

$$n = \min\{i > 0 | \exists Z, supp\, Z \subset E, (iC_1 + Z)\cdot E_j = 0\ \forall j\}$$

$$a = \min\{i > 0 | \exists Z, supp\, Z \subset E, ((n-i)C_1 + Z + C_0)\cdot E_j = 0\ \forall j\}$$

Resolving these systems of linear equations we get the familiar expression

$$\frac{n}{a} = [b_1, ..., b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \quad \vdots \atop b_{r-1} \quad - \cfrac{1}{b_r}}}$$

4. Taut singularities

We recall that a singularity (X, x) with Dynkin diagram D_X is taut if for every singularity (Y, y) such that $D_Y = D_X$ there exists an isomorphism $(Y, y) \simeq (X, x)$, in particular every automorphism of the Dynkin diagram of a taut singularity is induced by an analytic automorphism of the singular point.
Since smooth curves of fixed genus $g > 0$ have nontrivial moduli every irreducible exceptional curve of the minimal canonical resolution of a taut singularity is rational. The converse is false, in fact there exist rational singularities that are not taut ([Bri]).
In some particular case it is very easy to decide if a singularity is taut.

Example . Cones over rational projectively normal curves.
Let $X \subset \mathbb{A}^{n+1}$ be the affine cone over the rational curve of degree n in $\mathbb{P}^n$. The affine coordinate ring of X is

$$A_X = \oplus_{k \geq 0} H^0(\mathcal{O}_{\mathbb{P}^1}(kn)) = \mathbb{C}[x_0^n, x_0^{n-1}x_1, ..., x_1^n]$$

Note that A_X is the μ_n-invariant subring of $\mathbb{C}[x_0,x_1]$ where $\epsilon\in\mu_n$ acts by scalar multiplication and then $(X,0)$ is the cyclic singularity of type $\frac{1}{n}(1,1)$.
An explicit description of its minimal resolution is $(\mathcal{O}_{\mathbb{P}^1}(-n),E)\xrightarrow{\delta}(X,0)$ where for every integer r

$$\mathcal{O}_{\mathbb{P}^1}(r)=(\mathbb{C}^2-\{0\})\times\mathbb{C}/\sim \qquad (l_0,l_1,v)\sim(\lambda l_0,\lambda l_1,\lambda^r v)\quad \lambda\in\mathbb{C}^*$$

is the total space of the line bundle of degree r over $\mathbb{P}^1$ and the morphism δ is described in terms of the weighted homogeneous coordinates by

$$\delta(l_0,l_1,v)=(l_0^n v, l_0^{n-1}l_1 v,\ldots,l_1^n v)$$

Let $(S,E)\to(Y,y)$ be a resolution of a normal surface singularity such that $E=\mathbb{P}^1$, $E^2=-n$. This singularity is rational and, possibly shrinking S, there exists a line bundle $L\xrightarrow{\pi}S$ such that $L\cdot E=-1$ and E is the divisor of a section e of the line bundle $L^{\otimes n}$.
Let $\hat{S}\subset L$ be the smooth surface defined by the equation $z^n=e$, $z\in H^0(L,\pi^*L)$ is the tautological section, and let $B=div(z)\subset\hat{S}$. B is a smooth rational curve with $B^2=-1$ and then in a neighbourhood of B $\hat{S}=\mathcal{O}_{\mathbb{P}^1}(-1)$ is the blow up of $\mathbb{C}^2$.
S is the quotient of $\hat{S}$ by a cyclic group of order n acting trivially on B, applying Cartan lemma on $\mathbb{C}^2$ we find that, up to conjugation with a holomorphic automorphism, this action must be

$$\epsilon(l_0,l_1,w)=(l_0,l_1,\epsilon w)\qquad \epsilon^n=1,\ (l_0,l_1,w)\in\mathcal{O}_{\mathbb{P}^1}(-1)$$

and then S is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n)$.
Using similar ideas and some powerful results of Mumford [Mu] about the local fundamental group of a normal surface singularities, Brieskorn [Bri] proved that *every quotient singularity is taut* while Tyurina [Ty1] proved, by studying the obstruction to lifting automorphisms of infinitesimal neighbourhoods of the exceptional curve, that *every rational double or triple point is taut.*
Finally Laufer [La1], extending Tyurina's method, gave a complete classification of the Dynkin diagrams of taut singularities.

5. Projection formulas

Let $(S,E)\xrightarrow{\pi}(X,0)$ be a resolution of a normal surface singularity and denote by $E_1,\ldots,E_r$ the irreducible components of E and by $i\colon U=X-\{0\}\to X$ the inclusion.
For every locally free sheaf $\mathcal{F}$ on S there exists an exact sequence

$$0\longrightarrow\pi_*\mathcal{F}\xrightarrow{\alpha}i_*\mathcal{F}_U\longrightarrow H^1_E(S,\mathcal{F})\longrightarrow H^0(R^1\pi_*\mathcal{F})$$

where $H^1_E(S,\mathcal{F})$ is the direct limit of $\mathrm{Ext}^1_S(\mathcal{O}_Y,\mathcal{F})=H^0(Y,\mathcal{O}_Y(Y)\otimes\mathcal{F})$ over all the effective divisors Y supported on E (cf. [Ha2] 2.8).

The cokernel of α is naturally isomorphic to $H^1_{\{0\}}(\pi_*\mathcal{F})$ and the sheaf $\pi_*\mathcal{F}$ is reflexive if and only if α is an isomorphism. It is a well known fact (cf. [B-W]) that if π is the minimal resolution then $\pi_*\theta_S = \theta_X$ although in many cases (e.g. rational double points) the group $H^1_E(S, \theta_S)$ is different from 0.

Proposition 5.1. *In the previous notation if $\mathcal{L}$ is a line bundle on S such that for every effective divisor Y with support in E there exists a component $E_i \subset Y$ with $(Y+\mathcal{L})\cdot E_i < 0$ then $H^1_E(S,\mathcal{L}) = 0$ and $\pi_*\mathcal{L}$ is reflexive.*

Proof. Assume that $H^0(Z, \mathcal{O}_Z(Z+\mathcal{L})) \neq 0$ for some effective divisor supported on E end let Y minimal with this property. Then we may write $Y = Z + E_i$ with $(Y+\mathcal{L})\cdot E_i < 0$ and taking global sections associated to the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(Z+\mathcal{L}) \longrightarrow \mathcal{O}_Y(Y+\mathcal{L}) \longrightarrow \mathcal{O}_{E_i}(Y+\mathcal{L}) \longrightarrow 0$$

we get a contradiction. □

For every real number a we denote by $[a]$ its integral part, i.e. the greatest integer $\leq a$.

Corollary 5.2.([Sa2] 1.2) *In the previous notation let $\mathcal{L}$ be a line bundle on S and let $a_1, ..., a_r$ be rational numbers such that for every $i = 1,..,r$ $E_i \cdot (\mathcal{L} + \sum a_j E_j) = 0$. Then $\pi_*(\mathcal{L} + \sum [a_j] E_j)$ is reflexive.*

Proof. Let Y be an effective divisor supported on E and assume that $E_i{\cdot}(Y+\mathcal{L}+\sum [a_j]E_j) \geq 0$ for every irreducible component $E_i \subset Y$, we shall show that this gives a contradiction.
Without loss of generality we can assume that the irreducible components of Y are exactly $E_1, ..., E_s$, $s \leq r$. Considering the effective $\mathbb{Q}$-divisor $D = Y - \sum_{i \leq s}(a_i - [a_i])E_i$, we have

$$0 \leq D\cdot(Y + \mathcal{L} + \sum [a_i]E_i) = D\cdot(\mathcal{L} + \sum a_j E_j) + D\cdot(D - \sum_{i>s}(a_i - [a_i])E_i) < 0 \qquad \square$$

Corollary 5.3. *Let $(S,E) \xrightarrow{\pi} (X,0)$ be the minimal resolution of a normal surface singularity, then for every integer $n \leq 0$, $\pi_*\mathcal{O}_S(nK_S) = \mathcal{O}_X(nK_X)$.*

Proof. Since π is minimal $K_S \cdot E_i \geq 0$ for every irreducible component and then for $n \leq 0$, $Y\cdot(Y + nK_S) \leq Y^2 < 0$ for every effective divisor Y supported in E. □

Corollary 5.4. *Let $(S,E) \xrightarrow{\pi} (X,0)$ be the minimal resolution of a rational surface singularity, then $\pi_*\mathcal{O}_S(K_S) = \mathcal{O}_X(K_X)$.*

Proof. Let Y be an effective divisor supported on E, since the singularity is rational the arithmetic genus of Y is ≤ 0 and then $Y\cdot(Y+K_S) < 0$. □

A similar result holds for the sheaf of differentials, more precisely we have

Theorem 5.5.(Pinkham-Wahl) *Let $(S,E) \xrightarrow{\pi} (X,0)$ be the minimal resolution of a rational surface singularity, then $\pi_*\Omega^1_S = i_*\Omega^1_U$ is reflexive and the dimension of $H^1_E(S,\Omega^1_S)$ equals the number of irreducible components of E.*

For the proof we refer to ([Pi1] Appendice). □

Let X be a compact normal surface, we denote by $\mathrm{Div}(X)$ the group of Weil divisors on X and by $\mathrm{Div}(X,\mathbb{Q})$ the $\mathbb{Q}$-vector space $\mathrm{Div}(X)\otimes\mathbb{Q}$.

Let $S\xrightarrow{\delta}X$ be a resolution and $E=\cup E_i$ the irreducible decomposition of the exceptional locus of δ. We define a linear map $\delta^*\colon \mathrm{Div}(X,\mathbb{Q})\to\mathrm{Div}(S,\mathbb{Q})$ by setting, for D irreducible

$$\delta^*(D)=\delta^{-1}(D)+\sum\alpha_iE_i \qquad [\delta^*(D)]=\delta^{-1}(D)+\sum[\alpha_i]E_i$$

where $\delta^{-1}(D)$ is the strict transform of D by δ and α_i are rational numbers uniquely determined by the conditions $\delta^*(D)\cdot E_i=0\ \forall i$; then we extend δ^* by linearity. For any two $\mathbb{Q}$-divisors D,F the intersection number $D\cdot F$ is defined to be the rational number $\delta^*(D)\cdot\delta^*(F)$ (cf. [Mu] pag. 17).

According to projection formula 5.2 $\delta_*\mathcal{O}_S([\delta^*(D)])=\mathcal{O}_X(D)$ and then by Leray spectral sequence $\chi(\mathcal{O}_X(D))=\chi(\mathcal{O}_S([\delta^*(D)]))+h^0(R^1\delta_*\mathcal{O}_S([\delta^*(D)]))$.

Writing $\delta^*(D)=[\delta^*(D)]+D'$, $K_Y=\delta^*K_X+F$ we have by Riemann-Roch

$$\chi(\mathcal{O}_S([\delta^*(D)]))=\chi(\mathcal{O}_X)-h^0(R^1\delta_*\mathcal{O}_S)+\frac{1}{2}D\cdot(D-K_X)+\frac{1}{2}D'\cdot(D'+F)$$

If $p_1,...,p_s$ are the singular points of X we may write the above formula as

$$\chi(O_X(D))=\chi(\mathcal{O}_X)+\frac{1}{2}D\cdot(D-K_X)+\sum_{i=1}^{s}c(X,D,p_i)$$

where $c(X,D,p_i)$ is a local contribution depending only by the pair germ (X,D) at the point p_i. Note that if D is principal at p_i then $c(X,D,p_i)=0$ and for every divisor D the absolute value of $c(X,D,p_i)$ is bounded by a constant depending only from the singularity (X,p_i) (cf. [K-S] 2.19).

II. Normal bidouble covers and their deformations.

In this chapter we discuss some topics about deformations of complex spaces and analytic singularities, we assume the reader is familiar with the main results of deformation theory as reported, for example, in the introduction of Palamodov article [Pa].

In the first part we recall some general theorems, particular attention is given to Brieskorn-Tyurina theory of simultaneous resolution of rational double points.

In the second part we study deformations of normal surfaces Y with a G-action, where G is the group generated by two commuting involutions such that the quotient Y/G is smooth. In particular we study a particular class of deformations of the projection map $\pi: Y \to Y/G$ called natural deformations and we determine when they induce a complete family of deformations of Y.

Finally we apply these results to proving the stability under small deformations of simple bihyperelliptic surfaces of type $(a,b)(n,m)$ with $a > 2n$, $m > 2b$.

1. Some remarks on deformation theory.

We recall that every compact complex space X (resp. isolated singularity $(X,0)$) has a semiuniversal deformation (sometimes called effective versal or minimal versal), denote by $Def(X)$ (resp. $Def(X,0)$) its base space.

A deformation of X parametrized by $Spec(A)$ where A is a local Artinian $\mathbb{C}$-algebra is called an infinitesimal deformation, a deformation parametrized by $D = Spec(\mathbb{C}[\epsilon] = \mathbb{C}[t]/(t^2))$, $\epsilon \equiv t \ mod(t^2)$, is called a first order deformation. The set of first order deformations is usually denoted by $T^1(X)$ has a natural structure of complex vector space ([Sch]). If X has a semiuniversal deformation $\tilde{X} \to Def(X)$ then every first order deformation is induced by a unique map $D \to Def(X)$ and then there exists an isomorphism of vector spaces $T^1(X) = T_0 Def(X)$.

The study of infinitesimal deformations is considerably easier than the study of convergent ones, in fact, to give a deformation of X over the spectrum of A local Artinian is the same to give a sheaf on X of flat analytic A-algebras $\mathcal{F}$ such that $\mathcal{F} \otimes_A \mathbb{C} = \mathcal{O}_X$ and we can use the usual tools of cohomology theory.

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Fortunately in some cases we can obtain results on the convergent deformations by infinitesimal computations. We now discuss three typical examples of this situation.

A. The Kodaira-Spencer map.

Let $\tilde{X}\xrightarrow{f}S$ be a deformation of X, every morphism $D\to S$ induces a first order deformation of X and then it is defined a linear map $KS(f)\colon T_0S\to T^1(X)$ which is functorial in S, i.e. if $\tilde{X}'\xrightarrow{f'}S'$ is a deformation and f is induced from f' by a morphism $\phi\colon S\to S'$ then $KS(f)=KS(f')\circ d\phi$. If f' is the semiuniversal deformation then $KS(f')$ is an isomorphism and by implicit function theorem we get

Lemma 1.1. *If S is smooth, $KS(f)$ is surjective and X has a semiuniversal deformation then $Def(X)$ is isomorphic to an open subset of $T^1(X)$.*

If S is not smooth the understanding of the Kodaira-Spencer map is not sufficient to describe $Def(X)$, in this case it is necessary to study the general infinitesimal deformations.
Let $\mathcal{C}$ be the category of local Artinian $\mathbb{C}$-algebras, from now on by a functor of Artin rings we shall mean a covariant functor F from $\mathcal{C}$ to the category of sets with a distinguished point $*$ such that $F(\mathbb{C})=*$.
Examples of functors of Artin rings are the deformation functor Def_X for any complex space X

$$Def_X(A)=\{\text{ isomorphism classes of deformations of } X \text{ over } Spec(A)\}$$

and the representation functor h_T for any local $\mathbb{C}$-algebra T. $A\in\mathcal{C}$

$$h_T(A)=Hom_{\mathbb{C}-alg}(T,A)$$

A morphism $\phi\colon F\to G$ between functors of Artin rings is called smooth if for every surjection $A'\to A$ in $\mathcal{C}$ the natural map

$$F(A')\to F(A)\times_{G(A)}G(A')$$

is surjective. A functor F is smooth if for every surjection $A'\to A$ in $\mathcal{C}$ the map $F(A')\to F(A)$ is surjective.

Example .([Sch]) 1) If $R\to S$ is a homomorphism of local analytic algebras then the induced morphism $h_S\to h_R$ is smooth if and only if S is a convergent power series ring over R.
2) If $\tilde{X}\to Def(X)$ is a versal deformation of X and $S=\mathcal{O}_{Def(X),0}$ then the induced map $h_S\to Def_X$ is smooth.
In the category $\mathcal{C}$ there exist fiber products and for every functor of Artin rings F and every morphisms $A\to C, B\to C$ in $\mathcal{C}$ it is defined a natural map

$$\eta\colon F(A\times_C B)\to F(A)\times_{F(C)}F(B)$$

Definition . The functor F has a good deformation theory if satisfies the following two conditions:

H1: η is surjective whenever $B \to C$ is surjective.

H2: η is bijective when $C = \mathbb{C}$ and $B = \mathbb{C}[\epsilon]$.

Both the representation and deformation functors have a good deformation theory ([Sch]).

Note that if F satisfies H2 then the set $t_F = F(\mathbb{C}[\epsilon])$ has a natural structure of vector space and every morphism of functors $u: F \to G$ satisfying H2 induces a linear map $du: t_F \to t_G$. t_F is called the tangent space to F and du the differential of u.

For every small extension in $\mathcal{C}$, $\epsilon \in A$

$$0 \longrightarrow \mathbb{C}\epsilon \longrightarrow A \stackrel{p}{\longrightarrow} B \longrightarrow 0$$

there exists an isomorphism

$$A \times_{\mathbb{C}} \mathbb{C}[\epsilon] \stackrel{p'}{\longrightarrow} A \times_B A \qquad p'(a, a_0 + b\epsilon) = (a, a + b\epsilon)$$

where $a_0 \in \mathbb{C}$ is the valuation of $a \in A$ at the closed point and for every functor of Artin rings F with good deformation theory, p' induces a surjective map

$$F(A) \times t_F \stackrel{F(p')}{\longrightarrow} F(A) \times_{F(B)} F(A) \tag{1.2}$$

Proposition 1.3. *Let $F \stackrel{u}{\longrightarrow} G$, $G \stackrel{v}{\longrightarrow} H$ be morphisms of functors of Artin rings:*

1) If u, v are smooth then the composition vu is smooth.

2) If vu is smooth and u is surjective then v is smooth.

3) If vu is smooth, F, G have good deformation theory and $du: t_F \to t_G$ is surjective then u and v are smooth.

Proof. The proof is completely formal and is an easy consequence of the definition of smoothness and (1.2). Is left to the reader. Note that if H is the trivial functor then 3) is the formal analog of (1.1). ◻

B. Criteria for the existence of universal deformations.

We recall here only a simple sufficient condition for the existence of a universal deformation of a compact complex space X, for some stronger results we refer to ([Wav]).

Let $\tilde{X} \stackrel{f}{\longrightarrow} Def(X)$ be the semiuniversal deformation of X and let $S = \mathcal{O}_{Def(X),0}$, it is clear that f is universal if and only if the induced map of functors $h_S \longrightarrow Def_X$ is an isomorphism, in ([Sch]) it is proved the following

Theorem 1.4. *The map $h_S \to Def_X$ is bijective if and only if for every small extension $A \stackrel{p}{\longrightarrow} B$ and every deformation X_A of X over $Spec(A)$ the restriction map*

$$Aut(X_A) \to Aut(X_A \times_{Spec(A)} SpecB)$$

is surjective.

Since the kernel of the above map between automorphism groups is always isomorphic to $H^0(\theta_X) \otimes \ker(p)$ it follows by induction that if $H^0(\theta_X) = 0$ then every infinitesimal deformation of X has no automorphism and from theorem 1.4 follows immediately

Corollary 1.5. *If $H^0(\theta_X) = 0$ (e.g. $Aut(X)$ is finite) then X has a universal deformation.*

Clearly this condition is not necessary for the existence of a universal family (Example. Elliptic curves).

C. Globalization of deformations.

Let X be a compact complex space with a finite number of singular points $p_1, ..., p_n$. Every deformation of X induces by restriction deformations of the singularities $(X, p_1), .., (X, p_n)$ and then it is defined a germ of holomorphic function

$$Def(X) \xrightarrow{\Psi} \times_{i=1}^{n} Def(X, p_i)$$

In this situation denote by $S = \mathcal{O}_{Def(X)}$, $R = \mathcal{O}_{\times Def(X,p_i)}$ and by $\Psi^*: R \to S$ the algebra homomorphism induced by Ψ.

Definition . The morphism Ψ is called *smooth* if S is a convergent power series ring over R, i.e. if Ψ^* is the composition of two homomorphisms $R \xrightarrow{i} R\{z_1, ..., z_r\} \xrightarrow{j} S$ where i is the natural inclusion and j is an isomorphism, or equivalently if the induced morphism of functors $h_S \xrightarrow{\psi} h_R$ is smooth.

Note that a smooth morphism is in particular surjective, thus if in our situation Ψ is smooth then every deformation of the singular points of X can be globalized. We consider smoothness instead of surjectivity because smoothness can be checked formally.

There exists a commutative diagram of functors of Artin rings

$$\begin{array}{ccc} h_S & \longrightarrow & Def_X \\ \downarrow \psi & & \downarrow \phi \\ h_R & \longrightarrow & \times Def_{(X,p_i)} \end{array}$$

with the horizontal morphisms smooth. According to proposition 1.2 ψ is smooth if and only if ϕ is smooth.

We shall see next that the obstructions of ϕ to be smooth are in $H^2(\theta_X)$.

A similar situation is the following, $X \in \mathbb{P}^n$ is a projective variety and let $[X] \in Hilb^n$ be the point representing X in the Hilbert scheme. Then it is defined a germ of holomorphic map $\Psi: (Hilb^n, [X]) \to (Def(X), 0)$ and reasoning as in the previous case we see that Ψ is smooth if and only if the morphism of functors $Def_{X/\mathbb{P}^n} \to Def_X$ is smooth where $Def_{X/\mathbb{P}^n}$ is the functor of infinitesimal embedded deformations of X in $\mathbb{P}^n$.

2. Geometric interpretation of first order deformations.

For every singularity $(X,0) \subset (\mathbb{C}^n,0)$ defined by the ideal $I_X \subset \mathcal{O}_n$ $I_X = (f_1,...,f_r)$ there exists an isomorphism of vector spaces between $H^0(N_X) = \mathrm{Hom}(I_X/I_X^2, \mathcal{O}_X)$ and $T^1(X/\mathbb{C}^n)$ the space of first order embedded deformations.

We briefly recall here how this isomorphism is defined (for details [Ar3], [Laz]).

$T^1(X/\mathbb{C}^n)$ is the set of ideals $J \subset \mathcal{O}_n[\epsilon]$ $\epsilon^2 = 0$ satisfying the condition

(2.1) J is flat over $\mathbb{C}[\epsilon]$ and $J \otimes_{\mathbb{C}[\epsilon]} \mathbb{C} = I_X$.

Using a flatness criterion we see that (2.1) is equivalent to the existence of $g_1,..,g_r \in \mathcal{O}_n$ such that

(i) $f_i + \epsilon g_i$ $i = 1,...,r$ generate J.

(ii) For every relation $\sum r_i f_i = 0$ we have $\sum r_i g_i \in I_X$.

Moreover if the g_i's satisfy (i) and (ii) and $h_i \in \mathcal{O}_n$ then J is generated by $f_i + \epsilon h_i$ if and only if $g_i - h_i \in I_X$ for every $i = 1,..,r$.

Thus to every ideal $J = (f_i + \epsilon g_i)$ we associate the map $\phi: I_X/I_X^2 \to \mathcal{O}_X$ $\phi(f_i) \equiv g_i \, mod(I_X)$.

The natural morphism of functors $Def_{X/\mathbb{C}^n} \to Def_{(X,0)}$ is smooth ([Ar3] pag. 4), in particular the linear map $T^1(X/\mathbb{C}^n) \xrightarrow{\nu} T^1(X,0)$ is surjective and, with the above identification, there exists an exact sequence

$$Der_{\mathbb{C}}(\mathcal{O}_X,\mathcal{O}_X) \longrightarrow Der_{\mathbb{C}}(\mathcal{O}_n,\mathcal{O}_X) \xrightarrow{d^\vee} \mathrm{Hom}_{\mathcal{O}_X}(I_X/I_X^2,\mathcal{O}_X) \xrightarrow{\nu} T^1(X) \longrightarrow 0$$

If X is reduced the cokernel of $d^\vee$ is naturally isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$, if we think Ext^1 as the space of extensions of modules the morphism

$\mathrm{Hom}_{\mathcal{O}_X}(I_X/I_X^2,\mathcal{O}_X) \xrightarrow{\nu'} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$ is defined as follows:

Given $\phi: I_X/I_X^2 \to O_X$ there exists a commutative diagram with exact rows

$$\begin{array}{ccccccc} I_X/I_X^2 & \xrightarrow{d} & \Omega^1_{\mathbb{C}^n} \otimes \mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow 0 \\ \downarrow{\scriptstyle\phi} & & \downarrow & & \| & \\ \mathcal{O}_X & \xrightarrow{\alpha} & E & \longrightarrow & \Omega^1_X & \longrightarrow 0 \end{array}$$

where E is the push-out of ϕ and d. The kernel of α is supported in the singular locus of X and then since $\mathcal{O}_X$ is torsion free α is injective and the second row is the extension $\nu'(\phi)$.

Note that if $\phi(f_i) \equiv g_i$ $mod(I_X)$ and $Z \subset \mathbb{C}^n \times D$ is defined by $f_i(z_1,..,z_n) + \epsilon g_i(z_1,..,z_n) = 0$ $i = 1,..,r$ then

$$\Omega^1_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_X = \frac{\mathcal{O}_X[dz_1,...,dz_n,d\epsilon]}{(df_i + g_i d\epsilon)}$$

is exactly the push out of ϕ and d, and the isomorphism $T^1(X,0) = Ext^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$ is given by associating to every first order deformation $Z \to D$ of X the isomorphism class of the extension (exact sequence of differentials associated to the inclusion $X \subset Z$)

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_Z \otimes \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0 \tag{2.2}$$

The same isomorphism $T^1(X) = Ext^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$ holds for every reduced complex space X (cf. [Fl]). This follows essentially from the fact that (2.2) is well defined and that for any open covering $X = \cup U_i$ to give a first order deformation of X (resp.: an extension of Ω^1_X) is the same to give first order deformations of U_i (resp.: extensions of $\Omega^1_{U_i}$) and isomorphisms in the intersections U_{ij} satisfying the cocycle condition.

In other words there exists an exact commutative diagram with vertical isomorphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\theta_X) & \xrightarrow{ltriv} & T^1(X) & \xrightarrow{r} & H^0(\mathcal{T}^1_X) \\ & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(\theta_X) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) & \longrightarrow & H^0(\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)) \end{array}$$

The second row is the Ext spectral sequence, the image of $ltriv$ is the set of locally trivial deformations of X, $\mathcal{T}^1_X$ is the sheaf of local deformations (cf. also [Pa]) and r is the natural restriction map.

Example 2.3. If $(X,0) = \{f(z_1,...,z_n) = 0\}$ $f \in \mathcal{O}_n$ is an isolated hypersurface singularity then $T^1(X,0) = \mathcal{O}_X/Jac(f)$ where $Jac(f)$ is the ideal generated by all partial derivatives of f.

Thus if $g_1,...,g_\tau \in \mathcal{O}_n$ induce a basis of $\mathcal{O}_X/Jac(f)$ then the singularity $\tilde{X} \subset \mathbb{C}^n \times \mathbb{C}^\tau$ defined by $f + \sum \alpha_i g_i = 0$ is the semiuniversal deformation of X.

3. Simultaneous resolution of rational double points

Probably the best way to begin this section is to recalling the famous Atiyah construction. The affine variety $V \subset \mathbb{C}^4$ of equation $xy + z^2 = t^2$ can be considered as a flat family V_t of surfaces such that V_t is smooth for $t \neq 0$ and V_0 has an ordinary double point.

Let l_0, l_1 be homogeneous coordinates on $\mathbb{P}^1$ and consider $Y \subset \mathbb{C}^4 \times \mathbb{P}^1$ defined by equations

$$l_0(z+t) = l_1 x \qquad l_0 y = l_i(z-t)$$

The projection on the first factor gives a surjective map $Y \to V$ and it is easily verified that for every t, $Y_t \to V_t$ is the minimal resolution of singularities. In particular in Y_0 there is a (-2)-curve which doesn't appear in the other fibres.

We shall say that a family X_t, $t \in T$, i.e. a flat map $f\colon X \to T$, of normal surfaces admits a simultaneous resolution if there exists a complex space Y and a proper map $Y \to X$ such that the composition $Y \to T$ is flat and $Y_t \to X_t$ is the minimal resolution of singularities for every $t \in T$.

Note that if $T' \to T$ is a holomorphic map and $X \to T$ admits a simultaneous resolution then the induced family $X \times_T T' \to T'$ admits too. Therefore from Atiyah construction it follows that if $(X,0) \to (\mathbb{C}_t, 0)$ is a deformation of the RDP of type A_1 then the induced deformation $(X',0) \to (\mathbb{C}_s, 0)$, $t = s^2$, admits a simultaneous resolution.

This result has been generalized by Brieskorn and Tyurina [Ty2] to all rational double points and by O. Riemenschneider to cyclic singularities, the main result they proved is:

Theorem 3.1.(Brieskorn-Tyurina-Riemenschneider) *Let* $X \to T$ *be a flat family of normal surfaces each one with a finite number of singular points which are rational double points or cyclic triple points and such that the restriction* $\cup_t Sing(X_t) \to T$ *is proper.*
Then for every $p \in T$ *there exists a neighbourhood* $p \in U \subset T$ *and a finite surjective map* $U' \to U$ *such that the induced family* $X' \to U'$ *admits a simultaneous resolution.*

Proof. See [Ty2], [Rie] pag. 234. □

We note that the change of base is unavoidable, for example it is possible to prove that if $(X,0) \to (S,0)$ is a deformation of a rational double point $(X_0,0)$ admitting simultaneous resolution then the Kodaira-Spencer map $T_0S \to T^1(X_0,0)$ is zero ([B-W] 1.15).
Brieskorn-Tyurina results on simultaneous resolutions find useful application in deformation theory of surfaces of general type.
Given a smooth minimal surface of general type S its canonical ring is by definition

$$R = \oplus_{n\geq 0} R(n) = \oplus_{n\geq 0} H^0(K_S^{\otimes n})$$

After Bombieri work ([Bom]) it is known that R is a finitely generated $\mathbb{C}$-algebra and the surface $X = Proj(R)$ is called the canonical model of S. Moreover if $S \xrightarrow{f_n} X_n \subset \mathbb{P}(H^0(K_S^{\otimes n}))$ is the n-canonical map then for every $n \geq 5$, $X_n \simeq X$ is a normal projective surface with at most rational double points and f_n is the minimal resolution of singularities.
We can generalize this result to every deformation $S_A \to Spec(A)$ over the spectrum of a local Artinian $\mathbb{C}$-algebra A. It is defined the relative canonical ring

$$R_A = \oplus_{n\geq 0} R_A(n) = \oplus_{n\geq 0} H^0(K_A^{\otimes n})$$

where $K_A = K_{S_A/Spec(A)} = \bigwedge^2 \Omega^1_{S_A/Spec(A)}$ is the relative canonical line bundle. Then we have

Lemma 3.2. *R_A is a finitely generated A-algebra.*

Proof. We first note that for every n the A-module $R_A(n)$ is finite since the map $S_A \to Spec(A)$ is proper and it is sufficient to prove that the subalgebra $R'_A = \oplus_{n\neq 1} R_A(n)$ is finitely generated.
Since $H^1(K_S^{\otimes n}) = 0$ for $n > 1$ ([B-P-V] VII.5.5) it is easy to see that there exist homogeneous elements $f_1, ..., f_N \in R'_A$ such that their restriction to S generate $R' = \oplus_{n\neq 1} R(n)$. Denoting by $S_A \subset R_A$ the subalgebra generated by $f_1, ..., f_N$ we want to prove that $S_A = R'_A$.
By induction on the length of A we can assume that $S_B = R'_B$ for every small extension

$$0 \longrightarrow \mathbb{C}\epsilon \longrightarrow A \longrightarrow B \longrightarrow 0$$

By flatness there exists for every n an exact sequence of sheaves

$$0 \longrightarrow \epsilon K_A^{\otimes n} \longrightarrow K_A^{\otimes n} \longrightarrow K_B^{\otimes n} \longrightarrow 0$$

and an isomorphism $\epsilon K_A^{\otimes n} \simeq K_S^{\otimes n}$ commuting with the multiplication $K_A^{\otimes n} \xrightarrow{\epsilon} \epsilon K_A^{\otimes n}$ and the restriction map $K_A^{\otimes n} \to K_S^{\otimes n}$.

Taking global sections for $n > 1$ we get

$$0 \longrightarrow \epsilon R_A(n) \longrightarrow R_A(n) \xrightarrow{p} R_B(n)$$

Since $p(S_A(n)) = R_B(n)$ we have $\epsilon R_A(n) = \epsilon R_B(n) = \epsilon S_A(n) \subset S_A(n)$ and then $S_A(n) = R_A(n)$. □

The relative canonical model $X_A \to Spec(A)$ is then defined as $Proj_A(R_A)$. From the proof of lemma 3.2 it follows moreover that if $f_0, ..., f_N \in R_A(n)$ restrict to a basis of $R(n)$ then they generate the free (by Nakayama) A-module $R_A(n)$ and for n sufficiently large the relative canonical model is the image of the map $S_A \xrightarrow{(f_0,..,f_N)} \mathbb{P}_A^N$.

Lemma 3.3. *The relative canonical model X_A is flat over $T = Spec(A)$.*

Proof. We consider X_A as the image of the map $f = (f_0, ..., f_N)$ for a given basis of the A-module $R_A(n)$ $n >> 0$ and denote by $U_i = \{x \in S | f_i(x) \neq 0\}$.

Since $H^0(U_i, \mathcal{O}_{S_A})$ is A-flat (immediate consequence of the Cech cochain resolution over a finite affine cover of U_i, (cf. also [Wa4] 0.4)) then also the sheaf $f_* \mathcal{O}_{S_A}$ is A-flat and it is enough to prove that the natural map $\mathcal{O}_{X_A} \xrightarrow{\alpha} f_* \mathcal{O}_{S_A}$ is surjective. In fact if H_A is the kernel of α then the A-flatness of $f_* \mathcal{O}_{S_A}$ implies that $H_A \otimes_A \mathbb{C} = 0$ and then $H_A = 0$ by Nakayama.

We know that $f_* \mathcal{O}_S = \mathcal{O}_X$ and then the functions $\dfrac{f_j}{f_i}$ generate the $\mathbb{C}$-algebra $H^0(U_i, \mathcal{O}_S)$. X has at most rational singularities and by Leray spectral sequence $H^1(U_i, \mathcal{O}_S) = 0$, working exactly as in the proof of lemma 3.2 it follows that $\dfrac{f_j}{f_i}$ generate $H^0(U_i, \mathcal{O}_{S_A})$. □

The relative canonical model defines a morphism of functors of Artin rings $\beta \colon Def_S \to Def_X$ which is exactly the blow-down morphism of ([B-W] 2.3) defined by the property $f_* \mathcal{O}_{S_A} = \mathcal{O}_{X_A}$.

The morphism β extends to convergent deformations, in fact given a deformation $\tilde{S} \xrightarrow{f} T$ of S over a germ of complex space $(T, 0)$ and an integer $n \geq 5$ the sheaf $f_* K_{\tilde{S}/T}^{\otimes n}$ is locally free ([B-S] 3.3.9) and a system of free generators of it gives (possibly shrinking T) a map $\tilde{S} \to T \times \mathbb{P}^N$.

The flatness of the image $\tilde{X} \subset T \times \mathbb{P}$ follows from infinitesimal flatness (lemma 3.3) and Th. 22.3 of [Mat1].

Thus the map β is induced by a unique holomorphic map $\beta \colon Def(S) \to Def(X)$. From Brieskorn-Tyurina results on simultaneous resolution it follows that β is a finite surjective map.

The blow down map can be defined also in the following situation ([B-W]). Let V be a normal projective surface and let $\{V_i\}$ $i = 1, .., n$ be affine open subset of V such that every V_i contain exactly a singular point p_i which is a rational double point.

If $f\colon X \to V$ is the minimal resolution of $p_1, ..., p_n$ and $X_i = f^{-1}(V_i)$ there exists blow down maps $\beta\colon Def_X \to Def_V$, $\beta_i\colon Def_{X_i} \to Def_{V_i}$ with $d\beta_i = 0$ and a commutative diagram

$$\begin{array}{ccc} Def_X & \longrightarrow & \times Def_{X_i} \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\times\beta_i} \\ Def_V & \stackrel{r}{\longrightarrow} & \times Def_{V_i} \end{array}$$

where r is the natural restriction map. The main result ([B-W] 2.6) is

Theorem 3.4. *In the notation above Def_X is the fiber product of Def_V and $\times Def_{X_i}$. In particular $T^1(X) = \bigoplus_i T^1(X_i) \oplus \ker dr$ and Def_X is smooth if and only if r is smooth.*

A first consequence of theorem 3.4 is that for every minimal surface of general type S with singular canonical model X the blow-down morphism $\beta\colon Def(S) \to Def(X)$ is not an isomorphism.

4. Normal bidouble covers of surfaces and their natural deformations

For every point q in an algebraic variety X denote by $M_{q,X}$ the maximal ideal of the local ring of functions and by $T_{q,X} = (M_{q,X}/M_{q,X}^2)^\vee$ the Zariski tangent space at q.

Let X be a smooth algebraic surface and let $\pi\colon Y \to X$ be a Galois covering with group $G = (\mathbb{Z}/2\mathbb{Z})^2 = \{1, \sigma_1, \sigma_2, \sigma_3\}$. We assume that Y is a normal surface.

Let R_i be the divisorial part of $Fix(\sigma_i) = \{p \in Y | \sigma_i(p) = p\}$ and $D_i = \pi(R_i)$. By purity of branch locus the Weil divisor $R = R_1 \cup R_2 \cup R_3$ is the set of points where π is branched.

Since Y is normal the direct image sheaf $\pi_*\mathcal{O}_Y$ is locally free and we have a character decomposition

$$\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus (\oplus_i \mathcal{O}_X(-L_i))$$

where L_1, L_2, L_3 are line bundle on X and $\mathcal{O}_X \oplus \mathcal{O}_X(-L_i)$ is the σ_i-invariant subsheaf of $\pi_*\mathcal{O}_Y$.

We have (cf. [Ca1] §2)

$$D_k + L_k \equiv L_i + L_j \qquad 2L_i \equiv D_j + D_k \qquad \{i,j,k\} = \{1,2,3\}$$

where $\equiv$ means rational equivalence. If V is the vector bundle $L_1 \oplus L_2 \oplus L_3$ with fibres coordinates w_1, w_2, w_3, then we can realize Y in V as the zero locus of the ideal sheaf $I_Y \subset \mathcal{O}_V$ generated by the six equations

$$\begin{cases} w_i^2 - x_j x_k = 0 \\ w_k x_k - w_i w_j = 0 \end{cases} \{i,j,k\} = \{1,2,3\} \tag{4.1}$$

where $x_i \in H^0(\mathcal{O}_X(D_i))$ is a section defining D_i.

All these facts are proved in [Ca1], Catanese suppose that Y is a smooth surface but his proof is also valid in our more general situation. It's moreover easy to see that Y is smooth if and only if the curves D_i are smooth and the divisor $D = D_1 \cup D_2 \cup D_3$ has only ordinary double points as singularities (cf. also [Par]).

G acts on the fibres of V in the following way:

$$\sigma_i : w_i \to w_i \quad w_j \to -w_j \quad w_k \to -w_k$$

and R_i is the subset of Y defined by $x_i = w_j = w_k = 0$.

Proposition 4.2. *In the notation above are equivalent:*

a) $D_1 \cap D_2 \cap D_3 = \emptyset$

b) R_i *is a Cartier divisor for every* i

c) $\dim T_{q,Y} \le 4$ *for every* $q \in Y$

d) Y *is locally complete intersection in* V.

Proof. *a)* $\Rightarrow$ *d)* If $q \in Y$, $p = \pi(q)$ and $x_k(p) \neq 0$ then Y is locally defined by

$$\begin{cases} w_k = \dfrac{w_i w_j}{x_k} \\ w_i^2 = x_j x_k \\ x_i = \dfrac{w_j^2}{x_k} \end{cases} \tag{4.3}$$

$a \Rightarrow b)$, The ideal of R_i is generated by (w_j, w_k, x_i) and if, for example $q \in R_i$ $x_k(\pi(q)) \neq 0$ then from (4.3) it follows that the ideal or R_i is generated in Y by w_j.

b) $\Rightarrow$ *c)* If $q \notin R$ then $\dim T_{q,Y} = 2$. Suppose $q \in R_i$ and $\dim T_{q,Y} = 5$, then w_j, w_k are linearly independent in $T_{q,Y}^\vee$ and the ideal (w_j, w_k, x_i) cannot be principal at q.

c) $\Rightarrow$ *a)* If $q \in Y$ and $x_1(q) = x_2(q) = x_3(q) = 0$ then all the equation that define Y are in $M_{q,V}^2$, hence $T_{q,Y} = T_{q,V}$.

d) $\Rightarrow$ *a)* Take a point $q \in Y$ such that $x_i(q) = 0$ $i = 1,2,3$ and let's suppose $I_{Y,q} = (f_1, f_2, f_3)$, this will lead to a contradiction. Since the ideal of Y at q is contained in M^2, (here $M = M_{q,V}$), the vector subspace of M^2/M^3 generated by $I_{Y,q}$ has dimension at most equal to three, but it easy to see that the six equations (4.1) are linearly independent in M^2/M^3. □

Since in the applications we are principally interested to the case where Y has at most rational double points, from now on we always assume that $D_1 \cap D_2 \cap D_3 = \emptyset$.

Let $N_Y = (I_Y/I_Y^2)^\vee$ be the normal sheaf and let $p_i : \mathcal{O}_Y \to \mathcal{O}_{R_i}$ be the projection map.

Theorem 4.4. *If* $D_1 \cap D_2 \cap D_3 = \emptyset$ *then there exists a commutative diagram of* $\mathcal{O}_Y$*-modules with exact rows and columns.*

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \pi^*V & == & \pi^*V & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \theta_Y & \longrightarrow & \theta_V \otimes \mathcal{O}_Y & \xrightarrow{\eta} & N_Y & \xrightarrow{\mu} & \mathcal{T}_Y^1 & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \| & & \\
0 & \longrightarrow & \theta_Y & \xrightarrow{\alpha} & \pi^*\theta_X & \xrightarrow{\beta} & \oplus_i \mathcal{O}_{R_i}(\pi^*D_i) & \xrightarrow{\gamma} & \mathcal{T}_Y^1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array} \tag{4.5}$$

The proof of theorem 4.4 will be a consequence of the following two lemmas. We first note that $\theta_Y = Der(\mathcal{O}_Y, \mathcal{O}_Y), \pi^*\theta_X = Der(\pi^{-1}\mathcal{O}_X, \mathcal{O}_Y)$ and α is defined in the obvious way. Moreover α is an injective map because π is a finite morphism.

If u_1, u_2 are local coordinates on X we set

$$\varphi(\frac{\partial}{\partial w_i}) = 0, \varphi(\frac{\partial}{\partial u_i}) = \frac{\partial}{\partial u_i}$$

It's clear that $\pi^*V = \ker\varphi$. The upper row is a standard exact sequence [Ar3].

Lemma 4.6. *There exists a commutative diagram*

$$\begin{array}{ccccc} & \theta_V \otimes \mathcal{O}_Y & \xrightarrow{\eta} & \mathcal{H}om\mathcal{O}_Y I_Y/I_Y^2\mathcal{O}_Y = N_Y & \\ & \downarrow \varphi & & \downarrow \psi_i & \\ \pi^*\theta_X = & Der(\pi^{-1}\mathcal{O}_X, \mathcal{O}_Y) & \xrightarrow{\beta_i} & \mathcal{H}om\mathcal{O}_Y\pi^*\mathcal{O}_X(-D_i)\mathcal{O}_{R_i} & = \mathcal{O}_{R_i}(\pi^*D_i) \end{array} \tag{4.7}$$

Proof. For every $a \in Der(\pi^{-1}\mathcal{O}_X, \mathcal{O}_Y)$ and $f \in \mathcal{O}_X(-D_i)$ we define $\beta_i(a)(f) = p_i(a(f))$ and then we extend by $\mathcal{O}_Y$-linearity. β is a well defined map and $\beta_i \circ \alpha = 0$ since $\pi^*D_i = 2R_i$.
Let $r \in \mathcal{O}_Y$ be a local equation of R_i, if $f \in \mathcal{O}_X(-D_i)$ then $f \in I_Y + (r^2)$ and we can write $f = a + br^2$ with $a \in I_Y$. For $v \in N_Y$ we then define $\psi_i(v)(f) = p_i(v(a))$.
If s is another local equation of R_i and $f = c + ds^2$ then $p_i(v(a-c)) = 0$. In fact we have $s = hr + e$ with $e \in I_Y$ and $a - c = ds^2 - br^2 = r(dh^2r + 2dhe - br) + de^2$, since I_Y is a prime ideal necessarily $dh^2r + 2dhe - br \in I_Y$ and then $p_i(v(a-c)) = 0$.
In order to showing that (4.7) commutes it suffices to note that, if for example $x_k \neq 0$, then w_j is a local equation of R_i and $\psi_i(v)(x_i) = p_i(v(x_i - \frac{w_j^2}{x_k}))$. Thus

$$\psi_i(\frac{\partial}{\partial w_h})(x_i) = 0 \qquad h = 1,2,3$$

$$\psi_i(\frac{\partial}{\partial u_h})(x_i) = p_i(\frac{\partial x_i}{\partial u_h}) = \beta_i(\frac{\partial}{\partial u_h})(x_i) \qquad h = 1,2$$

□

Define $\beta = \oplus_i\beta_i$, $\psi = \oplus_i\psi_i$.

Lemma 4.8. *ψ is a surjective map and* $\ker\psi = \eta(\ker\varphi)$, *in particular* $\ker\psi \subset \ker\mu$ *and we can define γ as in (4.5).*

Proof. By lemma 4.6 $\eta(\ker\varphi) \subset \ker\psi$.
If $\psi(v) = 0$ and $x_k \neq 0$ then locally I_Y/I_S^2 is a free $\mathcal{O}_Y$-module generated by $(x_i - \frac{w_j^2}{x_k})$, $(x_j - \frac{w_i^2}{x_k}), (w_k - \frac{w_iw_j}{x_k})$. Moreover $v(x_i - \frac{w_j^2}{x_k}) = w_jh_i, v(x_i - \frac{w_j^2}{x_k}) = w_ih_j$.
if we set

$$v' = v + \frac{h_ix_k}{2}\frac{\partial}{\partial w_j} + \frac{h_jx_k}{2}\frac{\partial}{\partial w_i}$$

then $v'(x_i - \frac{w_j^2}{x_k}) = 0, v'(x_i - \frac{w_j^2}{x_k}) = 0$, $v'(w_k - \frac{w_iw_j}{x_k}) = h_k$ then $v' - h_k\frac{\partial}{\partial w_k} = 0$. □

Y is locally complete intersection in V, therefore there is an exact sequence

$$0 \longrightarrow I_Y/I_Y^2 \longrightarrow \Omega_V^1 \otimes \mathcal{O}_Y \longrightarrow \Omega_Y^1 \longrightarrow 0 \tag{4.9}$$

If we apply the functor $\mathcal{H}om$ we get the upper row of (4.5), if we apply Hom we get the exact sequence

$$H^0(\theta_V \otimes \mathcal{O}_Y) \xrightarrow{H^0(\eta)} H^0(N_Y) \xrightarrow{k} T_Y^1$$

If we apply the left exact functor H^0 to (1.3) we see that $\ker H^0(\psi) \subset Im H^0(\eta) = \ker k$ and there exist a map $\epsilon: H^0(\oplus_i \mathcal{O}_{R_i}(\pi^* D_i)) \to T^1(Y)$ such that $\epsilon \circ H^0(\psi) = k$.

Corollary 4.10. *If $H^1(\pi^*\theta_X) = 0$ then ϵ is surjective.*

Proof. If $\mathcal{F} = \ker \gamma$ then there exists a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
H^0(\pi^*\theta_X) & \longrightarrow & H^0(\mathcal{F}) & \longrightarrow & H^1(\theta_Y) & \longrightarrow & H^1(\pi^*\theta_X) \\
\| & & \downarrow & & \downarrow & & \\
H^0(\pi^*\theta_X) & \xrightarrow{H^0(\beta)} & H^0(\oplus_i \mathcal{O}_{R_i}(\pi^* D_i)) & \xrightarrow{\epsilon} & T^1(Y) & & \\
 & & \downarrow & & \downarrow & & \\
 & & H^0(\mathcal{T}_Y^1) & == & H^0(\mathcal{T}_Y^1) & & \\
 & & \downarrow & & \downarrow & & \\
H^1(\pi^*\theta_X) & \longrightarrow & H^1(\mathcal{F}) & \longrightarrow & H^2(\theta_Y) & &
\end{array} \tag{4.11}$$

where the right column is the first part of the cotangent spectral sequence. The conclusion follows by chasing through the diagram. □

We note that $\pi_* \mathcal{O}_{R_i} = \mathcal{O}_{D_i} \oplus \mathcal{O}_{D_i}(-L_i)$ and

$$H^0(\oplus_i \mathcal{O}_{R_i}(\pi^* D_i)) = \oplus_i (H^0(\mathcal{O}_{D_i}(D_i)) \oplus H^0(\mathcal{O}_{D_i}(D_i - L_i)))$$

moreover $H^1(\pi^*\theta_X) = H^1(\theta_X) \oplus (\oplus_i H^1(\theta_X(-L_i)))$.

More generally we can include the map ϵ into an exact sequence of cohomology groups, this can be done as follows. One first prove that $\Omega_{Y/X}^1 = \oplus_i \mathcal{O}_{R_i}(-R_i)$, then one consider the exact sequence

$$0 \longrightarrow \pi^*\Omega_X^1 \longrightarrow \Omega_Y^1 \longrightarrow \oplus_i \mathcal{O}_{R_i}(-R_i) \longrightarrow 0 \tag{4.12}$$

(recall that $\pi^*\Omega_X^1$ is locally free and $(\pi^*\Omega_X^1)^\vee = \pi^*\theta_X$). Applying the functor $\mathrm{Hom}_{\mathcal{O}_Y}(-, \mathcal{O}_Y)$ we get a long exact sequence

$$0 \longrightarrow H^0(\theta_Y) \longrightarrow H^0(\pi^*\theta_X) \longrightarrow \oplus_i \mathrm{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_{R_i}(-R_i), \mathcal{O}_Y) \longrightarrow T_Y^1 \longrightarrow H^1(\pi^*\theta_X) \longrightarrow \dots \tag{4.13}$$

Since R_i is a Cartier divisor its local equation is a regular element of $\mathcal{O}_Y$, using local commutative algebra ([Mat1] §18 lemma 2) we have for every $i \geq 0$

$$\mathcal{E}xt^{i+1}_{\mathcal{O}_Y}(\mathcal{O}_{R_i}(-R_i), \mathcal{O}_Y) = \mathcal{E}xt^i_{\mathcal{O}_{R_i}}(\mathcal{O}_{R_i}(-R_i), \mathcal{O}_{R_i}(R_i)) = \begin{cases} 0 & \text{if } i > 0 \\ \mathcal{O}_{R_i}(\pi^* D_i) & \text{if } i = 0 \end{cases}$$

and (4.13) becomes

$$H^0(\pi^*\theta_X)\longrightarrow \oplus_i H^0(\mathcal{O}_{R_i}(\pi^* D_i))\xrightarrow{\epsilon} T^1(Y)\xrightarrow{\sigma} H^1(\pi^*\theta_X)\longrightarrow \oplus_i H^1(\mathcal{O}_{R_i}(\pi^* D_i)) \qquad (4.14)$$

Let $Def(V/Y)$ be the space of embedded deformations of Y in V. It's well known that the natural map $\hat{k}\colon Def(V/Y) \to Def(Y)$ is holomorphic and its differential is $k\colon H^0(N_Y) \to T^1(Y)$.

In a neighbourhood of 0 is defined an analytic map

$$\xi\colon H = \oplus_i (H^0(\mathcal{O}_X(D_i)) \oplus H^0(\mathcal{O}_X(D_i - L_i))) \to Def_V(Y)$$

where $\xi(y_i, \gamma_i)$ is the surface in V defined by:

$$\begin{cases} w_i^2 = (x_j + y_j + \gamma_j w_j)(x_k + y_k + \gamma_k w_k) \\ w_j w_k = w_i(x_i + y_i + \gamma_i w_i) \end{cases} \qquad (4.15)$$

Definition . *We shall call the deformation of Y defined in* (4.15) *a natural deformation.*

Lemma 4.16. *Let $d\xi\colon \oplus_i (H^0(\mathcal{O}_X(D_i)) \oplus H^0(\mathcal{O}_X(D_i - L_i))) \to H^0(N_Y)$ be the differential of ξ. Then $H^0(\psi) \circ d\xi = \varrho$ where*

$$\varrho\colon \oplus_i (H^0(\mathcal{O}_X(D_i)) \oplus H^0(\mathcal{O}_X(D_i - L_i))) \to \oplus_i (H^0(\mathcal{O}_{D_i}(D_i)) \oplus H^0(\mathcal{O}_{D_i}(D_i - L_i)))$$

is the restriction map.

The proof is a straightforward verification and it is left to the reader.

If $H^1(\mathcal{O}_Y) = 0$ then $H^1(\mathcal{O}_X) = H^1(\mathcal{O}_X(-L_i)) = 0$ and ϱ is surjective, the kernel of ϵ has dimension $h^0(\pi^*\theta_X) - h^0(\pi^*\theta_Y)$ and since the parameter space H of natural deformations is smooth we have finally

Proposition 4.17. *If $H^1(\mathcal{O}_Y) = H^1(\pi^*\theta_X) = 0$ then $k \circ d\xi = \epsilon \circ \mathcal{H}^0(\psi) \circ d\xi = \epsilon \circ \varrho$ is surjective, the map $\hat{k} \circ \xi$ is smooth and $Def(Y)$ is smooth of dimension*

$$\sum_i (h^0(\mathcal{O}_X(D_i)) + h^0(\mathcal{O}_X(D_i - L_i)) - 1) - h^0(\pi^*\theta_X) + h^0(\theta_Y)$$

We remark that if the minimal resolution of Y is of general type then the group of automorphisms of Y is finite [Mat2] and $H^0(\theta_Y) = 0$.

Remark. . If $H^1(\pi^*\theta_X) \neq 0$ (this is true in particular if $H^1(\theta_X) \neq 0$) then in general ϵ is not surjective; in this case it may be useful to know $\operatorname{Im}\epsilon = \ker\sigma$. An exact sequence where σ appears is the following due to Ziv Ran [Ran]

$$\ldots \longrightarrow T^1_\pi \longrightarrow T^1(X) \oplus T^1(Y) \xrightarrow{\sigma'} \operatorname{Ext}^1_\pi(\Omega^1_X, \mathcal{O}_Y) \longrightarrow T^2_\pi \longrightarrow \ldots$$

where T^1_π is the space of first order deformation of the map π and $\operatorname{Ext}^n_\pi(\Omega^1_X, \mathcal{O}_Y)$ is defined as the limit of the spectral sequence $E_2^{p,n-p} = \operatorname{Ext}^p_{\mathcal{O}_Y}(L^{n-p}\pi^*\Omega^1_X, \mathcal{O}_Y)$. It is clear that in our case $\operatorname{Ext}^n_\pi(\Omega^1_X, \mathcal{O}_Y) = H^n(\pi^*\theta_X)$ and $\sigma(x) = \sigma'(0, x)$.

5. Stability of simple bihyperelliptic surfaces

In this section we apply the computation of §4 to a particular class of surfaces.

Denote $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{O}_X(a,b)$ be the line bundle on X whose sections are bihomogeneous polynomials of bidegree a, b. A minimal surface of general type is said to be simple bihyperelliptic of type $(a,b)(n,m)$ if its canonical model is defined in $\mathcal{O}_X(a,b) \oplus \mathcal{O}_X(n,m)$ by the equation

$$z^2 = f(x,y) \qquad w^2 = g(x,y) \tag{5.1}$$

where f, g are bihomogeneous polynomials of respective bidegree $(2a,2b),(2n,2m)$.

Let S be a simple bihyperelliptic surface of type $(a,b)(n,m)$ with $a,b,n,m \geq 3$ and let $\delta: S \to Y$ be the pluricanonical map onto its canonical model Y. Let (5.1) be the equation of Y.

In Y we have the following exact sequence (cf. (4.11)):

$$0 \longrightarrow H^1(\theta_Y) \longrightarrow T^1(Y) \longrightarrow H^0(\mathcal{T}^1_Y) \xrightarrow{ob} H^2(\theta_Y)$$

where *ob* is the obstruction to globalize a first order deformation of the singular points of Y.

As a consequence of Proposition 4.17 we have the following.

Theorem 5.2. *In the notation above $Def(Y)$ is smooth. $Def(S)$ is smooth if and only if $ob = 0$.*

Proof. Let $\pi: Y \to X = \mathbb{P}^1 \times \mathbb{P}^1$ be the projection, then

$$\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{O}_X(-a,-b) \oplus \mathcal{O}_X(-n,-m) \oplus \mathcal{O}_X(-a-n,-b-m)$$

$$\theta_X = \mathcal{O}_X(2,0) \oplus \mathcal{O}_X(0,2) \qquad \pi_*\pi^*\theta_X = \theta_X \otimes \pi_*\mathcal{O}_Y$$

Since $a,b,n,m \geq 3$ we have $h^1(\mathcal{O}_Y) = h^1(\pi^*\theta_X) = 0$ and by Proposition 4.17 $Def(Y)$ is smooth.

Denote by L_Y (resp.: D_Y) the functor of local (resp.: global) deformations of Y, since Y has a finite number of singular points which are R.D.P.'s L_Y is smooth with finite dimensional tangent space $H^0(\mathcal{T}^1_Y)$. Since $Def(Y)$ is smooth, the natural map $\Phi: D_Y \to L_Y$ is smooth if and only if its differential $T^1_Y \to H^0(\mathcal{T}^1_Y)$ is surjective. According to 3.4 the smoothness of $Def(S)$ is equivalent to the smoothness of Φ. □

Note that since we have a surjective map $H \to T^1(Y)$, the kernel of *ob* is exactly the subspace of $H^0(\mathcal{T}^1_Y)$ generated by the natural deformations of Y.

Theorem 5.3. *Simple bihyperelliptic surfaces of type $(a,b)(n,m)$ are stable under small deformations for $a > 2n, m > 2b$.*

Proof. Let $F: \mathcal{S} \to \Delta$ be a flat family over the complex disk with $S_0 = F^{-1}(0)$ simple bihyperelliptic of type $(a,b)(n,m)$.

Let $F'\colon \mathcal{Y} \to \Delta$ be the corresponding family of canonical models, then Y_0 is a normal bidouble cover of $X = \mathbb{P}^1 \times \mathbb{P}^1$ with, in the notation of §4, $L_1 = \mathcal{O}_X(n,m)$, $L_2 = \mathcal{O}_X(a,b)$, $L_3 = \mathcal{O}_X(a+n, b+m)$, $x_1 = f$, $x_2 = g$, $x_3 = 1$.

Then, for $a,b,n,m \geq 3$, the surface Y_0 satisfies the hypothesis of Proposition 4.17 and we can assume, possibly shrinking Δ, that F' is a natural deformation of Y_0.

The natural deformations of Y_0 are defined in $\mathcal{O}_X(a,b) \oplus \mathcal{O}_X(n,m)$ by

$$\begin{cases} z^2 = f'(x,y) + w\varphi(x,y) \\ w^2 = g'(x,y) + z\psi(x,y) \end{cases}$$

where $f' \in H^0(\mathcal{O}_X(2a,2b))$, $g' \in H^0(\mathcal{O}_X(2n,2m))$, $\varphi \in H^0(\mathcal{O}_X(2a-n, 2b-m))$, $\psi \in H^0(\mathcal{O}_X(2n-a, 2m-b))$. If $a > 2n$, $m > 2b$ then $\varphi = \psi = 0$ and the lemma is proved. $\square$

Example 5.4. Suppose $a > 2n$, $m > 2b$ and let (5.1) be the equations of Y. Denote $D_1 = div(f)$, $D_2 = div(g)$ and suppose moreover that $Sing(D_i) \cap D_j = \emptyset, \quad \{i,j\} = \{1,2\}$ and let $p \in D_1$ be a singular point.

Then $\pi^{-1}(p)$ contains exactly two singular points q_1, q_2 of Y and there exists an involution $\sigma \in G$ such that $\sigma(q_1) = q_2$. σ extends to every natural deformation, in particular every global deformation of Y gives by restriction isomorphic local deformations of (Y, q_1) and (Y, q_2) and Φ cannot be smooth.

More generally one can prove that if $ob = 0$ then the group G must act trivially on the vector space $H^0(\mathcal{T}^1_Y)$ and this is possible only if D_1 and D_2 are both smooth divisors.

III. Normal surfaces with anticanonical divisors.

A normal projective surface X has an anticanonical divisor if $-K_X$ is linearly equivalent to a nontrivial effective Weil divisor.

Every smooth rational surface $S \neq \mathbb{P}^2$ is obtained from a Segre-Hirzebruch surface after a finite sequence of blowings up $\mu\colon S \to \mathbb{F}_d$ and since P_{-1} decrease (if $\neq 0$) after a blow up at a generic point, smooth rational surfaces with anticanonical divisors and large Picard number can be considered quite "special".

After a computation of some cohomology groups in the Segre-Hirzebruch surfaces we shall see in section 2 that the condition $P_{-1} \geq 5$ gives strong constraint on the map μ. This is used in the proof of the main result of this section (Theorem 4.4) which is a classification theorem for normal projective surfaces with $\varrho = 1$, $P_{-1} \geq 5$ and at most rational singularities.

1. Tangent and cotangent vector fields on a Segre-Hirzebruch surface

We consider the following description of the Segre-Hirzebruch surface $\mathbb{F}_q$, $q \geq 0$ (cf. [Be],[Ha1] V.2, [B-P-V] V.4).

$$\mathbb{F}_q = (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\})/\sim$$

where $(l_0, l_1, t_0, t_1) \sim (\lambda l_0, \lambda l_1, \lambda^q \mu t_0, \mu t_1)$ for any $\lambda, \mu \in \mathbb{C}^*$.

From now on by the standard torus action on $\mathbb{F}_q$ we shall mean the faithful $(\mathbb{C}^*)^2$ action given by

$$(\mathbb{C}^*)^2 \ni (\xi, \eta)\colon (l_0, l_1, t_0, t_1) \to (l_0, \xi l_1, \eta t_0, t_1)$$

$\mathbb{F}_q$ is covered by four affine planes $\mathbb{C}^2 \simeq U_{i,j} = \{l_i t_j \neq 0\}$ which are invariant for the standard torus action. In this affine covering we define local coordinates according to the following table

Table 1.1.

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$U_{0,1}$	$z = \dfrac{l_1}{l_0}$	$s = \dfrac{t_0}{t_1 l_0^q}$
$U_{0,0}$	$z = \dfrac{l_1}{l_0}$	$s' = \dfrac{t_1 l_0^q}{t_0}$
$U_{1,0}$	$z' = \dfrac{l_0}{l_1}$	$y' = \dfrac{t_1 l_1^q}{t_0}$
$U_{1,1}$	$z' = \dfrac{l_0}{l_1}$	$y = \dfrac{t_0}{t_1 l_1^q}$

We shall call z, s principal affine coordinates and $U_{0,1}$ principal affine subset. The other pairs of affine coordinates are related to s, z by

$$z' = z^{-1} \qquad s' = s^{-1} \qquad y = sz^{-q} \qquad y' = s^{-1}z^q = y^{-1}$$

The map $\mathbb{F}_q \to \mathbb{P}^1$, $(l_0, l_1, t_0, t_1) \to (l_0, l_1)$ represents the Segre-Hirzebruch surface as a rational geometrically ruled surface where $\sigma_\infty = \{t_1 = 0\}$ is the unique section with negative selfintersection $\sigma_\infty^2 = -q$, $\sigma_0 = \{t_0 = 0\}$ is a section with $\sigma_0^2 = q$ and $f = \{l_1 = 0\}$ is a fibre. It is well known (cf. [Be]) that σ_0, f are a basis of $\mathrm{Pic}(\mathbb{F}_q)$, the canonical divisor is linearly equivalent to $-\sigma_0 - \sigma_\infty - 2f$ and the rational function y gives a rational equivalence $\sigma_\infty \sim \sigma_0 - qf$.

Our description of $\mathbb{F}_q$ is particularly useful for explicit computations of cohomology groups, for later use we prove here the following

Lemma 1.2. *For every* $p, q > 0$, $r \geq 0$ $h^0(\mathbb{F}_q, \Omega^1(p\sigma_0 + r\sigma_\infty)) = qp^2 - 1$.

Proof. $H^0(\Omega^1(p\sigma_0 + r\sigma_\infty))$ is the vector space of rational cotangent vector fields having at most poles of order p and r along σ_0 and σ_∞ respectively. The standard torus action induces an eigenspaces decomposition

$$H^0(\Omega^1(p\sigma_0 + r\sigma_\infty)) = \bigoplus_{a,b\in\mathbb{Z}} M_{a,b}$$

where $\omega \in M_{a,b}$ if and only if in the open set $U_{0,1}$ we have

$$\omega = \alpha_{a,b} z^{a-1} s^b dz + \beta_{a,b} z^a s^{b-1} ds$$

for some complex numbers $\alpha_{a,b}, \beta_{a,b}$.

The same ω is written in $U_{0,0}$ as

$$\omega = \alpha_{a,b} z^{a-1} s'^{-b} dz - \beta_{a,b} z^a s'^{-b-1} ds'$$

and in $U_{1,1}$

$$\omega = -(\alpha_{a,b} + q\beta_{a,b}) z'^{-(a+1+qb)} y^b dz' + \beta_{a,b} z'^{-(a+qb)} y^{b-1} dy$$

Note that $\sigma_0 \cap U_{0,1} = \{s = 0\}$, $\sigma_\infty \cap U_{0,1} = \emptyset$, $\sigma_0 \cap U_{0,0} = \emptyset$, $\sigma_\infty \cap U_{0,0} = \{s' = 0\}$, $\sigma_0 \cap U_{1,1} = \{y = 0\}$ and $\sigma_\infty \cap U_{1,1} = \emptyset$.

From the above local description of ω it follows immediately that $\omega \neq 0 \Rightarrow b < 0$ and then there exists an isomorphism $H^0(\Omega^1(p\sigma_0 + r\sigma_\infty)) = H^0(\Omega^1(p\sigma_0))$.
By reflexivity every section of $\Omega^1(p\sigma_0)$ on $U_{0,1} \cup U_{0,0} \cup U_{1,1}$ extends to a unique section on $\mathbb{F}_q$ and then the following set of rational cotangent vector fields

$$\begin{cases} z^{a-1}s^b dz & a \geq 1, \quad 0 \geq b \geq -p, \quad a+1+qb \leq 0 \\ z^a s^{b-1} ds & a \geq 0, \quad -1 \geq b \geq 1-p, \quad a+bq < 0 \\ -qz^{a-1}s^b dz + z^a s^{b-1} ds & -1 \geq b \geq 1-p, \quad a+bq = 0 \end{cases}$$

are $qp^2 - 1$ bihomogeneous sections of $\Omega^1(p\sigma_0)$ and an easy calculation that we omit shows that they are a basis. □

With a similar, but easier, proof it is possible to prove the following well known fact ([Ko2], [Ca6])

Lemma 1.3. *A bihomogeneous basis of $H^0(\mathbb{F}_q, \theta)$ is given in the open set $U_{0,1}$ by*

$$\begin{cases} \dfrac{\partial}{\partial z}, z\dfrac{\partial}{\partial z}, z^2\dfrac{\partial}{\partial z}, s\dfrac{\partial}{\partial s} \\ z^a \dfrac{\partial}{\partial s} \qquad 0 \leq a \leq q \end{cases}$$

Corollary 1.4. *For every $p, q, r > 0$, $h^1(\mathbb{F}_q, \theta) = q-1$, $h^1(\mathbb{F}_q, \Omega^1(p\sigma_0)) = 1$ and $h^2(\mathbb{F}_q, \theta) = h^2(\mathbb{F}_q, \Omega^1(p\sigma_0)) = h^1(\mathbb{F}_q, \Omega^1(p\sigma_0 + rf)) = 0$.*

Proof. By Hodge decomposition and Serre duality $h^0(\Omega^1) = h^2(\theta(K)) = 0$, $h^2(\Omega^1) = 0$ and since both $-K$ and $p\sigma_0$ are effective divisors also $h^2(\theta)$ and $h^2(\Omega^1(p\sigma_0))$ vanish.
By Riemann-Roch and previous lemmas we then get $h^1(\theta) = q-1$ and $h^1(\Omega^1(p\sigma_0)) = 1$.
For every $p, r > 0$ it follows from standard exact sequences

$$h^1(\Omega^1(p\sigma_0 + rf)) \leq h^1(\Omega^1(\sigma_0 + f)) = h^0(\Omega^1(\sigma_0 + f)) - q$$

and using the same method used in the proof of lemma 1.2 we easily see that $z^{a-1}s^{-1}dz$, $0 \leq a \leq q-1$ is a basis of $H^0(\Omega^1(\sigma_0 + f))$ and the above r.h.s. is 0. □

We end this section by recalling the vanishing theorem of line bundles on Segre-Hirzebruch surfaces.

Proposition 1.5. *In the surface $\mathbb{F}_q$ $q > 0$ we have:*

(i) *$H^0(a\sigma_0 + bf) \neq 0$ if and only if $a \geq 0$, $\quad aq + b \geq 0$.*

(ii) *The linear system $|a\sigma_0 + bf|$ contains a reduced divisor if and only if either $a > 0, b \geq -q$ or $a = 0, b > 0$.*

(iii) *$H^1(a\sigma_0 + bf) = 0$ if and only if either $a = -1$ or $a \geq 0, b \geq -1$ or $a \leq -2, b \leq q-1$.*

(iv) *For every pair of positive integers p, r the natural map*

$$H^0(p\sigma_0) \otimes H^0(r\sigma_0) \to H^0((p+r)\sigma_0)$$

is surjective, in particular the image of $\mathbb{F}_q$ *by the complete linear system* $|\sigma_0|$ *is projectively normal.*

(v) $P_{-1}(\mathbb{F}_q) = \max(9, q+6)$.

Proof. (i) and (ii) are clear since $|\sigma_0|$, $|f|$ are base point free and $\sigma_\infty \in |\sigma_0 - qf|$.
By Serre duality it is sufficient to study the vanishing of h^1 only for $a \geq -1$.
Using standard exact sequences and induction on $|b|$ we have for every integer b

$$h^1(-\sigma_0 + bf) = h^1(-\sigma_0) = 0$$

and if $b \geq -1$, by induction on $a \geq 0$ we have

$$h^1(a\sigma_0 + bf) \leq h^1(-\sigma_0 + bf) = 0$$

If $a \geq 0$ and $b \leq -2$ then we can write $a\sigma_0 + bf = \sigma_\infty + D$ where by (i) and Serre duality $h^2(D) = 0$, thus

$$h^1(a\sigma_0 + bf) \geq h^1(\mathcal{O}_{\sigma_\infty}(a\sigma_0 + bf) = h^1(\mathcal{O}_{\mathbb{P}^1}(b)) > 0$$

In the principal affine coordinates z, s a bihomogeneous basis of $H^0(p\sigma_0)$ is given by the monomials $s^{-a}z^b$ with $0 \leq a \leq p$, $0 \leq b \leq aq$, (iv) follows immediately.
For every $q \geq 0$ we have $-K = \sigma_0 + \sigma_\infty + 2f$ and $K^2 = 8$. If $q \leq 3$ by (iii) and Serre duality $H^1(-K) = H^2(-K) = 0$ and $P_{-1} = 9$ by Riemann-Roch. If $q \geq 3$ then $-K\cdot\sigma_\infty < 0$ and $P_{-1} = h^0(\sigma_0 + 2f) = q + 6$. □

2. Curves with negative self intersection in a rational surface

Let S be a smooth rational surface, then S does not contain any irreducible curve with negative self intersection if and only if $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$. From now on, by abuse of notation we shall denote by a rational surface a rational surface different from $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$.
Let S be such a rational surface, then there exists an integer $d \geq 1$ and a birational morphism $\mu: S \to \mathbb{F}_d$ such that μ is an isomorphism in a neighbourhood of the section σ_∞ with self intersection $-d$ (cf. for example [Be]) and by abuse of notation we denote by σ_∞ also its inverse image $\mu^{-1}(\sigma_\infty)$. We note that μ is the composition of $\varrho(S) - 2$ blowings-up.
Let $p: S \to \mathbb{P}^1$ be the fibration obtained by composing μ with the natural projection $\pi: \mathbb{F}_d \to \mathbb{P}^1$.
In order to simplify the presentation of next proofs we introduce some technical notation.

$(*)$ In the situation above let $r = \varrho(S) - 2$, let h be the number of degenerate fibres of p and let e be the number of (-1)-curves contained in the fibres of p. We note that $e \geq h$ and $r = \sum(b_2(f) - 1)$ where the summation is made over all degenerate fibres f of p.

Definition . In the notation above, a smooth irreducible curve $C \subset S$ is said to be μ-transversal or simply transversal if $C\cdot f > 0$ where f is a fibre of p.

Theorem 2.1. *Let S be a rational surface, $\mu: S \to \mathbb{F}_d$ a birational morphism which is an isomorphism in a neighbourhood of σ_∞ and $C \subset S$ a transversal curve $\neq \sigma_\infty$.*
A) If $h^0(-K_S) + min\{d,3\} \geq 8$ then $C^2 \geq -1$.
B) If $h^0(\theta_S) \geq 4$ then $C^2 \geq 0$.

Since μ is a composition of blowings up the first thing to do is to understand the behavior of tangent and anticanonical sheaves under blow up.

Lemma 2.2. *Let X be a smooth surface, $x \in X$ and $\tilde{X} \xrightarrow{f} X$ the blowing up of X at x. Then $R^1 f_* \theta_{\tilde{X}} = R^1 f_* \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}) = 0$ and there exist two exact sequences of sheaves on X*

$$0 \longrightarrow f_* \mathcal{O}_{\tilde{X}}(-K_{\tilde{X}}) \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \wedge^2 T_x X \longrightarrow 0$$

$$0 \longrightarrow f_* \theta_{\tilde{X}} \longrightarrow \theta_X \longrightarrow T_x X \longrightarrow 0$$

Proof. It is possible to give an elementary proof using local coordinates at x (cf. [Ca6] Lemma 9.22) but we prefer to give here a shorter proof that makes use of Leray spectral sequence. Let $M \subset \mathcal{O}_X$ be the ideal sheaf of the point x and let $E = f^{-1}(x)$ be the exceptional curve, since $E^2 < 0$ we have $H^0(N_{E/\tilde{X}}) = 0$ and then every tangent vector field s on a neighbourhood of E is tangent to E at every point $p \in E$, in particular it is well defined its direct image $f_* s \in H^0(M\theta_X)$ and there exists an exact sequence

$$0 \longrightarrow f_* \theta_{\tilde{X}} \longrightarrow \theta_X \longrightarrow V \longrightarrow 0$$

where V is a complex vector space of dimension ≥ 2.
Thus by Leray spectral sequence $\chi(\theta_X) = \chi(\theta_{\tilde{X}}) + \dim V + \dim R^1 f_* \theta_{\tilde{X}}$ and applying this formula to $X = \mathbb{P}^2, \tilde{X} = \mathbb{F}_1$ we get $\dim V + \dim R^1 f_* \theta_{\tilde{X}} = 10 - 8 = 2$.
The proof of the analog results for $-K$ is similar and it is omitted. □

Note in particular that the vector space $H^0(-K_{\tilde{X}})$ (resp. $H^0(\theta_{\tilde{X}})$) is naturally isomorphic to the space of sections of the anticanonical sheaf (resp. tangent sheaf) of X which vanish at x and $h^2(\theta)$ is a birational invariant of smooth surfaces.

Corollary 2.3. *Let S be a rational surface: if, in the notation above, $h^0(-K_S) + min\{d,3\} \geq 9$ and $C \in S$ is a transversal curve $\neq \sigma_\infty$, then $C^2 \geq 0$.*

Proof. The proof follows by considering the blowing up of S at a point of C. □

Theorem 2.1 cannot be improved. Let in fact S_d ($d \geq 1$) be a surface obtained by blowing up the surface $\mathbb{F}_d$ at $d+1$ generic points $p_0, ..., p_d$. These points lie on a section $\sigma_0 \subset \mathbb{F}_d$ such that $\sigma_0^2 = d$, let $C \subset S_d$ be the strict transform of σ_0: clearly $C^2 = -1$, and, recalling that

$$h^0(-K_{\mathbb{F}_d}) = \begin{cases} 9 & 1 \leq d \leq 3 \\ d+6 & d \geq 3 \end{cases}$$

it follows that $h^0(-K_S)+\min\{d,3\}=8$.
Similar examples show that also the inequality $h^0(\theta_X)\geq 4$ is the best possible (cf. Remark 4.5). Note moreover that the two conditions on P_{-1} and $h^0(\theta)$ are independent.

Lemma 2.4. *In the previous notation let S be a rational surface and let f be a generic fibre of p.*
Then $h^0(-K_S-f-\sigma_\infty)\geq h^0(-K_S)+min\{d,3\}-5$.

Proof. We have two exact sequences of sheaves

1) $$0\longrightarrow\mathcal{O}_S(-K_S-\sigma_\infty)\longrightarrow\mathcal{O}_S(-K_S)\longrightarrow\mathcal{O}_{\sigma_\infty}(-K_S)\longrightarrow 0$$

2) $$0\longrightarrow\mathcal{O}_S(-K_S-f-\sigma_\infty)\longrightarrow\mathcal{O}_S(-K_S-\sigma_\infty)\longrightarrow\mathcal{O}_f(-K_S-\sigma_\infty)\longrightarrow 0$$

By the genus formula $-K_S\cdot\sigma_\infty=2+\sigma_\infty^2=2-d$, thus $h^0(\mathcal{O}_{\sigma_\infty}(-K_S))=3-\min\{d,3\}$.
The proof follows by considering cohomology exact sequences associated to 1) and 2). □

Proof of theorem 2.1.A) If $S=\mathbb{F}_d$ we already know that σ_∞ is the only curve with negative self intersection, so we can assume that p has at least a degenerate fibre f_0.
If A is the irreducible component of f_0 which intersects σ_∞ then we have an exact sequence

$$0\longrightarrow\mathcal{O}_S(-K_S-f-\sigma_\infty-A)\longrightarrow\mathcal{O}_S(-K_S-\sigma_\infty-f)\longrightarrow\mathcal{O}_A(-K_S-\sigma_\infty-f)\longrightarrow 0$$

By the genus formula $(-K_S-f-\sigma_\infty)\cdot A=2+A^2-1\leq 0$ and by lemma 2.4 $h^0(-K_S-f-\sigma_\infty-A)\geq 2$.
Let $C\subset S$ be a transversal curve different from σ_∞ with $C^2\leq -2$; for every $D\in|-K_S-f-\sigma_\infty-A|$ we have

$$D\cdot C\leq 2+C^2-f\cdot C-\sigma_\infty\cdot C-A\cdot C<0$$

thus $D=C+E$ for some effective divisor E.
Moreover $E\cdot f=E\cdot\sigma_\infty=0$, in fact, by genus formula $D\cdot f=1$, $D\cdot\sigma_\infty=0$ and by hypothesis $C\cdot f>0$, $C\neq\sigma_\infty$. Therefore E is contained in the exceptional locus of μ but this is not possible because $\dim|D|=\dim|E|\geq 1$. □

Remark. 2.5. Looking at the proof of theorem 2.1 we note that if there exists a degenerate fibre f_0 such that the irreducible component A which intersects σ_∞ has self intersection $A^2\leq -2$ then theorem 2.1.A) holds under the less restrictive assumption $h^0(-K_S)+\min\{d,3\}\geq 7$.
We also note that the condition $A^2\leq -2$ holds in particular if f_0 contains exactly one (-1)-curve. In fact if

$$\mu\colon S\to S_1\xrightarrow{\mu_1}S_2\xrightarrow{\mu_2}\dots\xrightarrow{\mu_j}\mathbb{F}_d$$

is the decomposition of μ where $\mu_1,..,\mu_j$ are exactly the blowings up lying over $\mu(f_0)\backslash\sigma_\infty$ then the unique (-1)-curve must be the exceptional curve of μ_1.

Proof of Theorem 2.1.B). As before we can assume that S is not a Segre-Hirzebruch surface. We first note that since $\sigma_\infty^2 < 0$, every section of θ_S is tangent to σ_∞ and then there exists an exact sequence

$$0 \longrightarrow H^0(\theta_S(-\sigma_\infty)) \longrightarrow H^0(\theta_S) \xrightarrow{v} H^0(\theta_{\sigma_\infty})$$

The map v cannot be surjective, otherwise, in the set up of lemma 1.3, $H^0(\theta_S)$ must contain three sections $z^a \frac{\partial}{\partial z} + p_a(s,z)\frac{\partial}{\partial s}$ $a = 0,1,2$, p_a polynomials but, according to 2.2, this is clearly impossible since $\mu: S \to \mathbb{F}_d$ is not an isomorphism. In particular $h^0(\theta_S) \leq h^0(\theta_S(-\sigma_\infty)) + 2$. Note that $H^0(\theta_S(-\sigma_\infty))$ is contained in $H^0(\theta_{\mathbb{F}_d}(-\sigma_\infty))$ which is generated by $s\frac{\partial}{\partial s}, z^i\frac{\partial}{\partial s}$ for $i = 0,..,d$

Assume now that $C \subset S$ is an irreducible transversal curve with negative selfintersection, as before every tangent vector field on S is tangent to C and since $\frac{\partial}{\partial s}$ can be tangent to C only in a finite set of points it follows that $(s - \sum a_i z^i)\frac{\partial}{\partial s} \in H^0(\theta_S(-\sigma_\infty))$ only if $(s - \sum a_i z^i)$ vanishes on $\mu(C)$ and then $h^0(\theta_S(-\sigma_\infty)) \leq 1$, $h^0(\theta_S) \leq 3$. □

Lemma 2.6. *In the same notation of lemma 2.4, if $h^0(-K_S) + min\{d,3\} \geq 6$ then there exists at most one transversal curve $C \neq \sigma_\infty$ with $C^2 \leq -2$. If such a curve exists then $C \cdot f = 1$.*

Proof. By lemma 2.4 $h^0(-K_S - f - \sigma_\infty) \geq 1$, consider $D \in |-K_S - f - \sigma_\infty|$. By the genus formula

$$D \cdot C \leq 2 + C^2 - C \cdot f - C \cdot \sigma_\infty < 0$$

thus $D = C + B$ where B is an effective divisor. We note that $B \cdot f = 0$ and thus C is the only component of D such that $C \cdot f = D \cdot f = 1$. □

3. The weight of a rational surface

Let $p: X \to B$ a holomorphic map from a surface X to a smooth curve B. We shall say that p is a rational fibration with section (r.f.w.s. for short) if:

1) The generic fibre of p is a smooth rational curve.

2) It's given a section $s: B \to X$.

Without loss of generality we can obviously assume that $B \subset X$ and s is the embedding of B in X.

Definition . A r.f.w.s. $p: X \to B$ is minimal if every fibre contains no (-1)-curves disjoint from B.

Proposition 3.1. *In a minimal r.f.w.s $p: X \to B$ every fibre is smooth rational.*

Proof. The proof is essentially the same as Lemma III.8 of [Be]. □

Definition . The *weight* $w(S)$ of a rational surface $S \neq \mathbb{P}^2$ is the greatest integer n such that there exists a birational morphism $\mu: S \to \mathbb{F}_n$.

We note that $w(S) \leq h^1(\theta_{\mathbb{F}_{w(S)}}) + 1 \leq h^1(\theta_S) + 1$.

Let $\mathcal{C}$ be the set of irreducible curves $C \subset S$ such that there exists a smooth rational curve $f \subset S$ with $f^2 = 0$, $C \cdot f = 1$.

Theorem 3.2. *In the notation above* $w(S) = \max\{-C^2 | \ C \in \mathcal{C}\}$.

Proof. $\leq$ is trivial.

Conversely let $C \in \mathcal{C}$ such that $C^2 < 0$, we have to show that $-C^2 \leq w(S)$. Let f be a smooth rational curve such that $f^2 = 0$, $f \cdot C = 1$, then it's very easy to prove that the linear system $|f|$ is a base point free pencil. The associated morphism $p: S \to \mathbb{P}^1$ is a rational fibration with section C.

The conclusion follows from proposition 3.1 by considering the surface S' obtained by contracting all (-1)-curves contained in the degenerate fibres of p which are disjoint from C. □

4. Normal projective surfaces with $\varrho = 1$, $P_{-1} \geq 5$

We first observe that in this case, since X is normal projective, $P_n(X) = 0$ for every $n > 0$.

Lemma 4.1.(Sakai) *Let X be a normal projective surface with $\varrho(X) = 1$, $P_n(X) = 0$ for every $n > 0$. Then $q(X) = 0$.*

Proof. A proof of this lemma follows from the results of [Sa1] §4, for the reader's convenience we write here a direct proof. Let $\delta: Y \to X$ be the minimal resolution of X; since for every integer n the sheaf $\mathcal{O}_X(nK_X)$ is reflexive we have $P_n(Y) \leq P_n(X)$. In particular all the positive plurigenera of Y vanish and, by Enriques criterion, Y is a ruled surface.

By Serre duality $H^2(\mathcal{O}_X) = 0$ and by the Leray spectral sequence we get $q(Y) = q(X) + h(X)$ where, by definition, $h(X) = h^0(R^1\delta_*\mathcal{O}_Y)$. Let's assume $h(X) < q(Y)$ and let $p: Y \to B$ be the canonical ruled fibration onto a smooth curve B of genus $g = q(Y)$.

If D is an irreducible component of the exceptional divisor of δ then, by a general result (cf. [B-P-V] p. 74), $g(D) \leq h(X)$ and thus p is constant on D. We can thus factorize p to a ruled fibration $p': X \to B$, but this is impossible by the assumption $\varrho(X) = 1$. □

Theorem 4.2.(Badescu) *Let X be a normal projective surface such that $q(X) = P_n(X) = 0$ for every $n > 0$ and let $\delta: Y \to X$ be its minimal resolution. Then either*

1) The singularities of X are rational and Y is a rational surface, or

2) Y is a ruled surface of irregularity $q > 0$, X has precisely one non-rational singularity x of geometric genus q, the fibre of δ over x is composed by a section of the canonical ruled fibration $p: Y \to B$ and (possibly) by components of the degenerate fibres of p, the fibre of δ over a rational singularity of X is contained in a degenerate fibre of p.

Proof. [Ba1] Th. 2.3. □

Our goal is to give a structure theorem for surfaces X belonging to class 1) of Theorem 4.2 under the more restrictive assumption that $\varrho(X) = 1$, $P_{-1}(X) \geq 5$.

Definition . A normal projective surface $X \neq \mathbb{P}^2$ belongs to class (A) if:

A1) $\varrho(X) = 1$, $P_n(X) = 0$ $\forall n \geq 1$ and X has at most rational singularities.

A2) If $\delta: S \to X$ is the minimal resolution then S is a rational surface of weight $d \geq 2$.

A3) There exists a birational morphism $\mu: S \to \mathbb{F}_d$ such that the irreducible curves contracted by δ are exactly $\mu^{-1}\sigma_\infty$ and the components with self intersection ≤ -2 of degenerate fibres of $p = \pi \circ \mu: S \to \mathbb{P}^1$.

Let's denote, for every normal projective surface X with minimal resolution $\delta: Y \to X$, by $s(X)$ the number of singular points of X and by $b(X) = \max_{x \in X}\{b_2(\delta^{-1}(x))\}$

Proposition 4.3. *If X belongs to class (A) then:*

1) $s(X) \leq b(X)$

2) X has at most one non cyclic singularity.

3) If every singularity of X is cyclic then $s(X) \leq 3$.

Proof. Let $D \subset S$ be the exceptional divisor of δ, since the singularities of X are rational $\varrho(S) = 1 + b_2(D)$ (cf. I.2.5), this forces every degenerate fibre of p to contain exactly one (-1)-curve, in fact by easy considerations about ϱ we have, in the notation $(*)$ of section 2, $r + h = b_2(\overline{D \backslash \sigma_\infty}) + e$ and then $e = h$. In particular the components of degenerate fibres which intersect σ_∞ belong to D.

It's easy to see that if f_0 is a degenerate fibre, $E \subset f_0$ the (-1)-curve and $A \subset f_0$ the component intersecting σ_∞ then $\overline{f_0 \backslash E}$ has at most two connected component and the possible component that doesn't contain A is a string.

Thus it holds $s(X) \leq h + 1 \leq b(X)$ and, if (X, x) is a noncyclic singularity, then $\delta^{-1}(x)$ must be the connected component D' of D which contains σ_∞. This prove 1) and 2).

3) follows from the fact that D' is a string if and only if $h \leq 2$. □

We are now able to prove the following

Theorem 4.4. *Let X be a normal projective surface with $\varrho(X) = 1$, $P_{-1}(X) \geq 5$ with at most rational singularities. Then X belongs to class (A).*

Proof. Let $\delta: S \to X$ be the minimal resolution and let $D \subset S$ be the exceptional curve of δ. S is a rational surface of weight $d \geq 1$ and, according to I.5.3 $P_{-1}(S) = P_{-1}(X) \geq 5$.

We first note that, by lemma 2.6, for every $\mu: S \to \mathbb{F}_d$ there exists at most one transversal curve $C \subset D$ different from σ_∞ and then $e \leq h + 1$.

We first show by contradiction that $d \geq 2$. In fact if we assume $d = 1$ and $\mu: S \to \mathbb{F}_1$ is a birational morphism then $\varrho(S) = 1 + b_2(D)$ and there exists a transversal curve $C \subset D$, $C \neq \sigma_\infty$ with $C^2 \leq -2$. By lemma 2.6 $C \cdot f = 1$ and by theorem 3.2, $d \geq -C^2 \geq 2$.

If $P_{-1}(S) + \min\{d, 3\} \geq 8$ then for every birational morphism $\mu: S \to \mathbb{F}_d$ the curves on S with self intersection ≤ -2 are σ_∞ and some components of degenerate fibres. In this case the conclusion follows from easy considerations about the Picard number of S. This proves the theorem if $d \geq 3$ or $P_{-1} \geq 6$.

It remains to consider the case $d = 2$, $P_{-1}(S) = 5$. If, for some $\mu: S \to \mathbb{F}_2$, S contains a degenerate fibre f_0 such that $A^2 \leq -2$, where $A \subset f_0$ is the irreducible component which intersects σ_∞ (e.g. if $e < 2h$), then the proof follows by remark 2.4.

The remaining case is the following: $d = 2$, $P_{-1}(S) = 5$, for every birational morphism $\mu: S \to \mathbb{F}_2$ the composite fibration $p = \pi \circ \mu$ has only one degenerate fibre f_0, $e = 2$ and $A^2 = -1$ where $A \subset f_0$ is the component which intersects σ_∞. We prove that this case doesn't occur.

Let $\mu: S \to \mathbb{F}_2$ be a fixed morphism and write μ as a composition of blowings-up

$$S = S_r \xrightarrow{\mu_r} S_{r-1} \longrightarrow .... S_2 \xrightarrow{\mu_2} S_1 \xrightarrow{\mu_1} S_0 = \mathbb{F}_2$$

We note that $P_{-1}(S) = P_{-1}(\mathbb{F}_2) - 4$ thus $r \geq 4$. Let $p_i \in S_{i-1}$ be the base point of the blow up μ_i. p_i is exactly the image of the critical set of the composite map $S \to S_{i-1}$. If $i \leq j$ let $E_i \subset S_j$ be the strict transform of the exceptional curve of μ_i. We have $E_r^2 = -1$ and $E_i^2 \leq -2$ on S if $i < r$, in particular $p_i \in E_{i-1} \backslash A$ $\forall i > 1$.

Let's consider the surface Y obtained by contracting the curve σ_∞ in $\mathbb{F}_2$, It is a well known fact that $Y \subset \mathbb{P}^3$ is the cone over a smooth conic in $\mathbb{P}^2$.

We can consider the point $p_2 \in E_1 \backslash A$ as a tangent vector $v \in T_{p_1} Y$, let $\psi: Y - - - \to \mathbb{P}^1$ be the projection of centre the projective line L generated by v. Observe that L does not contain the vertex of Y and then the generic fibre of ψ is a smooth hyperplane section of Y. By elimination of indeterminacy we get a fibration $S_2 \to E_2$ which has $\sigma_\infty \cup A \cup E_1$ as unique degenerate fibre and then a fibration $\tau: S \to E_2$. The inclusion of E_2 in S gives a section for τ, in particular $E_2^2 \geq -w(S)$ which implies $E_2^2 = -2$.

By hypothesis τ has at most one degenerate fibre, then $p_3 \in E_1 \cap E_2$, in particular $E_2^2 = -2$ in S_3 and $p_4 \in E_3 \backslash E_2$ otherwise $E_2^2 < -2$ in S, therefore E_3 is the component of the degenerate fibre that intersects E_2 and $E_3^2 \leq -2$ contrary to the assumption. □

Remark. 4.5. It's no difficult to construct a normal projective surface X with $\varrho = 1$, $P_{-1} = 4$, $h^0(\theta_X) = 3$ and with three rational double points of type A_2, hence by proposition 4.3 X doesn't belong to class A. (One of the simplest examples is obtained by fixing a section $\sigma_0 \subset \mathbb{F}_2$ and two distinct fibres $f_0, f_1 \subset \mathbb{F}_2$ and performing 2 blowings up over the point $\sigma_0 \cap f_0$ and 3 blowings up over $\sigma_0 \cap f_1$ in such a way that the inverse image of $\sigma_0 \cup \sigma_\infty \cup f_0 \cup f_1$ contain exactly 3 (-1)-curves and 6 nodal curves).

5. Deformations of normal surfaces with anticanonical divisor.

For any algebraic algebraic variety $X \subset \mathbb{P}^n$ there exists a map of deformation functors $Hilb_X^n \xrightarrow{\phi} Def_X$ where $Hilb_X^n$ is the functor of embedded deformations of X in $\mathbb{P}^n$.

Lemma 5.1. *In the above notation, if $h^1(\mathcal{O}_X(1)) = h^2(\mathcal{O}_X) = 0$ then ϕ is smooth.*

Proof. Let $0\longrightarrow\mathbb{C}\epsilon\longrightarrow B\longrightarrow A\longrightarrow 0$ be a small extension of local Artinian $\mathbb{C}$-algebras and let $f\colon X_B\to Spec\,B$ be a deformation of X. According to the flatness of f we have an exact sequence on X

$$0\longrightarrow\epsilon\mathcal{O}_X\longrightarrow\mathcal{O}_{X_B}\longrightarrow\mathcal{O}_{X_A}\longrightarrow 0$$

where $X_A=f^{-1}Spec\,A\subset X_B$. Assume that $X_A\subset\mathbb{P}^n_A$ is an embedded deformation and let $L_A=\mathcal{O}_{X_A}(1)$ be the hyperplane line bundle.
The obstruction to extend L_A to a line bundle $L_B\to X_B$ lies in $H^2(\mathcal{O}_X)$. In fact let $X=\cup U_i$ be an affine covering of X where L_A trivialize and let g_{ij} its cocycle. Let $\tilde{g}_{ij}\in\Gamma(U_{ij},\mathcal{O}^*_{X_B})$ be a 1-cochain extending g_{ij} such that $\tilde{g}_{ij}\tilde{g}_{ji}=1$ for every i,j.
Then for every i,j,k $\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki}=1+\epsilon\delta_{ijk}$ and is easy to see that δ_{ijk} is a 2-cocycle and its cohomology class $\delta\in H^2(\mathcal{O}_X)$ is independent from the choice of $\tilde{g}_{ij}$'s.
δ is exactly the obstruction to get L_B, in fact if $\delta_{ijk}=h_{ij}+h_{jk}+h_{ki}$ then $\tilde{g}_{ij}(1-\epsilon h_{ij})$ is a cocycle defining the line bundle L_B. Note that in general L_B depends from the choice of h_{ij} but if $H^1(\mathcal{O}_X)=0$ then it is possible to prove that, up to isomorphism, L_B is unique.
L_B is f-flat and there exists an exact sequence

$$0\longrightarrow\epsilon\mathcal{O}_X(1)\longrightarrow L_B\longrightarrow\mathcal{O}_{X_A}(1)=L_A\longrightarrow 0$$

and since $H^1(\mathcal{O}_X(1))=0$ the $n+1$ homogeneous coordinates of $\mathbb{P}^n_A$ lift to $n+1$ sections of L_B and a standard computation shows that the linear system generated is f-very ample and define a closed embedding $X_B\subset\mathbb{P}^n_B$. □

In case X smooth lemma 5.1 is a particular case of Horikawa costability theorem ([Ho]III). This theorem asserts that if Y is a smooth variety, X is a smooth subvariety with ideal sheaf $I_X\subset\mathcal{O}_Y$ and $H^2(Y,I_X\theta_Y)=0$ then every deformation of X can be embedded in a deformation of Y. In the situation of 5.1 the vanishing of $H^2(\mathbb{P}^n,I_X\theta_{\mathbb{P}^n})$ follows from Euler exact sequence and every deformations of the projective space is trivial.
Assume now that X has a finite number of singular points $x_1,\dots,x_r$, we have then

Lemma 5.2. *If $H^2(\theta_X)=0$ then the restriction morphism of functors*

$$\Phi\colon Def_X\to\times_{i=1}^r Def_{X,x_i}$$

is smooth. In particular every deformation of the singular points can be globalized.

Proof. This results is very similar to the above costability theorem, a proof in the same spirit of Horikawa proof is given in ([Wa1] Prop 6.4). Here we give a proof that use general obstruction theory.
Let $T^*(X),\mathcal{T}^*_X$ be respectively the global and local cohomology of the cotangent complex of X ([Pa]), the groups $T^i(X)$ are related with the sheaves $\mathcal{T}^i_X$ by the spectral sequence

$$E_2^{p,q}=H^p(\mathcal{T}^q_X)\Rightarrow T^{p+q}(X)$$

For $i \geq 1$ the sheaf $\mathcal{T}^i_X$ is supported on $\{x_1, ..., x_r\}$, this implies that the natural map $r_i: T^i(X) \to H^0(\mathcal{T}^i_X)$ is surjective for $i = 1$ and injective for $i = 2$.
In fact $H^j(\mathcal{T}^i_X) = 0$ for $i, j \geq 1$ and, since $\mathcal{T}^0_X = \theta_X$, by assumption it follows $H^2(\mathcal{T}^0_X) = 0$.
The smoothness of Φ now follows by a standard criterion. □

If X belongs to class A then all the morphisms considered before are smooth, more generally we have

Proposition 5.3. *Let $X \subset \mathbb{P}^n$ be a normal projective surface with $q(X) = p_g(X) = 0$, $P_{-1}(X) > 0$ with at most rational singularities. Then $H^2(\theta_X) = H^1(\mathcal{O}_X(1)) = 0$.*

Proof. The minimal resolution $\delta: S \to X$ is a rational surface, in particular $p_g(S) = q(S) = H^0(\Omega^1_S) = 0$. Let $C \subset X$ be a smooth hyperplane section, then $C \cdot K_X < 0$ and from exact cohomology sequence associated to

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_C(1) \longrightarrow 0$$

we get immediately $H^1(\mathcal{O}_X(1)) = 0$.
Assume that $H^2(\theta_X)^\vee = \mathrm{Hom}(\theta_X, K_X) \neq 0$ then, since both θ and K are reflexive sheaves, $\mathrm{Hom}(\theta_X, K_X) = \mathrm{Hom}(\theta_U, K_U)$ where $U \subset X$ is the open set of regular points. Moreover K_U is an invertible sheaf and the composition bilinear map

$$\mathrm{Hom}(\theta_U, K_U) \times \mathrm{Hom}(K_U, \mathcal{O}_U) \to \mathrm{Hom}(\theta_U, \mathcal{O}_U)$$

is nonzero, thus $\mathrm{Hom}(\theta_U, \mathcal{O}_U) \neq 0$. This is a contradiction since, according to Theorem I.5.5, $\mathrm{Hom}(\theta_U, \mathcal{O}_U) = H^0(\Omega^1_U) = H^0(\Omega^1_S) = 0$. □

Example 5.4. *Deformations of the surface $\mathbb{F}_4$ with the negative self-intersection curve σ_∞ blow down.*
Let $f: \mathbb{F}_4 \to W_0$ the blowing down of the curve σ_∞, then $f_* \mathcal{O}_{\mathbb{F}_4}(\sigma_0)$ is a very ample line bundle and the associate complete linear system gives an isomorphism between W_0 and the projective cone over the smooth rational curve of degree 4 in $\mathbb{P}^4$.
Denoting by $x_0, ..., x_5$ the homogeneous coordinates of $\mathbb{P}^5$ the equation of W_0 is $rank(A) \leq 1$ where A is the matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}$$

Since $h^1(\theta_{W_0}) = h^2(\theta_{W_0}) = q(W_0) = 0$, $P_{-1}(W_0) = P_{-1}(\mathbb{F}_4) = 10$ we can apply the above results and we get an isomorphism

$$Def(W_0) = Def(W_0, w_0)$$

where $w_0 = (1, 0, 0, 0, 0, 0)$ is the vertex of the cone, moreover every deformation of W_0 can be obtained as an embedded deformation in $\mathbb{P}^5$.

The semiuniversal deformation $\tilde{W_0} \to Def(W_0)$ of W_0 is well understood (cf. for example [Rie] Satz 13 or [Ar3] pag. 77-78) and can be described in the following way:
The complex germ $Def(W_0)$ is reduced and can be represented in $(\mathbb{C}^4, 0)$ as the union of the line $T_1 = \{t_2 = t_3 = t_4 = 0\}$ and the hyperplane $T_2 = \{t_1 = 0\}$.
Let $W \subset \mathbb{P}^6$ be the projective cone over the Veronese surface $V \subset \mathbb{P}^5$ and let $\{H_t\}$ be a generic pencil of hyperplanes in $\mathbb{P}^6$ with the vertex w_0 of W belonging to H_0, then the family of projective surfaces $W_t = W \cap H_t$ is flat and then it is a deformation of the surface $W \cap H_0$=cone over the generic hyperplane section of $V = W_0$. This is precisely the component T_1 of $Def(W_0)$, note that for $t \neq 0$ $W \cap H_t$ is the Veronese surface.
If $t \in T_2$ then the corresponding deformation is given by $rank(A_t) \leq 1$ where A_t is the matrix

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 - t_2 x_0 & x_3 - t_3 x_0 & x_4 - t_4 x_0 & x_5 \end{pmatrix}$$

The restriction $\tilde{W_0} \to T_2 = \mathbb{C}^3$ admits a simultaneous resolution $\tilde{S} \xrightarrow{F} \tilde{W_0} \to T_2$. $\tilde{S}$ is the union of two copies $U_0, U_1 = \mathbb{C} \times \mathbb{P}^1 \times T_2$ with coordinates $u_0, (v_0, w_0), t_2, t_3, t_4$ (resp. $u_1, (v_1, w_1), t_2, t_3, t_4$) with the patching isomorphism $\{u_0 \neq 0\} \simeq \{u_1 \neq 0\}$ given by

$$u_0 u_1 = 1 \quad w_1 = w_0 \quad v_1 = u_0^4 v_0 + (t_2 u_0^3 + t_3 u_0^2 + t_4 u_0) w_0$$

The resolution map $F: \tilde{S} \to \tilde{W_0} \subset \mathbb{P}^5 \times T_2$ is given by $(F_0, ..., F_5, t_2, t_3, t_4)$ where

$$F_0 = w_0 = w_1 \qquad F_1 = v_0 = u_1^4 v_1 - (t_4 u_1^3 + t_3 u_1^2 + t_2 u_1) w_1$$

$$F_2 = u_0 v_0 + t_2 w_0 = u_1^3 v_1 - (t^4 u_1^2 + t_3 u_1) w_1, \; F_3 = u_0^2 v_0 + (t_2 u_0 + t_3) w_0 = u_1^2 v_1 - (t_4 u_1) w_1$$

$$F_4 = u_0^3 v_0 + (t_2 u_0^2 + t_3 u_0 + t_4) w_0 = u_1 v_1 \qquad F_5 = u_0^4 v_0 + (t_2 u_0^3 + t_3 u_0^2 + t_4 u_0) w_0 = v_1$$

We note incidentally that $\tilde{S} \to T_2$ is the semiuniversal deformation of the surface $\mathbb{F}_4$ (cf [Ca6] §6) and a direct computation (cf. [Ca2] §1, [Ko2] pag.72) shows that the surface $\tilde{S}_t$ is isomorphic to $\mathbb{F}_2$ for $t \neq 0, \Delta = t_3^2 - t_2 t_4 = 0$ and to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ for $\Delta \neq 0$.

IV. Degenerations of the complex projective plane.

For the reasons explained in the general introduction we are interested to investigate the structure of normal projective degenerations of rational surfaces, especially the case when the fibres have at most quotient singularities.

This chapter is devoted to a deep study of normal degenerations of $\mathbb{P}^2$, to be more precise we study the proper flat analytic maps $f\colon X \to \Delta$ where X is a reduced locally irreducible complex space of dimension three, $\Delta \subset \mathbb{C}$ is an open disk centered at 0, $X_t = f^{-1}(t)$ is isomorphic to $\mathbb{P}^2$ for $t \neq 0$ and X_0 is a normal surface.

For simplicity we study only the local structure of degenerations of $\mathbb{P}^2$, this means that we consider the map f equivalent to every degeneration obtained from f by shrinking Δ. Note that since Δ is smooth of dimension 1 the flatness of f is a consequence of the local irreducibility of X.

From now on, by abuse of language we shall say that a normal surface X_0 is a degeneration of $\mathbb{P}^2$ if, in the above notation, it is the central fibre of f.

It is a classical result ([H-K]) the fact that, in the above situation, if X_0 is smooth then it is the projective plane and in fact holds the stronger result that every compact complex surface with finite fundamental group and second Betti number $b_2 = 1$ is the projective plane ([B-P-V] V.1.1).

If we admits X_0 normal then the above result fails to be true, for example the cone over the rational curve of degree 4 in $\mathbb{P}^4$ deforms to both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (III.5.4).

This example of degeneration is only a particular case of a wider class of degenerations obtained by a classical construction called “sweeping out the cone with hyperplane sections”. More generally let $S \subset \mathbb{P}^n$ be a smooth surface and let $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a hyperplane. Let's suppose that the curve $Y = S \cap \mathbb{P}^{n-1}$ is projectively normal (this is true if Y is generic and S is arithmetically Cohen-Macaulay) and let $C(S,v) \subset \mathbb{P}^{n+1}$ be the projective cone over S with vertex $v \in \mathbb{P}^{n+1} \backslash \mathbb{P}^n$.

Let $\{H_t\}_{t\in\mathbb{P}^1}$ be the pencil of hyperplanes of $\mathbb{P}^{n+1}$ which contain $\mathbb{P}^{n-1}$, and set $X_t = H_t \cap C(S,v)$. This defines a flat projective family of surfaces.

If $v \in H_0$ then $X_t \simeq S$ for every $t \neq 0$ and X_0 is the cone over Y.

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For every $n > 0$ let $S_n \subset \mathbb{P}^N$, $N = \dfrac{n(n+3)}{2}$, be the image of $\mathbb{P}^2$ by the Veronese embedding of degree n^2. Since the generic hyperplane section of S_n is projectively normal we can operate the previous construction and we get a set $B = \{X_{0,n}\}$ of normal degenerations of $\mathbb{P}^2$.
The normal surface $X_{0,n}$ is the cone over a smooth curve of genus $p = \dfrac{(n-1)(n-2)}{2}$ and degree n^2 in $\mathbb{P}^{p+3n-2}$. In particular the surfaces $X_{0,1}$ and $X_{0,2}$ are the only ones in B with at most quotient singularities.
The first question we ask is whether the only normal degenerations of $\mathbb{P}^2$ are those of the set B, here we show that the answer is no, in fact even assuming X_0 with at most cyclic quotient singularity we will describe infinitely many examples of degenerations.
However the possibility for a normal surface to deform to the projective plane induces several restrictions on its geometry, our first result is the following (Th. 1.3):

Theorem A. *If X_0 is a normal degeneration of $\mathbb{P}^2$ then X_0 is a projective surface with $\varrho(X_0) = 1$ and $P_{-1}(X_0) \geq 10$.*

Therefore if X_0 has at most rational singularities we may apply the results proved in chapter III, especially prop. III.4.3, moreover in this case it is possible to prove also the stronger

Theorem B. *1) Let X_0 be a normal degeneration of $\mathbb{P}^2$ with at most quotient singularities, then the following properties hold:*

- *a) X_0 is projective algebraic.*
- *b) $q(X_0) = P_n(X_0) = 0 \ \forall n \geq 1$*
- *c) $\varrho(X_0) = 1$*
- *d) Every singularity of X_0 is cyclic of type $\dfrac{1}{n^2}(1, na-1)$ for some pair of positive integers a, n with $(a,n) = 1$ ((a,n) is the g.c.d. of a and n)*
- *e) If $p_1, p_2 \in X_0$ and the singularities (X_0, p_i) are cyclic of type $\dfrac{1}{n_i^2}(1, n_i a_i - 1)$ then the n_i's are not divisible by 3, moreover if $p_1 \neq p_2$ then $(n_1, n_2) = 1$*
- *f) X_0 has at most 3 singular points.*

2) Conversely if a normal surface X_0 satisfies a), b), c) and d) of 1) then X_0 is a degeneration of $\mathbb{P}^2$, in particular e) and f) hold too.

Theorem C. *Let X_0 be a normal degeneration of $\mathbb{P}^2$:*

- *(i) If X_0 has at most rational singularities then it has at most 4 singular points.*
- *(ii) If X_0 has at most quotient singularities then its weight can assume only the values 4, 7 or 10.*

The only degeneration of weight 4 is the "classical" cone over the rational curve of degree 4 in $\mathbb{P}^4$. Here we prove that there are infinitely many degenerations of weight 7 and we give a complete explicit classification of these (Cor. 4.3).
We prove that there are also infinitely many degenerations of weight 10, but in this case an explicit classification, although possible, is more complicated.

1. Preliminaries

Throughout this chapter by a surface we shall always mean a two dimensional irreducible reduced compact complex space with at most a finite number of isolated singularities and, unless otherwise stated, normal as local ringed space. By algebraic surface we shall always mean a projective algebraic surface.

We shall say that a map $f: Y_1 \to Y_2$ of complex spaces is projective if there exists a closed embedding $i: Y_1 \to Y_2 \times \mathbb{P}^n$ such that f is the composition of i with the projection on the first factor.

Let's consider now a normal surface X_0, by a smoothing of X_0 we shall mean a proper flat map $f: X \to \Delta$ smooth over $\Delta^* = \Delta - \{0\}$ where: X is a three dimensional reduced complex space, Δ is a small open disk in $\mathbb{C}$ centered at 0 and X_0 is isomorphic to $f^{-1}(0)$.

Under this setting if $t \in \Delta$ we set $X_t = f^{-1}(t)$. Since we are interested in the local properties of smoothings, from now on, all the assertions concerning f and X will be considered up to possible shrinking of Δ.

Lemma 1.1. *Let $f: X \to \Delta$ be a smoothing of a normal surface X_0. For every $t \in \Delta$, let $\mathrm{Pic}(X) \xrightarrow{r_t} \mathrm{Pic}(X_t)$ be the natural restriction map.*

If $q(X_0) = p_g(X_0) = 0$ then r_0 is bijective and r_t is injective for every $t \in \Delta$.

Proof. By a general fact of topology of complex spaces (cf. for example [B-P-V] Th. I.8.8) X_0 is a homotopic retract of X, in particular, the restriction map $H^2(X, \mathbb{Z}) \to H^2(X_0, \mathbb{Z})$ is an isomorphism.

By using semicontinuity we get, for every $t \in \Delta$ $q(X_t) = p_g(X_t) = 0$ (It is not necessary here to shrink Δ because q and p_g are topological invariants of the underlying oriented manifold).

The base change theorem gives $R^1 f_* \mathcal{O}_X = R^2 f_* \mathcal{O}_X = 0$ and by the Leray spectral sequence we get, since Δ is Stein, $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$.

Now the respective exponential sequences on X and X_0 give a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}) \\ \downarrow{\scriptstyle r_0} & & \| \\ \mathrm{Pic}(X_0) & \longrightarrow & H^2(X_0, \mathbb{Z}) \end{array}$$

Since the horizontal maps are isomorphisms r_0 is also an isomorphism.

If $\mathcal{L} \in \mathrm{Pic}(X)$ and we denote by $A \subset \Delta$ the set of t for which $\mathcal{L}_t = \mathcal{L} \otimes \mathcal{O}_{X_t}$ is the trivial sheaf, then the first part of this proof shows that A is open, furthermore since f is proper we have

$$A = \{t | h^0(\mathcal{L}_t) \geq 1,\ h^0(\mathcal{L}_t^{-1}) \geq 1\}$$

that is a closed set by semicontinuity. □

Remark. . In general it is not true that r_t is bijective for $t \neq 0$.

In the same situation of Lemma 1.1, if moreover X_0 is algebraic, then we shall show in the course of the proof of the next proposition that if $\mathcal{L}_0$ is a very ample sheaf on X_0 with

$H^1(\mathcal{L}_0) = 0$ (such sheaf always exists), then the unique extension $\mathcal{L}$ on X is relatively f-very ample and the morphism f is projective.

Proposition 1.2. *Let $X_0 \subset \mathbb{P}^n$ be a normal surface with anticanonical divisor and let $f: X \to \Delta$ be a smoothing of X_0.*
If $q(X_0) = p_g(X_0) = 0$ then there exists a closed embedding $X \xrightarrow{i} \mathbb{P}^n \times \Delta$ such that f is induced by the projection on the second factor.

Proof. This lemma is a consequence of lemma III.5.1, but it is instructive to give a direct proof that doesn't rely on the existence of the Kuranishi family and Hilbert scheme.
Let $C \subset X_0$ be a smooth hyperplane section not intersecting the singular locus of X_0 and set $\mathcal{L}_0 = \mathcal{O}_{X_0}(C)$.
We have $H^1(\mathcal{L}_0) = 0$, in fact there is an exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_{X_0}) \longrightarrow H^1(\mathcal{L}_0) \longrightarrow H^1(\mathcal{O}_C(C))$$

and by Serre duality and adjunction formula $H^1(\mathcal{O}_C(C)) = H^0(\mathcal{O}_C(K_{X_0})) = 0$ because $K_{X_0} \cdot C < 0$.
Let $\mathcal{L}$ be an invertible sheaf on X which extends $\mathcal{L}_0$, we have then an exact sequence

$$0 \longrightarrow \mathcal{L}(-X_0) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_0 \longrightarrow 0$$

since X_0 is linearly equivalent to 0 as a Cartier divisor we have $\mathcal{L}(-X_0) \sim \mathcal{L}$ as $\mathcal{O}_X$-module. In particular $\mathcal{L}(-X_0) \otimes \mathcal{O}_{X_0} \simeq \mathcal{L}_0$ and by semicontinuity, base change and Leray spectral sequence we get, eventually shrinking Δ, $H^1(\mathcal{L}(-X_0)) = 0$ and the restriction map $H^0(\mathcal{L}) \xrightarrow{\alpha} H^0(\mathcal{L}_0)$ is surjective.
Let $V_0 \subset H^0(\mathcal{L}_0)$ be the $(n+1)$-dimensional vector space generated by the homogeneous coordinates of $\mathbb{P}^n$ and let $V \subset H^0(\mathcal{L})$ be a subspace isomorphic to V_0 via α.
By further shrinking Δ, the linear system $|V|$ is base point free and we can define $i: X \to \mathbb{P}^n \times \Delta$ $i(x) = (v_0(x), ..., v_n(x), f(x))$ where $v_0, ..., v_n$ is a basis of V. It's easy to see that i gives the desired embedding. □

After this preparatory material we now are going to study more closely normal degenerations of $\mathbb{P}^2$.

Definition . We shall say that a normal surface X_0 is a (normal) degeneration of $\mathbb{P}^2$ if there exists a smoothing $f: X \to \Delta$ of X_0 such that $X_t \simeq \mathbb{P}^2$ for every $t \in \Delta^*$.
X_0 will be called a projective degeneration if in addition the map f can be chosen to be projective.
The main result of this section is the following:

Theorem 1.3. *Let X_0 be a normal degeneration of $\mathbb{P}^2$.*
Then X_0 is a projective degeneration with $q(X_0) = P_n(X_0) = 0$ $\forall n \geq 1$, $P_{-1}(X_0) \geq 10$ and $\varrho(X_0) = 1$.

Proof. Let $f: X \to \Delta$ be a smoothing of X_0 with $X_t \simeq \mathbb{P}^2$ for every $t \in \Delta^*$. Since X_0 is normal, the space X is normal and Cohen Macaulay, denote by ω_X its canonical sheaf and by $\omega_X^{(n)}$ the double dual of $\omega_X^{\otimes n}$.
By the adjunction formula we have $\omega_{X_0}^{(n)} = (\omega_X^{(n)} \otimes \mathcal{O}_{X_0})^{\vee\vee}$, $\omega_{X_t}^{(n)} = \omega_X^{(n)} \otimes \mathcal{O}_{X_t}$ $\forall t \neq 0$.
Since $\omega_X^{(n)}$ is reflexive it is flat over Δ and semicontinuity gives

$$h^0(X_0, \omega_X^{(n)} \otimes \mathcal{O}_{X_0}) \geq h^0(X_t, \omega_{X_t}^{(n)}) = h^0(\mathbb{P}^2, nK_{\mathbb{P}^2})$$

Moreover, $\omega_X^{(n)}$ is locally free on the regular locus of X and X_0 is Cartier, this implies (cf. for example [E-V] Lemma 2.1) that $\omega_X^{(n)} \otimes \mathcal{O}_{X_0} \subset \omega_{X_0}^{(n)}$, in particular

$$P_n(X_0) = h^0(X_0, \omega_{X_0}^{(n)}) \geq h^0(X_0, \omega_X^{(n)} \otimes \mathcal{O}_{X_0}) \geq P_n(\mathbb{P}^2) \qquad \forall n \in \mathbb{Z}$$

If $n > 0$ we have $P_{-n}(X_0) \geq P_{-n}(\mathbb{P}^2) = \binom{3n+2}{2}$. This proves that X_0 is Moishezon, i.e. $a(X_0) = 2$, and $-K_{X_0}$ is an effective Weil divisor.
Since $P_{-n}(X_0) > 1$ $\forall n > 0$ and $\omega_{X_0}^{(n)}$ is the reflexive extension of a nontrivial invertible sheaf, by compactness it follows that $P_n(X_0) = 0$, by Serre duality ([B-S] Chapitre 7) $p_g(X_0) = P_1(X_0) = 0$ and by the invariance of $\chi(\mathcal{O}_{X_t})$ $q(X_0) = 0$.
Since by Brenton's criterion of projectivity ([Bre]) every normal Moishezon surface with $p_g = 0$ is algebraic, using Prop. 1.2 we prove that X_0 is a projective degeneration.
The statement $\varrho(X_0) = 1$ follows from Lemma 1.1 ($\varrho(X_0) \leq \varrho(X_t) = 1$) and by the algebraicity of X_0. □

Remark. . A different proof of Theorem 1.3 which doesn't make use of Serre duality can be given by observing that $b_1(X_0) = 0$ ([G-S] Cor 3.1) and that for every normal Moishezon surface Y the group $\mathrm{Pic}^0(Y)$ is a torus (Appendix IV), hence $q(X_0) = 0$ and by the invariance of χ, $p_g(X_0) = 0$.
We refer to the paper of Badescu ([Ba1]) for some general results about normal projective surfaces Y with $q(Y) = P_n(Y) = 0$ $\forall n \geq 1$. In [Ba1] the author also gives a complete classification of normal projective Gorenstein degenerations of $\mathbb{P}^2$.

Corollary 1.4. *Under the same hypothesis of Theorem 1.3 there exists an integer $n > 0$ such that $\omega_X^{(n)}$ is an invertible sheaf, in particular $K_{X_0}^2 = K_{X_t}^2 = 9$.*

Proof. Let $\mathcal{L}$ be a non trivial invertible sheaf on X. Then there exists an integer n independent of t such that for every $t \in \Delta^*$, $\mathcal{L}_t$ is the sheaf associated to the divisor nH_t where H_t denotes the line divisor on $X_t \simeq \mathbb{P}^2$.
Therefore $\mathcal{F} = (\omega_X^{(n)} \otimes \mathcal{L}^{-3})$ is a reflexive sheaf with trivial restriction on X_t for every $t \neq 0$. We claim that $\mathcal{F}$ is the trivial sheaf, in fact by the Leray spectral sequence and Cartan's theorem A there exists a nonzero section s of $\mathcal{F}$, the divisor (s) must be a discrete collection of fibres of f, hence a Cartier divisor and the claim follows from Lemma 1.1.

According to intersection theory of invertible sheaves we have $n^2K^2_{X_0} = X_0\cdot(\omega_X^{(n)})^2 = X_t\cdot(\omega_X^{(n)})^2 = n^2K^2_{X_t} = 9n^2$. □

In general K^2 is not invariant under normal degenerations (example III.5.4) but is only upper semicontinuous. In fact we have already seen that for every integer n $\chi(\omega^{(n)}_{X_0}) \geq \chi(\omega^{(n)}_{X_t})$ and according to Riemann-Roch formula for Weil divisors (I.5, [K-S]) we have $K^2_{X_0} \geq K^2_{X_t}$.

2. The Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing of a two dimensional quotient singularity and applications to degenerations of $\mathbb{P}^2$

We start by recalling the notion of a smoothing of an irreducible isolated singularity $(V_0, 0)$ and of its associated Milnor fibre.

A smoothing of $(V_0, 0)$ is a flat map $f\colon V \to \Delta$ where V is a reduced complex space and $\Delta \subset \mathbb{C}$ is a small open disk centered at 0, such that $(f^{-1}(0), 0) \simeq (V_0, 0)$ and for every $t \in \Delta^*$ the fibre $V_t = f^{-1}(t)$ is nonsingular.

Suppose $(V_0, 0)$ is embedded in $(\mathbb{C}^N, 0)$: then there exists an embedding of $(V, 0)$ in $(\mathbb{C}^N \times \Delta, 0)$ such that the map f is induced by the projection on the second factor $\mathbb{C}^N \times \Delta \to \Delta$.

We fix now some further notation: if $r > 0$ we denote by $B_r = \{z \in \mathbb{C}^N | \; \|z\| < r\}$ and let $S_r = \partial B_r$. We shall say that S_r is a Milnor sphere for V_0 if for every $0 < r' \leq r$ the sphere $S_{r'}$ intersects V_0 transversally: a basic result ([Mi] Cor 2.9) asserts that every isolated embedded singularity admits a Milnor sphere.

Let S_r be a Milnor sphere for V_0, then (shrinking Δ if necessary) we can assume that $S_r \times \Delta$ intersects V_t transversally $\forall t \in \Delta$. In this situation we set

$$X = V \cap (B_r \times \Delta) \qquad X_t = V_t \cap X \qquad K_t = \partial X_t = V_t \cap (S_r \times \Delta)$$

By Ehresmann's fibration theorem we have $\partial X = \cup_{t \in \Delta} K_t \simeq K_0 \times \Delta$ and the map $f\colon X \setminus X_0 \to \Delta^*$ is a locally trivial C^∞ fibre bundle with fibre F diffeomorphic to X_t for $t \neq 0$.We call F (resp. $\overline{F}$) the Milnor fibre (resp. compact Milnor fibre) of the smoothing f.

The basic theory about Milnor fibre ([Lo2]) shows that the diffeomorphism class of F is independent of the embedding of V: in particular topological invariants of F are invariants of the smoothing.

Let n be the dimension of $(V_0, 0)$, since F is Stein, it has the homotopy type of a n-dimensional CW complex. Considering homology and cohomology we have $H_i(F, \mathbb{Z}) = 0$ for $i > n$ and $H_n(F, \mathbb{Z})$ is a finitely generated free abelian group.

Definition . The integer $\mu = \operatorname{rank} H_n(F, \mathbb{Z})$ is called the *Milnor number* of the smoothing.

The Lefschetz and Poincaré duality theorems give the following isomorphisms (in every ring of coefficients)

$$H^q_c(F) = H_{2n-q}(F) = H^q(\overline{F}, \partial F)$$

Using real coefficients the cup product induces a perfect pairing

$$H^n(\overline{F}) \times H^n(\overline{F}, \partial F) \xrightarrow{\cup} H^{2n}(\overline{F}, \partial F) = \mathbb{R}$$

which composed with the natural map $H^n(\overline{F}, \partial F) \to H^n(\overline{F})$ gives a symmetric bilinear form

$$H^n(\overline{F}, \partial F) \times H^n(\overline{F}, \partial F) \xrightarrow{q} \mathbb{R}$$

and we can write $\mu = \mu_0 + \mu_+ + \mu_-$ where μ_0 (resp.: μ_+, μ_-) is the number of zero (resp.: positive,negative) eigenvalues of q.

Let's consider now the case $n = 2$; by using Morse theory we see that $\overline{F}$ is obtained from ∂F up to homotopy by attaching a finite number of cells of dimension ≥ 2. This implies that the inclusion $\partial F \subset \overline{F}$ induces a surjection of the respective fundamental groups, moreover it is rather easy to prove, using the exact homotopy sequence of the fibration $f: X \backslash X_0 \to \Delta^*$, that the inclusion $F \subset X - \{0\}$ induces an isomorphism on π_1's (cf. [L-W] Lemma 5.1).

Definition 2.1. Let V_0 be a Stein representative of the surface singularity $(V_0, 0)$ and let $\pi: Z \to V_0$ be a resolution. The *geometric genus* of $(V_0, 0)$ is the integer $g(V_0, 0) = h^1(\mathcal{O}_Z) - \delta(V_0)$ where $\delta(V_0) = h^0(\pi_* \mathcal{O}_Z / \mathcal{O}_{V_0})$. For normal singularities $\delta = 0$ and this definition of genus is the same given in section I.1.

An important result of Steenbrink ([St2] Th. 2.24) is the following: given a smoothing of a two dimensional isolated surface singularity $(V_0, 0)$ we have $\mu_0 + \mu_+ = 2g(V_0, 0)$, in particular if the singularity is rational, that is, normal with geometric genus 0, then $\mu = \mu_-$.

We conclude this brief review by describing the homotopy type and the intersection form q of the Milnor fibre of a smoothing of a rational double point $(V_0, 0)$. This is easy, in fact by Brieskorn-Tyurina's result on simultaneous resolution the Milnor fibre is diffeomorphic to a neighbourhood of the exceptional curve in the minimal resolution of $(V_0, 0)$, thus if V_0 is a rational double point of type A_r, D_r or E_r then the Milnor fibre has the homotopy type of a bouquet of r spheres.

We now introduce the notion of a $\mathbb{Q}$-Gorenstein singularity.

Definition 2.2. Let $(Y, 0)$ be a normal Cohen Macaulay singularity with canonical divisor K_Y. We shall say that $(Y, 0)$ is $\mathbb{Q}$*-Gorenstein* of index n if there exists some nonzero integer n' such that the divisor $n' K_Y$ is principal and n is the smallest positive integer with this property.

Example . Let $(X, 0)$ be a $\mathbb{Q}$-Gorenstein singularity of dimension n and index r and let $(X, 0) \xrightarrow{\pi} (Y, 0)$ be the quotient of X by a finite group G acting freely in the complement of an analytic closed subset of codimension ≥ 2.

In this situation for every integer s $(\pi^* \mathcal{O}_Y(sK_Y))^{\vee\vee} = \mathcal{O}_X(sK_X)$ and then sK_Y is principal only if s is a multiple of r. Similarly if $r|s$, $\mathcal{O}_Y(sK_Y) = (\pi_* \mathcal{O}_X(sK_X))^G$ and then K_Y is $\mathbb{Q}$-Cartier if and only if there exists for some $s = rd$ an invertible G-invariant section of $\mathcal{O}_X(sK_X)$.

Fixing an isomorphism $\mathcal{O}_X(rK_X) \simeq \mathcal{O}_X$ we have an $\mathcal{O}_X$ morphism $\mathcal{O}_X(rK_X) \to \mathbb{C}$ which maps every section ω in its evaluation in 0. There exists then a character $det^r: G \to \mathbb{C}^*$ such that $(g\omega)(0) = det^r(g)(\omega(0))$ for every section ω of $\mathcal{O}_X(rK_X)$.

We claim that $(Y, 0)$ is $\mathbb{Q}$-Gorenstein and $index(Y) = index(X) order(det^r)$, in fact since the property of being normal and Cohen-Macaulay is stable under finite group quotient it is sufficient to show that K_Y is $\mathbb{Q}$-Cartier.

Taking a section w of $\mathcal{O}_X(rK_X)$ such that $w(0) \neq 0$ we consider the new section $\omega = \sum_{g\in G} det^r(g)^{-1}(gw)$, clearly $\omega(0) \neq 0$ and for every $g \in G$ $g\omega = det^r(g)\omega$ and if d is the order of det^r then ω^d is G-invariant and then drK_Y is principal. Conversely if $\omega' = f\omega^h$, $f \in \mathcal{O}_X^*$ is G-invariant then, since $gf(0) = f(0)$, h must be a multiple of d.

Let now $(X,0)\xrightarrow{\pi'}(\mathbb{C},0)$ be a $\mathbb{Q}$-Gorenstein smoothing of index n of a quotient surface singularity $(X_0,0)$ (i.e. a smoothing with $(X,0)$ $\mathbb{Q}$-Gorenstein of index n).

Using Mori's theorem on terminal three dimensional singularities Kollar and Shepherd Barron have proved ([K-S] Prop. 3.10) the following:

Theorem 2.3. *In the notation above, if $n=1$ then $(X_0,0)$ is a rational double point, if $n>1$ then $(X,0)$ is analytically isomorphic to the quotient $(Y,0)/G$ where:*

a) $(Y,0)\subset(\mathbb{C}^4,0)$ is an isolated hypersurface singularity defined by

$$F = uv + y^{dn} - t\varphi(u,v,y,t) = 0$$

for some $d>0$ and $\varphi \in \mathbb{C}\{u,v,y,t\}$.

b) $G \simeq \mu_n = \{$multiplicative group of n^{th} roots of $1\}$ acts linearly on $\mathbb{C}^4$ in the following way

$$\mu_n \ni \xi\colon (u,v,y,t)\longrightarrow(\xi u, \xi^{-1}v, \xi^a y, t)$$

for some integer a with $(a,n)=1$, moreover φ is invariant for this action.

c) The projection $\pi\colon (Y,0)\to(\mathbb{C},0)$ on the t-axis defines a smoothing of the rational double point of type A_{dn-1} $(Y_0,0)$. G acts, locally around 0, freely on $Y-\{0\}$ and π' is obtained from π by passing to the quotient.

In the notation of Theorem 2.3 $(X_0,0)$ is a cyclic singularity of type $\dfrac{1}{dn^2}(1,dna-1)$ (cf. [Wa1] Ex.5.9.1).

Remark. . A tedious but easy calculation shows that we can assume φ to be a polynomial in y^n of degree $<d$ with coefficients in $\mathbb{C}\{t\}$.

There are other proofs of Th. 2.3 (cf. [Ma2], [L-W]) but the presentation of the result given in [K-S] is the most convenient for our use.

We are now able to study more closely the Milnor fibre of such smoothings.

Proposition 2.4. *Let F be the Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing $(X,0)\to(\mathbb{C},0)$ of a cyclic singularity $(X_0,0)$ of type $\dfrac{1}{dn^2}(1,dna-1)$ with $(n,a)=1$, then:*

i) $b_2(F)=d-1$, $\pi_1(F)=\mathbb{Z}_n$

ii) $\pi_1(\partial F)=\mathbb{Z}_{dn^2}$

iii) The torsion subgroup of the Picard group $\mathrm{Pic}(F)$ of F is cyclic of order n and it is generated by the canonical bundle K_F.

Proof. i) From Theorem 2.3 it follows that F has an unramified connected covering F' of degree n which has the homotopy type of a bouquet of $dn-1$ spheres S^2, hence $\pi_1(F)=\mathbb{Z}_n$,

$b_1(F) = 0$ and $e(F) = 1 + b_2(F)$ where e denotes the topological Euler characteristic. Since $ne(F) = e(F') = 1 + b_2(F') = dn$ it follows the equality $b_2(F) = d - 1$.

ii) It follows from the fact that ∂F is diffeomorphic to the link of the cyclic singularity $(X_0, 0)$.

iii) Since F is Stein, from the exponential sequence we get

$$Tors\mathrm{Pic}(F) = TorsH^2(F) = TorsH_1(F) = \mathbb{Z}_n$$

According to Theorem 2.3 the index of $(X, 0)$ is n, which means that $K_{X-\{0\}}$ belongs to $\mathrm{Pic}(X - \{0\})$ and has order exactly n.

We claim that the natural restriction map $\alpha: Tors\mathrm{Pic}(X - \{0\}) \longrightarrow Tors\mathrm{Pic}(F)$ is an isomorphism: the proof will follow from the claim and the adjunction formula.

$(X, 0)$ is Cohen Macaulay, local cohomology theory implies $H^1(X - \{0\}, \mathcal{O}_{X-\{0\}}) = 0$ and by the exponential exact sequence

$$Tors\mathrm{Pic}(X - \{0\}) = TorsH^2(X - \{0\}, \mathbb{Z}) = \pi_1(X - \{0\}) = \pi_1(F) = \mathbb{Z}_n$$

Thus the claim can be proved either by using arguments of algebraic topology or in the following manner.

In the notation of Theorem 2.3 and of the proof of i) we have a commutative diagram

$$\begin{array}{ccc} F' & \subset & Y - \{0\} \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p} \\ F & \subset & X - \{0\} \end{array}$$

where p and $\hat{p}$ are unramified cyclic coverings. We have two canonical eigensheaves decompositions

$$\hat{p}_* \mathcal{O}_{F'} = \bigoplus_{i \in \mathbb{Z}_n} \hat{\mathcal{L}}_i \qquad p_* \mathcal{O}_{Y-\{0\}} = \bigoplus_{i \in \mathbb{Z}_n} \mathcal{L}_i$$

where $\mathcal{L}_i, \hat{\mathcal{L}}_i$ are the eigensheaves associated to the character $i: \mu_n \to \mathbb{C}^*$. Obviously $\alpha(\mathcal{L}_i) = \hat{\mathcal{L}}_i$ and since F' is connected we have $\hat{\mathcal{L}}_i \neq \hat{\mathcal{L}}_j$ if $i \neq j$, thus $Tors\mathrm{Pic}(F) = \{\hat{\mathcal{L}}_i\}_{i \in \mathbb{Z}_n}$. $\square$

Remark. . We observe that in the situation of Prop. 2.4 we can show that the intersection form on $H_2(F) = H_c^2(F)$ is negative definite without making use of Steenbrink's formula.

In fact if $p^*: H_c^2(F) \to H_c^2(F')$ is induced from the proper mapping $p: F' \to F$, then for any pair a, b of elements of $H_c^2(F)$ we have $(p^*a)\cdot(p^*b) = na\cdot b$.

We now are going to apply these results to investigate normal degenerations of the projective plane.

As before, let X_0 be a normal surface and let $f: X \to \Delta$ be a smoothing of X_0 with generic fibre isomorphic to $\mathbb{P}^2$. We note that the three dimensional space X is Cohen-Macaulay since every point of X belongs to a normal irreducible Cartier divisor (the fiber).

In section 1 we have seen that X_0 is algebraic and that X is a $\mathbb{Q}$-Gorenstein complex space. Let $\{p_1, ..., p_s\}$ be the singular points of X_0 and let F_i be the Milnor fibre of the smoothing f of the singularity (X_0, p_i).

Let $F \subset X_t$ be the disjoint union of the F_i's; then the natural homomorphism

$$i_*: H_2(F,\mathbb{Z}) = \overset{\perp}{\oplus} H_2(F_i,\mathbb{Z}) \longrightarrow H_2(\mathbb{P}^2,\mathbb{Z})$$

is an isometry, in particular if $b_2(F) = \mu_- + \mu_0 + \mu_+$; then we must have $\mu_- = 0$ and $\mu_+ \leq 1$.

Proposition 2.5. *In the notation above, if (X_0, p_i) is a rational singularity, then $b_2(F_i) = 0$ and p_i is a singular point of X.*

Proof. By Steenbrink's formula the intersection product in $H_2(F_i)$ is negative definite, thus $b_2(F_i) = 0$. Since a smoothing of an isolated hypersurface singularity $(V, 0)$ has Milnor number equal to 0 if and only if 0 is a regular point of V ([Mi] Th. 7.1), p_i cannot be a regular point of X. □

If the singularities of X_0 are quotient then we get more information on their structure and number.

Theorem 2.6. *In the above notation let $0 \leq r \leq s$ be an integer such that the singularities (X_0, p_i) are quotient singularities for $i = 1, ..., r$. If $1 \leq i \neq j \leq r$ then:*

1) *The singularity (X_0, y_i) is cyclic of type $\frac{1}{n_i^2}(1, n_i a_i - 1)$ for some pairs of relatively prime positive integers $n_i > a_i \geq 1$.*
2) *n_i is not divisible by 3.*
3) *n_i and n_j are relatively prime.*

Proof. 1) Trivial consequence of Th. 2.3 and Props. 2.4, 2.5.

2) Since F_i is an open subset of $\mathbb{P}^2$ we have $K_{F_i} = K_{\mathbb{P}^2 \mid F_i} = -3H_{\mid F_i}$ where $H \subset \mathbb{P}^2$ is the line. By Prop 2.4 K_{F_i} generates $\mathrm{Pic}(F_i) = \mathbb{Z}_{n_i}$ and thus necessarily $(n_i, 3) = 1$.

3) For every $i = 1, ..., r$ let's denote by N_i the closed set $\mathbb{P}^2 \backslash F_i$ and for every $1 \leq i_1 < ... < i_k \leq r$ $N_{i_1,..,i_k} = N_{i_1} \cap \quad \cap N_{i_k}$. We first prove the following lemma

Lemma 2.7. *Let's consider integral homology; for every $1 \leq i_1 < ... < i_k \leq r$ we have:*

1) *$H_1(N_{i_1,...,i_k}) = 0$*
2) *$H_2(N_{i_1,...,i_k}) = \mathbb{Z}$*
3) *The inclusion $N_{i_1,...,i_k} \subset \mathbb{P}^2$ induces an injection of the respective H_2's and the cokernel has order exactly equal to the product of $n'_{i_j}s$ $(j = 1, .., k)$.*

Proof. The proof of 2) and the equivalence of 1) and 3) follow easily, by excision and Prop. 2.5, from the homology long exact sequence of the pair $(\mathbb{P}^2, N_{i_1,...,i_k})$. We prove 1) by induction on k.

If $k > 0$ we have $N_{i_1,...,i_{k-1}} = N_{i_1,...,i_k} \cup \overline{F_{i_k}}$ and $N_{i_1,...,i_k} \cap \overline{F_{i_k}} = \partial F_{i_k}$. Mayer Vietoris gives

$$H_2(N_{i_1,...,i_{k-1}}) \longrightarrow H_1(\partial F_{i_k}) \longrightarrow H_1(F_{i_k}) \oplus H_1(N_{i_1,...,i_k}) \longrightarrow 0$$

and the thesis follows from Prop 2.4. □

Let's go back to proof of Theorem 2.6, the Mayer Vietoris homology exact sequence of the couple (N_i, N_j) gives

$$H_2(N_i) \oplus H_2(N_j) \xrightarrow{\alpha} H_2(\mathbb{P}^2) \longrightarrow H_1(N_{i,j}) = 0$$

the map $\alpha: \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is given by $\alpha(a,b) = n_i a - n_j b$ and it is surjective if and only if $(n_i, n_j) = 1$. □

3. The minimal good resolution of a two dimensional cyclic quotient singularity and degenerations of $\mathbb{P}^2$ with quotient singularities.

Assume that the normal surface X_0 is a degenerations of $\mathbb{P}^2$ with at most quotient singularities, according to theorem 1.3 X_0 belongs to the class (A) introduced in chapter III and then it is possible to describe its minimal resolution in a purely combinatorial way, for this we need first to well understand the Dynkin diagram of the cyclic singularities described in theorem 2.6.

We recall that a singularity is cyclic if and only if its Dynkin diagram is a string, i.e. of type

$$\underset{-b_1}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_2}{\bullet} \quad \cdots \quad -\!\!\!-\!\!\underset{-b_r}{\bullet} \qquad b_i \geq 2$$

If we set for every $b_1, ..., b_r$ integers ≥ 2

$$[b_1, ..., b_r] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \; b_{r-1} - \cfrac{1}{b_r}}} = \frac{n}{q}$$

where $n > q > 0$ are integers such that $(n, q) = 1$, then the corresponding cyclic singularity is of type $\frac{1}{n}(1, q)$.

Note that if $0 < q' < n$ and $qq' \equiv 1 \pmod{n}$ then $[b_r, ..., b_1] = \frac{n}{q'}$ according to the obvious isomorphism holding between the respective cyclic singularities of type $\frac{1}{n}(1, q)$, $\frac{1}{n}(1, q')$.

Let's define A to be the set of the symbols $[b_1, ..., b_r]$ where the b_i's are integers ≥ 2. There is an obvious bijection of A with the set of oriented Dynkin strings and, via the above partial fractions, with the set of rational numbers > 1.

For every $d > 0$ let $T_d \subset A$ be the following set of rational numbers

$$T_d = \left\{ \frac{dn^2}{dna - 1} | n, a \text{ integers }, n > a > 0, \quad (n, a) = 1 \right\}$$

The following theorem is very useful in order to detect a cyclic singularity of type $\frac{1}{dn^2}(1, dna-1)$ with $(n, a) = 1$ from its minimal resolution

Theorem 3.1. *With the above conventions:*

i) $\underset{-4}{\bullet} \in T_1$ *and* $\underset{-3}{\bullet}\!\!-\!\!\!-\!\!\underset{-2}{\bullet}\!\!-\!\!\!-\!\!\underset{-2}{\bullet} \quad \cdots \quad \underset{-2}{\bullet}\!\!-\!\!\!-\!\!\underset{-3}{\bullet}$ ($d \geq 2$ *vertices*) $\in T_d$

ii) *If* $\underset{-b_1}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_2}{\bullet} \quad \cdots \quad \underset{-b_{r-1}}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_r}{\bullet} \in T_d$ *then also*

$$\underset{-b_r}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_{r-1}}{\bullet} \quad \cdots \quad \underset{-b_2}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_1}{\bullet}$$

$$\underset{-b_1-1}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_2}{\bullet} \quad \cdots \quad \underset{-b_{r-1}}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_r}{\bullet}\!\!-\!\!\!-\!\!\underset{-2}{\bullet}$$

belong to T_d.

iii) *Every element of* T_d *is obtained starting from the one described in i) and iterating the steps described in ii).*

iv) *If* $\underset{-b_1}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_2}{\bullet} \quad \cdots \quad \underset{-b_{r-1}}{\bullet}\!\!-\!\!\!-\!\!\underset{-b_r}{\bullet} \in T_d$ *then* $\sum b_i = 3r + 2 - d$.

Proof. (cf. [Wa2] 2.8.2)

i) If $n = 2$ then $a = 1$ and the Dynkin strings corresponding to the numbers $\dfrac{4d}{2d-1}$ are exactly those described.

ii) We have already seen that

$$\frac{dm^2}{dma-1} = [b_1, \ldots, b_r] \iff [b_r, \ldots, b_1] = \frac{dm^2}{dm(m-a)-1}$$

therefore T_d is closed under orientation reversing. A little computation gives

$$\frac{dm^2}{dma-1} = [b_1, \ldots, b_r] \iff [b_1+1, \ldots, b_r, 2] = \frac{d(m+a)^2}{d(m+a)a-1}$$

which prove ii).

iii) Induction on n. Let us fix $\dfrac{dn^2}{dna-1} \in T_d$, if $n = 2$ then by i) there is nothing to prove. Suppose $n > 2$, by possibly considering $\dfrac{dn^2}{dn(n-a)-1}$ we can assume $a < \dfrac{n}{2}$ and setting $m = n - a$, according to ii) we can write

$$\frac{dn^2}{dna-1} = \frac{d(m+a)^2}{d(m+a)a-1} = [b_1+1, \ldots, b_r, 2]$$

where

$$[b_1, \ldots, b_r] = \frac{dm^2}{dma-1}$$

This proves the induction step.

iv) Trivial. □

Let $\delta\colon (S, E) \to (X, 0)$ be a resolution of a rational singularity with exceptional locus E. Writing $K_Z = \delta^* K_X + F$ where F is a $\mathbb{Q}$-divisor supported on E, we can consider the rational number $\beta = b_2(E) + F^2$.

Since after a blowing up of S at a point in E the second Betti number $b_2(E)$ increases by 1 and F^2 decreases by 1 it follows that β is an invariant of the singularity.

For a cyclic singularity of type $\dfrac{1}{n}(1, q)$ $\dfrac{n}{q} = [b_1, ..., b_r]$ a linear algebra computation shows that (cf [L-W] Prop 5.9)

$$\beta = r + 2 + \sum_{i=1}^{r} (2 - b_i) - \frac{q + q' + 2}{n}$$

In particular if $\frac{n}{q} \in T_d$, then

$$\beta = r + 1 + \sum_{i=1}^{r}(2 - b_i)$$

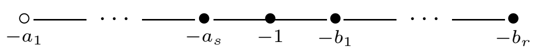
and by using Theorem 3.1.iv) we get readily that this invariant is $d-1$. Let as before X_0 be a surface with at most quotient singularities which is a degeneration of $\mathbb{P}^2$, denote by $\delta: S \to X_0$ its minimal resolution of singularities and let $\mu: S \to \mathbb{F}_w$, $2 \le w =$weight of X_0 be a birational morphism and let $p: S \to \mathbb{P}^1$ be the composition of μ with the canonical fibration $\mathbb{F}_w \to \mathbb{P}^1$.

Theorem 3.2. *In the notation above if s is the number of singular points of X_0 and h is the number of degenerate fibres of p then $h \le 2$, $s \le h+1$ and $w = 4 + 3h$.*

Proof. The two inequalities follows from Prop. III.4.3 since the singularities of X_0 are cyclic quotient, we now prove the relation $w = 4 + 3h$.

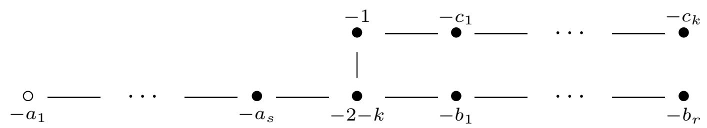
Let $f_0 \subset S$ be a degenerate fibre of p and let E be the (unique) (-1)-curve contained in f_0. Since $\overline{f_0 \setminus E}$ must be a disjoint union of strings, the dual intersection graph of f_0 must be one of the following two (the white circle denotes the irreducible component which intersects σ_∞):

(1)



$s, r > 0$

(2)



$c_1 = ... = c_k = 2$

$r, s > 0$

In both cases the relation $\sum a_i + \sum b_j = 3(s+r) - 2$ holds.

In case 1), since by Th. 3.1.iv) $\sum b_i = 3r + 1$, $\sum a_i$ must be equal to $3s - 3$.

In case 2), by 3.1.iv) it follows that $k = 0$, in fact if $k > 0$ then k would satisfy the relation $2k = \sum c_i = 3k + 1$. Therefore $k = 0$ and $\sum a_i + 2 + \sum b_j = 3(s + r + 1) - 3$.

In every case d satisfies the relation $d - 3 - 3h = 1$. □

For every $a = [a_1, ..., a_r] \in A$ we define its length to be the integer $l(a) = \sum_1^r a_i - r - 1$: we observe that $l(a) \ge 0$ and equality holds if and only if $a = [2]$.

We have 4 injective maps of A into itself defined below; if $a = [a_1, ..., a_r]$ we set

$d_1 a = [a_1, ..., a_r + 1]$

$d_2 a = [a_1, ..., a_r, 2]$

$s_1 a = [2, a_1, ..., a_r]$

$s_2 a = [a_1 + 1, ..., a_r]$

If $h \in \{1, 2\}$; then from Theorem 3.1 it follows that $d_h s_h a \in T_d$ if and only if $a \in T_d$.

Given any $a \in A$ of length l there exists exactly one sequence $i_1, ..., i_l$ with values in the set $\{1, 2\}$ such that $a = d_{i_1} \dots d_{i_l}[2]$; we then set $a' = s_{i_1} \dots s_{i_l}[2]$.

The mapping $a \to a'$ has the following geometric meaning. Let S be a smooth compact surface and let D be a global normal crossing divisor on S whose component are smooth rational curves with the following weighted dual graph

$$\underset{-a_1}{\bullet} — \cdots — \underset{-a_s}{\bullet} — \underset{-1}{\bullet} — \underset{-b_1}{\bullet} — \cdots — \underset{-b_r}{\bullet} \qquad (*)$$

where a_i, b_j are integers ≥ 2.

Let E be the (-1)-curve contained in D. We have two different types of blowing up of S with base point $p \in E$.

Type (a). This is the case if the base point p of the blowing up is a smooth point of D. The strict transform of D has the same properties of D with weighted dual graph

$$\underset{-a_1}{\bullet} — \cdots — \underset{-a_s}{\bullet} — \underset{-2}{\bullet} — \underset{-b_1}{\bullet} — \cdots — \underset{-b_r}{\bullet}$$

Type (b). This is the case if the base point p belong also to another component of D. The global transform of D has thus one of the following dual graphs.

$$\underset{-a_1}{\bullet} — \cdots — \underset{-a_s}{\bullet} — \underset{-2}{\bullet} — \underset{-1}{\bullet} — \underset{-b_1-1}{\bullet} — \cdots — \underset{-b_r}{\bullet}$$

$$\underset{-a_1}{\bullet} — \cdots — \underset{-a_s-1}{\bullet} — \underset{-1}{\bullet} — \underset{-2}{\bullet} — \underset{-b_1}{\bullet} — \cdots — \underset{-b_r}{\bullet}$$

It's very easy to see that if $a = [a_1, ..., a_s]$ and $b = [b_1, ..., b_r]$, then $a' = b$ if and only if the string $(*)$ is the global transform, by a finite sequence of blowings up of type (b), of the string $\underset{-2}{\bullet} — \underset{-1}{\bullet} — \underset{-2}{\bullet}$.

Lemma 3.3. *For every $a \in A$ and $h \in \{1, 2\}$ we have:*

1) $(d_h a)' = s_h a'$

2) $(s_h a)' = d_h a'$

3) $a'' = a$

Proof. 1) follows immediately from the definition of a'. We prove 2) and 3) by induction on $l(a)$; if $l(a) = 0$ the proof follows by a direct inspection.

Let's suppose $l(a) > 0$; then we have $a = d_k c$ for some $k = 1, 2$ and $c \in A$ with $l(c) = l(a) - 1$. Since s_h commutes with d_k, we have, by the induction hypothesis

$$(s_h a)' = (d_k s_h c)' = s_k (s_h c)' = d_h s_k c' = d_h a'$$

$$a'' = (d_k c)'' = (s_k c')' = d_k c'' = d_k c = a$$

□

Remark. . If $T'_d = \{a' | a \in T_d\}$, then a similar theorem to Theorem 3.1 holds for the sets T'_d. It is enough to exchange i) with

i$'$) $\underset{-2}{\bullet} — \underset{-d-1}{\bullet} — \underset{-2}{\bullet} \in T'_d$

while ii) and iii) remain unchanged. Since $[2, d+1, 2] = \dfrac{4d}{2d+1}$, from a calculation similar to that of the proof of Theorem 3.1.ii), it follows that $T'_d = \{\dfrac{dn^2}{dna+1} | 0 < a < n,\, (a, n) = 1\}$.

If $a=[a_1,...,a_s]$ and $b=[b_1,...,b_r]$ we define

$$[a,b]=[a_1,...,a_s,b_1,...,b_r] \qquad a*b=[a_1,...,a_s+1,b_1,...,b_r]$$

Corollary 3.4. *If $a,b\in A$, then:*

1) $(a*b)'=b'*a'$
2) $a*b*a'\in T_1 \Leftrightarrow b\in T_1$
3) $a*[b,2,b']\in T_1 \Rightarrow a*[a*b,2,b'*a']=a*[a*b,2,(a*b)']\in T_1$
4) $b'\in T_1$ *and* $a*b\in T_1 \Rightarrow a*b'*a'\in T_1$ *and* $a*a*b*a'=a*(a*b'*a')'\in T_1$

Proof. We first prove 1) and 2) by induction on $l(a)$.
If $a=[2]$ then $(a*b)'=(s_2s_1b)'=d_2d_1b'=b'*a'$, $a*b*a'=d_2s_2d_1s_1b\in T_1\Leftrightarrow b\in T_1$.
If $a=s_hc$ then, by the induction hypothesis, $(a*b)'=(s_hc*b)'=d_hb'*c'=b'*a'$,
$a*b*a'=s_hd_hc*b*c'\in T_1\Leftrightarrow b\in T_1$.
3) and 4) are trivial consequences of 1) and 2). □

4. Examples of normal degenerations of $\mathbb{P}^2$

In the introduction we have seen how to construct a countable family $\{X_{0,n}\}$ of degenerations of $\mathbb{P}^2$ obtained by sweeping out the cone of the general Veronese surfaces with hyperplane sections.
In this section we give further examples of degenerations of $\mathbb{P}^2$ with at most quotient singularities, this gives a negative answer to our first question.
Unfortunately in our examples it is very difficult to give explicitly the family $f\colon X\to\Delta$: we shall use the following theorem.

Theorem 4.1. *Let X_0 be a normal projective surface. Suppose the following conditions are satisfied:*

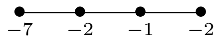
1) $q(X_0)=p_g(X_0)=0$
2) $P_{-1}(X_0)>0$.
3) $\varrho(X_0)=1$
4) X_0 *has at most cyclic singularities of type* $\frac{1}{n^2}(1,na-1)$, $(n,a)=1$.

Then X_0 is a normal projective degeneration of $\mathbb{P}^2$.

Proof. Every singularity of X_0 admits a $\mathbb{Q}$-Gorenstein smoothing, therefore, according to the globalization result of section III.5 there exists a projective $\mathbb{Q}$-Gorenstein smoothing $X\to\Delta$ of X_0. Semicontinuity gives $q(X_t)=P_n(X_t)=0\ \forall n>0$, and X_t is a rational surface.
Let D be the exceptional divisor of δ, we have $\varrho(S)=\varrho(X_0)+b_2(D)=1+b_2(D)$. From Noether's formula follows that $K_Y^2=10-\varrho(S)=9-b_2(D)$ and remembering that the invariant β of our singularities is 0 we get $K_{X_0}^2=9$.
Since X is $\mathbb{Q}$-Gorenstein $K_{X_t}^2=9$ for every t and the only rational surface satisfying this is the projective plane. □

The surface X_0 obtained from $\mathbb{F}_4$ by contracting the section σ_∞ satisfies the hypothesis of Theorem 4.1 and thus it is a normal degeneration of $\mathbb{P}^2$. We note that this surface is exactly the surface $X_{0,2}$ of the collection B (i.e. the cone over the rational curve of degree 4 in $\mathbb{P}^4$). By Theorem 3.2 it follows that this surface is the only degeneration of $\mathbb{P}^2$ with at most quotient singularities and weight 4.

We now try to find normal degeneration of $\mathbb{P}^2$ with quotient singularities and weight 7. For this we first operate two quadratic transforms of $\mathbb{F}_7$ such that the string $\sigma_\infty + f$ becomes



$(*)$

We now proceed by iterating blowings up of type (b) and possibly one, the last, of type (a) with respect to $(*)$ and its transforms.

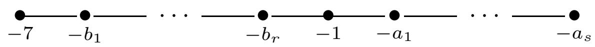
Let $\mu: S \to \mathbb{F}_7$ be the composition of these blowings up and let $D \subset \mu^{-1}(\sigma_\infty + f)$ be the union of the irreducible components with self-intersection < -1.

It's easy to see that $P_{-1}(S) > 0$, if fact $-K_{\mathbb{F}_7} = 2\sigma_\infty + 9f$ and by using adjunction formula we are able to write $-K_S$ as an effective divisor.

If D is a disjoint union of strings $\in T_1$, then the surface X_0 given by S contracting D satisfies the hypothesis of Theorem 4.1. In fact, by the Leray spectral sequence $q(X_0) = p_g(X_0) = 0$ and since $P_{-1}(S) \leq P_{-1}(X_0)$ it follows that $-K_{X_0}$ is effective.

We note that from Theorem 3.2 and its proof it follows that every degeneration of $\mathbb{P}^2$ with at most quotient singularities and weight 7 arises in this way.

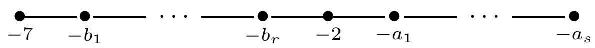
Given any $b = [b_1, ..., b_r] \in A$ there exists a (unique) finite sequence of blowings up of type (b) such that the global transform of $(*)$ becomes



$(**)$

If $[7, b_1, ..., b_r]$ and $[a_1, ..., a_s] = b'$ belong to T_1 then we can contract the corresponding curves and we obtain a surface X_0 with two cyclic singularities.

After a blowing up of type (a) with respect to $(**)$ the strict transform becomes



Thus if $[7, b, 2, b'] \in T_1$ then, by contraction, we obtain a surface X_0 with one cyclic singularity.

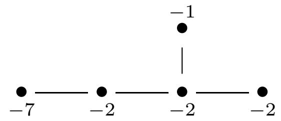
By using the combinatorics developed in the previous section we can write:

Proposition 4.2. *Given a $b \in A$, if $[6] * b, b' \in T_1$ (resp.: $[6] * [b, 2, b'] \in T_1$) then there exists a smooth rational surface S of weight 7, which is the minimal resolution of a normal projective degeneration of $\mathbb{P}^2$ with two cyclic singularities of respective types $[6] * b$, b' (resp.: one cyclic singularity of type $[6] * [b, 2, b']$).*

Given a $b \in A$ satisfying Prop. 4.2 we can find readily infinitely many others: in fact if $[6]*b, b' \in T_1$ (resp.: $[6]*[b,2,b'] \in T_1$), then $\hat{b} = [6]*b*[6]'$ (resp.: $\hat{b} = [6]*b$) has, by Cor. 3.4, the same properties.

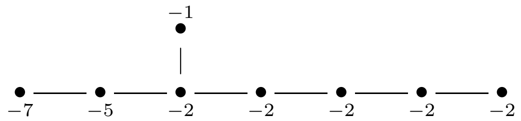
Four examples are the following ones

1)



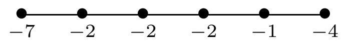
X_0 has a cyclic singularity of type $\frac{1}{25}(1,4)$.

2)



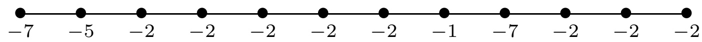
X_0 has a cyclic singularity of type $\frac{1}{13^2}(1,25)$.

3)



X_0 has a cyclic singularity of type $\frac{1}{25}(1,4)$ and one of type $\frac{1}{4}(1,1)$.

4)



X_0 has a cyclic singularity of type $\frac{1}{13^2}(1,25)$ and one of type $\frac{1}{25}(1,4)$.

As a consequence of these examples and Prop. 4.2 we have

Corollary 4.3. *Let $(n_i, a_i), (m_i, b_i)$ be the two sequences in $\mathbb{Z}^2$ defined as follows:*

$$\begin{cases} (n_0, a_0) = (5,1) \\ a_{i+1} = n_i \\ n_{i+1} = 7n_i - a_i \end{cases} \qquad \begin{cases} (m_0, b_0) = (2,1) \\ b_{i+1} = m_i \\ m_{i+1} = 7m_i - b_i \end{cases}$$

Then for every $i \in \mathbb{N}$ there exist four normal degenerations of $\mathbb{P}^2$ with the following singularities respectively:

1) A cyclic singularity of type $\frac{1}{n_i^2}(1, n_i a_i - 1)$.

2) A cyclic singularity of type $\frac{1}{m_i^2}(1, m_i b_i - 1)$.

3) Two cyclic singularities of respective types $\frac{1}{n_i^2}(1, n_i a_i - 1)$, $\frac{1}{m_i^2}(1, m_i b_i - 1)$.

4) Two cyclic singularities of respective types $\frac{1}{n_i^2}(1, n_i a_i - 1)$, $\frac{1}{m_{i+1}^2}(1, m_{i+1} b_{i+1} - 1)$.

Moreover every degeneration of $\mathbb{P}^2$ with at most quotient singularities and of weight 7 is one of these.

Proof. The first part follows from the above considerations and by observing that if $b = \frac{n^2}{nq-1} \in T_1$, then we have $[6]*b*[6]' = \frac{(7n-q)^2}{(7n-q)n-1}$. For the second part one can use an induction argument. We prove this result only for surfaces X_0 with one singular point x, the case where X_0 has two singular points, being similar, is left to the reader.

By Theorem 3.2 there exists a $b \in A$ such that the Dynkin diagram of (X_0, x) is $[6]*[b,2,b']$. Let's suppose $b \neq [2],[5]$, if we prove that $b = [6]*c$ for some $c \in A$, then the conclusion will follow by induction.

We first note that if $n, m \geq 2$ and $[n]*a*[m]' \in T_1$ for some $a \in A$ then $n = m$ (apply 3.1.iii)). We have three subcases:

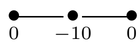
i) $b = [n]$, $n > 2$: then $[6]*[b,2,b'] = [6]*[n-1]*[n+1]'$ and by the previous remark $n+1 = 6$.

ii) $b = [2,c]$, $c \in A$: by Theorem 3.1.iii) this case cannot appear.

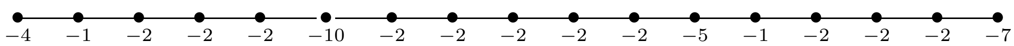
iii) $b = [n]*c$, $n \geq 2$, $c \in A$: then $[6]*[b,2,b'] = [6]*[b,2,c']*[n]'$ and $n = 6$ as required. □

Exercise. Prove directly that $G.C.D.(n_i, m_i) = G.C.D.(n_i, m_{i+1}) = 1$ for every $i > 0$. (Hint: first prove that n_i, m_i are not divisible by 3 and then compute the vector product $(n_i, a_i) \wedge (m_i, b_i)$).

By a similar construction we are able to describe some examples of minimal resolution of a normal degeneration of $\mathbb{P}^2$ by starting from the string $\sigma_\infty + f_1 + f_2 \subset \mathbb{F}_{10}$ and iterating blowings up. From Cor. 3.4 it follows that given such an example we can readily find infinitely many others.

Example. The following string can be obtained from 

by iterating a sequence of 14 blowings up.



Contracting all the components with self-intersection < -1 we obtain a normal degeneration of $\mathbb{P}^2$ with three cyclic singularities of respective types $\frac{1}{4}(1,1)$, $\frac{1}{25}(1,4)$, $\frac{1}{29^2}(1, 29\cdot 21 - 1)$.

5. Proof of theorems B and C.

Suppose first X_0 is a degeneration of $\mathbb{P}^2$ with at most quotient singularities. The properties a),...,f) are exactly those stated in the Theorems 1.3, 2.4 and 3.2.

Suppose now a), b), c) and d) hold, in order to apply Theorem 4.1 we have only to show that $P_{-1}(X_0) > 0$.

Let $Y \xrightarrow{\delta} X_0$ be the minimal resolution; the discussion made in the proof of Theorem 4.1 shows that Y is a rational surface and $K^2_{X_0} = 9$.

By the Serre duality theorem and the Riemann-Roch formula for Weil divisors on normal surfaces we get

$$P_{-1}(X_0) = P_{-1}(X_0) + P_2(X_0) \geq \chi(-K_{X_0}) = \chi(\mathcal{O}_{X_0}) + K^2_{X_0} + \sum_{i=1}^{s} c(X_0, p_i)$$

where $p_1, ..., p_s$ are the singular points of X_0 and, $\forall i$, $c(X_0, p_i) \in \mathbb{Q}$ is a local analytic invariant of the normal surface singularity (X_0, p_i). The proof will follow immediately from the following assertion.

Assertion. If a two dimensional normal surface singularity $(X_0, 0)$ admits a $\mathbb{Q}$-Gorenstein smoothing $(X, 0) \to (\mathbb{C}, 0)$ then $c(X_0, 0) \geq 0$.

This assertion is perhaps trivial for the experts, but we prove it here for completeness. According to Looijenga globalization theorem ([Lo1] Appendix) there exists a compact complex surface V_0 with distinguished point $0 \in V_0$, a reduced three dimensional complex space V and a proper flat map $F: V \to \Delta$ such that $F^{-1}(0) \simeq V_0$, $(V_0, 0) \simeq (X_0, 0)$, $(V, 0) \simeq (X, 0)$ and F is smooth in $V - \{0\}$. In particular V is $\mathbb{Q}$-Gorenstein.

We have

$$\chi(-K_{V_0}) \geq \chi(-K_V \otimes \mathcal{O}_{V_0}) = \chi(-K_{V_t}) = \chi(\mathcal{O}_{V_t}) + K_{V_t}^2 = \chi(\mathcal{O}_{V_0}) + K_{V_0}^2$$

On the other hand the Riemann-Roch formula in V_0 gives

$$\chi(-K_{V_0}) = \chi(\mathcal{O}_{V_0}) + K_{V_0}^2 + c(V_0, 0)$$

Since $c(V_0, 0) = c(X_0, 0) \geq 0$ the assertion is proved. □

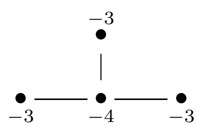
Let $f: X \to \Delta$ be a projective degeneration of $\mathbb{P}^2$ and assume that X_0 has at most rational singularities. Let $x_1, ..., x_s \in X_0$ be its singular points. We note that f is a smoothing of each (X_0, x_i). Denote by $D \subset \prod_{i=1}^s Def(X_0, x_i)$ the product of smoothing components which contain f and write $H = \phi^{-1} D$ where ϕ is the natural map $Def(X_0) \to \prod_{i=1}^s Def(X_0, x_i)$. By lemma III.5.2 ϕ is smooth and then H is an irreducible germ, since D is, moreover the projective plane is rigid and then every smooth surface corresponding to a point of H is isomorphic to $\mathbb{P}^2$. In particular for every $k \leq s$ if X_0^k is the surface obtained from X_0 by smoothing only the singularities (X_0, x_i) for $i = 1, ..., k$ then X_0^k is a normal projective degeneration of $\mathbb{P}^2$.

The proof of theorem C is now easy, in fact since X_0 belongs to class (A) it has at most one noncyclic singularity say at x_1 and the surface X_0^1 is then a degeneration of $\mathbb{P}^2$ with at most quotient singularities.

Actually we don't know any example of degeneration of $\mathbb{P}^2$ with some rational nonquotient singularity. The rational singularities which can appear in a normal degeneration of $\mathbb{P}^2$ are those admitting a $\mathbb{Q}$-Gorenstein smoothing with Milnor number 0 and there exist a lot of singularities with this property apart those described in 2.3.

Jonathan Wahl gives infinitely many examples ([Wa1] 5.9.2, [Wa5]) of rational quasi-homogeneous taut surface singularities admitting a $\mathbb{Q}$-Gorenstein smoothing with Milnor number equal to 0, the simplest of which has Dynkin diagram

5.1)



Let now X_0 be a normal projective surface with at most rational singularities, $\varrho = 1$, $P_{-1} \geq 5$ containing the above singularity 5.1. Then according to III.4.4 X_0 belong to class A, its minimal resolution is a rational surface of weight 4 and X_0 contains three rational double points of type A_2, in particular X_0 cannot be a degeneration of $\mathbb{P}^2$.

More generally, using the computation of the invariant β for the singularities of the class T_1 and theorem III.4.4, it is easy to see that the same conclusion holds for the other singularities in ([Wa1] 5.9.2).

Appendix IV. The Picard Variety of a Moishezon Surface

In this appendix we prove the following result used in the alternative proof of theorem IV.1.3.

Theorem A1. *Let X be a normal irreducible complex surface of algebraic dimension 2, then* $\mathrm{Pic}^0(X)$ *is an abelian variety.*

Let $S \xrightarrow{\pi} X$ be the minimal resolution of singularities with exceptional reduced divisor D, then $a(S) = 2$ and since S is smooth it is projective ([B-P-V] Cor. IV.5.5), in particular $\mathrm{Pic}^0(S)$ is an abelian variety. Our strategy of proof is to show that $\mathrm{Pic}^0(X)$ is isomorphic to a compact complex Lie subgroup of the Picard variety of S.

We begin with some preliminary results; let $C \subset S$ be a (possibly non reduced) curve, the exponential sequences on S and C give a commutative diagram

$$\begin{array}{ccccccc} H^1(S,\mathbb{Z}) & \xrightarrow{i} & H^1(\mathcal{O}_S) & \xrightarrow{e} & \mathrm{Pic}^0(S) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta_C} & & \downarrow{\scriptstyle\gamma} & & \\ H^1(C,\mathbb{Z}) & \xrightarrow{i} & H^1(\mathcal{O}_C) & \longrightarrow & \mathrm{Pic}^0(C) & \longrightarrow & 0 \end{array}$$

Lemma A2. *In the above notation $e(\ker \beta_C)$ is a compact complex Lie subgroup of* $\mathrm{Pic}^0(S)$.

Proof. Denote by $\Gamma = i(H^1(C,\mathbb{Z})) \subset H^1(\mathcal{O}_C)$, $\Delta = i(H^1(S,\mathbb{Z})) \subset H^1(\mathcal{O}_S)$, $E = Im\,\beta_C$, $\Gamma' = \Gamma \cap E$, $W_0 = \ker\,\beta_C$, $K = \ker\,\gamma$, $W = e^{-1}(K)$.

According to ([B-P-V] Prop. II.2.1), Γ is a closed discrete subgroup of $H^1(\mathcal{O}_C)$ and $\mathrm{Pic}^0(C)$ is Hausdorff, in particular K is a compact subgroup of $\mathrm{Pic}^0(S)$.

We claim that $e(W_0)$ is precisely the maximal connected subgroup K_0 of K, in fact there exists a (non canonical) isomorphism of topological groups $W = W_0 \oplus \Gamma'$ and W_0 is the path-connected component of W containing 0. K_0 is path-connected and e is a covering map, in particular by homotopy lifting property it follows easily that $e(W_0) = K_0$. □

The Leray's spectral sequence applied to π gives an exact sequence

$$0 \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_S) \xrightarrow{p} H^0(R^1\pi_*\mathcal{O}_S)$$

where p is the projective limits of the natural restriction maps $\beta_{nD}\colon H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_{nD})$.
Note that since $H^1(\mathcal{O}_S)$ is finite dimensional $\ker\,p = \ker\,\beta_{nD}$ for n sufficiently large.

Proof of theorem A1. Since X is normal $\pi_*\mathcal{O}_S = \mathcal{O}_X$ and $\pi_*\mathcal{O}_S^* = \mathcal{O}_X^*$ therefore we have a commutative exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathcal{O}_X) & \longrightarrow & H^1(\mathcal{O}_S) & \xrightarrow{p} & R^1\pi_*\mathcal{O}_S \\ & & \downarrow & & \downarrow{\scriptstyle e} & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \xrightarrow{\pi^*} & \mathrm{Pic}^0(S) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

and by lemma A2 $\pi^*(\mathrm{Pic}^0(X)) = e(\ker\,p)$ is a compact subgroup of $\mathrm{Pic}^0(S)$. □

V. General properties of moduli space of surfaces of general type.

Here we introduce the moduli space of surfaces of general type whose existence as quasiprojective variety was proved in 1976 by Gieseker and we list some properties of it. Then we introduce the problem of the connectedness of the moduli space of surfaces with fixed topological type and we shall show that the families of natural deformations of simple bihyperelliptic surfaces give examples of connected components.

1. What is the moduli space?

We shall say that two smooth surfaces S_1, S_2 are deformation each other in the large if there exists a proper flat family of smooth surfaces $f: X \to C$ where C is an irreducible smooth curve and there exists two fibres of f respectively isomorphic to S_1, S_2. Deformation in the large is a relation in the set of isomorphism classes of smooth surfaces and the equivalence relation generated is called *deformation equivalence* and will be denoted by $\stackrel{def}{\sim}$.
Let $X \to Y$ be a flat family of surfaces over an irreducible quasiprojective variety Y, then any two fibres of it are deformation equivalent, in fact by taking restriction to hyperplane sections we can assume that Y is a connected curve and then by transitivity we may reduce to the case where Y is an irreducible curve. Taking if necessary the normalization of Y we get a family over a smooth curve.
There are several properties of smooth surfaces which are invariant under deformation equivalence, here we list the most important ones.
a) By Ehresmann fibration theorem two deformation equivalent surfaces have the same differential structure, in particular all the topological and differential invariants of the underlying oriented 4-manifold are invariants under $\stackrel{def}{\sim}$. We recall that for a complex algebraic surface S the invariants $K_S^2, \chi(O_S)$ are topological invariants, more precisely

$$2\chi(\mathcal{O}_S) = 1 - b_1(S) + b_+, \quad K_S^2 = 12\chi(\mathcal{O}_S) - e(S) = b_+ - b_- + 8\chi(\mathcal{O}_S)$$

where $e(S)$ is the topological Euler-Poincaré characteristic and b_+, b_- are respectively the number of positive and negative eigenvalues of the intersection form on $H_2(S, \mathbb{Q})$.

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b) If $S_1 \overset{def}{\sim} S_2$ then S_1 and S_2 have the same Kodaira dimension. In fact holds the following stronger results proved by Iitaka as a (non trivial) consequence of Enriques-Kodaira classification of surfaces:

Theorem 1.1.([Ii]) *The positive plurigenera of a smooth compact complex surfaces are invariant under arbitrary holomorphic deformations.*

The deformations of Segre-Hirzebruch surfaces $\mathbb{F}_q$ (cf. [Ko2], [Ca6]) give examples where the negative plurigenera are not preserved.

The point b) is also a consequence of point a) and the fact that the Kodaira dimension is a differential invariant of smooth algebraic surfaces (this is the well known Van de Ven conjecture, proved recently by V.Pidstrigach and A.Tyurin using Donaldson theory and then simplified by C.Okonek and A.Teleman by using Seiberg-Witten invariants).

c) If $S_1 \overset{def}{\sim} S_2$ and S_1 is minimal of general type then also S_2 is minimal of general type.

This follows from Iitaka theorem because a surface of general type is minimal if and only if $P_2 = \chi + K^2$. The same result can be proved directly by using Kodaira theorem on stability of submanifolds [Ko1] (which is another essential tool used in the proof of Iitaka theorem).

Let $f\colon X \to C$ be a smooth family of surfaces over a smooth irreducible curve C and let $A \subset C$ the set of points whose fibres are minimal of general type. Since a surface S is minimal of general type if and only if $K_S^2 > 0$, $\chi(\mathcal{O}_S) > 0$, $H^1(2K) = H^2(2K) = 0$ by semicontinuity the set A is open. Let p a point in the closure of A, by semicontinuity of plurigenera $S_p = f^{-1}(p)$ is of general type and the proof is complete if we prove that it is minimal.

Assume S_p not minimal and let $E \subset S_p$ be a (-1)-curve, according to Kodaira stability theorem there exists a small open disk $D \subset C$ centered at p and a smooth subvariety $W \subset X$ such that $W \cap f^{-1}(q)$ is a (-1)-curve in $f^{-1}(q)$ for every $q \in D$, taking $q \in D \cap A$ we get a contradiction.

Note that without the assumption that the fibres of f are of general type it is false that A is closed, consider for example the deformation of the Segre-Hirzebruch surface $\mathbb{F}_3$ which deforms to the blow up of $\mathbb{P}^2$ at a point.

d) If S is a minimal surface of general type then $K_S^2 > 0$, in particular the canonical class $k_S = c_1(K_S) = -c_1(S)$ does not belong to the torsion subgroup of $H^2(S, \mathbb{Z})$ and then it is well defined its divisibility

$$r(S) = \max\{r \in \mathbb{N} | r^{-1} c_1(S) \in H^2(S, \mathbb{Z})\}$$

This is obviously a deformation invariant.

The definition of $\overset{def}{\sim}$ generalize in a natural way to the class of normal projective surfaces with at most rational double points. According to Brieskorn-Tyurina simultaneous resolution if two surfaces X_1, X_2 with at most RDP's are deformation equivalent the same holds for their minimal resolutions.

From now on, in order to avoid heavy notation, we shall call C-model every algebraic surface which is the canonical model of a minimal surface of general type. Let P be a set of properties of projective surfaces with at most RDP's which are invariant under $\stackrel{def}{\sim}$, an algebraic variety $\mathcal{M}(P)$ is called a *coarse moduli space* for C-models satisfying P if has the following properties:

M1) There exists a bijection between the set of closed points of $\mathcal{M}(P)$ and the set of isomorphism classes of C-models satisfying P.

M2) It is defined for every flat family $f\colon X \to T$ of C-models satisfying P a map $\mu(f)\colon T \to \mathcal{M}(P)$ such that for every closed point $t \in T$, $\mu(f)(t)$ is the closed point of $\mathcal{M}(P)$ corresponding to the isomorphism class of $f^{-1}(t)$. Moreover the maps μ must be compatible with base change, i.e. if a flat family $f'\colon X' \to T'$ is induced from f by a morphism $\phi\colon T' \to T$ then $\mu(f') = \mu(f) \circ \phi$.

M3) If $\mathcal{N}(\mathcal{P})$ is another algebraic variety which satisfy M1 and M2 with maps $\nu\colon T \to \mathcal{N}(\mathcal{P})$ then there exists a *unique* morphism of algebraic varieties $\Phi\colon \mathcal{M}(P) \to \mathcal{N}(\mathcal{P})$ such that for every family $f\colon X \to T$ $\nu(f) = \Phi \circ \mu(f)$.

It is clear that is a coarse moduli space exists then it is unique up to isomorphism, note that properties M3 is necessary in order to have unicity, in fact if $\mathcal{M}$ satisfy M1, M2 then the same is true for every product of $\mathcal{M}$ with a fat point.

The main result about the existence of coarse moduli space for surfaces is

Theorem 1.2.(Gieseker [Gi1])

a) For any pair x, y *of positive integers there exists a (possibly empty) quasiprojective variety* $\mathcal{M}_{x,y}$ *which is a coarse moduli space for canonical models* X *of surfaces of general type with* $\chi(\mathcal{O}_X) = x, K_X^2 = y$.

b) Two minimal surfaces of general type are deformation equivalent if and only if the isomorphism classes of their canonical models belongs to the same connected component of the moduli space.

Two surfaces of general type are birational if and only if they have the same canonical model, so roughly speaking, the moduli space $\mathcal{M} = \cup_{x,y}\mathcal{M}_{x,y}$ classify surfaces of general type up to birational equivalence and the space $\mathcal{M}$ is usually called the moduli space of surfaces of general type.

The reason of considering canonical model instead of minimal models for the construction of $\mathcal{M}$ is essentially technical and it will be clear in the next section. The statement 1.2.*b)* follows from the construction of the moduli space and not from its general functorial properties. In the next section we explain (without details) the construction of $\mathcal{M}$ and from this we deduce *b)* and the local analytic structure of $\mathcal{M}_{x,y}$.

Therefore the problem to determine if two surfaces are deformation equivalent is reduced to the (usually easier) problem to determine the connected components of moduli space.

2. Outline of the construction of the moduli space of surfaces of general type and its local analytic structure.

In this section we consider only C-models with fixed numerical invariant χ, K^2, this implies that all C-models have the same plurigenera $P_n = \chi + \frac{1}{2}n(n-1)K^2$ for every $n \geq 2$.

A *n-framed* C-model is the data of a C-model S together with a complete nondegenerate embedding $\nu\colon S \to \mathbb{P}^{P_n-1}$ such that $\omega_S^{\otimes n} = \nu^*\mathcal{O}(1)$. The general theory of pluricanonical maps tell us that for $n \geq 5$ every C-model has a $n-framing$.

Note that the group $SL(P_n, \mathbb{C})$, $n \geq 5$, acts via the projection $SL \to PGL$ in the set of n-framed C-model and the orbits of this action are the isomorphism classes of C-models.

There exists a natural concept of family of framed C-models, this is a consequence of the existence of the relative dualizing sheaf for a morphism.

More generally let $f\colon X \to Y$ be a flat family of normal surfaces and let $U \subset X$ be the (scheme theoretic) open subvariety of points where the map f is smooth. Then it is defined the relative dualizing sheaf $\omega_{X/Y}$ on X satisfying the following conditions ([Lip2] §3, [Wa2] 1.3):

(i) $\omega_{X/Y}$ is a coherent f-flat $\mathcal{O}_X$-module.

(ii) If $i\colon U \to X$ is the open immersion then $\omega_{X/Y} = i_*(\bigwedge^2 \Omega^1_{U/Y})$ where $\Omega^1_{U/Y}$ is the locally free sheaf of relative differentials.

(iii) The relative dualizing sheaf has the base change property, i.e. for every morphism $Y' \to Y$ if $\pi: X' = X \times_Y Y' \to X$ is the projection then $\omega_{X'/Y'} = \pi^*\omega_{X/Y}$.

(iv) If the fibres of f are Gorenstein (e.g. if f is a family of C-models) then $\omega_{X/Y}$ is locally free.

Definition . A *family of n-framed C-models* is the data of a family $f: X \to Y$ of C-models with a closed embedding $\nu: X \to Y \times \mathbb{P}^{P_n - 1}$ such that f is the composition of ν with the projection in the first factor and $\nu^*\mathcal{O}(1) = \omega_{X/Y}^{\otimes n}$.
Note that this is a good definition of families, in fact if $Y' \to Y$ is a morphism then, since the relative dualizing sheaf commutes with base change, the pull-back of the embedding ν gives an induced structure of n-framed family on the fiber product $X \times_Y Y'$. In particular it makes sense the definition of universal family.

Proposition 2.1.(Tankeev [Ta]) *For n sufficiently large there exists an universal family $Z_n \subset H_n \times \mathbb{P}^{P_n - 1}$ of n-framed C-models with H_n quasiprojective variety.*
Idea of proof. It is sufficient to take H_n the locally closed subscheme of the Hilbert scheme of irreducible nondegenerate surfaces S with at most RDP as singularities, $\mathcal{O}_S(nK_S) = \mathcal{O}_S(1)$ and Hilbert polynomial $h(d) = \chi(\mathcal{O}_S(dnK_S)) = \chi(O_S) + \frac{1}{2}dn(dn-1)K_S^2$. □
From the construction of the Hilbert scheme follows that there exists an embedding $H_n \subset \mathbb{P}^N$ as a quasiprojective variety and the natural action of $G = SL(P_n, \mathbb{C})$ on H_n is induced by a linear action of $\mathbb{P}^N$ (cf. [Gi1]).
The bulk of Gieseker paper [Gi1] is devoted to prove the following proposition (written in the language of geometric invariant theory ([Gi2],[Ne]))

Proposition 2.2. *In the notation above for n sufficiently large H_n is contained in the set of G-stable points of $\mathbb{P}^N$ and then there exists the geometric quotient $H_n/G = \mathcal{M}$ which is a quasiprojective variety.*
For reader convenience we recall here the properties which characterize geometric quotients.
Let H be an algebraic variety with a regular action of a linear algebraic group G. A geometric quotient is the data of an algebraic variety $\mathcal{M}$ and a surjective G-invariant affine morphism $\phi: H \to \mathcal{M}$ such that:
1) Every fibre of ϕ contains exactly one G-orbit.
2) $\mathcal{M}$ is a categorical quotient, this means that for every G-invariant morphism $\psi: H \to N$ there exists an unique morphism $\eta: \mathcal{M} \to N$ such that $\psi = \eta \circ \phi$.
3) For every open set $U \subset \mathcal{M}$ there exists an isomorphism

$$\phi^*: \Gamma(U, \mathcal{O}_{\mathcal{M}}) \to \Gamma(\phi^{-1}(U), \mathcal{O}_H)^G$$

.
4) If $W \subset H$ is a closed G-invariant subset then $\phi(W)$ is closed.

Let $\mathcal{M} = H_n/G$ the quotient as in proposition 2.2 and let $f: X \to Y$ be a family of C-models, since on every fibre X_y of f the group $H^1(nK_{X_y})$ vanishes, by semicontinuity and base change there exists an open covering $Y = \cup U_i$ and a structure of n-framed family on every restriction $X \to U_i$. H_n is universal and then there exists maps $\mu_i: U_i \to H_n$ inducing these families. Clearly their compositions with the projection map $H_n \to \mathcal{M}$ can be glued and we obtain finally a map $\mu: Y \to \mathcal{M}$. From this and from the general properties of geometric quotients it follows that $\mathcal{M} = H_n/G$ is a coarse moduli space for canonical models of surfaces of general type with fixed invariants.

Since for every regular action the dimension of the orbits is a lower semicontinuos function the fibres of the projection morphism $\phi: H_n \to \mathcal{M}$ are irreducible of constant dimension. In particular a closed subset $V \subset \mathcal{M}$ is irreducible if and only if $\phi^{-1}(V)$ is irreducible. If S_1, S_2 are canonical models of surfaces of general type belonging to the same irreducible component of $\mathcal{M}$ then there exists an irreducible component V of H_n such that S_1, S_2 are isomorphic to two fibres of the restriction to V of the universal family $Z_n \to H_n$ and then they are deformation equivalent.

Let $[X] \in H_n$ corresponding to a framed C-model $X \subset \mathbb{P}^{P_n - 1}$, the universal n-framed family $Z_n \to H_n$ induces a holomorphic map between germ of complex spaces $h: (H_n, [X]) \to (Def(X), 0)$.

Lemma 2.3. *The above map h is smooth and $h^{-1}(0)$ is the germ of the G-orbit of $[X]$.*

Proof. (sketch) Let $X_A \to Spec(A)$ be an infinitesimal deformation of X and let $p: A \to B$ be a small extension of local Artinian $\mathbb{C}$-algebras.

Since $H^1(\omega_X^{\otimes n}) = 0$ every section of $\omega^{\otimes n}_{X_B/Spec(B)}$ extends to a section of $\omega^{\otimes n}_{X_A/Spec(A)}$, extending the basis that gives the n-framing we can extend the framing to X_A.

Thus h is smooth and since there exists a factorization $(H_n, [X]) \xrightarrow{h} Def(X) \to \mathcal{M}$, $h^{-1}(0)$ is contained in the G-orbit. Conversely it follows from the definition of the G-action on H_n that the restriction of the universal family $Z_n \to H_n$ to every G-orbit is a locally trivial family of C-models and then $h^{-1}(0)$ contains the germ of the G-orbit. ☐

The stabilizer $Stab([X]) \subset PGL(P_n)$ of $[X] \in H_n$ is naturally isomorphic to the group of automorphisms of X and if $T \subset H_n$ is the image of a section of h then the induced action of Stab([X]) on T is compatible with the natural action of $Aut(X)$ on the base space of the Kuranishi family $Def(X)$, thus we have the following

Corollary 2.4. *Let X be the canonical model of a surface of general type, then the germ of $\mathcal{M}$ at $[X]$ is analytically isomorphic to the quotient $Def(X)/Aut(X)$.*

If S is a minimal surface of general type with canonical model X then the blow-down morphism defined in Chapter II, $Def(S) \to Def(X)$ is compatible with the actions of $Aut(X) = Aut(S)$ and then it is defined a natural map $Def(S)/Aut(S) \to \mathcal{M}$. This map is finite but from Burns and Wahl result (II.3.4) in some cases (e.g. K_S not ample and $Aut(S) = 0$) it is not an isomorphism.

Note that if X is a framed C-model with $q(X) = 0$ then locally at $[X]$, H_n is an open subscheme of the Hilbert scheme of $\mathbb{P}^{P_n - 1}$. In fact in this case if $X_A \subset \mathbb{P}^{P_n-1} \times Spec(A)$ is an infinitesimal embedded deformation of X then there exists at most one extension on X_A of the line bundle $O_X(1) = \omega_X^{\otimes n}$ (cf. III.5.1) and then $O_{X_A}(1) = \omega_{X_A/Spec(A)}^{\otimes n}$.

In next chapters we need to compute the closure of some subsets of the moduli space $\mathcal{M}$. The valuative criterion ([Ha1] pag 101, [Ne] pag 7) gives:

Let $N \subset \mathcal{M}$ be a locally closed subvariety and let X_0 be a C-model. Then $[X_0]$ belong to the closure of N if and only if there exists a flat family of C-models $f\colon X \to \Delta = \{t \in \mathbb{C} | \, |t| < \epsilon\}$ such that $[X_t] \in N$ for every $t \neq 0$.

This criterion is used in the proof ([Ca2] theorem 1.8) that for every finite group G the subset $\mathcal{M}^G \subset \mathcal{M}$ of minimal surfaces admitting a faithful regular G-action is a closed subvariety of $\mathcal{M}$. A similar result we shall need is the following:

Lemma 2.5. *Let $f\colon X \to \Delta$ be a flat family of C-models, G a finite group and for every $t \neq 0$ let $\varrho_t\colon G \to Aut(X_t)$ be a given faithful action.*

If for every $t \neq 0$ there exists an open neighbourhood $U \ni t$ and a regular G-action on $X_U \to U$ preserving fibres and inducing on every $t \in U$ the representation ϱ_t, then after a possible change of base $\Delta \xrightarrow{t^r} \Delta$ there exists a regular G-action on X preserving fibres and inducing the given G-action on every X_t, $t \neq 0$.

Moreover the quotient family $X/G \to \Delta$ is also flat.

Proof. (sketch) The minimal resolution of X_t is a surface of general type, in particular the group $Aut(X_t)$ is finite for every $t \in \Delta$ ([Mat2],[An]). By a monodromy argument it follows that after a possible change of base there exists a regular G-action on $f\colon X^* \to \Delta^* = \Delta - \{0\}$ inducing the desired G-action in the fibres and the same argument used in the proof of ([Ca2] 1.8, [F-P] 4.4) shows that this action extends to X. The flatness is a consequence of the local irreducibility of X and the flatness criteria for moduli over one-dimensional local regular rings ([Mat1] Exercise 11.8). $\square$

3. Digression: Obstructed deformations and everywhere nonreduced moduli spaces.

A question that had been unsolved for a long time was if every minimal model of surfaces of general type has a smooth complete family of deformations, in fact all the simplest surfaces (e.g. complete intersection, smooth ramified coverings), have this property and for a long time nobody was able to find any example of obstructed deformations.

The first examples of surfaces of general type with obstructed deformations were found independently by Burns-Wahl ([B-W]), Kas ([Kas]) and Horikawa ([Ho]). We have already seen the methods of Burns and Wahl and we have used it in Th. II.5.2 and example II.5.4.

Catanese [Ca7] also used the results of [B-W] for giving several examples and some general recipe to construct minimal surfaces of general type S with singular canonical model X and everywhere nonreduced Kuranishi family. Catanese method requires that the canonical model

X has unobstructed global deformations and then, although $Def(S)$ is nowhere smooth, the moduli space $\mathcal{M}$ is locally reduced and irreducible at S.

The examples of Horikawa are obtained in a completely different way as a consequence of some stability and costability theorems for deformations of holomorphic maps. One of the most interesting is the following (cf. [Ho]III,[Ca6]):

Example 3.1.(Horikawa-Mumford) Let $F \subset \mathbb{P}^3$ be a smooth cubic surface, let $C \subset F$ be a smooth curve linearly equivalent to $4H + 2E$ where H is the hyperplane section and E is a straight line contained in F and let $X \to \mathbb{P}^3$ be the blowing up of $\mathbb{P}^3$ with centre C. Let S be a very ample smooth divisor on X such that $H^1(\theta_X(-S)) = H^2(\theta_X(-S)) = H^1(\mathcal{O}_X(S)) = 0$.
Note that if S is sufficiently ample and general then K_S is very ample by adjunction formula, S is simply connected by Lefschetz theorem and $Aut(S) = 0$. In particular $Def(S)$ is analytically isomorphic to the moduli space $\mathcal{M}$ at the point $[S]$.

Horikawa ([Ho] III §10) claimed and proved that $Def(S)$ is obstructed but is easy to see that $Def(X)$ is everywhere nonreduced. This follows from the following two lemmas:

Lemma 3.2. *In the notation of example 3.1, $Def(X)$ is everywhere nonreduced.*

Proof. See [Ca6] §9. □

Lemma 3.3. *Let X be a smooth complex projective variety of dimension ≥ 3 with $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ and let S be a very ample smooth divisor such that $H^1(\theta_X(-S)) = H^2(\theta_X(-S)) = H^1(\mathcal{O}_X(S)) = 0$.*
Then there exists a noncanonical isomorphism of germs of complex spaces

$$(Def(S), 0) \simeq (Def(X) \times Coker(\psi), 0)$$

where ψ is the natural map $\psi: H^0(X, \theta_X) \to H^0(S, N_{S|X})$.

Proof. (cf. [Ch]) Assume X embedded in $\mathbb{P}^N$ by the complete linear system $|S|$ and let $V \subset H^0(\mathcal{O}_X(S))$ be a small neighbourhood of a section defining S, for every $v \in V$ let $H_v \subset \mathbb{P}^N$ the corresponding hyperplane.

Let $Hilb^N_X$ be the germ of the Hilbert scheme of $\mathbb{P}^N$ at the point X, then we have a smooth family of deformations of the pair (S, X) with base $Hilb^N_X \times V$ given by $\tilde{S} \subset \tilde{X} \times V$ where $\tilde{X} \to Hilb^N_X$ is the universal family and for $t \in Hilb^N_X, v \in V$ $S_{t,v} = X_t \cap H_v$.

Denote now by $Def_{S,X}$ the functor of deformation of the pair $S \subset X$, $Def_{S,X}$ has a good deformation theory and its tangent space is isomorphic to $H^1(\theta_X(-log\, S))$. We have natural maps of functors of Artin rings

$$Hilb^N_X \times V \xrightarrow{f} Def_{S,X} \xrightarrow{g} Def_X$$

$$Def_{S,X} \xrightarrow{h} Def_S$$

We claim that both f, g, h are smooth morphisms. The smoothness of h follows from the vanishing of $H^2(\theta_X(-S))$ and Horikawa costability theorem ([Ho] III.8.3,[Ran]). According to Lemma III.5.1 the composition gf is smooth and then the smoothness of f, g is equivalent to the surjectivity of df. We have an exact sequence

$$H^0(\theta_X) \xrightarrow{\psi} H^0(\mathcal{O}_S(S)) \longrightarrow H^1(\theta_X(-log S)) \xrightarrow{dg} H^1(\theta_X) \longrightarrow 0$$

and the surjectivity of df follows from the surjectivity of the maps $T_{[X]}Hilb_X^N \to H^1(\theta_X)$ and $H^0(\mathcal{O}_X(S)) \to H^0(\mathcal{O}_S(S))$.
From the exact sequence

$$H^1(\theta_X(-S)) \longrightarrow H^1(\theta_X(-log S)) \xrightarrow{dh} H^1(\theta_S) \longrightarrow H^2(\theta_X(-S))$$

it follows that dh is an isomorphism and then the proof follows from general properties of smooth functors. □

The general philosophy of the previous two lemmas is the following: Given a smooth curve C in a projective space $\mathbb{P}^n$ define X_C as the blown up of $\mathbb{P}^n$ with centre C, the proof of lemma 3.2 suggests that the map $C \to X_C$ induces a smooth morphism from the Hilbert scheme of $\mathbb{P}^n$ at C to the Kuranishi family of X_C, while lemma 3.3 can be generalized to complete intersections in X_C of $n-2$ sufficiently ample divisors. We then obtain regular surfaces of general type with ample canonical bundle and with Kuranishi family stably isomorphic to the Hilbert scheme of a curve in a projective space. The "converse map", from deformations of regular surfaces to embedded deformations of curves has been recently explored by B.Fantechi and R.Pardini ([F-P2]).

Corollary 3.4. *There exist everywhere singular irreducible components of the moduli space of surfaces of general type whose general member is a simply connected surface with very ample canonical bundle.*
In [Ch] Chang gives examples of threefolds X in $\mathbb{P}^5$ with $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ and obstructed deformations.

4. Deformation equivalent types of homeomorphic surfaces.

One of the first consequences of Gieseker theorem is that for every pair of positive integer x, y there exists a finite number $\delta(x,y)$ (=number of connected components of the quasiprojective variety $\mathcal{M}_{x,y}$) of deformation equivalence classes of minimal surfaces of general type with invariants $K^2 = y$, $\chi = x$. (More precisely, it is not necessary to assume Gieseker theorem in order to prove the finiteness of deformation equivalent types, but only the projectivity of the Hilbert schemes and Bombieri's results about pluricanonical maps.)
Contrary to the case of curves, where the genus classify completely the deformation equivalence classes, in the case of surfaces the number $\delta(x,y)$ is in general bigger than 1, in fact it is rather easy to show the existence of surfaces with the same invariants K^2, χ but with different homotopy groups. Therefore a more appropriate question is:

Given two homeomorphic minimal surfaces of general type, are they deformation equivalent? The first difficulty here is to determine when two surfaces are homeomorphic, in the case of simply connected surfaces this can be easily done by using Freedman results on the topology of four-manifolds.

For every simply connected compact oriented topological four-manifolds X the group $H^2(X,\mathbb{Z})$ is free of finite rank and the intersection product $q\colon H^2(X,\mathbb{Z})\times H^2(X,\mathbb{Z})\to\mathbb{Z}$ is a symmetric unimodular bilinear form.

Theorem 4.1.(Freedman [Fre] 1.5+addendum) *Let X_1, X_2 be two simply connected compact oriented smooth four-manifolds and let $f^*\colon H^2(X_2,\mathbb{Z}) \to H^2(X_1,\mathbb{Z})$ be an isometry with respect the intersection forms, then there exists a homeomorphism $f\colon X_1 \to X_2$ preserving orientation and inducing f^*.*

For every symmetric bilinear form $q\colon\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ its rank and its signature are defined respectively as the rank and the signature of the extended form $q_{\mathbb{R}}\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$. We shall say that the parity of q is even if $q(x,x)\in 2\mathbb{Z}$ for every $x\in\mathbb{Z}^n$, odd otherwise. A classical results (Eichler's theorem) states

Proposition 4.2.([Se] p.92,[Wall]) *Two unimodular indefinite symmetric bilinear forms defined over the integers are isometric if and only if they have the same rank, signature and parity.*

For definite forms this is not true but in the geometric case this doesn't give any problem, in fact we have:

Theorem 4.3.(Donaldson, [D-K] 1.3.1) *If the intersection form of a simply connected oriented compact smooth four-manifold X is definite positive then the intersection form q is represented by the identity matrix in some basis of $H^2(X,\mathbb{Z})$.*

Theorem 4.4.(Kodaira-Yau, see [B-P-V]) *The projective plane is the only simply connected compact complex surface with definite intersection product.*

Moreover if S is a simply connected algebraic surface then the (mod 2) reduction of $k_S \in H^2(S,\mathbb{Z})$ is exactly the Wu class ([M-S]) and then $q(k_S,x)\equiv q(x,x) \pmod 2$ and then if $k_S\neq 0$ the parity of q is equal to the parity of $r(S)$.

By Noether formula and index theorem K_S^2 and $\chi(\mathcal{O}_S)$ determine the rank and the signature of q_S, putting together all these fact we finally have

Corollary 4.5. *Two simply connected minimal surfaces of general type are orientedly homeomorphic if and only if they have the same K^2,χ and the same parity of r =divisibility of the canonical class.*

Fixing a minimal surface of general type S we define $\mathcal{M}^{top}(S)$ (resp.: $\mathcal{M}^{diff}(S)$) as the set of minimal surfaces of general type homeomorphic (resp.: diffeomorphic) to S, we put on $\mathcal{M}^{top}(S)$, $\mathcal{M}^{diff}(S)$ the topology induced in the natural way by the moduli space $\mathcal{M}$.

Question: Is the space $\mathcal{M}^{top}(S)$ (resp.: $\mathcal{M}^{diff}(S)$) connected?

Theorem 4.6.(Catanese [Ca4]) *The number of connected components of $\mathcal{M}(S)$ can be arbitrarily large.*

The idea of proof is elementary, since the divisibility $r(S)$ of the canonical class is invariant under deformations it is sufficient to find, for every $k>0$, k distinct homeomorphic minimal surfaces of general type $S_1,...,S_k$ with different $r(S_i)$.

Catanese takes as S_i simple bihyperelliptic surfaces (Chapt. II), since such surfaces can be considered as a composition of two double covers its invariants can be easily computed using the following two facts about double coverings.

Let $S \stackrel{\pi}{\longrightarrow} X$ be a double cover of smooth surfaces, denote by $R \subset S$ the ramification divisor and by $D \subset X$ the branching divisor. Note that R and D are both smooth.

Proposition 4.7. *If X is simply connected, $D^2>0$ and there exists a divisor $D_1 \in |D|$ which intersect transversally D then $\pi_1(X-D)$ is an abelian group generated by a small loop around D and S is simply connected.*

This is a well known fact, for a proof see [Ca1].

Proposition 4.8.([Ca4]) *The natural map $\pi^*: NS(X) \rightarrow NS(S)$ is injective. If $H_1(S,\mathbb{Z})=0$ then the image of π^* is a primitive subgroup, in particular if $R=\pi^* L$ then $r(S)$ is the divisibility of K_X+L in $NS(X)$.*

Therefore for a simple bihyperelliptic surface S of type $(a,b)(n,m)$ $a,b,n,m \geq 3$ we have

$$K_S^2 = 8(a+n-2)(b+m-2) \qquad \chi(\mathcal{O}_S) = \frac{1}{8}K_S^2 + ab + nm$$

$$r(S) = G.C.D.(a+n-2, b+m-2) \tag{4.9}$$

and theorem 4.6 is proved whenever we find k solutions a_i, b_i, n_i, m_i of an equation K^2=constant, χ=constant, $r \equiv$constant (mod2) giving k distinct integer values of r (see [Ca4] for details).

A very little in known about the space $\mathcal{M}^{diff}(S)$ because of the lack of simple criteria to determine whether two algebraic surfaces are diffeomorphic or not.

Conjecture. (Friedman-Morgan [F-M]) For every S minimal, $\mathcal{M}^{diff}(S)$ is connected.

Very recently E. Witten ([Wi]), using new differential invariants of smooth four-manifolds, proved that if $f: S_1 \longrightarrow S_2$ is a diffeomorphism of simply connected minimal surfaces of general type then $f^*(k_{S_2}) = \pm k_{S_1}$, in particular the divisibility r is a differential invariant. This result previously conjectured ([Ca4], [F-M]) was known to be true since 1988 for a large class of surfaces (e.g. complete intersections) and using this Friedman, Morgan and Moishezon ([F-M-M]) proved that in general $\mathcal{M}^{diff}(S) \neq \mathcal{M}^{top}(S)$. Later Salvetti ([Sal1],[Sal2]) using the same ideas but different examples proved that the number of homeomorphic algebraic surfaces of general type with different differentiable structures can be arbitrarily large.

In general, given a unimodular quadratic form of rank b and signature σ over an integral lattice Λ, a primitive vector $v \in \Lambda$ is called of characteristic type if $v \cdot x \equiv x^2 \pmod 2$ for every $x \in \Lambda$, otherwise it is called of ordinary type. Note that if the quadratic form is even than every primitive vector is of ordinary type.

A theorem of Wall ([Wall]) states that if $b-|\sigma| \geq 4$ then the group of isometric automorphism of Λ acts transitively on the set of primitive vectors of fixed norm and type. If $\Lambda = H^2(S,\mathbb{Z})$, S simply connected compact complex surface, the condition $b-|\sigma| \geq 4$ is equivalent to $\chi(\mathcal{O}_S) > 1$ and the primitive root of k_S is characteristic if and only if $r(S)$ is an odd integer.

In conclusion there exists a homeomorphism $f\colon S \to S'$ between simply connected algebraic surfaces with $\chi > 1$ matching up the canonical classes if and only if S, S' have the same invariants K^2, χ, r.

Define $\mathcal{M}_d(S) = \{[S'] \in \mathcal{M}^{top}(S) | r(S) = r(S'),\ S \text{ minimal }\}$, it is natural to ask if $\mathcal{M}_d(S)$ is connected and if its elements carry the same underlying differential structure. At this time (november 1995) the second question is still unsolved, in spite of the recent deep developments in the theory of four manifolds. In the next sections of this thesis we shall see that the first question has in general a negative answer.

5. Simple bihyperelliptic surfaces and examples of connected components of moduli space.

In chapter II, §5, we considered a particular class of surfaces called simple bihyperellitic surfaces. We recall here its definition:

Denote $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{O}_X(a,b)$ be the line bundle on X whose sections are bihomogeneous polynomials of bidegree a,b. A minimal surface of general type is said to be simple bihyperelliptic of type $(a,b)(n,m)$ if its canonical model is defined in $\mathcal{O}_X(a,b) \oplus \mathcal{O}_X(n,m)$ by the equation

$$z^2 = f(x,y) \qquad w^2 = g(x,y) \tag{5.1}$$

where f,g are bihomogeneous polynomials of respective bidegree $(2a,2b),(2n,2m)$.

If $a,b,n,m \geq 3$ simple bihyperelliptic surfaces are simply connected and its invariants are

$$K^2 = 8(a+n-2)(b+m-2) \qquad \chi(\mathcal{O}_S) = \frac{1}{8}K^2 + ab + nm$$

$$r(S) = G.C.D.(a+n-2, b+m-2) \tag{5.2}$$

If $a > 2n,\ m > 2b$ denote by $\hat{N} = \hat{N}_{(a,b)(n,m)}$ the subset of moduli space $\mathcal{M}$ of simple bihyperelliptic surfaces of type $(a,b)(n,m)$. According the stability theorem proved in chapter II and local structure of moduli space we have

Proposition 5.3. *For $a > 2n$, $m > 2b$ the subset $\hat{N}$ is open in the moduli space $\mathcal{M}$ and* $\dim \hat{N} = 4\chi - \frac{1}{2}K^2 + 2(a+b+n+m) - 6$.

If $N \subset \hat{N}$ is the subset of surfaces with smooth canonical model then clearly N is open in $\hat{N}$ and from 5.1 it follows immediately that it is a dense subset of $\hat{N}$ in the analytic topology of

$\mathcal{M}$. Therefore if for suitable values of a, b, n, m the closure $\overline{N}$ of N in $\mathcal{M}$ is contained in $\hat{N}$, then $\hat{N}$ is open and closed in $\mathcal{M}$ and then it is a connected component of moduli space. The subset $\overline{N}$ has been studied by Catanese ([Ca3]), he proved

Theorem 5.4. *If $a > 2n, m > 2b$ then the space $\overline{N}$ is contained in the set of surfaces which are minimal resolution of surfaces X with at most RDP that are bidouble cover of a Segre-Hirzebruch surface $\mathbb{F}_{2k}$ with*

$$k \leq \max(\frac{b}{a-1}, \frac{n}{m-1})$$

Theorem 5.5. *If $a \geq \max(2n+1, b+2)$, $m \geq \max(2b+1, n+2)$ then $\hat{N}_{(a,b)(n,m)}$ is a connected component of moduli space.*

We don't sketch here the proof of theorem 5.4 because the main ideas are used in the next chapters to study the closure of some other subsets of $\mathcal{M}$.

Note that the components $\hat{N}$ are irreducible and then is not too difficult to find criteria for distinguish two of then, for example by looking at their dimension. However for the components $\hat{N}$ we have the following beautiful result:

Proposition 5.6.([Ca1]) *If $a > 2n, m > 2b, n \geq 3, b \geq 3$ and $\hat{N}_{(a,b)(n,m)} = \hat{N}_{(c,d)(p,q)}$ then the 4-uple (c, d, p, q) is one of the following:*

$$(a, b, n, m) \quad (b, a, m, n) \quad (n, m, a, b) \quad (m, n, b, a)$$

Roughly speaking proposition 5.6 says that if a smooth surface S is defined in two ways as in 5.1 then these ways are obtained one from the other by changing the role of x and y or the role of z and w.

Catanese's proof is given by observing that the numbers a, b, n, m are uniquely determined up the above four permutations by the six numbers $\sigma_i(a, b, n, m)$, $i = 1, ..., 6$ where $\sigma_1, ..., \sigma_4$ are the symmetric functions, $\sigma_5 = an + bn$ and $\sigma_6 = am + bn$.

Then it is possible to recover the values of σ_i from the geometry of the canonical map $\phi: S \to \mathbb{P}^{p_g - 1}$ of the generic surface $[S] \in N_{(a,b)(n,m)}$, for example $4\sigma_6$ is exactly the number of points of inflection of ϕ.

Note that the deformation invariance of the inflectionary points of the canonical map is a very special feature of simple bihyperelliptic surfaces and is false for general surfaces with very ample canonical bundle.

We are now able to construct examples of distinct connected components of the space $\mathcal{M}_d(S)$.

Example 5.7. Let S_1, S_2 be two simple bihyperelliptic surfaces of respective types $(13, 4), (6, 13)$ and $(14, 5)(5, 12)$. Then these surfaces are homeomorphic, $r(S_1) = r(S_2) = 1$ and they belong to different connected components of $\mathcal{M}$.

The strategy used in example 5.7 is clear, we look for a pair of simple bihyperelliptic surfaces of respective types $(a, b)(n, m)$ and $(a+1, b+1)(n-1, m-1)$. Such surfaces have the same invariants K^2, χ and r if and only if $n+m=a+b+2$. It is then easy to construct infinite example of surfaces S where $\mathcal{M}_d(S)$ has at least 2 connected components.

In order two prove that the number of connected components is unbounded we need the following lemma proved in the appendix of [Ca1]

Lemma 5.8.(Bombieri) *Let* $1 > c > 3^{-\frac{1}{3}}$ *be a fixed real number,* M *a positive integer and let* $u_i v_i = M$ *be* k *distinct factorizations of* M *such that* $c\sqrt{M} < u_i < v_i < c^{-1}\sqrt{M}$.
Then there exist positive integers R, S, N *and* k *distinct pairs of integer* (z_i, w_i) *such that:*

$$w_i z_i - 2(u_i + v_i) = N, \qquad z_i + 4 < 2Rv_i < 3z_i - 2, \qquad w_i + 4 < 2Su_i < 3w_i - 2$$

Theorem 5.9. *For every* $k > 0$ *there exist simple bihyperelliptic surfaces* $S_1, ..., S_k$ *orientedly homeomorphic, with* $r(S_i) = r(S_j)$ *and any two of them are not deformation equivalent of each other.*

Proof. We have to find large positive integers $K^2, \chi(\mathcal{O}_S), r(S)$ such that (5.1) with the inequalities $a \geq \max(2n+1, b+2)$, $m \geq \max(2b+1, n+2)$ has at least k distinct solutions. Fix $1 > c > \max\{2^{-\frac{1}{2}}, 3^{-\frac{1}{3}}\}$ and let $u_i, v_i = M$ be k distinct factorizations with G.C.D.$(u_i, v_i) = 1$ such that $c\sqrt{M} < u_i < v_i < c^{-1}\sqrt{M}$. (We can take for example an integer h such that $\binom{2h}{h} > 2k$ and $M = p_1 p_2 .... p_{2h}$ where $p_1 < p_2 < .... < p_{2h}$ are prime numbers such that $p_1^h > c p_{2h}^h$).

Let R, S, N, w_i, z_i be as in lemma 5.8 and let S_i be a simple bihyperelliptic surface of type $(a_i, b_i)(n_i, m_i)$ where $a_i = 2RSu_i + Rw_i + 1, b_i = 2RSv_i - Sz_i + 1, n_i = 2RSu_i - Rw_i + 1, m_i = 2RSv_i + Sz_i + 1$.

A computation shows that for every $i = 1, ..., k$ $K^2_{S_i} = 128R^2S^2M$, $\chi(\mathcal{O}_{S_i}) = 24R^2S^2M - 2RSN + 2$, $r(S_i) = 4RS$ and $a_i \geq \max\{2n_i + 1, b + 2\}, m_i \geq \max\{2b_i + 1, n_i + 2\}$.

This surfaces belong to the same $\mathcal{M}_d$ but they are in distinct connected components by theorem 5.5. $\square$

VI. Iterated double covers and connected components of moduli spaces.

In the previous chapters we defined for every minimal surface of general type S the subset of moduli space $\mathcal{M}_d(S)=\{[S']\in\mathcal{M}^{top}(S)|\,r(S')=r(S)\}$.
Using simple bihyperelliptic surfaces and a numerical lemma we proved that the number $\delta(S)$ of connected components of $\mathcal{M}_d(S)$ can be arbitrarily large, here we prove that "in general" δ takes quite big values, more precisely we have

Theorem A. *For every real number $4\le\beta\le 8$ there exists a sequence S_n of simply connected surfaces of general type such that:*

a) $y_n=K^2_{S_n},x_n=\chi(\mathcal{O}_{S_n})\to\infty$ *as* $n\to\infty$.
b) $\lim\limits_{n\to\infty}\dfrac{y_n}{x_n}=\beta$.
c) $\delta(S_n)\ge y_n^{\frac{1}{5}\log y_n}$.

We note that the lower bound we achieve is considerably greater of the previous bounds and in particular we prove the impossibility of a polynomial upper bound of δ. Theorem A relies on the explicit description of the connected components in the moduli space of a wide class of surfaces of general type whose Chern numbers spread in all the region $\dfrac{1}{2}c_2\le c_1^2\le 2c_2$.

Definition B. A finite map between normal algebraic surfaces $p\colon X\to Y$ is called a *simple iterated double cover* associated to a sequence of line bundles $L_1,...,L_n\in\mathrm{Pic}(Y)$ if the following conditions hold:

1) There exist $n+1$ normal surfaces $X=X_0,...,X_n=Y$ and n flat double covers $\pi_i\colon X_{i-1}\to X_i$ such that $p=\pi_n\circ....\circ\pi_1$.

2) If $p_i\colon X_i\to Y$ is the composition of π_j's $j>i$ then we have for every $i=1,...,n$ the eigensheaves decomposition $\pi_{i*}\mathcal{O}_{X_{i-1}}=\mathcal{O}_{X_i}\oplus p_i^*(-L_i)$.

For any sequence $L_1,...,L_n\in\mathrm{Pic}(\mathbb{P}^1\times\mathbb{P}^1)$ define $N(\mathbb{P}^1\times\mathbb{P}^1,L_1,...,L_n)$ as the image in the moduli space of the set of surfaces of general type whose canonical model is a simple iterated double cover of $\mathbb{P}^1\times\mathbb{P}^1$ associated to $L_1,...,L_n$.

The main theme of this chapter is to determine sufficient conditions on the sequence $L_1,...,L_n$ in such a way that the set $N(\mathbb{P}^1\times\mathbb{P}^1,L_1,..,L_n)$ has "good" properties; the condition we find are summarized in the following definition:

Marco Manetti: Degenerations of Algebraic Surfaces and applications to Moduli problems.

Definition C. A sequence $L_1, ..., L_n$, $L_i = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a_i, b_i)$ $n \geq 2$ of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ is called a *good sequence* if satisfies the following conditions.

C1) $a_i, b_i \geq 3$ for every $i = 1, ..., n$.

C2) $\max_{j<i} \min(2a_i - a_j, 2b_i - b_j) < 0$.

C3) $a_n \geq b_n + 2$, $b_{n-1} \geq a_{n-1} + 2$.

C4) a_i, b_i are even for $i = 2, ..., n$.

C5) For every $i < n$ $2a_i - a_{i+1} \geq 2, 2b_i - b_{i+1} \geq 2$.

The main result we prove is:(Th.'s 4.1, 4.2 and 4.7)

Theorem D. *Let $L_1, ..., L_n$ be a good sequence in sense of definition C, then:*

a) $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ is a nonempty connected component of the moduli space.

b) $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ is reduced, irreducible and unirational. (for a) and b) the condition C5 is not necessary).

c) The generic $[S] \in N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ is a surface with ample canonical bundle and $Aut(S) = \mathbb{Z}/2\mathbb{Z}$.

d) If $M_1, ..., M_m$ is another good sequence and $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n) = N(\mathbb{P}^1 \times \mathbb{P}^1, M_1, ..., M_m)$ then $n = m$ and $L_i = M_i$ for every $i = 1, ..., n$. $\blacksquare$

Simple iterated double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to good sequences are simply connected (because of C1, according to [Ca1] Th. 1.8) and by two of them are homeomorphic if and only if they have the same invariants K^2, χ and r mod2.

It is clear that the proof of theorem A reduces to counting the number of good sequences giving the same invariants K^2, χ and r.

Theorem D gives us some new interesting examples of homeomorphic but not deformation equivalent surfaces of general type.

Example E. *Two deformation not equivalent surfaces S_1, S_2 homeomorphic with the same divisibility which are double covers of the same surface S_0*

Define $S_0 \xrightarrow{p} \mathbb{P}^1 \times \mathbb{P}^1$ a simple iterated double cover associated to $L_1 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(8, 12)$, $L_2 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(8, 4)$; by adjunction formula $K_{S_0} = p^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(14, 14)$.

Let $a \neq b$ be integer ≥ 17 and let $D_1 \in |p^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2a, 2b)|$, $D_2 \in |p^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2b, 2a)|$ be two smooth divisors, the double cover S_1, S_2 of S_0 with branching divisors D_1, D_2 respectively have the required properties. Note that $D_1^2 = D_2^2$, $K_{S_0} \cdot D_1 = K_{S_0} \cdot D_2$ and D_1, D_2 have the same genus.

It is worth to mention here another interesting fact (Cor. 4.8), if

$$X = X_0 \xrightarrow{\pi_1} X_1 \longrightarrow ... \xrightarrow{\pi_n} X_n = \mathbb{P}^1 \times \mathbb{P}^1$$

is a simple iterated double cover associated to a good sequence then the surfaces X_1 and the map π_1 (and then by induction X_i and π_i for all $i = 1, .., n$) are uniquely determined by

X. In fact, assume for simplicity that $[X] \in N(L_1, .., L_n)$ is generic, then by Theorem D.c) X has only a nontrivial automorphism τ and then X_1 is the quotient X/τ.

Using the same idea we prove $D.d)$ as a consequence of $D.a), D.b)$ and $D.c)$.

Every simple iterated double cover X associated to $L_1, .., L_n \in \mathrm{Pic}(Y)$ can be embedded in the total space of the vector bundle $V = L_1 \oplus \ldots \oplus L_n \xrightarrow{p} Y$, e.g. in the case $n = 2$ the equations of X are

$$z_1^2 = f_1 + z_2 g_1, \qquad z_2^2 = f_2$$

with $z_i \in H^0(V, p^* L_i)$ the tautological section, $f_i \in H^0(Y, 2L_i)$ and $g_1 \in H^0(Y, 2L_1 - L_2)$.

Thus simple iterated double covers are naturally parametrized by a Zariski open subset of a finite dimensional vector space and then the proof of the openess of $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ reduces to showing the surjectivity of a Kodaira-Spencer map.

In order to prove the closure of $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ in the moduli space we must show that for every 1-parameter family of simple iterated double covers degenerate to a surface of general type X_0 then $[X_0] \in N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$.

Here the main trouble is to prove that the flatness of all covering maps is preserved under specialization. The section 3 is devoted to prove this fact under some special and at a first sight very strange assumption (e.g. C4). The key result is the classification of involutions acting on smoothings of rational double points (Prop. 3.2), from this it follows that if a family of smooth double covers $X_t \to Y_t$, $t \in \Delta^*$ degenerate to a nonflat double cover $X_0 \to Y_0$ and X_0 has at most rational double points then Y_0 has at least one cyclic singularity at y_0 and the Milnor fibre F_t of the smoothing $(Y, y_0) \to (\Delta, 0)$ has the canonical class in $H^2(F_t, \mathbb{Z})$ not divisible by 2. In particular if $r(Y_t)$ is even then the inclusion $F_t \subset Y_t$ gives a contradiction.

The proof of $D.c)$ (§4) use a degeneration argument.

1. Preliminaries and conventions

Let $f: X \to Y$ be a morphism between complex algebraic varieties. If $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{G}$ is an $\mathcal{O}_Y$-module the natural sheaf morphisms $\mathcal{G} \to f_*f^*\mathcal{G}$, $f^*f_*\mathcal{F} \to \mathcal{F}$ induce isomorphisms
$f_*\mathcal{H}om\mathcal{O}_X f^*\mathcal{G}\mathcal{F} \simeq \mathcal{H}om\mathcal{O}_Y\mathcal{G}f_*\mathcal{F}$ and $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$ (cf.[Ha1] pag. 110).

Lemma 1.1. *In the notation above assume $\mathcal{F},\mathcal{G}$ coherent:*
a) If f is flat (i.e. f^* is an exact functor) then there exists a convergent spectral sequence of vector spaces

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{O}_Y}(\mathcal{G}, R^qf_*\mathcal{F}) \quad \Rightarrow \quad \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F})$$

b) If f is finite then there exists a convergent spectral sequence of $\mathcal{O}_Y$-modules

$$E_2^{p,q} = f_*\mathcal{E}xt^p_{\mathcal{O}_X}(L^qf^*\mathcal{G},\mathcal{F}) \quad \Rightarrow \quad \mathcal{E}xt^{p+q}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

c) If f is finite flat then for every $i \geq 0$ we have

$$\mathrm{Ext}^i_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F}) = \mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \qquad f_*\mathcal{E}xt^i_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F}) = \mathcal{E}xt^i_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

Proof. a) Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module, from the exactness of the functor f^* and formula $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{I})$ it follows that the direct image $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module.
The functor $\mathcal{F} \to \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G},\mathcal{F})$ is the composition $\mathcal{F} \to f_*\mathcal{F} \to \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$ and the sequence in a) is the Grothendieck spectral sequence associated to this composition.
b) The proof is similar to a), we only recall that since f is finite f_* is an exact functor from coherent sheaves on X to coherent sheaves on Y and the $\mathcal{E}xt$'s can be computed applying the contravariant $\mathcal{H}om$ to locally free resolutions. (cf. [Ha1] III.6.5).
c) is an obvious consequence of a) and b). □

Remark. . The condition f flat in the point 1.1.a) cannot be deleted, in fact if $\phi: A \to B$ is a morphism of commutative rings, M an A-module and N a B-module then there exists a spectral sequence (composition of $-\otimes B$ and $\mathrm{Hom}_B(-,N)$)

$$E_2^{p,q} = \mathrm{Ext}^p_B(Tor^A_q(M,B),N) \quad \Rightarrow \quad \mathrm{Ext}^{p+q}_A(M,N).$$

In particular ϕ is flat if and only if every injective B-module is an injective A-module.
Throughout all this paper by a *tower* of height n we shall mean the data of $n+1$ irreducible algebraic varieties of the same dimension $X_0,...,X_n$ and n finite flat morphisms $\pi_i: X_{i-1} \to X_i$. A tower is smooth (resp.: normal) if every X_i is smooth (resp.: normal).

A *deformation of the tower* (X_i, π_i) parametrized by a germ of complex space $(S, 0)$ is a commutative diagram

$$\begin{array}{ccccccccc} X_0 & \xrightarrow{\pi_1} & X_1 & \longrightarrow & \dots & \longrightarrow & X_n & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \tilde{X}_0 & \xrightarrow{\tilde{\pi}_1} & \tilde{X}_1 & \longrightarrow & \dots & \longrightarrow & \tilde{X}_n & \longrightarrow & S \end{array}$$

such that for every $i = 0, ..., n$ the induced diagram

$$\begin{array}{ccc} X_i & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \tilde{X}_i & \longrightarrow & S \end{array}$$

is a deformation of X_i parametrized by S. Note that for tower of height 1 this is the usual definition of deformations of maps ([Ran]).

Denote by $Def(X_i, \pi_i)$ the functor of isomorphism classes of deformations of the tower (X_i, π_i) and, for $j = 0, ...n$, by $r_j : Def(X_i, \pi_i) \to Def(X_j)$ the induced morphism of functors.

Let now $\pi : X \to Y$ be a finite flat map between irreducible reduced algebraic varieties, by Lemma 1.1 we have an isomorphism

$$\Phi : \mathrm{Ext}^1_{\mathcal{O}_X}(\pi^*\Omega^1_Y, \mathcal{O}_X) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \pi_*\mathcal{O}_X)$$

and the natural maps $\pi^*\Omega^1_Y \to \Omega^1_X$, $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$ induce maps of Ext groups

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) & \xrightarrow{\alpha} & \mathrm{Ext}^1_{\mathcal{O}_X}(\pi^*\Omega^1_Y, \mathcal{O}_X) \\ & & \downarrow \Phi \\ \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) & \xrightarrow{\beta} & \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \pi^*\mathcal{O}_X) \end{array}$$

where if $e \in \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)$ is the isomorphism class of the extension

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow E \longrightarrow \Omega^1_Y \longrightarrow 0$$

then $\Phi^{-1}\beta(e)$ is the isomorphism class of the extension

$$0 \longrightarrow \mathcal{O}_X = \pi^*\mathcal{O}_Y \longrightarrow \pi^*E \longrightarrow \pi^*\Omega^1_Y \longrightarrow 0$$

The maps α and $\Phi^{-1}\beta$ have an interesting interpretation in terms of obstruction to deforming the map π.

We recall that if Z is a reduced variety and T^1_Z is the vector space of deformations of Z over the double point $D = Spec(\mathbb{C}[t]/(t^2))$ there exists an isomorphism $T^1_Z = \mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega^1_Z, \mathcal{O}_Z)$ which to the deformation $Z \subset \tilde{Z} \to D$ associates the extension

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \Omega^1_{\tilde{Z}} \otimes \mathcal{O}_Z \longrightarrow \Omega^1_Z \longrightarrow 0$$

If T^1_π is the space of first order deformations of the map π then there exists a commutative diagram

$$\begin{array}{ccc} T^1_\pi & \xrightarrow{r_X} & T^1_X = \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \\ \downarrow{\scriptstyle r_Y} & & \downarrow{\scriptstyle \alpha} \\ T^1_Y = \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) & \xrightarrow{\Phi^{-1}\beta} & \mathrm{Ext}^1_{\mathcal{O}_X}(\pi^*\Omega^1_Y, \mathcal{O}_X) \end{array}$$

where r_X and r_Y are the natural forgetting maps. In fact if $\tilde{X}\xrightarrow{\tilde{\pi}}\tilde{Y}$ is a deformation of π over the double point then by standard flatness criterion ([Mat1] Th. 22.3) it's easy to see that $\tilde{\pi}$ is flat and the relation $\alpha r_X(\tilde{\pi}) = \Phi^{-1}\beta r_Y(\tilde{\pi})$ follows from the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X = \tilde{\pi}^*\mathcal{O}_Y & \longrightarrow & \tilde{\pi}^*\Omega^1_{\tilde{Y}} \otimes \mathcal{O}_X & \longrightarrow & \pi^*\Omega^1_Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega_{\tilde{X}} \otimes \mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \end{array}$$

Sometimes, especially in §2, if $(S,0)$ is a germ of complex vector space we consider $(S,0)$ as a covariant functor from the category of local artinian $\mathbb{C}$-algebras to the category of sets defined in the following way:

$$(S,0)(A) = \{\text{morphisms}\, \varphi\colon (Spec\, A, 0) \to (S,0)\}$$

where $0 \in Spec\, A$ is the closed point.

2. Deformations of iterated double covers.

From now on by a surface we mean a complex projective surface. Let X be a normal surface and let $\pi\colon X \to Y$ be the quotient of X by an involution τ.

Lemma 2.1. *In the above notation the following conditions are equivalent:*

i) *π is flat.*

ii) *There exists a line bundle $\pi\colon L \to Y$ and a section $f \in H^0(Y, 2L)$ such that the pair X, τ is isomorphic to the subvariety of L defined by the equation $z^2 = f$, $z \in H^0(L, \pi^*L)$ is the tautological section, and the involution obtained by multiplication for -1 in the fibres of L.*

iii) *The fixed subvariety $R =$ Fix (τ) is a Cartier divisor.*

Moreover if X is smooth then π is flat if and only if Y is smooth.

Proof. The proof is standard, we give a sketch.

i) $\Rightarrow$ *ii)* If π is flat then the group $G = \{1, \tau\}$ acts on the rank 2 locally free sheaf $\pi_*\mathcal{O}_X$ and yields a character decomposition $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$ for some $L \in \mathrm{Pic}(Y)$. X depends only by the $\mathcal{O}_Y$ algebra structure of $\pi_*\mathcal{O}_X$ which is uniquely determined by a map $f\colon \mathcal{O}_Y(-2L) \to \mathcal{O}_Y$, $f \in H^0(Y, 2L)$.

ii) $\Rightarrow$ *iii)* is clear since R is the divisor of a section of π^*L.

iii) $\Rightarrow$ *i)* Let p be a fixed point of τ, then G acts on the local $\mathbb{C}$-algebra $B = \mathcal{O}_{X,p}$. Let $A = B^G$ be the subring of invariant functions and let I be the ideal of R, by definition I is the ideal of B generated by $\tau f - f$, all $f \in B$.

If I is a principal ideal, it is easy to see using Nakayama lemma that there exists a generator h of I such that $\tau h = -h$ and then B is a free A-module generated by $1, h$.

If X is smooth, by $i) \Leftrightarrow iii)$ it follows that π is flat if and only if τ has not isolated fixed point, i.e. if and only if Y is smooth. (note that if Y is smooth then π is always flat). $\square$

In this section we investigate the deformations of X under the hypothesis of π to be flat. Consider thus $X \subset L \xrightarrow{\pi} Y$ defined by the equation $z^2 = f(y)$. Denote $D = div(f) \subset Y$, $R = div(z) \subset X$.

Note that $\pi^* D = 2R$ and X is normal if and only if Y is normal and D is reduced. If K_X, K_Y are the Weil canonical divisors of X and Y respectively we have the adjunction formula $K_X = \pi^* K_Y + R$, this follows from the usual Hurwitz formula for smooth varieties and from the reflexivity of canonical sheaves on normal varieties. In particular if Y is Gorenstein then also X is Gorenstein (cf. [Mat1] 23.4).

Let $\tilde{X}$ be the variety defined in $L \times H^0(Y, D)$ by

$$\tilde{X} = \{(z, y, h) |\, z^2 = f(y) + h(y)\}$$

clearly $\tilde{X}$ is a double flat cover of $Y \times H^0(Y, D)$ hence the second projection $\tilde{X} \to H^0(Y, D)$ is flat and defines a map of functors $Nat_\pi \colon (H^0(Y, D), 0) \to Def(X)$.

Definition 2.2. The image of the map Nat_π is called the set of *natural deformations* of X associated to π.

Proposition 2.3. *In the above notation let $\tilde{X} \to \tilde{Y} \to H$ be a deformation of the map π parametrized by a smooth germ $(H, 0)$ and let $r_X \colon (H, 0) \to Def(X)$, $r_Y \colon (H, 0) \to Def(Y)$ be the induced maps. Assume:*

i) r_Y is smooth.

ii) The image of r_X contains the natural deformations.

iii) $\mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, -L) = 0$, $H^1(\mathcal{O}_Y) = 0$.

Then $\dim T^1_X = \dim T^1_Y + h^0(\mathcal{O}_Y(D)) + h^0(\theta_X) - h^0(\theta_Y) - h^0(\theta_Y(-L)) - 1$ and the map r_X is smooth.

We prove this proposition after some lemmas.

Lemma 2.4. *There exists an exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \pi^* \Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \mathcal{O}_R(-R) \longrightarrow 0 \tag{1}$$

Proof. Let $i \colon X \to L$ be the inclusion as in lemma 2.1, since $L \xrightarrow{\pi} Y$ is locally a product there exists an obvious inclusion of sheaves $\pi^{-1}\Omega^1_Y \subset i^{-1}\Omega^1_L$, tensoring with the flat module O_X we get an injection $\pi^*\Omega^1_Y \longrightarrow \Omega^1_L \otimes \mathcal{O}_X$.

The sheaf $\Omega^1_{L/Y}$ is clearly locally free and it is the O_L dual of the sheaf of vertical vector fields and therefore it is naturally isomorphic to $\pi^*(-L)$.

We have the following first and second exact sequences of differentials

$$0 \longrightarrow \pi^*\Omega^1_Y \longrightarrow \Omega^1_L \otimes \mathcal{O}_X \longrightarrow \Omega^1_{L/Y} \otimes \mathcal{O}_X = \mathcal{O}_X(-R) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(-\pi^*D) = \mathcal{O}_X(-X) \longrightarrow \Omega^1_L \otimes \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow 0 \tag{2}$$

and (1) is obtained by applying the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-\pi^*D) & \longrightarrow & \Omega^1_L \otimes \mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathcal{O}_X(-L) & == & \mathcal{O}_X(-L) & & & & \end{array}$$

□

The proof of 2.4 shows also that there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-X) & \longrightarrow & \Omega^1_L \otimes \mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X(-X) & \longrightarrow & \mathcal{O}_X(-R) & \longrightarrow & \mathcal{O}_R(-R) & \longrightarrow & 0 \end{array}$$

If we apply $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ to the above diagram we get the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-X), \mathcal{O}_X) & \xrightarrow{\delta} & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \\ \| & & \uparrow \epsilon \\ \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-X), \mathcal{O}_X) & \xrightarrow{\gamma} & \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_R(-R), \mathcal{O}_X) \end{array}$$

Lemma 2.5.*In the notation above, if $H^1(\mathcal{O}_Y) = 0$ then the image of ϵ is the vector space of first order natural deformations.*

Proof. We know that δ is the natural map from first order embedded deformations of X in L to T^1_X (cf. [Ar3]) and then the set of first order natural deformations is the image of the composite map

$$H^0(\mathcal{O}_Y(D)) \xrightarrow{\pi^*} H^0(\mathcal{O}_X(\pi^*D)) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-X), \mathcal{O}_X) \xrightarrow{\delta} T^1_X$$

Thus in order to prove the lemma it's enough to show that $\gamma \circ \pi^*$ is surjective.
Since R is a locally principal divisor in the normal surface X we have (cf. II.4.13, II.4.14) $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_R(-R), \mathcal{O}_X) = H^0(\mathcal{O}_R(2R))$ and, since $\pi_*\mathcal{O}_R = \mathcal{O}_D$, we also have $H^0(\mathcal{O}_R(2R)) = H^0(\mathcal{O}_D(D))$ and the restriction map $H^0(\mathcal{O}_Y(D)) \to H^0(\mathcal{O}_D(D))$ is surjective if $H^1(\mathcal{O}_Y) = 0$.
□

Proof of proposition 2.3 We have a commutative diagram

$$\begin{array}{ccc} T_0H & \xrightarrow{dr_X} & \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \\ \downarrow dr_Y & & \downarrow \alpha \\ \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) & \xrightarrow{\Phi^{-1}\beta} & \mathrm{Ext}^1_{\mathcal{O}_X}(\pi^*\Omega^1_Y, \mathcal{O}_X) \end{array}$$

By lemma 1.1 and hypothesis iii) the map $\Phi^{-1}\beta$ is bijective. The kernel of α is the set of natural deformations and by ii) is contained in the image of dr_X. It is now trivial to observe that dr_Y surjective implies dr_X surjective and since H is smooth this is sufficient to prove that r_X is smooth and $\dim T^1_X = \dim T^1_Y + \dim Im\epsilon$. □

If X has the universal deformation then Prop. 2.3 remains true without assuming H smooth. In fact, the condition 2.3.iii) implies that the trivial involution τ acts trivialy on the universal deformation of X (cf. [F-P]) and then it is defined in a natural way a morphism of functors $Def_X \to Def_Y$. The conclusion now follows by II.1.3 and the surjectivity of dr_X.

Definition 2.6.

a) A normal tower (X_i, π_i) of height n is said to be *simple* if for every i, $\pi_i: X_{i-1} \to X_i$ is a flat double cover and there exist line bundles $L_1, ..., L_n \in \mathrm{Pic}(X_n)$ such that $\pi_{i*}\mathcal{O}_{X_{i-1}} = \mathcal{O}_{X_i} \oplus p_i^*(-L_i)$ where p_i is the composition of π_j's $j > i$.

b) If (X_i, π_i, L_i) is a simple tower we call the surface $X = X_0$ a *simple iterated double cover* of Y associated to $L_1, ..., L_n \in \mathrm{Pic}(Y)$ and the involution $\tau: X \to X$ such that $X/\tau = X_1$ the *trivial involution*.

Clearly the trivial involution depends on the simple tower and in general X does not determine τ.

It is important to observe that if (X_i, π_i, L_i) is a smooth simple tower and $\mathrm{Pic}(X_n)$ is without torsion then the maps $p_i^*: \mathrm{Pic}(X_n) \to \mathrm{Pic}(X_i)$ are injective and the line bundles $L_1, ..., L_n$ are uniquely determined by the maps $\pi_1, ..., \pi_n$.

Theorem 2.7. *Let (X_i, π_i, L_i) be a simple tower of height n and let $(H, 0)$ be a smooth germ parametrizing a deformation of the tower. Denote $X = X_0, Y = X_n$ and let $r_i: (H, 0) \to Def(X_i)$ be the induced maps. Assume:*

i) $H^1(\mathcal{O}_Y) = 0$.

ii) $r_n: (H, 0) \to Def(Y)$ *is smooth.*

iii) The natural deformations of $\pi_{i+1}: X_i \to X_{i+1}$ are contained in the image of r_i.

iv) For every sequence $1 \le j_1 < j_2 < ... < j_h \le n$, $h > 0$

$$\mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \sum_{s=1}^{h} -L_{j_s}) = 0 \qquad H^1(Y, \sum_{s=1}^{h} -L_{j_s}) = 0$$

v) For every $i \in \{2, ..., n\}$ and for every subset $\{j_1, ..., j_h\} \subset \{1, ..., i-1, i+1, ..., n\}$ with $h > 0$ and $j_1 < i$

$$H^0(Y, 2L_i - \sum_{s=1}^{h} L_{j_s}) = 0$$

Then $r_0: H \to Def(X)$ is smooth.

Note. If $H^0(L_i) \neq 0$ for every i then the condition v) is equivalent to

vi) for every $j < i$ $H^0(Y, 2L_i - L_j) = 0$.

Proof. Induction on n, for $n = 1$ is just proposition 2.3.

Assuming the theorem true for towers of height $n-1$ it suffices to prove that conditions i),...,v) hold for the surface $Z = X_{n-1}$ and the line bundles $M_i = \pi_n^* L_i$ $i = 1, ..., n-1$.

The only nontrivial condition to check is the part of iv) concerning Ext's. Let $R \subset Z$, $D \subset Y$ be respectively the ramification and branching divisors of π_n.
Applying $\mathrm{Hom}_{\mathcal{O}_Z}(-, \sum_{s=1}^h -M_{j_s})$ to the exact sequence

$$0 \longrightarrow \pi_n^* \Omega_Y^1 \longrightarrow \Omega_Z^1 \longrightarrow \mathcal{O}_R(-R) \longrightarrow 0$$

we get

$$H^0(\mathcal{O}_D(2L_n - \sum_{s=1}^h -L_{j_s})) = \mathrm{Ext}^1_{\mathcal{O}_Z}(\mathcal{O}_R(-R), \sum_{s=1}^h -M_{j_s}) \to \mathrm{Ext}^1_{\mathcal{O}_Z}(\Omega_Z^1, \sum_{s=1}^h -M_{j_s}) \to$$

$$\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_Z}(\pi_n^*\Omega_Y^1, \sum_{s=1}^h -M_{j_s}) = \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega_Y^1, \sum_{s=1}^h -L_{j_s}) \oplus \mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega_Y^1, \sum_{s=1}^h -L_{j_s} - L_n)$$

and the vector space on the left belong to the exact sequence

$$H^0(\mathcal{O}_Y(2L_n - \sum_{s=1}^h -L_{j_s})) \longrightarrow H^0(\mathcal{O}_D(2L_n - \sum_{s=1}^h -L_{j_s})) \longrightarrow H^1(\mathcal{O}_D(\sum_{s=1}^h -L_{j_s}))$$

□

Corollary 2.8. *Let Y be a rigid (i.e. $T_Y^1 = 0$) normal surface and let X be a simple iterated double cover of Y associated to $L_1, ..., L_n \in \mathrm{Pic}(Y)$. If conditions i), iv) and v) of theorem 2.7 are satisfied then $Def(X)$ is smooth.*

Proof. X is the top of a simple tower (X_i, π_i, L_i) of height n thus according to theorem 2.7 it's enough to show the existence of a smooth family of deformations of the tower satisfying conditions 2.7.ii) and 2.7.iii).
By lemma 2.1 applied n times we can embed X in the vector bundle $V = L_1 \oplus ... \oplus L_n \xrightarrow{p} Y$ by the equations

$$z_i^2 = f_i \quad i = 1, ..., n$$

where $z_i \colon V \to p^*L_i$ tautological section and $f_i \in H^0(X_i, p_i^* 2L_i)$ where X_i is the surface in $L_{i+1} \oplus ... \oplus L_n$ of equations $z_j^2 = f_j$, $j > i$ and π_i is the restriction to X_{i-1} of the natural projection $L_i \oplus ... \oplus L_n \to L_{i+1} \oplus ... \oplus L_n$. Note that there exists a natural identification of vector spaces

$$H^0(X_i, p_i^* 2L_i) = \bigoplus_{h=0}^{n-i} \bigoplus_{\{j_1,...,j_h\} \subset \{i+1,...,n\}} z_{j_1} ... z_{j_h} H^0(Y, 2L_i - L_{j_1} - ... - L_{j_h})$$

Take $H = \bigoplus_{i=1}^{n} H^0(X_i, p_i^* 2L_i)$ and the map $H \to Def(X_i, \pi_i)$ is given by

$$(h_1, ..., h_n) \to X' = \{z_i^2 = f_i + h_i\} \qquad (*)$$

Clearly $H \to Def(Y) = 0$ is smooth and the image of r_i contains the natural deformations of each π_i. □

The deformations of X defined by the equation $(*)$ are called *natural deformations* of X associated to the simple tower (X_i, π_i, L_i). Note that the trivial involution $\tau: z_1 \to -z_1$ extends to every natural deformation of the tower, therefore if the family of natural deformation is complete (e.g. Cor. 2.8) then τ acts trivially on T^1_X.

Example 2.9. If $Y = \mathbb{P}^2$ and $deg\, L_i = a_i$ then the hypotheses of Cor. 2.8 are satisfied if for every i $a_i \geq 4$ and $a_i > 2a_{i+1}$.

As in the introduction define $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ the subset of moduli space of surfaces of general type whose canonical model is a simple iterated double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to $L_1, .., L_n$ and by $N_0(\mathbb{P}^1 \times \mathbb{P}^1, L_1, .., L_n)$ the subset of $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ of surfaces whose canonical model is nonsingular, it is clear that N_0 is an open subset of N.

Corollary 2.10. *If $Y = \mathbb{P}^1 \times \mathbb{P}^1$, $L_i = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a_i, b_i)$ with $a_i, b_i \geq 3$ and for every $j < i$ $Min(2a_i - a_j, 2b_i - b_j) < 0$ then $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ and $N_0(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ are open subsets of the moduli space $\mathcal{M}$.*

Proof. Take $[S] \in N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ and let (X_i, π_i, L_i) be a tower with bottom $\mathbb{P}^1 \times \mathbb{P}^1$ and top the canonical model X of S.

It is easy to show that $L_1, ..., L_n$ satisfy the conditions of corollary 2.8 and then we have a surjective map of germs of complex spaces $(H, 0) \to (Def(X), 0) \to (\mathcal{M}, [S])$ where H is the parameter space of natural deformations associated to the tower. The thesis now follows immediately since by explicit construction of natural deformations the image of $(H, 0)$ is contained in $(N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, .., L_n), [S])$. □

The next result will be used in chapter VII.

Corollary 2.11. *Let $X \longrightarrow Y$ be a simple iterated double cover associated to a sequence $L_1, ..., L_n \in \mathrm{Pic}(Y)$. Assume that Y and $L_1, .., L_n$ satisfy conditions 2.7.i), 2.7.iv), 2.7.v) and assume moreover that:*

(a) $Def(Y)$ is smooth.

(b) $L_1, .., L_n$ extends to a complete deformation of Y.

(c) For every $0 < i < j_1 < ... < j_h \leq n$, $h \geq 0$

$$H^1(Y, 2L_i - \sum_{s=1}^{h} L_{j_s}) = 0$$

then $Def(X)$ is smooth.

Proof. Let (X_i, π_i, L_i) be a simple tower with $X_0 = X$, $X_n = Y$, $X_{n-1} = Z$. We already proved that the surface Z and the line bundles $M_1, ..., M_{n-1}$, $M_i = \pi_n^* L_i$ satisfy 2.7.i), iv) and v). By induction on n it is sufficient to prove that they satisfy (a), (b) and (c).

Let $\tilde{Y} \longrightarrow Def(Y)$ be the Kuranishi family of Y and let $\tilde{L_i} \in \mathrm{Pic}(\tilde{Y})$ be the extension of L_i. Since $H^1(Y, 2L_n) = 0$ by semicontinuity and base change theorems there exists a subspace $V \subset H^0(\tilde{Y}, 2\tilde{L_n})$ such that the natural restriction $V \longrightarrow H^0(Y, 2L_n)$ is an isomorphism.

Consider then the flat double cover $\tilde{Z}\xrightarrow{\tilde{\pi}}\tilde{Y}\times V$ defined by equation

$$z_n^2 = v(y) \qquad y \in \tilde{Y}, \quad v \in V$$

By construction the flat maps

$$\tilde{Z}\xrightarrow{\tilde{\pi}}\tilde{Y}\times V\longrightarrow Def(Y)\times V$$

are a deformation of the double cover $Z\xrightarrow{\pi_n}Y$ and satisfy the hypotheses of 2.3. Therefore $Def(Z)$ is smooth and it is clear that $\tilde{M_i} = \tilde{\pi}^*\tilde{L_i}$ extends M_i to a complete family. The verification of (c) is easy. □

Note that if $L_1, ..., L_n$ satisfies the hypotheses of Corollary 2.10 then in general they don't satisfy the condition 2.11.c and then 2.11 cannot used in the proof of 2.10.

3. Degenerations of iterated double covers.

Let $f: X \to \Delta = \{t \in \mathbb{C} | \, |t| < 1\}$ be a proper flat family of normal projective surfaces and let $\tau: X \to X$ be an involution preserving f. Let $\pi: X \to Y = X/\tau$ be the projection to quotient and assume that $\pi_t: X_t \to Y_t$ is flat for every $t \neq 0$.

In general $\pi_0: X_0 \to Y_0$ is not flat, this section is almost entirely devoted to prove the following theorem which gives a sufficient condition for the map π_0 to be flat.

Theorem 3.1. *In the above situation suppose that:*

i) X_t, Y_t are smooth surfaces for $t \neq 0$.

ii) X_0 has at most rational double points (RDP) as singularities.

iii) The divisibility of the canonical class of Y_t is even for $t \neq 0$.

Then Y_0 has at most RDP's and the map $\pi: X \to Y$ is flat.

Since flatness is a local property we need to investigate quotient of smoothing of RDP.

Proposition 3.2. *Let $f: (X,0) \to (\mathbb{C},0)$ be a smoothing of a rational double point X_0 and let $f': (Y,0) \to \Delta$ be the quotient of $(X,0)$ by an involution τ preserving f.*

Suppose that $(Y,0)$ is a smoothing of the normal singularity Y_0 and let $F_t \subset Y_t$ be the associated Milnor fibre. Then either one of the following possibilities holds:

i) Y_0 is a RDP and the quotient projection $\pi: (X,0) \to (Y,0)$ is flat.

ii) Y_0 is cyclic of type $\frac{1}{2d+1}(1, 2d-1)$ and the intersection form on $H_2(F_t, \int)$ is odd and negative definite.

iii) f' is a $\mathbb{Q}$-Gorenstein smoothing of the cyclic singularity of type $\frac{1}{4d}(1, 2d-1)$, the torsion subgroup of $H^2(F_t, \int)$ has order 2 and is generated by the canonical class.

Proof of Theorem 3.1. It's enough to prove that the map $Y \to \Delta$ cannot be locally of type ii) or iii) described in prop 3.2. Let $p \in Y$ be a singular point: (Y, p) cannot be of type ii) above since the inclusion $F_t \subset Y_t$ induces an isometry $H_2(F_t, \int) \to H_2(Y_t, \int)$ with respect the intersection forms and the intersection form of Y_t is even by Wu's formula.

If (Y,p) is of type iii) above and if $r\colon H^2(Y_t,\int) \to H^2(F_t,\int)$ is the natural restriction then $r(c_1(K_Y))$ generates the torsion subgroup of $H^2(F)$ which is $\int/2\int$ but this gives a contradiction since $c_1(K_X)$ is 2-divisible. □

We point out that, according to IV.2.6, the cyclic singularity of type $\frac{1}{4}(1,1)$ is the unique singularity described in the statement of 3.2 which can appear in a normal degeneration of the complex projective plane.

The proof of 3.1 shows that the condition $r(Y_t)$ even is essential in order to have Y_0 with at most rational double points. In fact in chapter VII we shall construct examples of degeneratins where the divisibility $r(Y_t)$ is odd and Y_0 has singularities of type 3.2.iii).

Our strategy of proof of proposition 3.2 divides in two steps. The first step is the classification of all conjugacy classes of involutions acting on a RDP; this computation is already done by Catanese and the result is illustrated in the next two tables.

Table 1. Equations of RDP's in $\mathbb{C}^3$.

E_8	$z^2+x^3+y^5=0$
E_7	$z^2+x(y^3+x^2)=0$
E_6	$z^2+x^3+y^4=0$
D_n, $n\geq 4$	$z^2+x(y^2+x^{n-2})=0$
A_n	$z^2+x^2+y^{n+1}=0$ or $uv+y^{n+1}=0$
smooth	$x=0$

Table 2. ([Ca3] Th. 2.1) Conjugacy classes of involutions acting on the RDP's defined as in table 1.

$a)$	$y\to -y$	E_6, D_n, A_{2n+1}
$b)$	$y\to -y$, $z\to -z$	smooth, E_6, D_n, A_{2n+1}
$c)$	$(u,v,y)\to(-u,v,-y)$	A_{2n}
$d)$	$x\to -x$, $z\to -z$	A_n
$e)$	$(u,v,y)\to(-u,-v,-y)$	A_{2n+1}
$f)$	$z\to -z$	all RDP's

Corollary 3.3. *Let $X\to Y$ be a flat double cover of normal surfaces.*
If X is smooth then Y is smooth.
If X has at most RDP's then Y has at most RDP's.

Proof. According to table 2 the only involutions whose fixed locus is a Cartier divisor are exactly of types a) and f). □

The second step in the proof of proposition 3.2 is to give a (very rough) classification of the smoothing of the involutions of table 2 according to the following definition.

Definition 3.4. Let $(X_0,0)$ be a singularity and $g_0\in Aut(X_0,0)$. A smoothing of g_0 is the data of a smoothing $(X,0)\xrightarrow{t}(\mathbb{C},0)$ of $(X_0,0)$ and an automorphism g of $(X,0)$ preserving the map t such that g_0 is the restriction of g to X_0 and the quotient $(Y,0)=(X/g,0)\xrightarrow{t}(\mathbb{C},0)$ is a smoothing of (X_0/g_0).

The following Cartan-type Lemma will be very useful for our purposes.

Lemma 3.5. *Let* $(X,0)\xrightarrow{t}(\mathbb{C},0)$ *be a morphism of germs of analytic singularities and let* $G \subset Aut(X,0)$ *be a finite subgroup preserving* t.

Assume the group G *acts linearly on a finite dimensional* $\mathbb{C}$*-vector space* V *and let* $i_o: (X_0,0) \to (V,0)$ *be a* G*-embedding, then there exists a* G*-embedding* $i: (X,0) \to (V \times \mathbb{C}, 0)$ *extending* i_o *and such that* $t = p \circ i$ *where* p *is the projection on the second factor.*

Moreover if t *is flat and* $f_1(z),...,f_k(z)$ *are the equations of* i_0X_0 *in* V *such that* $g(f_i) = \chi_i(g)f_i$ *for characters* $\chi_1,...,\chi_k$ *then we can choose equations* $F_i(z,t)$ *of* $i(X)$ *in* $V \times \mathbb{C}$ *such that* $F_i(z,0) = f_i(z)$ *and* $gF_i = \chi_i(g)F$ *for every* i, g.

Proof. Let m, m_0 be respectively the maximal ideals of $\mathcal{O}_X, \mathcal{O}_{X_0}$. According to classical Cartan Lemma ([Car]) if $V' \subset V$ is the Zariski tangent space of $(i_0(X_0),0)$ then there exists a G-equivariant analytic automorphism α of $(V,0)$ such that $\alpha(i_0(X_0))$ is contained in V' and then we can assume without loss of generality that V is G-isomorphic to $(m_0/m_0^2)^\vee$, the Zariski tangent space of X_0 at 0.

If $z_1,...,z_n$ is a basis of $V^\vee$ and let $i_0^*: \mathbb{C}\{z_1,...,z_n\} \to \mathcal{O}_{X_0}$ be the induced surjective G-equivariant morphism of algebras, the germs of function $i_0(z_i)$, $i = 1,...,n$ are a basis of m_0/m_0^2.

The ideal $I = m^2 + (t) \subset m$ is clearly G-stable and since G is finite there exists a G-stable vector space $H \subset m$ such that $m = I \oplus H$. The restriction of the natural projection $\mathcal{O}_X \to \mathcal{O}_{X_0}$ to H induces a G-isomorphism $H \simeq m_0/m_0^2$ and then there exists a G-lifting of i_0^*, say $\eta^*: \mathbb{C}\{z_1,...,z_n\} \to \mathcal{O}_X$.

It is now easy to prove that the map $i: (X,0) \to (V \times \mathbb{C}, 0)$ associate to the local homomorphism of analytic algebras $i^*: \mathbb{C}\{z_1,...,z_n,t\} \to \mathcal{O}_X$ $i^*(t) = t$, $i^*(z_i) = \eta^*(z_i)$ is the desired embedding.

Let now f_i be as in the statement, then using the linear reductivity of G we can find functions $F_i \in \mathbb{C}\{z_1,...,z_n,t\}$ in the ideal I_X of $i(X)$ such that $F_i(z,0) = f_i(z)$ and $gF_i = \chi_i(g)F_i$. The flatness of t implies that the F_i's generate I_X (cf. II.2.1). □

Lemma 3.6. *The involutions of types* $b)$ *and* $d)$ *are not smoothable.*

Proof. There are several cases to investigate, here we made only a particular case for illustrating the idea, for the other cases the proof is similar.

Let $X_0 = D_n$ and τ involution of type $b)$ and assume that the action of τ extends to a smoothing $(X,0)\xrightarrow{t}(\mathbb{C},0)$. By lemma 3.5 we can assume that $(X,0)$ is defined in $\mathbb{C}^4$ by the equation

$$z^2 + x(y^2 + x^{n-2}) + t\varphi(x,y,z,t) = 0$$

$\tau(x,y,z,t) = (x,-y,-z,t)$ and φ is τ-invariant.

The fixed locus of τ is the germ of curve of equation $x^{n-1} + t\varphi(x,0,0,t)$ contained in the plane $y = z = 0$ and then for $|t| << 1$ τ has a finite number of fixed points on X_t and then the quotient X_t/τ is singular. □

Lemma 3.7. *Let* $(X,0) \xrightarrow{t} (\mathbb{C},0)$ *be a smoothing of a RDP and let* τ *be an involution of* $(X,0)$ *preserving* t. *If* $\tau_{|X_0}$ *is of type a) or f) then* X_0/τ *is a RDP and the projection to* $(Y,0) = (X/\tau, 0)$ *is flat.*

Proof. In case a) by lemma 3.5 we can assume $(X,0) \subset (\mathbb{C}^4, 0)$ defined by the equation

$$f(x,y^2,z) + t\varphi(x,y^2,z,t) = 0$$

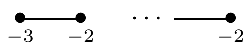
and $\tau(x,y,z,t) = (x,-y,z,t)$. Thus the equation of $(Y,0)$ is

$$f(x,s,z) + t\varphi(x,s,z,t) = 0$$

and $(X,0)$ is defined in $(Y \times \mathbb{C}_y, 0)$ by the equation $y^2 = s$. The case of involution of case f) is similar. □

Proof of Proposition 3.2. By lemma 3.6 the restriction of τ to X_0 can be only of type a),c),e),f). In cases a) and f) by lemma 3.7 the situation 3.2.i) holds.

The quotient Y of the rational double point A_{2n} by the involution of type c) is the cyclic singularity of type $\frac{1}{2n+1}(1, 2n-1)$ ([Ca3] Th. 2.4) and its dynkin diagram is



n vertices

Thus the selfintersection of the fundamental cycle is -3 and then it is also a rational triple point. According to II.3.1 every smoothing of Y admits after base change simultaneous resolution and then its Milnor fibre is diffeomorphic to its minimal resolution.

In case e) $(Y,0)$ is a smoothing of a cyclic singularity of type $\frac{1}{4d}(1, 2d-1)$ ([Ca3] Th. 2.5). Since $Y - \{0\}$ is smooth τ must act freely on $X - \{0\}$ and then Y is $\mathbb{Q}$-Gorenstein of order 2. The statement about the Milnor fibre is proved in IV.2.4. □

Lemma 3.8. *Let* $X \to \Delta$ *be a proper flat family of normal irreducible surfaces and let* $\mathcal{L}$ *be a line bundle on* X.

If $\mathcal{L}_t = \mathcal{L} \otimes \mathcal{O}_{X_t}$ *is trivial for every* $t \neq 0$ *then* $\mathcal{L}_0$ *is trivial. If moreover* $h^1(\mathcal{O}_{X_0}) = 0$ *and* X_t *is smooth for* $t \neq 0$ *then* $\mathcal{L}$ *is trivial.*

Proof. The first part follows from semicontinuity since $h^0(\mathcal{L}_0) > 0$ and $h^0(\mathcal{L}_0^{-1}) > 0$.

If $h^1(\mathcal{O}_{X_0}) = 0$ then by semicontinuity and base change $H^1(\mathcal{O}_X) = 0$. According to ([B-P-V] I.8.8) X_0 is a deformation retract of some open neighbourhood, therefore if X_t is smooth for $t \neq 0$ then the restriction map $H^2(X, \mathbb{Z}) \to H^2(X_0, \mathbb{Z})$ is bijective. From the exponential sequences it follows that the restriction map $Pic(X) \to Pic(X_0)$ is injective (cf. IV.1.1). □

Corollary 3.9. *In the situation of the beginning of §3 assume that X_t is smooth for $t \neq 0$, X_0 has at most RDP's and $Y_t = \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$.*
If for $t \neq 0$ $\pi_{t*}\mathcal{O}_{X_t} = \mathcal{O}_{Y_t} \oplus \mathcal{O}_{Y_t}(a,b)$ with $a \neq b$ (this condition is independent of the particular isomorphism from Y_t to $\mathbb{P}^1 \times \mathbb{P}^1$) then Y_0 is a Segre-Hirzebruch surface $\mathbb{F}_{2k}$.

Proof. By theorem 3.1 Y_0 has at most RDP's and the map $\pi: X \to Y$ is a flat double cover and we have $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L}$, $\mathcal{L}$ line bundle.
If Y_0 is smooth then it is well known that it is a surface $\mathbb{F}_{2k}$ for some $k \geq 0$. If Y_0 is singular its minimal resolution of singularities is $\mathbb{F}_2$ (this follows from Brieskorn-Tyurina theory on simultaneous resolution) and Y_0 is the irreducible singular quadric in $\mathbb{P}^3$ whose Picard group is generated by the hyperplane section $\mathcal{O}_{Y_0}(1)$.
But if $\mathcal{L}_0 = n\cdot\mathcal{O}_{Y_0}(1)$ then $\mathcal{L}_t = \mathcal{O}_{Y_t}(n,n)$ contrary to the assumption. □

Theorem 3.10. *Let $f: X \to \Delta$ be a proper flat map from a normal 3-dimensional complex space X to the unit disk such that:*
1) X_0 has at most rational double points as singularities.
2) $f: X^* \to \Delta^* = \Delta - \{0\}$ is a family of iterated smooth double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to line bundles $L_1, ..., L_n \in Pic(\mathbb{P}^1 \times \mathbb{P}^1)$.
3) $L_i = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a_i, b_i)$ with $a_i, b_i \geq 3$, $a_n \geq b_n + 2$ and a_i, b_i even for $i = 2, ..., n$.
Then if $f': Z \to \Delta$ is the relative canonical model of X there exists a factorization of f' $Z \xrightarrow{\pi} Y \to \Delta$ such that π is finite flat, $\pi_t: Z_t \to Y_t$ is an iterated flat double cover for every t, $Y_t = \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$ and $Y_0 = \mathbb{F}_{2k}$.

Proof. Induction on n. Case $n = 1$. The action of the involution τ on X^* extends to a biregular action on Z (cf. [Ca2] Th. 1.8) and taking quotient we have a factorization $Z \xrightarrow{\pi} Y = Z/\tau \longrightarrow \Delta$ where $Y_t = \mathbb{P}^1 \times \mathbb{P}^1$ for $t \neq 0$. The thesis follows from corollary 3.9.
Case $n > 1$. As in case $n = 1$ there exists an involution acting on Z preserving fibres and a factorization

$$Z \xrightarrow{\pi_1} V = Z/\tau \longrightarrow \Delta$$

where for $t \neq 0$ V_t is a smooth iterated double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to line bundles $L_2, ..., L_n$. By adjunction formula the divisibility of the canonical class of V_t is even and by Th. 3.1 π_1 is flat and V_0 has at most rational double points.
By induction we have then a factorization

$$Z \xrightarrow{\pi} V \xrightarrow{\delta} W \xrightarrow{\pi_2} Y \longrightarrow \Delta$$

where W is the relative canonical model of V. Then we complete the proof by proving that δ is an isomorphism.
By normality of V_0 and W_0 the fibres of δ are connected. Assume that there exists an irreducible curve $C \subset V_0$ contracted by δ and let $D \subset Z_0$ be the strict transform of C.

Since π_1 is flat we have $\pi_{1*}\mathcal{O}_Z = \mathcal{O}_V \oplus M$ for a line bundle M such that for $t \neq 0$ $M_t = \delta^*\pi_2^* L_1$. By lemma 3.8 if $\mathcal{L}$ is the line bundle on Y such that $\mathcal{L}_t = L_1$ then $M = \delta^*\pi_2^*\mathcal{L}$ and $C\cdot M = 0$.

Using adjunction formula $K_{Z_0} = \pi_1^*(K_{V_0} + M_0)$ and $D\cdot K_{Z_0} = 0$ which is impossible since K_{Z_0} is ample. □

Proposition 3.11. *In the same hypotheses of Th. 3.10 if in addition $n \geq 2$ and $2b_{n-1} > b_n+2$ then $Y_0 = \mathbb{F}_{2k}$ with*

$$k \leq max(\frac{a_{n-1}}{b_{n-1}-1}, \frac{2a_{n-1}-a_n}{2b_{n-1}-b_n-2})$$

In particular if $b_{n-1} \geq a_{n-1}+2$ then $Y_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. Without loss of generality we can assume $n = 2$ and $k > 0$.

Let σ_0, F be the standard basis of $\mathrm{Pic}(\mathbb{F}_{2k})$ $(\sigma_0^2 = 2k, F^2 = 0, F\cdot\sigma_0 = 1)$ and let $\sigma_\infty \subset \mathbb{F}_{2k}$ be the "section to infinity" (i.e. the unique effective divisor linearly equivalent to $\sigma_0 - 2kF$). We recall that for an effective divisor $D \sim a\sigma_0 + bF$ if $b < -2k$ then $2\sigma_\infty \subset D$ and in particular D is not reduced.

In our situation we have 2 line bundles L_1, L_2 on $\mathbb{F}_{2k}$ such that Z_0 is isomorphic to a surface in $L_1 \oplus L_2$ defined by the equations

$$\begin{cases} z^2 = & f \quad f \in H^0(2L_2) \\ w^2 = & g + zh \quad g \in H^0(2L_1) \quad h \in H^0(2L_1 - L_2) \end{cases}$$

Since L_i deform to the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a_i, b_i)$ we have

$$(1)\begin{cases} L_1 = & b_1\sigma_0 + (a_1 - b_1k)F \\ L_2 = & b_2\sigma_0 + (a_2 - b_2k)F \end{cases} \quad \text{or} \quad (2)\begin{cases} L_1 = & a_1\sigma_0 + (b_1 - a_1k)F \\ L_2 = & a_2\sigma_0 + (b_2 - a_2k)F \end{cases}$$

Since $a_2 \geq b_2 + 2$ and the divisor of f is reduced holds necessarily possibility (1). Moreover since $2\sigma_\infty$ is not contained in both the divisors of g and h we have

$$2(a_1 - b_1k) \geq -2k \quad \text{or} \quad (2a_1 - a_2) - (2b_1 - b_2)k \geq -2k$$

□

4. Automorphisms of iterated double covers.

Theorem 4.1. *Let $L_1, ..., L_n$ be fixed line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$. $L_i = \mathcal{O}_{\mathbb{P}\times\mathbb{P}^1}(a_i, b_i)$. If $n \geq 2$, $a_i, b_i \geq 3$, a_i, b_i even for $i \geq 2$, $a_n \geq b_n + 2$, $b_{n-1} \geq a_{n-1} + 2$ and*

$$\max_{j<i} \min(2a_i - a_j, 2b_i - b_j) < 0$$

then $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ is a connected component of the moduli space $\mathcal{M}$, irreducible and unirational.

Proof. By Cor. 2.10 it's enough to prove that $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ contains the closure of $N_0(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ in $\mathcal{M}$, but this is a consequence of Th. 3.10 and Prop. 3.11. □

Here we study the group of automorphisms of the generic element of the irreducible component $N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$. Clearly if $[S] \in N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ then there exists at least one involution acting on the canonical model of X and then $Aut(S)$ always contain a subgroup of order 2. Our main result is the following.

Theorem 4.2. *If $L_1, ..., L_n$ is a good sequence (in sense of definition C) of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ then there exists a nonempty Zariski open subset $U \subset N(\mathbb{P}^1 \times \mathbb{P}^1, L_1, ..., L_n)$ such that for every $[S] \in U$ $Aut(S)$ has order exactly 2.*

We prove this theorem later on, after some preparatory material. The first lemma is the particular case $n = 1$ of theorem 4.2.

Lemma 4.3. *If $a, b \geq 3$ then for generic $f \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, (2a, 2b))$ the only nontrivial automorphism of the surface S of equation $z^2 = f$ is the involution $\tau: z \to -z$.*

Proof. For generic f the divisor $D = div(f)$ is a smooth curve and does not exist any nontrivial automorphism h of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $h(D) = D$.

The divisor $R = div(z) \subset S$ is the set of critical points of the canonical map and then for every $g \in Aut(S)$ $g(R) = R$. This implies that for every $p \in R$ $g^{-1}\tau g(p) = p$ and since the stabiliser of R is cyclic (Easy consequence of Cartan lemma, cf. [Ca1] Prop. 1.1) $g^{-1}\tau g = \tau$. Thus g induces the identity on S/τ and then $g = Id$ or $g = \tau$. □

Lemma 4.4. *Let S be a surface of general type and assume that its canonical model X has at least one rational double point of type E_7 or E_8 at a point p.*

Then there exists at most one involution τ of X such that $\tau(p) = p$.

Proof. Let $G \subset Aut(X) = Aut(S)$ be the subgroup generated by the involutions leaving p fixed, since $Aut(S)$ is finite ([Mat2]) G is finite and by (I.3.2) G is cyclic. □

Lemma 4.5. *Let $X \to Y$ be a double cover with X canonical model of a surface of general type and Y smooth.*

If X has at least one rational double point of type E_7 or E_8 then every automorphism of X commutes with the trivial involution τ.

Proof. Let $\{p_1, ..., p_s\}$ be the (nonempty) set of singular points of X which are RDP of type E_7 or E_8. Since Y is smooth $p_1, ..., p_s$ belong to the fixed locus of τ and therefore for every $g \in Aut(X)$ and every $i = 1, ..., s$ $g^{-1}\tau g(p_i) = p_i$. The conclusion now follows from lemma 4.4 $g\tau = \tau g$. □

Lemma 4.6. *If $L_1, ..., L_n$ is a good sequence of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ then there exists an iterated flat double cover*

$$p: X \to X_1 \to .... \to X_n = \mathbb{P}^1 \times \mathbb{P}^1$$

associated to $L_1, ..., L_n$ *such that* X_1 *is smooth and* X *has exactly* 2^{n-2} *rational double points of type* E_8.

Moreover the branching divisor $D_1 \subset X_1$ *of* $p_1: X \to X_1$ *is not invariant for the trivial involution of* $p_2: X_1 \to X_2$.

Proof. We look for a surface X of equations

$$\begin{cases} z_1^2 = f_1 + z_2 h_1 \\ z_2^2 = f_2 \\ \cdot \\ \cdot \\ z_n^2 = f_n \end{cases}$$

with $f_i \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, 2L_i)$ and $h_1 \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, 2L_1 - L_2)$.

We first fix $f_2, ..., f_n$ such that the divisors $D_i = div(f_i)$ and the surface $X_1 = \{z_i^2 = f_i \; i > 1\}$ are smooth.

Take $u \in D_2 - \cup_{i>2} D_i$ and $l \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, (1,1))$ such that $E = div(l)$ is the tangent line of D_2 at u and fix $h_1 = l^2 k$ with $k(u) \neq 0$.

We now claim that for generic $f_1 \in H^0(\mathcal{M}_u^3 \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2L_1))$ (here $\mathcal{M}_u \subset \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the ideal sheaf of $\{u\}$) the surface X has the required properties.

By Bertini theorem for generic f_1 the surface X is smooth outside $p^{-1}(u)$ and $\frac{\partial^3 f_1}{\partial x^3} \neq 0$ where x, y are local coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ at u such that $y = f_2$.

If $v \in p^{-1}(u)$ then x, y are local coordinates of X_2 at v and the local equation of X is

$$\begin{cases} z_1^2 = f_1(x,y) + z_2(ay^2 + h(x,y)) \\ z_2^2 = y \end{cases}$$

with $a \neq 0$ and $h \in \mathcal{M}^3$. We can rewrite the equation as

$$z_1^2 = x^3 e(x, z_2) + x^2 z_2^2 \phi_1(z_2) + x z_2^4 \phi_2(z_2) + z_2^5 \phi_3(z_2)$$

with $e(0,0) \neq 0$ and $\phi_3(0) \neq 0$. By the computation of ([B-P-V] pag. 63-64) it follows that this is the equation of a rational double point of type E_8. □

Proof of Theorem 4.2 We prove the theorem by induction on n. The case $n = 1$ is proved in Lemma 4.3 thus we can assume that there exists a nonempty Zariski open subset $V \subset N(L_2, ..., L_n)$ such that for $[S] \in V \;\; Aut(S) = \int /2 \int$.

For every finite group G define

$$N^G = \{[S] \in N(L_1, .., L_n) | G \text{ is isomorphic to a subgroup of } Aut(S)\}$$

By ([Ca2] Th. 1.8) N^G is closed in $N = N(L_1, ..., L_n)$ and since K_S^2 is constant on N, $N^G = \emptyset$ if $ord(G) >> 0$ ([An],[Cor]). Clearly U is the complement of the union of N^G's for $ord(G) > 2$, so we only need to show that $U \neq \emptyset$.

For a fixed integer $m \geq 5$ and for every group G we may write ([Ca2] proof of Th. 1.8) N^G as a finite union of closed subset $N^{G,\varrho}$ where ϱ belong to a (finite) set of representatives of isomorphism classes of faithful representation $G \subset GL(P_m(S), \mathbb{C})$ and $N^{G,\varrho}$ is the intersection of N with the image of the natural map $H^\varrho \to \mathcal{M}$ where H^ϱ is the Hilbert scheme of the ϱ-invariant m-canonical images of surfaces of general type in $\mathbb{P}^{P_m-1}$.

Assume that for some G, ϱ, $N^{G,\varrho} = N$ and let $X \to Z = X/\tau \to \Delta$ be a family of flat iterated double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ with X_0 as in lemma 4.6 and $Z_t \in V \subset N(L_2, ..., L_n)$ for $t \neq 0$.

After a possible base change $\Delta \xrightarrow{t^s} \Delta$ the group G acts on X preserving fibres (cf. V.2.5, [F-P]). Our goal is to prove that the only possible nontrivial element of G is the trivial involution τ, we first note that we can assume without loss of generality that $\tau \in G$.

Let $g \neq 1$ be a fixed element of G and consider $q = g^{-1}\tau g \tau \in G$, according to 4.5 q is the trivial automorphism in X_0 and since G acts faithfully on every fibre we have $g\tau = \tau g$ in G. Thus g induces an automorphism g' on Z preserving fibres and then by the inductive hypothesis either $g' = 1, g = \tau$ or $g' = \tau'$, where τ' is the trivial involution of Z. Since γ' preserves ov every fibre the fixed locus of τ the second possibility cannot occur and then $g = \tau$. □

Corollary 4.7. *Let $L_1, ..., L_n, M_1, ..., M_m$ be two good sequences of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ with $L_1, .., L_n$ good and $M_1, .., M_m$ satisfying conditions C1,C2.*

If $N(L_1, ..., L_n) \cap N(M_1, ..., M_m) \neq \emptyset$ then $n = m$ and $L_i = f^* M_i$ for every i and some $f \in Aut(\mathbb{P}^1 \times \mathbb{P}^1)$.

Proof. By Cor. 2.10 and Th. 4.1 $N(M_1, .., M_m)$ is an open subset of $N(L_1, ..., L_n)$. By Theorem 4.2 applied to the good sequence $L_1, ..., L_n$ there exists an iterated smooth double cover

$$p: X \to X_1 \to .... \to X_n = \mathbb{P}^1 \times \mathbb{P}^1$$

with $[X] \in N(M_1, ..., M_m)$ such that for every $i < n$ $Aut(X_i) = \{1, \tau_i\}$ and $X_{i+1} = X_i/\tau_i$. Since X_i is of general type for every $i < n$ we must have $n = m$.

Moreover we have already seen that the sequence L_i is uniquely determined by the maps $\pi_i: X_{i-1} \to X_i$ and then up to automorphisms $L_i = M_i$ for every i. □

Corollary 4.8. *Let X be a simple iterated double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ associated to a good sequence with at most rational double points. Then X determines the trivial involution τ.*

Proof. Let $v: Aut(X) \to Aut(T^1_X)$ be the homomorphism induced by the natural action of $Aut(X)$ in the space of first order deformations and denote $G = \ker v$. Since $\tau \in G$ it's enough to prove that $G = \mathbb{Z}/2\mathbb{Z}$.

$Aut(X)$ is finite and then there exists the universal deformation of X ([Sch] 3.12) $f: \tilde{X} \to (S, 0)$. Moreover there exist a natural action of $Aut(X)$ on the germ $(S, 0)$ and we have $(\mathcal{M}, [X]) = (S, 0)/Aut(X)$.

By Cartan lemma G acts trivially on $(S,0)$ and then the action of G on X extends to an action on every fibre of f, the thesis follows from theorem 4.2. □

5. Invariants and a lower bound for the number of connected components.

We begin with a general formula for the computation of Chern numbers of simple iterated double covers, for this it is convenient to introduce for every algebraic surface S its *index* $I_S = K_S^2 - 8\chi(\mathcal{O}_S)$.

Lemma 5.1. *Let $p: X \to Y$ be a smooth simple iterated double cover associated to a sequence $L_1, ..., L_n \in \mathrm{Pic}(Y)$. Then:*
(a) $K_X^2 = 2^n(K_Y + \sum_{i=1}^n L_i)^2$
(b) $I_X = 2^n(I_Y - \sum_{i=1}^n L_i^2)$

Proof. (a) is a simple application of Hurwitz formula, we left the details to the reader, we prove *(b)* by induction on n being the formula trivially true for $n = 0$.
Assume $n > 0$ and consider a factorization

$$p: X \xrightarrow{\pi} Z \xrightarrow{q} Y$$

with q simple iterated double cover associated to $L_2, ..., L_n$ and $\pi_* \mathcal{O}_X = \mathcal{O}_Z \oplus \mathcal{O}_Z(-q^* L_1)$. Thus

$$K_X^2 = 2(K_Z + q^* L_1)^2 \qquad \chi(\mathcal{O}_X) = \chi(\mathcal{O}_Z) + \chi(-q^* L_1) = 2\chi(\mathcal{O}_Z) + \frac{1}{2} q^* L_1 (K_Z + q^* L_1)$$

and then $I_X = 2I_Z - 2(q^* L_1)^2 = 2I_Z - 2^n L_1^2$. □

For a smooth simple iterated double cover $p: X \to \mathbb{P}^1 \times \mathbb{P}^1$ associated to the sequence $L_1, ..., L_n$ $L_i = \mathcal{O}(a_i, b_i)$ with $a_i, b_i \geq 3$ we have:
$\pi_1(X) = 0$ ([Ca1] Th.1.8).
$K_X^2 = 2^{n+1}(\sum a_i - 2)(\sum b_i - 2)$.
$\chi(\mathcal{O}_X) = 1 + h^0(K_X) = 1 + \sum_{h=1}^n \sum_{j_1 < .. < j_h} (a_{j_1} + ... + a_{j_h} - 1)(b_{j_1} + ... + b_{j_h} - 1)$.
$r(X) = \max\{r \in \mathbb{N} | r^{-1} c_1(X) \in H^2(X, \mathbb{Z}) = G.C.D.(\sum a_i - 2, \sum b_i - 2)$ ([Ca4]).

Remark. 5.2. If $a_i = a =$constant then K^2, χ and r depend only on n, a and $T = \sum b_i$. In fact, according to 5.1, we have:

$$K^2 = 2^{n+1}(na - 2)(T - 2)$$

$$r = G.C.D.(na - 2, T - 2)$$

$$\chi = \frac{K^2}{8} + 2^{n-2} aT$$

Proof of Theorem A. We keep the notation used in the statement of theorem A. We first set $T_n = 8 \cdot 3^n$ and we choose a sequence of integers d_n such that

i) $6 \leq d_n \leq n^2$.
ii) $\lim_{n\to\infty} \frac{\gamma_n}{\gamma_n+1} = \frac{8}{\beta} - 1$ where $\gamma_n = \frac{d_n}{6n-2}$.

Let q_n be the cardinality of the set

$$Q_n = \{ \text{ good sequences } L_1, ..., L_n | \, L_i = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(6, b_i), \, \sum_{i=1}^{n} b_i = T_n \}$$

The second step is to choose for every n an iterated smooth double cover $X_n \xrightarrow{\pi} \mathbb{P}^1 \times \mathbb{P}^1$ associated to an element of Q_n.

By adjunction formula, corollary 4.7 and Remark 5.2 we have:

$K_{X_n} = \pi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(6n-2, T_n-2)$.

$\delta(X_n) \geq q_n$.

$\lim_{n\to\infty} \alpha_n = 1 \quad$ where $\quad \alpha_n = \frac{8\chi(\mathcal{O}_{X_n})}{K^2_{X_n}}$.

The last step is to define S_n as a smooth double cover of X_n associated to the line bundle $M_n = \pi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d_n, n(T_n-2))$. It is clear that for every $(L_1, ..., L_n) \in Q_n$ the sequence $M_n, L_1, ..., L_n$ is good and the invariant of S_n are independent of the particular choice of $L_1, ..., L_n$.

In fact an easy calculation shows

$$y_n = K^2_{S_n} = 2(1+\gamma_n)(1+n)K^2_{X_n}$$

$$\frac{8x_n}{y_n} = \frac{8\chi(\mathcal{O}_{S_n})}{y_n} = 1 + \frac{n\gamma_n + \alpha_n - 1}{(1+\gamma_n)(1+n)}$$

Therefore we have $\delta(S_n) \geq q_n$ and $\lim \frac{y_n}{x_n} = \beta$.

Claim. $q_n \geq 3^{\frac{1}{2}(n-1)^2}$.

Proof of Claim. We have an injective map $\phi: P_n \to Q_n$ where

$$P_n = \{(c_2, ..., c_n) \in \mathbb{N}^{n-1} | \, c_n = 2, c_2 \leq 3^n, c_i > 2c_{i+1}\}$$

and $\phi(c_2, ..., c_n) = (L_1, ..., L_n)$ where $L_i = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(6, 2c_i)$ for every $i \geq 2$ and $L_1 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(6, T_n - 2\sum_{i\geq 2} c_i)$.

If p_n is the cardinality of P_n we have $p_2 = 1$ and for $n \geq 3$

$$q_n \geq p_n \geq 3^{n-1}p_{n-1} \geq 3^{(n-1)+(n-2)+...+2} = 3^{\frac{1}{2}n(n-1)-1} \geq 3^{\frac{1}{2}(n-1)^2}$$

□

Note that $y_n \leq Cn^3 6^{n-1}$ where $C > 0$ is a constant independent on n and since $log_3 6 < 5/3$ we have for $n >> 0$, $y_n \leq 3^{\frac{5}{3}(n-1)}$ and then

$$\delta(S_n) \geq q_n \geq y_n^{\frac{9}{50}\log_3 y_n} \geq y_n^{\frac{1}{5}\log y_n}$$

□

VII. Simple iterated double covers of the projective plane.

In the previous chapter we gave the definition of simple iterated double cover and we proved some general facts about them. Here we want specialize to iterated double covers of $\mathbb{P}^2$ and give other examples of connected components of moduli space of surfaces of general type.

Given $L_1, ..., L_n \in \mathrm{Pic}(\mathbb{P}^2)$ define $N = N(\mathbb{P}^2, L_1, ..., L_n) \subset \mathcal{M}$ as the subset of surfaces whose canonical model is a simple iterated double cover of $\mathbb{P}^2$ associated to the sequence of line bundles $L_1, ..., L_n$. We already know that, denoting by l_i the degree of L_i, if for every i, $l_i \geq 4$ and $l_i > 2l_{i+1}$ then $N(\mathbb{P}^2, L_1, ..., L_n)$ is open in the moduli space $\mathcal{M}$ (VI.2.9).

Since $r(\mathbb{P}^2)$ is odd we cannot apply Theorem VI.3.1 in the proof of the closure of N, in fact we shall see that in general (but not always) the set N is not closed in $\mathcal{M}$. However, in view of prop VI.3.2 it is reasonable, at least for some special values of l_i, to give a complete classification of surfaces belonging in the closure of N.

In the case $n = 1$ the situation is well summarized in the statement of the following theorem which strongly relies on the classification of degenerations of $\mathbb{P}^2$ made in chapter IV.

Theorem A. *The subset $N = N(\mathbb{P}^2, \mathcal{O}(h))$, $h \geq 4$ is a connected component of moduli space if and only if h is even.*

If h is odd then the closure of N in the moduli space is a connected component.

For a general simple iterated double cover $X_0 \to X_1 \to \ldots \to X_n = \mathbb{P}^2$, $n \geq 2$, associated to $L_1, ..., L_n$, keeping in mind the proofs of previous chapter, it is reasonable to expect that the easiest situation to study is when the divisibility of the canonical classes of $X_0, ..., X_{n-1}$ is even, we give thus the following:

Definition B. A sequence of line bundles $L_1, ..., L_n$, $L_i = \mathcal{O}_{\mathbb{P}^2}(l_i)$, is called a *good sequence* if satisfies the following 3 conditions:

B1) $l_i \geq 4$ for every $i = 1, ..., n$.

B2) $l_i > 2l_{i+1}$ for every $i = 1, ..., n-1$.

B3) l_n is odd, l_i is even for $i = 1, ..., n-1$.

A *good* simple iterated double cover of $\mathbb{P}^2$ is, by definition, a simple iterated double cover associated to a good sequence.

The main result we prove is the following

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Theorem C. *Let $L_1, ..., L_n \in \mathrm{Pic}(\mathbb{P}^2)$ be a good sequence of line bundles and let X be the canonical model of a surface belonging to the closure of $N(\mathbb{P}^2, L_1, ..., L_n)$, then:*

(i) X is either a simple iterated double cover of $\mathbb{P}^2$ or X is singular, X is a simple iterated double cover of Y where Y is a nonflat double cover of the cone over the nondegenerate rational curve of degree 4 in $\mathbb{P}^4$.

(ii) The Kuranishi family of X is smooth and the closure of $N(\mathbb{P}^2, L_1, ..., L_n)$ is a connected component of moduli space.

Using the same proofs (with some inessential changes) used in section VI.4 we can prove easily that for the generic minimal surface S belonging to $N(\mathbb{P}^2, L_1, ..., L_n)$ the canonical bundle is ample and the only nontrivial automorphism of S is the trivial involution, thus the sequence $L_1, ..., L_n$ is uniquely determined by S and then we have the following

Corollary D. *Two good simple iterated double cover of $\mathbb{P}^2$ are deformation equivalent if and only if they are associated to the same good sequence.*

In section 5 we shall see how, using only good simple iterated double covers of $\mathbb{P}^2$, it is possible to prove a lower bound of type $\delta \geq (K^2)^{c \log K^2}$ for a positive constant c.

1. Degenerations of double covers of the projective plane

Throughout all this chapter we denote by $W_0 \subset \mathbb{P}^5$ the projective cone over the nondegenerate rational curve of degree 4 in $\mathbb{P}^4$ and by $w_0 \in W_0$ its singular point.

Lemma-Definition 1.1. *Let $\sigma \subset W_0 \subset \mathbb{P}^5$ be a generic hyperplane section. Then σ is a generator of $\mathrm{Pic}(W_0) = \mathbb{Z}$.*

If $W \to \Delta$ is a deformation of W_0 such that $W_t = \mathbb{P}^2$ for every $t \neq 0$ every line bundle on W_0 extends to a line bundle on W and if L is a line bundle on W such that $L_0 = a\sigma$ then $L_t = \mathcal{O}_{\mathbb{P}^2}(2a)$ for $t \neq 0$.

Proof. Let $X = \mathbb{F}_4 \xrightarrow{\gamma} W_0$ be the minimal resolution, since σ doesn't contain the vertex w_0 of the cone, $\gamma^{-1}(\sigma)$ is the section σ_0. The singularity at w_0 is rational and then $\mathrm{Pic}(W_0)$ is identified with the set of line bundle L_0 on X such that $L_0 \cdot \sigma_\infty = 0$. Since $q(W_0) = p_g(W_0) = 0$ the restriction $\mathrm{Pic}(W) \to \mathrm{Pic}(W_0)$ is an isomorphism by IV.1.1. After a possible restriction of the family $W \to \Delta$ to an open disk $0 \in \Delta' \subset \Delta$ of smaller radius we can assume W embedded in $\mathbb{P}^5 \times \Delta$ (cf. IV.1.2) and the restriction of $\mathcal{O}_{\mathbb{P}^5}(a)$ to W_t, $t \neq 0$, is a very ample line bundle with selfintersection $4a^2$. The conclusion is now trivial. $\square$

Lemma 1.2. *Let $f: Y \to \Delta$ be a proper flat family of normal surfaces such that for every $t \neq 0$ Y_t is a smooth surface and Y_0 has at most R.D.P.'s as singularities.*

Let $\tau: Y \to Y$ be an involution preserving f such that $Y_t/\tau = \mathbb{P}^2$ for every $t \neq 0$. Then either:

(i) $Y_0/\tau = \mathbb{P}^2$, or

(ii) $Y_0/\tau = W_0$. *The double cover* $Y_0 \xrightarrow{\pi} W_0$ *is branched exactly over the vertex* $w_0 \in W_0$ *and over a divisor* $D' \sim (2a-1)\sigma$ *with* $w_0 \notin D'$. *For* $t \neq 0$, $Y_t \to Y_t/\tau = \mathbb{P}^2$ *is branched over* $D'_t \sim \mathcal{O}(4a-2)$ *and* $r(Y_t)$ *is even.*

Proof. Y_0/τ is a normal degeneration of $\mathbb{P}^2$ with at most singular points of the three types described in proposition VI.3.2 and therefore according to the results of chapter IV either $Y_0/\tau = \mathbb{P}^2$ or $Y_0/\tau = W_0$.

Assume $Y_0/\tau = W_0$, then, since (W_0, w_0) is not a rational double point, $y_0 = \pi^{-1}(w_0)$ is a fixed point of the involution τ.

According to Proposition VI.3.2 and its proof, the singularity (Y_0, y_0) is a simple node defined by the equation $x_0^2 + x_1^2 + x_2^2 = 0$ and $\tau(x_i) = -x_i$, in particular y_0 is an isolated fixed point of the involution.

Let $S \xrightarrow{\delta} Y_0$ be the resolution of the node (Y_0, y_0) and let $E = \delta^{-1}(y_0) \subset S$ be the corresponding nodal curve. The action of τ can be lifted to an action on S (cf. the example in I.4) and it is easy to see that $S/\tau = X = \mathbb{F}_4$.

Moreover the flat double cover $\pi\colon S \to \mathbb{F}_4$ is branched over $D = \sigma_\infty \cup D'$, $\sigma_\infty \cap D' = \emptyset$ and since this divisor must be 2-divisible in $NS(\mathbb{F}_4)$, $D' \sim (2a-1)\sigma_0$ and $\frac{1}{2}(\sigma_\infty \cup D') = a\sigma_0 - 2f$ where f denote the fibre of $\mathbb{F}_4$.

The study of surfaces Y_0 as before plays an important role in our proof of theorem C, we give the following

Definition 1.3. Let $a \geq 3$ be an integer and let $S \xrightarrow{\pi} \mathbb{F}_4$ be the double cover associated to $L = a\sigma_0 - 2f$ branched over the disjoint union of σ_∞ and a divisor $D' \sim (2a-1)\sigma_0$ with at most simple singularities ([B-P-V] II.8). $E = \pi^{-1}(\sigma_\infty)$ is a nodal curve, taking its contraction $\delta\colon S \to Y_0$ we get a surface with at most rational double points as singularities which is a double cover of the cone W_0. We shall call Y_0 a *degenerate double cover of* $\mathbb{P}^2$. The number a determines $K^2_{Y_0} = 8(a-2)^2$ and will be called the *discrete building data* of Y_0.

Theorem 1.4. *The set* $N = N(\mathbb{P}^2, \mathcal{O}(h))$, $h \geq 4$ *is a connected component of moduli space if* h *is even. If* h *is odd then the set* $\overline{N} - N$ *is contained in the set of degenerate double covers of* $\mathbb{P}^2$ *with discrete building data* $a = \dfrac{h+1}{2}$.

Proof. According to VI.2.9 N is open in the moduli space and if N_0 denotes the subspace of surfaces with smooth canonical model then N_0 and N have the same closure in the moduli space.

If $[S_0] \in \overline{N_0}$ then by valuative criterion there exists a deformation of S_0 $f\colon S \to \Delta$ with $[S_t] \in N_0$ for every $t \neq 0$ and an involution τ acting on the relative canonical model $Y \to \Delta$ such that $Y_t/\tau = \mathbb{P}^2$ for every $t \neq 0$. The thesis follows from lemma 1.2. □

2. Vanishing theorems for degenerate double covers of $\mathbb{P}^2$ and deformations locally trivial at the vertex.

Throughout this section a is a fixed integer ≥ 3. Let X be the Segre-Hirzebruch surface $\mathbb{F}_4$ and let $S \xrightarrow{\pi} X$ be the double cover ramified over $D = \sigma_\infty \cup D'$ with D' reduced divisor linearly equivalent to $(2a-1)\sigma_0$. We assume that S has at most rational double points as singularities and let $R \subset S$ be the ramification divisor.
We have $\pi_* \mathcal{O}_S = \mathcal{O}_X \oplus \mathcal{O}_X(-L)$ where $L = a\sigma_0 - 2f$ and $E = \pi^{-1}(\sigma_\infty)$ is a nodal curve, i.e. a smooth rational curve with selfintersection $E^2 = -2$. Denote by $\delta\colon S \to Y_0$ the contraction of E, Y_0 is a surface with at most rational double points and ample canonical bundle. We shall call $\delta(E) = y_0$ the vertex of the degenerate double cover Y_0.
By abuse of notation we denote with the same letter σ the line bundles $\sigma_0 \in Pic(X)$, $\pi^*\sigma_0 \in Pic(S)$ and $\delta_*\pi^*\sigma_0 \in Pic(Y_0)$. By Hurwitz formula $K_S = \pi^*(K_X + L) = (a-2)\sigma$.

Lemma 2.1. *$H^1(Y_0, p\sigma) = 0$ for every integer p.*

Proof. According to Leray spectral sequence we have

$$H^1(Y_0, p\sigma) = H^1(S, p\sigma) = H^1(X, p\sigma) \oplus H^1(X, (p-a)\sigma + 2f)$$

and the thesis follows from proposition III.1.5.iii). $\square$

Lemma 2.2. *For every smooth curve C contained in a smooth surface S, $H^1_C(\Omega^1_S) \neq 0$.*

Proof. For any locally free sheaf $\mathcal{F}$ on S there exists an inclusion $H^0(\mathcal{F} \otimes \mathcal{O}_C(C)) \subset H^1_C(\mathcal{F})$ (this is proved in [B-W] 1.5 for the tangent sheaf but the same proof works for any locally free sheaf, cf. also I.5) and according to the exact sequence of differentials $H^0(\mathcal{O}_C) \subset H^0(\Omega^1_S \otimes \mathcal{O}_C(C))$. $\square$

Lemma 2.3. *If $p \geq 2a$ then $h^1(S, \Omega^1_S(K_S + p\sigma)) \leq 1$.*

Proof. We consider the exact sequence on S (VI.2.4)

$$0 \longrightarrow \pi^*(\Omega^1_X(K_X + L + p\sigma)) \longrightarrow \Omega^1_S(K_S + p\sigma) \longrightarrow \mathcal{O}_R(\pi^*(K_X + p\sigma)) \longrightarrow 0$$

where $R \subset S$ is the ramification divisor.
Using the results of section III.1, we get for $p \geq 2a$

$$h^1(\mathcal{O}_D(K_X + p\sigma)) \leq h^1(X, (p-2)\sigma + 2f) + h^2(X, K_X + (p-2a)\sigma + 4f) = 0$$

$$h^1(\pi^*\Omega^1_X(K_X + L + p\sigma)) = h^1(\Omega^1_X(K_X + L + p\sigma)) + h^1(\Omega^1_X(K_X + p\sigma)) = 1$$

and the proof follows from the equality $h^1(\mathcal{O}_R(\pi^*(K_X + p\sigma))) = h^1(\mathcal{O}_D(K_X + p\sigma))$. $\square$

Theorem 2.4. *In the notation above $\mathrm{Ext}^1_{Y_0}(\Omega^1_{Y_0}, -p\sigma) = 0$ for every $p \geq 2a$.*

Proof. Y_0 is a Gorenstein surface, in particular $K_{Y_0} + p\sigma$ is a Cartier divisor and by Serre duality ([Ha1] pag. 243)

$$\mathrm{Ext}^1_{Y_0}(\Omega^1_{Y_0}, -p\sigma)^\vee = \mathrm{Ext}^1_{Y_0}(\Omega^1_{Y_0}(K_{Y_0} + p\sigma), K_{Y_0})^\vee H^1(\Omega^1_{Y_0}(K_{Y_0} + p\sigma))$$

We use the following exact sequence of sheaves on Y_0 ([Kas],[Pi2])

$$0\longrightarrow\Omega^1_{Y_0}\longrightarrow\delta_*\Omega^1_S\stackrel{\alpha}{\longrightarrow}\mathbb{C}_{y_0}\longrightarrow 0$$

where for every open subset $E\subset U\subset S$ and every $\omega\in H^0(U,\Omega^1_S)$, $\alpha(\omega)=0$ if and only if the holomorphic two-form $d\omega$ vanishes in E. It is immediate to observe that $\Omega^1_{Y_0}$, being locally generated by closed 1-form, is contained in the kernel of α; the converse inclusion requires some computation ([Kas] p. 55). Note moreover that, according to I.5.5, the sheaf $\delta_*\Omega^1_S$ is reflexive and then the exactness of the above sequence is equivalent to the equality $H^1_{\{y_0\}}(Y_0,\Omega^1_{Y_0})=\mathbb{C}$.

Twisting the above exact sequence by $K_{Y_0}+p\sigma=\delta_*(K_S+p\sigma)$ we get

$$0\longrightarrow\Omega^1_Y(K_{Y_0}+p\sigma)\longrightarrow\delta_*\Omega^1_S(K_S+p\sigma)\stackrel{\alpha}{\longrightarrow}\mathbb{C}_{y_0}\longrightarrow 0$$

Our first step is to prove that, for $p\geq 2a$, $H^1(\Omega^1_{Y_0}(K_{Y_0}+p\sigma))=H^1(\delta_*\Omega^1_S(K_S+p\sigma))$, i.e. that α is surjective on the global sections. Actually holds the following stronger result

Lemma 2.5. *In the above notation if $p\geq 2$ then the composition of $H^0(\alpha)$ with the pullback map $\pi^*: H^0(\Omega^1_X(K_X+p\sigma))\to H^0(\Omega^1_S(K_S+p\sigma))$ is surjective.*

Proof. Let s,z be the principal affine coordinates on $X=\mathbb{F}_4$ (cf. III.1) and consider $\omega= s^{-2}dz(dz\wedge ds)\in H^0(\Omega^1_X(K_X+p\sigma))$.

In the open set $U_{0,0}\subset X$ with coordinates z,s', $\omega=dz(ds'\wedge dz)$, $\sigma_\infty=\{s'=0\}$ and locally S is the double cover of X defined by equation $\xi^2=s'$ and then $\pi^*\omega=2\xi dz(d\xi\wedge dz)$.

Now $d\xi\wedge dz$ extends to a holomorphic invertible section of K_S in a neighbourhood of E and then, up to nonzero scalar multiplication, $\alpha(\pi^*\omega)=\alpha(\xi dz)\neq 0$ since $d(\xi dz)=d\xi\wedge dz$. □

The Leray spectral sequence gives an exact sequence

$$0\longrightarrow H^1(\delta_*\Omega^1_S(K_S+p\sigma))\longrightarrow H^1(\Omega^1_S(K_S+p\sigma))\stackrel{r}{\longrightarrow}H^0(R^1\delta_*\Omega^1_S(K_S+p\sigma))$$

and if $r\neq 0$ then by lemma 2.3 the proof is complete.

For any open set $E\subset U\subset S$ there exists an exact sequence

$$0\longrightarrow H^0(U,\Omega^1_S(K_S+p\sigma))\stackrel{\beta}{\longrightarrow}H^0(U-E,\Omega^1_S(K_S+p\sigma))\stackrel{d}{\longrightarrow}$$

$$\stackrel{d}{\longrightarrow}H^1_E(\Omega^1_S(K_S+p\sigma))\stackrel{r_U}{\longrightarrow}H^1(U,\Omega^1_S(K_S+p\sigma))$$

On the open set $V=\delta(U)\subset Y$, according to I.5.5, the coherent sheaf $\delta_*\Omega^1_S(K_S+p\sigma)$ is reflexive, in particular the above map β is an isomorphism and the map r_U is injective.

Since $H^1_E(\Omega^1_S(K_S+p\sigma))=H^1_E(\Omega^1_S)\neq 0$ the above inclusion factors as

$$H^1_E(\Omega^1_S)\subset H^1(S,\Omega^1_S(K_S+p\sigma))\stackrel{r_U}{\longrightarrow}H^1(U,\Omega^1_S(K_S+p\sigma))$$

and then $r=\varinjlim r_U\neq 0$. □

As in the previous chapters we shall call natural deformations of S the deformations obtained by deforming the branch divisor $D = \sigma_\infty \cup D' \sim 2a\sigma - 4f$. Since σ_∞ is a fixed part of the linear system $|D|$ the natural deformations are parametrized by $H^0(X,(2a-1)\sigma)$.

The singularity (Y_0, y_0) is rational and therefore, as in chapter II, it is possible to define the blow-down morphism $\beta\colon Def_S \longrightarrow Def_{Y_0}$. It is clear that every (infinitesimal) natural deformation of S is trivial in a neighbourhood of E and its blow-down is a deformation of Y_0 locally trivial at y_0.

Therefore, taking first order deformations, we get a commutative diagram

$$\begin{array}{ccc} H^0(X,(2a-1)\sigma) & \xrightarrow{Nat} & T^1_S \\ \downarrow \varrho & & \downarrow \beta \\ T^1LT(Y_0,y_0) & \longrightarrow & T^1_{Y_0} \end{array}$$

Where β is the blow-down map described in chapter II and $T^1LT(Y_0,y_0)$ is the kernel of the natural restriction map $T^1_{Y_0} \to T^1_{(Y_0,y_0)}$. Note that the natural deformations never give a complete family of deformations of S, since the nontrivial contribution of the nodal curve E to the space T^1_S ([B-W]).

Theorem 2.6. *The above map ϱ is surjective and the blow down of the family of natural deformations of S is a complete family of deformations of Y_0, locally trivial at the vertex, with smooth base space.*

Proof. According to the results of VI.2 there exists an exact sequence

$$H^0(\mathcal{O}_R(\pi^* D)) \xrightarrow{\epsilon} \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S) \xrightarrow{\sigma} H^1(\theta_X) \oplus H^1(\theta_X(-L))$$

and the image of ϵ is the set of first order natural deformations. Given an open subset $V \subset X$ the inclusion $\pi^*\Omega^1_X \to \Omega^1_S$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S) & \longrightarrow & \mathrm{Ext}^1_{\pi^{-1}(V)}(\Omega^1_{\pi^{-1}(V)}, \mathcal{O}_{\pi^{-1}(V)}) \\ \downarrow \sigma & & \downarrow \\ H^1(\theta_X) \oplus H^1(\theta_X(-L)) & \xrightarrow{\gamma_V} & H^1(\theta_V) \oplus H^1(\theta_V(-L)) \end{array}$$

Lemma 2.7. *In the above set up, if $\sigma_\infty \subset V$, then the map γ_V is injective.*

Proof. It is clearly sufficient to prove that the two natural maps

$$\gamma_1\colon H^1(\theta_X) \to H^1(\theta_X \otimes \mathcal{O}_{\sigma_\infty}) \quad \gamma_2\colon H^1(\theta_X(-L)) \to H^1(\theta_X(-L) \otimes \mathcal{O}_{\sigma_\infty})$$

are isomorphisms.

Note first that $h^1(\theta_X \otimes \mathcal{O}_{\sigma_\infty}) = 3$, $h^1(\theta_X(-L) \otimes \mathcal{O}_{\sigma_\infty}) = 1$ and by corollary III.1.4 $h^1(\theta_X) = 3$, $h^1(\theta_X(-L)) = h^1(\Omega^1_X((a-2)\sigma_0)) = 1$, $h^2(\theta_X(-\sigma_\infty)) = h^0(\Omega^1_X(-\sigma_0 - 2f)) = 0$.

Thus γ_1 is surjective and then it is an isomorphism, in order to show that γ_2 is surjective we prove that the natural map $H^2(\theta_X(-L-\sigma_\infty)) \to H^2(\theta_X(-L))$ or its Serre dual $H^0(\Omega^1_X((a-$

$2)\sigma_0)) \to H^0(\Omega^1_X((a-2)\sigma_0+\sigma_\infty))$ is an isomorphism but this is exactly the result of lemma III.1.2. □

Returning to the proof of theorem 2.6 we note that the open sets $\pi^{-1}(V)$, $\sigma_\infty \subset V$ are a fundamental system of neighbourhoods of E. Thus from lemma 2.7 it follows that for every open subset $U \subset S$ with $E \subset U$, the kernel of the natural map

$$\alpha\colon \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S) \to \mathrm{Ext}^1_U(\Omega^1_U, \mathcal{O}_U)$$

is contained in the set of first order natural deformations $\ker\sigma = Im\,\epsilon$.

We now apply this fact to a smooth open subset $E \subset U$ such that $\delta(U)$ is an affine open neighbourhood of y_0. According to the Cartesian diagram (cf. Chapter II)

$$\begin{array}{ccc} T^1_S & \xrightarrow{\alpha} & H^1(U,\theta_U) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_U} \\ T^1_{Y_0} & \xrightarrow{r} & T^1_{Y_0,y_0} \end{array}$$

we have $\beta(\ker\alpha) = \ker r = T^1LT(Y_0,y_0)$ and since $\varrho = \beta\circ\epsilon$ the first part of the theorem is proved.

For the second part we introduce the functor of Artin rings $LT(Y_0,y_0)$ of deformations of Y_0 which are locally trivial at the point y_0.

More generally for every complex space Z with isolated singularities and for every finite subset $\{z_1,...,z_n\} \subset Z$ we can define the functor D of deformations of Z which are locally trivial at the points $z_1,...,z_n$. This functor has been studied by several authors, in [G-K] it is proved that:

i) D satisfies the Schlessinger conditions H1,H2 and H3.

ii) There exists a closed analytic subgerm (possibly nonreduced) V of $Def(Z)$ such that the restriction of the semiuniversal deformation of Z to V is a complete family of deformations locally trivial at $z_1,...,z_n$.

iii) The Zariski tangent space of V is the kernel of the differential of the natural morphism $Def(Z) \to \Pi Def(Z,z_i)$.

Applying these results to the functor $LT(Y_0,y_0)$ we conclude the proof. □

3. The Kuranishi family of a degenerate double cover.

Let $Y_0 \xrightarrow{\pi} W_0$ be a degenerate double cover of $\mathbb{P}^2$ ramified over the union of the vertex w_0 and a divisor $D' \sim (2a-1)\sigma$ with $a \geq 3$. Here we construct explicitly a smooth complete family of deformations of Y_0, this will imply in particular that the moduli space at Y_0 is locally irreducible and then the closure on the moduli space of the set $N(\mathbb{P}^2, \mathcal{O}(h))$ is a connected component for every $h \geq 4$.

The idea is to describe deformations of Y_0 as canonical coverings of suitable deformations of the cone W_0 and then prove that they give a complete family.

We first recall some well known facts about cyclic coverings associated to $\mathbb{Q}$-Cartier divisors. For every normal complex space X we denote by $\mathcal{M}_X$ the sheaf of meromorphic functions on X and for every analytic Weil divisor $D \subset X$ we denote by $\mathcal{O}_X(D)$ the reflexive subsheaf of $\mathcal{M}_X$ of meromorphic functions f such that $div(f)+D \geq 0$. We keep this explicit description of $O_X(D)$ throughout all this section.

Let L be a Weil divisor on a normal irreducible variety X such that nL is Cartier and let $s \in H^0(X, nL)$ be a meromorphic function such that the divisor $D = div(s) + nL$ is reduced and is contained in the set of points where L is Cartier.

The multiplication by s gives a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(-nL) \to \mathcal{O}_X$ and we may define in a natural way a coherent analytic reflexive $\mathcal{O}_X$-algebra (cf. [Reid] 3.6, [E-V] 1.4)

$$\mathcal{A}(L,s) = \bigoplus_{i=0}^{n-1} \mathcal{A}_i = \bigoplus_{i=0}^{n-1} \mathcal{O}_X(-iL)$$

If (X, x) is a normal analytic singularity, its local analytic class group is by definition the quotient of the free Abelian group generated by the germs of analytic Weil divisors modulo the subgroup of principal divisors. For a twodimensional rational singularity it is a finite group naturally isomorphic to the first homology group of the link of X ([Bri]).

Lemma 3.1. *Let n, L, s, D be as above, if $x \notin D$ then the local analytic $\mathcal{O}_x$-algebra $\mathcal{A}_x(L,s)$ depends, up to isomorphism, only by the class of L in the local analytic class group of the analytic singularity (X, x).*

Proof. Let n, L', s', D' be another set of data with $x \notin D'$ and assume $L - L'$ principal at x. This means that there exists an analytic open neighbourhood U of x and a meromorphic function f on U such that $L = L' + div(f)$ and $div(s)_{|U} = -nL$, $div(s')_{|U} = -nL'$. Therefore $s^{-1}s'f^{-n}$ is an invertible holomorphic function on U and, possibly shrinking U, it admits a n-th root g. Thus $s' = s(fg)^n$ and the multiplication map $(fg)^i\colon \mathcal{O}_U(-iL') \longrightarrow \mathcal{O}_U(-iL)$ gives the required isomorphism. □

On the algebra $\mathcal{A}$ acts the cyclic group μ_n

$$\mu_n \times \mathcal{A}_i \ni (\xi, h) \to \xi^{-i} h \in \mathcal{A}_i$$

and then the finite map

$$Z = Specan_X(\mathcal{A}(L,s)) \xrightarrow{\pi} X$$

is a cyclic covering of normal varieties (*Specan* ([Fi] 1.14) is the analytic spectrum, if X is projective then by GAGA principles is the same of the usual algebraic spectrum ([Ha1] II, Ex. 5.17)).

According to lemma 3.1 if $x \notin div(s) + nL$ the germ of the covering over the point x is independent from s.

Corollary 3.2. *In the above set-up assume X compact and let T be a sufficiently small analytic open neighbourhood of s in $H^0(X, nL)$.*

Let $Z_T \xrightarrow{\pi} X \times T$ *be the cyclic covering of degree* n *associated to the Weil divisor* $L \times T$ *and multiplication given by* $s(x,t) = t(x)$, $t \in T$.
If $X \longrightarrow S$ *is a flat map such that the composition* $Z \longrightarrow X \longrightarrow S$ *is flat then also the composition* $Z_T \longrightarrow X \times T \longrightarrow S \times T$ *is flat.*

Proof. Let $U \subset X$ be the open subset where L is Cartier, if T is sufficiently small then $s_t(x) = 0$ for some $t \in T$ implies that $x \in U$. Therefore if $x \notin U$ then by lemma 3.1 the germ of Z_T over (x,s) is locally isomorphic to $Z \times T$. On the other hand the map $U \times T \longrightarrow S \times T$ is flat and the restriction of the algebra $\mathcal{A}$ over $U \times T$ is locally free and then the restriction of π over $U \times T$ is a flat map. □

Therefore, in case $S = point$, we have a morphism from deformations of s to deformations of Z. Consider for example the hypersurface $Z \subset \mathbb{P}^3 \times \mathbb{C}$ of equation $z_1 z_2 - z_3^2 = t z_0^2$, $t \in \mathbb{C}$ and the involution $\tau: Z \to Z$, $\tau(t, z_0, z_1, z_2, z_3) = (t, z_0, -z_1, -z_2, -z_3)$.
Let $t: Z \to \mathbb{C}$ be the projection on the coordinate t and let Z_t the projective subvariety of Z of points with fixed t. It is immediate to observe that Z_t is a smooth quadric for $t \neq 0$, Z_0 is the cone over a nonsingular conic and t gives the semiuniversal deformation of the isolated singularity $(Z_0, (1,0,0,0,0))$.
The quotient Z/τ is the variety $W \subset \mathbb{P}^5 \times \mathbb{C}$ defined by the equation

$$(3.3) \qquad rank \begin{pmatrix} x_1 & x_2 & x_3 + t x_0 \\ x_2 & x_3 & x_4 \\ x_3 + t x_0 & x_4 & x_5 \end{pmatrix} \leq 1$$

where $x_0 = z_0^2$, $x_1 = z_1^2$, $x_2 = z_1 z_3$, $x_3 = z_3^2$, $x_4 = z_2 z_3$, $x_5 = z_2^2$.
The quotient family $W \longrightarrow \mathbb{C}$, $(x,t) \to t$ is a deformation of W_0 and is exactly the degeneration of $\mathbb{P}^2$ obtained by sweeping out the cone over the Veronese surface $V \subset \mathbb{P}^5$. To see this let $C(V,v) \subset \mathbb{P}^6$ be the projective cone over the image of the map $\mathbb{P}^2_u \to \mathbb{P}^5_x$, $x_1 = u_0^2$, $x_2 = u_0 u_1$, $x_3 = u_1^2$, $x_4 = u_1 u_2$, $x_5 = u_2^2$, $x_6 = u_0 u_2 - u_1^2$. It is defined by the equation

$$(3.4) \qquad rank \begin{pmatrix} x_1 & x_2 & x_3 + x_6 \\ x_2 & x_3 & x_4 \\ x_3 + x_6 & x_4 & x_5 \end{pmatrix} \leq 1$$

V is the intersection of $C(V,v)$ with the hyperplane $x_0 = 0$ and the vertex v is the point of homogeneous coordinates $(1,0,0,0,0,0,0)$.
Let $H_t \subset \mathbb{P}^6$, $t \in \mathbb{C}$ be the hyperplane of equation $x_6 - t x_0 = 0$, then $H_t \cap V = V \cap \{x_6 = 0\}$ is a smooth hyperplane section and the surface $W_t = C(V,v) \cap H_t$ is exactly the surface defined in (3.3).
Let $H \subset W$ be the Weil divisor defined by the equation $x_2 = x_3 = x_4 = 0$. Then $\mathcal{O}_W(-H)$ is the ideal sheaf of H and $2H$ is the hyperplane section $x_3 = 0$ of W. In fact the closed subset

$\{x_1 = x_3 = x_5 = 0\}$ has codimension 3 in W and then it is sufficient to prove the equality $2H = div(x_3)$ on its complement. An easy computation then shows that on every affine subset $W \cap \{x_i \neq 0\}$ $i = 1,3,5$ holds the ideals equality $(x^2x_i^{-1}, x^3x_i^{-1}, x^4x_i^{-1})^2 = (x^3x_i^{-1})$. Note that $\pi_*\mathcal{O}_Z = \mathcal{O}_W \oplus \frac{z_0}{z_3}\mathcal{O}_W(-H)$ and then there exists an isomorphism of $\mathcal{O}_W$-algebras $\pi_*\mathcal{O}_Z = \mathcal{O}_W \oplus \mathcal{O}_W(-H)$ where the algebra structure in the right side is induced by the multiplication morphism $\frac{x_0}{x_3}\colon \mathcal{O}_W(-2H) \longrightarrow \mathcal{O}_W$.

Let now $Y_0 \xrightarrow{\pi_0} W_0 \subset \mathbb{P}^5$ be a fixed degenerate double cover, then, according to III.1.5, W_0 is projectively normal in $\mathbb{P}^5$ and then there exists a section $s_0 \in H^0(\mathbb{P}^5, \mathcal{O}(2a-1))$ such that π_0 is ramified over w_0 and over the divisor of the restriction of s_0 to W_0.

Let T be a small open neighbourhood of s_0 and consider the double covers

$$Y_T = Specan_{W\times T}(\mathcal{O}_{W\times T} \oplus \mathcal{O}_{W\times T}(-(2a-1)H \times T))) \longrightarrow W \times T$$

where the algebra structure is induced by the section $s(x,t) = s_t(x)$ $s_t \in T, x \in W$. This makes sense since $2H \times T$ is a Cartier divisor linearly equivalent to $\{s(x,t) = 0\}$.

By previous results (3.1, 3.2) it follows that:

(i) The map $Y_T \longrightarrow T$ is a deformation of the space

$$Y = Specan_W(\mathcal{O}_W \oplus \mathcal{O}_W(-(2a-1)H)))$$

with the algebra structure induced by s_0.

(ii) Over the vertex w_0 the space Y is isomorphic to the above space Z and then the composition $Y \longrightarrow W \longrightarrow \mathbb{C}$ gives a complete deformation of the node (Y_0, y_0).

It is now easy to prove the following

Theorem 3.5. *In the above notation the composition*

$$f\colon Y_T \longrightarrow W \times T \longrightarrow \mathbb{C} \times T$$

is a smooth complete family of deformations of Y_0.

Proof. We need to prove that $f^{-1}(0, s_0) = Y_0$ and that the Kodaira-Spencer map of the family is surjective.

By definition $f^{-1}(0,s_0) = Spec_{W_0}(\mathcal{O}_{W_0} \oplus (\mathcal{O}_W(-(2a-1)H) \otimes \mathcal{O}_{W_0}))$ while from the definition and the normality of Y_0 we have $Y_0 = Spec_{W_0}(\mathcal{O}_{W_0} \oplus \mathcal{O}_{W_0}(-L))$ where $L = a\sigma - 2l$, $l \subset W_0$ is a line through w_0.

Note that all lines through w_0 are linearly equivalent, L is linearly equivalent to $(4a-2)l$, the intersection $H_0 = H \cap W_0$ is the union the two lines $l_1 = \{x_1 = x_2 = x_3 = x_4 = 0\}$, $l_2 = \{x_5 = x_2 = x_3 = x_4 = 0\}$ and then the natural map $\mathcal{O}_W(nH) \otimes \mathcal{O}_{W_0} \xrightarrow{j_n} \mathcal{O}_{W_0}(2nl)$ is an isomorphism over $W_0 - \{w_0\}$ for every integer n.

In a neighbourhood of the vertex w_0, since the sheaf $\mathcal{O}_W(nH)$ is reflexive on W and invertible for n even, according to ([E-V] 2.1, cf. also the proof of IV.1.3) the map j_n is injective for every n and an isomorphism for n even, moreover the ideal of $H_0 \subset W_0$ is generated by $x_2x_0^{-1}, x_3x_0^{-1}, x_4x_0^{-1}$ and then j_{-1} is also surjective. Tensoring with the line bundle $\mathcal{O}_W(2pH)$, $p \in \mathbb{Z}$, we get the surjectivity of j_n for every integer n. In particular since j_{1-2a} is an isomorphism Y_0 is a fibre of f.

By (ii) the composition of the Kodaira-Spencer map of f with the natural map $T^1(Y_0) \xrightarrow{r} T^1(Y_0, y_0)$ is surjective, therefore it is sufficient to prove that Y_T contains every deformation locally trivial at the vertex. But this is an immediate consequence of Theorem 2.6 and the surjectivity of the map $H^0(\mathbb{P}^5, \mathcal{O}(2a-1)) \to H^0(W_0, (2a-1)\sigma) = H^0(\mathbb{F}_4, (2a-1)\sigma)$.
□

Corollary 3.6. *Every degenerate double cover deforms to a smooth double cover of $\mathbb{P}^2$, in particular for h odd ≥ 5 the subset $N(\mathbb{P}^2, \mathcal{O}(h))$ is not closed in the moduli space.*

Corollary 3.7. *The line bundle σ of Y_0 can be extended to every deformation of Y_0.*

Proof. The pull back of the hyperplane section $2H$ to Y_T is an extension of σ to a complete family. □

Proof of theorem A: the case h even follows from 1.4. If h is odd then N is open and irreducible in the moduli space but, according to 3.6, it is not closed in the moduli space. Again by 1.4 and 3.5 the moduli space at every point of $\overline{N}$ is locally irreducible and then $\overline{N}$ is open. □

4. Proof of theorem C.

For $n = 1$ theorem C is an immediate consequence of theorems 1.4 and 3.5, for $n \geq 2$ part *(i)* is a consequence of the following

Proposition 4.1. *Let $L_1, ..., L_n \in \mathrm{Pic}(\mathbb{P}^2)$ be a good sequence (def. B), $L_i = \mathcal{O}(l_i)$ and let X_0 be the canonical model of a surface belonging to the closure of $N(\mathbb{P}^2, L_1, ..., L_n)$. Then either X_0 is a simple iterated double cover of $\mathbb{P}^2$ associated to $L_1, ..., L_n$ or there exists a degenerate double cover Y_0 of $\mathbb{P}^2$ of discrete building data $a = \dfrac{l_n + 1}{2}$ such that X_0 is a simple iterated double cover of Y_0 associated to the sequence $M_1, ..., M_{n-1}$, $M_i = \dfrac{l_i}{2}\sigma$.*

Proof. The proof is similar to the proof of VI.3.10 and then we give only a sketch. Let $f: X \longrightarrow \Delta$ be a deformation of X_0 such that for $t \neq 0$ X_t is a simple iterated double cover of $\mathbb{P}^2$ associated to $L_1, ..., L_n$.

We now prove by induction on n that, up to base change, there exists a factorization

$$f: X \xrightarrow{p} Y \xrightarrow{g} \Delta$$

where g is a deformation of a (possibly degenerated) double cover Y_0 of the projective plane with Y_t, $t \neq 0$, smooth double cover associated to L_n and p is a simple iterated double cover of Y associated to $\tilde{M}_1, ..., \tilde{M}_{n-1}$ with $\tilde{M}_i$ the unique extension of M_i to Y (if Y_0 is not degenerate we set $M_i = g_0^* L_i$.

This is trivially true if $n = 1$, if $n > 1$ we consider the action of the trivial involution τ on X and, by induction we get a factorization of f

$$X \xrightarrow{\pi} Z = X/\tau \xrightarrow{\delta} Z_{can} \xrightarrow{p} Y \xrightarrow{g} \Delta$$

with π flat double cover (since $r(Z_t)$ is even) and p, g as before.

Now working exactly as in the proof of VI.3.10 we prove that δ is an isomorphism and $\pi_* \mathcal{O}_X = \mathcal{O}_Z \oplus p^* \mathcal{O}_Z(-\tilde{M}_1)$. □

Part *(ii)* of theorem C follows from

Proposition 4.2. *Let Y_0 be a degenerate double cover of $\mathbb{P}^2$ of degree $a \geq 3$ and let $L_1, ..., L_n$ be line bundle on Y with $L_i = p_i \sigma$, $p_i > 2p_{i+1}, p_n \geq 2a$.*

Then every simple iterated double cover of Y associated to $L_1, ..., L_n$ has unobstructed deformations.

Proof. According to 2.1, 2.4, 3.5 and 3.7 the surface Y_0 and the line bundles $L_1, ..., L_n$ satisfy the hypotheses of corollary VI.2.11. □

5. Numerical examples.

In this section we want to find examples, using simple iterated double covers, of surfaces belonging to different connected components of the same $\mathcal{M}_d$ with self-intersection of the canonical class as small as possible. Unfortunately even in this cases our surfaces will have the topological Euler characteristic of the order of thousands and then any attempt to find global handle decomposition or to apply Kirby calculus seems quite prohibitive.

Since K^2 and the index are algebraic functions on the parameters of the branching divisor it is natural to expect that, in order to find examples, we need at least 3 parameters, i.e. we must consider 4-fold covers of $\mathbb{P}^1 \times \mathbb{P}^1$ and 8-fold covers of $\mathbb{P}^2$.

Example 5.1. Let X, X' be simple iterated double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ associated respectively to the sequences $L_1 = \mathcal{O}(6,9)$, $L_2 = \mathcal{O}(6,4)$ and $L_1' = \mathcal{O}(6,10)$, $L_2' = \mathcal{O}(6,3)$. X, X' have the same invariants $K^2 = 880$, $I = -624$, $c_2 = (K^2 - 3I)/2 = 1376$, $r = 1$ and according to corollary VI.4.7 X is not deformation equivalent to X'.

Example 5.2. If X is a simple iterated double cover of $\mathbb{P}^2$ associated to a sequence $L_i = \mathcal{O}(l_i)$ then according to VI.5.1 the invariants K_X^2, I_X and $r(X)$ depends only by $\sum l_i$ and $\sum l_i^2$.

For $n = 3$ we can consider the pairs of sequences

$$l_1 = 3T - 24,\, l_2 = T,\, l_3 = 5 \qquad l_1' = 3T - 22,\, l_2' = T - 6,\, l_3' = 9$$

Then $\sum l_i = \sum l'_i$, $\sum l_i^2 = \sum {l'_i}^2$ and $L_i = \mathcal{O}(l_i)$, $L'_i = \mathcal{O}(l'_i)$ are good sequences for every even number $T \geq 26$.

For $T = 26$ the associated simple iterated double covers have $K^2 = 53792$, $I = -28928$, $c_2 = 70288$, $r = 82$.

Example 5.3. Let $X \to \mathbb{P}^2$, $Y \to \mathbb{P}^1 \times \mathbb{P}^1$ be simple iterated double covers associated to $L_1 = \mathcal{O}(26), L_2 = \mathcal{O}(12), L_3 = \mathcal{O}(5)$ and $L_1 = \mathcal{O}(20,40), L_2 = \mathcal{O}(22,2)$. A calculation shows that X and Y belong to the same $\mathcal{M}_d$ and it is not difficult to see that X, Y are not deformation equivalent.

In fact the equation of a generic Y is

$$\begin{cases} z^2 = & f + wh \qquad f \in H^0(\mathcal{O}(40,80)),\ h \in H^0(\mathcal{O}(18,78)) \\ w^2 = & g \qquad g \in H^0(\mathcal{O}(44,4)) \end{cases}$$

with f, g, h generic and the same arguments used in section VI.4 show that the unique automorphism of Y is the trivial involution $z \to -z$ and its quotient is the surface $Y_1 = \{w^2 = g\}$. Since the invariants of Y_1 are different from the invariants of elements of $N(\mathbb{P}^2, L_2, L_3)$, Y cannot belong to $N(\mathbb{P}^2, L_1, L_2, L_3)$.

Although is not easy to find explicitly simple iterated double covers of $\mathbb{P}^2$ with the same invariants it is not difficult to see that, using these surfaces, we can prove again a lower bound for the number of connected components of type $\delta \geq (K^2)^{c \log K^2}$ with c positive constant.

In fact for n sufficiently big if q_n is the number of of sequences $l_1, ..., l_n$ such that $\sum l_i = T_n = 8{\cdot}3^n + 3$, $l_n \geq 5$ odd, l_i even for $i < n$ and $l_i > 2l_{i+1}$ then $\log q_n \geq an^2$ for a positive constant a independent on n.

For every one of the above q_n sequences its quadratic sum $\sum l_i^2$ is smaller than T_n^2 and then there exists at least q_n/T_n^2 good sequences giving simple iterated double covers with the same invariants $K^2 = 2^n T_n^2$ and $I = 2^n(1 - \sum l_i^2)$. An easy computation gives the required lower bound of δ.

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