Scuola Normale Superiore, corso di Geometria Algebrica 1996-97.

The Kuranishi family of a stable map.

21 February 1997

In this note we give an alternative, almost selfcontained, proof of theorem 2 of "W. Fulton, R. Pandharipande: *Notes on stable maps and quantum cohomology.* Alg-geom/9608011".

A smooth projective variety Y over $\mathbb{C}$ is called *convex* if for every map $\mu\colon\mathbb{P}^1\to Y$, the vector bundle μ^*T_Y is generically generated by global sections. We want to prove the following

Theorem 1. $\mu\colon(C,\{p_i\})\to Y$ be a stable map (i.e. without infinitesimal automorphisms) from an n-pointed quasistable curve $(C,\{p_i\})$ of genus 0 with $N\geq 0$ nodes into a convex variety Y and denote $\mu_*[C]=\beta\in H_2(Y,\mathbb{Z})$.
Let B be the base space of the universal deformations of the data $(C,\{p_i\},\mu)$ which leave Y fixed and denote by $B_j\subset B$ the subvariety corresponding to curves with at least j nodes. Then:

a) B is smooth of dimension $\dim Y+\int_\beta c_1(T_Y)+n-3$.

b) If $N\geq 1$, B_1 is a normal crossing divisor.

c) If $j\leq N$, B_j has pure codimension j.

1. Deformations of maps with fixed smooth target

Let $\mu\colon X\to Y$ be a morphism of schemes of finite type over an algebraically closed field k with Y smooth, denote by Art the category of local Artinian k-algebras and let $Def_{\mu/Y}\colon Art\to Set$ be the functor of deformations of the map μ leaving Y fixed. Denote also by Def_X the functor of deformations of X and by $\nu\colon Def_{\mu/Y}\to Def_X$ the natural transformation which forgets μ. Let θ_X,θ_Y be respectively the tangent sheaves of X and Y.

Theorem 2. In the notation above, if $H^1(X,\mu^*\theta_Y)=0$ then ν is smooth with relative tangent space $t_\nu=coker(H^0(X,\theta_X)\to H^0(X,\mu^*\theta_Y))$.

Remark For the notion of smooth morphism and relative tangent space, see e.g. "B.Fantechi, M. Manetti: *Obstruction calculus for functors of Artin rings, I* Journal of Algebra **202** (1998) 541-576".

Proof. Let $0\longrightarrow k\stackrel{\epsilon}{\longrightarrow}B\longrightarrow A\longrightarrow 0$ be a small extension of Artinian k-algebras, $X_B\longrightarrow Spec(B)$ a deformation of X and $\mu_A\colon X_A=X_B\times_{Spec(B)}Spec(A)\to Y$ a map, we want to prove that the obstruction to extending μ_A to X_B lies in $H^1(X,\mu^*\theta_Y)$.

Step 1. X affine, $Y=\mathbb{A}^n=Spec(P)$, $P=k[x_1,....,x_n]$.
Let $X=Spec(\mathcal{O})$ and the morphism μ induced by $\mu^*\colon P\to\mathcal{O}$. It is well known that every deformation of an affine scheme is still affine and then we have $X_B=Spec(\mathcal{O}_B)$, $\mu_A^*\colon P\to$

$\mathcal{O}_A = \mathcal{O}_B \otimes_B A$. As P is a free algebra it is possible to lift μ_A^* to a morphism $\mu_B^*\colon P \to \mathcal{O}_B$. Note that two liftings differ by a derivation $\delta \in Der_k(P, \epsilon\mathcal{O})$, where the P-module structure of $\mathcal{O}$ is induced by μ^*.

Step 2. $X = Spec(\mathcal{O})$ affine, $Y \subset \mathbb{A}^n$ closed subscheme.

Let $I \subset P$ be the ideal of Y, $Y = Spec(\mathcal{O}_Y)$, $\mathcal{O}_Y = P/I$. By Step 1 there exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & P & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi_{|I}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mu_A^*} & & \\ 0 & \longrightarrow & \epsilon\mathcal{O} & \longrightarrow & \mathcal{O}_B & \longrightarrow & \mathcal{O}_A & \longrightarrow & 0 \end{array}$$

Since $\psi(I) \subset \epsilon\mathcal{O}$ and $\epsilon^2 = 0$, $\psi(I^2) = 0$ and therefore the restriction of ψ to I gives a $\psi_{|I} \in \mathrm{Hom}_{\mathcal{O}_Y}(I/I^2, \epsilon\mathcal{O})$. By assumption Y is a closed smooth subscheme of $\mathbb{A}^n$, in particular, for every $\mathcal{O}_Y$ module M, the natural morphism

$$Der_k(P, M) \to \mathrm{Hom}_{\mathcal{O}_Y}(I/I^2, M)$$

is surjective. Let $\delta \in Der_k(P, \epsilon\mathcal{O})$ be a lifting of $\psi_{|I}$, then $\psi - \delta$ factors through a morphism $\mu_B^*\colon \mathcal{O}_Y \to \mathcal{O}_B$ which lifts μ_A^*.

As above two liftings of μ_A^* differ by an element of $Der_k(\mathcal{O}_Y, \epsilon\mathcal{O}) = \epsilon H^0(X, \mu^*\theta_Y)$.

Step 3. General case.

The morphism of schemes μ is the data of a morphism $\mu\colon |X| \to |Y|$ between underlying topological spaces and a sheaf homomorphism $\mu^*\colon \mu^{-1}\mathcal{O}_Y \to \mathcal{O}$. It is useful to consider the deformation X_B as a sheaf $\mathcal{O}_B$ of flat B-algebras over $|X|$ and then prove that the obstruction to lift a morphism $\mu_A^*\colon \mu^{-1}\mathcal{O}_Y \to \mathcal{O}_A = \mathcal{O}_B \otimes_B A$ to $\mathcal{O}_B$ lies in $H^1(X, \mu^*\theta_Y)$.

Fix open affine coverings $X = \cup U_i$, $Y = \cup V_i$ such that $\mu(U_i) \subset V_i$; by step 2 for every i there exists liftings $\mu_{B,i}^*$ of the restriction of μ_A^* to the open subset U_i and therefore for every pair i,j there exists a $\varphi_{ij} \in H^0(U_{ij}, \mu^*\theta_Y)$ such that, over U_{ij}, $\epsilon\varphi_{ij} = \mu_{B,i}^* - \mu_{B,j}^*$.

It is clear from the definition, that $\{\varphi_{ij}\}$ is a cocycle and that its class $[\varphi_{ij}] \in H^1(X, \mu^*\theta_Y)$ is independent from the choice of the liftings $\mu_{B,i}^*$ and the affine coverings. If $[\varphi_{ij}] = 0$ there exist $\varphi_i \in H^0(U_i, \mu^*\theta_Y)$ such that $\varphi_{ij} = \varphi_i - \varphi_j$ and then the morphisms $\mu_{B,i}^* - \epsilon\varphi_i$ define a global $\mu_B^*\colon \mu^{-1}\mathcal{O}_Y \to \mathcal{O}_B$.

We note that in Step 3 we have only used the existence of the local liftings $\mu_{B,i}$ and not the smoothness of Y.

Remark. The cocycle $\{\varphi_{ij}\}$ does not change if we compose the $\mu_{B,i}^*$'s with B-algebra automorphisms of $\mathcal{O}_B(U_i)$ inducing the identity on $\mathcal{O}(U_i)$, therefore the class $[\varphi_{ij}]$ depends only by the isomorphism classes of the deformations X_B, $\mu_A\colon X_A \to Y$ and the above construction gives a linear complete obstruction theory for the morphism ν.

We now compute the relative tangent space, i.e. the set of isomorphism classes of liftings $\psi\colon \mu^{-1}\mathcal{O}_Y \to \mathcal{O} \otimes k[\epsilon]$, $\epsilon^2 = 0$. Every such a lifting can be written as $\mu^* + \epsilon\eta$ where $\eta \in H^0(X, \mu^*\theta_Y)$, two liftings $\mu^* + \epsilon\eta_1$, $\mu^* + \epsilon\eta_2$ are in the same isomorphism class if and only if there exists $\delta \in H^0(X,\theta_X)$ such that $\mu^* + \epsilon\eta_1 = (1+\epsilon\delta)(\mu^* + \epsilon\eta_2)$, i.e. $\eta_1 = \delta\mu^* + \eta_2$. The conclusion follows. □

Given $p_1, ..., p_m \in X$ regular points, we define $Def_{\mu/Y,\{p_i\}}$ as the functor of deformations of the m-pointed map $\mu\colon (X, p_1, ..., p_m) \to Y$. This means that for an Artinian k-algebra A, $Def_{\mu/Y,\{p_i\}}(A)$ is the set of isomorphism classes of the following data:

i) a deformation of X, $X_A \to Spec(A)$.

ii) m sections $p_{A,i}\colon Spec(A) \to X_A$ such that $p_{A,i}(Spec(k)) = p_i$.

iii) a map $\mu_A\colon X_A \to Y$.

In a similar way we define the functor $Def_{X,\{p_i\}}$.

Exercise. Convince yourself that if the pointed scheme $(X, \{p_i\})$ has no infinitesimal automorphism and $H^1(X,\mu^*\theta_Y) = 0$ then the natural morphism $Def_{\mu/Y,\{p_i\}} \to Def_{X,\{p_i\}}$ is smooth of relative dimension $h^0(X,\mu^*\theta_Y)$.

Lemma 3. Let $q \in X$ be a smooth point, then the natural morphism

$$f : Def_{\mu/Y,\{p_i\},q} \to Def_{\mu/Y,\{p_i\}}$$

is smooth. If moreover the differential of μ is injective at q, then the relative tangent space of f is isomorphic to the Zariski tangent space of X at q.

Proof. The smoothness of f follows from the fact that every deformation $X_A \to Spec(A)$, being flat, is also formally smooth at q. If $d\mu(q)$ is injective, then every infinitesimal automorphism of μ is the identity in a neighbourhood of q and the relative tangent space is exactly the space of section $q\colon Spec(k[\epsilon]) \to X \times Spec(k[\epsilon])$ which is canonically isomorphic to the Zariski tangent space at q. □

2. Application to stable maps

Lemma 4. Let $C, p_1, ..., p_n$ be a n-pointed quasistable curve of genus 0, $(\chi(\mathcal{O}_C) = 1)$ and let $\mu\colon (C, p_1, ..., p_n) \to Y$ be a map (not necessarily stable) into a convex variety.

Then $h^1(C,\mu^*\theta_Y) = 0$, $h^0(C,\mu^*\theta_Y) = \dim Y + \deg \mu^*\theta_Y = rank(\mu^*\theta_Y)\chi(\mathcal{O}_C) + \deg \mu^*\theta_Y$.

Proof. By Riemann-Roch and Serre duality it is sufficient to prove that if $\phi \in \mathrm{Hom}(\mu^*\theta_Y, \omega_C)$ then $\phi = 0$.

Let $C = A \cup B$ be a decomposition such that $B = \mathbb{P}^1$ and $A{\cdot}B \le 1$, we first prove that $\phi_{|B} = 0$. The restriction of ϕ to B give $\mu^*\theta_Y \otimes \mathcal{O}_B \to \omega_C \otimes \mathcal{O}_B = \mathcal{O}_{\mathbb{P}^1}(A{\cdot}B - 2)$ and by the convexity assumption this morphism is 0. As the restriction of ϕ over B is 0, its restriction over A gives, by adjunction $\mu^*\theta_Y \otimes \mathcal{O}_A \to \omega_C \otimes \mathcal{O}_A(-B) = \omega_A$ and then by induction on the number of irreducible components of A this morphism is trivial. □

Proof of Theorem 1.

a) The Kuranishi family is a hull for the functor $Def_{\mu/Y,\{p_i\}}$, it is therefore sufficient to prove the smoothness and compute the dimension of the tangent space of the above functor.

It is possible to find a finite set of points $q_1, ..., q_m \in C$ such that the pointed curve $C, \{p_i\} \cup \{q_j\}$ is stable and the differential of μ is injective at q_j for every j. We have a commutative diagram

$$\begin{array}{ccc} Def_{\mu/Y,\{p_i\}\cup\{q_j\}} & \xrightarrow{\alpha} & Def_{\mu/Y,\{p_i\}} \\ \downarrow \beta & & \downarrow \gamma \\ Def_{C,\{p_i\}\cup\{q_j\}} & \xrightarrow{\delta} & Def_C \end{array}$$

Since $(C, \{p_i\}, \{q_j\})$ is stable, $Def_{C,\{p_i\},\{q_j\}}$ is smooth of dimension $n+m-3$, by theorem 2 β is smooth of relative dimension $h^1(C, \mu^*\theta_Y)$ and by lemma 3 α is smooth of relative dimension m. Therefore $Def_{\mu/Y,\{p_i\}}$ is smooth of dimension $n+m-3+h^1(\mu^*\theta_Y)-m = \dim Y + deg_C \mu^*\theta_Y + n - 3$.

b) and c). Again by theorem 2, the morphism γ is smooth and the conclusion follows from the analogous properties of the base space of the Kuranishi family of the curve C. □

3. Further remarks on theorem 2

In this section we extend the result of theorem 2 in the more general case of Y reduced "along the image of μ". The precise statement is

Theorem 5. Let $\mu: X \to Y$ be a morphism of schemes of finite type over k such that $\mu^{-1}(Sing(Y))$ is nowhere dense in X.

Let $\nu: Def_{\mu/Y} \to Def_X$ be the morphism defined in §1, then ν admits a linear complete obstruction theory whose space is $\mathrm{Ext}^1_X(\mu^*\Omega_Y, \mathcal{O}_X)$.

Proof. The proof is just a continuation of the proof of theorem 2.

Step 4. $X = Spec(\mathcal{O})$, $Y = Spec(\mathcal{O}_Y)$, $\mathcal{O}_Y = P/I$, $P = k[x_1, ..., x_n]$.