Deformations of singularities via differential graded Lie algebras

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1 Introduction

Let $\mathbb{K}$ be a fixed algebraically closed field of characteristic 0, $X \subset \mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ a closed subscheme. Denote by **Art** the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$.

Definition 1.1. *An* infinitesimal deformation *of X over $A \in$ **Art** is a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X_A \\ \downarrow & & \downarrow{\scriptstyle f_A} \\ \mathrm{Spec}(\mathbb{K}) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

such that f_A is flat and the induced morphism $X \to X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{K})$ is an isomorphism.

It is not difficult to see (cf. [1]) that X_A is affine and more precisely it is isomorphic to a closed subscheme of $\mathbb{A}^n \times \mathrm{Spec}(A)$. Two deformations $X \xrightarrow{i} X_A \xrightarrow{f_A} \mathrm{Spec}(A)$, $X \xrightarrow{j} \tilde{X}_A \xrightarrow{g_A} \mathrm{Spec}(A)$ are isomorphic if there exists a commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X_A \\ {\scriptstyle j}\downarrow & \swarrow{\scriptstyle \theta} & \downarrow{\scriptstyle f_A} \\ \tilde{X}_A & \xrightarrow[g_A]{} & \mathrm{Spec}(A) \end{array}$$

It is easy to prove that necessarily θ is an isomorphism (cf. [8]). Since flatness commutes with base change, for every deformations $X \xrightarrow{i} X_A \xrightarrow{f_A} \mathrm{Spec}(A)$ and every morphism $A \to B$ in the category **Art**, the diagram

$$\begin{array}{ccc} X & \longrightarrow & X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{K}) & \longrightarrow & \mathrm{Spec}(B) \end{array}$$

is a deformation of X over $\operatorname{Spec}(B)$; it is then defined a covariant functor $\operatorname{Def}_X\colon \mathbf{Art}\to\mathbf{Set}$,

$$\operatorname{Def}_X(A)=\{\text{ isomorphism classes of deformations of } X \text{ over } A\,\}.$$

The set $\operatorname{Def}_X(\mathbb{K})$ contains only one point.

In a similar way we can define the functor $\operatorname{Hilb}_X\colon \mathbf{Art}\to\mathbf{Set}$ of embedded deformations of X into $\mathbb{A}^n$: $\operatorname{Hilb}_X(A)$ is the set of closed subschemes $X_A\subset\mathbb{A}^n\times\operatorname{Spec}(A)$ such that the restriction to X_A of the projection on the second factor is a flat map $X_A\to\operatorname{Spec}(A)$ and $X_A\cap(\mathbb{A}^n\times\operatorname{Spec}(\mathbb{K}))=X\times\operatorname{Spec}(\mathbb{K})$.

In these notes we give a recipe for the construction of two differential graded Lie algebras $\mathcal{H}$, $\mathcal{L}$ together two isomorphism of functors

$$\operatorname{Def}_{\mathcal{L}}=\frac{MC_{\mathcal{L}}}{\text{gauge}}\to\operatorname{Def}_X,\qquad \operatorname{Def}_{\mathcal{H}}=\frac{MC_{\mathcal{H}}}{\text{gauge}}\to\operatorname{Hilb}_X\,.$$

The DGLAs $\mathcal{L}$, $\mathcal{H}$ are unique up to quasiisomorphism and their cohomology can be computed in terms of the cotangent complex of X. For the notion of differential graded Lie algebra, Maurer-Cartan functors and gauge equivalence we refer to [7], [8], [3], [6].

Moreover we can choose $\mathcal{H}$ as a differential graded Lie subalgebra of $\mathcal{L}$ such that $\mathcal{H}^i=\mathcal{L}^i$ for every $i>0$.

2 Flatness and relations

In this section $A\in\mathbf{Art}$ is a fixed local artinian $\mathbb{K}$-algebra with residue field $\mathbb{K}$.

Lemma 2.1. *Let M be an A-module, if $M\otimes_A\mathbb{K}=0$ then $M=0$.*

Proof. If M is finitely generated this is Nakayama's lemma. In the general case consider a filtration of ideals $0=I_0\subset I_1\subset\ldots\subset I_n=A$ such that $I_{i+1}/I_i=\mathbb{K}$ for every i. Applying the right exact functor $\otimes_A M$ to the exact sequences of A-modules

$$0{\longrightarrow}\mathbb{K}=\frac{I_{i+1}}{I_i}{\longrightarrow}\frac{A}{I_i}{\longrightarrow}\frac{A}{I_{i+1}}{\longrightarrow}0$$

we get by induction that $M\otimes_A(A/I_i)=0$ for every i. □

The following is a special case of the *local flatness criterion* [9, Thm. 22.3]

Theorem 2.2. *For an A-module M the following conditions are equivalent:*

1. *M is free.*
2. *M is flat.*
3. *$\operatorname{Tor}_1^A(M,\mathbb{K})=0$.*

Proof. The only nontrivial assertion is 3) $\Rightarrow$ 1). Assume $\mathrm{Tor}_1^A(M,\mathbb{K}) = 0$ and let F be a free module such that $F \otimes_A \mathbb{K} = M \otimes_A \mathbb{K}$. Since $M \to M \otimes_A \mathbb{K}$ is surjective there exists a morphism $\alpha\colon F \to M$ such that its reduction $\overline{\alpha}\colon F \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is an isomorphism. Denoting by K the kernel of α and by C its cokernel we have $C \otimes_A \mathbb{K} = 0$ and then $C = 0$; $K \otimes_A \mathbb{K} = \mathrm{Tor}_1^A(M,\mathbb{K}) = 0$ and then $K = 0$. ☐

Corollary 2.3. *Let $h\colon P \to L$ be a morphism of flat A-modules, $A \in$ **Art**. If $\overline{h}\colon P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is injective (resp.: surjective) then also h is injective (resp.: surjective).*

Proof. Same proof of Theorem 2.2. ☐

Corollary 2.4. *Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules with N flat. Then:*

1. *$M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective $\Rightarrow P$ flat.*
2. *P flat $\Rightarrow M$ flat and $M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective.*

Proof. Take the associated long $\mathrm{Tor}_*^A(-,\mathbb{K})$ exact sequence and apply 2.2 and 2.3. ☐

Corollary 2.5. *Let*

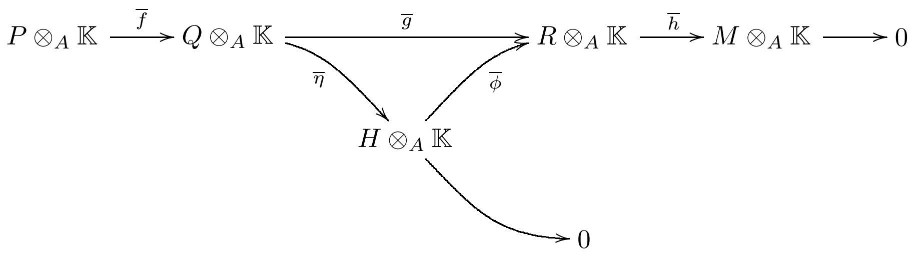
$$P \xrightarrow{f} Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0 \tag{1}$$

be a complex of A-modules such that:

1. *P, Q, R are flat.*
2. *$Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0$ is exact.*
3. *$P \otimes_A \mathbb{K} \xrightarrow{\overline{f}} Q \otimes_A \mathbb{K} \xrightarrow{\overline{g}} R \otimes_A \mathbb{K} \xrightarrow{\overline{h}} M \otimes_A \mathbb{K} \longrightarrow 0$ is exact.*

Then M is flat and the sequence (1) is exact.

Proof. Denote by $H = \ker h = \operatorname{Im} g$ and $g = \phi\eta$, where $\phi\colon H \to R$ is the inclusion and $\eta\colon Q \to H$; by assumption we have an exact diagram



which allows to prove, after an easy diagram chase, that $\overline{\phi}$ is injective. According to Corollary 2.4 H and M are flat modules. Denoting $L = \ker g$ we have, since H is flat, that also L is flat and $L \otimes_A K \to Q \otimes_A \mathbb{K}$ injective. This implies that $P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is surjective. By Corollary 2.3 $P \to L$ is surjective. ☐

Corollary 2.6. *Let $n > 0$ and*

$$0 \longrightarrow I \longrightarrow P_0 \xrightarrow{d_1} P_1 \longrightarrow \ldots \xrightarrow{d_n} P_n,$$

a complex of A-modules with $P_0, \ldots, P_n$ flat. Assume that

$$0 \longrightarrow I \otimes_A \mathbb{K} \longrightarrow P_0 \otimes_A \mathbb{K} \xrightarrow{\overline{d_1}} P_1 \otimes_A \mathbb{K} \longrightarrow \ldots \xrightarrow{\overline{d_n}} P_n \otimes_A \mathbb{K}$$

is exact; then $I, \operatorname{coker}(d_n)$ are flat modules and the natural morphism $I \to \ker(P_0 \otimes_A \mathbb{K} \to P_1 \otimes_A \mathbb{K})$ is surjective.

Proof. Induction on n and Corollary 2.5. □

3 Differential graded algebras, I

Unless otherwise specified by the symbol $\otimes$ we mean the tensor product $\otimes_{\mathbb{K}}$ over the field $\mathbb{K}$. We denote by:

- **G** the category of $\mathbb{Z}$-graded $\mathbb{K}$-vector space; given an object $V = \oplus V_i$, $i \in \mathbb{Z}$, of **G** and a homogeneous element $v \in V_i$ we denote by $\overline{v} = i$ its degree.
- **DG** the category of $\mathbb{Z}$-graded differential $\mathbb{K}$-vector space (also called complexes of vector spaces).

Given (V, d) in **DG** we denote as usual by $Z(V) = \ker d$, $B(V) = d(V)$, $H(V) = Z(V)/B(V)$.

Given an integer n, the shift functor $[n] \colon \mathbf{DG} \to \mathbf{DG}$ is defined by setting $V[n] = \mathbb{K}[n] \otimes V$, $V \in \mathbf{DG}$, $f[n] = Id_{\mathbb{K}[n]} \otimes f$, $f \in \operatorname{Mor}_{\mathbf{DG}}$, where

$$\mathbb{K}[n]_i = \begin{cases} \mathbb{K} & \text{if } i + n = 0 \\ 0 & \text{otherwise} \end{cases}$$

More informally, the complex $V[n]$ is the complex V with degrees shifted by n, i.e. $V[n]_i = V_{i+n}$, and differential multiplied by $(-1)^n$.

Given two graded vector spaces V, W, the "graded Hom" is the graded vector space

$$\operatorname{Hom}^*_{\mathbb{K}}(V, W) = \oplus_n \operatorname{Hom}^n_{\mathbb{K}}(V, W) \in \mathbf{G},$$

where by definition $\operatorname{Hom}^n_{\mathbb{K}}(V, W)$ is the set of $\mathbb{K}$-linear map $f \colon V \to W$ such that $f(V_i) \subset W_{i+n}$ fore every $i \in \mathbb{Z}$. Note that $\operatorname{Hom}^0_{\mathbb{K}}(V, W) = \operatorname{Hom}_{\mathbf{G}}(V, W)$ is the space of morphisms in the category **G** and there exist natural isomorphisms

$$\operatorname{Hom}^n_{\mathbb{K}}(V, W) = \operatorname{Hom}_{\mathbf{G}}(V[-n], W) = \operatorname{Hom}_{\mathbf{G}}(V, W[n]).$$

A morphism in **DG** is called a quasiisomorphism if it induces an isomorphism in homology. A commutative diagram in **DG**

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ C & \longrightarrow & D \end{array}$$

is called cartesian if the morphism $A \to C \times_D B$ is an isomorphism; it is an easy exercise in homological algebra to prove that if f is a surjective (resp.: injective) quasiisomorphism, then g is a surjective (resp.: injective) quasiisomorphism.

Definition 3.1. *A graded (associative, $\mathbb{Z}$-commutative) algebra is a graded vector space $A = \oplus A_i \in \mathbf{G}$ endowed with a product $A_i \times A_j \to A_{i+j}$ satisfying the properties:*

1. $a(bc) = (ab)c$.
2. $a(b+c) = ab + ac$, $(a+b)c = ac + bc$.
3. *(Koszul sign convention)* $ab = (-1)^{\overline{a}\,\overline{b}} ba$ *for* a, b *homogeneous.*

The algebra A is unitary if there exists $1 \in A_0$ such that $1a = a1 = a$ for every $a \in A$.

Let A be a graded algebra, then A_0 is a commutative $\mathbb{K}$-algebra in the usual sense; conversely every commutative $\mathbb{K}$-algebra can be considered as a graded algebra concentrated in degree 0. If $I \subset A$ is a homogeneous left (resp.: right) ideal then I is also a right (resp.: left) ideal and the quotient A/I has a natural structure of graded algebra.

Example 3.2. *Polynomial algebras.* Given a set $\{x_i\}$, $i \in I$, of homogeneous indeterminates of integral degree $\overline{x_i} \in \mathbb{Z}$ we can consider the graded algebra $\mathbb{K}[\{x_i\}]$. As a $\mathbb{K}$-vector space $\mathbb{K}[\{x_i\}]$ is generated by monomials in the indeterminates x_i. Equivalently $\mathbb{K}[\{x_i\}]$ can be defined as the symmetric algebra $\bigoplus_{n\geq 0} \bigodot^n V$, where $V = \oplus_{i\in I} \mathbb{K}\, x_i \in \mathbf{G}$. In some cases, in order to avoid confusion about terminology, for a monomial $x_{i_1}^{\alpha_1} \dots x_{i_n}^{\alpha_n}$ it is defined:

- The *internal degree* $\sum_h \overline{x_{i_h}} \alpha_h$.
- The *external degree* $\sum_h \alpha_h$.

In a similar way it is defined $A[\{x_i\}]$ for every graded algebra A.

Definition 3.3. *A* dg-algebra *(differential graded algebra) is the data of a graded algebra A and a $\mathbb{K}$-linear map $s \colon A \to A$, called* differential*, with the properties:*

1. $s(A_n) \subset A_{n+1}$, $s^2 = 0$.
2. *(graded Leibniz rule)* $s(ab) = s(a)b + (-1)^{\overline{a}} a s(b)$.

A morphism of dg-algebras is a morphism of graded algebras commuting with differentials; the category of dg-algebras is denoted by **DGA**.

In the sequel, for every dg-algebra A we denote by $A_\sharp$ the underlying graded algebra.

Exercise 3.4. Let (A, s) be a unitary dg-algebra; prove:

1. $1 \in Z(A)$.
2. $1 \in B(A)$ if and only if $H(A) = 0$.
3. $Z(A)$ is a graded subalgebra of A and $B(A)$ is a homogeneous ideal of $Z(A)$.
4. If A is local with maximal ideal M then $s(M) \subset M$ if and only if $H(A) \neq 0$.

△

A differential ideal of a dg-algebra (A, s) is a homogeneous ideal I of A such that $s(I) \subset I$; there exists an obvious bijection between differential ideals and kernels of morphisms of dg-algebras.

On a polynomial algebra $\mathbb{K}[\{x_i\}]$ a differential s is uniquely determined by the values $s(x_i)$.

Example 3.5. Let t, dt be inderminates of degrees $\overline{t} = 0$, $\overline{dt} = 1$; on the polynomial algebra $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ there exists an obvious differential d such that $d(t) = dt$, $d(dt) = 0$. Since $\mathbb{K}$ has characteristic 0, we have $H(\mathbb{K}[t, dt]) = \mathbb{K}$. More generally if (A, s) is a dg-algebra then $A[t, dt]$ is a dg-algebra with differential $s(a \otimes p(t)) = s(a) \otimes p(t) + (-1)^{\overline{a}} a \otimes p'(t)dt$, $s(a \otimes q(t)dt) = s(a) \otimes q(t)dt$.

Definition 3.6. *A morphism of dg-algebras $B \to A$ is a* quasiisomorphism *if the induced morphism $H(B) \to H(A)$ is an isomorphism.*

Given a morphism of dg-algebras $B \to A$ the space $\mathrm{Der}^n_B(A, A)$ of B-derivations of degree n is by definition

$$\mathrm{Der}^n_B(A, A) = \{\phi \in \mathrm{Hom}^n_{\mathbb{K}}(A, A) \mid \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b),\ \phi(B) = 0\}.$$

We also consider the graded vector space

$$\mathrm{Der}^*_B(A, A) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}^n_B(A, A) \in \mathbf{G}.$$

There exists a structure of differential graded Lie algebra on $\mathrm{Der}^*_B(A, A)$ with differential

$$d\colon \mathrm{Der}^n_B(A, A) \to \mathrm{Der}^{n+1}_B(A, A), \qquad d\phi = d_A\phi - (-1)^n \phi d_A$$

and bracket

$$[f, g] = fg - (-1)^{\overline{f}\,\overline{g}} gf.$$

Exercise 3.7. Verify that $d[f, g] = [df, g] + (-1)^{\overline{f}}[f, dg]$. △

4 The DG-resolvent

Let $X \subset \mathbb{A}^n$ be a closed subscheme, $R_0 = \mathbb{K}[x_1, \ldots, x_n]$ the ring of regular functions on $\mathbb{A}^n$, $I_0 \subset R_0$ the ideal of X and $\mathcal{O}_X = R_0/I$ the function ring of X.

Our aim is to construct a dg-algebra (R, d) and a quasiisomorphism $R \to \mathcal{O}_X$ such that $R = R_0[y_1, y_2, \ldots]$ is a countably generated graded polynomial R_0-algebra, every indeterminate y_i has negative degree and, if $R = \oplus_{i\leq 0} R_i$, then R_i is a finitely generated free R_0 module.

Choosing a set of generators $f_1, \ldots, f_{s_1}$ of the ideal I_0 we first consider the graded-commutative polynomial dg-algebra

$$R(1) = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_{s_1}] = R_0[y_1, \ldots, y_{s_1}], \qquad \overline{x_i} = 0, \quad \overline{y_i} = -1$$

with differential d defined by $dx_i = 0$, $dy_j = f_j$. Note that $(R(1), d)$, considered as a complex of R_0 modules, is the Koszul complex of the sequence $f_1, \ldots, f_{s_1}$. By construction the complex of R_0-modules

$$\ldots \longrightarrow R(1)_{-2} \xrightarrow{d} R(1)_{-1} \xrightarrow{d} R_0 \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0$$

is exact in R_0 and $\mathcal{O}_X$. If $(R(-1), d) \to \mathcal{O}_X$ is a quasiisomorphism of dg-algebras (e.g. if X is a complete intersection) the construction is done. Otherwise let $f_{s_1+1}, \ldots, f_{s_2} \in \ker d \cap R(1)_{-1}$ be a set of generators of the R_0 module $(\ker d \cap R(1)_{-1})/dR(1)_{-2}$ and define

$$R(2) = R(1)[y_{s_1+1}, \ldots, y_{s_2}], \qquad \overline{y_j} = -2, \quad dy_j = f_j, \quad j = s_1 + 1, \ldots, s_2.$$

Repeating in a recursive way the above argument (step by step killing cycles) we get a chain of polynomial dg-algebras

$$R_0 = R(0) \subset R(1) \subset \ldots \subset R(i) \subset \ldots$$

such that $(R(i), d) \to \mathcal{O}_X$ is a quasiisomorphism in degree $> -i$. Setting

$$R = \cup R(i) = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m, \ldots] = \bigoplus_{i\leq 0} R_i,$$

the projection $\pi \colon R \to \mathcal{O}_X$ is a quasiisomorphism of dg-algebras; in particular

$$\ldots \xrightarrow{d} R_{-i} \xrightarrow{d} \ldots \xrightarrow{d} R_{-2} \xrightarrow{d} R_{-1} \xrightarrow{d} R_0 \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0$$

is a free resolution of the R_0 module $\mathcal{O}_X$.

We denote by:

1. $Z_i = \ker d \cap R_i$.
2. $\mathcal{L} = \mathrm{Der}^*_{\mathbb{K}}(R, R)$.
3. $\mathcal{H} = \mathrm{Der}^*_{R_0}(R, R) = \{g \in \mathcal{L} \mid g(R_0) = 0\}$.

It is clear that, since $gR_i \subset R_{i+j}$ for every $g \in \mathcal{L}^j$, $\mathcal{L}^i = \mathcal{H}^i$ for every $i > 0$ and then the DGLAs $\mathcal{L}$, $\mathcal{H}$ have the same Maurer-Cartan functor $MC_{\mathcal{H}} = MC_{\mathcal{L}}$. Moreover R is a free graded algebra and then $\mathcal{L}^j$ is in bijection with the maps of "degree j" $\{x_i, y_h\} \to R$.

Consider a fixed $\eta \in MC_{\mathcal{H}}(A)$. Recalling the definition of $MC_{\mathcal{H}}$ we have that $\eta = \sum \eta_i \otimes a_i \in \mathrm{Der}^1_{R_0}(R, R) \otimes m_A$ and the A-derivation

$$d + \eta\colon R \otimes A \to R \otimes A, \qquad (d+\eta)(x \otimes a) = dx \otimes a + \sum \eta_i(x) \otimes a_i a$$

is a differential. Denoting by $\mathcal{O}_A$ the cokernel of $d + \eta\colon R_{-1} \otimes A \to R_0 \otimes A$ we have by Corollary 2.5 that $(R \otimes A, d + \eta) \to \mathcal{O}_A$ is a quasiisomorphism, $\mathcal{O}_A$ is flat and $\mathcal{O}_A \otimes \mathbb{K} = \mathcal{O}_X$. Therefore we have natural transformations of functors

$$MC_{\mathcal{H}} = MC_{\mathcal{L}} \to \mathrm{Hilb}_X \to \mathrm{Def}_X\,.$$

Lemma 4.1. *The above morphisms of functors are surjective.*

Proof. Let $\mathcal{O}_A$ be a flat A-algebra such that $\mathcal{O}_A \otimes_A \mathbb{K} = \mathcal{O}_X$; since R_0 is a free $\mathbb{K}$-algebra, the projection $R_0 \xrightarrow{\pi} \mathcal{O}_X$ can be extended to a morphism of flat A-algebras $R_0 \otimes A \xrightarrow{\pi_A} \mathcal{O}_A$. According to Corollary 2.3 π_A is surjective; this proves that $\mathrm{Hilb}_X(A) \to \mathrm{Def}_X(A)$ is surjective (in effect it is possible to prove directly that $\mathrm{Hilb}_X \to \mathrm{Def}_X$ is smooth, cf. [1]). An element of $\mathrm{Hilb}_X(A)$ gives an exact sequence of flat A-modules

$$R_0 \otimes A \xrightarrow{\pi_A} \mathcal{O}_A \longrightarrow 0.$$

Denoting by $I_{0,A} \subset R_0 \otimes A$ the kernel of π_A we have that $I_{0,A}$ is A-flat and the projection $I_{0,A} \to I_0$ is surjective. We can therefore extend the restriction to $R(1)$ of the differential d to a differential d_A on $R(1) \otimes A$ by setting $d_A(y_j) \in I_{0,A}$ a lifting of $d(y_j)$, $j = 1, \dots, s_1$. Again by local flatness criterion the kernel $Z_{-1,A}$ of $R_{-1} \otimes A = R(1)_{-1} \otimes A \xrightarrow{d_A} R_0 \otimes A$ is flat and surjects onto Z_{-1}. The same argument as above, with $I_{0,A}$ replaced by $Z_{-1,A}$ shows that d can be extended to a differential d_A on $R(2)$ and then by induction to a differential d_A on $R \otimes A$ such that $(R \otimes A, d_A) \to \mathcal{O}_A$ is a quasiisomorphism. If $a_1, \dots, a_r$ is a $\mathbb{K}$-basis of the maximal ideal of A we can write $d_A(x \otimes 1) = dx \otimes 1 + \sum \eta_i(x) \otimes a_i$ and then $\eta = \sum \eta_i \otimes a_i \in MC_{\mathcal{H}}(A)$. $\square$

If $\xi \in \mathrm{Der}^0_{R_0}(R, R) \otimes m_A$, $A \in \mathbf{Art}$, then $e^{\xi}\colon R \otimes A \to R \otimes A$ is an automorphism inducing the identity on R and $R_0 \otimes A$. Therefore the morphism $MC_{\mathcal{H}}(A) \to \mathrm{Hilb}_X(A)$ factors through $\mathrm{Def}_{\mathcal{H}}(A) \to \mathrm{Hilb}_X(A)$. Similarly the morphism $MC_{\mathcal{L}}(A) \to \mathrm{Def}_X(A)$ factors through $\mathrm{Def}_{\mathcal{L}}(A) \to \mathrm{Def}_X(A)$.

Theorem 4.2. *The natural transformations*

$$\mathrm{Def}_{\mathcal{H}} \to \mathrm{Hilb}_X, \qquad \mathrm{Def}_{\mathcal{L}} \to \mathrm{Def}_X$$

are isomorphisms of functors.

Proof. We have already proved the surjectivity. The injectivity follows from the following lifting argument. Given $d_A, d'_A\colon R\otimes A\to R\otimes A$ two liftings of the differential d and $f_0\colon R_0\otimes A\to R_0\otimes A$ a lifting of the identity on R_0 such that $f_0 d_A(R_{-1}\otimes A)\subset d'_A(R_{-1}\otimes A)$ there exists an isomorphism $f\colon (R\otimes A, d_A)\to (R\otimes A, d'_A)$ extending f_0 and the identity on R. This is essentially trivial because $R\otimes A$ is a free $R_0\otimes A$ graded algebra and $(R\otimes A, d'_A)$ is exact in degree <0. Thinking f as an automorphism of the graded algebra $R\otimes A$ we have, since $\mathbb{K}$ has characteristic 0, that $f=e^{\xi}$ for some $\xi\in\mathcal{L}^0$ and $\xi\in\mathcal{H}^0$ if and only if $f_0=Id$. By the definition of gauge action $d'_A-d=exp(\xi)(d_A-d)$; the injectivity follows. ∎

Proposition 4.3. *If $I\subset R_0$ is the ideal of $X\subset\mathbb{A}^n$ then:*

1. $H^i(\mathcal{H})=H^i(\mathcal{L})=0$ *for every* $i<0$.
2. $H^0(\mathcal{H})=0$, $H^0(\mathcal{L})=\mathrm{Der}_{\mathbb{K}}(\mathcal{O}_X,\mathcal{O}_X)$.
3. $H^1(\mathcal{H})=\mathrm{Hom}_{\mathcal{O}_X}(I/I^2,\mathcal{O}_X)$ *and* $H^1(\mathcal{L})$ *is the cokernel of the natural morphism*

$$\mathrm{Der}_{\mathbb{K}}(R_0,\mathcal{O}_X)\xrightarrow{\alpha}\mathrm{Hom}_{\mathcal{O}_X}(I/I^2,\mathcal{O}_X).$$

Proof. There exists a short exact sequence of complexes

$$0\longrightarrow\mathcal{H}\longrightarrow\mathcal{L}\longrightarrow\mathrm{Der}^*_{\mathbb{K}}(R_0,R)\longrightarrow 0.$$

Since R_0 is free and R is exact in degree <0 we have:

$$H^i(\mathrm{Der}^*_{\mathbb{K}}(R_0,R))=\begin{cases}0 & i\neq 0,\\ \mathrm{Der}_{\mathbb{K}}(R_0,\mathcal{O}_X) & i=0.\end{cases}$$

Moreover $\mathrm{Der}_{\mathbb{K}}(\mathcal{O}_X,\mathcal{O}_X)$ is the kernel of α and then it is sufficient to compute $H^i(\mathcal{H})$ for $i\le 1$.

Every $g\in Z^i(\mathcal{H})$, $i\le 0$, is a R_0-derivation $g\colon R\to R$ such that $g(R)\subset\oplus_{i<0}R_i$ and $gd=\pm dg$. As above R is free and exact in degree <0, a standard argument shows that g is a coboundary. If $g\in Z^1(\mathcal{H})$ then $g(R_{-1})\subset R_0$ and, since $gd+dg=0$, g induces a morphism

$$\overline{g}\colon\frac{R_{-1}}{dR_{-2}}=I\longrightarrow\frac{R_0}{dR_{-1}}=\mathcal{O}_X.$$

The easy verification that $Z^1(\mathcal{H})\to\mathrm{Hom}_{R_0}(I,\mathcal{O}_X)$ induces an isomorphism $H^1(\mathcal{H})\to\mathrm{Hom}_{R_0}(I,\mathcal{O}_X)$ is left to the reader. ∎

5 Differential graded algebras, II

Lemma 5.1. *Let A be graded algebra: if every $a \neq 0$ is invertible then $A = A_0$ is a field.*

Proof. Assume that there exists $a \in A_i$, $a \neq 0$, $i > 0$. Then $1 - a \neq 0$ and by assumption we have

$$1 = (1-a)\sum_{j=-n}^{n} a_j, \qquad a_j \in A_j.$$

This is equivalent to the system of equations

$$\begin{cases} a_{-n} = 0 \\ a_{i-j} - aa_{-j} = \delta_{ij}, \qquad j < n \end{cases}$$

The solution is $a_j = 0$ for $j < 0$, $a_j = a^j$ for $j > 0$; in particular $a^{n+1} = 0$ and then a is not invertible. □

Lemma 5.2. *Let A be a graded algebra and let $I \subset A$ be a left ideal. Then the following conditions are equivalent:*

1. *I is the unique left maximal ideal.*
2. *A_0 is a local ring with maximal ideal M and $I = M \oplus_{i\neq 0} A_i$.*

Proof. 1⇒2: For every $t \in \mathbb{K}$, $t \neq 0$, the morphism $\phi\colon A \to A$, $x \to xt^{\overline{x}}$, is an isomorphism of graded algebras, in particular $\phi(I) = I$ and the Vandermonde's argument shows that I is homogeneous and then bilateral. By Lemma 5.1 the quotient A/I is a field and $I = M \oplus_{i\neq 0} A_i$ with $M \subset A_0$ maximal. Let $a \in A_0 - M$, then $a \notin I$ and a is invertible in A; since $a^{-1} \in A_0$ a is also invertible in A_0 and then A_0 is a local ring. 2⇒1: Let $J \subset A$ be a proper left ideal, then $J \cap A_0 \subset M$ and therefore $J \subset M \oplus_{i\neq 0} A_i = I$. □

Let A be a graded algebra, if $A \to B$ is a morphism of graded algebras then B has a natural structure of A-algebra. Given two A-algebras B, C it is defined their tensor product $B \otimes_A C$ as the quotient of $B \otimes_{\mathbb{K}} C = \oplus_{n,m} B_n \otimes_{\mathbb{K}} C_m$ by the ideal generated by $ba \otimes c - b \otimes ac$ for every $a \in A$, $b \in B$, $c \in C$. $B \otimes_A C$ has a natural structure of graded algebra with degrees $\overline{b \otimes c} = \overline{b} + \overline{c}$ and multiplication $(b \otimes c)(\beta \otimes \gamma) = (-1)^{\overline{c}\,\overline{\beta}} b\beta \otimes c\gamma$. Note in particular that $A[\{x_i\}] = A \otimes_{\mathbb{K}} \mathbb{K}[\{x_i\}]$.

Given a dg-algebra A and $h \in \mathbb{K}$ it is defined an evaluation morphism $e_h\colon A[t, dt] \to A$, $e_h(a \otimes p(t)) = ap(h)$, $e_h(a \otimes q(t)dt) = 0$.

Lemma 5.3. *For every dg-algebra A the evaluation map $e_h\colon A[t, dt] \to A$ induces an isomorphism $H(A[t, dt]) \to H(A)$ independent from $h \in \mathbb{K}$.*

Proof. Let $\imath\colon A \to A[t, dt]$ be the inclusion, since $e_h\imath = Id_A$ it is sufficient to prove that $\imath\colon H(A) \to H(A[t, dt])$ is bijective. For every $n > 0$ denote $B_n = At^n \oplus At^{n-1}dt$; since $d(B_n) \subset B_n$ and $A[t, dt] = \imath(A) \bigoplus_{n>0} B_n$ it is sufficient to prove that $H(B_n) = 0$ for every n. Let $z \in Z_i(B_n)$, $z = at^n + nbt^{n-1}dt$, then $0 = dz = dat^n + ((-1)^i a + db)nt^{n-1}dt$ which implies $a = (-1)^{i-1}db$ and then $z = (-1)^{i-1}d(bt^n)$. □

Definition 5.4. *Given two morphisms of dg-algebras $f, g\colon A \to B$, a* homotopy *between f and g is a morphism $H\colon A \to B[t, dt]$ such that $H_0 := e_0 \circ H = f$, $H_1 := e_1 \circ H = g$. We denote by $[A, B]$ the quotient of* $\mathrm{Hom}_{\mathbf{DGA}}(A, B)$ *by the equivalence relation $\sim$ generated by homotopy. If $B \to C$ is a morphism of dg-algebras with kernel J, a homotopy $H\colon A \to B[t, dt]$ is called* constant on C *if the image of H is contained in $B \oplus_{j\geq 0} (Jt^{j+1} \oplus Jt^j dt)$. Two dg-algebras A, B are said to be* homotopically equivalent *if there exist morphisms $f\colon A \to B$, $g\colon B \to A$ such that $fg \sim Id_B$, $gf \sim Id_A$.*

According to Lemma 5.3 homotopic morphisms induce the same morphism in homology.

Lemma 5.5. *Given morphisms of dg-algebras,*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \underset{l}{\overset{h}{\rightrightarrows}} C\ ,$$

if $f \sim g$ and $h \sim l$ then $hf \sim lg$.

Proof. It is obvious from the definitions that $hg \sim lg$. For every $a \in \mathbb{K}$ there exists a commutative diagram

$$\begin{array}{ccc} B \otimes \mathbb{K}[t, dt] & \xrightarrow{h\otimes Id} & C \otimes \mathbb{K}[t, dt] \\ \downarrow{\scriptstyle e_a} & & \downarrow{\scriptstyle e_a} \\ B & \xrightarrow{\quad h \quad} & C \end{array}\ .$$

If $F\colon A \to B[t, dt]$ is a homotopy between f and g, then, considering the composition of F with $h \otimes Id$, we get a homotopy between hf and hg. □

Example 5.6. Let A be a dg-algebra, $\{x_i\}$ a set of indeterminates of integral degree and consider the dg-algebra $B = A[\{x_i, dx_i\}]$, where dx_i is an indeterminate of degree $\overline{dx_i} = \overline{x_i} + 1$ and the differential d_B is the unique extension of d_A such that $d_B(x_i) = dx_i$, $d_B(dx_i) = 0$ for every i. The inclusion $i\colon A \to B$ and the projection $\pi\colon B \to A$, $\pi(x_i) = \pi(dx_i) = 0$ give a homotopy equivalence between A and B. In fact $\pi i = Id_A$; consider now the homotopy $H\colon B \to B[t, dt]$ given by

$$H(x_i) = x_i t, \quad H(dx_i) = dH(x_i) = dx_i t + (-1)^{\overline{x_i}} x_i dt, \quad H(a) = a,\ \forall a \in A.$$

Taking the evaluation at $t = 0, 1$ we get $H_0 = ip$, $H_1 = Id_B$.

Exercise 5.7. Let $f, g\colon A \to C$, $h\colon B \to C$ be morphisms of dg-algebras. If $f \sim g$ then $f \otimes h \sim g \otimes h\colon A \otimes_{\mathbb{K}} B \to C$. △

Remark 5.8. *In view of future geometric applications, it seems reasonable to define the spectrum of a unitary dg-algebra A as the usual spectrum of the commutative ring $Z_0(A)$.*

If $S \subset Z_0(A)$ is a multiplicative part we can consider the localized dg-algebra $S^{-1}A$ with differential $d(a/s) = da/s$. Since the localization is an exact functor in the category of $Z_0(A)$ modules we have $H(S^{-1}A) = S^{-1}H(A)$. If $\phi\colon A \to C$ is a morphism of dg-algebras and $\phi(s)$ is invertible for every $s \in S$ then there is a unique morphism $\psi\colon S^{-1}A \to C$ extending ϕ. Moreover if ϕ is a quasiisomorphism then also ψ is a quasiisomorphism (easy exercise).

If $\mathcal{P} \subset Z_0(A)$ is a prime ideal, then we denote as usual $A_{\mathcal{P}} = S^{-1}A$, where $S = Z_0(A) - \mathcal{P}$. It is therefore natural to define Spec(A) *as the ringed space $(X, \tilde{A})$, where X is the spectrum of A and $\tilde{A}$ is the (quasi coherent) sheaf of dg-algebras with stalks $A_{\mathcal{P}}$, $\mathcal{P} \in X$.*

6 Differential graded modules

Let (A, s) be a fixed dg-algebra, by an A-dg-module we mean a differential graded vector space (M, s) together two associative distributive multiplication maps $A \times M \to M$, $M \times A \to M$ with the properties:

1. $A_iM_j \subset M_{i+j}, \quad M_iA_j \subset M_{i+j}$.
2. $am = (-1)^{\overline{a}\,\overline{m}}ma$, for homogeneous $a \in A$, $m \in M$.
3. $s(am) = s(a)m + (-1)^{\overline{a}}as(m)$.

If $A = A_0$ we recover the usual notion of complex of A-modules.

If M is an A-dg-module then $M[n] = \mathbb{K}[n] \otimes_{\mathbb{K}} M$ has a natural structure of A-dg-module with multiplication maps

$$(e \otimes m)a = e \otimes ma, \qquad a(e \otimes m) = (-1)^{n\overline{a}} e \otimes am, \qquad e \in \mathbb{K}[n],\ m \in M,\ a \in A.$$

The tensor product $N \otimes_A M$ is defined as the quotient of $N \otimes_{\mathbb{K}} M$ by the graded submodules generated by all the elements $na \otimes m - n \otimes am$.

Given two A-dg-modules $(M, d_M), (N, d_N)$ we denote by

$$\operatorname{Hom}^n_A(M, N) = \{ f \in \operatorname{Hom}^n_{\mathbb{K}}(M, N) \mid f(ma) = f(m)a,\ m \in M, a \in A\}$$

$$\operatorname{Hom}^*_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}^n_A(M, N).$$

The graded vector space $\operatorname{Hom}^*_A(M, N)$ has a natural structure of A-dg-module with left multiplication $(af)(m) = af(m)$ and differential

$$d\colon \operatorname{Hom}^n_A(M, N) \to \operatorname{Hom}^{n+1}_A(M, N), \qquad df = [d, f] = d_N \circ f - (-1)^n f \circ d_M.$$

Note that $f \in \mathrm{Hom}^0_A(M,N)$ is a morphism of A-dg-modules if and only if $df = 0$. A *homotopy* between two morphism of dg-modules $f, g\colon M \to N$ is a $h \in \mathrm{Hom}^{-1}_A(M,N)$ such that $f - g = dh = d_N h + h d_M$. Homotopically equivalent morphisms induce the same morphism in homology.

Morphisms of A-dg-modules $f\colon L \to M$, $h\colon N \to P$ induce, by composition, morphisms $f^*\colon \mathrm{Hom}^*_A(M,N) \to \mathrm{Hom}^*_A(L,N)$, $h_*\colon \mathrm{Hom}^*_A(M,N) \to \mathrm{Hom}^*_A(M,P)$;

Lemma 6.1. *In the above notation if f is homotopic to g and h is homotopic to l then f^* is homotopic to g^* and l_* is homotopic to h_*.*

Proof. Let $p \in \mathrm{Hom}^{-1}_A(L,M)$ be a homotopy between f and g, It is a straightforward verification to see that the composition with p is a homotopy between f^* and g^*. Similarly we prove that h_* is homotopic to l_*. ☐

Lemma 6.2. *Let $A \to B$ be a morphism of unitary dg-algebras, M an A-dg-module, N a B-dg-modules. Then there exists a natural isomorphism of B-dg-modules*

$$\mathrm{Hom}^*_A(M,N) \simeq \mathrm{Hom}^*_B(M \otimes_A B, N).$$

Proof. Consider the natural maps:

$$\mathrm{Hom}^*_A(M,N) \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{Hom}^*_B(M \otimes_A B, N)\ ,$$

$$Lf(m \otimes b) = f(m)b, \qquad Rg(m) = g(m \otimes 1).$$

We left as exercise the easy verification that $L, R = L^{-1}$ are isomorphism of B-dg-modules. ☐

Given a morphism of dg-algebras $B \to A$ and an A-dg-module M we set:

$$\mathrm{Der}^n_B(A,M) = \{\phi \in \mathrm{Hom}^n_{\mathbb{K}}(A,M) | \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b),\ \phi(B) = 0\}$$

$$\mathrm{Der}^*_B(A,M) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}^n_B(A,M).$$

As in the case of Hom^*, there exists a structure of A-dg-module on $\mathrm{Der}^*_B(A,M)$ with product $(a\phi)(b) = a\phi(b)$ and differential

$$d\colon \mathrm{Der}^n_B(A,M) \to \mathrm{Der}^{n+1}_B(A,M), \qquad d\phi = [d,\phi] = d_M\phi - (-1)^n \phi d_A.$$

Given $\phi \in \mathrm{Der}^n_B(A,M)$ and $f \in \mathrm{Hom}^m_A(M,N)$ their composition $f\phi$ belongs to $\mathrm{Der}^{n+m}_B(A,N)$.

Proposition 6.3. *Let $B \to A$ be a morphisms of dg-algebras: there exists an A-dg-module $\Omega_{A/B}$ together a closed derivation $\delta\colon A \to \Omega_{A/B}$ of degree 0 such that, for every A-dg-module M, the composition with δ gives an isomorphism*

$$\mathrm{Hom}^*_A(\Omega_{A/B}, M) = \mathrm{Der}^*_B(A,M).$$

Proof. Consider the graded vector space

$$F_A = \bigoplus A\delta x, \quad x \in A \text{ homogeneous}, \qquad \overline{\delta x} = \overline{x}.$$

F_A is an A-dg-module with multiplication $a(b\delta x) = ab\delta x$ and differential

$$d(a\delta x) = da\delta x + (-1)^{\overline{a}} a\delta(dx).$$

Note in particular that $d(\delta x) = \delta(dx)$. Let $I \subset F_A$ be the homogeneous submodule generated by the elements

$$\delta(x+y) - \delta x - \delta y, \quad \delta(xy) - x(\delta y) - (-1)^{\overline{x}\,\overline{y}} y(\delta x), \quad \delta(b), b \in B,$$

Since $d(I) \subset I$ the quotient $\Omega_{A/B} = F_A/I$ is still an A-dg-module. By construction the map $\delta\colon A \to \Omega_{A/B}$ is a derivation of degree 0 such that $d\delta = d_\Omega\delta - \delta d_A = 0$. Let $\circ\delta\colon \operatorname{Hom}^*_A(\Omega_{A/B}, M) \to \operatorname{Der}^*_B(A, M)$ be the composition with δ:

a) L is a morphism of A-dg-modules. In fact $(af)\circ\delta = a(f\circ\delta)$ for every $a \in A$ and

$$\begin{aligned} d(f\circ\delta)(x) &= d_M(f(\delta x)) - (-1)^{\overline{f}} f\delta(dx) = \\ &= d_M(f(\delta x)) - (-1)^{\overline{f}} f(d(\delta x)) = df\circ\delta. \end{aligned}$$

b) $\circ\delta$ is surjective. Let $\phi \in \operatorname{Der}^n_B(A, M)$; define a morphism $f \in \operatorname{Hom}^n_A(F_A, M)$ by the rule $f(a\delta x) = (-1)^{n\overline{a}} a\phi(x)$; an easy computation shows that $f(I) = 0$ and then f factors to $f \in \operatorname{Hom}^n_A(\Omega_{A/B}, M)$: by construction $f\circ\delta = \phi$.

c) $\circ\delta$ is injective. In fact the image of δ generate $\Omega_{A/B}$.

□

When $B = \mathbb{K}$ we denote for notational simplicity $\operatorname{Der}^*(A, M) = \operatorname{Der}^*_{\mathbb{K}}(A, M)$, $\Omega_A = \Omega_{A/\mathbb{K}}$. Note that if $C \to B$ is a morphism of dg-algebras, then the natural map $\Omega_{A/C} \to \Omega_{A/B}$ is surjective and $\Omega_{A/C} = \Omega_{A/B}$ whenever $C \to B$ is surjective.

Definition 6.4. *The module $\Omega_{A/B}$ is called the module of relative* Kähler differentials *of A over B and δ the* universal derivation.

By the universal property, the module of differential and the universal derivation are unique up to isomorphism.

Example 6.5. If $A_\sharp = \mathbb{K}[\{x_i\}]$ is a polynomial algebra then $\Omega_A = \oplus_i A\delta x_i$ and $\delta\colon A \to \Omega_A$ is the unique derivation such that $\delta(x_i) = \delta x_i$.

Proposition 6.6. *Let $B \to A$ be a morphism of dg-algebras and $S \subset Z_0(A)$ a multiplicative part. Then there exists a natural isomorphism $S^{-1}\Omega_{A/B} = \Omega_{S^{-1}A/B}$.*

Proof. The closed derivation $\delta\colon A \to \Omega_{A/B}$ extends naturally to $\delta\colon S^{-1}A \to S^{-1}\Omega_{A/B}$, $\delta(a/s) = \delta a/s$, and by the universal property there exists a unique morphism of $S^{-1}A$ modules $f\colon \Omega_{S^{-1}A/B} \to S^{-1}\Omega_{A/B}$ and a unique morphism of A modules $g\colon \Omega_{A/B} \to \Omega_{S^{-1}A/B}$. The morphism g extends to a morphism of $S^{-1}A$ modules $g\colon S^{-1}\Omega_{A/B} \to \Omega_{S^{-1}A/B}$. Clearly these morphisms commute with the universal closed derivations and then $gf = Id$. On the other hand, by the universal property, the restriction of fg to $\Omega_{A/B}$ must be the natural inclusion $\Omega_{A/B} \to S^{-1}\Omega_{A/B}$ and then also $fg = Id$. □

7 Projective modules

Definition 7.1. *An A-dg-module P is called* projective *if for every surjective quasiisomorphism $f\colon M \to N$ and every $g\colon P \to N$ there exists $h\colon P \to M$ such that $fh = g$.*

$$\begin{array}{ccccccc} & & M & \Rightarrow & & & M \\ & & \Big\downarrow{\scriptstyle f,\ qis} & & & \nearrow^{h} & \Big\downarrow{\scriptstyle f,\ qis} \\ P & \xrightarrow[g]{} & N & \Rightarrow & P & \xrightarrow[g]{} & N \end{array}.$$

Exercise 7.2. Prove that if $A = A_0$ and $P = P_0$ then P is projective in the sense of 7.1 if and only if P_0 is projective in the usual sense. △

Lemma 7.3. *Let P be a projective A-dg-module, $f\colon P \to M$ a morphism of A-dg-modules and $\phi\colon M \to N$ a surjective quasiisomorphism. If ϕf is homotopic to 0 then also f is homotopic to 0.*

Proof. We first note that there exist natural isomorphisms $\mathrm{Hom}^i_A(P, M[j]) = \mathrm{Hom}^{i+j}_A(P, M)$. Let $h\colon P \to N[-1]$ be a homotopy between ϕf and 0 and consider the A-dg-modules $M \oplus N[-1]$, $M \oplus M[-1]$ endowed with the differentials

$$d\colon M_n \oplus N_{n-1} \to M_{n+1} \oplus N_n, \quad d(m_1, n_2) = (dm_1, f(m_1) - dn_2),$$

$$d\colon M_n \oplus M_{n-1} \to M_{n+1} \oplus M_n, \quad d(m_1, m_2) = (dm_1, m_1 - dm_2).$$

The map $Id_M \oplus f\colon M \oplus M[-1] \to M \oplus N[-1]$ is a surjective quasiisomorphism and $(\phi, h)\colon P \to M \oplus N[-1]$ is morphism of A-dg-modules. If $(\phi, l)\colon P \to M \oplus M[-1]$ is a lifting of (ϕ, h) then l is a homotopy between ϕ and 0. □

Lemma 7.4. *Let $f\colon M \to N$ be a morphism of A-dg-modules, then there exist morphisms of A-dg-modules $\pi\colon L \to M$, $g\colon L \to N$ such that g is surjective, π is a homotopy equivalence and g is homotopically equivalent to $f\pi$.*

Proof. Consider $L = M \oplus N \oplus N[-1]$ with differential

$$d\colon M_n \oplus N_n \oplus N_{n-1} \to M_{n+1} \oplus N_{n+1} \oplus N_n, \quad d(m, n_1, n_2) = (dm, dn_1, n_1 - dn_2).$$

We define $g(m, n_1, n_2) = f(m) + n_1$, $\pi(m, n_1, n_2) = m$ and $s\colon M \to L$, $s(m) = (m, 0, 0)$. Since $gs = f$ and $\pi s = Id_M$ it is sufficient to prove that $s\pi$ is homotopic to Id_L. Take $h \in \mathrm{Hom}_A^{-1}(L, L)$, $h(m, n_1, n_2) = (0, n_2, 0)$; then

$$d(h(m, n_1, n_2)) + hd(m, n_1, n_2) = (0, n_1, n_2) = (Id_L - s\pi)(m, n_1, n_2).$$

□

Theorem 7.5. *Let P be a projective A-dg-module: For every quasiisomorphism $f\colon M \to N$ the induced map $\mathrm{Hom}_A^*(P, M) \to \mathrm{Hom}_A^*(P, N)$ is a quasiisomorphism.*

Proof. By Lemma 7.4 it is not restrictive to assume f surjective. For a fixed integer i we want to prove that $H^i(\mathrm{Hom}_A^*(P, M)) = H^i(\mathrm{Hom}_A^*(P, N))$. Replacing M and N with $M[i]$ and $N[i]$ it is not restrictive to assume $i = 0$. Since $Z^0(\mathrm{Hom}_A^*(P, N))$ is the set of morphisms of A-dg-modules and P is projective, the map

$$Z^0(\mathrm{Hom}_A^*(P, M)) \to Z^0(\mathrm{Hom}_A^*(P, N))$$

is surjective. If $\phi \in Z^0(\mathrm{Hom}_A^*(P, M))$ and $f\phi \in B^0(\mathrm{Hom}_A^*(P, N))$ then by Lemma 7.3 also ϕ is a coboundary. □

A projective resolution of an A-dg-module M is a surjective quasiisomorphism $P \to M$ with P projective. We will show in next section that projective resolutions always exist. This allows to define for every pair of of A-dg-modules M, N

$$\mathrm{Ext}^i(M, N) = H^i(\mathrm{Hom}_A^*(P, N)),$$

where $P \to M$ is a projective resolution.

Exercise 7.6. Prove that the definition of Ext's is independent from the choice of the projective resolution. △

8 Semifree resolutions

From now on K is a fixed dg-algebra.

Definition 8.1. *A K-dg-algebra (R, s) is called* semifree *if:*

1. *The underlying graded algebra R is a polynomial algebra over K $K[\{x_i\}]$, $i \in I$.*

2. *There exists a filtration* $\emptyset = I(0) \subset I(1) \subset \dots\ , \cup_{n\in\mathbb{N}} I(n) = I$, *such that* $s(x_i) \in R(n)$ *for every* $i \in I(n+1)$, *where by definition* $R(n) = K[\{x_i\}]$, $i \in I(n)$.

Note that $R(0) = K$, $R(n)$ *is a dg-subalgebra of* R *and* $R = \cup R(n)$.

Let $R = K[\{x_i\}] = \cup R(n)$ be a semifree K-dg-algebra, S a K-dg-algebra; to give a morphism $f\colon R \to S$ is the same to give a sequence of morphisms $f_n\colon R(n) \to S$ such that f_{n+1} extends f_n for every n. Given a morphism $f_n\colon R(n) \to S$, the set of extensions $f_{n+1}\colon R(n+1) \to S$ is in bijection with the set of sequences $\{f_{n+1}(x_i)\}$, $i \in I(n+1) - I(n)$, such that $s(f_{n+1}(x_i)) = f_n(s(x_i))$, $\overline{f_{n+1}(x_i)} = \overline{x_i}$.

Example 8.2. $\mathbb{K}[t, dt]$ is semifree with filtration $\mathbb{K} \oplus \mathbb{K}\,dt \subset \mathbb{K}[t, dt]$. For every dg-algebra A and every $a \in A_0$ there exists a unique morphism $f\colon \mathbb{K}[t, dt] \to A$ such that $f(t) = a$.

Exercise 8.3. Let (V, s) be a complex of vector spaces, the differential s extends to a unique differential s on the symmetric algebra $\bigodot V$ such that $s(\bigodot^n V) \subset \bigodot^n V$ for every n. Prove that $(\bigodot V, s)$ is semifree. △

Exercise 8.4. The tensor product (over K) of two semifree K-dg-algebras is semifree. △

Proposition 8.5. *Let* $(R = K[\{x_i\}], s)$, $i \in \cup I(n)$, *be a semifree* K*-dg-algebra: for every surjective quasiisomorphism of* K*-dg-algebras* $f\colon A \to B$ *and every morphism* $g\colon R \to B$ *there exists a lifting* $h\colon R \to A$ *such that* $fh = g$. *Moreover any two of such liftings are homotopic by a homotopy constant on* B.

Proof. Assume by induction on n that it is defined a morphism $h_n\colon R(n) \to A$ such that fh_n equals the restriction of g to $R(n) = \mathbb{K}[\{x_i\}]$, $i \in I(n)$. Let $i \in I(n+1) - I(n)$, we need to define $h_{n+1}(x_i)$ with the properties $fh_{n+1}(x_i) = g(x_i)$, $dh_{n+1}(x_i) = h_n(dx_i)$ and $\overline{h_{n+1}(x_i)} = \overline{x_i}$. Since $dh_n(dx_i) = 0$ and $fh_n(dx_i) = g(dx_i) = dg(x_i)$ we have that $h_n(dx_i)$ is exact in A, say $h_n(dx_i) = da_i$; moreover $d(f(a_i) - g(x_i)) = f(da_i) - g(dx_i) = 0$ and, since $Z(A) \to Z(B)$ is surjective there exists $b_i \in A$ such that $f(a_i + b_i) = g(x_i)$ and then we may define $h_{n+1}(x_i) = a_i + b_i$. The inverse limit of h_n gives the required lifting.
Let $h, l\colon R \to A$ be liftings of g and denote by $J \subset A$ the kernel of f; by assumption J is acyclic and consider the dg-subalgebra $C \subset A[t, dt]$,

$$C = A \oplus_{j\geq 0} (Jt^{j+1} \oplus Jt^j dt).$$

We construct by induction on n a coherent sequence of morphisms $H_n\colon R(n) \to C$ giving a homotopy between h and l. Denote by $N \subset \mathbb{K}[t, dt]$ the differential ideal generated by $t(t-1)$; there exists a direct sum decomposition $\mathbb{K}[t, dt] = \mathbb{K} \oplus \mathbb{K}\,t \oplus \mathbb{K}\,dt \oplus N$. We may write:

$$H_n(x) = h(x) + (l(x) - h(x))t + a_n(x)dt + b_n(x, t),$$

with $a_n(x) \in J$ and $b_n(x,t) \in J \otimes N$. Since $dH_n(x) = H_n(dx)$ we have for every $x \in R(n)$:

$$(-1)^{\overline{x}}(l(x) - h(x)) + d(a_n(x)) = a_n(dx), \quad d(b_n(x,t)) = b_n(dx,t). \tag{2}$$

Let $i \in I(n+1) - I(n)$, we seek for $a_{n+1}(x_i) \in J$ and $b_{n+1}(x_i, t) \in J \otimes N$ such that, setting

$$H_{n+1}(x_i) = h(x_i) + (l(x_i) - h(x_i))t + a_{n+1}(x_i)dt + b_{n+1}(x_i, t),$$

we want to have

$$\begin{aligned} 0 &= dH_{n+1}(x_i) - H_n(dx_i) \\ &= ((-1)^{\overline{x_i}}(l(x_i) - h(x_i)) + da_{n+1}(x_i) - a_n(dx_i))dt + db_{n+1}(x_i, t) - b_n(dx_i, t). \end{aligned}$$

Since both J and $J \otimes N$ are acyclic, such a choice of $a_{n+1}(x_i)$ and $b_{n+1}(x_i, t)$ is possible if and only if $(-1)^{\overline{dx_i}}(l(x_i) - h(x_i)) + a_n(dx_i)$ and $b_n(dx_i, t)$ are closed. According to Equation 2 we have

$$\begin{aligned} d((-1)^{\overline{dx_i}}(l(x_i) - h(x_i) + a_n(dx_i)) &= (-1)^{\overline{dx_i}}(l(dx_i) - h(dx_i)) + d(a_n(dx_i)) \\ &= a_n(d^2x_i) = 0, \\ db_n(dx_i, t) &= b_n(d^2x_i, t) = 0. \end{aligned}$$

□

Definition 8.6. *A* K-semifree resolution *(also called resolvent) of a K-dg-algebra A is a surjective quasiisomorphism $R \to A$ with R semifree K-dg-algebra.*

By 8.5 if a semifree resolution exists then it is unique up to homotopy.

Theorem 8.7. *Every K-dg-algebra admits a K-semifree resolution.*

Proof. Let A be a K-dg-algebra, we show that there exists a sequence of K-dg-algebras $K = R(0) \subset R(1) \subset \ldots \subset R(n) \subset \ldots$ and morphisms $f_n \colon R(n) \to A$ such that:

1. $R(n+1) = R(n)[\{x_i\}]$, $dx_i \in R(n)$.
2. f_{n+1} extends f_n.
3. $f_1 \colon Z(R(1)) \to Z(A)$, $f_2 \colon R(2) \to A$ are surjective.
4. $f_n^{-1}(B(A)) \cap Z(R(n)) \subset B(R(n+1)) \cap R(n)$, for every $n > 0$.

It is then clear that $R = \cup R(n)$ and $f = \varinjlim f_n$ give a semifree resolution. $Z(A)$ is a graded algebra and therefore there exists a polynomial graded algebra $R(1) = K[\{x_i\}]$ and a surjective morphism $f_1 \colon R(1) \to Z(A)$; we set the trivial differential $d = 0$ on $R(1)$. Let v_i be a set of homogeneous generators of the ideal $f_1^{-1}(B(A))$, if $f_1(v_i) = da_i$ it is not restrictive to assume that the a_i's generate A. We then define $R(2) = R(1)[\{x_i\}]$, $f_2(x_i) = a_i$ and $dx_i = v_i$. Assume now by induction that we have defined $f_n \colon R(n) \to A$ and let $\{v_j\}$ be a set of generators of $f_n^{-1}(B(A)) \cap Z(R(n))$, considered as an ideal of $Z(R(n))$; If $f_n(v_j) = da_j$ we define $R(n+1) = R(n)[\{x_j\}]$, $dx_j = v_j$ and $f_{n+1}(x_j) = a_j$. □

Remark 8.8. *It follows from the above proof that if $K_i = A_i = 0$ for every $i > 0$ then there exists a semifree resolution $R \to A$ with $R_i = 0$ for every $i > 0$.*

Exercise 8.9. If, in the proof of Theorem 8.7 we choose at every step $\{v_i\} = f_n^{-1}(B(A)) \cap Z(R(n))$ we get a semifree resolution called *canonical*. Show that every morphism of dg-algebras has a natural lift to their canonical resolutions. △

Given two semifree resolutions $R \to A$, $S \to A$ we can consider a semifree resolution $P \to R \times_A S$ of the fibred product and we get a commutative diagram of semifree resolutions

$$\begin{array}{ccc} P & \longrightarrow & R \\ \downarrow & \searrow & \downarrow \\ R & \longrightarrow & A. \end{array}$$

Definition 8.10. *An A-dg-module F is called* semifree *if $F = \oplus_{i \in I} A m_i$, $\overline{m_i} \in \mathbb{Z}$ and there exists a filtration $\emptyset = I(0) \subset I(1) \subset \ldots \subset I(n) \subset \ldots$ such that*

$$i \in I(n+1) \Rightarrow dm_i \in F(n) = \oplus_{i \in I(n)} A m_i.$$

A semifree resolution *of an A-dg-module M is a surjective quasiisomorphism $F \to M$ with F semifree.*

The proof of the following two results is essentially the same of 8.5 and 8.7:

Proposition 8.11. *Every semifree module is projective.*

Theorem 8.12. *Every A-dg-module admits a semifree resolution.*

Exercise 8.13. An A-dg-module M is called *flat* if for every quasiisomorphism $f \colon N \to P$ the morphism $f \otimes Id \colon N \otimes M \to P \otimes M$ is a quasiisomorphism. Prove that every semifree module is flat. △

Example 8.14. If $R = K[\{x_i\}]$ is a K-semifree algebra then $\Omega_{R/K} = \oplus R \delta x_i$ is a semifree R-dg-module.

9 The cotangent complex

Proposition 9.1. *Assume it is given a commutative diagram of K-dg-algebras*

$$\begin{array}{ccccc} R & \xrightarrow{f} & S & \xleftarrow{g} & R \\ & {\scriptstyle p}\searrow & \downarrow & \swarrow{\scriptstyle p} & \\ & & A & & \end{array}$$

If there exists a homotopy between f and g, constant on A, then the induced morphisms of A-dg-modules

$$f, g\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A$$

are homotopic.

Proof. Let $J \subset S$ be the kernel of $S \to A$ and let $H\colon R \to S \oplus_{j\geq 0} (Jt^{j+1} \oplus Jt^j\, dt)$ be a homotopy between f and g; the first terms of H are

$$H(x) = f(x) + t(g(x) - f(x)) + dt\, q(x) + \dots .$$

From $dH(x) = H(dx)$ we get $g(x) - f(x) = q(dx) + dq(x)$ and from $H(xy) = H(x)H(y)$ follows $q(xy) = q(x)f(y) + (-1)^{\overline{x}} f(x)q(y)$. Since $f(x) - g(x), q(x) \in J$ for every x, the map

$$q\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A, \quad q(\delta x \cdot r \otimes a) = \delta(q(x))f(r) \otimes a,$$

is a well defined element of $\mathrm{Hom}^{-1}_A(\Omega_{R/K} \otimes_R A, \Omega_{S/K} \otimes_S A)$. By definition $f, g\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A$ are defined by

$$f(\delta x \cdot r \otimes a) = \delta(f(x))f(r) \otimes a, \quad g(\delta x \cdot r \otimes a) = \delta(g(x))g(r) \otimes a = \delta(g(x))f(r) \otimes a.$$

A straightforward verification shows that $dq = f - g$. □

Definition 9.2. *Let $R \to A$ be a K-semifree resolution, the A-dg-module $\mathbb{L}_{A/K} = \Omega_{R/K} \otimes_R A$ is called the* relative cotangent complex *of A over K. By 9.1 the homotopy class of $\mathbb{L}_{A/K}$ is independent from the choice of the resolution. For every A-dg-module M the vector spaces*

$$T^i(A/K, M) = H^i(\mathrm{Hom}^*_A(\mathbb{L}_{A/K}, M)) = \mathrm{Ext}^i_A(\mathbb{L}_{A/K}, M),$$

$$T_i(A/K, M) = H_i(\mathbb{L}_{A/K} \otimes M)) = \mathrm{Tor}^A_i(\mathbb{L}_{A/K}, M),$$

are called respectively the cotangent and tangent cohomolgy of the morphism $K \to A$ with coefficient on M.

Lemma 9.3. *Let $p\colon R \to S$ be a surjective quasiisomorphism of semifree dg-algebras: consider on S the structure of R-dg-module induced by p. Then:*

1. $p_*\colon \operatorname{Der}^*(R,R) \to \operatorname{Der}^*(R,S)$, $f \to pf$, *is a surjective quasiisomorphism.*

2. $p^*\colon \operatorname{Der}^*(S,S) \to \operatorname{Der}^*(R,S)$, $f \to fp$, *is an injective quasiisomorphism.*

Proof. A derivation on a semifree dg-algebra is uniquely determined by the values at its generators, in particular p_* is surjective and p^* is injective. Since Ω_R is semifree, by 7.5 the morphism $p_*\colon \operatorname{Hom}^*_R(\Omega_R, R) \to \operatorname{Hom}^*_R(\Omega_R, S)$ is a quasiisomorphism. By base change $\operatorname{Der}^*(R,S) = \operatorname{Hom}^*_S(\Omega_R \otimes_R S, S)$ and, since $p\colon \Omega_R \otimes_R S \to \Omega_S$ is a homotopy equivalence, also p^* is a quasiisomorphism. □

Every morphism $f\colon A \to B$ of dg-algebras induces a morphism of B modules $\mathbb{L}_A \otimes_A B \to \mathbb{L}_B$ unique up to homotopy. In fact if $R \to A$ and $P \to B$ are semifree resolution, then there exists a lifting of f, $R \to P$, unique up to homotopy constant on B. The morphism $\Omega_R \to \Omega_P$ induce a morphism $\Omega_R \otimes_R B = \mathbb{L}_A \otimes_A B \to \Omega_P \otimes_P B = \mathbb{L}_B$. If B is a localization of A we have the following

Theorem 9.4. *Let A be a dg-algebra, $S \subset Z_0(A)$ a multiplicative part: then the morphism*

$$\mathbb{L}_A \otimes_A S^{-1}A \to \mathbb{L}_{S^{-1}A}$$

is a quasiisomorphism of $S^{-1}A$ modules.

Proof. (sketch) Denote by $f\colon R \to A$, $g\colon P \to S^{-1}A$ two semifree resolutions and by

$$H = \{x \in Z_0(R) \,|\, f(x) \in S\}, \qquad K = \{x \in Z_0(P) \,|\, g(x) \text{ is invertible }\}.$$

The natural morphisms $H^{-1}R \to S^{-1}A$, $K^{-1}P \to S^{-1}A$ are both surjective quasiisomorphisms. By the lifting property of semifree algebras we have a chain of morphisms

$$R \xrightarrow{\alpha} P \xrightarrow{\beta} H^{-1}R \xrightarrow{\gamma} K^{-1}P$$

with γ the localization of α. Since $\beta\alpha$ and $\gamma\beta$ are homotopic to the natural inclusions $R \to H^{-1}R$, $P \to K^{-1}P$, the composition of morphisms

$$\Omega_R \otimes_R S^{-1}A \xrightarrow{\alpha} \Omega_P \otimes_P S^{-1}A \xrightarrow{\beta} \Omega_{H^{-1}R} \otimes_{H^{-1}R} S^{-1}A = \Omega_R \otimes_R S^{-1}A,$$

$$\Omega_P \otimes_P S^{-1}A \xrightarrow{\beta} \Omega_{H^{-1}R} \otimes_{H^{-1}R} S^{-1}A \xrightarrow{\gamma} \Omega_{K^{-1}P} \otimes_{K^{-1}P} S^{-1}A = \Omega_P \otimes_P S^{-1}A$$

are homotopic to the identity and hence quasiisomorphisms. □

Example 9.5. Hypersurface singularities.
Let $X = V(f) \subset \mathbb{A}^n$, $f \in \mathbb{K}[x_1, \dots, x_n]$, be an affine hypersurface and denote by $A = \mathbb{K}[X] = \mathbb{K}[x_1, \dots, x_n]/(f)$ its structure ring. A DG-resolvent of A is given by $R = \mathbb{K}[x_1, \dots, x_n, y]$, where y has degree -1 and the differential is given by

$s(y) = f$. The R-module Ω_R is semifreely generated by $dx_1, \dots, dx_n, dy$, with the differential

$$s(dy) = d(s(y)) = df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

The cotangent complex $\mathbb{L}_A$ is therefore

$$0 \longrightarrow Ady \xrightarrow{s} \bigoplus_{i=1}^{n} Adx_i \longrightarrow 0.$$

In particular $T^i(A/\mathbb{K}, A) = \operatorname{Ext}^i(\mathbb{L}_A, A) = 0$ for every $i \neq 0, 1$. The cokernel of s is isomorphic to Ω_A and then $T^0(A/\mathbb{K}, A) = \operatorname{Ext}^0(\mathbb{L}_A, A) = \operatorname{Der}_{\mathbb{K}}(A, A)$. If f is reduced then s is injective, $\mathbb{L}_A$ is quasiisomorphic to Ω_A and then $T^1(A/\mathbb{K}, A) = \operatorname{Ext}^1(\Omega_A, A)$.

Exercise 9.6. In the set-up of Example 9, prove that the A-module $T^1(A/\mathbb{K}, A)$ is finitely generated and supported in the singular locus of X. △

10 The controlling differential graded Lie algebra

Let $p\colon R \to S$ be a surjective quasiisomorphism of semifree algebras and let $I = \ker p$. By the lifting property of S there exists a morphism of dg-algebras $e\colon S \to R$ such that $pe = Id_S$. Define

$$L_p = \{f \in \operatorname{Der}^*(R, R) \mid f(I) \subset I\}.$$

It is immediate to verify that L_p is a dg-Lie subalgebra of $\operatorname{Der}^*(R, R)$. We may define a map

$$\theta_p\colon L_p \to \operatorname{Der}^*(S, S), \qquad \theta_p(f) = p \circ f \circ e.$$

Since $pf(I) = 0$ for every $f \in L_p$, the definition of θ_p is independent from the choice of e.

Lemma 10.1. *θ_p is a morphism of DGLA.*

Proof. For every $f, g \in L_p$, $s \in S$, we have:

$$d(\theta_p f)(s) = dpfe(s) - (-1)^{\overline{f}} pfe(ds) = pdfe(s) - (-1)^{\overline{f}} pfd(e(s)) = \theta_p(df)(s).$$

Since $pfep = pf$ and $pgep = pg$

$$[\theta_p f, \theta_p g] = pfepge - (-1)^{\overline{f}\,\overline{g}} pgepfe = p(fg - (-1)^{\overline{f}\,\overline{g}} gf)e = \theta_p([f, g]).$$

□

Theorem 10.2. *The following is a cartesian diagram of quasiisomorphisms of DGLA*

$$\begin{array}{ccc} L_p & \xhookrightarrow{\imath_p} & \mathrm{Der}^*(R,R)\ , \\ \Big\downarrow{\scriptstyle \theta_p} & & \Big\downarrow{\scriptstyle p_*} \\ \mathrm{Der}^*(S,S) & \xrightarrow[p^*]{} & \mathrm{Der}^*(R,S) \end{array}$$

where $\imath_p$ is the inclusion.

We recall that cartesian means that it is commutative and that L_p is isomorphic to the fibred product of p_* and p^*.

Proof. Since $pfep = pf$ for every $f \in L_p$ we have $p^*\theta_p(f) = pfep = pf = p_*f$ and the diagram is commutative. Let

$$K = \{(f, g) \in \mathrm{Der}^*(R, R) \times \mathrm{Der}^*(S, S) \,|\, pf = gp\}$$

be the fibred product; the map $L_p \to K$, $f \to (f, \theta_p(f))$, is clearly injective. Conversely take $(f, g) \in K$ and $x \in I$, since $pf(x) = gp(x) = 0$ we have $f(I) \subset I$, $f \in L_p$. Since p is surjective g is uniquely determined by f and then $g = \theta_p(f)$. This proves that the diagram is cartesian. By 9.3 p_* (resp.: p^*) is a surjective (resp.: injective) quasiisomorphism, by a standard argument in homological algebra also θ_p (resp.: $\imath_p$) is a surjective (resp.: injective) quasiisomorphism. □

Corollary 10.3. *Let $P \to A$, $Q \to A$ be semifree resolutions of a dg-algebra. Then $\mathrm{Der}^*(P, P)$ and $\mathrm{Der}^*(Q, Q)$ are quasiisomorphic DGLA.*

Proof. There exists a third semifree resolution $R \to A$ and surjective quasiisomorphisms $p \colon R \to P$, $q \colon R \to Q$. Then there exists a sequence of quasiisomorphisms of DGLA

$$\begin{array}{ccccccc} & L_p & & & & L_q & \\ {\scriptstyle \theta_p}\swarrow & & \searrow{\scriptstyle \imath_p} & & {\scriptstyle \imath_q}\swarrow & & \searrow{\scriptstyle \theta_q} \\ \mathrm{Der}^*(P,P) & & & \mathrm{Der}^*(R,R) & & & \mathrm{Der}^*(Q,Q). \end{array}$$

□

Remark 10.4. *If $R \to A$ is a semifree resolution then*

$$H^i(\mathrm{Der}^*(R, R)) = H^i(\mathrm{Hom}_R(\Omega_R, R)) = H^i(\mathrm{Hom}_R(\Omega_R, A)) =$$

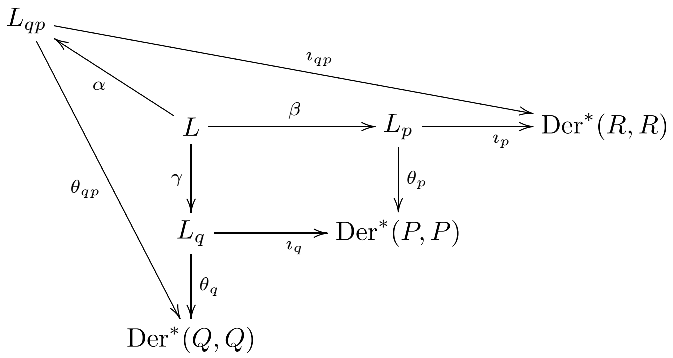
$$= H^i(\mathrm{Hom}_A(\Omega_R \otimes_R A, A)) = \mathrm{Ext}^i(\mathbb{L}_A, A).$$

Unfortunately, contrarily to what happens to the cotangent complex, the application $R \to \mathrm{Der}^*(R, R)$ is quite far from being a functor: it only earns some functorial properties when composed with a suitable functor $\mathbf{DGLA} \to \mathbf{D}$.

Let $\mathbf{D}$ be a category and $\mathcal{F}\colon \mathbf{DGLA} \to \mathbf{D}$ be a functor which sends quasiisomorphisms into isomorphisms of $\mathbf{D}$[1]. By 10.3, if $P \to A$, $Q \to A$ are semifree resolutions then $\mathcal{F}(\mathrm{Der}^*(P, P)) \simeq \mathcal{F}(\mathrm{Der}^*(Q, Q))$; now we prove that the recipe of the proof of 10.3 gives a NATURAL isomorphism independent from the choice of P, p, q. For notational simplicity denote $\mathcal{F}(P) = \mathcal{F}(\mathrm{Der}^*(P, P))$ and for every surjective quasiisomorphism $p\colon R \to P$ of semifree dg-algebras, $\mathcal{F}(p) = \mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1}\colon \mathcal{F}(R) \to \mathcal{F}(P)$.

Lemma 10.5. *Let $p\colon R \to P$, $q\colon P \to Q$ be surjective quasiisomorphisms of semifree dg-algebras, then $\mathcal{F}(qp) = \mathcal{F}(q)\mathcal{F}(p)$.*

Proof. Let $I = \ker p$, $J = \ker q$, $H = \ker qp = p^{-1}(J)$, $e\colon P \to R$, $s\colon Q \to P$ sections. Note that $e(J) \subset H$. Let $L = L_q \times_{\mathrm{Der}^*(P,P)} L_p$, if $(f, g) \in L$ and $x \in H$ then $pg(x) = pg(ep(x)) = f(x) \in J$ and then $g(x) \in H$, $g \in L_{qp}$; denoting $\alpha\colon L \to L_{qp}$, $\alpha(f, g) = g$, we have a commutative diagram of quasiisomorphisms of DGLA



and then

$$\mathcal{F}(qp) = \mathcal{F}(\theta_{qp})\mathcal{F}(\imath_{qp})^{-1} = \mathcal{F}(\theta_q)\mathcal{F}(\gamma)\mathcal{F}(\alpha)^{-1}\mathcal{F}(\alpha)\mathcal{F}(\beta)^{-1}\mathcal{F}(\imath_p)^{-1} =$$

$$= \mathcal{F}(\theta_q)\mathcal{F}(\imath_q)^{-1}\mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1} = \mathcal{F}(q)\mathcal{F}(p).$$

□

Let P be a semifree dg-algebra $Q = P[\{x_i, dx_i\}] = P \otimes_{\mathbb{K}} \mathbb{K}[\{x_i, dx_i\}]$, $i\colon P \to Q$ the natural inclusion and $\pi\colon Q \to P$ the projection $\pi(x_i) = \pi(dx_i) = 0$: note that i, π are quasiisomorphisms. Since P, Q are semifree we can define

[1]The examples that we have in mind are the associated deformation functor and the homotopy class of the corresponding L_∞-algebra

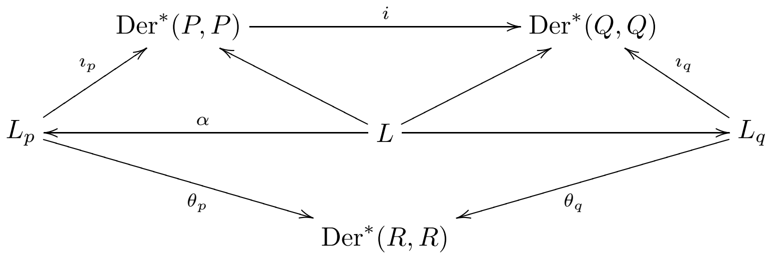
a morphism of DGLA

$$\begin{aligned} i\colon \operatorname{Der}^*(P,P) &\longrightarrow \operatorname{Der}^*(Q,Q),\\ (if)(x_i) &= (if)(dx_i) = 0,\\ (if)(p) &= i(f(p)),\ p\in P. \end{aligned}$$

Since $\pi_* i = \pi^*\colon \operatorname{Der}^*(P,P) \to \operatorname{Der}^*(Q,P)$, according to 9.3 i is an injective quasiisomorphism.

Lemma 10.6. *Let P, Q as above, let $q\colon Q \to R$ a surjective quasiisomorphism of semifree algebras. If $p = qi\colon P \to R$ is surjective then $\mathcal{F}(p) = \mathcal{F}(q)\mathcal{F}(i)$.*

Proof. Let $L = \operatorname{Der}^*(P,P) \times_{\operatorname{Der}^*(Q,Q)} L_q$ be the fibred product of i and $\imath_q$; if $(f,g) \in L$ then $g = if$ and for every $x \in \ker p$, $i(f(x)) = g(i(x)) \in \ker q \cap i(P) = i(\ker p)$. Denoting $\alpha\colon L \to L_p$, $\alpha(f,g) = f$, we have a commutative diagram of quasiisomorphisms



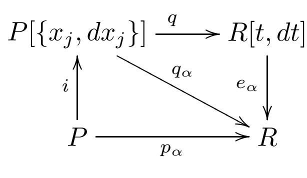
and then $\mathcal{F}(q)\mathcal{F}(i) = \mathcal{F}(\theta_q)\mathcal{F}(\imath_q)^{-1}\mathcal{F}(i) = \mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1}$. □

Lemma 10.7. *Let $p_0, p_1\colon P \to R$ be surjective quasiisomorphisms of semifree algebras. If p_0 is homotopic to p_1 then $\mathcal{F}(p_0) = \mathcal{F}(p_1)$.*

Proof. We prove first the case $P = R[t,dt]$ and $p_i = e_i$, $i = 0,1$, the evaluation maps. Denote by

$$L = \{f \in \operatorname{Der}^*(P,P) \mid f(R) \subset R, f(t) = f(dt) = 0\}.$$

Then $L \subset L_{e_\alpha}$ for every $\alpha = 0,1$, $\theta_{e_\alpha}\colon L \to \operatorname{Der}^*(P,P)$ is an isomorphism not depending from α and $L \subset L_{e_\alpha} \subset \operatorname{Der}^*(R,R)$ are quasiisomorphic DGLA. This proves that $\mathcal{F}(e_0) = \mathcal{F}(e_1)$. In the general case we can find commutative diagrams, $\alpha = 0,1$,



with q surjective quasiisomorphism. We then have $\mathcal{F}(p_0) = \mathcal{F}(q_0)\mathcal{F}(i)^{-1} = \mathcal{F}(e_0)\mathcal{F}(q)\mathcal{F}(i)^{-1} = \mathcal{F}(e_1)\mathcal{F}(q)\mathcal{F}(i)^{-1} = \mathcal{F}(q_1)\mathcal{F}(i)^{-1} = \mathcal{F}(p_1)$. □

We are now able to prove the following

Theorem 10.8. *Let*

$$\begin{array}{ccc} R & \xrightarrow{p} & P \\ {\scriptstyle q}\downarrow & & \downarrow \\ Q & \longrightarrow & A \end{array}$$

be a commutative diagram of surjective quasiisomorphisms of dg-algebras with P, Q, R *semifree. Then* $\Psi = \mathcal{F}(p)\mathcal{F}(q)^{-1} \colon \mathcal{F}(Q) \to \mathcal{F}(P)$ *does not depend from* R, p, q.

Proof. Consider two diagrams as above

$$\begin{array}{ccc} R_0 & \xrightarrow{p_0} & P \\ {\scriptstyle q_0}\downarrow & & \downarrow \\ Q & \longrightarrow & A, \end{array} \qquad \begin{array}{ccc} R_1 & \xrightarrow{p_1} & P \\ {\scriptstyle q_1}\downarrow & & \downarrow \\ Q & \longrightarrow & A. \end{array}$$

There exists a commutative diagram of surjective quasiisomorphisms of semifree algebras

$$\begin{array}{ccc} T & \xrightarrow{t_1} & R_1 \\ {\scriptstyle t_0}\downarrow & & \downarrow{\scriptstyle q_1} \\ R_0 & \xrightarrow[q_0]{} & Q. \end{array}$$

By Lemma 10.5 $\mathcal{F}(q_0)\mathcal{F}(t_0) = \mathcal{F}(q_1)\mathcal{F}(t_1)$. According to 8.5 the morphisms $p_0t_0, p_1t_1 \colon T \to P$ are homotopic and then $\mathcal{F}(p_0)\mathcal{F}(t_0) = \mathcal{F}(p_1)\mathcal{F}(t_1)$. This implies that $\mathcal{F}(p_0)\mathcal{F}(q_0)^{-1} = \mathcal{F}(p_1)\mathcal{F}(q_1)^{-1}$. □

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