$\mathbb{Q}$-GORENSTEIN SMOOTHINGS OF QUOTIENT SINGULARITIES.

MARCO MANETTI

We give a topological proof of the following theorem:

Theorem. *Let $(X_0, 0)$ be a two dimensional quotient singularity, if $(X_0, 0)$ admits a $\mathbb{Q}$-Gorenstein smoothing over the disk, then either $(X_0, 0)$ is a rational double point or it is a cyclic singularity of type $\frac{1}{dn^2}(1, dna-1)$ for some integers a, n, d with a and n relatively prime.*

We give also an explicit classification of such smoothings and we study their Milnor fibres.

§0 . Introduction

The goal of this paper is to study some particular classes of deformations of quotient singularities of dimension 2.

Let $G \subset Aut(\mathbb{C}^n, 0)$ be a finite subgroup of germs of holomorphic automorphisms of $\mathbb{C}^n$ in 0, according to a classical result due to H.Cartan [C] there exists a new coordinates system where the action of G linearizes, i.e. $G \subset GL(n, \mathbb{C})$. A quotient singularity is the germ $(X, 0)$ where $X = \mathbb{C}^n/G$ and $0 \in X$ is the representative of the G-orbit $\{0\}$.

By a well known theorem of Schlessinger every quotient singularity of dimension $n \neq 2$ is rigid, so it makes sense to study deformations only in the two dimensional case.

We recall that a quotient singularity $\mathbb{C}^2/G$ is said to be a rational double point (R.D.P. for short) if $G \subset SL(2, \mathbb{C})$. Given $G \subset GL(2, \mathbb{C})$ if $G' = G \cap SL(2, \mathbb{C})$ then $H = G/G'$ is a finite subgroup of $\mathbb{C}^*$ thus it is cyclic, in particular every quotient singularity of dimension two is the quotient of a R.D.P. by an automorphism of finite order.

This simple remark is the idea which has been inspirating this work. In fact we show here that some deformations of two dimensional quotient singularities are the quotient, by an automorphism of finite order, of some deformations of R.D.P.'s and we consequently use this information in order to give a classification result.

Roughly speaking a deformation of a normal singularity is called a $\mathbb{Q}$-Gorenstein smoothing if some multiple of the canonical divisor is locally principal and the generic fibre is smooth (see §1 and §2 for a precise definition). The main result stated here is the following:

Main Theorem. *Let $(X,0) \to (\mathbb{C},0)$ be a $\mathbb{Q}$-Gorenstein smoothing of a two dimensional quotient singularity $(X_0,0)$. Then either $(X_0,0)$ is a rational double point or there exists an integer $n>1$ such that $(X,0)$ is analytically isomorphic to $(Y,0)/G$ where:*

a) $(Y,0) \subset (\mathbb{C}^4,0)$ is an isolated hypersurface singularity defined by

$$F = uv + y^{dn} - t^b - \varphi_1(t)y^n - \ldots - \varphi_{d-1}(t)y^{(d-1)n} = 0$$

for some positive integers d,b.

The φ_i's are convergent power series satisfying $\varphi_i \in \mathbb{C}\{t\}$, $\varphi_i(0)=0$ and such that the projection on the t-axis $(Y,0)\xrightarrow{\pi}(\mathbb{C},0)$ is a smoothing of $(Y_0,0)$.

b) $G \simeq \mu_n$ acts on $\mathbb{C}^4$ in the following way

$$\mu_n \ni \xi\colon (u,v,y,t) \to (\xi u, \xi^{-1}v, \xi^a y, t) \qquad \text{with } (a,n)=1$$

c) π' is obtained from π by passing to the quotient.

Conversely all such smoothings are $\mathbb{Q}$-Gorenstein.

A proof of this theorem can be obtained by joining Cor. 3.6 and Prop. 3.10 of [K-S]. Kollar and Shepherd-Barron's proof relies over some recent advances in three dimensional geometry and their applications to deformations of surface singularities.

We note that a weaker version of the main theorem (valid in the hypothesis that $(X_0,0)$ is a cyclic quotient singularity) can be easily proved by studying the self-intersection K^2 of the canonical divisor in the minimal resolution of $(X_0,0)$. This study is essentially contained in [Wa] and we refer to ([L-W] Prop 5.9) for a detailed exposition.

In this paper, following the previously described idea, we give a new proof of the main theorem which is essentially different from the other ones and which we believe to be easier and more elementary.

The paper is organized as follows:
In §1 we recall some definitions and some basic facts about smoothing of normal singularities.
In §2 we prove some preparatory material and some fact which we believe to be of independent interest. We prove that the torsion part of the Picard group of the Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing of a quotient singularity is cyclic and that it is generated by the canonical bundle.
Finally §3 is entirely dedicated to the proof of the main theorem.
This work is essentially contained in [Ma2] where the results proved here find application in the study of degenerations of algebraic surfaces. We refer to [Ma1] for an useful application to investigation of normal degenerations of the projective plane.
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§1 . Generalities about Milnor fibre and analytic singularities

We shall always work over the field of the complex numbers $\mathbb{C}$. By a surface singularity we shall mean a two dimensional irreducible germ $(V_0, 0)$ of analytic space which has 0 as an isolated singular point. A smoothing of $(V_0, 0)$ is a flat map $f: V \to \Delta$ where V is a reduced complex space, $\Delta \subset \mathbb{C}$ is a small open disk centered at 0, $f^{-1}(0) \simeq V_0$ and for every $t \in \Delta^* = \Delta - \{0\}$ the fibre $V_t = f^{-1}(t)$ is nonsingular.
Let's suppose $(V_0, 0)$ embedded in $(\mathbb{C}^n, 0)$, then there exists a closed embedding of $(V, 0)$ in $(\mathbb{C}^n \times \Delta, 0)$ such that the map f is induced by the projection on the second factor $\mathbb{C}^n \times \Delta \to \Delta$.
Let's fix now some further notation: if $r > 0$ we denote by $B_r = \{z \in \mathbb{C}^N | \; \|z\| < r\}$ and let $S_r = \partial B_r$. We shall call S_r a Milnor sphere for V_0 if for every $0 < r' \leq r$ the sphere $S_{r'}$ intersects V_0 transversally : according to a basic result ([Mi] Cor 2.9) every isolated embedded singularity admits a Milnor sphere.
Let S_r be a Milnor sphere for V_0, then (shrinking Δ if necessary) we can assume that $S_r \times \Delta$ intersects V_t transversally $\forall t \in \Delta$. In this set-up we denote

$$X = V \cap (B_r \times \Delta) \qquad X_t = V_t \cap X \qquad K_t = \partial X_t = V_t \cap (S_r \times \Delta)$$

By the Ehresmann's fibration theorem we have $\partial X = \cup_{t \in \Delta} K_t \simeq K_0 \times \Delta$ and the map $f: X \backslash X_0 \to \Delta^*$ is a locally trivial C^∞ fibre bundle with fibre F diffeomorphic to X_t for $t \neq 0$.We call F (resp. $\overline{F}$) the Milnor fibre (resp. compact Milnor fibre) of the smoothing f.

The basic theory about Milnor fibres shows that the diffeomorphism class of F is independent of the embedding of V, in particular topological invariants of F are invariants of the smoothing.

Since F is Stein, it has the homotopy type of a two dimensional CW complex and $\overline{F}$ is obtained from ∂F up to homotopy by attaching a finite number of cells of dimension ≥ 2. This implies that the inclusion $\partial F \subset \overline{F}$ induces a surjection of the respective fundamental groups, moreover, it is rather easy to prove, by using the exact homotopy sequence of the fibration $f \colon X \backslash X_0 \to \Delta^*$, that the inclusion $F \subset X - \{0\}$ induces an isomorphism of the π_1's (cf. [L-W] Lemma 5.1).

Let's consider now homology and cohomology: we have $H_i(F, \mathbb{Z}) = 0$ for $i > 2$ and $H_2(F, \mathbb{Z})$ is a finitely generated free abelian group.

Definition. The integer $\mu = \text{rank} H_2(F, \mathbb{Z})$ is called the *Milnor number* of the smoothing.

The Lefschetz and Poincaré duality theorems give the following isomorphisms (in every ring of coefficients)

$$H^q_c(F) = H_{4-q}(F) = H^q(\overline{F}, \partial F)$$

With the real coefficients the cup product induces a perfect pairing

$$H^2(\overline{F}) \times H^2(\overline{F}, \partial F) \xrightarrow{\cup} H^4(\overline{F}, \partial F) = \mathbb{R}$$

which composed with the natural map $H^2(\overline{F}, \partial F) \to H^2(\overline{F})$ gives a symmetric bilinear form

$$H^2(\overline{F}, \partial F) \times H^2(\overline{F}, \partial F) \xrightarrow{q} \mathbb{R}$$

We can thus write $\mu = \mu_0 + \mu_+ + \mu_-$ where μ_0 (resp.: μ_+, μ_-) is the number of zero (resp.: positive,negative) eigenvalues of q.

Definition. Let X_0 be a Stein representative of the surface singularity $(X_0, 0)$ and let $\pi \colon Z \to X_0$ be a resolution of singularities. The *geometric genus* of $(X_0, 0)$ is the integer $p_g(X_0) = h^1(\mathcal{O}_Z) - \delta(X_0)$ where $\delta(X_0) = h^0(\pi_* \mathcal{O}_Z / \mathcal{O}_{X_0})$.

An important result of Steenbrink ([St] Th. 2.24) is the following: given a smoothing of a two dimensional isolated surface singularity $(X_0, 0)$ we have $\mu_0 + \mu_+ = 2p_g$; in particular if the singularity is rational, that is normal with geometric genus 0, then it follows that $\mu = \mu_-$.

For later use we recall also the following result of Greuel and Steenbrink ([G-S] Th. 2): if F is the Milnor fibre of a smoothing of a normal surface singularity then $b_1(F) = 0$.

We conclude this brief review by describing the homotopy type and the intersection form q of the Milnor fibre of a smoothing of a rational double point X_0. This is easy, in fact by Brieskorn-Tyurina's result on simultaneous resolution (Tyurina [Ty] Th. 1) the Milnor fibre is diffeomorphic to a neighbourhood of the exceptional curve in the minimal resolution of X_0, in particular if X_0 is a rational double point of type A_r, D_r or E_r then the Milnor fibre has the homotopy type of a bouquet of r spheres.

Let now $(Y,0)\xrightarrow{\pi}(\mathbb{C},0)$ be a smoothing of an isolated surface singularity $(Y_0,0)$ and let G be a finite group acting on $(Y,0)$, freely on $Y-\{0\}$, and such that $\pi g(y) = \pi(y)$ for every $g \in G$ and $y \in Y$. In this way the map π is a G-map where the action of G in $(\mathbb{C},0)$ is the trivial one.

Let $X = Y/G$ be the quotient singularity, we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & \mathbb{C} \\ \downarrow{\scriptstyle p} & & \| \\ X & \xrightarrow{\pi'} & \mathbb{C} \end{array}$$

where p is the natural projection.

If $t \neq 0$, G acts freely on Y_t, thus π' is a smoothing of the singularity $X_0 = Y_0/G$. In order to study the Milnor fibres of π and π' we give a G-embedding of $(Y,0)$ using the following simple lemma which generalizes the well known Cartan lemma ([C] pag. 97).

Lemma 1. *Let $(Y,0)$ be an analytic singularity and let $G \subset Aut(Y,0)$ be a finite group of holomorphic automorphisms of $(Y,0)$. Let us suppose moreover that the same group G acts linearly on a finite dimensional $\mathbb{C}$-vector space E.*

If $\pi\colon (Y,0) \to (E,0)$ is a G-map with respect to the above actions such that the tangent map $\pi_*\colon T \to E$ on the corresponding tangent spaces is surjective, then there exists a G-embedding $(Y,0)\xrightarrow{\phi}(T,0)$ where G acts on T in the natural way, such that the diagram

$$\begin{array}{ccc} (Y,0) & \xrightarrow{\phi} & (T,0) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_*} \\ (E,0) & = & (E,0) \end{array}$$

is commutative.

Proof. Let $R = \mathcal{O}_{(Y,0)}$ be the analytic algebra of $(Y,0)$ and let $\mathcal{M} \subset R$ be the maximal ideal. Choosing a basis $z_1, \dots, z_r$ of $E^\vee$, we have a G-equivariant homomorphism of analytic algebras $\pi^*\colon \mathbb{C}\{z_1, \dots, z_r\} \to R$.
Let $U \subset \mathcal{M}$ be the vector space generated by $\pi^*(z_1), \dots, \pi^*(z_r)$, by hypothesis U is G-stable and its image on $\mathcal{M}/\mathcal{M}^2$ is a vector space of dimension r.
Let us observe now that for giving an embedding $(Y,0)\xrightarrow{\phi}(T,0)$ it suffices to give a vector space $V \subset \mathcal{M}$ such that $\mathcal{M} = \mathcal{M}^2 \oplus V$, moreover ϕ is a G-embedding if and only if V is G-stable. Since G is finite and the characteristic of $\mathbb{C}$ is zero there exists a G-stable vector space $V \subset \mathcal{M}$ such that $\mathcal{M} = \mathcal{M}^2 \oplus V$, on the other side $U \cap \mathcal{M}^2 = 0$, we can then choose $U \subset V$ and the conclusion follows immediately.□

If n is the dimension of the tangent space of Y_0 at 0, by Lemma 1 we can choose a G-embedding $(Y,0)\xrightarrow{\varphi}(\mathbb{C}^n \times \Delta, 0)$ where G acts linearly on $\mathbb{C}^n$ and trivially on Δ.
Choosing a proper embedding $\mathbb{C}^n/G \subset \mathbb{C}^m$ we have a commutative diagram

$$\begin{array}{ccccccc}
\Delta & \xleftarrow{\pi} & Y & \xrightarrow{\varphi} & \mathbb{C}^n \times \Delta & = & \mathbb{C}^n \times \Delta \\
\| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
\Delta & \xleftarrow{\pi'} & X & \xrightarrow{\varphi'} & (\mathbb{C}^n/G) \times \Delta & \subset & \mathbb{C}^m \times \Delta
\end{array}$$

If $S_r \subset \mathbb{C}^n$ is a Milnor sphere for Y_0, then there exists a Milnor sphere $S_\epsilon \subset \mathbb{C}^m$ for X_0 and $0 < r' < r$ such that (possibly shrinking Δ)

$$Y \cap (B_{r'} \times \Delta) \subset Y \cap p_2^{-1}(B_\epsilon \times \Delta) \subset Y \cap (B_r \times \Delta)$$

hence the Milnor fibre F' of π' is diffeomorphic to the quotient, by G, of the Milnor fibre F of π (see [Lo] Chapter 2). In particular $\pi_1(F')/\pi_1(F) \simeq G$ and $\chi(F) = |G|\chi(F')$ where χ denotes the topological Euler Poincaré characteristic.
If Y_0 is a normal surface singularity we have $b_2(F) + 1 = |G|(b_2(F') + 1)$, in fact $X_0 = Y_0/G$ is also normal and $b_1(F) = b_1(F') = 0$.
In the following example we shall study a particular class of smoothings which will be the main object of interest in this paper. To this purpose we first need some definitions.

Definition. Let μ_n be the cyclic multiplicative group of n^{th} roots of unity and suppose that μ_n acts linearly on $\mathbb{C}^2$. After a linear base change we can assume that μ_n acts diagonally, i.e.

$$\mu_n \ni \xi\colon (z_1, z_2) \longrightarrow (\xi^a z_1, \xi^b z_2)$$

where $0 \le a, b < n$. The couple of rational numbers $\frac{1}{n}(a,b)$ is called the *type* of the cyclic quotient singularity $X = \mathbb{C}^2/\mu_n$.

Remark. The type determines completely, up to isomorphism, the singularity but not conversely: for example, every two dimensional cyclic singularity is isomorphic to one of type $\frac{1}{n}(1,q)$ whit g.c.d.$(n,q) = 1$.

Example 1. Fixing relatively prime integers $0 < a < n$ we have the following action of μ_n on $\mathbb{C}^4$

$$\mu_n \ni \xi\colon (u,v,y,t) \to (\xi u, \xi^{-1} v, \xi^a y, t)$$

Let $(Y,0) \subset (\mathbb{C}^4, 0)$ be a G-stable hypersurface singularity defined by the equation

$$uv + y^{dn} = t\varphi(u,v,y,t) \qquad d > 0$$

where φ is a G-invariant holomorphic germ at $0 \in \mathbb{C}^4$.
Suppose that the map $\pi\colon (Y,0) \to (\mathbb{C},0)$ induced from projection on the t-axis by restriction is a smoothing of the surface singularity $(Y_0, 0) = \{uv + y^{dn} = 0\}$. This is a rational double point of type A_{dn-1} and the Milnor fibre of π has therefore the homotopy type of a bouquet of $dn-1$ sphere S^2.
Let's suppose moreover that G acts freely on $Y - \{0\}$, in this case we have a quotient smoothing

$$Y \xrightarrow{p} X = Y/G \xrightarrow{\pi'} \Delta$$

Proposition. 2. *In the above situation $(X_0, 0)$ is a cyclic quotient singularity of type $\frac{1}{dn^2}(1, dna-1)$.*

Proof. Let $Z_0 = \mathbb{C}^2/\mu_{dn^2}$ be the quotient by the action

$$\mu_{dn^2} \ni \xi : (z_1, z_2) \to (\xi z_1, \xi^{dna-1} z_2)$$

and let $0 \longrightarrow \mu_{dn} \longrightarrow \mu_{dn^2} \longrightarrow \mu_n \longrightarrow 0$ be the natural exact sequence of abelian groups.∎
We can write $Z_0 = (\mathbb{C}^2/\mu_{dn})/\mu_n$ and a simple calculation gives $\mathbb{C}^2/\mu_{dn} \simeq Y_0$.
The analytic algebra of $(Y_0, 0)$ is

$$\mathbb{C}\{z_1^{dn}, z_2^{dn}, z_1 z_2\} = \mathbb{C}\{u,v,y\}/(uv + y^{dn})$$

where $u = z_1^{dn}$, $v = -z_2^{dn}$, $y = z_1 z_2$. The action of μ_{dn^2} on $\mathcal{O}_{Y_0,0}$ is

$$\mu_{dn^2} \ni \xi : (u,v,y) \to (\xi^{dn} u, \xi^{-dn} v, \xi^{dna} y)$$

The proof follows by considering this as a μ_n-action. □

The topological invariants of the smoothing π' are now of immediate computation.

Proposition. 3. *If F is the Milnor fibre of the smoothing $X \xrightarrow{\pi'} \Delta$ of Example 1 we have:*

1) $\pi_1(X - \{0\}) = \pi_1(F) = \mathbb{Z}_n$
2) $b_2(F) = d - 1$

Proof. It follows trivially from the fact that F has an unramified covering of degree n which has the homotopy type of a bouquet of $dn - 1$ spheres S^2. □

§2 . Canonical coverings of singularities

It is well known that a normal hypersurface singularity is Gorenstein, that is Cohen-Macaulay with locally principal canonical divisor. If $V \subset \mathbb{C}^{n+1}$ is a normal hypersurface defined by $F(x_0, .., x_n) = 0$, then, supposing $\dfrac{\partial F}{\partial x_0} \not\equiv 0$, we have a meromorphic n-form on V by setting

$$s = \frac{dx_1 \wedge \ldots \wedge dx_n}{\dfrac{\partial F}{\partial x_0}}$$

and the divisor (s) is canonical.
The properties normal and Cohen Macaulay are stable by passing to quotient by finite group action, the same does not hold in general for the Gorenstein property: for example a two dimensional quotient singularity is Gorenstein if and only if it is a rational double point.

Definition. A normal Cohen Macaulay singularity $(V, 0)$ is called $\mathbb{Q}$-*Gorenstein* if some nonzero power of the canonical divisor K_V is principal. The smallest positive integer n such that nK_V is principal is called the *index* of the singularity.
The singularity $(X, 0)$ of Example 1 is $\mathbb{Q}$-Gorenstein of index n. In fact, since it is a quotient by a finite group of a hypersurface singularity, it is normal and C.M.. A meromorphic 3-form on Y is

$$s = \frac{du \wedge dv \wedge dy}{\dfrac{\partial F}{\partial t}} \qquad F = uv + y^{dn} - t\varphi(u, v, y, t)$$

we have $\mu_n \ni \xi \colon s \to \xi^a s$ and, since $(a, n) = 1$, the index of X is exactly n.

Let V be a connected complex space and let $\mu_n \subset Aut(V)$ be a finite cyclic subgroup of holomorphic automorphisms of V which acts freely .
Let $V \xrightarrow{\pi} U = V/\mu_n$ be the projection to the quotient, then μ_n acts on the direct image sheaf $\pi_* \mathcal{O}_V$

$$\mu_n \ni g \colon f \longrightarrow f \circ g^{-1} = g(f) \qquad \forall f \in \pi_* \mathcal{O}_V$$

and we have a decomposition

$$\pi_* \mathcal{O}_V = \bigoplus_{\alpha \in \mathbb{Z}_n} \mathcal{L}_\alpha$$

where $\mathcal{L}_\alpha = \{f \in \pi_* \mathcal{O}_V |\ g(f) = g^{-\alpha} f \ \ \forall g \in \mu_n\}$ is the invertible sheaf associated to the character α.
The sheaves $\mathcal{L}_\alpha$ have the following properties:

- $\mathcal{L}_0 \simeq \mathcal{O}_U$
- $\mathcal{L}_\alpha \otimes \mathcal{L}_\beta \simeq \mathcal{L}_\alpha \cdot \mathcal{L}_\beta = \mathcal{L}_{\alpha+\beta}$
- $\mathcal{L}_\alpha \neq \mathcal{L}_\beta$ if $\alpha \neq \beta$
- $\pi^* \mathcal{L}_\alpha \simeq \mathcal{O}_V$

Conversely, given an invertible $\mathcal{O}_U$-module $\mathcal{L}$ belonging to the torsion subgroup of Pic(U), if n is the order of $\mathcal{L}$ then there exists a cyclic connected unramified covering $V \xrightarrow{\pi} U$ of degree n such that $\pi_* \mathcal{O}_V \simeq \bigoplus_{i=0}^{n-1} \mathcal{L}^{-i}$ as a $\mathcal{O}_U$-module.
Let now (V, p) be a normal singularity, a Weil divisor D on V is said to be $\mathbb{Q}$-Cartier if some nonzero power nD is locally principal at p and the smallest positive n with this property is called the index of D at p.
Possibly shrinking V we can assume that nD is principal, so that the divisorial sheaf $\mathcal{O}_V(nD)$ is the trivial sheaf $\mathcal{O}_V$.
Choosing a section $s \colon \mathcal{O}_V \xrightarrow{\sim} \mathcal{O}_V(nD)$, for every integer r it is defined an isomorphism $\mathcal{O}_V(rD) \xrightarrow{s} \mathcal{O}_V((r+n)D)$ and we have a structure of a coherent $\mathcal{O}_V$-algebra on
$\mathcal{A} = \bigoplus_{i=0}^{n-1} \mathcal{O}_V(-iD)$ by extending the natural products

$$\mathcal{O}_X(-iD) \times \mathcal{O}_X(-jD) \to \mathcal{O}_X(-(i+j)D) \qquad i+j<n$$
$$\mathcal{O}_X(-iD) \times \mathcal{O}_X(-jD) \to \mathcal{O}_X(-(i+j)D) \xrightarrow{s} \mathcal{O}_X(-(i+j-n)D) \ \ i+j \geq n$$

There exists a μ_n-action on $\mathcal{A}$ given by $\mu_r \ni \xi \colon f_i \to \xi^{-i} f_i$ for $f_i \in \mathcal{O}_X(-iD)$.
The μ_n-invariant subalgebra of $\mathcal{A}$ is exactly $\mathcal{O}_V$, thus the analytic spectrum of $\mathcal{A}$

$$V' = Specan_{\mathcal{O}_V} \mathcal{A} \xrightarrow{\pi} V$$

is a cyclic covering ramified over the locus where D is not principal.

$\pi^{-1}(Reg(V))$ is connected, hence $\pi^{-1}(p)$ consists of one point. $\mathcal{A}$ is reflexive and the morphism π is finite, this implies that V' is normal. Moreover $\pi^*\mathcal{O}(-D)_{|\pi^{-1}(Reg(V))}$ is the trivial sheaf and the divisor π^*D is principal.

V' is called the cyclic covering associated to divisor D. The cyclic covering associated to canonical divisor of a $\mathbb{Q}$-Gorenstein singularity is usually called the canonical covering.

Caution: The canonical covering of a $\mathbb{Q}$-Gorenstein singularity is a normal singularity with principal canonical divisor but, if the dimension is ≥ 3, in general it is not Cohen Macaulay (see for example [L-W] 5.11)

Lemma 4. *With the above notation (V,p) is a quotient singularity if and only if $(V',0)$ it is.*

Proof. Suppose first (V,p) is quotient, we can write $(V,p)=(\mathbb{C}^n,0)/G$ where G is a finite subgroup of $GL(n,\mathbb{C})$ and, according to Chevalley theorem, we can assume that the locus $E\subset\mathbb{C}^n$ where G does not act freely is a finite union of linear subspaces of codimension ≥ 2.

Let B be a small ball of centre 0 in some G-invariant norm and let $B\xrightarrow{\delta}V$ be the projection, then $\delta\colon B-E\to\delta(B-E)\subset V$ is the universal covering and by Riemann-Hartogs extension theorem there exists a subgroup $G'\subset G$ and a commutative diagram

$$\begin{array}{ccc} B/G' & \xrightarrow{f} & V' \\ \uparrow p & & \downarrow \pi \\ B & \xrightarrow{\delta} & V \end{array}$$

where f is a biholomorphism on $p(B-E)$. Both B/G and V' are normal, this implies that f is biholomorphic on $p(B)$. The converse is easy. □

Corollary. 5. *The canonical covering of a two dimensional quotient singularity is a rational double point.*

Proof. It follows from the fact that the R.D.P.'s are quotient Gorenstein and they are the only surface singularities with this property. □

The next step is to show that a cyclic covering of a smoothing of a two dimensional quotient singularity is again a smoothing of a quotient singularity. In order to show this we first need some preparatory material that we think to be of independent interest.

Let $f: X \to \Delta$ be a smoothing of a normal surface singularity $(X_0, 0)$ and let D be a Weil divisor on X which is $\mathbb{Q}$-Cartier of index n.
Let's denote by $Y = Specan_{\mathcal{O}_X}\mathcal{A}$, where $\mathcal{A} = \overset{n-1}{\underset{i=0}{\oplus}} \mathcal{O}_X(-iD)$, the cyclic covering associated to D. We have seen that Y is a normal singularity and, by the base change property of analytic spectrum, Y is a smoothing of $Y_0 = Specan_{\mathcal{O}_{X_0}}(\mathcal{A}\otimes_{\mathcal{O}_X} \mathcal{O}_{X_0})$.

Lemma 6. *In the above notation* $(Y_0, 0)$ *is isolated, reduced and irreducible.*

Proof. Y_0 is a generically reduced Cartier divisor on the normal space Y and this proves that it is reduced. To prove that Y_0 is irreducible is the same as to prove that $Y_0 - \{0\}$ is connected. We have a commutative diagram of continuous mappings

$$\begin{array}{ccc} Y_0 - \{0\} & \xrightarrow{j} & Y - \{0\} \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p} \\ X_0 - \{0\} & \xrightarrow{i} & X - \{0\} \end{array}$$

p is a regular covering of degree n and $Y_0 - \{0\}$ is the fibred product of $Y_0 - \{0\}$ and $X_0 - \{0\}$. Since the immersion i induces a surjection of the respective fundamental groups, the index of $\pi_1(Y_0 - \{0\})$ in $\pi_1(X_0 - \{0\})$ is exactly n. The proof follows by an elementary argument of algebraic topology. $\square$

Choosing possibly a linearly equivalent divisor we can assume that the support of D does not contain X_0. Setting D_0 the restriction of D to the normal singularity X_0 we have

$$\mathcal{A}_0 = \overset{n-1}{\underset{i=0}{\oplus}} \mathcal{O}_{X_0}(-iD_0) = (\mathcal{A} \otimes \mathcal{O}_{X_0})^{\vee\vee}$$

and we infer from the following lemma that the natural map $\mathcal{A} \otimes \mathcal{O}_{X_0} \to \mathcal{A}_0$ is injective.

Lemma 7. *Let X be a normal complex space and let $f: X \to \mathbb{C}$ be a holomorphic map such that $X_0 = f^{-1}(0)$ is normal and the closed set $X_0 \cap Sing(X)$ has codimension ≥ 2 in X_0.*
If $\mathcal{F}$ is a coherent reflexive sheaf on X, locally free on $Reg(X)$, then $\mathcal{F}_0 = \mathcal{F} \otimes \mathcal{O}_{X_0}$ is torsion free.

Proof. This is a well known result, we give a proof for completeness' sake.
Denoting by $i: Reg(X) \to X$ the inclusion and by $i_0: Reg(X) \cap X_0 \to X_0$ its restriction we have three exact sequences

$$\mathcal{F}(-X_0) = \mathcal{F} \otimes \mathcal{O}_X(-X_0) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

$$0\longrightarrow i^*\mathcal{F}(-X_0)\longrightarrow i^*\mathcal{F}\longrightarrow i_0^*\mathcal{F}_0\longrightarrow 0$$

$$0\longrightarrow i_*i^*\mathcal{F}(-X_0)\longrightarrow i_*i^*\mathcal{F}\longrightarrow i_{0*}i_0^*\mathcal{F}_0$$

The exactness of the second follows from the flatness of $\mathcal{F}$ on $Reg(X)$ and the third is obtained from the previous sequence by applying the left exact functor i_*. Reflexivity is a local property, therefore the tensor product of a reflexive sheaf by an invertible sheaf is again reflexive and the last sequence becomes

$$0\longrightarrow\mathcal{F}(-X_0)\longrightarrow\mathcal{F}\longrightarrow i_{0*}i_0^*\mathcal{F}_0$$

Comparing this sequence with the first one we get finally $\mathcal{F}_0\subset i_{0*}i_0^*\mathcal{F}_0$. □

It is clear that the map $\mathcal{A}\otimes\mathcal{O}_{X_0}\to\mathcal{A}_0$ is an isomorphism if and only if Y_0 is normal, or equivalently if and only if Y is Cohen-Macaulay. In general $Y_0'=Specan_{\mathcal{O}_{X_0}}\mathcal{A}_0$ is the normalization of Y_0.

We also observe that D_0 is a $\mathbb{Q}$-Cartier divisor on X_0 of index n, in fact it is obvious that nD_0 is principal and if rD_0 is principal for some $0<r<n$ then $Y_0'-\{0\}=Y_0-\{0\}$ will be not connected.

Theorem 8. *Let $X\to\Delta$ be a smoothing of a two dimensional quotient singularity X_0 and let $Y\to X$ be the cyclic covering associated to some $\mathbb{Q}$-Cartier divisor, then Y_0 is a quotient singularity.*

Proof. It suffices to show that Y_0 is normal, let Y_0' be its normalization and Z a resolution of Y_0'

$$Z\xrightarrow{\pi}Y_0'\longrightarrow Y_0$$

The geometric genus of Y_0 is by definition $p_g(Y_0)=h^1(\mathcal{O}_Z)-h^0(\mathcal{O}_{Y_0'}/\mathcal{O}_{Y_0})$ and, since Y_0 is smoothable, by using Steenbrink's formula we get $2p_g(Y_0)=\mu_0+\mu_+\geq 0$. Now Y_0' is a quotient singularity hence $h^1(\mathcal{O}_Z)=0$ and then $h^1(\mathcal{O}_Z)=h^0(\mathcal{O}_{Y_0'}/\mathcal{O}_{Y_0})=0$. □

Corollary. 9. *Let $X\xrightarrow{f}\Delta$ be a $\mathbb{Q}$-Gorenstein smoothing of index n of a quotient singularity.*

Then the torsion part of the Picard group of the Milnor fibre is finite cyclic of order n and it is generated by the canonical sheaf.

Proof. Let $Y\xrightarrow{\pi}X$ be the canonical covering of X, by Theorem 8 Y is a smoothing of a rational double point and the Milnor fibre of the smoothing $f\circ\pi$ is simply connected.

If we denote by F the Milnor fibre of X and by $Pic(F)_0, H^i(F)_0$ the torsion part of the respective groups then, since F is Stein

$$Pic(F)_0 = H^2(F)_0 = H_1(F)_0 = \pi_1(F) = \mathbb{Z}_n$$

In order to prove the assertion about K_F, by adjunction formula, it is sufficient to show that the restriction map $Pic(X - \{0\})_0 \to Pic(F)_0$ is an isomorphism.
X is Cohen Macaulay, by using local cohomology and exponential exact sequence we get

$$0 = H^1(\mathcal{O}_{X-\{0\}}) \longrightarrow Pic(X - \{0\}) \longrightarrow H^2(X - \{0\}) \longrightarrow H^2(\mathcal{O}_{X-\{0\}})$$

and since the group on the right is torsion free

$$Pic(X - \{0\})_0 = H^2(X - \{0\})_0 = H_1(X - \{0\})_0 = \pi_1(X - \{0\}) = \mathbb{Z}_n$$

We observe that the elements of $Pic(X - \{0\})_0$ (resp $Pic(F)_0$) are exactly the eigensheaves of $\pi_*\mathcal{O}_{Y-\{0\}}$ (resp $\pi_*\mathcal{O}_{\pi^{-1}(F)}$) and the conclusion is now trivial. □

Remark. In some situation the Corollary 9 gives a necessary condition to have the Milnor fibre F of a $\mathbb{Q}$-Gorenstein smoothing of a quotient singularity contained as an open set in a smooth surface S.
In fact, if $K_S = rD$ for some $r > 0$ and $D \in Pic(S)$, since $K_{S|F}$ generate $Pic(F)_0$, r must be relatively prime to the index of the smoothing (see [Ma1] for applications).

§3 . Statement and proof of the main theorem

Let $(X, 0) \xrightarrow{\pi'} (\mathbb{C}, 0)$ be a $\mathbb{Q}$-Gorenstein smoothing of index n of a two dimensional quotient singularity $(X_0, 0)$. If $n = 1$, that is X Gorenstein, then also $(X_0, 0)$ is Gorenstein and hence a rational double point. Since for every R.D.P. its semiuniversal deformation is well known, we fix our attention to the case $n > 1$.

Theorem 10. *In the notation above if $n > 1$ then $(X, 0)$ is analytically isomorphic to $(Y, 0)/G$ where:*

a) $(Y, 0) \subset (\mathbb{C}^4, 0)$ is an isolated hypersurface singularity defined by

$$F = uv + y^{dn} - t^b - \varphi_1(t)y^n - \ldots - \varphi_{d-1}(t)y^{(d-1)n} = 0$$

for some integers d, b.

The φ_i's are convergent power series satisfying $\varphi_i \in \mathbb{C}\{t\}$, $\varphi_i(0) = 0$ and such that the projection on the t-axis $(Y,0) \xrightarrow{\pi} (\mathbb{C},0)$ is a smoothing of $(Y_0,0)$.
b) $G \simeq \mu_n$ acts on $\mathbb{C}^4$ in the following way

$$\mu_n \ni \xi\colon (u,v,y,t) \to (\xi u, \xi^{-1} v, \xi^a y, t) \qquad \textit{with } (a,n) = 1$$

c) π' is obtained from π by passing to the quotient.

Remark. If $(Y,0)$ and the G-action are defined as in a) and b), then G acts, locally around 0, freely on $Y - \{0\}$, hence π' defines a smoothing of $(X_0,0) = (Y_0,0)/G$ which, as we have seen, is a cyclic singularity of type $\dfrac{1}{dn^2}(1, dna-1)$.

Proof. Define $(Y,0)$ as the canonical covering of $(X,0)$, if $d > 0$ is the topological Euler Poincaré characteristic of the Milnor fibre of π' then $(Y_0,0)$ is a R.D.P. of type A_{dn-1}, D_{dn-1} or E_{dn-1}.
By Lemma 1 we can assume that $(Y,0) \subset (\mathbb{C}^3 \times \mathbb{C}, 0)$ is a hypersurface singularity defined by

$$F = f(x,y,z) + t\varphi(x,y,z,t) = 0$$

where f is the equation of the R.D.P. $(Y_0,0)$ and the group $G \simeq \mu_n$ acts linearly on $\mathbb{C}^3 \times \mathbb{C}$, diagonally on $\mathbb{C}^3$ and trivially on $\mathbb{C}$.
We now are going to determine all the possible μ_n-actions on $(Y,0)$ which satisfy this condition. The main tool necessary for such computation is the classification of finite groups of automorphisms of R.D.P's.
Let $X = \mathbb{C}^2/G$, $G \subset SL(2,\mathbb{C})$ be a rational double point and let τ be an automorphism of $(X,0)$. Since $\mathbb{C}^2 - \{0\} \to X - \{0\}$ is a regular covering with group G, $\tau_{|X-\{0\}}$ lifts to exactly $|G|$ automorphisms of $\mathbb{C}^2 - \{0\}$ which, by Riemann-Hartogs extension theorem, extend to automorphisms of $(\mathbb{C}^2,0)$.
If $H \subset Aut(X,0)$ is a finite subgroup and $\Gamma \subset Aut(\mathbb{C}^2,0)$ is the set of lifted automorphisms, we have an exact sequence of groups

$$0 \longrightarrow G \longrightarrow \Gamma \longrightarrow H \longrightarrow 0$$

By Lemma 1 we can assume that $\Gamma \subset GL(2,\mathbb{C})$ and we can study the action of Γ in the polynomial ring $\mathbb{C}[w_1,w_2]$.
The group $\Gamma/G \simeq H$ acts faithfully on $\mathbb{C}[w_1,w_2]^G$, moreover every $\gamma \in \Gamma$ defines a graded automorphism of $\mathbb{C}[w_1,w_2]$. Thus H is contained in the group, which we denote by $\hat{Aut}(X,0)$, of graded automorphisms of $\mathbb{C}[w_1,w_2]^G$. (see [Ca] for details).

We see in particular that $\hat{Aut}(X,0)$ is the quotient, by G, of the normalizer of G in $GL(2,\mathbb{C})$. The computation of $\hat{Aut}(X,0)$ is elementary and not difficult, but very useful in the study of quotient singularities. In ([Ca] Th 1.2) Catanese has computed the groups $\hat{Aut}(X,0)$ for every R.D.P. $(X,0)$.

In order to describe these groups and to make our classification we shall consider every R.D.P. in the form $f=0$ where $f \in \mathbb{C}[x,y,z]$ is the equation given in the following list

Table.

Type	Equation	
E_6	$z^2+x^3+y^4$	
E_7	$z^2+x(y^3+x^2)$	
E_8	$z^2+x^3+y^5$	
D_n	$z^2+x(y^2+x^{n-2})$	$n\geq 4$
A_{n-1}	$z^2+x^2+y^n$	$n\geq 2$
A_{n-1}	$uv+y^n$	$n\geq 2$

The second equation for A_{n-1} is obtained from the first after the linear coordinates change

$$\begin{cases} u = & z+ix \\ v = & z-ix \end{cases}$$

Let us consider now the following problem.

Problem 11: Let $(Y_0,0)\subset(\mathbb{C}^3,0)$ be a fixed rational double point defined by the equation $f(x,y,z)=0$ and let g be an automorphism of $(Y_0,0)$ induced by an element $g\in GL(3,\mathbb{C})$ of finite order.

Does it exist a holomorphic germ $\varphi(x,y,z,t)$ such that the automorphism g, extended by $gt=t$, acts on the hypersurface singularity $(Y,0)\subset(\mathbb{C}^3\times\Delta,0)$ defined by

$$F(x,y,z,t)=f(x,y,z)+t\varphi(x,y,z,t)=0$$

and gives, locally around 0, a free action on $Y-\{0\}$?

Let us suppose to have the following μ_n-action

$$\mu_n\ni\xi\colon(x,y,z,t,f)\to(\xi^a x,\xi^b y,\xi^c z,t,\xi^e f)$$

then we have two necessary conditions to extend this action freely on $Y-\{0\}$.

i) e must be equal to 0 modulo n.

In fact $(Y,0)$ is μ_n-stable: moreover the line $R=\{x=y=z=0\}$ is μ_n-invariant, thus $R\cap Y=0$ and $F(0,0,0,t)\not\equiv 0$.

ii) a, b, c must be invertible in $\mathbb{Z}_n$.
In fact, if $E \subset \mathbb{C}^4$ denotes the locus where the action is not free, then $E \cap Y = 0$ implies, for dimensional reasons, that E is the union of a finite number of lines.
Consider now the various cases separately. In each case $(Y_0, 0)$ denotes the R.D.P. of corresponding type defined as in the previous list.
1) Type $E_8 = E_{dn-1}$

$$f = z^2 + x^3 + y^5 \qquad\qquad dn = 8 + 1 = 9$$

$\hat{Aut}(Y_0, 0) = \mathbb{C}^*$ where

$$\mathbb{C}^* \ni \lambda\colon (x, y, z, f) \to (\lambda^{10}x, \lambda^6 y, \lambda^{15}z, \lambda^{30}f)$$

If $\mu_n \subset \hat{Aut}(Y_0, 0)$ then there are two subcases

$$d = 1 \qquad \mu_9 \ni \xi\colon (x, y, z, f) \to (\xi x, \xi^6 y, \xi^6 z, \xi^3 f)$$
$$d = 3 \qquad \mu_3 \ni \xi\colon (x, y, z, f) \to (\xi x, y, z, f)$$

Both of them don't satisfy i) and ii).
2) Type E_7

$$f = z^2 + x(y^3 + x^2) \qquad\qquad dn = 8$$

$\hat{Aut}(Y_0, 0) = \mathbb{C}^*$ where

$$\mathbb{C}^* \ni \lambda\colon (x, y, z, f) \to (\lambda^6 x, \lambda^4 y, \lambda^9 z, \lambda^{18} f)$$

We have three subcases

$$d = 1 \qquad \mu_8 \ni \xi\colon (x, y, z, f) \to (\xi^6 x, \xi^4 y, \xi z, \xi^2 f)$$
$$d = 2 \qquad \mu_4 \ni \xi\colon (x, y, z, f) \to (\xi^2 x, y, \xi z, \xi^2 f)$$
$$d = 4 \qquad \mu_2 \ni \xi\colon (x, y, z, f) \to (x, y, \xi z, f)$$

Both of them don't satisfy i) and ii).
3) Type E_6

$$f = z^2 + x^3 + y^4 \qquad\qquad dn = 7$$

$\hat{Aut}(Y_0, 0) = \mathbb{C}^* \times \mathbb{Z}_2$ where

$$\mathbb{C}^* \ni \lambda\colon (x, y, z, f) \to (\lambda^4 x, \lambda^3 y, \lambda^6 z, \lambda^{12} f)$$

and $\mathbb{Z}_2$ is generated by the involution $\tau\colon (x, y, z) \to (x, y, -z)$.

the only subcase is

$$d = 1 \qquad \mu_7 \ni \xi \colon (x, y, z, f) \to (\xi^4 x, \xi^3 y, \xi^6 z, \xi^5 f)$$

which does not satisfy i).

4) Type D_4

$$f = z^2 + x(y^2 + x^2) \qquad\qquad dn = 5$$

$\hat{Aut}(Y_0, 0)$ is the subgroup of $GL(2, \mathbb{C}) \times \mathbb{C}^*$ of all the elements (γ, λ) such that

$$\gamma^*(x(y^2 + x^2)) = \lambda^2(x(y^2 + x^2))$$

where γ acts on $\begin{pmatrix} x \\ y \end{pmatrix}$ and λ on z.

If we denote $H \simeq \mathbb{C}^* \subset \hat{Aut}(Y_0, 0)$ the subgroup of elements

$$\left(\begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix}, t^3 \right) \qquad t \in \mathbb{C}^*$$

then $\hat{Aut}(Y_0, 0)/H \simeq \sum_3 =$ symmetric group on three elements.

Since $|\sum_3| = 6$, it must then be $\mu_5 \subset H$, and the action is

$$\mu_5 \ni \xi \colon (x, y, z, f) \to (\xi^2 x, \xi^2 y, \xi^3 z, \xi f)$$

which does not satisfy i).

5) Type D_r $\qquad r \geq 5$

$$f = z^2 + x(y^2 + x^{r-2}) \qquad\qquad dn = r + 1$$

$\hat{Aut}(Y_0, 0) = \mathbb{C}^* \times \mathbb{Z}_2$ where

$$\mathbb{C}^* \ni \lambda \colon (x, y, z, f) \to (\lambda^2 x, \lambda^{r-2} y, \lambda^{r-1} z, \lambda^{2r-2} f)$$

and $\mathbb{Z}_2$ is generated by the involution $\tau \colon (x, y, z) \to (x, y, -z)$.

In every case the action is diagonal and we have

$$\mu_n \ni \xi \colon (x, f) \to (\xi^2 x, \xi^{-4} f)$$

We must distinguish two subcases:

i) $n \neq 2, 4$: i) is not satisfied.

ii) $n = 2, 4$: ii) is not satisfied.

6) Type A_1 $\qquad n=2$

If f is a quadratic form defining $(Y_0,0)$ we have

$$\hat{Aut}(Y_0,0)=\{g\in GL(3,\mathbb{C})|g^*f=\pm f\}$$

The only diagonal automorphism of $\mathbb{C}^3$ satisfying ii) is $-Id$ and we can choose as f every nondegenerate quadratic form, for example $f=uv+y^2$.

7) Type A_r $\qquad r\geq 2$

$$f=uv+y^{r+1} \qquad\qquad dn=r+1$$

$\hat{Aut}(Y_0,0)=(\mathbb{C}^*)^2\prod\mathbb{Z}_2$ where

$$(\mathbb{C}^*)^2\ni(\lambda_1,\lambda_2)\colon(u,v,y)\to(\lambda_2u,\lambda_1^{r+1}\lambda_2^{-1}v,\lambda_1y)$$

and $\mathbb{Z}_2$ is generated by the involution τ which exchanges u and v.

The product of $(\mathbb{C}^*)^2$ and $\mathbb{Z}_2$ is semidirect and the composition law is the following

$$\tau\cdot(\lambda_1,\lambda_2)\cdot\tau=(\lambda_1,\lambda_1^{r+1}\lambda_2^{-1})$$

If $\mu_n\subset\hat{Aut}(Y_0,0)$ and ϵ is a generator of μ_n then either $\epsilon=(\xi_1,\xi_2)$ or $\epsilon=(\xi_1,\xi_2)\cdot\tau$.

In the first subcase ξ_1 and ξ_2 are n^{th} roots of unity and the action is diagonal

$$\epsilon\colon(u,v,y)\to(\xi_2u,\xi_2^{-1}v,\xi_1y)$$

according to condition ii) ξ_2 must be primitive and we can write

$$\mu_n\ni\xi\colon(u,v,y)\to(\xi u,\xi^{-1}v,\xi^ay)$$

where a is an integer relatively prime to n.

In the second subcase, by composition law it follows immediately that ξ_1 is a n^{th} root of 1. By making the linear base change

$$\begin{cases} x=v+\xi_2u\\ z=-\xi_2^{-1}v+u\end{cases}$$

we have in the new system $f=\dfrac{1}{4\xi_2}(x^2-\xi_2^2z^2)+y^{r+1}$, and $\epsilon\colon(x,y,z)\to(x,-z,\xi_1y)$ which does not satisfy ii).

Summarizing, we have shown that $F = uv + y^{dn} + t\varphi(u, v, y, t)$ with $\varphi(0, 0, 0, t) \not\equiv 0$ and $G \simeq \mu_n$ acts as in b). The conclusion of the proof follows now from some standard manipulation in singularity theory which we omit.

For example, by generalised Morse lemma there exists a new system of coordinates of $\mathbb{C}^4$ such that F becomes $uv + y^{dn} + t\varphi(y, t)$ and the coordinate change can be taken in a G-equivariant way (see [Ma2] for details).

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Dr. Marco Manetti
Scuola Normale Superiore
Piazza Cavalieri 7
56100 Pisa, ITALY