A VOYAGE ROUND COALGEBRAS

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Abstract. I found the most ready way of explaining my employment was to ask them how it was that they themselves were not curious concerning earthquakes and volcanos? - why some springs were hot and others cold? - why there were mountains in Chile, and not a hill in La Plata? These bare questions at once satisfied and silenced the greater number; some, however (like a few in England who are a century behindhand), thought that all such inquiries were useless and impious; and that it was quite sufficient that God had thus made the mountains.
Charles Darwin: The voyage of the Beagle.

Through all the chapter we work over a fixed field $\mathbb{K}$ of characteristic 0. Unless otherwise specified all the tensor products are made over $\mathbb{K}$.

Notation. $\mathbf{G}$ is the category of graded vector spaces over $\mathbb{K}$.

The *tensor algebra* generated by $V \in \mathbf{G}$ is by definition the graded vector space

$$T(V) = \bigoplus_{n\geq 0} \bigotimes^n V$$

endowed with the associative product $(v_1 \otimes \cdots \otimes v_p)(v_{p+1} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$.

Let $V, W \in \mathbf{G}$. The *twist map* $\mathtt{tw}\colon V \otimes W \to W \otimes V$ is defined by the rule $\mathtt{tw}(v \otimes w) = (-1)^{\overline{v}\,\overline{w}} w \otimes v$, for every pair of homogeneous elements $v \in V$, $w \in W$.

The following convention is adopted in force: let V, W be graded vector spaces and $F\colon T(V) \to T(W)$ a linear map. We denote by

$$F^i\colon T(V) \to \bigotimes^i W, \quad F_j\colon \bigotimes^j V \to T(W), \quad F^i_j\colon \bigotimes^j V \to \bigotimes^i W$$

the compositions of F with the inclusion $\bigotimes^j V \to T(V)$ and/or the projection $T(W) \to \bigotimes^i W$.

1. Graded coalgebras

Definition 1.1. A coassociative $\mathbb{Z}$-graded coalgebra is the data of a graded vector space $C = \oplus_{n\in\mathbb{Z}} C^n \in \mathbf{G}$ and of a coproduct $\Delta\colon C \to C \otimes C$ such that:

- Δ is a morphism of graded vector spaces.
- (coassociativity) $(\Delta \otimes \mathrm{Id}_C)\Delta = (\mathrm{Id}_C \otimes \Delta)\Delta\colon C \to C \otimes C \otimes C$.

For simplicity of notation, from now on with the term *graded coalgebra* we intend a $\mathbb{Z}$-graded coassociative coalgebra.

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Definition 1.2. Let (C,Δ) and (B,Γ) be graded coalgebras. A *morphism* of graded coalgebras $f\colon C\to B$ is a morphism of graded vector spaces that commutes with coproducts, i.e.

$$\Gamma f=(f\otimes f)\Delta\colon C\to B\otimes B.$$

The category of graded coalgebras is denoted by $\mathbf{GC}$.

Example 1.3. Let $C=\mathbb{K}[t]$ be the polynomial ring in one variable t (of degree 0). The linear map

$$\Delta\colon \mathbb{K}[t]\to\mathbb{K}[t]\otimes\mathbb{K}[t],\qquad \Delta(t^n)=\sum_{i=0}^{n}t^i\otimes t^{n-i},$$

gives a coalgebra structure (exercise: check coassociativity).
For every sequence $f_n\in\mathbb{K}$, $n>0$, it is associated a morphism of coalgebras $f\colon C\to C$ defined as

$$f(1)=1,\qquad f(t^n)=\sum_{s=1}^{n}\sum_{\substack{(i_1,\dots,i_s)\in\mathbb{N}^s\\ i_1+\cdots+i_s=n}}f_{i_1}f_{i_2}\cdots f_{i_s}t^s.$$

The verification that $\Delta f=(f\otimes f)\Delta$ can be done in the following way: Let $\{x^n\}\subset C^\vee=\mathbb{K}[[x]]$ be the dual basis of $\{t^n\}$. Then for every $a,b,n\in N$ we have:

$$\langle x^a\otimes x^b,\Delta f(t^n)\rangle=\sum_{i_1+\cdots+i_a+j_1+\cdots+j_b=n}f_{i_1}\cdots f_{i_a}f_{j_1}\cdots f_{j_b},$$

$$\langle x^a\otimes x^b,f\otimes f\Delta(t^n)\rangle=\sum_{s}\sum_{i_1+\cdots+i_a=s}\sum_{j_1+\cdots+j_b=n-s}f_{i_1}\cdots f_{i_a}f_{j_1}\cdots f_{j_b}.$$

Note that the sequence $\{f_n\}$, $n\geq 1$, can be recovered from f by the formula $f_n=\langle x,f(t^n)\rangle$.

Example 1.4. Let A be a graded associative algebra with product $\mu\colon A\otimes A\to A$ and C a graded coassociative coalgebra with coproduct $\Delta\colon C\to C\otimes C$.
Then $\mathrm{Hom}^*(C,A)$ is a graded associative algebra by the *convolution product*

$$fg=\mu(f\otimes g)\Delta.$$

We left as an exercise the verification that the product in $\mathrm{Hom}^*(C,A)$ is associative.
In particular $\mathrm{Hom}_{\mathbf{G}}(C,A)=\mathrm{Hom}^0(C,A)$ is an associative algebra and $C^\vee=\mathrm{Hom}^*(C,\mathbb{K})$ is a graded associative algebra.

Remark 1.5. The above example shows in particular that the dual of a coalgebra is an algebra. In general the dual of an algebra is not a coalgebra (with some exceptions, see e.g. Example 2.3). Heuristically, this asymmetry comes from the fact that, for an infinite dimensional vector space V, there exist a natural map $V^\vee\otimes V^\vee\to(V\otimes V)^\vee$, while does not exist any natural map $(V\otimes V)^\vee\to V^\vee\otimes V^\vee$.

Example 1.6. The dual of the coalgebra $C=\mathbb{K}[t]$ (Example 1.3) is exactly the algebra of formal power series $A=\mathbb{K}[[x]]=C^\vee$. Every coalgebra morphism $f\colon C\to C$ induces a local homomorphism of $\mathbb{K}$-algebras $f^t\colon A\to A$. The morphism f^t is uniquely determined by the power series $f^t(x)=\sum_{n>0}f_nx^n$ and then every morphism of coalgebras $f\colon C\to C$ is uniquely determined by the sequence $f_n=\langle f^t(x),t^n\rangle=\langle x,f(t^n)\rangle$.
The map $f\mapsto f^t$ is functorial and then preserves the composition laws.

Definition 1.7. Let (C,Δ) be a graded coalgebra; the iterated coproducts $\Delta^n\colon C\to C^{\otimes n+1}$ are defined recursively for $n\geq 0$ by the formulas

$$\Delta^0=\mathrm{Id}_C,\qquad \Delta^n\colon C\xrightarrow{\Delta}C\otimes C\xrightarrow{\mathrm{Id}_C\otimes\Delta^{n-1}}C\otimes C^{\otimes n}=C^{\otimes n+1}.$$

Lemma 1.8. *Let (C,Δ) be a graded coalgebra. Then:*

(1) *For every $0\le a\le n-1$ we have*

$$\Delta^n=(\Delta^a\otimes\Delta^{n-1-a})\Delta\colon C\to\bigotimes^{n+1}C.$$

(2) *For every $s\ge 1$ and every $a_0,\dots,a_s\ge 0$ we have*

$$(\Delta^{a_0}\otimes\Delta^{a_1}\otimes\cdots\otimes\Delta^{a_s})\Delta^s=\Delta^{s+\sum a_i}.$$

(3) *If $f\colon(C,\Delta)\to(B,\Gamma)$ is a morphism of graded coalgebras then, for every $n\ge 1$ we have*

$$\Gamma^n f=(\otimes^{n+1}f)\Delta^n\colon C\to\bigotimes^{n+1}B.$$

Proof. [1] If $a=0$ or $n=1$ there is nothing to prove, thus we can assume $a>0$ and use induction on n. we have:

$$(\Delta^a\otimes\Delta^{n-1-a})\Delta=((\mathrm{Id}_C\otimes\Delta^{a-1})\Delta\otimes\Delta^{n-1-a})\Delta=$$

$$=(\mathrm{Id}_C\otimes\Delta^{a-1}\otimes\Delta^{n-1-a})(\Delta\otimes\mathrm{Id}_C)\Delta=$$

$$=(\mathrm{Id}_C\otimes\Delta^{a-1}\otimes\Delta^{n-1-a})(\mathrm{Id}_C\otimes\Delta)\Delta=(\mathrm{Id}_C\otimes(\Delta^{a-1}\otimes\Delta^{n-1-a})\Delta)\Delta=\Delta^n.$$

[2] Induction on s, being the case $s=1$ proved in item 1. If $s\ge 2$ we can write

$$(\Delta^{a_0}\otimes\Delta^{a_1}\otimes\cdots\otimes\Delta^{a_s})\Delta^s=(\Delta^{a_0}\otimes\Delta^{a_1}\otimes\cdots\otimes\Delta^{a_s})(\mathrm{Id}_C\otimes\Delta^{s-1})\Delta=$$

$$(\Delta^{a_0}\otimes(\Delta^{a_1}\otimes\cdots\otimes\Delta^{a_s})\Delta^{s-1})\Delta=(\Delta^{a_0}\otimes\Delta^{s-1+\sum_{i>0}a_i})\Delta=\Delta^{s+\sum a_i}.$$

[3] By induction on n,

$$\Gamma^n f=(\mathrm{Id}_B\otimes\Gamma^{n-1})\Gamma f=(f\otimes\Gamma^{n-1}f)\Delta=(f\otimes(\otimes^n f)\Delta^{n-1})\Delta=(\otimes^{n+1}f)\Delta^n.$$

□

Definition 1.9. Let (C,Δ) be a graded coalgebra and $p\colon C\to V$ a morphism of graded vector spaces. We shall say that p is a *system of cogenerators* of C if for every $c\in C$ there exists $n\ge 0$ such that $(\otimes^{n+1}p)\Delta^n(c)\ne 0$ in $\bigotimes^{n+1}V$.

Example 1.10. In the notation of Example 1.3, the natural projection $\mathbb{K}[t]\to\mathbb{K}\oplus\mathbb{K}t$ is a system of cogenerators.

Proposition 1.11. *Let $p\colon B\to V$ be a system of cogenerators of a graded coalgebra (B,Γ).*
Then every morphism of graded coalgebras $\phi\colon(C,\Delta)\to(B,\Gamma)$ is uniquely determined by its composition $p\phi\colon C\to V$.

Proof. Let $\phi,\psi\colon(C,\Delta)\to(B,\Gamma)$ be two morphisms of graded coalgebras such that $p\phi=p\psi$. In order to prove that $\phi=\psi$ it is sufficient to show that for every $c\in C$ and every $n\ge 0$ we have

$$(\otimes^{n+1}p)\Gamma^n(\phi(c))=(\otimes^{n+1}p)\Gamma^n(\psi(c)).$$

By Lemma 1.8 we have $\Gamma^n\phi=(\otimes^{n+1}\phi)\Delta^n$ and $\Gamma^n\psi=(\otimes^{n+1}\psi)\Delta^n$. Therefore

$$(\otimes^{n+1}p)\Gamma^n\phi=(\otimes^{n+1}p)(\otimes^{n+1}\phi)\Delta^n=(\otimes^{n+1}p\phi)\Delta^n=$$
$$=(\otimes^{n+1}p\psi)\Delta^n=(\otimes^{n+1}p)(\otimes^{n+1}\psi)\Delta^n=(\otimes^{n+1}p)\Gamma^n\psi.$$

□

Definition 1.12. Let (C,Δ) be a graded coalgebra. A linear map $d\in \mathrm{Hom}^n(C,C)$ is called a *coderivation of degree* n if it satisfies the *coLeibniz rule*

$$\Delta d=(d\otimes \mathrm{Id}_C+\mathrm{Id}_C\otimes d)\Delta.$$

A coderivation d is called a *codifferential* if $d^2=d\circ d=0$.
More generally, if $\theta\colon C\to D$ is a morphism of graded coalgebras, a morphism of graded vector spaces $d\in\mathrm{Hom}^n(C,D)$ is called a coderivation of degree n (with respect to θ) if

$$\Delta_D d=(d\otimes\theta+\theta\otimes d)\Delta_C.$$

In the above definition we have adopted the Koszul sign convention: i.e. if $x,y\in C$, $f,g\in\mathrm{Hom}^*(C,D)$, $h,k\in\mathrm{Hom}^*(B,C)$ are homogeneous then $(f\otimes g)(x\otimes y)=(-1)^{\overline{g}\,\overline{x}}f(x)\otimes g(y)$ and $(f\otimes g)(h\otimes k)=(-1)^{\overline{g}\,\overline{h}}fh\otimes gk$.

The coderivations of degree n with respect a coalgebra morphism $\theta\colon C\to D$ form a vector space denoted $\mathrm{Coder}^n(C,D;\theta)$.
For simplicity of notation we denote $\mathrm{Coder}^n(C,C)=\mathrm{Coder}^n(C,C;Id)$.

Lemma 1.13. *Let* $C\xrightarrow{\theta}D\xrightarrow{\rho}E$ *be morphisms of graded coalgebras. The compositions with* θ *and* ρ *induce linear maps*

$$\rho_*\colon \mathrm{Coder}^n(C,D;\theta)\to\mathrm{Coder}^n(C,E;\rho\theta),\qquad f\mapsto\rho f;$$
$$\theta^*\colon \mathrm{Coder}^n(D,E;\rho)\to\mathrm{Coder}^n(C,E;\rho\theta),\qquad f\mapsto f\theta.$$

Proof. Immediate consequence of the equalities

$$\Delta_E\rho=(\rho\otimes\rho)\Delta_D,\qquad \Delta_D\theta=(\theta\otimes\theta)\Delta_C.$$

□

Lemma 1.14. *Let* $C\xrightarrow{\theta}D$ *be morphisms of graded coalgebras and let* $d\colon C\to D$ *be a* θ*-coderivation. Then:*

(1) *For every* n

$$\Delta_D^n\circ d=(\sum_{i=0}^n\theta^{\otimes i}\otimes d\otimes\theta^{\otimes n-i})\circ\Delta_C^n.$$

(2) *If* $p\colon D\to V$ *is a system of cogenerators, then* d *is uniquely determined by its composition* $pd\colon C\to V$.

Proof. The first item is a straightforward induction on n, using the equalities $\Delta^n=\mathrm{Id}\otimes\Delta^{n-1}$ and $\theta^{\otimes i}\Delta_C^{i-1}=\Delta_D^{i-1}\theta$.
For item 2, we need to prove that $pd=0$ implies $d=0$. Assume that there exists $c\in C$ such that $dc\neq 0$, then there exists n such that $p^{\otimes n+1}\Delta_D^n dc\neq 0$. On the other hand

$$p^{\otimes n+1}\Delta_D^n dc=(\sum_{i=0}^n(p\theta)^{\otimes i}\otimes pd\otimes(p\theta)^{\otimes n-i})\circ\Delta_C^n c=0.$$

□

Exercise 1.15. A *counity* of a graded coalgebra is a morphism of graded vector spaces $\epsilon\colon C\to\mathbb{K}$ such that $(\epsilon\otimes\mathrm{Id}_C)\Delta=(\mathrm{Id}_C\otimes\epsilon)\Delta=\mathrm{Id}_C$. Prove that if a counity exists, then it is unique (Hint: $(\epsilon\otimes\epsilon')\Delta=?$).

Exercise 1.16. Let (C,Δ) be a graded coalgebra. A graded subspace $I\subset C$ is called a *coideal* if $\Delta(I)\subset C\otimes I+I\otimes C$. Prove that a subspace is a coideal if and only if is the kernel of a morphism of coalgebras.

Exercise 1.17. Let C be a graded coalgebra and $d \in \operatorname{Coder}^1(C,C)$ a codifferential of degree 1. Prove that the triple $(L, \delta, [,])$, where:

$$L = \oplus_{n\in\mathbb{Z}} \operatorname{Coder}^n(C,C), \quad [f,g] = fg - (-1)^{\overline{g}\,\overline{f}} gf, \quad \delta(f) = [d,f]$$

is a differential graded Lie algebra.

2. Connected coalgebras

Definition 2.1. A graded coalgebra (C,Δ) is called *nilpotent* if $\Delta^n = 0$ for $n >> 0$. It is called *locally nilpotent* if it is the direct limit of nilpotent graded coalgebras or equivalently if $C = \cup_n \ker \Delta^n$.

Example 2.2. The vector space

$$\overline{\mathbb{K}[t]} = \{p(t) \in \mathbb{K}[t] \mid p(0) = 0\} = \bigoplus_{n>0} \mathbb{K}t^n$$

with the coproduct

$$\Delta\colon \overline{\mathbb{K}[t]} \to \overline{\mathbb{K}[t]} \otimes \overline{\mathbb{K}[t]}, \qquad \Delta(t^n) = \sum_{i=1}^{n-1} t^i \otimes t^{n-i},$$

is a locally nilpotent coalgebra. The projection $\mathbb{K}[t] \to \overline{\mathbb{K}[t]}$, $p(t) \to p(t) - p(0)$, is a morphism of coalgebras.

Example 2.3. Let $A = \oplus A_i$ be a finite dimensional graded associative commutative $\mathbb{K}$-algebra and let $C = A^\vee = \operatorname{Hom}^*(A, \mathbb{K})$ be its graded dual.
Since A and C are finite dimensional, the pairing $\langle c_1 \otimes c_2, a_1 \otimes a_2\rangle = (-1)^{\overline{a_1}\,\overline{c_2}}\langle c_1, a_1\rangle\langle c_2, a_2\rangle$ gives a natural isomorphism $C \otimes C = (A \otimes A)^\vee$ commuting with the twisting maps T; we may define Δ as the transpose of the multiplication map $\mu\colon A \otimes A \to A$.
Then (C,Δ) is a coassociative cocommutative coalgebra. Note that C is nilpotent if and only if A is nilpotent.

Exercise 2.4. Let (C,Δ) be a graded coalgebra. Prove that for every $a, b \geq 0$

$$\Delta^a(\ker \Delta^{a+b}) \subset \bigotimes^{a+1} (\ker \Delta^b).$$

(Hint: prove first that $\Delta^a(\ker \Delta^{a+b}) \subset \ker \Delta^b \otimes C^{\otimes a}$.)

Exercise 2.5. Let (C,Δ) be a locally nilpotent graded coalgebra. Prove that every projection $p\colon C \to \ker \Delta$ is a system of cogenerators.

Definition 2.6 ([8, p. 282])**.** A graded coalgebra (C,Δ) is called *connected* if there is an element $1 \in C$ such that $\Delta(1) = 1 \otimes 1$ (in particular $\deg(1) = 0$) and $C = \cup_{r=0}^{+\infty} F_rC$, where F_rC is defined recursively by the formulas

$$F_0C = \mathbb{K}\,1, \qquad F_{r+1}C = \{x \in C \mid \Delta(x) - 1 \otimes x - x \otimes 1 \in F_rC \otimes F_rC\}.$$

Example 2.7. Every locally nilpotent coalgebra is connected (with $1 = 0$, see Exercise 2.4). If $f\colon C \to D$ is a surjective morphism of coalgebras and C is connected, then also D is connected.

Lemma 2.8. *Let C be a connected coalgebra and $e \in C$ such that $\Delta(e) = e \otimes e$. Then either $e = 0$ or $e = 1$.*
In particular the idempotent 1 as in Definition 2.6 is determined by C.

Proof. Let r be the minimum integer such that $e \in F_rC$. If $r = 0$ then $e = t1$ for some $t \in \mathbb{K}$; if $1 \neq 0$ then $t^2 = t$ and $t = 0, 1$.
If $r > 0$ we have

$$(e-1)\otimes(e-1) = \Delta(e) - 1\otimes e - e\otimes 1 + 1\otimes 1 \in F_{r-1}C \otimes F_{r-1}C$$

and then $e - 1 \in F_{r-1}C$ which is a contradiction. □

The *reduction* of a connected coalgebra C is defined as its quotient $\overline{C} = C/\mathbb{K}1$; it is a locally nilpotent coalgebra.

3. The reduced tensor coalgebra

Given a graded vector space V, we denote $\overline{T(V)} = \bigoplus_{n>0} \bigotimes^n V$. When considered as a subset of $T(V)$ it becomes an ideal of the tensor algebra generated by V.
The *reduced tensor coalgebra* generated by V is the graded vector space $\overline{T(V)}$ endowed with the coproduct $\mathfrak{a}\colon \overline{T(V)} \to \overline{T(V)} \otimes \overline{T(V)}$:

$$\mathfrak{a}(v_1\otimes\cdots\otimes v_n) = \sum_{r=1}^{n-1}(v_1\otimes\cdots\otimes v_r)\otimes(v_{r+1}\otimes\cdots\otimes v_n).$$

We can also write

$$\mathfrak{a} = \sum_{n=2}^{+\infty}\sum_{a=1}^{n-1}\mathfrak{a}_{a,n-a},$$

where $\mathfrak{a}_{a,n-a}\colon \bigotimes^n V \to \bigotimes^a V \otimes \bigotimes^{n-a} V$ is the inverse of the multiplication map.

The coalgebra $(\overline{T(V)}, \mathfrak{a})$ is coassociative, it is locally nilpotent and the projection $p^1\colon \overline{T(V)} \to V$ is a system of cogenerators: in fact, for every $s > 0$,

$$\mathfrak{a}^{s-1}(v_1\otimes\cdots\otimes v_n) = \sum_{1\le i_1<i_2<\cdots<i_s=n}(v_1\otimes\cdots\otimes v_{i_1})\otimes\cdots\otimes(v_{i_{s-1}+1}\otimes\cdots\otimes v_n)$$

and then

$$\ker\mathfrak{a}^{s-1} = \bigoplus_{i=1}^{s-1} V^{\otimes i}, \qquad (\otimes^s p^1)\mathfrak{a}^{s-1} = p^s\colon \overline{T(V)} \to V^{\otimes s}.$$

Exercise 3.1. Let $\mu\colon \bigotimes^s \overline{T(V)} \to \overline{T(V)}$ be the multiplication map. Prove that for every $v_1, \ldots, v_n \in V$

$$\mu\mathfrak{a}^{s-1}(v_1\otimes\cdots\otimes v_n) = \binom{n-1}{s-1} v_1\otimes\cdots\otimes v_n.$$

For every morphism of graded vector spaces $f\colon V \to W$ the induced morphism of graded algebras

$$T(f)\colon \overline{T(V)} \to \overline{T(W)}, \qquad T(f)(v_1\otimes\cdots\otimes v_n) = f(v_1)\otimes\cdots\otimes f(v_n)$$

is also a morphism of graded coalgebras.
If (C, Δ) is a locally nilpotent graded coalgebra then, for every $c \in C$, there exists $n > 0$ such that $\Delta^n(c) = 0$ and then it is defined a morphism of graded vector spaces

$$\frac{1}{1-\Delta} = \sum_{n=0}^{\infty}\Delta^n\colon C \to \overline{T(C)}.$$

Proposition 3.2. *Let (C, Δ) be a locally nilpotent graded coalgebra, then:*

(1) *The map $\dfrac{1}{1-\Delta} = \sum_{n\ge 0}\Delta^n\colon C \to \overline{T(C)}$ is a morphism of graded coalgebras.*

(2) *For every graded vector space V and every morphism of graded coalgebras $\phi\colon C \to \overline{T(V)}$, there exists a unique morphism of graded vector spaces $f\colon C \to V$ such that ϕ factors as*

$$\phi = T(f)\frac{1}{1-\Delta} = \sum_{n=1}^{\infty}(\otimes^n f)\Delta^{n-1}\colon C \to \overline{T(C)} \to \overline{T(V)}.$$

Proof. [1] We have

$$\left(\left(\sum_{n\ge 0}\Delta^n\right)\otimes\left(\sum_{n\ge 0}\Delta^n\right)\right)\Delta = \sum_{n\ge 0}\sum_{a=0}^{n}(\Delta^a\otimes\Delta^{n-a})\Delta$$
$$= \sum_{n\ge 0}\sum_{a=0}^{n}\mathfrak{a}_{a+1,n+1-a}\Delta^{n+1} = \mathfrak{a}\left(\sum_{n\ge 0}\Delta^n\right)$$

where in the last equality we have used the relation $\mathfrak{a}\Delta^0 = 0$.
[2] The unicity of f is clear, since by the formula $\phi = T(f)(\sum_{n\ge 0}\Delta^n)$ it follows that $f = p^1\phi$.
To prove the existence of the factorization, take any morphism of graded coalgebras $\phi\colon C \to \overline{T(V)}$, denote by $f = p^1\phi$ and by $\psi\colon C \to \overline{T(V)}$ the coalgebra morphism $\psi = T(f)(1-\Delta)^{-1}$. Since $p^1\psi = p^1\phi$ and p^1 is a system of cogenerators we have $\phi = \psi$. □

It is useful to restate part of the Proposition 3.2 in the following form

Corollary 3.3. *Let V be a fixed graded vector space; for every locally nilpotent graded coalgebra C the composition with the projection $p^1\colon \overline{T(V)} \to V$ induces a bijection*

$$\mathrm{Hom}_{\mathbf{GC}}(C, \overline{T(V)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{G}}(C, V).$$

In other words, every morphism of graded vector spaces $C \to V$ has a unique lifting to a morphism of graded coalgebras $C \to \overline{T(V)}$.

When C is a reduced tensor coalgebra, Proposition 3.2 takes the following more explicit form

Corollary 3.4. *Let U, V be graded vector spaces. the projection. Given $f\colon \overline{T(U)} \to V$, the linear map $F\colon \overline{T(U)} \to \overline{T(V)}$:*

$$F(v_1\otimes\cdots\otimes v_n) = \sum_{s=1}^{n}\sum_{1\le i_1<i_2<\cdots<i_s=n} f(v_1\otimes\cdots\otimes v_{i_1})\otimes\cdots\otimes f(v_{i_{s-1}+1}\otimes\cdots\otimes v_{i_s}),$$

is the morphism of graded coalgebras lifting f.

Example 3.5. Let A be an associative graded algebra. Consider the projection $p\colon \overline{T(A)} \to A$, the multiplication map $\mu\colon \overline{T(A)} \to A$ and its conjugate

$$\mu^* = -\mu T(-1), \qquad \mu^*(a_1\otimes\cdots\otimes a_n) = (-1)^{n-1}\mu(a_1\otimes\cdots\otimes a_n) = (-1)^{n-1}a_1a_2\cdots a_n.$$

The two coalgebra morphisms $\overline{T(A)} \to \overline{T(A)}$ induced by μ and μ^* are isomorphisms, the one inverse of the other.
In fact, the coalgebra morphism $F\colon \overline{T(A)} \to \overline{T(A)}$

$$F(a_1\otimes\cdots\otimes a_n) = \sum_{s=1}^{n}\sum_{1\le i_1<i_2<\cdots<i_s=n}(a_1a_2\cdots a_{i_1})\otimes\cdots\otimes(a_{i_{s-1}+1}\cdots a_{i_s})$$

is induced by μ (i.e. $pF=\mu$), $\mu^*F(a)=a$ for every $a\in A$ and for every $n\geq 2$

$$\mu^*F(a_1\otimes\cdots\otimes a_n)=\sum_{s=1}^{n}(-1)^{s-1}\sum_{1\leq i_1<i_2<\cdots<i_s=n}a_1a_2\cdots a_n=$$

$$=\sum_{s=1}^{n}(-1)^{s-1}\binom{n-1}{s-1}a_1a_2\cdots a_n=\left(\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}\right)a_1a_2\cdots a_n=0.$$

This implies that $\mu^*F=p$ and therefore, if $F^*\colon\overline{T(A)}\to\overline{T(A)}$ is induced by μ^* then $pF^*F=\mu^*F=p$ and by Corollary 3.3 F^*F is the identity.

Proposition 3.6. *Let (C,Δ) be a locally nilpotent graded coalgebra, V a graded vector space and*

$$\theta=\sum_{n=1}^{\infty}(\otimes^n f)\Delta^{n-1}\colon C\to\overline{T(V)}$$

the morphism of coalgebras induced by $p\theta=f\in\operatorname{Hom}^0(C,V)$. For every n and every $q\in\operatorname{Hom}^k(C,V)$, the linear map

$$Q=\sum_{n=0}^{\infty}(\sum_{i=0}^{n}(f^{\otimes i}\otimes q\otimes f^{\otimes n-i})\Delta^n\colon C\to\overline{T(V)}$$

is the θ-coderivation induced by $pQ=q$. In particular the map

$$\operatorname{Coder}^k(C,\overline{T(V)};\theta)\to\operatorname{Hom}^k(C,V),\qquad Q\mapsto pQ,$$

is bijective.

Proof. The map Q is the composition of the coalgebra morphism $\sum\Delta^n\colon C\to\overline{T(C)}$ and the map

$$R\colon\overline{T(C)}\to\overline{T(V)},\qquad R=\sum_{i,j\geq 0}f^{\otimes i}\otimes q\otimes f^{\otimes j}.$$

It is therefore sufficient to prove that R is a $T(f)$-coderivation, i.e. that satisfies the coLeibniz rule

$$(R\otimes T(f)+T(f)\otimes R)\mathfrak{a}=\mathfrak{a}R.$$

Denoting $R_n=\sum_{i+j=n-1}f^{\otimes i}\otimes q\otimes f^{\otimes j}$ we have, for every a,n

$$\mathfrak{a}_{a,n-a}R_n=(R_a\otimes f^{\otimes n-a}+f^{\otimes a}\otimes R_{n-a})\mathfrak{a}_{a,n-a}.$$

Taking the sum over $a,n-a$ we get the proof. □

Corollary 3.7. *Let V be a graded vector space. Every $q\in\operatorname{Hom}^k(\overline{T(V)},V)$ induce a coderivation $Q\in\operatorname{Coder}^k(\overline{T(V)},\overline{T(V)})$ given by the explicit formula*

$$Q(a_1\otimes\cdots\otimes a_n)=$$
$$=\sum_{i,l}(-1)^{k(\overline{a_1}+\cdots+\overline{a_i})}a_1\otimes\cdots\otimes a_i\otimes q(a_{i+1}\otimes\cdots\otimes a_{i+l})\otimes\cdots\otimes a_n.$$

Proof. Apply Proposition 3.6 with the map $f\colon\overline{T(V)}\to V$ equal to the projection (and then $\theta=\mathrm{Id}$). □

Exercise 3.8. Let $p\colon T(V)\to\overline{T(V)}$ be the projection with kernel $\mathbb{K}=\bigotimes^0 V$ and $\phi\colon T(V)\to T(V)\otimes T(V)$ the unique homomorphism of graded algebras such that $\phi(v)=v\otimes 1+1\otimes v$ for every $v\in V$. Prove that $p\phi=\mathfrak{a}p$.

Exercise 3.9. Let A be an associative graded algebra over the field $\mathbb{K}$, for every local homomorphism of $\mathbb{K}$-algebras $\gamma\colon \mathbb{K}[[x]] \to \mathbb{K}[[x]]$, $\gamma(x) = \sum \gamma_n x^n$, we can associate a coalgebra morphism $F_\gamma\colon \overline{T(A)} \to \overline{T(A)}$ induced by the linear map

$$f_\gamma\colon \overline{T(A)} \to A, \qquad f(a_1 \otimes \cdots \otimes a_n) = \gamma_n a_1 \cdots a_n.$$

Prove the composition formula $F_{\gamma\delta} = F_\delta F_\gamma$. (Hint: Example 1.6.)

Exercise 3.10. A graded coalgebra morphism $F\colon \overline{T(U)} \to \overline{T(V)}$ is surjective (resp.: injective, bijective) if and only if $F_1^1\colon U \to V$ is surjective (resp.: injective, bijective). (Hint: F preserves the filtrations of kernels of iterated coproducts.)

4. Rooted trees

Definition 4.1. An *unreduced*[1] *rooted forest* is the data of a finite set of *vertices* V and a *flow* map $f\colon V \to V$ such that:

$$\mathrm{Fix}(f) = \bigcap_{n>0} f^n(V - \mathrm{Fix}(f)),$$

where $\mathrm{Fix}(f) = \{v \in V \mid f(v) = v\}$ is the subset of fixed points of f.

The vertices of an unreduced rooted forest (V, f) are divided into three disjoint classes:

- $V_r = \{\text{root vertices}\} = \mathrm{Fix}(f)$.
- $V_t = \{\text{tail vertices}\} = V - f(V)$.
- $V_i = \{\text{internal vertices}\} = f(V) - \{\text{root vertices}\}$.

Every unreduced rooted forest (V, f) can be described by a directed graph with set of vertices V and oriented edges $v \to f(v)$ for every $v \notin \{\text{root vertices}\}$. In our pictures internal vertices will be denoted by a black dot, while tail and root vertices will be denoted by a circle.
As an example, the pair (V, f), where $V = \{1, 2, 3, 4\}$ and $f(i) = \min(4, i + 1)$ is an almost rooted forest described by the oriented graph

$$\underset{\text{tail}}{\circ} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \underset{\text{root}}{\circ}$$

Note that the map $f\colon V_t \cup V_i \to V_i \cup V_r$ is surjective and then the number of tail vertices is always greater than or equal to the number of root vertices.

The set of edges $\{(v, f(v)) \mid v \notin \{\text{roots}\}\}$ is divided into types. An edge $(v, f(v))$ is called a *root edge* if $f(v)$ is a root vertex; it is called a *tail edge* if v is a tail vertex and it is called an *internal edge* if both $v, f(v)$ are both internal vertices. Notice that an edge may be tail and root at the same time.

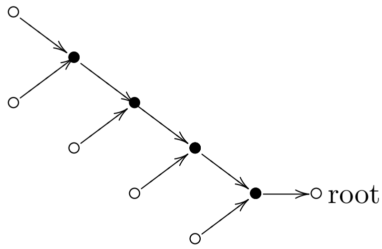
The *arity* (also called valence in literature) $|v|$ of a vertex v is the number of incoming edges; equivalently

$$|v| = |\{w \neq v \mid f(w) = v\}|.$$

A *rooted forest* is an almost rooted forest such that every root has arity 1 and every internal vertex has arity ≥ 2.
A *rooted tree* is a rooted forest with exactly one root.

[1]Here unreduced means not necessarily reduced.

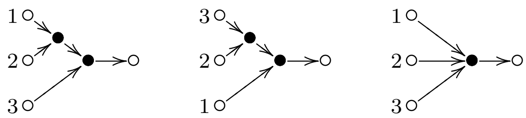
Every rooted forest is then a disjoint union of rooted trees; the following picture represents a rooted tree with 5 tail vertices and 4 internal vertices.



An automorphism of a rooted forest (V, f) is a bijective map $\phi\colon V \to V$ such that $f\phi = \phi f$. The group of automorphisms will be denoted by $\mathrm{Aut}(V, f)$.

Definition 4.2. Let (V, f) be a rooted forest. An *orientation* of (V, f) is a total ordering $\leq$ on the set V_t of tail vertices such that if $v \leq u \leq w$ and $f^k(v) = f^h(w)$, for some $h, k \geq 0$, then there exists $l \geq 0$ such that $f^k(v) = f^l(u) = f^h(w)$.
It is often convenient to describe an orientation $\leq$ by the order-preserving bijection $\nu\colon \{1, \ldots, n\} \to V_t$, where $|V_t| = n$. Therefore, an *oriented rooted forest* is a triple (V, f, ν) where ν is an orientation of (V, f).

For instance, there are (up to isomorphism) exactly three oriented rooted trees with 3 tails:



Lemma 4.3. *Let V be a rooted tree. Then the number of isomorphism classes of orientations on V is equal to*

$$\frac{1}{|\,\mathrm{Aut}(V)|} \prod_{v \in V_i} |v|!$$

Proof. The group of automorphisms of V acts freely on the set of orientations. We note that every orientation is uniquely determined by:

(1) A total ordering of root edges.
(2) For every internal vertex, a total ordering of incoming edges.

Therefore the product $\prod_{v \in V_i} |v|!$ is equal to the number of orientations on the tree V. $\square$

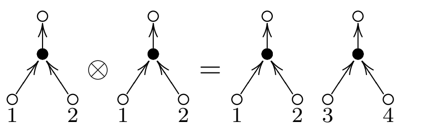
Denote by $F(n, m)$ the set of isomorphism classes of oriented rooted forests with n tails and m roots. Notice that $F(n, m) = \emptyset$ for $m > n$ and $F(n, n)$ contains only one element, denoted by $\mathbb{I}_n$.

There are defined naturally two binary operations:

$$\circ\colon F(l, m) \times F(n, l) \to F(n, m) \qquad \text{composition}$$

$$\otimes\colon F(n, m) \times F(a, b) \to F(n + a, m + b) \qquad \text{tensor product}$$

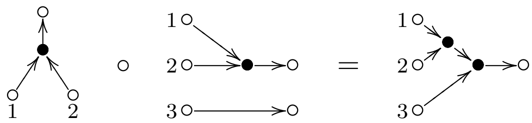
The tensor product $V \otimes W$ is the disjoint union of V and W with the orientation $\{1, 2, \ldots\} \to W_t$ shifted by the number of tail vertices of V. For instance $\mathbb{I}_a \otimes \mathbb{I}_b = \mathbb{I}_{a+b}$ and



Given $(V, f) \in F(n, l)$ and $(W, g) \in F(l, m)$ we define $W \circ V$ in the following way: first we take the unique bijection $\eta\colon V_r \to W_t$ such that for $i >> 0$ the map

$$V_t \xrightarrow{\eta f^i} W_t$$

is nondecreasing. Then we use η to annihilate the root vertices of V with the tail vertices of W. For instance, for every $V \in F(n,m)$ we have $V = \mathbb{I}_m \circ V = V \circ \mathbb{I}_n$ and



The operations $\circ$ and $\otimes$ are associative and satisfy the *interchange law* [6]: this means that

$$(V \otimes W) \circ (A \otimes B) = (V \circ A) \otimes (W \circ B)$$

holds whenever the composites $V \circ A$ and $W \circ B$ are defined. By convention we set $\mathbb{I}_0 = \emptyset \in F(0,0)$ and then $\mathbb{I}_0 \otimes V = V \otimes \mathbb{I}_0 = V$ for every V.

Exercise 4.4. Given $V \in F(n,m)$ denote by

$$w(V) = \max\{a \mid \exists\, W \in F(n-a, m-a) \text{ such that } V = \mathbb{I}_a \otimes W\}.$$

We shall say that a composition $V_1 \circ V_2 \circ \cdots \circ V_r$ is *monotone* if $w(V_1) \leq w(V_2) \leq \cdots \leq w(V_r)$. Prove that every oriented rooted forest $V \in F(n,m)$ can be written uniquely as a monotone composition of oriented rooted forests with one internal vertex.

5. Automorphisms of $\overline{T(V)}$ and inversion formula.

For every graded vector space V we can define binary operations

$$\circ\colon \operatorname{Hom}^*(V^{\otimes l}, V^{\otimes m}) \times \operatorname{Hom}^*(V^{\otimes n}, V^{\otimes l}) \to \operatorname{Hom}^*(V^{\otimes n}, V^{\otimes m}) \qquad (f,g) \mapsto f \circ g,$$

$$\otimes\colon \operatorname{Hom}^*(V^{\otimes n}, V^{\otimes m}) \times \operatorname{Hom}^*(V^{\otimes a}, V^{\otimes b}) \to \operatorname{Hom}^*(V^{\otimes n+a}, V^{\otimes m+b}) \qquad (f,g) \mapsto f \otimes g.$$

By a representation of $\mathcal{F} = \cup_{n,m} F(n,m)$ we shall mean a map

$$Z\colon \mathcal{F} \to \bigcup_{n,m} \operatorname{Hom}^*(V^{\otimes n}, V^{\otimes m})$$

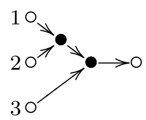
such that $Z_{\mathbb{I}_n} = \mathrm{Id}_{V^{\otimes n}}$ and commutes with the operations $\circ$ and $\otimes$.
Every representation Z is determined by its value on the irreducible trees $\mathbb{T}_n$. Conversely, for every sequence of maps $f_n \in \operatorname{Hom}^*(V^{\otimes n}, V)$, $n \geq 2$, there exists an unique representation

$$Z(f_i)\colon \mathcal{F} \to \bigcup_{n,m} \operatorname{Hom}^*(V^{\otimes n}, V^{\otimes m})$$

such that

$$Z_{\mathbb{T}_n}(f_i) = f_n.$$

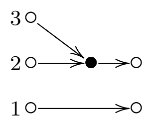
For instance, the oriented rooted tree

$\Gamma =$ 

gives

$$Z_\Gamma(f_i)(v_1 \otimes v_2 \otimes v_3) = f_2(f_2(v_1 \otimes v_2) \otimes v_3),$$

while the oriented rooted forest

$\Gamma =$ 

gives

$$Z_\Gamma(f_i)(v_1\otimes v_2\otimes v_3)=(-1)^{\deg(v_1)\deg(f_2)}v_1\otimes f_2(v_2\otimes v_3).$$

Definition 5.1. For every n,m let $S(n,m)\subset F(n,m)$ be the subset of (isomorphism classes of) oriented rooted forests without internal edges and denote $\mathcal{S}=\bigcup_{n,m}S(n,m)$.

Equivalently $\Gamma\in\mathcal{S}$ if and only if Γ is the tensor product of irreducible oriented rooted trees.

Lemma 5.2. *For every sequence $g_n\in \operatorname{Hom}^0(V^{\otimes n},V)$, $n\ge 2$, the maps*

$$G=\sum_{\Gamma\in\mathcal{S}}Z_\Gamma(g_i)\colon \overline{T(V)}\to\overline{T(V)}$$

$$F=\sum_{\Gamma\in\mathcal{F}}Z_\Gamma(g_i)\colon \overline{T(V)}\to\overline{T(V)}$$

are morphism of graded coalgebras.

Proof. Denote by $f_n^m=\sum_{\Gamma\in\ F(n,m)}Z_\Gamma(g_i)$. According to Corollary 3.4, G is a coalgebra morphism, while F is a coalgebra morphism if and only if

$$f_n^m=\sum_{\substack{(i_1,\dots,i_m)\in\mathbb{N}^m\\ i_1+\cdots+i_m=n}}f_{i_1}^1\otimes\cdots\otimes f_{i_m}^1.$$

On the other hand, every $\Gamma\in F(n,m)$ can be written uniquely as a tensor product of m oriented trees, i.e. the map

$$\bigcup_{\substack{(i_1,\dots,i_m)\in\mathbb{N}^m\\ i_1+\cdots+i_m=n}}F(i_1,1)\times\cdots\times F(i_m,1)\to F(n,m),\quad (\Gamma_1,\cdots,\Gamma_m)\mapsto\Gamma_1\otimes\cdots\otimes\Gamma_m,$$

is bijective. The conclusion follows from the fact that

$$Z_{\Gamma_1\otimes\cdots\otimes\Gamma_m}(f_i)=Z_{\Gamma_1}(f_i)\otimes\cdots\otimes Z_{\Gamma_m}(f_i).$$

□

Lemma 5.3. *Given $g\in\operatorname{Hom}^0(W,V)$ and a sequence of maps $g_n\in\operatorname{Hom}^0(V^{\otimes n},V)$, $n\ge 2$, for every $n,m\ge 1$ denote*

$$f_n^m=\sum_{\Gamma\in\ F(n,m)}Z_\Gamma(g_i)\circ(\otimes^n g)\colon W^{\otimes n}\to V^{\otimes m}.$$

Then, for every $n\ge 0$

$$f_n^1=\sum_{a=2}^n g_a\circ f_n^a.$$

Proof. Every $\Gamma\in F(n,1)$ has a unique decomposition of the form $\Gamma=\mathbb{T}_a\circ\Gamma'$, with $\Gamma'\in F(n,a)$ and then

$$\sum_{1<a\le n}\sum_{\Gamma'\in F(n,a)}Z_{\mathbb{T}_a\circ\Gamma'}(g_i)=\sum_{\Gamma\in F(n,1)}Z_\Gamma(g_i).$$

Composing with $(\otimes^n g)$ we get the equality $f_n^1=\sum_{a=2}^n g_a\circ f_n^a$. □

Theorem 5.4 (Inversion formula)**.** *For every sequence $g_n\in\operatorname{Hom}^0(V^{\otimes n},V)$, $n\ge 2$, the morphisms*

$$H=\sum_{\Gamma\in\mathcal{S}}Z_\Gamma(-g_i)\colon\overline{T(V)}\to\overline{T(V)},\qquad F=\sum_{\Gamma\in\mathcal{F}}Z_\Gamma(g_i)\colon\overline{T(V)}\to\overline{T(V)}$$

are isomorphisms and $F=H^{-1}$.

Proof. We first note that $H(v) = v$ for every $v \in V$ and we can write

$$H = \mathrm{Id} + \sum_{m<n} h_n^m, \qquad h_n^m = \sum_{\Gamma\in S(n,m)} Z_\Gamma(-g_i) \colon V^{\otimes n} \to V^{\otimes m}$$

Denoting $K = \mathrm{Id} - H$, we have $\cup_n \ker(K^n) = \overline{T(V)}$ and then H is invertible with inverse

$$H^{-1} = \mathrm{Id} + \sum_{n=1}^{\infty} K^n.$$

Writing

$$H^{-1} = \sum_{m\le n} f_n^m,$$

we have, since H^{-1} is a coalgebra morphism and $H \circ H^{-1} = \mathrm{Id}$ we have $f_n^n = \mathrm{Id}$ and for every $m < n$

$$f_n^m = \sum_{\substack{(i_1,\dots,i_m)\in\mathbb{N}^m \\ i_1+\cdots+i_m=n}} f_{i_1}^1 \otimes\cdots\otimes f_{i_m}^1 = -\sum_{m<i\le n} h_i^m \circ f_n^i.$$

Let $n > 0$ and assume that

$$f_a^m = \sum_{\Gamma\in F(a,m)} Z_\Gamma(g_i) :$$

For every $m \le a < n$. We want to prove that for every $m \le n$ we have

$$f_n^m = \sum_{\Gamma\in F(n,m)} Z_\Gamma(g_i).$$

Since F is a morphism of coalgebras it is not restrictive to assume $m = 1$ and then

$$f_n^1 = -\sum_{1<a\le n} h_a^1 \circ f_n^a = \sum_{1<a\le n} g_a \circ \sum_{\Gamma\in F(n,a)} Z_\Gamma(g_i).$$

By Lemma 5.3, with $g = \mathrm{Id}$, we get

$$f_n^1 = \sum_{1<a\le n}\sum_{\Gamma'\in F(n,a)} Z_{\mathbb{T}_a\circ\Gamma'}(g_i) = \sum_{\Gamma\in F(n,1)} Z_\Gamma(g_i).$$

□

Exercise 5.5. Denote by t_n the number of oriented rooted trees with n tail vertices ($t_n = |F(n,1)|$) and b_n the number of oriented binary rooted trees (a binary rooted tree is a rooted tree where every internal vertex has two incoming edges). Prove the following series expansion identities:

$$\sum_{n>0} t_n x^n = \frac{x+1-\sqrt{1-6x+x^2}}{4}, \qquad \sum_{n>0} b_n x^n = \frac{1-\sqrt{1-4x}}{2}.$$

(Hint: denote

$$f(y) = y - y^2, \qquad g(y) = \frac{y(1-2y)}{(1-y)} = y - y^2 - y^3 - \cdots.$$

Use inversion formula in case $V = \mathbb{K}$ to prove that $f(\sum_{n>0} b_n x^n) = g(\sum_{n>0} t_n x^n) = x$.)

6. Koszul sign, symmetrization and unshuffles

Dor every set A we denote by $\Sigma(A)$ the group of permutations of A and by $\Sigma_n = \Sigma(\{1,\ldots,n\})$.

The action of the twist map on $\bigotimes^2 V$ extends naturally, for every $n\ge 0$, to an action of the symmetric group Σ_n on the graded vector space $\bigotimes^n V$. Notice that

$$\sigma_{\mathtt{tw}}(v_1\otimes\cdots\otimes v_n)=\pm(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}).$$

Definition 6.1. The *Koszul sign* $\epsilon(V,\sigma;v_1,\ldots,v_n)=\pm 1$ is defined by the relation

$$\sigma_{\mathtt{tw}}^{-1}(v_1\otimes\cdots\otimes v_n)=\epsilon(V,\sigma;v_1,\ldots,v_n)(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})$$

For notational simplicity we shall write $\epsilon(\sigma;v_1,\ldots,v_n)$ or $\epsilon(\sigma)$ when there is no possible confusion about V and $v_1,\ldots,v_n$.

Remark 6.2. The twist action on $\bigotimes^n(\mathrm{Hom}^*(V,W))$ is compatible with the conjugate of the twist action on $\mathrm{Hom}^*(V^{\otimes n},W^{\otimes n})$. This means that

$$\sigma_{\mathtt{tw}}(f_1\otimes\cdots\otimes f_n)=\sigma_{\mathtt{tw}}\circ f_1\otimes\cdots\otimes f_n\circ\sigma_{\mathtt{tw}}^{-1}.$$

Define the linear map $N\colon \bigotimes^n V\to\bigotimes^n V$

$$\begin{aligned}N(v_1\otimes\cdots\otimes v_n)&=\sum_{\sigma\in\Sigma_n}\epsilon(\sigma;v_1,\ldots,v_n)(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})\\&=\sum_{\sigma\in\Sigma_n}\sigma_{\mathtt{tw}}(v_1\otimes\cdots\otimes v_n),\quad v_1,\ldots,v_n\in V.\end{aligned}$$

Denoting by $(\bigotimes^n V)^{\Sigma_n}\subset\bigotimes^n V$ the subspace of twist-invariant tensors, we have that the map

$$\frac{1}{n!}N\colon\bigotimes^n V\to(\bigotimes^n V)^{\Sigma_n}$$

is a projection and then

$$\bigotimes^n V=(\bigotimes^n V)^{\Sigma_n}\oplus\ker(N).$$

Lemma 6.3. *In the notation above, the kernel of N is the subspace generated by all the vectors $v-\sigma_{\mathtt{tw}}(v)$, $\sigma\in\Sigma_n$, $v\in\bigotimes^n V$.*

Proof. Denote by W the subspace generated by the vectors $v-\sigma_{\mathtt{tw}}(v)$: it is clear that $N(W)=0$ and therefore it is sufficient to prove that $\mathrm{Im}(N)+W=\bigotimes^n V$. For every $v\in\bigotimes^n V$ we can write

$$v=\frac{N}{n!}v+\left(v-\frac{N}{n!}v\right)=\frac{N}{n!}v+\frac{1}{n!}\sum_{\sigma\in\Sigma_n}(v-\sigma_{\mathtt{tw}}v).$$

□

Definition 6.4. The set of *unshuffles* of type (p,q) is the subset $S(p,q)\subset\Sigma_{p+q}$ of permutations σ such that $\sigma(i)<\sigma(i+1)$ for every $i\neq p$.

Since $\sigma\in S(p,q)$ if and only if the restrictions $\sigma\colon\{1,\ldots,p\}\to\{1,\ldots,p+q\}$, $\sigma\colon\{p+1,\ldots,p+q\}\to\{1,\ldots,p+q\}$, are increasing maps, it follows easily that the unshuffles are a set of representatives for the left cosets of the canonical embedding of $\Sigma_p\times\Sigma_q$ inside Σ_{p+q}. More precisely for every $\eta\in\Sigma_{p+q}$ there exists a unique decomposition $\eta=\sigma\tau$ with $\sigma\in S(p,q)$ and $\tau\in\Sigma_p\times\Sigma_q$.

Lemma 6.5. *For every $v_1,\dots,v_n\in V$ and every $a=0,\dots,n$ we have*
$$N(v_1\otimes\cdots\otimes v_n)=\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)N(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(a)})\otimes N(v_{\sigma(a+1)}\otimes\cdots\otimes v_{\sigma(n)}).$$

Proof.
$$\begin{aligned}N(v_1\otimes\cdots\otimes v_n)&=\sum_{\eta\in\Sigma_n}\eta_{\mathtt{tw}}^{-1}v_1\otimes\cdots\otimes v_n\\&=\sum_{\sigma\in S(a,n-a)}\sum_{\tau\in\Sigma_a\times\Sigma_{n-a}}\tau_{\mathtt{tw}}^{-1}\sigma_{\mathtt{tw}}^{-1}v_1\otimes\cdots\otimes v_n\\&=\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)\sum_{\tau\in\Sigma_a\times\Sigma_{n-a}}\tau_{\mathtt{tw}}^{-1}v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}\\&=\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)N(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(a)})\otimes N(v_{\sigma(a+1)}\otimes\cdots\otimes v_{\sigma(n)}).\end{aligned}$$
□

Consider two graded vector spaces V,M, a positive integer n, two maps
$$f\in\operatorname{Hom}^0(V,M),\qquad q\in\operatorname{Hom}^k(\bigotimes^l V,M)$$
and define
$$Q=\sum_{i=0}^{n-l}f^{\otimes i}\otimes q\otimes f^{\otimes n-l-i}\in\operatorname{Hom}^k(\bigotimes^n V,\bigotimes^{n-l+1}M).$$
More explicitly
$$Q(a_1\otimes\cdots\otimes a_n)=$$
$$=\sum_{i=0}^{n-l}(-1)^{k(\overline{a_1}+\cdots+\overline{a_i})}f(a_1)\otimes\cdots\otimes f(a_i)\otimes q(a_{i+1}\otimes\cdots\otimes a_{i+l})\otimes f(a_{i+l+1})\otimes\cdots\otimes f(a_n).$$

Lemma 6.6. *In the notation above*
$$\begin{aligned}&QN(a_1\otimes\cdots\otimes a_n)=\\&=\sum_{\sigma\in S(l,n-l)}\epsilon(\sigma)N(qN(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(l)})\otimes f(a_{\sigma(l+1)})\otimes\cdots\otimes f(a_{\sigma(n)}))\\&=N\left(\sum_{\sigma\in S(l,n-l)}\epsilon(\sigma)qN(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(l)})\otimes f(a_{\sigma(l+1)})\otimes\cdots\otimes f(a_{\sigma(n)})\right).\end{aligned}$$
and then
$$Q\circ N=\frac{1}{l!(n-l)!}N\circ(qN\otimes\operatorname{Id}^{\otimes n-l})\circ N.$$

Proof. Denote
$$H=\{\sigma\in\Sigma_n\mid\sigma(l+1)<\sigma(l+2)<\cdots<\sigma(n)\}$$
and for every $j=0,\dots,n-l$ choose permutations $\tau^j\in\Sigma(\{0,\dots,n-l\})$, $\eta^j\in\Sigma_n$ such that
$$\tau^j(0)=j,\qquad\tau^j_{\mathtt{tw}}\circ(q\otimes f^{\otimes n-l})\circ\eta^j_{\mathtt{tw}}=f^{\otimes j}\otimes q\otimes f^{\otimes n-l-j}.$$
We have
$$Q(a_1\otimes\cdots\otimes a_n)=\sum_j\tau^j_{\mathtt{tw}}\circ(q\otimes f^{\otimes n-l})\circ\eta^j_{\mathtt{tw}}(a_1\otimes\cdots\otimes a_n)$$

and then

$$QN(a_1\otimes\cdots\otimes a_n)=\sum_j \tau_{\mathtt{tw}}^j\circ(q\otimes f^{\otimes n-l})\circ N(a_1\otimes\cdots\otimes a_n).$$

On the other side, since $\Sigma(\{0,\ldots,n-l\})=\cup_j\tau^j\Sigma_{n-l}$, we have

$$\begin{aligned}\sum_{\sigma\in S(l,n-l)}&\epsilon(\sigma)N(qN(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(l)})\otimes f(a_{\sigma(l+1)})\otimes\cdots\otimes f(a_{\sigma(n)}))=\\ &=\sum_{\sigma\in H}\epsilon(\sigma)N(q(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(l)})\otimes f(a_{\sigma(l+1)})\otimes\cdots\otimes f(a_{\sigma(n)}))\\ &=\sum_j\sum_{\sigma\in\Sigma_n}\epsilon(\sigma)\tau_{\mathtt{tw}}^j(q(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(l)})\otimes f(a_{\sigma(l+1)})\otimes\cdots\otimes f(a_{\sigma(n)}))\\ &=\sum_j\sum_{\sigma\in\Sigma_n}\tau_{\mathtt{tw}}^j\circ(q\otimes f^{\otimes n-l})\circ\sigma_{\mathtt{tw}}^{-1}(a_1\otimes\cdots\otimes a_n)\\ &=\sum_j\tau_{\mathtt{tw}}^j\circ(q\otimes f^{\otimes n-l})\circ N(a_1\otimes\cdots\otimes a_n).\end{aligned}$$

□

Given two oriented rooted forests Γ,Ω we shall write $\Gamma\sim\Omega$ is Γ and Ω are isomorphic as rooted forests, i.e. if they differ only by the orientation.
We have seen that the cardinality of the equivalence class of a oriented rooted tree T is

$$\frac{1}{|\operatorname{Aut}(T)|}\prod_{v\in T_i}|v|!$$

Lemma 6.7. *Let $\Omega\in F(n,m)$ and $q_i\in\operatorname{Hom}^0(V^{\otimes i},V)$, $n\geq 2$. Then we have*

$$\sum_{\Gamma\sim\Omega}Z_\Gamma(q_i)\circ N=\frac{1}{|\operatorname{Aut}(\Omega)|}N\circ Z_\Omega(q_iN)\circ N.$$

In particular, if $\Gamma,\Omega\in F(n,1)$ and $\Gamma\sim\Omega$, then

$$Z_\Gamma(q_iN)\circ N=Z_\Omega(q_iN)\circ N.$$

Proof. Assume that $\Omega=(V,f,\nu)$, where (V,f) is a rooted forest and $\nu\colon\{1,\ldots,n\}\to V_t$ is a numbering. Define

$$G_\Omega=\{\sigma\in\Sigma_n\mid\nu\circ\sigma^{-1}\text{ is an orientation }\}$$

and, for every $\sigma\in G_\Omega$ denote by

$$\sigma\Omega=(V,f,\nu\circ\sigma^{-1}).$$

The group $\operatorname{Aut}(\Omega)$, when interpreted as a subgroup of $\Sigma(V_t)$, acts freely on G_Ω and there is a bijection

$$G_\Omega/\operatorname{Aut}(\Omega)\simeq\{\Gamma\sim\Omega\}.$$

Therefore the lemma is equivalent to the equality

$$\sum_{\sigma\in G_\Omega}Z_{\sigma\Omega}(q_i)\circ N=N\circ Z_\Omega(q_iN)\circ N.$$

If $n=m$, then $\Omega=\mathbb{I}_n$, $G_\Omega=\Sigma_n$ and the formula becomes $N^2=n!N$ that is trivially verified.
By induction we may assume that the formula holds for every $\Omega\in F(a,b)$ with $a^2-b^2<n^2-m^2$. Assume first that $m>1$, therefore we have

$$\Omega=T_1\otimes\cdots\otimes T_m,\qquad T_i\in F(n_i,1).$$

Since $\sum_i(n_i^2-1)\le n^2-m^2$ the symmetrization formula holds for every tree T_i.
Denote by $R=\Sigma_{n_1}\times\cdots\times\Sigma_{n_m}\subset\Sigma_n$ and by $S\subset\Sigma_n$ a set of representatives for the left cosets of R.
Define also

$$K=R\cap G_\Omega=G_{T_1}\times\cdots\times G_{T_n}.$$

By the inductive formula, applied to trees T_i

$$\sum_{\sigma\in K}\sum_{\eta\in R}Z_{\sigma\Omega}(q_i)\circ\eta_{\mathtt{tw}}^{-1}=\sum_{\eta\in R}Z_\Omega(q_iN)\circ\eta_{\mathtt{tw}}^{-1}.$$

and then

$$\sum_{\sigma\in K}Z_{\sigma\Omega}(q_i)\circ N=\sum_{\rho\in S}\sum_{\sigma\in K}\sum_{\eta\in R}Z_{\sigma\Omega}(q_i)\circ\eta_{\mathtt{tw}}^{-1}\circ\rho_{\mathtt{tw}}^{-1}=$$

$$=\sum_{\rho\in S}\sum_{\eta\in R}Z_\Omega(q_iN)\circ\eta_{\mathtt{tw}}^{-1}\circ\rho_{\mathtt{tw}}^{-1}=Z_\Omega(q_iN)\circ N.$$

For every $\tau\in\Sigma_m$ denote by $\hat\tau\in G_\Omega$ the unique element satisfying

$$\hat\tau\Omega=T_{\tau(1)}\otimes\cdots\otimes T_{\tau(m)}.$$

Notice that for every $\tau\in\Sigma_m$ and every $\kappa\in K$ we have $\hat\tau\in G_{\sigma\Omega}$ and

$$G_\Omega=\bigcup_{\tau\in\Sigma_m}\hat\tau K.$$

Since every operator q_i has even degree we have

$$Z_{\hat\tau\Omega}(q_i)=\tau_{\mathtt{tw}}^{-1}\circ Z_\Omega(q_i)\circ\hat\tau_{\mathtt{tw}}.$$

and more generally, for every $\kappa\in K$

$$Z_{\hat\tau\kappa\Omega}(q_i)=\tau_{\mathtt{tw}}^{-1}\circ Z_{\kappa\Omega}(q_i)\circ\hat\tau_{\mathtt{tw}}.$$

Therefore

$$\sum_{\sigma\in G_\Omega}Z_{\sigma\Omega}(q_i)\circ N=\sum_{\tau\in\Sigma_m}\sum_{\kappa\in K}Z_{\hat\tau\kappa\Omega}(q_i)\circ N=\sum_{\tau\in\Sigma_m}\sum_{\kappa\in K}\tau_{\mathtt{tw}}^{-1}\circ Z_{\kappa\Omega}(q_i)\circ\hat\tau_{\mathtt{tw}}\circ N$$

$$=\sum_{\tau\in\Sigma_m}\tau_{\mathtt{tw}}^{-1}\circ Z_\Omega(q_iN)\circ N=N\circ Z_\Omega(q_iN)\circ N.$$

Assume now $m=1$ and decompose Ω as

$$\Omega=\mathbb{T}_m\circ\Theta,\qquad \Theta\in F(n,m).$$

We have $G_\Omega=G_\Theta$ and

$$\sigma\Omega=\mathbb{T}_m\circ\sigma\Theta,\qquad \sigma\in G_\Omega=G_\Theta.$$

By inductive assumption

$$\sum_\sigma Z_{\sigma\Omega}(q_i)\circ N=q_m\circ\sum_\sigma Z_{\sigma\Theta}(q_i)\circ N=q_mN\circ Z_\Theta(q_iN)\circ N=Z_\Omega(q_iN)\circ N.$$

□

Definition 6.8. A graded coalgebra (C,Δ) is called *cocommutative* if $\mathtt{tw}\circ\Delta=\Delta$.

Lemma 6.9. *Let (C,Δ) be a graded coassociative cocommutative coalgebra. Then the image of Δ^{n-1} is contained in the set of Σ_n-invariant elements of $\bigotimes^n C$.*

Proof. The twist action of Σ_n on $\bigotimes^n C$ is generated by the operators $\mathtt{tw}_a = \mathrm{Id}_{\bigotimes^a C} \otimes \mathtt{tw} \otimes \mathrm{Id}_{\bigotimes^{n-a-2} C}$, $0 \le a \le n-2$, and, if $\mathtt{tw} \circ \Delta = \Delta$ then, according to Lemma 1.8

$$\begin{aligned}\mathtt{tw}_a \Delta^{n-1} &= \mathtt{tw}_a(\mathrm{Id}_{\bigotimes^a C} \otimes \Delta \otimes \mathrm{Id}_{\bigotimes^{n-a-2} C})\Delta^{n-2}\\ &=(\mathrm{Id}_{\bigotimes^a C} \otimes \Delta \otimes \mathrm{Id}_{\bigotimes^{n-a-2} C})\Delta^{n-2} = \Delta^{n-1}.\end{aligned}$$

□

Exercise 6.10. Prove that a coalgebra C is cocommutative if and only if the algebra $\mathrm{Hom}^*(C, A)$ is commutative for every commutative algebra A.

Exercise 6.11. Let C be a cocommutative graded coalgebra and L a graded Lie algebra. Prove that $\mathrm{Hom}^*(C, L)$ is a graded Lie algebra.

7. Symmetric algebras

Let V be a graded vector space, $T(V)$ its tensor algebra and denote by $I \subset \bigodot^*(V)$ be the homogeneous ideal generated by the elements $x \otimes y - \mathtt{tw}(x \otimes y)$, $x, y \in V$.
The *symmetric algebra* generated by V is by definition the quotient

$$S(V) = \frac{T(V)}{I} = \bigoplus_{n\ge 0} \bigodot^n V, \qquad \bigodot^n V = \frac{\bigotimes^n V}{\bigotimes^n V \cap I}.$$

The product in $S(V)$ is denoted by $\odot$. In particular if $\pi\colon T(V) \to S(V)$ is the projection to the quotient then for every $v_1, \dots, v_n \in V$, $v_1 \odot \cdots \odot v_n = \pi(v_1 \otimes \cdots \otimes v_n)$.

If σ is a permutation of $\{1, \dots, n\}$, then for every $v \in \bigotimes^n V$ we have $v - \sigma_{\mathtt{tw}} v \in I$ and then $\pi(v) = \pi\sigma_{\mathtt{tw}}(v)$. More explicitly

$$v_1 \odot \cdots \odot v_n = \epsilon(\sigma; v_1, \dots, v_n)(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(n)}).$$

The map $N\colon \bigotimes^n V \to \bigotimes^n V$ factors to

$$N\colon \bigodot^n V \to \bigotimes^n V, \qquad N(v_1 \odot \cdots \odot v_n) = N(v_1 \otimes \cdots \otimes v_n)$$

and the composition $\bigodot^n V \xrightarrow{N} \bigotimes^n V \xrightarrow{\pi} \bigodot^n V$ is $n!\,\mathrm{Id}$.

For every morphism of graded vector spaces $f\colon V \to W$ we denote by

$$S(f)\colon S(V) \to S(W), \qquad S(f)(v_1 \odot \cdots \odot v_n) = f(v_1) \odot \cdots \odot f(v_n)$$

the induced morphism of algebras.

Remark 7.1. For every differential graded vector space W there exists a natural inclusion

$$\mathrm{Hom}^*(V^{\odot n}, W) \subseteq \mathrm{Hom}^*(V^{\otimes n}, W):$$

given $f \in \mathrm{Hom}^*(V^{\odot n}, W)$ we set

$$f(v_1 \otimes \cdots \otimes v_n) = f(v_1 \odot \cdots \odot v_n).$$

Conversely, a map $f \in \mathrm{Hom}^*(V^{\otimes n}, W)$ belongs to $\mathrm{Hom}^*(V^{\odot n}, W)$ if and only if $f = f \circ \sigma_{\mathtt{tw}}$ for every permutation $\sigma \in \Sigma_n$.
As an example, if $\Gamma \in F(n, 1)$ is an oriented rooted tree, then for every sequence $f_i \in \mathrm{Hom}^0(V^{\otimes i}, V)$ we have

$$Z_\Gamma(f_i) \circ N \in \mathrm{Hom}^0(V^{\odot n}, V),$$

and the second part of Lemma 6.7 implies that, if $f_i \in \mathrm{Hom}^0(V^{\odot i}, V)$, then

$$Z_\Gamma(f_i) \circ N = Z_\Omega(f_i) \circ N$$

for every $\Omega \sim \Gamma$.

8. The reduced symmetric coalgebra

For every graded vector space V denote $\overline{S(V)} = \bigoplus_{n>0} \bigodot^n V$.

Lemma 8.1. *The map* $\mathfrak{l}\colon \overline{S(V)} \to \overline{S(V)} \otimes \overline{S(V)}$,

$$\mathfrak{l}(v_1 \odot \cdots \odot v_n) = \sum_{a=1}^{n-1} \sum_{\sigma\in S(a,n-a)} \epsilon(\sigma)(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(a)}) \otimes (v_{\sigma(a+1)} \odot \cdots \odot v_{\sigma(n)})$$

is a cocommutative coproduct and the map

$$N\colon (\overline{S(V)}, \mathfrak{l}) \to (\overline{T(V)}, \mathfrak{a})$$

is an injective morphism of coalgebras.

Proof. The cocommutativity of $\mathfrak{l}$ is clear from definition. Since N is injective, we only need to prove that $\mathfrak{a}N = (N \otimes N)\mathfrak{l}$. According to Lemma 6.5, for every a

$$\mathfrak{a}_{a,n-a}N(v_1 \odot \cdots \odot v_n) = N \otimes N \sum_{\sigma\in S(a,n-a)} \epsilon(\sigma)(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(a)}) \otimes (v_{\sigma(a+1)} \otimes \cdots \otimes v_{\sigma(n)})$$

and then

$$\mathfrak{a}N(v_1 \odot \cdots \odot v_n) = \sum_{a=1}^{n-1} \mathfrak{a}_{a,n-a}N(v_1 \odot \cdots \odot v_n) = N \otimes N\mathfrak{l}(v_1 \odot \cdots \odot v_n).$$

□

Definition 8.2. The reduced symmetric coalgebra generated by V is the graded vector space $\overline{S(V)}$ with the coproduct $\mathfrak{l}$ defined in Lemma 8.1

$$\mathfrak{l}(v_1 \odot \cdots \odot v_n) = \sum_{a=1}^{n-1} \sum_{\sigma\in S(a,n-a)} \epsilon(\sigma)(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(a)}) \otimes (v_{\sigma(a+1)} \odot \cdots \odot v_{\sigma(n)}).$$

It is often convenient to think the reduced symmetric coalgebra as a subset of the tensor coalgebra, via the identification provided by N. In particular $\overline{S(V)}$ is locally nilpotent and the projection $\overline{S(V)} \to V$ is a system of cogenerators.
Moreover, since N is an injective morphism of coalgebras we have

$$\ker \mathfrak{l}^n = N^{-1}(\ker \mathfrak{a}^n) = N^{-1}(\oplus_{i=1}^n V^{\otimes i}) = \oplus_{i=1}^n V^{\odot i}.$$

For every morphism of graded vector spaces $f\colon V \to W$ we have

$$N \circ S(f) = T(f) \circ N\colon S(V) \to T(W)$$

and then $S(f)\colon \overline{S(V)} \to \overline{S(W)}$ is a morphism of graded coalgebras.

Exercise 8.3. Assume V finite dimensional with basis $\partial_1, \dots, \partial_m$ of degree 0. Prove that

$$\mathfrak{l}(\partial_1^{n_1} \cdots \partial_m^{n_m}) = \sum_{a_1,\dots,a_m} \binom{n_1}{a_1} \cdots \binom{n_m}{a_m} \partial_1^{a_1} \cdots \partial_m^{a_m} \otimes \partial_1^{n_1-a_1} \cdots \partial_m^{n_m-a_m}$$

and deduce that the dual algebra $\overline{S(V)}^\vee$ is isomorphic to the maximal ideal of the power series ring $\mathbb{K}[[x_1, \dots, x_m]]$, with pairing

$$\langle \partial_1^{n_1} \cdots \partial_m^{n_m}, f(x)\rangle = \frac{\partial^{n_1+\cdots+n_m} f}{\partial x_1^{n_1} \cdots \partial x_m^{n_m}}(0) = (\prod_i n_i!)\cdot(\text{coefficient of } x_1^{n_1} \cdots x_m^{n_m} \text{ in } f(x)).$$

Proposition 8.4. *Let V be a graded vector space; for every locally nilpotent cocommutative graded coalgebra (C,Δ) the composition with the projection ($\Gamma\colon \overline{S(V)}\to V$) $\mathcal{P}\colon \overline{S(V)}\to V$, gives a bijective map*

$$\mathrm{Hom}_{\mathbf{GC}}(C,\overline{S(V)})\longrightarrow \mathrm{Hom}_{\mathbf{G}}(C,V),\qquad f\mapsto \mathcal{P}f,$$

with inverse

$$f\mapsto \mathcal{P}^*f=\sum_{n=1}^{+\infty}\frac{S(f)\circ\pi}{n!}\Delta^{n-1}=\sum_{n=1}^{+\infty}\frac{\pi\circ T(f)}{n!}\Delta^{n-1}\colon C\to\overline{S(V)},$$

where $\pi\colon T(C)\to S(C)$, $\pi\colon T(V)\to S(V)$ are the projections.

Notice that

$$S(f)\circ\pi(c_1\otimes\cdots\otimes c_n)=\pi\circ T(f)(c_1\otimes\cdots\otimes c_n)=m(c_1)\odot\cdots\odot m(c_n)$$

.

Proof. Since $\mathcal{P}\mathcal{P}^*(f)=f$, $\mathcal{P}\colon\overline{S(V)}\to V$ is a system of cogenerators and N is an injective morphism of coalgebras, it is sufficient to prove that $N\circ\mathcal{P}^*(f)\colon C\to\overline{T(V)}$ is a morphism of graded coalgebras. According to Lemma 6.9 the image of Δ^n is contained in the subspace of symmetric tensors and therefore

$$\Delta^{n-1}=N\circ\frac{\pi}{n!}\Delta^{n-1},$$

$$N\theta(m)=\sum_{n=1}^{+\infty}\frac{N\circ S(f)\circ\pi}{n!}\Delta^{n-1}=\sum_{n=1}^{+\infty}\frac{T(f)\circ N\circ\pi}{n!}\Delta^{n-1}=\sum_{n=1}^{+\infty}T(f)\circ\Delta^{n-1}$$

and the conclusion follows from Proposition 3.2. ☐

Corollary 8.5. *Let C be a locally nilpotent cocommutative graded coalgebra, and V a graded vector space. A morphism $\theta\in\mathrm{Hom}_{\mathbf{G}}(C,\overline{S(V)})$ is a morphism of graded coalgebras if and only if there exists $m\in\mathrm{Hom}_{\mathbf{G}}(C,V)\subset\mathrm{Hom}_{\mathbf{G}}(C,\overline{S(V)})$ such that*

$$\theta=\exp(m)-1\ =\sum_{n=1}^{\infty}\frac{1}{n!}m^n,$$

being the n-th power of m is considered with respect to the algebra structure on $\mathrm{Hom}_{\mathbf{G}}(C,\overline{S(V)})$ (Example 1.4).

Proof. An easy computation gives the formula $m^n=S(m)\pi\Delta^{n-1}$ for the product defined in Example 1.4. ☐

Proposition 8.6. *Let V be a graded vector space and C a locally nilpotent cocommutative coalgebra. Then for every coalgebra morphism $\theta\colon C\to\overline{S(V)}$ and every integer k, the composition with $N\colon\overline{S(V)}\to\overline{T(V)}$ gives an isomorphism*

$$\mathrm{Coder}^k(C,\overline{S(V)};\theta)\simeq\mathrm{Coder}^k(C,\overline{T(V)};N\theta).$$

Proof. We need to prove that if $Q\colon C\to\overline{T(V)}$ is a coderivation with respect to some morphism $\eta=N\theta$, then $Q=NP$ for some $P\colon C\to\overline{S(V)}$. According to Proposition 3.6 we have

$$Q=\sum_{n=0}^{\infty}\sum_{i=0}^{n}(f^{\otimes i}\otimes q\otimes f^{\otimes n-i})\Delta^n\colon C\to\overline{T(V)}$$

for some $f\in\mathrm{Hom}^0(C,V)$ and $q\in\mathrm{Hom}^k(C,V)$. Since C is cocommutative we have $N\Delta^n=(n+1)!\Delta^n$ and then

$$Q = \sum_{n=0}^{\infty}\sum_{i=0}^{n}(f^{\otimes i}\otimes q\otimes f^{\otimes n-i})\Delta^n = \sum_{n=0}^{\infty}\sum_{i=0}^{n}(f^{\otimes i}\otimes q\otimes f^{\otimes n-i})\frac{N}{(n+1)!}\Delta^n.$$

By Lemma 6.6

$$Q = \sum_{n=0}^{\infty}\frac{1}{n!}N(q\otimes f^{\otimes n})\frac{N}{(n+1)!}\Delta^n = N\sum_{n=0}^{\infty}\frac{1}{n!}(q\otimes f^{\otimes n})\Delta^n.$$

□

Corollary 8.7. *Let V be a graded vector space and (C,Δ) a locally nilpotent cocommutative coalgebra. Then for every coalgebra morphism $\theta\colon C\to\overline{S(V)}$ and every integer n, the composition with the projection $\mathcal{P}\colon\overline{S(V)}\to V$ gives a bijective map*

$$\mathrm{Coder}^n(C,\overline{S(V)};\theta)\to\mathrm{Hom}^n(C,V),\qquad Q\mapsto\mathcal{P}Q,$$

with inverse

$$q\mapsto\sum_{n=1}^{+\infty}\frac{\pi}{n!}(q\otimes(\theta^1)^{\otimes n})\Delta^n.$$

Proof. Immediate consequence of Propositions 3.6 and the same computation made in the proof of Proposition 8.6. □

Corollary 8.8. *Let V be a graded vector space, $\overline{S(V)}$ its reduced symmetric coalgebra. The application $Q\mapsto\{Q^1_k\}$ gives an isomorphism of vector spaces*

$$\mathrm{Coder}^n(\overline{S(V)},\overline{S(V)})\to\prod_{k=1}^{+\infty}\mathrm{Hom}^n(V^{\odot k},V)$$

whose inverse D is given by the formula

$$D(q_i)(v_1\odot\cdots\odot v_n)=\sum_{k=1}^{n}\sum_{\sigma\in S(k,n-k)}\epsilon(\sigma)q_k(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(k)})\odot v_{\sigma(k+1)}\odot\cdots\odot v_{\sigma(n)}.$$

In particular for every coderivation Q we have $Q^i_j=0$ for every $i>j$ and then the subcoalgebras $\bigoplus_{i=1}^r\bigodot^i V$ are preserved by Q.

Proof. As above we only need to prove that $D(q_i)$ is a coderivation. By linearity it is not restrictive to assume that $q_i=0$ for every $i\neq l$. Let $r\in\mathrm{Hom}^n(\bigotimes^l V,V)$ such that $rN=_l$ and let $R\in\mathrm{Coder}^n(\overline{T(V)},\overline{T(V)})$ the coderivation such that $R^1=r$; we will show that $R\circ N=N\circ D(q_i)$. According to Corollary 3.7

$$R(a_1\otimes\cdots\otimes a_n)= \\ =\sum_{i,l}(-1)^{k(\overline{a_1}+\cdots+\overline{a_i})}a_1\otimes\cdots\otimes a_i\otimes r(a_{i+1}\otimes\cdots\otimes a_{i+l})\otimes\cdots\otimes a_n.$$

and then, by Lemma 6.6

$$RN(a_1\odot\cdots\odot a_n)= \\ =N\left(\sum_{\sigma\in S(l,n-l)}\epsilon(\sigma)rN(a_{\sigma(1)}\odot\cdots\odot a_{\sigma(l)})\odot a_{\sigma(l+1)})\odot\cdots\odot a_{\sigma(n)}\right) \\ =N\left(\sum_{\sigma\in S(l,n-l)}\epsilon(\sigma)Q^1_a(a_{\sigma(1)}\odot\cdots\odot a_{\sigma(l)})\odot a_{\sigma(l+1)}\odot\cdots\odot a_{\sigma(n)}\right)$$

□

9. Q-manifolds

Definition 9.1 ([5, 4.3])**.** A *formal graded pointed Q-manifold* is the data $(V, q_1, q_2, \ldots)$ of a graded vector space V and a sequence of maps

$$q_n \in \operatorname{Hom}^1(V^{\odot n}, V), \quad n \geq 1,$$

such that the coderivation $D(q_n)$ (defined in Corollary 8.8) is a codifferential of the reduced symmetric coalgebra $\overline{S(V)}$.

For notational simplicity, from now we shall simply say Q-manifolds, omitting the adjectives formal, graded and pointed.

Lemma 9.2. *Let V be a graded vector space and $q_n \in \operatorname{Hom}^1(V^{\odot n}, V)$, for $n \geq 1$, be a sequence of maps. Then $D(q_n)$ is a codifferential, i.e. $D(q_n) \circ D(q_n) = 0$, if and only if for every $n > 0$ and every $v_1, \ldots, v_n \in V$*

$$\sum_{k+l=n+1} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma; v_1, \ldots, v_n) q_l(q_k(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}) = 0.$$

Proof. Denote $P = D(q_n) \circ D(q_n) = \frac{1}{2}[D(q_n), D(q_n)]$: since P is a coderivation we have that $P = 0$ if and only if $P^1 = D(q_n)^1 \circ D(q_n) = 0$. According to Corollary 8.8

$$D(q_n)(v_1 \odot \cdots \odot v_n) = \sum_{k=1}^{n} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma) q_k(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)}.$$

and then $P^1(v_1 \odot \cdots \odot v_n)$ is equal to the expression in the statement. □

In particular, if $(V, q_1, q_2, \ldots)$ is an Q-manifold, then (V, q_1) is a differential graded vector space.

Definition 9.3. A *morphism* $f_\infty \colon (V, q_i) \to (W, r_i)$ of Q-manifolds is a linear map

$$f_\infty \in \operatorname{Hom}^0(\overline{S(V)}, W)$$

such that the morphism $\mathcal{P}^* f_\infty \colon \overline{S(V)} \to \overline{S(W)}$ (defined in Proposition 8.4) is a morphism of differential graded coalgebras, i.e. $D(r_i)\mathcal{P}^* f_\infty = \mathcal{P}^* f_\infty D(q_i)$.

The composition of two morphisms $f_\infty \in \operatorname{Hom}^0(\overline{S(V)}, W)$, $g_\infty \in \operatorname{Hom}^0(\overline{S(U)}, V)$ is defined as

$$f_\infty \circ g_\infty = f_\infty(\mathcal{P}^* g_\infty) \in \operatorname{Hom}^0(\overline{S(U)}, W).$$

The category of Q-manifolds is equivalent to the full subcategory of **DGC** (differential graded coalgebras). If C is a differential graded coalgebra and $\mathfrak{g} = (V, q_i)$ is a Q-manifold we denote by

$$\operatorname{Mor}_{\mathbf{DGC}}(C, \mathfrak{g}) = \operatorname{Mor}_{\mathbf{DGC}}(C, (\overline{S(V)}, D(q_i))).$$

Remark 9.4. In Definition 9.3 it is sufficient to require $(\sum r_i)\mathcal{P}^* f_\infty = f_\infty D(q_n)$. In fact $D(r_i)\mathcal{P}^* f_\infty$ and $\mathcal{P}^* f_\infty D(q_i)$ are both $\mathcal{P}^* f_\infty$-coderivations and then $(\sum r_i)\mathcal{P}^* f_\infty = f_\infty D(q_n)$ if and only if $D(r_i)(\mathcal{P}^* f_\infty) = (\mathcal{P}^* f_\infty) D(q_i)$.

Given two Q-manifolds $\mathfrak{g}_1 = (V, q_1, q_2, \ldots)$, $\mathfrak{g}_2 = (W, r_1, r_2, \ldots)$ we denote

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 = (V \oplus W, q_1 \oplus r_1, q_2 \oplus r_2, \ldots)$$

where

$$q_n \oplus r_n(x) = \begin{cases} q_n(x) & \text{if } x \in V^{\odot n} \\ r_n(x) & \text{if } x \in W^{\odot n} \\ 0 & \text{if } x \in V^{\odot i} \otimes W^{\odot n-i} \text{ and } 0 < i < n. \end{cases}$$

It is immediate from Lemma 9.2 that $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Q-manifold.

The next sections will be devoted to the proof of the following important result.

Theorem 9.5. *Let $(V, q_1, q_2, \dots)$ be a Q-manifold and let $i\colon (H, d) \to (V, q_1)$ be an* ***injective quasiisomorphism*** *of complexes. Then there exist a Q-manifold structure $(H, r_1, r_2, \dots)$ and two morphisms of Q-manifolds*

$$\imath_\infty\colon (H, r_1, r_2, \dots) \to (V, q_1, q_2, \dots), \qquad \pi_\infty\colon (V, q_1, q_2, \dots) \to (H, r_1, r_2, \dots)$$

such that $r_1 = d$, $\imath_1 = \imath$ and $\pi_\infty \circ \imath_\infty = \mathrm{Id}$.

Remark 9.6. In the situation of Theorem 9.5 The Q-manifold structure $(H, r_1, r_2, \dots)$ is unique up to (non canonical) isomorphism. In fact if $(H, s_1, s_2, \dots)$, j_∞ and p_∞ is another triple, then

$$p_\infty \circ \imath_\infty\colon (H, r_1, r_2, \dots) \to (H, s_1, s_2, \dots)$$

is an isomorphism.

The proof will goes as follows: since $\imath$ is an injective quasiisomorphism there exists $h \in \mathrm{Hom}^{-1}(V, V)$ such that $\mathrm{Id}_V + [q_1, h]$ is a projection onto the image of $\imath$. Then we give an explicit construction, in terms of q_i, $\imath$ and h, of the maps $r_n, \imath_n$: this is done by using rooted tree formalism. Lastly we prove the existence of π_∞ and the unicity properties using an analog of the decomposition theorem of Q-manifolds.

10. Contractions

Definition 10.1 (Eilenberg and Mac Lane [1, p. 81])**.** A *contraction* is the data

$$(\ M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N\ , h)$$

where M, N are differential graded vector spaces, $h \in \mathrm{Hom}^{-1}(N, N)$ and $\imath, \pi$ are cochain maps such that:

(1) (deformation retraction) $\pi\imath = \mathrm{Id}_M$, $\imath\pi - \mathrm{Id}_N = d_N h + h d_N$,
(2) (annihilation properties) $\pi h = h\imath = h^2 = 0$.

The maps $\imath, \pi$ and h are referred as the *inclusion, projection* and *homotopy* of the contraction.

Definition 10.2. A *morphism* of contractions

$$f\colon (\ M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N\ , h) \to (\ A \underset{p}{\overset{i}{\rightleftarrows}} B\ , k)$$

is a morphism of differential graded vector spaces $f\colon N \to B$ such that $fh = kf$.

It is an easy exercise to prove that if

$$f\colon (\ M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N\ , h) \to (\ A \underset{p}{\overset{i}{\rightleftarrows}} B\ , k)$$

is a morphism of contractions then there exists an unique morphism of complexes $f'\colon M \to B$ such that $f'\pi = pf$ and $if' = f\imath$.

Remark 10.3. If $(\ M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N\ , h)$ is a contraction, then $h^2 = h + hd_N h = 0$. Conversely, every $h \in \operatorname{Hom}^{-1}(N, N)$ satisfying $h^2 = h + hd_N h = 0$ gives a contraction $(\ M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N\ , h)$ where $M = \ker(d_N h + hd_N)$, $\imath\colon M \to N$ is the inclusion and $\pi = \imath^{-1}(\operatorname{Id}_N + d_N h + hd_N)$.

Example 10.4.

$$\left(\ \mathbb{K} \underset{e_0}{\overset{\imath}{\rightleftarrows}} \mathbb{K}[t, dt]\ , -\int_0\right)$$

is a contraction, where e_0 is the evaluation at 0 and $\imath$ is the inclusion.

Example 10.5.

$$\left(\ \mathbb{K} \oplus \mathbb{K}t \oplus \mathbb{K}dt \underset{\pi}{\overset{\imath}{\rightleftarrows}} \mathbb{K}[t, dt]\ , t\int_0^1 - \int_0\right)$$

is a contraction, where

$$\pi(q(t) + p(t)dt) = tq(1) + (1-t)q(0) + \left(\int_0^1 p(s)ds\right) dt$$

and $\imath$ is the inclusion.

Lemma 10.6. *Let $\imath\colon M \hookrightarrow N$ be an injective morphism of differential graded vector spaces. Then $\imath$ is the inclusion of a contraction if and only if $\imath\colon H^*(M) \to H^*(N)$ is an isomorphism.*

Proof. One implication is clear: if $(\ M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N\ , h)$ is a contraction, then h is a homotopy between $\imath\pi$ and the identity on N.

Conversely, it is not restrictive to assume M a subcomplex of N and $\imath$ the inclusion; assume $H^*(M) = H^*(N)$ and denote by d the differential of N. Since $H^*(M) \to H^*(N)$ is injective we have

$$M \cap dN = Z(M) \cap dN = dM$$

and we can find a direct sum decomposition

$$dN = dM \oplus B, \quad B \cap M = \emptyset.$$

Moreover $H^*(M) \to H^*(N)$ is surjective and then

$$Z(N) = Z(M) + dN = Z(M) \oplus B.$$

Choosing a direct sum decomposition

$$d^{-1}(B) = Z(N) \oplus C$$

we have $(M \oplus B) \cap C = 0$. In fact, if $c = m + b$ with $c \in C$, $m \in M$ and $b \in B$, then $dc = dm \in B \cap M = 0$ and therefore $c \in Z(N) \cap C = 0$. Let now $n \in N$, there exist $m \in M$ such that $dn - dm \in B$ and then $n - m \in d^{-1}(B)$. We can write $n - m = a + c$, with $a \in Z(N) \subset M \oplus B$ and $c \in C$. Therefore $N = M + B + C$ and we have proved

$$N = M \oplus B \oplus C, \qquad d\colon C \xrightarrow{\simeq} B.$$

Define therefore $\pi\colon N \to M$ as the projection with kernel $C \oplus B$ and

$$h(m + b + c) = d^{-1}(b) \in C.$$

□

Definition 10.7. Given two contractions $(M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N , h)$ and $(N \underset{p}{\overset{i}{\rightleftarrows}} P , k)$, their *composition* is the contraction defined as

$$(M \underset{\pi p}{\overset{i\imath}{\rightleftarrows}} P , k + ihp).$$

Example 10.8. Given two contractions $(M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N , h)$ and $(A \underset{p}{\overset{i}{\rightleftarrows}} B , k)$ we define their tensor product as

$$(M \otimes A \underset{\pi\otimes p}{\overset{\imath\otimes i}{\rightleftarrows}} N \otimes B , h * k), \qquad h * k = \imath\pi \otimes k + h \otimes \mathrm{Id}_B .$$

Denoting by $\hat{d} = d \otimes \mathrm{Id}_B + \mathrm{Id}_N \otimes d$ the differential on $N \otimes B$, we have

$$\begin{aligned}
(h * k \circ \hat{d} + \hat{d} \circ h * k)(x \otimes y) = & \\
= h * k(dx \otimes y + (-1)^{\overline{x}} x \otimes dy) + \hat{d}(hx \otimes y + (-1)^{\overline{x}} \imath\pi(x) \otimes ky) & \\
= hdx \otimes y - (-1)^{\overline{x}} d\imath\pi(x) \otimes ky + (-1)^{\overline{x}} hx \otimes dy + \imath\pi(x) \otimes kdy + & \\
+ dhx \otimes y - (-1)^{\overline{x}} hx \otimes dy + (-1)^{\overline{x}} d\imath\pi(x) \otimes ky + \imath\pi(x) \otimes dky & \\
= (hd + dh)x \otimes y + \imath\pi(x) \otimes (kd + dk)y & \\
= \imath\pi x \otimes y - x \otimes y + \imath\pi(x) \otimes ip(y) - \imath\pi(x) \otimes y = (\imath\pi \otimes ip - \mathrm{Id}_N \otimes \mathrm{Id}_A)x \otimes y. &
\end{aligned}$$

It is straightforward to verify the annihilation properties of $h * k$ and the associativity of such tensor product.

Example 10.9. Given a contraction $(M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N , h)$, its tensor nth power is

$$\bigotimes\nolimits_R^n (M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N , h) = (M^{\otimes n} \underset{\pi^{\otimes n}}{\overset{\imath^{\otimes n}}{\rightleftarrows}} N^{\otimes n} , T^n h),$$

where

$$T^n h = \sum_{i=1}^{n} (\imath\pi)^{\otimes i-1} \otimes h \otimes \mathrm{Id}_N^{\otimes n-i} .$$

Since the differential on $N^{\otimes n}$ commutes with the twist action of the symmetric group Σ_n, we can take the simmetrization of $T^n h$

$$S^n h = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_{\mathtt{tw}} \circ T^n h \circ \sigma_{\mathtt{tw}}^{-1} .$$

In order to prove that $(M^{\otimes n} \underset{\pi^{\otimes n}}{\overset{\imath^{\otimes n}}{\rightleftarrows}} N^{\otimes n} , S^n h)$ is a contraction, the only non trivial condition to verify is $(S^n h)^2 = 0$. More generally we have that $T^n h \circ \sigma_{\mathtt{tw}} \circ T^n h \circ \sigma_{\mathtt{tw}}^{-1} = 0$ for every permutation σ: this is an exercise about Koszul rule of signs and it is left to the reader.

Exercise 10.10. Prove that if N contracts to M, then $\bigodot^n N$ contracts to $\bigodot^n M$.

11. Contracting Q-manifolds: recursive formulas

In this section $(V, q_1, q_2, \ldots)$ is a fixed Q-manifold and denote by

$$Q = D(q_n) \colon \overline{S(V)} \to \overline{S(V)}$$

the induced codifferential of degree 1. Denote also

$$q_+ = \sum_{i \geq 2} q_i \colon \overline{S(V)} \to V,$$

so that $Q^1 = q_1 + q_+$.
Assume to have a graded vector space W and a coderivation $\hat{Q} \colon \overline{S(W)} \to \overline{S(W)}$ of degree 1 such that $(W, \hat{Q}_1^1)$ is a differential graded vector space. Assume moreover to have two morphisms of differential graded vector spaces

$$\varphi_1^1 \colon W \to V, \qquad \pi \colon V \to W$$

and a homotopy $K \in \operatorname{Hom}^{-1}(V, V)$ between $\varphi_1^1 \circ \pi$ and Id_V, i.e.

$$q_1 \varphi_1^1 = \varphi_1^1 \hat{Q}_1^1, \qquad \pi q_1 = \hat{Q}_1^1 \pi, \qquad q_1 K + K q_1 = \varphi_1^1 \pi - \operatorname{Id}_V.$$

Theorem 11.1. *In the above set-up, assume that $\varphi \colon \overline{S(W)} \to \overline{S(V)}$ is a morphism of graded coalgebras lifting φ_1^1. If*

$$\varphi^1 = \varphi_1^1 + K q_+ \varphi, \qquad \hat{Q}^1 = \hat{Q}_1^1 + \pi q_+ \varphi, \tag{1}$$

then, denoting by $\hat{Q}$ the coderivation indeuced by $\hat{Q}^1$, we have

$$Q\varphi = \varphi \hat{Q}, \qquad \hat{Q}\hat{Q} = 0.$$

Remark 11.2. Using the projection operators $\mathcal{P}$ we have $\varphi_1^1 = \varphi\mathcal{P}$, $\varphi^1 = \mathcal{P}\varphi$ and then the equations 1 may be written as

$$\mathcal{P}\varphi = \varphi\mathcal{P} + K(\mathcal{P}Q - Q\mathcal{P})\varphi, \qquad \mathcal{P}\hat{Q} = \hat{Q}\mathcal{P} + \pi(\mathcal{P}Q - Q\mathcal{P})\varphi,$$

or, in a more compact form,

$$[\mathcal{P}, \varphi] = K[\mathcal{P}, Q]\varphi, \qquad [\mathcal{P}, \hat{Q}] = \pi[\mathcal{P}, Q]\varphi.$$

Proof. (D. Fiorenza [2]) We first prove that

$$(Q\varphi - \varphi\hat{Q})^1 = K q_+ (Q\varphi - \varphi\hat{Q}).$$

We have

$$\begin{aligned}
(Q\varphi - \varphi\hat{Q})^1 &= Q^1\varphi - \varphi^1\hat{Q} = q_1\varphi^1 + q_+\varphi - \varphi^1\hat{Q} \\
&= q_1\varphi_1^1 + q_1 K q_+\varphi + q_+\varphi - \varphi_1^1\hat{Q}^1 - K q_+\varphi\hat{Q} \\
&= q_1\varphi_1^1 + (\varphi_1^1\pi - \operatorname{Id}_V - K q_1) q_+\varphi + q_+\varphi - \varphi_1^1\hat{Q}^1 - K q_+\varphi\hat{Q} \\
&= q_1\varphi_1^1 + \varphi_1^1\pi q_+\varphi - K q_1 q_+\varphi - \varphi_1^1\hat{Q}^1 - K q_+\varphi\hat{Q} \\
&= q_1\varphi_1^1 + \varphi_1^1\pi q_+\varphi - K q_1 q_+\varphi - \varphi_1^1\hat{Q}_1^1 - \varphi_1^1\pi q_+\varphi - K q_+\varphi\hat{Q} \\
&= (q_1\varphi_1^1 - \varphi_1^1\hat{Q}_1^1) - K q_1 q_+\varphi - K q_+\varphi\hat{Q} \\
&= -K q_1 q_+\varphi - K q_+\varphi\hat{Q}.
\end{aligned}$$

Since $0 = Q^1 Q = q_1 Q^1 + q_+ Q = q_1 q_+ + q_+ Q$ we have $q_1 q_+ = -q_+ Q$ and therefore

$$(Q\varphi)^1 - (\varphi\hat{Q})^1 = -K q_1 q_+ \varphi - K q_+ \varphi\hat{Q} = K q_+ (Q\varphi - \varphi\hat{Q}).$$

The map

$$\delta = Q\varphi - \varphi\hat{Q} \colon \overline{S(W)} \to \overline{S(V)}$$

is a φ-derivation and then, in order to prove that $\delta = 0$, it is sufficient to show that $\delta^1 = 0$. We shall prove by induction on n that δ^1 vanishes on $\bigodot^n W$; for $n = 0$ there is nothing to prove. Let's assume $n > 0$ and $\delta^1(\bigodot^i W) = 0$ for every $i < n$, then by coLeibniz rule, for every $w \in \bigodot^n W$ we have $\delta(w) = \delta^1(w) \in V$ and therefore

$$\delta^1(w) = Kq_+\delta(w) = Kq_+\delta^1(w) = 0.$$

We also have

$$(\hat{Q}\hat{Q})^1 = \hat{Q}^1\hat{Q} = \hat{Q}^1_1\hat{Q} + \pi q_+\varphi\hat{Q} =$$
$$= \hat{Q}^1_1\hat{Q}^1 + \pi q_+ Q\varphi = \hat{Q}^1_1\pi q_+\varphi + \pi q_+ Q\varphi = \pi(q_1q_+ + q_+Q)\varphi.$$

We have already noticed that $q_1q_+ = -q_+Q$ and then $(\hat{Q}\hat{Q})^1 = 0$. □

Remark 11.3. For later use, we point out that, since $(q_+\varphi)^1_1 = 0$, the equalities $\varphi^1 = \varphi^1_1 + Kq_+\varphi$ and $\hat{Q}^1 = \hat{Q}^1_1 + \pi q_+\varphi$ of Theorem 11.1, are equivalent to

$$\varphi^1_n = K\sum_{i=2}^n q_i\varphi^i_n, \qquad \hat{Q}^1_n = \pi\sum_{i=2}^n q_i\varphi^i_n, \quad \forall\, n \geq 2.$$

According to Corollary 3.4, every φ^i_n depends only of $\varphi^1_1, \varphi^1_2, \ldots, \varphi^1_{n-i+1}$ and then the hypothesis of Theorem 11.1 implies that φ and $\hat{Q}$ are recursively determined by φ^1_1, π, K and q_n for $n \geq 1$.

Corollary 11.4. *Let $(V, q_1, q_2, \ldots)$ be a Q-manifold and let $\varphi^1_1\colon (W, r_1) \to (V, q_1)$ be an injective quasiisomorphism of differential graded vector spaces. Then (W, r_1) can be extended to a Q-manifold $(W, r_1, r_2, \ldots)$ and φ^1_1 can be lifted to a morphism of Q-manifolds.*

Proof. According to Lemma 10.6, we can find a morphism of complexes $\pi\colon (V, q_1) \to (W, r_1)$ and a homotopy $K \in \operatorname{Hom}^{-1}(V, V)$ such that

$$q_1K + Kq_1 = \varphi^1_1\pi - \mathrm{Id}_V, \qquad \pi\varphi^1_1 = \mathrm{Id}_W.$$

It is sufficient to define recursively $\varphi^1_n = \sum_{i=2}^n (Kq_i)\varphi^i_n$ as in Remark 11.3; then define $r_n = \sum_{i=2}^n (\pi q_i)\varphi^i_n$ and apply Theorem 11.1. □

Remark 11.5. The formulas of Corollary 11.4 commutes with composition of contractions. Given two contractions $(M \underset{\pi}{\overset{\imath}{\rightleftarrows}} N , h)$, $(N \underset{p}{\overset{i}{\rightleftarrows}} P , k)$, their composition $(M \underset{\pi p}{\overset{i\imath}{\rightleftarrows}} P , k + ihp)$ and a codifferential $Q\colon \overline{S(P)} \to \overline{S(P)}$ there exists two morphisms of graded coalgebras $\varphi\colon \overline{S(N)} \to \overline{S(P)}$, $\psi\colon \overline{S(M)} \to \overline{S(N)}$ and two codifferentials $\hat{Q}\colon \overline{S(N)} \to \overline{S(N)}$, $\tilde{Q}\colon \overline{S(M)} \to \overline{S(M)}$ uniquely defined by the system of equations

$$[\mathcal{P}, \varphi] = k[\mathcal{P}, Q]\varphi, \quad [\mathcal{P}, \hat{Q}] = p[\mathcal{P}, Q]\varphi, \quad \varphi\mathcal{P} = i,$$

$$[\mathcal{P}, \psi] = h[\mathcal{P}, \hat{Q}]\psi, \quad [\mathcal{P}, \tilde{Q}] = \pi[\mathcal{P}, \hat{Q}]\psi, \quad \psi\mathcal{P} = \imath.$$

Then

$$\begin{aligned}[\mathcal{P}, \varphi\psi] &= [\mathcal{P}, \varphi]\psi + \varphi[\mathcal{P}, \psi] = k[\mathcal{P}, Q]\varphi\psi + \varphi h[\mathcal{P}, \hat{Q}]\psi \\ &= k[\mathcal{P}, Q]\varphi\psi + \varphi hp[\mathcal{P}, Q]\varphi\psi = k[\mathcal{P}, Q]\varphi\psi + ihp[\mathcal{P}, Q]\varphi\psi \\ &= (k + ihp)[\mathcal{P}, Q]\varphi\psi.\end{aligned}$$

$$[\mathcal{P}, \tilde{Q}] = \pi[\mathcal{P}, \hat{Q}]\psi = \pi p[\mathcal{P}, Q]\varphi\psi.$$

Corollary 11.6. *Let $(V, q_1, q_2, q_3, \ldots)$ be an acyclic Q-manifold, where acyclic means that the complex (V, q_1) is acyclic. Then $(V, q_1, q_2, q_3, \ldots)$ is isomorphic to $(V, q_1, 0, 0, \ldots)$.*

Proof. Apply the theorem with $W = V$, $\varphi_1^1 = \mathrm{Id}_V$, $\pi = 0$ and K any homotopy between 0 and Id_V. □

12. Contracting Q-manifolds: global formulas

In this section we will give a description of the morphism φ and the coderivation $\hat{Q}$ of Theorem 11.1 as a sum over rooted trees. We first need the analog of Lemma 5.2 for reduced symmetric coalgebras. Notice that, since $\mathrm{Hom}^0(V^{\odot n}, V) \subseteq \mathrm{Hom}^0(V^{\otimes n}, V)$ (see Remark 7.1), it makes sense to consider the operators $Z_\Gamma(h_i) \in \mathrm{Hom}^0(V^{\otimes n}, V^{\otimes m})$ for every oriented rooted forest Γ and every sequence $h_n \in \mathrm{Hom}^0(V^{\odot n}, V)$.

Lemma 12.1. *Let V, W be graded vector spaces. Given $\imath \in \mathrm{Hom}^0(W, V)$ and a sequence of maps $h_n \in \mathrm{Hom}^0(V^{\odot n}, V)$, $n \geq 2$.*
Then, for every $n, m \geq 1$ there exists $f_n^m \in \mathrm{Hom}^0(W^{\odot n}, V^{\odot m})$ such that

$$N \circ f_n^m = \sum_{\Gamma \in \frac{F(n,m)}{\sim}} \frac{1}{|\mathrm{Aut}(\Gamma)|} Z_\Gamma(h_i) \circ (\otimes^n \imath) \circ N \colon W^{\odot n} \to V^{\otimes m}.$$

Moreover

$$\sum_{n,m \geq 1} f_n^m \colon \overline{S(V)} \to \overline{S(V)}$$

is a morphism of graded coalgebras and, for every $n \geq 1$

$$f_n^1 = \sum_{\Gamma \in \frac{F(n,1)}{\sim}} \frac{1}{|\mathrm{Aut}(\Gamma)|} Z_\Gamma(h_i) \circ (\otimes^n \imath) \circ N = \sum_{a=2}^{n} h_a \circ f_n^a.$$

Proof. For every $n \geq 2$ let $g_n \in \mathrm{Hom}^0(V^{\otimes n}, V)$ be such that $h_n = g_n N$ (e.g. $g_n = h_n/n!$). By Lemma 5.2 the morphism

$$\sum F_n^m \colon \overline{T(W)} \to \overline{T(V)}, \qquad F_n^m = \sum_{\Gamma \in F(n,m)} Z_\Gamma(g_i) \circ (\otimes^n \imath)$$

is a morphism of graded coalgebras. According to Lemma 6.7

$$F_n^m \circ N = \sum_{\Gamma \in F(n,m)} Z_\Gamma(g_i) \circ N \circ (\odot^n \imath) =$$

$$= \sum_{\Gamma \in \frac{F(n,m)}{\sim}} \frac{1}{|\mathrm{Aut}(\Gamma)|} N \circ Z_\Gamma(g_i N) \circ N \circ (\odot^n \imath)$$

$$= N \circ \sum_{\Gamma \in \frac{F(n,m)}{\sim}} \frac{1}{|\mathrm{Aut}(\Gamma)|} Z_\Gamma(h_i) \circ (\otimes^n \imath) \circ N.$$

Therefore there exists f_n^m such that

$$N \circ f_n^m = F_n^m \circ N$$

and then the f_n^m are the components of a morphism of graded symmetric coalgebras. By Lemma 5.3 we have

$$F_n^1 = \sum_{a=2}^{n} g_a \circ F_n^a,$$

and then

$$f_n^1 = F_n^1 \circ N = \sum_{a=2}^{n} g_a \circ F_n^a \circ N = \sum_{a=2}^{n} g_a \circ N \circ f_n^a = \sum_{a=2}^{n} h_a \circ f_n^a.$$

□

It is now convenient to introduce a new formalism. Assume there are given $K \in \mathrm{Hom}^{-1}(V,V)$ and a sequence of maps $q_n \in \mathrm{Hom}^1(V^{\odot n}, V)$, $n \geq 2$.
For every **oriented tree** $\Gamma \in F(n,1)$, denote by

$$Z_\Gamma(K, q_i) \in \mathrm{Hom}^1(V^{\otimes n}, V)$$

the composite operator described by the tree Γ, where every internal vertex of arity k is decorated by q_k and every internal edge is decorated by K.
The relation between $Z_\Gamma(K, q_i)$ and $Z_\Gamma(Kq_i)$ is easy to describe: in fact if $n > 1$ then $Z_\Gamma(Kq_i) = K \circ Z_\Gamma(K, q_i)$, while if $\Gamma = \mathbb{T}_k \circ \Omega$ with $\Omega \in F(n,k)$, then $Z_\Gamma(K, q_i) = q_k \circ Z_\Gamma(Kq_i)$.

It is now easy to prove the following theorem.

Theorem 12.2. *Let $(V, q_1, q_2, \ldots)$ be a Q-manifold and let*

$$\pi \colon (V, q_1) \to (H, r_1), \qquad \imath \colon (H, r_1) \to (V, q_1)$$

be two morphism of complexes such that $\pi\imath = \mathrm{Id}_H$.
Assume that there exists $K \in \mathrm{Hom}^{-1}(V,V)$ such that

$$\mathrm{Id}_V + q_1 K + K q_1 = \imath\pi.$$

Then $(H, r_1, r_2, \ldots)$ is a Q-manifold, where for every $n \geq 2$

$$r_n(a_1 \odot \cdots \odot a_n) = \sum_{\Gamma \in \frac{F(n,1)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) \pi Z_\Gamma(K, q_i)(\imath(a_{\sigma(1)}) \otimes \cdots \otimes \imath(a_\sigma(n))),$$

and $\imath_\infty \colon (H, r_1, r_2, \ldots) \to (V, q_1, q_2, \ldots)$ is a morphism of Q-manifold, where $\imath_1 = \imath$ and, for $n \geq 2$

$$\imath_n(a_1 \odot \cdots \odot a_n) = \sum_{\Gamma \in \frac{F(n,1)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) K Z_\Gamma(K, q_i)(\imath(a_{\sigma(1)}) \otimes \cdots \otimes \imath(a_\sigma(n))).$$

Proof. We define i_n and r_n as in Corollary 11.4 and then we only need to prove the explicit formulas. According to Lemma 12.1 we have for every $n \geq 2$

$$\imath_n = \sum_{\Gamma \in \frac{F(n,1)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} Z_\Gamma(Kq_i) \circ N \circ S(\imath) = K \circ \sum_{\Gamma \in \frac{F(n,1)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} Z_\Gamma(K, q_i) \circ N \circ S(\imath).$$

Again by Lemma 12.1 we have

$$N \circ \imath_n^m = N \circ \sum_{\Gamma \in \frac{F(n,m)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} Z_\Gamma(Kq_i) \circ (\otimes^n \imath) \circ N.$$

Therefore

$$r_n = \sum_{m=2}^{n} (\pi q_m) \imath_n^m = \sum_{m=2}^{n} \pi \frac{q_m}{m!} \circ N \circ \imath_n^m =$$

$$\sum_{m=2}^{n} \pi \frac{q_m}{m!} \circ N \circ \sum_{\Gamma \in \frac{F(n,m)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} Z_\Gamma(Kq_i) \circ (\otimes^n \imath) \circ N$$

$$= \sum_{m=2}^{n} \pi q_m \circ \sum_{\Gamma \in \frac{F(n,m)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} Z_\Gamma(Kq_i) \circ (\otimes^n \imath) \circ N$$

$$= \pi \sum_{\Gamma \in \frac{F(n,1)}{\sim}} \frac{1}{|\operatorname{Aut}(\Gamma)|} Z_\Gamma(K, q_i) \circ (\otimes^n \imath) \circ N.$$

□

Exercise 12.3. Use Exercise 8.3, inversion formula 5.4 and symmetrization to prove the tree formula for reversion of power series of [9] (if you haven't full text article it is sufficient to consult Math. Reviews).

13. Homotopy classification of Q-manifolds

Definition 13.1. A morphism $\{f_n\}: (V, q_1, q_2, \ldots) \to (W, r_1, r_2, \ldots)$ of Q-manifolds is called:

(1) *linear* (sometimes *strict*) if $f_n = 0$ for every $n > 1$.
(2) *quasiisomorphism* if $f_1: (V, q_1) \to (W, r_1)$ is a quasiisomorphism of complexes.

Given two Q-manifolds $\mathfrak{g}_1 = (V, q_1, q_2, \ldots)$, $\mathfrak{g}_2 = (W, r_1, r_2, \ldots)$ we denote

$$\mathfrak{g}_1 \oplus \mathfrak{g}_2 = (V \oplus W, q_1 \oplus r_1, q_2 \oplus r_2, \ldots)$$

where

$$q_n \oplus r_n(x) = \begin{cases} q_n(x) & \text{if } x \in V^{\odot n} \\ r_n(x) & \text{if } x \in W^{\odot n} \\ 0 & \text{if } x \in V^{\odot i} \otimes W^{\odot n-i} \text{ and } 0 < i < n. \end{cases}$$

It is immediate from Lemma 9.2 that $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Q-manifold. The natural inclusions

$$i_1: \mathfrak{g}_1 \to \mathfrak{g}_1 \oplus \mathfrak{g}_2, \qquad i_2: \mathfrak{g}_1 \to \mathfrak{g}_2 \oplus \mathfrak{g}_2$$

and the natural projections

$$p_1: \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_1, \qquad p_2: \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_2$$

are linear morphisms.

Proposition 13.2. *In the notation above, the diagram*

$$\begin{array}{ccc} \mathfrak{g}_1 \oplus \mathfrak{g}_2 & \xrightarrow{p_1} & \mathfrak{g}_1 \\ \downarrow{\scriptstyle p_2} & & \\ \mathfrak{g}_2 & & \end{array}$$

is a product in the category of locally nilpotent cocommutative differential graded coalgebras.

Proof. Assume that C is a locally nilpotent cocommutative differential graded coalgebra and let

$$F: C \to \mathfrak{g}_1, \qquad G: C \to \mathfrak{g}_2$$

be two morphisms of differential graded coalgebras. According to Proposition 8.4 there exists an unique morphism of graded coalgebras

$$H: C \to \overline{S(V \oplus W)}$$

such that

$$H^1 = F^1 \oplus G^1: C \to V \oplus W$$

and then $p_1 H = F$, $p_2 H = G$. Denoting by d the codifferential of C we have

$$H^1 \circ d = (F^1 \circ d) \oplus (G^1 \circ d) = D(q_i)^1 \circ F \oplus D(r_i)^1 \circ G = D(q_i \times r_i)^1 \circ H$$

and then $Hd = D(q_i \times r_i)H$. □

Proposition 13.3. *Let (C,Δ,d) be a differential graded cocommutative coalgebra and $B\subset C$ a differential graded subcoalgebra such that $\Delta(C)\subset B\otimes B$ and the complex C/B is acyclic. Then for every Q-manifold $\mathfrak{g}$ the restriction map*

$$\mathrm{Mor}_{\mathbf{DGC}}(C,\mathfrak{g})\to\mathrm{Mor}_{\mathbf{DGC}}(B,\mathfrak{g})$$

is surjective.

Proof. Assume $\mathfrak{g}=(V,q_1,q_2,\ldots)$ and let $f\colon(B,d)\to(\overline{S(V)},D(q_i))$ be a morphism of differential graded coalgebras. Choosing any lifting of $f^1\colon B\to V$ to a morphism $g^1\colon C\to V$, we get a morphism of graded coalgebras $g\colon C\to\overline{S(V)}$ extending f.
The morphism

$$\psi:=D(q_i)g-gd\colon C\to\overline{S(V)}$$

is a g-coderivation. Since $\psi(B)=0$ and $\Delta(C)\subset B$, by Corollary 8.7 we have $\psi(C)\subset V$ and then we have a factorization

$$\psi\colon\frac{C}{B}\to V.$$

Since

$$0=D(q_i)\psi+\psi d=q_1\psi+\psi d$$

and C/B is acyclic, there exists $\phi\colon C\to V$ such that $\phi(B)=0$ and $q_1\phi-\phi d=\psi$. Denote by $h\colon C\to\overline{S(V)}$ the coalgebra such that $h^1=g^1-\phi$. It is now straightforward to check that $h=g-\phi$ and h is a morphism of differential graded coalgebras. □

Definition 13.4. An Q-manifold $(V,q_1,q_2,\ldots)$ is called *linear contractible* if (V,q_1) is an acyclic complex and $q_j=0$ for every $j>1$.

Lemma 13.5. *Let $\mathfrak{u}=(U,d,0,\ldots)$ be a linear contractible Q-manifold and*

$$f_\infty\colon\mathfrak{g}\to\mathfrak{h}=(W,r_1,r_2,\ldots)$$

be a morphism of Q-manifolds. Then for every morphism of complexes $j\colon(U,d)\to(W,r_1)$ there exists a morphism of Q-manifolds

$$g_\infty\colon\mathfrak{g}\oplus\mathfrak{u}\to\mathfrak{h}$$

such that $g_{\infty|\mathfrak{g}}=f_\infty$ and $g_1(u)=j(u)$ for every $u\in U$.

Proof. Suppose $\mathfrak{g}=(V,q_1,q_2,\ldots)$ and consider the filtration of differential subcoalgebras

$$C_n=\overline{S(V)}\oplus\oplus_{i=1}^n(V\oplus U)^{\odot i}\subset\overline{S(V\oplus U)}.$$

We have $\Delta(C_n)\subset C_{n-1}\times C_{n-1}$; the quotient C_n/C_{n-1} is isomorphic to $\oplus_{i=1}^n U^{\odot i}\otimes V^{\odot n-i}$ and then it is acyclic by Künneth formula. We can apply Proposition 13.3. □

Theorem 13.6. *Let*

$$f_\infty\colon(H,r_1,r_2,\ldots)\to(V,q_1,q_2,\ldots)$$

be a morphism of Q-manifolds such that $f_1\colon(H,r_1)\to(V,q_1)$ is an ***injective quasi-isomorphism*** *of complexes. Then there exist a morphism*

$$p_\infty\colon(V,q_1,q_2,\ldots)\to(H,r_1,r_2,\ldots)$$

such that $p_\infty\circ f_\infty=\mathrm{Id}$.

Proof. By Lemma 10.6 we have a direct sum decomposition $V=f_1(H)\oplus U$, with U acyclic subcomplex of (V,q_1). According to Lemma 13.5 the morphism f_∞ extents to an isomorphism

$$g_\infty\colon(H,r_1,r_2,\ldots)\oplus(U,q_1,0,\ldots)\to(V,q_1,q_2,\ldots).$$

We can take p_∞ the composition of the inverse of g_∞ with the projection onto the first factor. □

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