CHAPTER 8

Totalization of semicosimplicial DG-vector spaces

8.1. Simplicial objects

Let $\mathbf{\Delta}$ be the category of finite ordinals: the objects are objects are $[0] = \{0\}$, $[1] = \{0,1\}$, $[2] = \{0,1,2\}$ ecc. and morphisms are the non decreasing maps.

Finally $\mathbf{\Delta}_{\text{mon}}$ is the category with the same objects as above and whose morphisms are order-preserving injective maps among them.

In order to avoid heavy notations it is convenient to denote also $[n] = \emptyset$ for every $n < 0$ and write

$$M(n,m) = \mathrm{Mor}_{\mathbf{\Delta}}([n],[m]) = \{f\colon \{0,1,\dots,n\} \to \{0,1,\dots,m\} \mid f(i) \le f(i+1)\},$$
$$I(n,m) = \mathrm{Mor}_{\mathbf{\Delta}_{\text{mon}}}([n],[m]) = \{f\colon \{0,1,\dots,n\} \to \{0,1,\dots,m\} \mid f(i) < f(i+1)\}.$$

Every morphism in $\mathbf{\Delta}_{\text{mon}}$, different from the identity, is a finite composition of **coface** morphisms:

$$\partial_k\colon [i-1] \to [i], \qquad \partial_k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } k \le p \end{cases}, \qquad k = 0,\dots,i.$$

Equivalently ∂_k is the unique strictly monotone map whose image misses k.

More generally, every morphism in $\mathbf{\Delta}$ is a finite composition of coface morphisms and **codegeneracy** morphisms

$$s_k\colon [i+1] \to [i], \qquad s_k(p) = \begin{cases} p & \text{if } p \le k \\ p-1 & \text{if } k > p \end{cases}, \qquad k = 0,\dots,i.$$

Equivalently s_k is the unique surjective monotone hitting k twice.

The relations about compositions of cofaces and codegeneracies are generated by the **cosimplicial identities** (see e.g. [36]):

(1) $\partial_l\partial_k = \partial_{k+1}\partial_l$ for every $l \le k$;
(2) $\partial_l s_k = s_{k+1}\partial_l$ for every $l \le k$;
(3) $s_k\partial_k = s_k\partial_{k+1} = Id$;
(4) $\partial_l s_k = s_k\partial_{l+1}$ for every $l > k$;
(5) $s_l s_k = s_k s_{l+1}$ for every $k \le l$.

Definition 8.1.1 ([123])**.** Let $\mathbf{C}$ be a category:

(1) A *cosimplicial* object in $\mathbf{C}$ is a covariant functor $A^{\Delta}\colon \mathbf{\Delta} \to \mathbf{C}$.
(2) A *semicosimplicial* object in $\mathbf{C}$ is a covariant functor $A^{\Delta}\colon \mathbf{\Delta}_{\text{mon}} \to \mathbf{C}$.
(3) A *simplicial* object in $\mathbf{C}$ is a contravariant functor $A_{\Delta}\colon \mathbf{\Delta} \to \mathbf{C}$.
(4) A *semisimplicial* object in $\mathbf{C}$ is a contravariant functor $A_{\Delta}\colon \mathbf{\Delta}_{\text{mon}} \to \mathbf{C}$.

Example 8.1.2. Giving a semicosimplicial object A^{Δ} is the same of giving a diagram

$$A_0 \rightrightarrows A_1 \overset{\longrightarrow}{\rightrightarrows} A_2 \overset{\rightrightarrows}{\rightrightarrows} \cdots,$$

where each A_i is in $\mathbf{C}$, and, for each $i > 0$, there are $i+1$ morphisms

$$\partial_k\colon A_{i-1} \to A_i, \qquad k = 0,\dots,i,$$

such that $\partial_l\partial_k = \partial_{k+1}\partial_l$, for any $l \le k$.

Example 8.1.3. Let $\mathbb{K}$ be a field. Define the standard n-simplex over $\mathbb{K}$ as the affine space

$$\Delta^n = \{(t_0,\dots,t_n) \in \mathbb{K}^{n+1} \mid t_0 + t_1 + \cdots + t_n = 1\}.$$

The vertices of Δ^n are the points

$$e_0 = (1,0,\dots,0), \quad e_1 = (0,1,\dots,0),\dots, e_n = (0,0,\dots,1).$$

Then the family $\{\Delta^n\}$, $n \geq 0$, is a cosimplicial affine space, where for every monotone map $f\colon [n] \to [m]$ we set $f\colon \Delta^n \to \Delta^m$ as the affine map such that $f(e_i) = e_{f(i)}$. Equivalently $f(t_0, \dots, t_n) = \sum t_i e_{f(i)} = (u_0, \dots, u_m)$, where

$$u_i = \sum_{\{j \mid f(j)=i\}} t_j \qquad \text{(we intend that } \sum_{\emptyset} t_j = 0).$$

In particular, for $m = n + 1$ we have

$$\partial_k(t_0, \dots, t_n) = (t_0, \dots, t_{k-1}, 0, t_k, \dots, t_n),$$

and this explain why ∂_k is called face map.

Example 8.1.4 ([22])**.** For every $0 \leq p \leq n$, let Ω_n^p be the vector space of polynomial differential p-forms on the standard n-simplex Δ^n. Then, the space of polynomial differential forms on the standard n-simplex

$$\Omega_n = \bigoplus_{p=0}^{n} \Omega_n^p = \frac{\mathbb{K}[t_0, \dots, t_n, dt_0, \dots, dt_n]}{(1 - \sum t_i, \sum dt_i)}$$

is a differential graded algebra. Notice that there exists a natural isomorphism of differential graded algebras

$$\mathbb{K}[t_1, \dots, t_n, dt_1, \dots, dt_n] \to \Omega_n.$$

Since every affine map $f\colon \Delta^n \to \Delta^m$ induce by pull-back a morphism of differential graded algebra $f^*\colon \Omega_m \to \Omega_n$ we have that the sequence $\Omega_\bullet = \{\Omega_n\}$ is a simplicial DG-algebra.

In particular the face maps $\partial_k^*\colon \Omega_n^p \to \Omega_{n-1}^p$, $k = 0, \dots, n$, are given by pull-back of forms under the inclusion of standard simplices

$$(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{k-1}, 0, t_k, \dots, t_{n-1}).$$

Let $X = \{X_n\}$ be a simplicial set and for every $f \in M(n, m)$ denote by $f^*\colon X_m \to X_n$ the corresponding map. In particular, dualizing the first cosimplicial identity we obtain

$$\partial_i^* \partial_j^* = \partial_{j-1}^* \partial_i^*, \qquad \text{for every} \quad i < j.$$

In particular, for $n \geq 2$, $x \in X_n$ and $x_i = \partial_i^* x \in X_{n-1}$ we have $\partial_i^* x_j = \partial_{j-1}^* x_i$ for every $i < j$.

Definition 8.1.5. A simplicial set $\{X_n\}$ is called an **acyclic Kan complex** if:

(1) the map $X_1 \to X_0 \times X_0$, $x \mapsto (\partial_0^* x, \partial_1^* x)$, is surjective;
(2) for every $n \geq 2$ and every sequence $x_0, \dots, x_n \in X_{n-1}$ such that

$$\partial_i^* x_j = \partial_{j-1}^* x_i \qquad \text{for every} \quad i < j,$$

there exists $x \in X_n$ such that $\partial_i^* x = x_i$ for every i.

Theorem 8.1.6. *The simplicial DG-algebra $\Omega_\bullet$ is an acyclic Kan complex.*

Proof. See [22]. □

8.2. Integration and Stokes formula

Lemma 8.2.1. *Let $\mathbb{K}$ be a field of characteristic 0, then there exists a unique sequence of linear maps*

$$\int_{\Delta^n} \colon \Omega_n \to \mathbb{K}, \qquad n \geq 0,$$

such that:

(1) $\displaystyle\int_{\Delta^n} \eta = 0$ *if $\eta \in \Omega_n^p$ and $p \neq n$.*

(2) $\displaystyle\int_{\Delta^0} \colon \Omega_0^0 = \frac{\mathbb{K}[t_0]}{(t_0 - 1)} \to \mathbb{K}, \qquad \int_0 p(t_0) = p(1).$

(3) $\displaystyle\int_{\Delta^n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{k_0! k_1! \cdots k_n!}{(k_0 + k_1 + \cdots + k_n + n)!}.$

(4) *(Stokes formula) For every $n > 0$ and $\omega \in \Omega_n^{n-1}$, we have*

$$\int_{\Delta^n} d\omega = \sum_{k=0}^{n} (-1)^k \int_{\Delta^{n-1}} \partial_k^* \omega.$$

Proof. The unicity follows from the first two conditions. To prove the existence, define

$$\int_{\Delta^n} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{k_1! \cdots k_n!}{(k_1 + \cdots + k_n + n)!}$$

and extend by $\mathbb{K}$ linearity to a map $\int_{\Delta^n} : \Omega_n^n \to \mathbb{K}$. We first prove by induction on k_0 the formula

$$\int_{\Delta^n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{k_0! k_1! \cdots k_n!}{(k_0 + k_1 + \cdots + k_n + n)!}.$$

Assume $k_0 > 0$ and denote $a = (k_0 - 1)! k_1! \cdots k_n!$, $b = k_0 + k_1 + \cdots + k_n + n$. Since

$$t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} = t_0^{k_0 - 1} t_1^{k_1} \cdots t_n^{k_n} (1 - \sum_{i=1}^{n} t_i),$$

by induction hypothesis, we have

$$\int_{\Delta^n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{a}{(b-1)!} - \sum_{i=1}^{n} \frac{a}{b!}(k_i + 1)$$

$$= \frac{a}{(b-1)!} - \frac{a}{b!}(b - k_0) = \frac{ab - a(b - k_0)}{b!} = \frac{k_0 a}{b!}.$$

Notice that the symmetric group $\mathfrak{S}_{n+1}$ acts on Ω_n by permutation of indices and, for every $\sigma \in \mathfrak{S}_{n+1}$, we have

$$\int_{\Delta^n} \sigma(\omega) = (-1)^{\sigma} \int_{\Delta^n} \omega.$$

(It is sufficient to check the above identity for transpositions).

By linearity, it is sufficient to prove Stokes formula for ω of type

$$\omega = t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n.$$

Up to permutation of indices, we may assume $i = n$. Assume first $k_n = 0$, i.e.,

$$\omega = t_1^{k_1} \cdots t_{n-1}^{k_{n-1}} dt_1 \wedge \cdots \wedge dt_{n-1}.$$

In this case, $d\omega = 0$, $\partial_k^* \omega = 0$ for every $k \neq 0, n$, and

$$\partial_0^* \omega = t_0^{k_1} \cdots t_{n-2}^{k_{n-1}} dt_0 \wedge \cdots \wedge dt_{n-2} = (-1)^{n-1} t_0^{k_1} \cdots t_{n-2}^{k_{n-1}} dt_1 \wedge \cdots \wedge dt_{n-1},$$

$$\partial_n^* \omega = t_1^{k_1} \cdots t_{n-1}^{k_{n-1}} dt_1 \wedge \cdots \wedge dt_{n-1};$$

therefore

$$\int_{\Delta^{n-1}} \partial_0^* \omega + (-1)^n \int_{\Delta^{n-1}} \partial_n^* \omega = 0.$$

Next, assume $k_n > 0$, then $\partial_k^* \omega = 0$ for every $k \neq 0$, and

$$\int_{\Delta^n} d\omega = \int_{\Delta^n} (-1)^{n-1} k_n t_1^{k_1} \cdots t_n^{k_n - 1} dt_1 \wedge \cdots \wedge dt_n = \frac{(-1)^{n-1} k_1! \cdots k_n!}{(k_1 + \cdots + k_n + n - 1)!},$$

$$\int_{\Delta^{n-1}} \partial_0^* \omega = \int_{\Delta^{n-1}} t_0^{k_1} \cdots t_{n-1}^{k_n} dt_0 \wedge \cdots \wedge dt_{n-2}$$

$$= (-1)^{n-1} \int_{\Delta^{n-1}} t_0^{k_1} \cdots t_{n-1}^{k_n} dt_1 \wedge \cdots \wedge dt_{n-1} = \frac{(-1)^{n-1} k_1! \cdots k_n!}{(k_1 + \cdots + k_n + n - 1)!}.$$

□

Exercise Prove that for $\mathbb{K} = \mathbb{R}$ the operator $\int_{\Delta^n}$ is equal to the usual integration on the topological simplex $\Delta^n \cap \{t_i \geq 0 \ \forall i\}$.

8.3. Homotopy operators

For every $n \geq -1$, consider the affine space

$$C^n = \{(s, t_0, t_1, \ldots, t_n) \in \mathbb{K}^{n+2} \mid s + \sum t_i = 1\}.$$

The identity on $\mathbb{K}^{n+2}$ induces an isomorphism $c\colon \Delta^{n+1} \to C^n$ and therefore an integration operator

$$\int_{C^n} \colon \frac{\mathbb{K}[s, t_0, \ldots, t_n, ds, dt_0, \ldots, dt_n]}{(s + \sum t_i - 1, ds + \sum dt_i)} \to \mathbb{K}, \qquad \int_{C^n} \eta = \int_{\Delta^n} c^*\eta.$$

We have affine maps

$$i\colon \Delta^n \to C^n, \qquad i(t_0, \ldots, t_n) = (0, t_0, \ldots, t_n)$$

and for every $f \in M(n, m)$ we also denote

$$f\colon C^n \to C^m, \qquad f(1, 0, \ldots, 0) = (1, 0, \ldots, 0), \quad f(e_i) = e_{f(i)},\ i \geq 0.$$

$$\widehat{f}\colon C^n \times \Delta^m \to \Delta^m, \quad \widehat{f}((s, t_0, \ldots, t_n), v) = sv + \sum t_i e_{f(i)},$$

$$\widetilde{f}\colon \Delta^n \times \Delta^m \to \Delta^m, \quad \widetilde{f}(u, v) = \widehat{f}(i(u), v).$$

Finally define for every $k = 0, \ldots, n$

$$\widehat{f}_k\colon C^{n-1} \times \Delta^m \to \Delta^m, \quad \widehat{f}_k(u, v) = \widehat{f}(\partial_k u, v).$$

Lemma 8.3.1. *In the notation above:*

(1) $\widehat{f}_k = \widehat{f\partial_k}$,
(2) $\widetilde{f}$ *is the composition of the projection* $\Delta^n \times \Delta^m \to \Delta^n$ *and* $f\colon \Delta^n \to \Delta^m$.

PROOF. Trivial. □

Lemma 8.3.2. *In the notation above, for every* $g \in M(m, p)$ *we have a commutative diagram*

$$\begin{array}{ccc} C^n \times \Delta^m & \xrightarrow{\widehat{f}} & \Delta^m \\ \Big\downarrow{\scriptstyle Id \times g} & & \Big\downarrow{\scriptstyle g} \\ C^n \times \Delta^p & \xrightarrow{\widehat{gf}} & \Delta^p \end{array}$$

PROOF. Trivial. □

Passing to differential forms we have morphisms for differential graded alebras

$$\widehat{f}^*\colon \Omega_m \to B_n \otimes \Omega_m,$$

where

$$B_m = \frac{\mathbb{K}[s, t_0, \ldots, t_n, ds, dt_0, \ldots, dt_n]}{(s + \sum t_i - 1, ds + \sum dt_i)}$$

is the de Rham algebra of C^n.

Definition 8.3.3. For every $n \geq -1$, $m \geq 0$ and $f \in M(n, m)$ define the operator $h_f \in \operatorname{Hom}^{-n-1}(\Omega_m, \Omega_m)$ as the composition

$$h_f\colon \Omega_m \xrightarrow{\widehat{f}^*} B_n \otimes \Omega_m \xrightarrow{\int_{C^n} \otimes Id} \Omega_m.$$

Notice that for $n = -1$ the above operator equals the identity.

Lemma 8.3.4. *For every* $n \geq 0$, $m \geq 0$, $f \in M(n, m)$ *and* $\eta \in \Omega_m$ *we have*

$$[h_f, d](\eta) = h_f(d\eta) + (-1)^n d h_f(\eta) = \int_{\Delta^n} f^*\eta - \sum_{k=0}^{n} (-1)^k h_{f\partial_k}(\eta).$$

In particular, for $n = 0$ *we have* $h_f(d\eta) + dh_f(\eta) = \eta(e_{f(0)}) - \eta$ *and then the evaluation at a vertex is homotopic to the identity.*

PROOF. For every $\beta \in B_n$ we have by Stokes formula

$$\int_{C^n} d\beta = \int_{\Delta^n} i^*\beta - \sum_{k=0}^{n} (-1)^k \int_{C^{n-1}} \partial_k^* \beta.$$

Writing

$$\widehat{f}^*\eta = \sum_i \beta_i \otimes \alpha_i, \qquad \beta_i \in B_n,\ \alpha_i \in A_m$$

we have

$$dh_f(\eta) = d\sum_i \left(\int_{C^n} \beta_i\right) \alpha_i = \sum_i \left(\int_{C^n} \beta_i\right) d\alpha_i\ ,$$

$$\widehat{f}^*(d\eta) = d\widehat{f}^*(\eta) = \sum_i d\beta_i \otimes \alpha_i + \sum_i (-1)^{\overline{\beta_i}} \beta_i \otimes d\alpha_i,$$

$$h_f(d\eta) = \sum_i \left(\int_{C^n} d\beta_i\right) \otimes \alpha_i + (-1)^{n+1} \sum_i \left(\int_{C^n} \beta_i\right) \otimes d\alpha_i,$$

Therefore

$$\begin{aligned}
h_f(d\eta) + (-1)^n dh_f(\eta) &= \sum_i \left(\int_{C^n} d\beta_i\right) \otimes \alpha_i \\
&= \sum_i \left(\int_{\Delta^n} i^*\beta_i\right) \otimes \alpha_i - \sum_{k=0}^{n} (-1)^k \sum_i \left(\int_{C^{n-1}} \partial_k^* \beta_i\right) \otimes \alpha_i \\
&= \left(\int_{\Delta^n} \otimes Id\right) (i^* \otimes Id)\widehat{f}^*(\eta) - \sum_{k=0}^{n} (-1)^k \left(\int_{C^{n-1}} \otimes Id\right) (\partial_k^* \otimes Id)\widehat{f}^*(\eta) \\
&= \left(\int_{\Delta^n} \otimes Id\right) \widetilde{f}^*(\eta) - \sum_{k=0}^{n} (-1)^k \left(\int_{C^{n-1}} \otimes Id\right) \widehat{f\partial_k}^*(\eta) \\
&= \int_{\Delta^n} f^*\eta - \sum_{k=0}^{n} (-1)^k h_{f\partial_k}(\eta).
\end{aligned}$$

□

Lemma 8.3.5. *Given $f \in M(n,m)$, $g \in M(m,p)$ and $\eta \in \Omega_p$ we have:*

$$g^* h_{gf}(\eta) = h_f(g^*\eta).$$

PROOF. Immediate consequence of the commutative diagram

$$\begin{array}{ccccc}
A_p & \xrightarrow{\widehat{gf}^*} & B_n \otimes A_p & \xrightarrow{\int_{C^n} \otimes Id} & A_p \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle Id \otimes g^*} & & \downarrow{\scriptstyle g^*} \\
A_m & \xrightarrow{\widehat{f}^*} & B_n \otimes A_m & \xrightarrow{\int_{C^n} \otimes Id} & A_m
\end{array}$$

□

8.4. Whitney elementary forms

Definition 8.4.1. For every $f \in M(n,m)$ define the *elementary form*

$$\omega_f = n! \sum_{i=0}^{n} (-1)^i t_{f(i)} dt_{f(0)} \wedge \cdots \wedge \widehat{dt_{f(i)}} \wedge \cdots \wedge dt_{f(n)} \in \Omega_m^n.$$

Denote by $W_m \subset \Omega_m$ the graded subspace generated by the elementary forms.

Notice that $\omega_f \neq 0$ if and only if f is injective.

Lemma 8.4.2. *We have:*

(1) *For every $f\in M(n,m)$ and every $g\in M(p,m)$ we have*

$$g^*\omega_f=\sum_{\{h\in M(n,p)\mid f=gh\}}\omega_h.$$

In particular for $n=p$ we have $g^*\omega_f\neq 0$ if and only if $f=g$.

(2) *For every $f\in M(n,m)$*

$$d\omega_f=\sum_k(-1)^k\sum_{\{g\mid g\partial_k=f\}}\omega_g.$$

(3) *For every $f\in I(n,m)$ we have*

$$\int_{\Delta^n}f^*\omega_f=1.$$

In particular $\{W_m\}$ is a simplicial differential graded subspace of $\{\Omega_m\}$

Proof. The first item is easy and left as an exercise. More generally, for every finite sequence $0\le i_0,i_1,\dots,i_n\le m$ denote

$$\omega_{i_0,\dots,i_n}=n!\sum_{k=0}^n(-1)^k t_{i_k}dt_{i_0}\wedge\cdots\wedge\widehat{dt_{i_k}}\wedge\cdots\wedge dt_{i_n},$$

then

$$d\omega_{i_0,\dots,i_n}=\sum_{i=0}^m\omega_{i,i_0,\dots,i_n}.$$

In fact

$$d\omega_{i_0,\dots,i_n}=n!\sum_{k=0}^n dt_{i_0}\wedge\cdots\wedge dt_{i_k}\wedge\cdots\wedge dt_{i_n}=(n+1)!dt_{i_0}\wedge\cdots\wedge dt_{i_k}\wedge\cdots\wedge dt_{i_n}.$$

and

$$\begin{aligned}\sum_{i=0}^m\omega_{i,i_0,\dots,i_n}&=(n+1)!\sum_{i=0}^m t_idt_{i_0}\wedge\cdots\wedge dt_{i_k}\wedge\cdots\wedge dt_{i_n}-(n+1)\sum_{i=0}^m dt_i\wedge\omega_{i_0,\dots,i_n}\\&=(n+1)!dt_{i_0}\wedge\cdots\wedge dt_{i_k}\wedge\cdots\wedge dt_{i_n}\end{aligned}$$

It is now sufficient to observe that for $f\in M(n,m)$ we have

$$\sum_{i=0}^m\omega_{i,f(0),\dots,f(n)}=\sum_{k=0}^n(-1)^k\sum_{f(k-1)<i<f(k)}\omega_{f(0),\dots,f(k-1),i,f(k),\dots,f(n)}=\sum_k(-1)^k\sum_{\{g\mid g\partial_k=f\}}\omega_g.$$

Since

$$f^*\omega_f=n!\sum_{k=0}^n(-1)^k t_kdt_0\wedge\cdots\wedge\widehat{dt_k}\wedge\cdots\wedge dt_n,$$

using the equalities $dt_0=-\sum_{i>0}dt_i$, $\sum_i t_i=1$ we obtain

$$\begin{aligned}f^*\omega_f&=n!\left(t_0dt_1\wedge\cdots\wedge dt_n-\sum_{k=1}^n(-1)^k t_kdt_k\wedge\cdots\wedge\widehat{dt_k}\wedge\cdots\wedge dt_n\right)\\&=n!(t_0+\cdots+t_n)dt_1\wedge\cdots\wedge dt_n=n!\;dt_1\wedge\cdots\wedge dt_n\end{aligned}$$

and then

$$\int_{\Delta^n}f^*\omega_f=n!\int_{\Delta^n}dt_1\wedge\cdots\wedge dt_n=1.$$

□

Remark 8.4.3. For later use we point out that

$$\bigcap_{k=0}^m\ker(\partial_k^*\colon W_m\to W_{m-1})=W_m^m.$$

Definition 8.4.4. For every $m\geq 0$ define the operators

$$\pi_m\colon \Omega_m\to W_m,\qquad \pi_m(\eta)=\sum_{n=0}^m\sum_{f\in I(n,m)}\left(\int_{\Delta_n}f^*\eta\right)\omega_f$$

$$K_m\colon \Omega_m\to \Omega_m,\qquad K_m(\eta)=\sum_{n=0}^m\sum_{f\in I(n,m)}\omega_f\wedge h_f(\eta).$$

Theorem 8.4.5. *In the above notation we have:*

(1) π_m *is a projector, i.e.* $\pi_m^2=\pi_m$*;*

(2)
$$K_m d+dK_m=\pi_m-Id;$$

(3)
$$K_p g^*=g^*K_m,\qquad \pi_p g^*=g^*\pi_m,\qquad \text{for every } g\in M(p,m).$$

Proof. The first item is trivial. For the second we have

$$K_m(d\eta)+dK_m(\eta)=$$

$$\sum_{n=0}^m\sum_{f\in I(n,m)}d\omega_f\wedge h_f(\eta)+\sum_{n=0}^m\sum_{f\in I(n,m)}\omega_f\wedge((-1)^n dh_f(\eta)+h_f(d\eta))$$

$$=\sum_{n=0}^m\sum_{f\in I(n,m)}d\omega_f\wedge h_f(\eta)+\sum_{n=0}^m\sum_{f\in I(n,m)}\omega_f\wedge\left(\int_{\Delta^n}f^*\eta-\sum_{k=0}^n(-1)^k h_{f\partial_k}(\eta)\right)$$

Since $h_\emptyset=Id$ and $\sum_{f\in I(0,m)}\omega_f=\sum_{i=0}^m t_i=1$ we have

$$K_m(d\eta)+dK_m(\eta)-\pi_m(\eta)+\eta=\sum_{n=0}^m\sum_{f\in I(n,m)}d\omega_f\wedge h_f(\eta)-\sum_{n=1}^m\sum_{f\in I(n,m)}\omega_f\wedge\sum_{k=0}^n(-1)^k h_{f\partial_k}(\eta).$$

The vanishing of the right side follows from the equations

$$\sum_{n=0}^m\sum_{f\in I(n,m)}d\omega_f\wedge h_f(\eta)=\sum_{n=0}^{m-1}\sum_{f\in I(n,m)}d\omega_f\wedge h_f(\eta)=$$

$$=\sum_{n=0}^{m-1}\sum_{f\in I(n,m)}\sum_{k=0}^n(-1)^k\sum_{\{g|f=g\partial_k\}}\omega_g\wedge h_{g\partial_k}(\eta)=\sum_{n=1}^m\sum_{g\in I(n,m)}\sum_{k=0}^n(-1)^k\omega_g\wedge h_{g\partial_k}(\eta).$$

For the last item it is sufficient to prove that $K_p g^*=g^*K_m$;

$$g^*K_m(\eta)=\sum_{n=0}^m\sum_{f\in I(n,m)}g^*(\omega_f)\wedge g^*h_f(\eta)=\sum_{n=0}^m\sum_{f\in I(n,m)}\sum_{\{h\in M(n,p)|f=gh\}}\omega_h\wedge g^*h_f(\eta)=$$

$$=\sum_{n=0}^m\sum_{h\in I(n,p)}\omega_h\wedge g^*h_{gh}(\eta)=\sum_{n=0}^m\sum_{h\in I(n,p)}\omega_h\wedge h_h(g^*\eta)=K_p(g^*\eta).$$

□

8.5. Cochains and normalized cochains

Given a double complex $C^{i,j}$, $i,j\in\mathbb{Z}$, of vector spaces, with differentials

$$d_1\colon C^{i,j}\to C^{i+1,j},\quad d_2\colon C^{i,j}\to C^{i,j+1},\quad d_1^2=d_2^2=d_1d_2+d_2d_1=0$$

we can define their **total complexes** as the DG-vector spaces:

$$\mathrm{Tot}^{\oplus}(C^{*,*})=\bigoplus_{n\in\mathbb{Z}}\mathrm{Tot}(C^{*,*})^n,\quad \mathrm{Tot}^{\oplus}(C^{*,*})^n=\bigoplus_{i+j=n}C^{i,j},\quad d=d_1+d_2,$$

$$\mathrm{Tot}^{\Pi}(C^{*,*})=\bigoplus_{n\in\mathbb{Z}}\mathrm{Tot}(C^{*,*})^n,\quad \mathrm{Tot}^{\Pi}(C^{*,*})^n=\prod_{i+j=n}C^{i,j},\quad d=d_1+d_2.$$

The above two constructions have different behaviour with respect spectral sequences.

Lemma 8.5.1. *Let $f\colon C^{*,*}\to D^{*,*}$ be a morphism of double complexes. Assume that:*

1. $C^{i,*}=D^{i,*}=0$ *for every* $i<0$,
2. $f\colon (C^{i,*},d_2)\to(D^{i,*},d_2)$ *is a quasiisomorphism for every* i.

Then $f\colon \operatorname{Tot}^{\prod}(C^{*,*})\to\operatorname{Tot}^{\prod}(D^{*,*})$ *is a quasiisomorphism.*

PROOF. Exercise. □

Example 8.5.2. The above lemma is generally false for the total complex $\operatorname{Tot}^{\oplus}$. Consider for instance the double complex $C^{i,j}=\mathbb{K}$ for $i+j=0,1$, $i\ge 0$, and $C^{i,j}=0$ otherwise, with both differentials d_1,d_2 equal to the identity for $i+j=0$ and 0 otherwise. Then $\operatorname{Tot}^{\prod}(C^{*,*})$ is acyclic, while $H^1(\operatorname{Tot}^{\oplus}(C^{*,*}))=\mathbb{K}$.

Lemma 8.5.3. *Let $f\colon C^{*,*}\to D^{*,*}$ be a morphism of double complexes. Assume that:*

1. $C^{i,*}=D^{i,*}=0$ *for every* $i<0$,
2. $H^j(C^{i,*},d_2)=H^j(D^{i,*},d_2)=0$ *for every i and every* $j<0$,
3. $f\colon (C^{*,j},d_1)\to(D^{*,j},d_1)$ *is a quasiisomorphism for every* j.

Then $f\colon \operatorname{Tot}^{\prod}(C^{*,*})\to\operatorname{Tot}^{\prod}(D^{*,*})$ *is a quasiisomorphism.*

PROOF. Exercise. Hint: use the Lemma above and truncations. □

Let

$$V^{\Delta}:\qquad V_0 \rightrightarrows V_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; V_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots,$$

be a semicosimplicial DG-vector space. Then the graded vector space $\bigoplus_{n\ge0}V_n[-n]$ has two differentials

$$d=\sum_n(-1)^n d_n,\qquad \text{where}\quad d_n \text{ is the differential of } V_n,$$

and

$$\partial=\sum_i(-1)^i\partial_i,\qquad \text{where}\quad \partial_i \text{ are the face maps.}$$

More explicitly, if $v\in V_n^i$, then the degree of v is $i+n$ and

$$d(v)=(-1)^n d_n(v)\in V_n^{i+1},\qquad \partial(v)=\partial_0(v)-\partial_1(v)+\cdots+(-1)^{n+1}\partial_{n+1}(v)\in V_{n+1}^i.$$

Since $d^2=\partial^2=d\partial+\partial d=0$ the following definition makes sense:

Definition 8.5.4. The **cochain complex** of V^{Δ} is the differential graded vector space

$$C(V^{\Delta})=\left(\prod_{n\ge0}V_n[-n],d+\partial\right).$$

More explicitely,

$$C(V^{\Delta})=\bigoplus_{p\in\mathbb{Z}}C(V^{\Delta})^p,\qquad C(V^{\Delta})^p=\prod_{n\ge0}V_n^{p-n}.$$

Corollary 8.5.5. *Let $f\colon V^{\Delta}\to W^{\Delta}$ be a morphism of cosimplicial DG-vector spaces. If $f\colon V_n\to W_n$ is a quasiisomorphism for every $n\ge0$, then also the map*

$$f\colon C(V^{\Delta})\to C(W^{\Delta})$$

is a quasiisomorphism.

8.6. The Thom-Whitney-Sullivan construction

Here we consider only the semicosimplicial case; the same results holds, with minor modification also in the cosimplicial case.

Definition 8.6.1. The (Thom-Whitney-Sullivan) semicosimplicial **totalization** of a semicosimplicial DG-vector space

$$V^\Delta: \qquad V_0 \rightrightarrows V_1 \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\ V_2 \ \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\ \cdots,$$

is

$$\operatorname{Tot}(V^\Delta) = \left\{ (x_n) \in \prod_{n\ge 0} \Omega_n \otimes V_n \;\middle|\; (\partial_k^* \otimes Id)x_n = (Id \otimes \partial_k)x_{n-1} \text{ for every } 0 \le k \le n \right\}.$$

Theorem 8.6.2 (Whitney)**.** *The map*

$$\oint\colon \operatorname{Tot}(V^\Delta) \to C(V^\Delta)$$

defined componentwise as

$$\operatorname{Tot}(V^\Delta)^p \xrightarrow{\textit{inclusion}} \prod_{n\ge 0}(\bigoplus_i \Omega_n^{p-i} \otimes V_n^i) \xrightarrow{\prod_n \int_{\Delta^n} \otimes Id_{V_n}} \prod_n V_n^{p-n} = C(V^\Delta)^p$$

is a quasiisomorphism of differential graded vector spaces.

Proof. Consider the subspace

$$W(V^\Delta) = \left\{ (x_n) \in \prod_{n\ge 0} W_n \otimes V_n \;\middle|\; (\partial_k^* \otimes Id)x_n = (Id \otimes \partial_k)x_{n-1} \text{ for every } 0 \le k \le n \right\}.$$

Since the operators K_m and π_n are simplicial we have

$$K = \prod_n (K_n \otimes Id_{V_n})\colon \operatorname{Tot}(V^\Delta) \to \operatorname{Tot}(V^\Delta),$$

$$\pi = \prod_n (\pi_n \otimes Id_{V_n})\colon \operatorname{Tot}(V^\Delta) \to \operatorname{Tot}(V^\Delta),$$

and the equality $dK + Kd = \pi - Id$. This implies that π is a quasiisomorphism of DG-vector spaces. Consider now the morphism

$$\phi\colon W(V^\Delta) \to C(V^\Delta)$$

defined componentwise as

$$W(V^\Delta)^p \xrightarrow{\textit{inclusion}} \prod_{n\ge 0}(\bigoplus_i W_n^{p-i} \otimes V_n^i) \xrightarrow{\prod_n \int_{\Delta^n} \otimes Id_{V_n}} \prod_n V_n^{p-n} = C(V^\Delta)^p$$

In order to conclude the proof we will show that ϕ is an isomorphism and $\oint = \phi \circ \pi$.

For every $n \ge 0$ consider the map $E\colon C(V^\Delta) \to \prod_n W_n \otimes V_n$ defined componentwise as

$$E_n\colon C(V^\Delta) \to W_n \otimes V_n, \qquad E_n(\{v_p\}) = \sum_{p=0}^{n} \sum_{f\in I(p,n)} \omega_f \otimes f(v).$$

For every $g \in I(n, m)$ we have

$$(g^* \otimes Id)E_m(v) = \sum_{f\in I(p,m)} g^*\omega_f \otimes f(v) = \sum_{f\in I(p,m)} \sum_{\{h | f=gh\}} \omega_h \otimes gh(v) =$$

$$= \sum_{h\in I(p,n)} \omega_h \otimes gh(v) = (Id \otimes g)E_n(v).$$

It is obvious that $\phi \circ E = Id$ and if $\phi(x_n) = 0$ then $x_p = 0$ and if $x_n = \sum_{f\in I(p,n)} \omega_f \otimes v_f$ then $(f^* \otimes Id)(x_n) = f^*\omega_f \otimes v_f = (Id \otimes f)(x_p) = 0$ and then $v_f = 0$. This proves that ϕ is bijective. As easy application od Stokes formula show that $\partial\phi = \phi d$. □

Corollary 8.6.3. *Let $f\colon V^\Delta \to W^\Delta$ be a morphism of semicosimplicial DG-vector spaces. If $f\colon V_n \to W_n$ is a quasiisomorphism for every $n \ge 0$, then also the map $f\colon \operatorname{Tot}(V^\Delta) \to \operatorname{Tot}(W^\Delta)$ is a quasiisomorphism.*

Theorem 8.6.4. *Let* $0 \to K^\Delta \to V^\Delta \xrightarrow{f} W^\Delta \to 0$ *be a sequence of morphisms of semicosimplicial DG-vector spaces such that for every* n *the sequence*

$$0 \to K_n \to V_n \xrightarrow{f} W_n \to 0$$

is exact. Then the sequence

$$0 \to \mathrm{Tot}(K^\Delta) \to \mathrm{Tot}(V^\Delta) \xrightarrow{f} \mathrm{Tot}(W^\Delta) \to 0$$

is exact.

Proof. The only non trivial assertion is the surjectivity of $\mathrm{Tot}(V^\Delta) \xrightarrow{f} \mathrm{Tot}(W^\Delta)$. Let $(w_0, w_1, \ldots) \in \mathrm{Tot}(W^\Delta)$ and assume that for some n we have $(v_1, \ldots, v_{n-1}) \in \prod_{i<n} \Omega_i \otimes V_i$ such that

$$f(v_i) = w_i, \qquad \partial_k v_i = \partial_k^* v_{i+1}.$$

Let $z \in \Omega_n \otimes V_n$ such that $f(z) = w_n$ and consider the elements

$$k_i = \partial_i^* z - \partial_i v_{n-1} \in \Omega_{n-1} \otimes K_n, \qquad i = 0, \ldots, n.$$

For every $0 \le i < j \le n$ we have:

$$\partial_i^* k_j = \partial_i^* \partial_j^* z - \partial_i^* \partial_j v_{n-1} = \partial_i^* \partial_j^* z - \partial_j \partial_i^* v_{n-1} = \partial_i^* \partial_j^* z - \partial_j \partial_i v_{n-2}.$$

Similarly we have $\partial_{j-1}^* k_i = \partial_{j-1}^* \partial_i^* z - \partial_i \partial_{j-1} v_{n-2}$ and then $\partial_i^* k_j = \partial_{j-1}^* k_i$ for every $i < j$. Since $\Omega_\bullet \otimes K_n$ is an acyclic Kan complex there exists $k \in K_n$ such that $\partial_i^* k = k_i$ and then

$$f(z - k) = w_n, \qquad \partial_i^*(z - k) = \partial_i v_{n-1}.$$

We set $v_n = z - k$ and proceed by induction. □

8.7. The cosimplicial case

Definition 8.7.1. Let V^Δ be a cosimplicial DG-vector space. The **normalized cochain complex** of V^Δ is the graded subspace $N(V^\Delta) \subset C(V^\Delta)$ defined as $N(V^\Delta) = (\prod_{n \ge 0} K_n[-n], d + \partial)$ where $K_0 = V_0$ and

$$K_n = \bigcap_{f \in M(n, n-1)} \ker(f \colon V_n \to V_{n-1}), \qquad n > 0.$$

Theorem 8.7.2. *In the notation above* $N(V^\Delta)$ *is a DG-vector subspace of* $C(V^\Delta)$ *and the inclusion* $N(V^\Delta) \to C(V^\Delta)$ *is a quasiisomorphism.*

Proof. See e.g. [**18**, **36**]. □

The cosimplicial totalization of a cosimplicial DG-vector space is defined as

$$\mathrm{Tot}(V^\Delta) = \left\{ (x_n) \in \prod_{n \ge 0} \Omega_n \otimes V_n \;\middle|\; (f^* \otimes Id) x_n = (Id \otimes f) x_m \ \forall\, n, m,\ f \colon [m] \to [n] \right\}.$$

In this case the integration map $\oint$ is a surjective quasiisomorphism onto the normalized cochain complex $N(V^\Delta)$: the proof is completely similar to the semicosimplicial case.