The cotangent complex in characteristic 0

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We use the same notation and conventions of [8]; in particular $\mathbb{K}$ will be a fixed field of characteristic 0.

1 Homotopy of differential graded algebras

Let A be a graded algebra, if $A \to B$ is a morphism of graded algebras then B has a natural structure of A-algebra. Given two A-algebras B, C it is defined their tensor product $B \otimes_A C$ as the quotient of $B \otimes_{\mathbb{K}} C = \oplus_{n,m} B_n \otimes_{\mathbb{K}} C_m$ by the ideal generated by $ba \otimes c - b \otimes ac$ for every $a \in A$, $b \in B$, $c \in C$. $B \otimes_A C$ has a natural structure of graded algebra with degrees $\overline{b \otimes c} = \overline{b} + \overline{c}$ and multiplication $(b \otimes c)(\beta \otimes \gamma) = (-1)^{\overline{c}\,\overline{\beta}} b\beta \otimes c\gamma$. Note in particular that $A[\{x_i\}] = A \otimes_{\mathbb{K}} \mathbb{K}[\{x_i\}]$.

Given a dg-algebra A and $h \in \mathbb{K}$ it is defined an evaluation morphism $e_h \colon A[t, dt] \to A$, $e_h(a \otimes p(t)) = ap(h)$, $e_h(a \otimes q(t)dt) = 0$.

Lemma 1.1. *For every dg-algebra A the evaluation map $e_h \colon A[t, dt] \to A$ induces an isomorphism $H(A[t, dt]) \to H(A)$ independent from $h \in \mathbb{K}$.*

Proof. Let $\imath \colon A \to A[t, dt]$ be the inclusion, since $e_h \imath = Id_A$ it is sufficient to prove that $\imath \colon H(A) \to H(A[t, dt])$ is bijective. For every $n > 0$ denote $B_n = At^n \oplus At^{n-1}dt$; since $d(B_n) \subset B_n$ and $A[t, dt] = \imath(A) \bigoplus_{n>0} B_n$ it is sufficient to prove that $H(B_n) = 0$ for every n. Let $z \in Z_i(B_n)$, $z = at^n + nbt^{n-1}dt$, then $0 = dz = dat^n + ((-1)^i a + db)nt^{n-1}dt$ which implies $a = (-1)^{i-1}db$ and then $z = (-1)^{i-1}d(bt^n)$. $\square$

Definition 1.2. *Given two morphisms of dg-algebras $f, g \colon A \to B$, a* homotopy *between f and g is a morphism $H \colon A \to B[t, dt]$ such that $H_0 := e_0 \circ H = f$, $H_1 := e_1 \circ H = g$. We denote by $[A, B]$ the quotient of $\mathrm{Hom}_{\mathbf{DGA}}(A, B)$ by the equivalence relation $\sim$ generated by homotopy. If $B \to C$ is a morphism of dg-algebras with kernel J, a homotopy $H \colon A \to B[t, dt]$ is called* constant on C *if the image of H is contained in $B \oplus_{j \geq 0} (Jt^{j+1} \oplus Jt^j dt)$. Two dg-algebras A, B are said to be* homotopically equivalent *if there exist morphisms $f \colon A \to B$, $g \colon B \to A$ such that $fg \sim Id_B$, $gf \sim Id_A$.*

According to Lemma 1.1 homotopic morphisms induce the same morphism in homology.

Lemma 1.3. *Given morphisms of dg-algebras,*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "l"'] & C \end{tikzcd},$$

if $f \sim g$ and $h \sim l$ then $hf \sim lg$.

Proof. It is obvious from the definitions that $hg \sim lg$. For every $a \in \mathbb{K}$ there exists a commutative diagram

$$\begin{tikzcd} B \otimes \mathbb{K}[t,dt] \arrow[r, "h\otimes Id"] \arrow[d, "e_a"] & C \otimes \mathbb{K}[t,dt] \arrow[d, "e_a"] \\ B \arrow[r, "h"] & C \end{tikzcd} .$$

If $F\colon A \to B[t, dt]$ is a homotopy between f and g, then, considering the composition of F with $h \otimes Id$, we get a homotopy between hf and hg. □

Example 1.4. Let A be a dg-algebra, $\{x_i\}$ a set of indeterminates of integral degree and consider the dg-algebra $B = A[\{x_i, dx_i\}]$, where dx_i is an indeterminate of degree $\overline{dx_i} = \overline{x_i}+1$ and the differential d_B is the unique extension of d_A such that $d_B(x_i) = dx_i$, $d_B(dx_i) = 0$ for every i. The inclusion $i\colon A \to B$ and the projection $\pi\colon B \to A$, $\pi(x_i) = \pi(dx_i) = 0$ give a homotopy equivalence between A and B. In fact $\pi i = Id_A$; consider now the homotopy $H\colon B \to B[t, dt]$ given by

$$H(x_i) = x_i t, \quad H(dx_i) = dH(x_i) = dx_i t + (-1)^{\overline{x_i}} x_i dt, \quad H(a) = a,\ \forall a \in A.$$

Taking the evaluation at $t = 0, 1$ we get $H_0 = ip$, $H_1 = Id_B$.

Exercise 1.5. Let $f, g\colon A \to C$, $h\colon B \to C$ be morphisms of dg-algebras. If $f \sim g$ then $f \otimes h \sim g \otimes h\colon A \otimes_{\mathbb{K}} B \to C$. △

Remark 1.6. *In view of future geometric applications, it seems reasonable to define the spectrum of a unitary dg-algebra A as the usual spectrum of the commutative ring $Z_0(A)$.*

If $S \subset Z_0(A)$ is a multiplicative part we can consider the localized dg-algebra $S^{-1}A$ with differential $d(a/s) = da/s$. Since the localization is an exact functor in the category of $Z_0(A)$ modules we have $H(S^{-1}A) = S^{-1}H(A)$. If $\phi\colon A \to C$ is a morphism of dg-algebras and $\phi(s)$ is invertible for every $s \in S$ then there is a unique morphism $\psi\colon S^{-1}A \to C$ extending ϕ. Moreover if ϕ is a quasiisomorphism then also ψ is a quasiisomorphism (easy exercise).

If $\mathcal{P} \subset Z_0(A)$ is a prime ideal, then we denote as usual $A_{\mathcal{P}} = S^{-1}A$, where $S = Z_0(A) - \mathcal{P}$. It is therefore natural to define Spec(A) *as the ringed space $(X, \tilde{A})$, where X is the spectrum of A and $\tilde{A}$ is the (quasi coherent) sheaf of dg-algebras with stalks $A_{\mathcal{P}}$, $\mathcal{P} \in X$.*

2 Differential graded modules

Let (A, s) be a fixed dg-algebra, by an A-dg-module we mean a differential graded vector space (M, s) together two associative distributive multiplication maps $A \times M \to M$, $M \times A \to M$ with the properties:

1. $A_i M_j \subset M_{i+j}$, $\quad M_i A_j \subset M_{i+j}$.
2. $am = (-1)^{\overline{a}\,\overline{m}} ma$, for homogeneous $a \in A$, $m \in M$.
3. $s(am) = s(a)m + (-1)^{\overline{a}} as(m)$.

If $A = A_0$ we recover the usual notion of complex of A-modules.

If M is an A-dg-module then $M[n] = \mathbb{K}[n] \otimes_{\mathbb{K}} M$ has a natural structure of A-dg-module with multiplication maps

$$(e \otimes m)a = e \otimes ma, \qquad a(e \otimes m) = (-1)^{n\overline{a}} e \otimes am, \qquad e \in \mathbb{K}[n],\ m \in M,\ a \in A.$$

The tensor product $N \otimes_A M$ is defined as the quotient of $N \otimes_{\mathbb{K}} M$ by the graded submodules generated by all the elements $na \otimes m - n \otimes am$.

Given two A-dg-modules $(M, d_M), (N, d_N)$ we denote by

$$\mathrm{Hom}^n_A(M,N) = \{\, f \in \mathrm{Hom}^n_{\mathbb{K}}(M,N) \mid f(ma) = f(m)a,\ m \in M, a \in A\}$$

$$\mathrm{Hom}^*_A(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n_A(M,N).$$

The graded vector space $\mathrm{Hom}^*_A(M,N)$ has a natural structure of A-dg-module with left multiplication $(af)(m) = af(m)$ and differential

$$d\colon \mathrm{Hom}^n_A(M,N) \to \mathrm{Hom}^{n+1}_A(M,N), \qquad df = [d,f] = d_N \circ f - (-1)^n f \circ d_M.$$

Note that $f \in \mathrm{Hom}^0_A(M,N)$ is a morphism of A-dg-modules if and only if $df = 0$. A *homotopy* between two morphism of dg-modules $f, g\colon M \to N$ is a $h \in \mathrm{Hom}^{-1}_A(M,N)$ such that $f - g = dh = d_N h + h d_M$. Homotopically equivalent morphisms induce the same morphism in homology.

Morphisms of A-dg-modules $f\colon L \to M$, $h\colon N \to P$ induce, by composition, morphisms $f^*\colon \mathrm{Hom}^*_A(M,N) \to \mathrm{Hom}^*_A(L,N)$, $h_*\colon \mathrm{Hom}^*_A(M,N) \to \mathrm{Hom}^*_A(M,P)$;

Lemma 2.1. *In the above notation if f is homotopic to g and h is homotopic to l then f^* is homotopic to g^* and l_* is homotopic to h_*.*

Proof. Let $p \in \mathrm{Hom}^{-1}_A(L,M)$ be a homotopy between f and g, It is a straightforward verification to see that the composition with p is a homotopy between f^* and g^*. Similarly we prove that h_* is homotopic to l_*. □

Lemma 2.2. *Let $A \to B$ be a morphism of unitary dg-algebras, M an A-dg-module, N a B-dg-modules. Then there exists a natural isomorphism of B-dg-modules*

$$\mathrm{Hom}^*_A(M,N) \simeq \mathrm{Hom}^*_B(M \otimes_A B, N).$$

Proof. Consider the natural maps:

$$\mathrm{Hom}^*_A(M,N) \underset{R}{\overset{L}{\rightleftarrows}} \mathrm{Hom}^*_B(M \otimes_A B, N)\ ,$$

$$Lf(m \otimes b) = f(m)b, \qquad Rg(m) = g(m \otimes 1).$$

We left as exercise the easy verification that $L, R = L^{-1}$ are isomorphism of B-dg-modules. □

Given a morphism of dg-algebras $B \to A$ and an A-dg-module M we set:

$$\mathrm{Der}^n_B(A,M) = \{\phi \in \mathrm{Hom}^n_{\mathbb{K}}(A,M) | \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b),\ \phi(B) = 0\}$$

$$\mathrm{Der}^*_B(A,M) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Der}^n_B(A,M).$$

As in the case of Hom^*, there exists a structure of A-dg-module on $\mathrm{Der}^*_B(A,M)$ with product $(a\phi)(b) = a\phi(b)$ and differential

$$d\colon \mathrm{Der}^n_B(A,M) \to \mathrm{Der}^{n+1}_B(A,M), \qquad d\phi = [d,\phi] = d_M\phi - (-1)^n \phi d_A.$$

Given $\phi \in \mathrm{Der}^n_B(A,M)$ and $f \in \mathrm{Hom}^m_A(M,N)$ their composition $f\phi$ belongs to $\mathrm{Der}^{n+m}_B(A,N)$.

Proposition 2.3. *Let $B \to A$ be a morphisms of dg-algebras: there exists an A-dg-module $\Omega_{A/B}$ together a closed derivation $\delta\colon A \to \Omega_{A/B}$ of degree 0 such that, for every A-dg-module M, the composition with δ gives an isomorphism*

$$\mathrm{Hom}^*_A(\Omega_{A/B}, M) = \mathrm{Der}^*_B(A, M).$$

Proof. Consider the graded vector space

$$F_A = \bigoplus A\delta x, \quad x \in A \text{ homogeneous}, \qquad \overline{\delta x} = \overline{x}.$$

F_A is an A-dg-module with multiplication $a(b\delta x) = ab\delta x$ and differential

$$d(a\delta x) = da\delta x + (-1)^{\overline{a}} a\delta(dx).$$

Note in particular that $d(\delta x) = \delta(dx)$. Let $I \subset F_A$ be the homogeneous submodule generated by the elements

$$\delta(x+y) - \delta x - \delta y, \quad \delta(xy) - x(\delta y) - (-1)^{\overline{x}\,\overline{y}} y(\delta x), \quad \delta(b), b \in B,$$

Since $d(I) \subset I$ the quotient $\Omega_{A/B} = F_A/I$ is still an A-dg-module. By construction the map $\delta\colon A \to \Omega_{A/B}$ is a derivation of degree 0 such that $d\delta = d_\Omega \delta - \delta d_A = 0$. Let $\circ\delta\colon \mathrm{Hom}^*_A(\Omega_{A/B}, M) \to \mathrm{Der}^*_B(A, M)$ be the composition with δ:

a) L is a morphism of A-dg-modules. In fact $(af) \circ \delta = a(f \circ \delta)$ for every $a \in A$ and

$$\begin{aligned} d(f \circ \delta)(x) &= d_M(f(\delta x)) - (-1)^{\overline{f}} f\delta(dx) = \\ &= d_M(f(\delta x)) - (-1)^{\overline{f}} f(d(\delta x)) = df \circ \delta. \end{aligned}$$

b) $\circ\delta$ is surjective. Let $\phi \in \mathrm{Der}^n_B(A, M)$; define a morphism $f \in \mathrm{Hom}^n_A(F_A, M)$ by the rule $f(a\delta x) = (-1)^{n\overline{a}} a\phi(x)$; an easy computation shows that $f(I) = 0$ and then f factors to $f \in \mathrm{Hom}^n_A(\Omega_{A/B}, M)$: by construction $f \circ \delta = \phi$.

c) $\circ\delta$ is injective. In fact the image of δ generate $\Omega_{A/B}$.

□

When $B = \mathbb{K}$ we denote for notational simplicity $\mathrm{Der}^*(A, M) = \mathrm{Der}^*_{\mathbb{K}}(A, M)$, $\Omega_A = \Omega_{A/\mathbb{K}}$. Note that if $C \to B$ is a morphism of dg-algebras, then the natural map $\Omega_{A/C} \to \Omega_{A/B}$ is surjective and $\Omega_{A/C} = \Omega_{A/B}$ whenever $C \to B$ is surjective.

Definition 2.4. *The module $\Omega_{A/B}$ is called the module of relative* Kähler differentials *of A over B and δ the* universal derivation.

By the universal property, the module of differential and the universal derivation are unique up to isomorphism.

Example 2.5. If $A_\sharp = \mathbb{K}[\{x_i\}]$ is a polynomial algebra then $\Omega_A = \oplus_i A\delta x_i$ and $\delta\colon A \to \Omega_A$ is the unique derivation such that $\delta(x_i) = \delta x_i$.

Proposition 2.6. *Let $B \to A$ be a morphism of dg-algebras and $S \subset Z_0(A)$ a multiplicative part. Then there exists a natural isomorphism $S^{-1}\Omega_{A/B} = \Omega_{S^{-1}A/B}$.*

Proof. The closed derivation $\delta\colon A \to \Omega_{A/B}$ extends naturally to $\delta\colon S^{-1}A \to S^{-1}\Omega_{A/B}$, $\delta(a/s) = \delta a/s$, and by the universal property there exists a unique morphism of $S^{-1}A$ modules $f\colon \Omega_{S^{-1}A/B} \to S^{-1}\Omega_{A/B}$ and a unique morphism of A modules $g\colon \Omega_{A/B} \to \Omega_{S^{-1}A/B}$. The morphism g extends to a morphism of $S^{-1}A$ modules $g\colon S^{-1}\Omega_{A/B} \to \Omega_{S^{-1}A/B}$. Clearly these morphisms commute with the universal closed derivations and then $gf = Id$. On the other hand, by the universal property, the restriction of fg to $\Omega_{A/B}$ must be the natural inclusion $\Omega_{A/B} \to S^{-1}\Omega_{A/B}$ and then also $fg = Id$. □

3 Projective modules

Definition 3.1. *An A-dg-module P is called* projective *if for every surjective quasiisomorphism $f\colon M \to N$ and every $g\colon P \to N$ there exists $h\colon P \to M$ such that $fh = g$.*

$$\begin{tikzcd} & M \arrow[d, "f"', "qis"] \\ P \arrow[r, "g"'] & N \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} & M \arrow[d, "f"', "qis"] \\ P \arrow[r, "g"'] \arrow[ur, "h"] & N \end{tikzcd} .$$

Exercise 3.2. Prove that if $A = A_0$ and $P = P_0$ then P is projective in the sense of 3.1 if and only if P_0 is projective in the usual sense. △

Lemma 3.3. *Let P be a projective A-dg-module, $f\colon P \to M$ a morphism of A-dg-modules and $\phi\colon M \to N$ a surjective quasiisomorphism. If ϕf is homotopic to 0 then also f is homotopic to 0.*

Proof. We first note that there exist natural isomorphisms $\mathrm{Hom}^i_A(P, M[j]) = \mathrm{Hom}^{i+j}_A(P, M)$. Let $h\colon P \to N[-1]$ be a homotopy between ϕf and 0 and consider the A-dg-modules $M \oplus N[-1]$, $M \oplus M[-1]$ endowed with the differentials

$$d\colon M_n \oplus N_{n-1} \to M_{n+1} \oplus N_n, \quad d(m_1, n_2) = (dm_1, f(m_1) - dn_2),$$

$$d\colon M_n \oplus M_{n-1} \to M_{n+1} \oplus M_n, \quad d(m_1, m_2) = (dm_1, m_1 - dm_2).$$

The map $Id_M \oplus f\colon M \oplus M[-1] \to M \oplus N[-1]$ is a surjective quasiisomorphism and $(\phi, h)\colon P \to M \oplus N[-1]$ is morphism of A-dg-modules. If $(\phi, l)\colon P \to M \oplus M[-1]$ is a lifting of (ϕ, h) then l is a homotopy between ϕ and 0. □

Lemma 3.4. *Let $f\colon M \to N$ be a morphism of A-dg-modules, then there exist morphisms of A-dg-modules $\pi\colon L \to M$, $g\colon L \to N$ such that g is surjective, π is a homotopy equivalence and g is homotopically equivalent to $f\pi$.*

Proof. Consider $L = M \oplus N \oplus N[-1]$ with differential

$$d\colon M_n \oplus N_n \oplus N_{n-1} \to M_{n+1} \oplus N_{n+1} \oplus N_n, \quad d(m, n_1, n_2) = (dm, dn_1, n_1 - dn_2).$$

We define $g(m, n_1, n_2) = f(m) + n_1$, $\pi(m, n_1, n_2) = m$ and $s\colon M \to L$, $s(m) = (m, 0, 0)$. Since $gs = f$ and $\pi s = Id_M$ it is sufficient to prove that $s\pi$ is homotopic to Id_L. Take $h \in \mathrm{Hom}^{-1}_A(L, L)$, $h(m, n_1, n_2) = (0, n_2, 0)$; then

$$d(h(m, n_1, n_2)) + hd(m, n_1, n_2) = (0, n_1, n_2) = (Id_L - s\pi)(m, n_1, n_2).$$

□

Theorem 3.5. *Let P be a projective A-dg-module: For every quasiisomorphism $f\colon M \to N$ the induced map $\mathrm{Hom}^*_A(P, M) \to \mathrm{Hom}^*_A(P, N)$ is a quasiisomorphism.*

Proof. By Lemma 3.4 it is not restrictive to assume f surjective. For a fixed integer i we want to prove that $H^i(\mathrm{Hom}^*_A(P, M)) = H^i(\mathrm{Hom}^*_A(P, N))$. Replacing M and N with $M[i]$ and $N[i]$ it is not restrictive to assume $i = 0$. Since $Z^0(\mathrm{Hom}^*_A(P, N))$ is the set of morphisms of A-dg-modules and P is projective, the map

$$Z^0(\mathrm{Hom}^*_A(P, M)) \to Z^0(\mathrm{Hom}^*_A(P, N))$$

is surjective. If $\phi \in Z^0(\mathrm{Hom}^*_A(P, M))$ and $f\phi \in B^0(\mathrm{Hom}^*_A(P, N))$ then by Lemma 3.3 also ϕ is a coboundary. □

A projective resolution of an A-dg-module M is a surjective quasiisomorphism $P \to M$ with P projective. We will show in next section that projective resolutions always exist. This allows to define for every pair of of A-dg-modules M, N

$$\mathrm{Ext}^i(M,N) = H^i(\mathrm{Hom}^*_A(P,N)),$$

where $P \to M$ is a projective resolution.

Exercise 3.6. Prove that the definition of Ext's is independent from the choice of the projective resolution. △

4 Semifree resolutions

From now on K is a fixed dg-algebra.

Definition 4.1. *A K-dg-algebra (R, s) is called* semifree *if:*

1. *The underlying graded algebra R is a polynomial algebra over K $K[\{x_i\}]$, $i \in I$.*
2. *There exists a filtration $\emptyset = I(0) \subset I(1) \subset \dots$, $\cup_{n\in\mathbb{N}} I(n) = I$, such that $s(x_i) \in R(n)$ for every $i \in I(n+1)$, where by definition $R(n) = K[\{x_i\}]$, $i \in I(n)$.*

Note that $R(0) = K$, $R(n)$ is a dg-subalgebra of R and $R = \cup R(n)$.

Let $R = K[\{x_i\}] = \cup R(n)$ be a semifree K-dg-algebra, S a K-dg-algebra; to give a morphism $f\colon R \to S$ is the same to give a sequence of morphisms $f_n\colon R(n) \to S$ such that f_{n+1} extends f_n for every n. Given a morphism $f_n\colon R(n) \to S$, the set of extensions $f_{n+1}\colon R(n+1) \to S$ is in bijection with the set of sequences $\{f_{n+1}(x_i)\}$, $i \in I(n+1) - I(n)$, such that $s(f_{n+1}(x_i)) = f_n(s(x_i))$, $\overline{f_{n+1}(x_i)} = \overline{x_i}$.

Example 4.2. $\mathbb{K}[t, dt]$ is semifree with filtration $\mathbb{K} \oplus \mathbb{K}\, dt \subset \mathbb{K}[t, dt]$. For every dg-algebra A and every $a \in A_0$ there exists a unique morphism $f\colon \mathbb{K}[t, dt] \to A$ such that $f(t) = a$.

Exercise 4.3. Let (V, s) be a complex of vector spaces, the differential s extends to a unique differential s on the symmetric algebra $\bigodot V$ such that $s(\bigodot^n V) \subset \bigodot^n V$ for every n. Prove that $(\bigodot V, s)$ is semifree. △

Exercise 4.4. The tensor product (over K) of two semifree K-dg-algebras is semifree. △

Proposition 4.5. *Let $(R = K[\{x_i\}], s)$, $i \in \cup I(n)$, be a semifree K-dg-algebra: for every surjective quasiisomorphism of K-dg-algebras $f\colon A \to B$ and every morphism $g\colon R \to B$ there exists a lifting $h\colon R \to A$ such that $fh = g$. Moreover any two of such liftings are homotopic by a homotopy constant on B.*

Proof. Assume by induction on n that it is defined a morphism $h_n\colon R(n) \to A$ such that fh_n equals the restriction of g to $R(n) = \mathbb{K}[\{x_i\}]$, $i \in I(n)$. Let $i \in I(n+1) - I(n)$, we need to define $h_{n+1}(x_i)$ with the properties $fh_{n+1}(x_i) = g(x_i)$, $dh_{n+1}(x_i) = h_n(dx_i)$ and $\overline{h_{n+1}(x_i)} = \overline{x_i}$. Since $dh_n(dx_i) = 0$ and $fh_n(dx_i) = g(dx_i) = dg(x_i)$ we have that $h_n(dx_i)$ is exact in A, say $h_n(dx_i) = da_i$; moreover $d(f(a_i) - g(x_i)) = f(da_i) - g(dx_i) = 0$ and, since $Z(A) \to Z(B)$ is surjective there exists $b_i \in A$ such that $f(a_i + b_i) = g(x_i)$ and then we may define $h_{n+1}(x_i) = a_i + b_i$. The inverse limit of h_n gives the required lifting.
Let $h, l\colon R \to A$ be liftings of g and denote by $J \subset A$ the kernel of f; by assumption J is acyclic and consider the dg-subalgebra $C \subset A[t, dt]$,

$$C = A \oplus_{j\geq 0} (Jt^{j+1} \oplus Jt^j dt).$$

We construct by induction on n a coherent sequence of morphisms $H_n\colon R(n) \to C$ giving a homotopy between h and l. Denote by $N \subset \mathbb{K}[t, dt]$ the differential ideal generated by $t(t-1)$; there exists a direct sum decomposition $\mathbb{K}[t, dt] = \mathbb{K} \oplus \mathbb{K}\,t \oplus \mathbb{K}\,dt \oplus N$. We may write:

$$H_n(x) = h(x) + (l(x) - h(x))t + a_n(x)dt + b_n(x, t),$$

with $a_n(x) \in J$ and $b_n(x, t) \in J \otimes N$. Since $dH_n(x) = H_n(dx)$ we have for every $x \in R(n)$:

$$(-1)^{\overline{x}}(l(x) - h(x)) + d(a_n(x)) = a_n(dx), \quad d(b_n(x, t)) = b_n(dx, t). \tag{1}$$

Let $i \in I(n + 1) - I(n)$, we seek for $a_{n+1}(x_i) \in J$ and $b_{n+1}(x_i, t) \in J \otimes N$ such that, setting

$$H_{n+1}(x_i) = h(x_i) + (l(x_i) - h(x_i))t + a_{n+1}(x_i)dt + b_{n+1}(x_i, t),$$

we want to have

$$\begin{aligned} 0 &= dH_{n+1}(x_i) - H_n(dx_i) \\ &= ((-1)^{\overline{x_i}}(l(x_i) - h(x_i)) + da_{n+1}(x_i) - a_n(dx_i))dt + db_{n+1}(x_i, t) - b_n(dx_i, t). \end{aligned}$$

Since both J and $J \otimes N$ are acyclic, such a choice of $a_{n+1}(x_i)$ and $b_{n+1}(x_i, t)$ is possible if and only if $(-1)^{\overline{dx_i}}(l(x_i) - h(x_i)) + a_n(dx_i)$ and $b_n(dx_i, t)$ are closed.
According to Equation 1 we have

$$\begin{aligned} d((-1)^{\overline{dx_i}}(l(x_i) - h(x_i) + a_n(dx_i)) &= (-1)^{\overline{dx_i}}(l(dx_i) - h(dx_i)) + d(a_n(dx_i)) \\ &= a_n(d^2x_i) = 0, \\ db_n(dx_i, t) &= b_n(d^2x_i, t) = 0. \end{aligned}$$

□

Definition 4.6. *A K-semifree resolution (also called resolvent) of a K-dg-algebra A is a surjective quasiisomorphism $R \to A$ with R semifree K-dg-algebra.*

By 4.5 if a semifree resolution exists then it is unique up to homotopy.

Theorem 4.7. *Every K-dg-algebra admits a K-semifree resolution.*

Proof. Let A be a K-dg-algebra, we show that there exists a sequence of K-dg-algebras $K = R(0) \subset R(1) \subset \ldots \subset R(n) \subset \ldots$ and morphisms $f_n\colon R(n) \to A$ such that:

1. $R(n + 1) = R(n)[\{x_i\}]$, $dx_i \in R(n)$.
2. f_{n+1} extends f_n.
3. $f_1\colon Z(R(1)) \to Z(A)$, $f_2\colon R(2) \to A$ are surjective.
4. $f_n^{-1}(B(A)) \cap Z(R(n)) \subset B(R(n + 1)) \cap R(n)$, for every $n > 0$.

It is then clear that $R = \cup R(n)$ and $f = \underleftarrow{\lim} f_n$ give a semifree resolution. $Z(A)$ is a graded algebra and therefore there exists a polynomial graded algebra $R(1) = K[\{x_i\}]$ and a surjective morphism $f_1\colon R(1) \to Z(A)$; we set the trivial differential $d = 0$ on $R(1)$. Let v_i be a set of homogeneous generators of the ideal $f_1^{-1}(B(A))$, if $f_1(v_i) = da_i$ it is not restrictive to assume that the a_i's generate A. We then define $R(2) = R(1)[\{x_i\}]$, $f_2(x_i) = a_i$ and $dx_i = v_i$. Assume now by induction that we have defined $f_n\colon R(n) \to A$ and let $\{v_j\}$ be a set of generators of $f_n^{-1}(B(A)) \cap Z(R(n))$, considered as an ideal of $Z(R(n))$; If $f_n(v_j) = da_j$ we define $R(n + 1) = R(n)[\{x_j\}]$, $dx_j = v_j$ and $f_{n+1}(x_j) = a_j$. □

Remark 4.8. *It follows from the above proof that if $K_i = A_i = 0$ for every $i > 0$ then there exists a semifree resolution $R \to A$ with $R_i = 0$ for every $i > 0$.*

Exercise 4.9. If, in the proof of Theorem 4.7 we choose at every step $\{v_i\} = f_n^{-1}(B(A)) \cap Z(R(n))$ we get a semifree resolution called *canonical*. Show that every morphism of dg-algebras has a natural lift to their canonical resolutions. △

Given two semifree resolutions $R \to A$, $S \to A$ we can consider a semifree resolution $P \to R \times_A S$ of the fibred product and we get a commutative diagram of semifree resolutions

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr] & R \arrow[d] \\ R \arrow[r] & A. \end{tikzcd}$$

Definition 4.10. *An A-dg-module F is called* semifree *if $F = \oplus_{i\in I} Am_i$, $\overline{m_i} \in \mathbb{Z}$ and there exists a filtration $\emptyset = I(0) \subset I(1) \subset \ldots \subset I(n) \subset \ldots$ such that*

$$i \in I(n+1) \Rightarrow dm_i \in F(n) = \oplus_{i \in I(n)} Am_i.$$

A semifree resolution *of an A-dg-module M is a surjective quasiisomorphism $F \to M$ with F semifree.*

The proof of the following two results is essentially the same of 4.5 and 4.7:

Proposition 4.11. *Every semifree module is projective.*

Theorem 4.12. *Every A-dg-module admits a semifree resolution.*

Exercise 4.13. An A-dg-module M is called *flat* if for every quasiisomorphism $f\colon N \to P$ the morphism $f \otimes Id\colon N \otimes M \to P \otimes M$ is a quasiisomorphism. Prove that every semifree module is flat. △

Example 4.14. If $R = K[\{x_i\}]$ is a K-semifree algebra then $\Omega_{R/K} = \oplus R\delta x_i$ is a semifree R-dg-module.

5 The cotangent complex

Proposition 5.1. *Assume it is given a commutative diagram of K-dg-algebras*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "p"'] & S \arrow[d] & R \arrow[l, "g"'] \arrow[dl, "p"] \\ & A & \end{tikzcd}$$

If there exists a homotopy between f and g, constant on A, then the induced morphisms of A-dg-modules

$$f, g\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A$$

are homotopic.

Proof. Let $J \subset S$ be the kernel of $S \to A$ and let $H\colon R \to S \oplus_{j\geq 0} (Jt^{j+1} \oplus Jt^j dt)$ be a homotopy between f and g; the first terms of H are

$$H(x) = f(x) + t(g(x) - f(x)) + dt\, q(x) + \dots .$$

From $dH(x) = H(dx)$ we get $g(x) - f(x) = q(dx) + dq(x)$ and from $H(xy) = H(x)H(y)$ follows $q(xy) = q(x)f(y) + (-1)^{\overline{x}} f(x)q(y)$. Since $f(x) - g(x), q(x) \in J$ for every x, the map

$$q\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A, \quad q(\delta x {\cdot} r \otimes a) = \delta(q(x))f(r) \otimes a,$$

is a well defined element of $\mathrm{Hom}^{-1}_A(\Omega_{R/K} \otimes_R A, \Omega_{S/K} \otimes_S A)$. By definition $f, g\colon \Omega_{R/K} \otimes_R A \to \Omega_{S/K} \otimes_S A$ are defined by

$$f(\delta x {\cdot} r \otimes a) = \delta(f(x))f(r) \otimes a, \quad g(\delta x {\cdot} r \otimes a) = \delta(g(x))g(r) \otimes a = \delta(g(x))f(r) \otimes a.$$

A straightforward verification shows that $dq = f - g$. □

Definition 5.2. *Let $R \to A$ be a K-semifree resolution, the A-dg-module $\mathbb{L}_{A/K} = \Omega_{R/K} \otimes_R A$ is called the* relative cotangent complex *of A over K. By 5.1 the homotopy class of $\mathbb{L}_{A/K}$ is independent from the choice of the resolution. For every A-dg-module M the vector spaces*

$$T^i(A/K, M) = H^i(\mathrm{Hom}^*_A(\mathbb{L}_{A/K}, M)) = \mathrm{Ext}^i_A(\mathbb{L}_{A/K}, M),$$

$$T_i(A/K, M) = H_i(\mathbb{L}_{A/K} \otimes M)) = \mathrm{Tor}^A_i(\mathbb{L}_{A/K}, M),$$

are called respectively the cotangent and tangent cohomolgy of the morphism $K \to A$ with coefficient on M.

Lemma 5.3. *Let $p\colon R \to S$ be a surjective quasiisomorphism of semifree dg-algebras: consider on S the structure of R-dg-module induced by p. Then:*

1. *$p_*\colon \mathrm{Der}^*(R, R) \to \mathrm{Der}^*(R, S)$, $f \to pf$, is a surjective quasiisomorphism.*

2. *$p^*\colon \mathrm{Der}^*(S, S) \to \mathrm{Der}^*(R, S)$, $f \to fp$, is an injective quasiisomorphism.*

Proof. A derivation on a semifree dg-algebra is uniquely determined by the values at its generators, in particular p_* is surjective and p^* is injective. Since Ω_R is semifree, by 3.5 the morphism $p_*\colon \mathrm{Hom}^*_R(\Omega_R, R) \to \mathrm{Hom}^*_R(\Omega_R, S)$ is a quasiisomorphism. By base change $\mathrm{Der}^*(R, S) = \mathrm{Hom}^*_S(\Omega_R \otimes_R S, S)$ and, since $p\colon \Omega_R \otimes_R S \to \Omega_S$ is a homotopy equivalence, also p^* is a quasiisomorphism. □

Every morphism $f\colon A \to B$ of dg-algebras induces a morphism of B modules $\mathbb{L}_A \otimes_A B \to \mathbb{L}_B$ unique up to homotopy. In fact if $R \to A$ and $P \to B$ are semifree resolution, then there exists a lifting of f, $R \to P$, unique up to homotopy constant on B. The morphism $\Omega_R \to \Omega_P$ induce a morphism $\Omega_R \otimes_R B = \mathbb{L}_A \otimes_A B \to \Omega_P \otimes_P B = \mathbb{L}_B$. If B is a localization of A we have the following

Theorem 5.4. *Let A be a dg-algebra, $S \subset Z_0(A)$ a multiplicative part: then the morphism*

$$\mathbb{L}_A \otimes_A S^{-1}A \to \mathbb{L}_{S^{-1}A}$$

is a quasiisomorphism of $S^{-1}A$ modules.

Proof. (sketch) Denote by $f\colon R\to A$, $g\colon P\to S^{-1}A$ two semifree resolutions and by

$$H=\{x\in Z_0(R)\,|\,f(x)\in S\},\qquad K=\{x\in Z_0(P)\,|\,g(x)\text{ is invertible }\}.$$

The natural morphisms $H^{-1}R\to S^{-1}A$, $K^{-1}P\to S^{-1}A$ are both surjective quasiisomorphisms. By the lifting property of semifree algebras we have a chain of morphisms

$$R\xrightarrow{\alpha}P\xrightarrow{\beta}H^{-1}R\xrightarrow{\gamma}K^{-1}P$$

with γ the localization of α. Since $\beta\alpha$ and $\gamma\beta$ are homotopic to the natural inclusions $R\to H^{-1}R$, $P\to K^{-1}P$, the composition of morphisms

$$\Omega_R\otimes_R S^{-1}A\xrightarrow{\alpha}\Omega_P\otimes_P S^{-1}A\xrightarrow{\beta}\Omega_{H^{-1}R}\otimes_{H^{-1}R}S^{-1}A=\Omega_R\otimes_R S^{-1}A,$$

$$\Omega_P\otimes_P S^{-1}A\xrightarrow{\beta}\Omega_{H^{-1}R}\otimes_{H^{-1}R}S^{-1}A\xrightarrow{\gamma}\Omega_{K^{-1}P}\otimes_{K^{-1}P}S^{-1}A=\Omega_P\otimes_P S^{-1}A$$

are homotopic to the identity and hence quasiisomorphisms. □

Example 5.5. Hypersurface singularities.
Let $X=V(f)\subset\mathbb{A}^n$, $f\in\mathbb{K}[x_1,\dots,x_n]$, be an affine hypersurface and denote by $A=\mathbb{K}[X]=\mathbb{K}[x_1,\dots,x_n]/(f)$ its structure ring. A DG-resolvent of A is given by $R=\mathbb{K}[x_1,\dots,x_n,y]$, where y has degree -1 and the differential is given by $s(y)=f$. The R-module Ω_R is semifreely generated by $dx_1,\dots,dx_n,dy$, with the differential

$$s(dy)=d(s(y))=df=\sum_i\frac{\partial f}{\partial x_i}dx_i.$$

The cotangent complex $\mathbb{L}_A$ is therefore

$$0\longrightarrow Ady\xrightarrow{s}\bigoplus_{i=1}^n Adx_i\longrightarrow 0.$$

In particular $T^i(A/\mathbb{K},A)=\operatorname{Ext}^i(\mathbb{L}_A,A)=0$ for every $i\neq 0,1$. The cokernel of s is isomorphic to Ω_A and then $T^0(A/\mathbb{K},A)=\operatorname{Ext}^0(\mathbb{L}_A,A)=\operatorname{Der}_{\mathbb{K}}(A,A)$. If f is reduced then s is injective, $\mathbb{L}_A$ is quasiisomorphic to Ω_A and then $T^1(A/\mathbb{K},A)=\operatorname{Ext}^1(\Omega_A,A)$.

Exercise 5.6. In the set-up of Example 5, prove that the A-module $T^1(A/\mathbb{K},A)$ is finitely generated and supported in the singular locus of X. △

6 The controlling differential graded Lie algebra

Let $p\colon R\to S$ be a surjective quasiisomorphism of semifree algebras and let $I=\ker p$. By the lifting property of S there exists a morphism of dg-algebras $e\colon S\to R$ such that $pe=Id_S$. Define

$$L_p=\{f\in\operatorname{Der}^*(R,R)\,|\,f(I)\subset I\}.$$

It is immediate to verify that L_p is a dg-Lie subalgebra of $\operatorname{Der}^*(R,R)$. We may define a map

$$\theta_p\colon L_p\to\operatorname{Der}^*(S,S),\qquad \theta_p(f)=p\circ f\circ e.$$

Since $pf(I)=0$ for every $f\in L_p$, the definition of θ_p is independent from the choice of e.

Lemma 6.1. *θ_p is a morphism of DGLA.*

Proof. For every $f, g \in L_p$, $s \in S$, we have:

$$d(\theta_p f)(s) = dpfe(s) - (-1)^{\overline{f}} pfe(ds) = pdfe(s) - (-1)^{\overline{f}} pfd(e(s)) = \theta_p(df)(s).$$

Since $pfep = pf$ and $pgep = pg$

$$[\theta_p f, \theta_p g] = pfepge - (-1)^{\overline{f}\,\overline{g}} pgepfe = p(fg - (-1)^{\overline{f}\,\overline{g}} gf)e = \theta_p([f, g]).$$

□

Theorem 6.2. *The following is a cartesian diagram of quasiisomorphisms of DGLA*

$$\begin{tikzcd} L_p \arrow[r, hook, "\imath_p"] \arrow[d, "\theta_p"] & \operatorname{Der}^*(R,R) \arrow[d, "p_*"] \\ \operatorname{Der}^*(S,S) \arrow[r, "p^*"'] & \operatorname{Der}^*(R,S) \end{tikzcd} ,$$

where $\imath_p$ is the inclusion.

We recall that cartesian means that it is commutative and that L_p is isomorphic to the fibred product of p_* and p^*.

Proof. Since $pfep = pf$ for every $f \in L_p$ we have $p^*\theta_p(f) = pfep = pf = p_* f$ and the diagram is commutative. Let

$$K = \{(f, g) \in \operatorname{Der}^*(R, R) \times \operatorname{Der}^*(S, S) \,|\, pf = gp\}$$

be the fibred product; the map $L_p \to K$, $f \to (f, \theta_p(f))$, is clearly injective. Conversely take $(f, g) \in K$ and $x \in I$, since $pf(x) = gp(x) = 0$ we have $f(I) \subset I$, $f \in L_p$. Since p is surjective g is uniquely determined by f and then $g = \theta_p(f)$. This proves that the diagram is cartesian. By 5.3 p_* (resp.: p^*) is a surjective (resp.: injective) quasiisomorphism, by a standard argument in homological algebra also θ_p (resp.: $\imath_p$) is a surjective (resp.: injective) quasiisomorphism. □

Corollary 6.3. *Let $P \to A$, $Q \to A$ be semifree resolutions of a dg-algebra. Then* $\operatorname{Der}^*(P, P)$ *and* $\operatorname{Der}^*(Q, Q)$ *are quasiisomorphic DGLA.*

Proof. There exists a third semifree resolution $R \to A$ and surjective quasiisomorphisms $p \colon R \to P$, $q \colon R \to Q$. Then there exists a sequence of quasiisomorphisms of DGLA

$$\begin{tikzcd} & L_p \arrow[dl, "\theta_p"'] \arrow[dr, "\imath_p"] & & L_q \arrow[dl, "\imath_q"'] \arrow[dr, "\theta_q"] & \\ \operatorname{Der}^*(P,P) & & \operatorname{Der}^*(R,R) & & \operatorname{Der}^*(Q,Q). \end{tikzcd}$$

□

Remark 6.4. *If $R \to A$ is a semifree resolution then*

$$H^i(\operatorname{Der}^*(R, R)) = H^i(\operatorname{Hom}_R(\Omega_R, R)) = H^i(\operatorname{Hom}_R(\Omega_R, A)) =$$

$$= H^i(\operatorname{Hom}_A(\Omega_R \otimes_R A, A)) = \operatorname{Ext}^i(\mathbb{L}_A, A).$$

Unfortunately, contrarily to what happens to the cotangent complex, the application $R \to \mathrm{Der}^*(R, R)$ is quite far from being a functor: it only earns some functorial properties when composed with a suitable functor $\mathbf{DGLA} \to \mathbf{D}$.

Let $\mathbf{D}$ be a category and $\mathcal{F}\colon \mathbf{DGLA} \to \mathbf{D}$ be a functor which sends quasiisomorphisms into isomorphisms of $\mathbf{D}$[1]. By 6.3, if $P \to A$, $Q \to A$ are semifree resolutions then $\mathcal{F}(\mathrm{Der}^*(P, P)) \simeq \mathcal{F}(\mathrm{Der}^*(Q, Q))$; now we prove that the recipe of the proof of 6.3 gives a NATURAL isomorphism independent from the choice of P, p, q. For notational simplicity denote $\mathcal{F}(P) = \mathcal{F}(\mathrm{Der}^*(P, P))$ and for every surjective quasiisomorphism $p\colon R \to P$ of semifree dg-algebras, $\mathcal{F}(p) = \mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1}\colon \mathcal{F}(R) \to \mathcal{F}(P)$.

Lemma 6.5. *Let $p\colon R \to P$, $q\colon P \to Q$ be surjective quasiisomorphisms of semifree dg-algebras, then $\mathcal{F}(qp) = \mathcal{F}(q)\mathcal{F}(p)$.*

Proof. Let $I = \ker p$, $J = \ker q$, $H = \ker qp = p^{-1}(J)$, $e\colon P \to R$, $s\colon Q \to P$ sections. Note that $e(J) \subset H$. Let $L = L_q \times_{\mathrm{Der}^*(P,P)} L_p$, if $(f, g) \in L$ and $x \in H$ then $pg(x) = pg(ep(x)) = f(x) \in J$ and then $g(x) \in H$, $g \in L_{qp}$; denoting $\alpha\colon L \to L_{qp}$, $\alpha(f, g) = g$, we have a commutative diagram of quasiisomorphisms of DGLA

$$\begin{tikzcd}
L_{qp} \arrow[rrr, "\imath_{qp}"] \arrow[ddr, "\theta_{qp}"'] & & & \mathrm{Der}^*(R,R) \\
& L \arrow[ul, "\alpha"'] \arrow[r, "\beta"] \arrow[d, "\gamma"] & L_p \arrow[ur, "\imath_p"'] \arrow[d, "\theta_p"] & \\
& L_q \arrow[r, "\imath_q"'] \arrow[d, "\theta_q"] & \mathrm{Der}^*(P,P) & \\
& \mathrm{Der}^*(Q,Q) & &
\end{tikzcd}$$

and then

$$\mathcal{F}(qp) = \mathcal{F}(\theta_{qp})\mathcal{F}(\imath_{qp})^{-1} = \mathcal{F}(\theta_q)\mathcal{F}(\gamma)\mathcal{F}(\alpha)^{-1}\mathcal{F}(\alpha)\mathcal{F}(\beta)^{-1}\mathcal{F}(\imath_p)^{-1} =$$

$$= \mathcal{F}(\theta_q)\mathcal{F}(\imath_q)^{-1}\mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1} = \mathcal{F}(q)\mathcal{F}(p).$$

□

Let P be a semifree dg-algebra $Q = P[\{x_i, dx_i\}] = P \otimes_{\mathbb{K}} \mathbb{K}[\{x_i, dx_i\}]$, $i\colon P \to Q$ the natural inclusion and $\pi\colon Q \to P$ the projection $\pi(x_i) = \pi(dx_i) = 0$: note that i, π are quasiisomorphisms. Since P, Q are semifree we can define a morphism of DGLA

$$\begin{aligned}
i\colon \mathrm{Der}^*(P, P) &\longrightarrow \mathrm{Der}^*(Q, Q),\\
(if)(x_i) &= (if)(dx_i) = 0,\\
(if)(p) &= i(f(p)),\ p \in P.
\end{aligned}$$

Since $\pi_* i = \pi^*\colon \mathrm{Der}^*(P, P) \to \mathrm{Der}^*(Q, P)$, according to 5.3 i is an injective quasiisomorphism.

Lemma 6.6. *Let P, Q as above, let $q\colon Q \to R$ a surjective quasiisomorphism of semifree algebras. If $p = qi\colon P \to R$ is surjective then $\mathcal{F}(p) = \mathcal{F}(q)\mathcal{F}(i)$.*

[1]The examples that we have in mind are the associated deformation functor and the homotopy class of the corresponding L_∞-algebra

Proof. Let $L = \mathrm{Der}^*(P,P) \times_{\mathrm{Der}^*(Q,Q)} L_q$ be the fibred product of i and $\imath_q$; if $(f,g) \in L$ then $g = if$ and for every $x \in \ker p$, $i(f(x)) = g(i(x)) \in \ker q \cap i(P) = i(\ker p)$. Denoting $\alpha\colon L \to L_p$, $\alpha(f,g) = f$, we have a commutative diagram of quasiisomorphisms

$$\begin{tikzcd}
 & \mathrm{Der}^*(P,P) \arrow[rr, "i"] & & \mathrm{Der}^*(Q,Q) & \\
L_p \arrow[ur, "\imath_p"] \arrow[drr, "\theta_p"'] & & L \arrow[ll, "\alpha"'] \arrow[ul] \arrow[ur] \arrow[rr] & & L_q \arrow[ul, "\imath_q"'] \arrow[dll, "\theta_q"] \\
 & & \mathrm{Der}^*(R,R) & &
\end{tikzcd}$$

and then $\mathcal{F}(q)\mathcal{F}(i) = \mathcal{F}(\theta_q)\mathcal{F}(\imath_q)^{-1}\mathcal{F}(i) = \mathcal{F}(\theta_p)\mathcal{F}(\imath_p)^{-1}$. □

Lemma 6.7. *Let $p_0, p_1\colon P \to R$ be surjective quasiisomorphisms of semifree algebras. If p_0 is homotopic to p_1 then $\mathcal{F}(p_0) = \mathcal{F}(p_1)$.*

Proof. We prove first the case $P = R[t, dt]$ and $p_i = e_i$, $i = 0, 1$, the evaluation maps. Denote by

$$L = \{f \in \mathrm{Der}^*(P,P) \mid f(R) \subset R, f(t) = f(dt) = 0\}.$$

Then $L \subset L_{e_\alpha}$ for every $\alpha = 0, 1$, $\theta_{e_\alpha}\colon L \to \mathrm{Der}^*(P,P)$ is an isomorphism not depending from α and $L \subset L_{e_\alpha} \subset \mathrm{Der}^*(R,R)$ are quasiisomorphic DGLA. This proves that $\mathcal{F}(e_0) = \mathcal{F}(e_1)$. In the general case we can find commutative diagrams, $\alpha = 0, 1$,

$$\begin{tikzcd}
P[\{x_j, dx_j\}] \arrow[r, "q"] \arrow[dr, "q_\alpha"] & R[t, dt] \arrow[d, "e_\alpha"] \\
P \arrow[u, "i"] \arrow[r, "p_\alpha"'] & R
\end{tikzcd}$$

with q surjective quasiisomorphism. We then have $\mathcal{F}(p_0) = \mathcal{F}(q_0)\mathcal{F}(i)^{-1} = \mathcal{F}(e_0)\mathcal{F}(q)\mathcal{F}(i)^{-1} = \mathcal{F}(e_1)\mathcal{F}(q)\mathcal{F}(i)^{-1} = \mathcal{F}(q_1)\mathcal{F}(i)^{-1} = \mathcal{F}(p_1)$. □

We are now able to prove the following

Theorem 6.8. *Let*

$$\begin{tikzcd}
R \arrow[r, "p"] \arrow[d, "q"'] & P \arrow[d] \\
Q \arrow[r] & A
\end{tikzcd}$$

be a commutative diagram of surjective quasiisomorphisms of dg-algebras with P, Q, R semifree. Then $\Psi = \mathcal{F}(p)\mathcal{F}(q)^{-1}\colon \mathcal{F}(Q) \to \mathcal{F}(P)$ does not depend from R, p, q.

Proof. Consider two diagrams as above

$$\begin{tikzcd}
R_0 \arrow[r, "p_0"] \arrow[d, "q_0"'] & P \arrow[d] \\
Q \arrow[r] & A,
\end{tikzcd}
\qquad
\begin{tikzcd}
R_1 \arrow[r, "p_1"] \arrow[d, "q_1"'] & P \arrow[d] \\
Q \arrow[r] & A.
\end{tikzcd}$$

There exists a commutative diagram of surjective quasiisomorphisms of semifree algebras

$$\begin{tikzcd} T \arrow[r, "t_1"] \arrow[d, "t_0"'] & R_1 \arrow[d, "q_1"] \\ R_0 \arrow[r, "q_0"'] & Q. \end{tikzcd}$$

By Lemma 6.5 $\mathcal{F}(q_0)\mathcal{F}(t_0) = \mathcal{F}(q_1)\mathcal{F}(t_1)$. According to 4.5 the morphisms $p_0t_0, p_1t_1 \colon T \to P$ are homotopic and then $\mathcal{F}(p_0)\mathcal{F}(t_0) = \mathcal{F}(p_1)\mathcal{F}(t_1)$. This implies that $\mathcal{F}(p_0)\mathcal{F}(q_0)^{-1} = \mathcal{F}(p_1)\mathcal{F}(q_1)^{-1}$. □

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