CHAPTER 1

Differential graded coalgebras

1.1. Graded coalgebras

Definition 1.1.1. A (coassociative) **graded coalgebra** is the data of a graded vector space $C = \oplus_{n\in\mathbb{Z}} C^n$ and of a morphism of graded vector spaces $\Delta\colon C \to C \otimes C$, called **coproduct**, whic satifies the coassociativity equation:

$$(\Delta \otimes \mathrm{Id}_C)\Delta = (\mathrm{Id}_C \otimes \Delta)\Delta\colon C \to C \otimes C \otimes C\,.$$

Definition 1.1.2. Let (C, Δ) and (B, Γ) be graded coalgebras. A **morphism** of graded coalgebras $f\colon C \to B$ is a morphism of graded vector spaces that commutes with coproducts, i.e.

$$\Gamma f = (f \otimes f)\Delta\colon C \to B \otimes B.$$

Example 1.1.3. Let $C = \mathbb{K}\,[t]$ be the polynomial ring in one variable t (of degree 0). The linear map

$$\Delta\colon \mathbb{K}\,[t] \to \mathbb{K}\,[t] \otimes \mathbb{K}\,[t], \qquad \Delta(t^n) = \sum_{i=0}^{n} t^i \otimes t^{n-i},$$

gives a coalgebra structure (exercise: check coassociativity).
For every sequence $f_n \in \mathbb{K}$, $n > 0$, it is associated a morphism of coalgebras $f\colon C \to C$ defined as

$$f(1) = 1, \qquad f(t^n) = \sum_{s=1}^{n} \sum_{\substack{(i_1,\ldots,i_s)\in\mathbb{N}^s\\ i_1+\cdots+i_s=n}} f_{i_1} f_{i_2} \cdots f_{i_s} t^s.$$

The verification that $\Delta f = (f \otimes f)\Delta$ can be done in the following way: Let $\{x^n\} \subset C^\vee = \mathbb{K}\,[[x]]$ be the dual basis of $\{t^n\}$. Then for every $a, b, n \in N$ we have:

$$\langle x^a \otimes x^b, \Delta f(t^n)\rangle = \sum_{i_1+\cdots+i_a+j_1+\cdots+j_b=n} f_{i_1} \cdots f_{i_a} f_{j_1} \cdots f_{j_b},$$

$$\langle x^a \otimes x^b, f \otimes f\Delta(t^n)\rangle = \sum_{s} \sum_{i_1+\cdots+i_a=s} \sum_{j_1+\cdots+j_b=n-s} f_{i_1} \cdots f_{i_a} f_{j_1} \cdots f_{j_b}.$$

Note that the sequence $\{f_n\}$, $n \geq 1$, can be recovered from f by the formula $f_n = \langle x, f(t^n)\rangle$.

Definition 1.1.4. A graded coalgebra (C, Δ) is called **cocommutative** if $\mathtt{tw} \circ \Delta = \Delta$, where $\mathtt{tw}\colon C \otimes C \to C \otimes C$ is the twist map.

Example 1.1.5. The polynomial coalgebra of Example 1.1.3 is cocommutative.

Example 1.1.6. Let C be a graded coalgebra with coproduct $\Delta\colon C \to C \otimes C$. Then the **convolution product** defined as

$$\mathrm{Hom}^*_{\mathbb{K}}(C, \mathbb{K}\,) \times \mathrm{Hom}^*_{\mathbb{K}}(C, \mathbb{K}\,) \to \mathrm{Hom}^*_{\mathbb{K}}(C, \mathbb{K}\,), \qquad (f, g) \mapsto \mu(f \otimes g)\Delta,$$

where $\mu\colon \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ is the product, is an associative product. Thus the dual of a coalgebra is an algebra.

In general the dual of an algebra is not a coalgebra (with some exceptions, see e.g. Example 1.1.16). Heuristically, this asymmetry comes from the fact that, for an infinite dimensional vector space V, there exist a natural map $V^\vee \otimes V^\vee \to (V \otimes V)^\vee$, while does not exist any natural map $(V \otimes V)^\vee \to V^\vee \otimes V^\vee$.

Example 1.1.7. The dual of the coalgebra $C = \mathbb{K}[t]$ (Example **??**) is exactly the algebra of formal power series $A = \mathbb{K}[[x]] = C^\vee$. Every coalgebra morphism $f\colon C \to C$ induces a local homomorphism of $\mathbb{K}$-algebras $f^t\colon A \to A$. The morphism f^t is uniquely determined by the power series $f^t(x) = \sum_{n>0} f_n x^n$ and then every morphism of coalgebras $f\colon C \to C$ is uniquely determined by the sequence $f_n = \langle f^t(x), t^n\rangle = \langle x, f(t^n)\rangle$.
The map $f \mapsto f^t$ is functorial and then preserves the composition laws.

Definition 1.1.8. Let (C, Δ) be a graded coalgebra; the iterated coproducts $\Delta^n\colon C \to C^{\otimes n+1}$ are defined recursively for $n \geq 0$ by the formulas

$$\Delta^0 = \mathrm{Id}_C, \qquad \Delta^n\colon C \xrightarrow{\Delta} C \otimes C \xrightarrow{\mathrm{Id}_C \otimes \Delta^{n-1}} C \otimes C^{\otimes n} = C^{\otimes n+1}.$$

Lemma 1.1.9. *Let (C, Δ) be a graded coalgebra. Then:*

(1) *For every $0 \leq a \leq n-1$ we have*

$$\Delta^n = (\Delta^a \otimes \Delta^{n-1-a})\Delta\colon C \to \bigotimes^{n+1} C.$$

(2) *For every $s \geq 1$ and every $a_0, \ldots, a_s \geq 0$ we have*

$$(\Delta^{a_0} \otimes \Delta^{a_1} \otimes \cdots \otimes \Delta^{a_s})\Delta^s = \Delta^{s+\sum a_i}.$$

(3) *If $f\colon (C, \Delta) \to (B, \Gamma)$ is a morphism of graded coalgebras then, for every $n \geq 1$ we have*

$$\Gamma^n f = (\otimes^{n+1} f)\Delta^n\colon C \to \bigotimes^{n+1} B.$$

Proof. [1] If $a = 0$ or $n = 1$ there is nothing to prove, thus we can assume $a > 0$ and use induction on n. we have:

$$(\Delta^a \otimes \Delta^{n-1-a})\Delta = ((\mathrm{Id}_C \otimes \Delta^{a-1})\Delta \otimes \Delta^{n-1-a})\Delta =$$
$$= (\mathrm{Id}_C \otimes \Delta^{a-1} \otimes \Delta^{n-1-a})(\Delta \otimes \mathrm{Id}_C)\Delta =$$
$$= (\mathrm{Id}_C \otimes \Delta^{a-1} \otimes \Delta^{n-1-a})(\mathrm{Id}_C \otimes \Delta)\Delta = (\mathrm{Id}_C \otimes (\Delta^{a-1} \otimes \Delta^{n-1-a})\Delta)\Delta = \Delta^n.$$

[2] Induction on s, being the case $s = 1$ proved in item 1. If $s \geq 2$ we can write

$$(\Delta^{a_0} \otimes \Delta^{a_1} \otimes \cdots \otimes \Delta^{a_s})\Delta^s = (\Delta^{a_0} \otimes \Delta^{a_1} \otimes \cdots \otimes \Delta^{a_s})(\mathrm{Id}_C \otimes \Delta^{s-1})\Delta =$$
$$(\Delta^{a_0} \otimes (\Delta^{a_1} \otimes \cdots \otimes \Delta^{a_s})\Delta^{s-1})\Delta = (\Delta^{a_0} \otimes \Delta^{s-1+\sum_{i>0} a_i})\Delta = \Delta^{s+\sum a_i}.$$

[3] By induction on n,

$$\Gamma^n f = (\mathrm{Id}_B \otimes \Gamma^{n-1})\Gamma f = (f \otimes \Gamma^{n-1} f)\Delta = (f \otimes (\otimes^n f)\Delta^{n-1})\Delta = (\otimes^{n+1} f)\Delta^n.$$

□

Lemma 1.1.10. *Let (C, Δ) be a graded coalgebra. Then for every $n \geq 0$ we have*

$$\ker \Delta^{n+1} = \{x \in C \mid \Delta(x) \in (\ker \Delta^n) \otimes (\ker \Delta^n)\}.$$

Proof. The formula

$$\Delta^{n+1} = (\Delta^n \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta^n)\Delta\,.$$

implies the inclusion $\supset$. Conversely, notice that $\Delta(x) = 0$ if and only if every homogeneous component of x belongs to $\ker \Delta^{n+1}$ Let $x \in \ker \Delta^{n+1}$ homogeneous and write $\Delta(x) = \sum_{i=1}^r x_i \otimes y_i$ with r minimum. Then the vectors x_i are linearly independent and the same holds for the vectors y_i. The conclusion is now immediate fro the above formula. □

Definition 1.1.11. Let (C, Δ) be a graded coalgebra. A morphism of graded vector spaces $p\colon C \to V$ is called a **cogenerator** of C if for every $c \in C$ there exists $n \geq 0$ such that $(\otimes^{n+1} p)\Delta^n(c) \neq 0$ in $\bigotimes^{n+1} V$. Equivalently, $p\colon C \to V$ is a cogenerator of C is the map

$$C \to \prod_{n \geq 0} \bigotimes^{n+1} V, \qquad c \mapsto (c, \Delta c, \Delta^2 c, \ldots),$$

is injective.

Example 1.1.12. In the notation of Example 1.1.3, the natural projection $\mathbb{K}[t] \to \mathbb{K} \oplus \mathbb{K}\,t$ is a cogenerator.

Proposition 1.1.13. *Let $p\colon B \to V$ be a cogenerator of a graded coalgebra (B, Γ). Then every morphism of graded coalgebras $\phi\colon (C, \Delta) \to (B, \Gamma)$ is uniquely determined by its composition $p\phi\colon C \to V$.*

Proof. Let $\phi,\psi\colon (C,\Delta)\to(B,\Gamma)$ be two morphisms of graded coalgebras such that $p\phi=p\psi$. In order to prove that $\phi=\psi$ it is sufficient to show that for every $c\in C$ and every $n\geq 0$ we have

$$(\otimes^{n+1}p)\Gamma^n(\phi(c))=(\otimes^{n+1}p)\Gamma^n(\psi(c)).$$

By Lemma 1.1.9 we have $\Gamma^n\phi=(\otimes^{n+1}\phi)\Delta^n$ and $\Gamma^n\psi=(\otimes^{n+1}\psi)\Delta^n$. Therefore

$$(\otimes^{n+1}p)\Gamma^n\phi=(\otimes^{n+1}p)(\otimes^{n+1}\phi)\Delta^n=(\otimes^{n+1}p\phi)\Delta^n=$$
$$=(\otimes^{n+1}p\psi)\Delta^n=(\otimes^{n+1}p)(\otimes^{n+1}\psi)\Delta^n=(\otimes^{n+1}p)\Gamma^n\psi.$$

□

Definition 1.1.14. A graded coalgebra (C,Δ) is called **nilpotent** if $\Delta^n=0$ for $n>>0$. It is called **locally nilpotent** if it is the direct limit of nilpotent graded coalgebras or equivalently if $C=\cup_n\ker\Delta^n$.

Example 1.1.15. The vector space

$$\overline{\mathbb{K}[t]}=\{p(t)\in\mathbb{K}[t]\mid p(0)=0\}=\bigoplus_{n>0}\mathbb{K}\,t^n$$

with the coproduct

$$\Delta\colon\overline{\mathbb{K}[t]}\to\overline{\mathbb{K}[t]}\otimes\overline{\mathbb{K}[t]},\qquad \Delta(t^n)=\sum_{i=1}^{n-1}t^i\otimes t^{n-i},$$

is a locally nilpotent coalgebra. The projection $\mathbb{K}[t]\to\overline{\mathbb{K}[t]}$, $p(t)\to p(t)-p(0)$, is a morphism of coalgebras.

Example 1.1.16. Let $A=\oplus A_i$ be a finite dimensional graded associative $\mathbb{K}$-algebra and let $C=A^\vee=\operatorname{Hom}^*(A,\mathbb{K})$ be its graded dual. Since A and C are finite dimensional, the pairing $\langle c_1\otimes c_2,a_1\otimes a_2\rangle=(-1)^{\overline{a_1}\,\overline{c_2}}\langle c_1,a_1\rangle\langle c_2,a_2\rangle$ gives a natural isomorphism $C\otimes C=(A\otimes A)^\vee$ and we may define Δ as the transpose of the multiplication map $\mu\colon A\otimes A\to A$. Then (C,Δ) is a graded coalgebra. Note that C is nilpotent if and only if A is nilpotent.

Lemma 1.1.17. *Let (C,Δ) be a locally nilpotent graded coalgebra. Then every projection $p\colon C\to\ker\Delta$ is a cogenerator of C.*

Proof. □

Definition 1.1.18 ([104, p. 282])**.** A graded coalgebra (C,Δ) is called **connected** if there is an element $e\in C$ such that $\Delta(e)=e\otimes e$ (in particular $\deg(e)=0$) and $C=\cup_{r=0}^{+\infty}F_rC$, where F_rC is defined recursively in the following way:

$$F_0C=\mathbb{K}\,e,\qquad F_{r+1}C=\{x\in C\mid \Delta(x)-e\otimes x-x\otimes e\in F_rC\otimes F_rC\}.$$

Example 1.1.19. In the notation of the above definition, according to Lemma 1.1.10 we have $e=0$ if and only if $F_rC=\ker\Delta^r$. In particular every locally nilpotent coalgebra is connected.

1.2. Comodules and coderivations

Definition 1.2.1. Let (C,Δ) be a graded coalgebra. A C-comodule is the data of a graded vector space M and two morphisms of graded vector spaces

$$\phi\colon M\to M\otimes C,\qquad \psi\colon M\to C\otimes M$$

such that:

(1) $(\mathrm{Id}_M\otimes\Delta)\phi=(\phi\otimes\mathrm{Id}_C)\phi\colon M\to M\otimes C\otimes C$,
(2) $(\Delta\otimes\mathrm{Id}_M)\psi=(\mathrm{Id}_C\otimes\psi)\psi\colon M\to C\otimes C\otimes M$,
(3) $(\psi\otimes\mathrm{Id}_C)\phi=(\mathrm{Id}_C\otimes\phi)\psi\colon M\to C\otimes M\otimes C$.

Example 1.2.2. If $F\colon(D,\Gamma)\to(C,\Delta)$ is a morphism of graded coalgebras, then the maps

$$\phi=(\mathrm{Id}_D\otimes F)\Gamma\colon D\to D\otimes C,\qquad \psi=(F\otimes\mathrm{Id}_D)\Gamma\colon D\to C\otimes D,$$

give a structure of C-comodule on D.

Definition 1.2.3. Let (C,Δ) be a graded coalgebra and

$$\phi\colon M\to M\otimes C,\qquad \psi\colon M\to C\otimes M$$

a C-comodule. A linear map $d\in \operatorname{Hom}^n_{\mathbb{K}}(M,C)$ is called a **coderivation** of degree n if it satisfies the **coLeibniz rule**

$$\Delta d=(d\otimes \mathrm{Id}_C)\phi+(\mathrm{Id}_C\otimes d)\psi.$$

In the above definition we have adopted the Koszul sign convention: i.e. if $x,y\in C$, $f,g\in \operatorname{Hom}^*(C,D)$, $h,k\in\operatorname{Hom}^*(B,C)$ are homogeneous then $(f\otimes g)(x\otimes y)=(-1)^{\overline{g}\,\overline{x}}f(x)\otimes g(y)$ and $(f\otimes g)(h\otimes k)=(-1)^{\overline{g}\,\overline{h}}fh\otimes gk$.

In these notes we are mainly intersted to C-comodules structure induced by a morphism of graded coalgebras $F\colon (D,\Gamma)\to(C,\Delta)$. In this case, a morphism of graded vector spaces $d\in\operatorname{Hom}^n(C,D)$ is a coderivation of degree n if and only if

$$\Delta d=(d\otimes F+F\otimes d)\Gamma.$$

The coderivations of degree n with respect a coalgebra morphism $F\colon C\to D$ form a vector space denoted $\operatorname{Coder}^n(C,D;F)$. For simplicity of notation we denote $\operatorname{Coder}^n(C,C)=\operatorname{Coder}^n(C,C;Id)$. In other terms

$$\operatorname{Coder}^n(C,C)=\{f\in\operatorname{Hom}^n_{\mathbb{K}}(C,C)\mid \Delta f=(f\otimes\mathrm{Id}_C+\mathrm{Id}_C\otimes f)\Delta\}.$$

Lemma 1.2.4. *Let C be a graded coalgebra. Then $\operatorname{Coder}^*(C,C)=\bigoplus_n\operatorname{Coder}^n(C,C)$ is a graded Lie subalgebra of $\operatorname{Hom}^*_{\mathbb{K}}(C,C)$.*

PROOF. We only need to prove that $\operatorname{Coder}^*(C,C)$ is closed under the graded commutator. This is straightforward and left to the reader. ☐

Example 1.2.5. For every $k\geq -1$ consider the differential operator

$$f_k\colon \mathbb{K}[t]\to\mathbb{K}[t],\qquad f_k=t\left(\frac{d}{dt}\right)^{k+1}.$$

Then every f_k is a coderivation with respect the coproduct

$$\tilde{\Delta}\colon\mathbb{K}[t]\to\mathbb{K}[t]\otimes\mathbb{K}[t],\qquad \tilde{\Delta}(t^n)=\sum_{i=0}^n\binom{n}{i}t^i\otimes t^{n-i}.$$

Using the definition of binomial coefficicnts

$$\binom{n}{k}=\frac{1}{k!}\prod_{i=0}^{k-1}(n-i),\qquad \binom{n}{0}=1,$$

we have for every $n\geq 0$ and every $k\geq 0$

$$\frac{\tilde{\Delta}(f_{k-1}(t^n))}{k!}=\tilde{\Delta}(\binom{n}{k}t^{n-k+1})=\sum_{i\geq 0}\binom{n}{k}\binom{n-k+1}{i}t^i\otimes t^{n-k-i+1},$$

$$\frac{(f_{k-1}\otimes Id)\tilde{\Delta}(t^n)}{k!}=\sum_{j\geq k}\binom{n}{j}\binom{j}{k}t^{j-k+1}\otimes t^{n-j}=\sum_{i\geq 0}\binom{n}{i+k-1}\binom{i+k-1}{k}t^i\otimes t^{n-k-i+1},$$

$$\frac{(Id\otimes f_{k-1})\tilde{\Delta}(t^n)}{k!}=\sum_{i\geq 0}\binom{n}{i}\binom{n-i}{k}t^i\otimes t^{n-k-i+1},$$

and the conclusion follows from the straightforward equality

$$\binom{n}{k}\binom{n-k+1}{i}=\binom{n}{i+k-1}\binom{i+k-1}{k}+\binom{n}{i}\binom{n-i}{k}.$$

Notice that

$$[f_n,f_m]=f_n\circ f_m-f_m\circ f_n=(n-m)f_{n+m}.$$

Notice that the Lie subalgebra generated by f_k is the same of the Lie algebra generated by the derivations $g_h=z^{h+1}\frac{d}{dz}$ of $\mathbb{K}[z]$.

Lemma 1.2.6. *Let $C\xrightarrow{\theta}D\xrightarrow{\rho}E$ be morphisms of graded coalgebras. The compositions with θ and ρ induce linear maps*

$$\rho_*\colon \operatorname{Coder}^n(C,D;\theta)\to\operatorname{Coder}^n(C,E;\rho\theta),\qquad f\mapsto \rho f;$$
$$\theta^*\colon \operatorname{Coder}^n(D,E;\rho)\to\operatorname{Coder}^n(C,E;\rho\theta),\qquad f\mapsto f\theta.$$

Proof. Immediate consequence of the equalities

$$\Delta_E\rho=(\rho\otimes\rho)\Delta_D,\qquad \Delta_D\theta=(\theta\otimes\theta)\Delta_C.$$

□

Lemma 1.2.7. *Let $C\xrightarrow{\theta}D$ be morphisms of graded coalgebras and let $d\colon C\to D$ be a coderivation (with respect to the comodule structure induced by θ). Then:*

1. *For every n*
$$\Delta_D^n\circ d=\left(\sum_{i=0}^n\theta^{\otimes i}\otimes d\otimes\theta^{\otimes n-i}\right)\circ\Delta_C^n.$$
2. *If $p\colon D\to V$ is a cogenerator, then d is uniquely determined by its composition $pd\colon C\to V$.*

Proof. The first item is a straightforward induction on n, using the equalities $\Delta^n=\operatorname{Id}\otimes\Delta^{n-1}$ and $\theta^{\otimes i}\Delta_C^{i-1}=\Delta_D^{i-1}\theta$.
For item 2, we need to prove that $pd=0$ implies $d=0$. Assume that there exists $c\in C$ such that $dc\neq 0$, then there exists n such that $p^{\otimes n+1}\Delta_D^n dc\neq 0$. On the other hand

$$p^{\otimes n+1}\Delta_D^n dc=\left(\sum_{i=0}^n(p\theta)^{\otimes i}\otimes pd\otimes(p\theta)^{\otimes n-i}\right)\circ\Delta_C^n c=0.$$

□

For later use we point out that if $\alpha\colon C\to C$ be a nilpotent coderivation of degree 0. Then the map

$$e^\alpha=\sum_{n\geq 0}\frac{\alpha^n}{n!}\colon C\to C$$

is a morphism of coalgebras, as follows immediately from the easy formula

$$e^\alpha\otimes e^\alpha=\sum_{n\geq 0}\frac{1}{n!}(\alpha\otimes\operatorname{Id}+\operatorname{Id}\otimes\alpha)^n\in\operatorname{Hom}^0(C\otimes C,C\otimes C).$$

Definition 1.2.8. A **differential graded coalgebra** is the data of a differential graded algebra C together a coderivation $d_C\in\operatorname{Coder}^1(C,C)$, called differential, such that $d_C^2=0$. A morphism of differential graded coalgebras is a morphism of graded coalgebras commuting with differentials.

1.3. The reduced tensor coalgebra

Given a graded vector space V, we denote $\overline{T}(V)=\bigoplus_{n>0}\bigotimes^n V$ and by $p\colon\overline{T}(V)\to V$ the projection with kernel $\bigoplus_{n\geq 2}\bigotimes^n V$.

The **reduced tensor coalgebra** generated by V is the graded vector space $\overline{T}(V)$ endowed with the coproduct $\mathfrak{a}\colon\overline{T}(V)\to\overline{T}(V)\otimes\overline{T}(V)$:

$$\mathfrak{a}(v_1\otimes\cdots\otimes v_n)=\sum_{r=1}^{n-1}(v_1\otimes\cdots\otimes v_r)\otimes(v_{r+1}\otimes\cdots\otimes v_n).$$

We can also write

$$\mathfrak{a}=\sum_{n=2}^{+\infty}\sum_{a=1}^{n-1}\mathfrak{a}_{a,n-a},$$

where

$$\mathfrak{a}_{a,b}\colon\bigotimes^{a+b}V\to\bigotimes^{a}V\otimes\bigotimes^{n-a}V,\qquad \mathfrak{a}_{a,b}(v_1\otimes\cdots\otimes v_a\otimes w_1\otimes\cdots\otimes w_b)=(v_1\otimes\cdots\otimes v_a)\otimes(w_1\otimes\cdots\otimes w_b),$$

The coalgebra $(\overline{T}(V), \mathfrak{a})$ is coassociative, locally nilpotent and the projection $p\colon \overline{T}(V) \to V$ is a cogenerator: in fact, for every $s > 0$,

$$\mathfrak{a}^{s-1}(v_1 \otimes \cdots \otimes v_n) = \sum_{1\le i_1<i_2<\cdots<i_s=n} (v_1 \otimes \cdots \otimes v_{i_1}) \otimes \cdots \otimes (v_{i_{s-1}+1} \otimes \cdots \otimes v_n)$$

and then

$$\ker \mathfrak{a}^{s-1} = \bigoplus_{i=1}^{s-1} V^{\otimes i}, \qquad (\otimes^s p)\mathfrak{a}^{s-1}(v_1 \otimes \cdots \otimes v_s) = v_1 \otimes \cdots \otimes v_s.$$

Exercise 1.3.1. Let $\mu\colon \bigotimes^s \overline{T(V)} \to \overline{T(V)}$ be the multiplication map. Prove that for every $v_1, \ldots, v_n \in V$

$$\mu\mathfrak{a}^{s-1}(v_1 \otimes \cdots \otimes v_n) = \binom{n-1}{s-1} v_1 \otimes \cdots \otimes v_n.$$

For every morphism of graded vector spaces $f\colon V \to W$ the induced morphism of graded algebras

$$T(f)\colon \overline{T(V)} \to \overline{T(W)}, \qquad T(f)(v_1 \otimes \cdots \otimes v_n) = f(v_1) \otimes \cdots \otimes f(v_n)$$

is also a morphism of graded coalgebras.

If (C, Δ) is a locally nilpotent graded coalgebra then, for every $c \in C$, there exists $n > 0$ such that $\Delta^n(c) = 0$ and then it is defined a morphism of graded vector spaces

$$\sum_{n=0}^{\infty} \Delta^n \colon C \to \overline{T}(C).$$

Proposition 1.3.2. *Let (C, Δ) be a locally nilpotent graded coalgebra, then:*

1. *The map $\sum_{n\ge 0} \Delta^n \colon C \to \overline{T}(C)$ is a morphism of graded coalgebras.*
2. *For every graded vector space V and every morphism of graded vector spaces $f\colon C \to V$ there exists a unique morphism of graded coalgebras $F\colon C \to \overline{T}(V)$ such that $pF = f$. Moreover*

$$F = \sum_{n=1}^{\infty} (\otimes^n f)\Delta^{n-1} \colon C \to \overline{T}(C) \to \overline{T}(V).$$

PROOF. [1] We have

$$\left(\left(\sum_{n\ge 0} \Delta^n\right) \otimes \left(\sum_{n\ge 0} \Delta^n\right)\right) \Delta = \sum_{n\ge 0}\sum_{a=0}^{n} (\Delta^a \otimes \Delta^{n-a})\Delta$$
$$= \sum_{n\ge 0}\sum_{a=0}^{n} \mathfrak{a}_{a+1,n+1-a}\Delta^{n+1} = \mathfrak{a}\left(\sum_{n\ge 0} \Delta^n\right)$$

where in the last equality we have used the relation $\mathfrak{a}\Delta^0 = 0$.
[2] The unicity of F is clear since the projection p is a cogenerator. For the existence it is sufficicnt to consider F as the composition of the morphisms of graded coalgebras

$$\sum_{n\ge 0} \Delta^n \colon C \to \overline{T}(C), \qquad T(f)\colon \overline{T}(C) \to \overline{T}(V).$$

□

Corollary 1.3.3. *Let U, V be graded vector spaces. Given a morphism $f\colon \overline{T}(U) \to V$ of graded vector spaces, the linear map $F\colon \overline{T}(U) \to \overline{T}(V)$:*

$$F(v_1 \otimes \cdots \otimes v_n) = \sum_{s=1}^{n} \sum_{1\le i_1<i_2<\cdots<i_s=n} f(v_1 \otimes \cdots \otimes v_{i_1}) \otimes \cdots \otimes f(v_{i_{s-1}+1} \otimes \cdots \otimes v_{i_s}),$$

is the unique morphism of graded coalgebras lifting f.

Example 1.3.4. Let A be an associative graded algebra. Consider the projection $p\colon \overline{T(A)} \to A$, the multiplication map $\mu\colon \overline{T(A)} \to A$ and its conjugate

$$\mu^* = -\mu T(-1), \qquad \mu^*(a_1 \otimes \cdots \otimes a_n) = (-1)^{n-1}\mu(a_1 \otimes \cdots \otimes a_n) = (-1)^{n-1} a_1 a_2 \cdots a_n.$$

The two coalgebra morphisms $\overline{T(A)} \to \overline{T(A)}$ induced by μ and μ^* are isomorphisms, the one inverse of the other.
In fact, the coalgebra morphism $F\colon \overline{T(A)} \to \overline{T(A)}$

$$F(a_1 \otimes \cdots \otimes a_n) = \sum_{s=1}^{n} \sum_{1\le i_1<i_2<\cdots<i_s=n} (a_1a_2\cdots a_{i_1}) \otimes \cdots \otimes (a_{i_{s-1}+1}\cdots a_{i_s})$$

is induced by μ (i.e. $pF = \mu$), $\mu^*F(a) = a$ for every $a \in A$ and for every $n \geq 2$

$$\mu^*F(a_1 \otimes \cdots \otimes a_n) = \sum_{s=1}^{n}(-1)^{s-1} \sum_{1\le i_1<i_2<\cdots<i_s=n} a_1a_2\cdots a_n =$$

$$= \sum_{s=1}^{n}(-1)^{s-1}\binom{n-1}{s-1}a_1a_2\cdots a_n = \left(\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}\right) a_1a_2\cdots a_n = 0.$$

This implies that $\mu^*F = p$ and therefore, if $F^*\colon \overline{T(A)} \to \overline{T(A)}$ is induced by μ^* then $pF^*F = \mu^*F = p$ and then F^*F is the identity.

Proposition 1.3.5. *Let (C, Δ) be a locally nilpotent graded coalgebra, V a graded vector space and*

$$F = \sum_{n=1}^{\infty}(\otimes^n f)\Delta^{n-1}\colon C \to \overline{T}(V)$$

the morphism of coalgebras lifting $f \in \mathrm{Hom}^0(C, V)$. For every $q \in \mathrm{Hom}^k(C, V)$, the linear map

$$Q = \sum_{n=0}^{\infty}(\sum_{i=0}^{n}(f^{\otimes i} \otimes q \otimes f^{\otimes n-i})\Delta^n\colon C \to \overline{T}(V)$$

is the unique F-coderivation lifting q, i.e. $q = pQ$. In particular the map

$$\mathrm{Coder}^*(C, \overline{T}(V); F) \to \mathrm{Hom}^*(C, V), \qquad Q \mapsto pQ,$$

is an isomorphism of vector graded vector spaces.

Proof. The map Q is the composition of the coalgebra morphism $\sum \Delta^n\colon C \to \overline{T(C)}$ and the map

$$R\colon \overline{T}(C) \to \overline{T}(V), \qquad R = \sum_{i,j\geq 0} f^{\otimes i} \otimes q \otimes f^{\otimes j}.$$

It is therefore sufficient to prove that R is a $T(f)$-coderivation, i.e. that satisfies the coLeibniz rule

$$(R \otimes T(f) + T(f) \otimes R)\mathfrak{a} = \mathfrak{a}R.$$

Denoting $R_n = \sum_{i+j=n-1} f^{\otimes i} \otimes q \otimes f^{\otimes j}$ we have, for every a, n

$$\mathfrak{a}_{a,n-a}R_n = (R_a \otimes f^{\otimes n-a} + f^{\otimes a} \otimes R_{n-a})\mathfrak{a}_{a,n-a}.$$

Taking the sum over $a, n-a$ we get the proof. □

Corollary 1.3.6. *Let V be a graded vector space. Every $q \in \mathrm{Hom}^k(\overline{T}(V), V)$ lifts to a coderivation $Q \in \mathrm{Coder}^k(\overline{T}(V), \overline{T}(V))$ given by the explicit formula*

$$Q(a_1 \otimes \cdots \otimes a_n) =$$
$$= \sum_{i,l}(-1)^{k(\overline{a_1}+\cdots+\overline{a_i})}a_1 \otimes \cdots \otimes a_i \otimes q(a_{i+1} \otimes \cdots \otimes a_{i+l}) \otimes \cdots \otimes a_n.$$

Proof. Apply Proposition 1.3.5 with the map $f = p\colon \overline{T}(V) \to V$ equal to the projection (and then $F = \mathrm{Id}$). □

Remark 1.3.7. Let Q be the coderivation of $\overline{T}(V)$ lifting a morphism $q \in \mathrm{Hom}^*(\overline{T}(V), V)$. It is a immediate consequence of the above corollary that $Q(\bigotimes^n V) \subset \bigoplus_{k=1}^n \bigotimes^k V$.

Moreover if $q = (q_1, q_2, q_3, \ldots)$ with $q_k\colon \bigotimes^k V \to V$ and $q_k = 0$ for every $k \leq r$, then $Q(\bigotimes^n V) \subset \bigoplus_{k=1}^{n-r} \bigotimes^k V$.

Definition 1.3.8. Given a graded vector space V the **Gerstenhaber product**

$$\mathrm{Hom}^*_{\mathbb{K}}(\overline{T}(V),V)\times \mathrm{Hom}^*_{\mathbb{K}}(\overline{T}(V),V)\to \mathrm{Hom}^*_{\mathbb{K}}(\overline{T}(V),V),\qquad (f,g)\mapsto f\circ g,$$

is defined as $f\circ g=fG$, where $G\in \mathrm{Coder}^*(\overline{T}(V),\overline{T}(V))$ is the unique coderivation lifting g.

The **Gerstenhaber bracket** is defined as

$$[f,g]=f\circ g-(-1)^{\overline{f}\,\overline{g}}g\circ f,\qquad f,g\in \mathrm{Hom}^*_{\mathbb{K}}(\overline{T}(V),V).$$

Notice that if $F\in \mathrm{Coder}^*(\overline{T}(V),\overline{T}(V))$ is the coderivation lifting f, then $pFG=f\circ g$, $pgF=g\circ f$ and then $p[F,G]=[f,g]$. Therefore the isomorphism $\mathrm{Coder}^*(\overline{T}(V),\overline{T}(V))\simeq \mathrm{Hom}^*_{\mathbb{K}}(\overline{T}(V),V)$ commutes with brackets and then the Gerstenhaber bracket gives a structure of graded Lie algebra.

Given $f\in \mathrm{Hom}^a_{\mathbb{K}}(V^{\otimes n+1},V)$ and $g\in \mathrm{Hom}^b_{\mathbb{K}}(V^{\otimes m+1},V)$, considered as elements of $\mathrm{Hom}^*_{\mathbb{K}}(\overline{T}(V),V)$ via the natural inclusion $V^{\otimes n}\subset \overline{T}(V)$ and $V^{\otimes m}\subset \overline{T}(V)$ we have $f\circ g\in \mathrm{Hom}^{a+b}_{\mathbb{K}}(V^{\otimes n+m},V)$,

$$f\circ g(v_0\otimes\cdots\otimes v_{n+m})=\sum_{i=0}^{n}(-1)^{b(\overline{v_0}+\cdots+\overline{v_{i-1}})}f(v_1\otimes\cdots\otimes v_{i-1}\otimes g(v_i\otimes\cdots\otimes v_{i+m})\otimes\cdots\otimes v_{n+m}).$$

1.4. Symmetrization and unshuffles

Given a graded vector space V, the twist map extends naturally, for every $n\geq 0$, to an action of the symmetric group Σ_n on the graded vector space $\bigotimes^n V$. More explicitely, for $v_1,\ldots,v_n$ homogeneous vectors and $\sigma\in\Sigma_n$ we have:

$$\sigma_{\mathtt{tw}}(v_1\otimes\cdots\otimes v_n)=\pm(v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}),$$

where the sign is the signature of the restriction of σ to the subset of indices i such that v_i has odd degree.

Definition 1.4.1. The **Koszul sign** $\epsilon(V,\sigma;v_1,\ldots,v_n)=\pm1$ is defined by the relation

$$\sigma_{\mathtt{tw}}^{-1}(v_1\otimes\cdots\otimes v_n)=\epsilon(V,\sigma;v_1,\ldots,v_n)(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})$$

For notational simplicity we shall write $\epsilon(\sigma;v_1,\ldots,v_n)$ or $\epsilon(\sigma)$ when there is no possible confusion about V and $v_1,\ldots,v_n$.

Remark 1.4.2. The twist action on $\bigotimes^n(\mathrm{Hom}^*(V,W))$ is compatible with the conjugate of the twist action on $\mathrm{Hom}^*(V^{\otimes n},W^{\otimes n})$. This means that

$$\sigma_{\mathtt{tw}}(f_1\otimes\cdots\otimes f_n)=\sigma_{\mathtt{tw}}\circ(f_1\otimes\cdots\otimes f_n)\circ\sigma_{\mathtt{tw}}^{-1},$$

where $\circ$ is the composition product.

Definition 1.4.3. The symmetric powers of a graded vector space V are defined as

$$\bigodot^n V=\frac{\bigotimes^n V}{I},$$

where I is the subspace generated by all the vectors $v-\sigma_{\mathtt{tw}}(v)$, $\sigma\in\Sigma_n$, $v\in\bigotimes^n V$. We will denote by $\pi\colon \bigotimes^n V\to\bigodot^n V$ the natural projection and

$$v_1\odot\cdots\odot v_n=\pi(v_1\otimes\cdots\otimes v_n).$$

Definition 1.4.4. Denote by $N\colon \bigodot^n V\to\bigotimes^n V$ the map (see next Lemma 1.4.5):

$$\begin{aligned}N(v_1\odot\cdots\odot v_n)&=\sum_{\sigma\in\Sigma_n}\epsilon(\sigma;v_1,\ldots,v_n)(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})\\&=\sum_{\sigma\in\Sigma_n}\sigma_{\mathtt{tw}}(v_1\otimes\cdots\otimes v_n),\quad v_1,\ldots,v_n\in V.\end{aligned}$$

Lemma 1.4.5. *The map N is well defined, it is injective and its image is the subspace $(\bigotimes^n V)^{\Sigma_n}$ of twist-invariant tensors.*

Proof. Consider the map $N'\colon \bigotimes^n V \to \bigotimes^n V$:

$$N'(v_1\otimes\cdots\otimes v_n)=\sum_{\sigma\in\Sigma_n}\sigma_{\mathtt{tw}}(v_1\otimes\cdots\otimes v_n),\quad v_1,\ldots,v_n\in V.$$

It is clear that

$$\frac{1}{n!}N'\colon \bigotimes^n V\to (\bigotimes^n V)^{\Sigma_n}$$

is a projection and then

$$\bigotimes^n V=(\bigotimes^n V)^{\Sigma_n}\oplus\ker(N').$$

Denote as above by I the subspace generated by all the vectors $v-\sigma_{\mathtt{tw}}(v)$, $\sigma\in\Sigma_n$, $v\in\bigotimes^n V$. Since $N'(v)=N'(\sigma_{\mathtt{tw}}v)$ we have $I\subset\ker(N')$. For every $v\in\bigotimes^n V$ we can write

$$v=\frac{N'}{n!}v+\left(v-\frac{N'}{n!}v\right)=\frac{N'}{n!}v+\frac{1}{n!}\sum_{\sigma\in\Sigma_n}(v-\sigma_{\mathtt{tw}}v).$$

This shows that $\mathrm{Im}(N')+I=\bigotimes^n V$ and this implies that $\ker N'=I$ and $N'=N\pi$. □

Lemma 1.4.6. *Let (C,Δ) be a graded cocommutative coalgebra. Then the image of Δ^{n-1} is contained in the set of Σ_n-invariant elements of $\bigotimes^n C$ and therefore*

$$\Delta^{n-1}=N\frac{\pi}{n!}\Delta^{n-1}.$$

Proof. The twist action of Σ_n on $\bigotimes^n C$ is generated by the operators $\mathtt{tw}_a=\mathrm{Id}_{\bigotimes^a C}\otimes\mathtt{tw}\otimes\mathrm{Id}_{\bigotimes^{n-a-2}C}$, $0\le a\le n-2$; since $\mathtt{tw}\circ\Delta=\Delta$, according to Lemma 1.1.9 we have:

$$\begin{aligned}\mathtt{tw}_a\Delta^{n-1}&=\mathtt{tw}_a(\mathrm{Id}_{\bigotimes^a C}\otimes\Delta\otimes\mathrm{Id}_{\bigotimes^{n-a-2}C})\Delta^{n-2}\\&=(\mathrm{Id}_{\bigotimes^a C}\otimes\Delta\otimes\mathrm{Id}_{\bigotimes^{n-a-2}C})\Delta^{n-2}=\Delta^{n-1}.\end{aligned}$$

□

Definition 1.4.7. The set of **unshuffles** of type (p,q) is the subset $S(p,q)\subset\Sigma_{p+q}$ of permutations σ such that $\sigma(i)<\sigma(i+1)$ for every $i\neq p$. Equivalently

$$S(p,q)=\{\sigma\in\Sigma_{p+q}\mid\sigma(1)<\sigma(2)<\ldots<\sigma(p),\quad\sigma(p+1)<\sigma(p+2)<\ldots<\sigma(p+q)\}.$$

The unshuffles are a set of representatives for the left cosets of the canonical embedding of $\Sigma_p\times\Sigma_q$ inside Σ_{p+q}. More precisely for every $\eta\in\Sigma_{p+q}$ there exists a unique decomposition $\eta=\sigma\tau$ with $\sigma\in S(p,q)$ and $\tau\in\Sigma_p\times\Sigma_q$.

Lemma 1.4.8. *For every $v_1,\ldots,v_n\in V$ and every $a=0,\ldots,n$ we have*

$$N(v_1\odot\cdots\odot v_n)=\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)N(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(a)})\otimes N(v_{\sigma(a+1)}\odot\cdots\odot v_{\sigma(n)}).$$

Proof.

$$\begin{aligned}N(v_1\odot\cdots\odot v_n)&=\sum_{\eta\in\Sigma_n}\eta_{\mathtt{tw}}^{-1}(v_1\otimes\cdots\otimes v_n)\\&=\sum_{\sigma\in S(a,n-a)}\sum_{\tau\in\Sigma_a\times\Sigma_{n-a}}\tau_{\mathtt{tw}}^{-1}\sigma_{\mathtt{tw}}^{-1}(v_1\otimes\cdots\otimes v_n)\\&=\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)\sum_{\tau\in\Sigma_a\times\Sigma_{n-a}}\tau_{\mathtt{tw}}^{-1}(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)})\\&=\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)N(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(a)})\otimes N(v_{\sigma(a+1)}\odot\cdots\odot v_{\sigma(n)}).\end{aligned}$$

□

Consider now two graded vector spaces V,M, a positive integer l and two maps

$$f\in\mathrm{Hom}^0(V,M),\qquad b\in\mathrm{Hom}^k(V^{\otimes l},M).$$

Denoting by $q=bN\in\mathrm{Hom}^k(V^{\odot l},M)$, for every integer $n\ge l$ define the maps

$$B\in\mathrm{Hom}^k(V^{\otimes n},M^{\otimes n-l+1}),\qquad Q\in\mathrm{Hom}^k(V^{\odot n},M^{\odot n-l+1}),$$

by the formulas:

$$B(v_1\otimes\cdots\otimes v_n)=$$
$$=\sum_{i=0}^{n-l}(-1)^{k(\overline{v_1}+\cdots+\overline{v_i})}f(v_1)\otimes\cdots\otimes f(v_i)\otimes b(v_{i+1}\otimes\cdots\otimes v_{i+l})\otimes f(v_{i+l+1})\otimes\cdots\otimes f(v_n).$$

$$Q(v_1\odot\cdots\odot v_n)=\sum_{\sigma\in S(l,n-l)}\epsilon(\sigma)q(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(l)})\odot f(v_{\sigma(l+1)})\odot\cdots\odot f(v_{\sigma(n)}).$$

Lemma 1.4.9. *In the notation above we have*

$$BN=NQ\in \operatorname{Hom}^k(V^{\odot n},M^{\otimes n-l+1}).$$

Proof. Easy and left to the reaader. ☐

1.5. The reduced symmetric coalgebra

For every graded vector space V we will denote $\overline{S}(V)=\bigoplus_{n>0}\bigodot^n V$, while $\pi\colon \overline{T}(V)\to\overline{S}(V)$ is the projection to the quotient and $N\colon\overline{S}(V)\to\overline{T}(V)$ is the direct sum of the maps of Definition 1.4.4.

Lemma 1.5.1. *The map* $\mathfrak{l}\colon\overline{S(V)}\to\overline{S(V)}\otimes\overline{S(V)}$,

$$\mathfrak{l}(v_1\odot\cdots\odot v_n)=\sum_{a=1}^{n-1}\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(a)})\otimes(v_{\sigma(a+1)}\odot\cdots\odot v_{\sigma(n)})$$

is a cocommutative coproduct and the map

$$N\colon(\overline{S(V)},\mathfrak{l})\to(\overline{T(V)},\mathfrak{a})$$

is an injective morphism of coalgebras.

Proof. The cocommutativity of $\mathfrak{l}$ is clear from definition. Since N is injective, we only need to prove that $\mathfrak{a}N=(N\otimes N)\mathfrak{l}$. According to Lemma 1.4.8, for every a

$$\mathfrak{a}_{a,n-a}N(v_1\odot\cdots\odot v_n)=N\otimes N\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(a)})\otimes(v_{\sigma(a+1)}\otimes\cdots\otimes v_{\sigma(n)})$$

and then

$$\mathfrak{a}N(v_1\odot\cdots\odot v_n)=\sum_{a=1}^{n-1}\mathfrak{a}_{a,n-a}N(v_1\odot\cdots\odot v_n)=N\otimes N\mathfrak{l}(v_1\odot\cdots\odot v_n).$$

☐

Definition 1.5.2. The reduced symmetric coalgebra generated by V is the graded vector space $\overline{S}(V)$ with the coproduct $\mathfrak{l}$ defined in Lemma 1.5.1

$$\mathfrak{l}(v_1\odot\cdots\odot v_n)=\sum_{a=1}^{n-1}\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)(v_{\sigma(1)}\odot\cdots\odot v_{\sigma(a)})\otimes(v_{\sigma(a+1)}\odot\cdots\odot v_{\sigma(n)}).$$

It is often convenient to think the reduced symmetric coalgebra as a subset of the tensor coalgebra, via the identification provided by N. In particular $\overline{S}(V)$ is locally nilpotent and the projection $p\colon\overline{S}(V)\to V$ with kernel $\oplus_{n>1}V^{\odot n}$ is a cogenerator.

Moreover, since N is an injective morphism of coalgebras we have

$$\ker\mathfrak{l}^n=N^{-1}(\ker\mathfrak{a}^n)=N^{-1}(\oplus_{i=1}^n V^{\otimes i})=\oplus_{i=1}^n V^{\odot i}.$$

For every morphism of graded vector spaces $f\colon V\to W$ we have

$$N\circ S(f)=T(f)\circ N\colon S(V)\to T(W)$$

and then $S(f)\colon\overline{S(V)}\to\overline{S(W)}$ is a morphism of graded coalgebras.

Proposition 1.5.3. *Let* (C,Δ) *be a locally nilpotent graded cocommutative coalgebra, then:*

(1) *The map*

$$\sum_{n>0} \frac{\pi}{n!}\Delta^{n-1} \colon C \to \overline{S}(C)$$

is a morphism of graded coalgebras.

(2) *For every graded vector space V and every morphism of graded vector spaces $f \colon C \to V$ there exists a unique morphism of graded coalgebras $F \colon C \to \overline{S}(V)$ such that $pF = f$. Moreover*

$$F = \sum_{n=1}^{\infty} \frac{\pi}{n!}(\otimes^n f)\Delta^{n-1} \colon C \to \overline{S}(C) \to \overline{S}(V).$$

Proof. According to Lemma 1.4.6 we have

$$\sum_{n>0} \Delta^{n-1} = N\left(\sum_{n>0} \frac{\pi}{n!}\Delta^{n-1}\right)$$

and the the first item is an immediate consequence of the fact that N is an injective morphism of graded coalgebras. Similarly for every morphism of graded vector spaces $f \colon C \to V$ we have

$$\sum_{n>0} (\otimes^n f)\Delta^{n-1} = N\left(\sum_{n>0} \frac{\pi}{n!}(\otimes^n f)\Delta^{n-1}\right).$$

□

Proposition 1.5.4. *Let V be a graded vector space and C a locally nilpotent cocommutative coalgebra. Then for every coalgebra morphism $F \colon C \to \overline{S}(V)$ and every integer k, the composition with $N \colon \overline{S}(V) \to \overline{T}(V)$ gives an isomorphism*

$$\operatorname{Coder}^k(C, \overline{S}(V); F) \simeq \operatorname{Coder}^k(C, \overline{T}(V); NF).$$

Proof. We need to prove that if $B \colon C \to \overline{T}(V)$ is a coderivation with respect to the morphism NF, then $B = NP$ for some $P \colon C \to \overline{S}(V)$. According to Proposition 1.3.5 we have

$$B = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (f^{\otimes i} \otimes b \otimes f^{\otimes n-i})\Delta^n \colon C \to \overline{T}(V)$$

where $f = pNF = pF$ and $b \in \operatorname{Hom}^k(C, V)$. According to Lemmas 1.4.6 and 1.4.9 the image of B is contained in the image of N. □

Corollary 1.5.5. *Let V be a graded vector space. Then for every integer k, the composition with $p \colon \overline{S}(V) \to V$ gives an isomorphism of vector spaces*

$$\operatorname{Coder}^k(\overline{S}(V), \overline{S}(V)) \to \prod_{i=1}^{+\infty} \operatorname{Hom}^k(V^{\odot i}, V).$$

More explicitely, for every sequence $q_i \in \operatorname{Hom}^n(V^{\odot k}, V)$, $i > 0$, the map $Q \in \operatorname{Hom}^n_{\mathbb{K}}(\overline{S}(V), \overline{S}(V))$ defined as

$$Q(v_1 \odot \cdots \odot v_n) = \sum_{i=1}^{n} \sum_{\sigma \in S(i,n-i)} \epsilon(\sigma) q_i(v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \cdots \odot v_{\sigma(n)},$$

is the unique coderivation of $\overline{S}(V)$ such that $pQ = \sum_i q_i$.

Proof. We only need to prove that the map Q is a coderivation. By linearity it is not restrictive to assume that $q_i = 0$ for every $i \neq l$. Let $b \in \operatorname{Hom}^n(\bigotimes^l V, V)$ be any map such that $bN = q_l$ (e.g. $b = \pi q_l/n!$) and let $B \in \operatorname{Coder}^n(\overline{T}(V), \overline{T}(V))$ be the coderivation such that $pB = b$. According to Corollary 1.3.6

$$B(v_1 \otimes \cdots \otimes v_n) = \sum_i (-1)^{k(\overline{v_1} + \cdots + \overline{v_i})} v_1 \otimes \cdots \otimes v_i \otimes b(v_{i+1} \otimes \cdots \otimes v_{i+l}) \otimes \cdots \otimes v_n,$$

and then Lemma 1.4.9 gives $RN = NQ$. □

Remark 1.5.6. The above results show in particular that:

(1) if $F \colon \overline{S}(V) \to \overline{S}(W)$ is a morphism of graded coalgebras, then $F(V^{\odot n}) \subset \sum_{i \le n} W^{\odot i}$;

(2) if $Q \colon \overline{S}(V) \to \overline{S}(V)$ is a coderivation, then $Q(V^{\odot n}) \subset \sum_{i \le n} V^{\odot i}$.

Definition 1.5.7. Given a graded vector space V the **symmetric Gerstenhaber product**

$$\operatorname{Hom}^*_{\mathbb{K}}(\overline{S}(V), V) \times \operatorname{Hom}^*_{\mathbb{K}}(\overline{S}(V), V) \to \operatorname{Hom}^*_{\mathbb{K}}(\overline{S}(V), V), \qquad (f,g) \mapsto f \circ g,$$

is defined as $f \circ g = fG$, where $G \in \operatorname{Coder}^*(\overline{S}(V), \overline{S}(V))$ is the unique coderivation lifting g.

The **symmetric Gerstenhaber bracket** is defined as

$$[f,g] = f \circ g - (-1)^{\overline{f}\,\overline{g}} g \circ f, \qquad f, g \in \operatorname{Hom}^*_{\mathbb{K}}(\overline{S}(V), V).$$

Given $f \in \operatorname{Hom}^a_{\mathbb{K}}(V^{\odot n+1}, V)$ and $g \in \operatorname{Hom}^b_{\mathbb{K}}(V^{\odot m+1}, V)$ we have $f \circ g \in \operatorname{Hom}^{a+b}_{\mathbb{K}}(V^{\odot n+m+1}, V)$,

$$f \circ g(v_0 \odot \cdots \odot v_{n+m}) = \sum_{\sigma \in S(m+1,n)} \epsilon(\sigma) f(g(v_{\sigma(0)} \odot \cdots \odot v_{\sigma(m)}) \odot v_{\sigma(m+1)} \odot \cdots \odot v_{\sigma(m+n)}).$$

1.6. Exercises

Exercise 1.6.1. A **counity** of a graded coalgebra (C, Δ) is a morphism of graded vector spaces $\epsilon \colon C \to \mathbb{K}$ such that $(\epsilon \otimes \operatorname{Id}_C)\Delta = (\operatorname{Id}_C \otimes \epsilon)\Delta = \operatorname{Id}_C$. Prove that if a counity exists, then it is unique (Hint: $(\epsilon \otimes \epsilon')\Delta = ?$).

Exercise 1.6.2. Let (C, Δ) be a graded coalgebra. A graded subspace $I \subset C$ is called a **coideal** if $\Delta(I) \subset C \otimes I + I \otimes C$. Prove that a subspace is a coideal if and only if is the kernel of a morphism of coalgebras.

Exercise 1.6.3. Let (C, Δ) be a graded coalgebra. Prove that for every $a, b \geq 0$

$$\Delta^a(\ker \Delta^{a+b}) \subset \bigotimes^{a+1}(\ker \Delta^b).$$

Exercise 1.6.4. Let C be a graded coalgebra and $d \in \operatorname{Coder}^1(C, C)$ a codifferential of degree 1. Prove that the triple $(L, \delta, [,])$, where:

$$L = \oplus_{n \in \mathbb{Z}} \operatorname{Coder}^n(C, C), \quad [f,g] = fg - (-1)^{\overline{g}\,\overline{f}} gf, \quad \delta(f) = [d, f]$$

is a differential graded Lie algebra.

Exercise 1.6.5. Let $p \colon T(V) \to \overline{T(V)}$ be the projection with kernel $\mathbb{K} = \bigotimes^0 V$ and $\phi \colon T(V) \to T(V) \otimes T(V)$ the unique homomorphism of graded algebras such that $\phi(v) = v \otimes 1 + 1 \otimes v$ for every $v \in V$. Prove that $p\phi = \mathfrak{a}p$.

Exercise 1.6.6. Let A be an associative graded algebra over the field $\mathbb{K}$. For every local homomorphism of $\mathbb{K}$-algebras $\gamma \colon \mathbb{K}[[x]] \to \mathbb{K}[[x]]$, $\gamma(x) = \sum \gamma_n x^n$, let $F_\gamma \colon \overline{T}(A) \to \overline{T}(A)$ be the unique morphism of graded coalgebras lifting the map

$$f_\gamma \colon \overline{T}(A) \to A, \qquad f(a_1 \otimes \cdots \otimes a_n) = \gamma_n a_1 \cdots a_n.$$

Prove the validity of the composition formula $F_{\gamma\delta} = F_\delta F_\gamma$. (Hint: Example 1.1.7.)

Exercise 1.6.7. Prove that a graded coalgebra morphism $F \colon \overline{S}(U) \to \overline{S}(V)$ is surjective (resp.: injective, bijective) if and only if the composition $U \xrightarrow{\imath} \overline{S}(U) \xrightarrow{F} \overline{S}(V) \xrightarrow{p} V$ is surjective (resp.: injective, bijective). (Hint: F preserves the filtrations of kernels of iterated coproducts.)

Exercise 1.6.8. Assume V finite dimensional with basis $\partial_1, \ldots, \partial_m$ of degree 0. Prove that

$$\mathfrak{l}(\partial_1^{n_1} \cdots \partial_m^{n_m}) = \sum_{a_1, \ldots, a_m} \binom{n_1}{a_1} \cdots \binom{n_m}{a_m} \partial_1^{a_1} \cdots \partial_m^{a_m} \otimes \partial_1^{n_1 - a_1} \cdots \partial_m^{n_m - a_m}$$

and deduce that the dual algebra $\overline{S(V)}^\vee$ is isomorphic to the maximal ideal of the power series ring $\mathbb{K}[[x_1, \ldots, x_m]]$, with pairing

$$\langle \partial_1^{n_1} \cdots \partial_m^{n_m}, f(x) \rangle = \frac{\partial^{n_1 + \cdots + n_m} f}{\partial x_1^{n_1} \cdots \partial x_m^{n_m}}(0) = (\prod_i n_i!) \cdot (\text{coefficient of } x_1^{n_1} \cdots x_m^{n_m} \text{ in } f(x)).$$