Deformation theory, homological algebra, and mirror symmetry

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0.1 Introduction.

In this article the author would like to explain a relation of deformation theory to mirror symmetry. Deformation theory or theory of moduli is related to mirror symmetry in many ways. We discuss only one part of it. The part we want to explain here is related to rather abstract and formal point of the theory of moduli, which was much studied in 50's and 60's. They are related to the definition of scheme, stack and its complex analytic analogue, and also to various parts of homological and homotopical algebra. Recently those topics again call attention of several people working in areas closely related to mirror symmetry.

The author met them twice. He first met them when he was working [34] with K.Ono on the construction of Gromov-Witten invariant of general symplectic manifolds and the study of periodic orbit of periodic Hamiltonian system. There we found that a C^∞ analogue of the notion of scheme and stack is appropriate to attach transversality problem. The transversality problem we met was one on the moduli space of holomorphic maps from Riemann surface. Later the author learned that in the algebraic geometry side, the same problem is studied by using stacks [72, 8]. This point however is not our main concern in this article. Our main focus is the next point.

The author met the relation of homological algebra to the theory of moduli while he, together with Y.G.Oh, H.Ohta, K.Ono [33], was trying to find a good formulation of Floer homology of the Lagrangian submanifold. There we first met a trouble that Floer homology of Lagrangian submanifold is not defined in general. So studying the condition when it is defined becomes an interesting problem. For this purpose, we developed an obstruction theory for the Floer homology to be defined. We next found that the Floer homology, even when it is defined, is not independent of the various choices involved. [1]

An example of this phenomenon is as follows. Let us consider a Lagrangian submanifold L in a symplectic manifold M. A problem, which is related to the definition of Floer homology, is to count the number of holomorphic maps $\varphi : D^2 \to M$ such that $\varphi(\partial D^2) \subset L$.[2] Then the trouble is the number thus defined depends on the various choices involved. For example it is *not* independent of the deformation of (almost) complex structure of M. So unless clarifying in which sense the number of holomorphic disks is invariant of various choices, it does not make mathematical sense to count it. It is this essential point where we need deformation theory

[1] This problem is quite similar to the case of Donaldson's Gauge theory invariant of 4 manifolds with $b_2^+ = 1$ [17]. (Here $b_2^+ = 1$ is the number of positive eigenvalue of the intersection matrix on the second homology.)

[2] Definitions of several notions we need in symplectic geometry will be given at the beginning of §3.2.

and homological algebra. Namely we construct an algebraic structure using the number of disks and homotopy types of this algebraic structure is independent of the perturbation.

We thus developed a moduli theory of the deformations of Floer homology. Namely we defined a moduli space $\mathcal{M}(L)$ for each Lagrangian submanifold L and Floer homology is defined as a family of graded vector spaces parametrized by $\mathcal{M}(L_1) \times \mathcal{M}(L_2)$. The moduli space $\mathcal{M}(L)$ is related to the actual deformation of Lagrangian submanifolds but the relation is rather delicate. The algebraic machinery to construct such moduli space is one of A_∞ algebra and Maurer-Cartan equation.

The A_∞ algebra we use there is a version of one found by the author in [23][3]. Using this A_∞ structure, M.Kontsevich [64, 68] discovered a very interesting version of mirror symmetry conjecture which he called homological mirror symmetry conjecture. There it is conjectured that Lagrangian submanifold corresponds to a coherent sheaf on the mirror bundle.

After developing the theory of deformation of and obstruction to Floer homology of Lagrangian submanifold, we could make homological mirror symmetry conjecture more precise. For example, we now conjecture that the moduli space $\mathcal{M}(L)$ will become a moduli space of holomorphic vector bundles on the mirror. It is the purpose of this article to explain the formulation of homological mirror symmetry based on homological algebra and deformation theory.

During the conference "Geometry and Physics of Branes", the author leaned that recently there are several works by Physicists which seems to be closely related to the story we had been developed. For example, the obstruction phenomenon seems to be rediscovered. The fact that our Maurer-Cartan equation which control the deformation of Floer homology (see §3.4) is inhomogeneous and 0 is not its solution seems to be related to what is called "Tachyon condensation"[4]. The author does not quote references of the papers by Physicists on those points, since he expects that it is included in other parts of this book and since it is hard for the author to find correct choice of the papers to be quoted. It seems interesting to find a good dictionary between Physics side and Mathematics side of the works. The author hopes that this volume is helpful for this purpose.

As we already mentioned, the main purpose of this article is to describe a version of homological mirror symmetry precisely. For example, we want

[3] The original motivation of the author to introduce A_∞ structure on Floer homology was to use it to study gauge theory Floer homology of 3-manifolds with boundary. (The author was inspired by Segal and Donaldson to use category theory for this purpose.) The research toward this original direction is still on progress ([25, 26]) and the author believes that the relation of A_∞ structure of Floer homology of Lagrangian submanifolds to gauge theory of 3-manifolds with boundary, would be related to some kind of duality in future.

[4] The author does not yet understand precise relation of our story to ones developed by Physicists.

to state precisely the conjectured coincidence of the moduli spaces mentioned above. For this purpose, we need to review various basic aspects of moduli theory (especially its local version, the deformation theory). Hence the classical theory of deformation of the holomorphic structures of vector bundles on complex manifolds (together with proofs of various parts of it) is included in this article.

Chapter 1 of this article thus is devoted to theory of deformations. Deformation theory or local theory of moduli is a classical topic initiated by Riemann in the case of moduli space of complex structures of Riemann surface. Kodaira-Spencer [61, 63], generalized it to higher dimension and studied the deformation theory of complex structures of complex manifolds of higher dimension. It was further amplified by many people, for example, [71, 19, 105, 85]. There are many versions of the deformation theory, that is deformation theory of holomorphic vector bundles, deformation theory of complex submanifolds, deformation theory of holomorphic maps etc.. But, as far as the points mentioned in Chapter 1 concern, the difference among them are rather minor. So we mostly restrict ourselves to the case of vector bundles. In this article, we are taking analytic point of view and use (nonlinear) partial differential equation. There is algebraic theory of deformation (and of moduli). Some of basic references of it are [20, 45, 4, 93, 79]. The author tried to make Chapter 1 selfcontained. Especially he tried to explain several abstract notions which are popular among algebraic geometers but not so much popular among researchers in the other fields. For example, we explain a notion of analytic space (complex analytic analogue of scheme), the relation of category theory to the problem of moduli, especially notion of "Functor from Artin ring", which is basic to study formal moduli. (Here formal moduli means that we consider formal power series solutions of defining equation of moduli space.) The proof of several of the main results of the local theory of moduli (existence of Kuranishi family, its completeness, versality, etc.) are postponed to Chapter 2, where we give a proof of them based on homotopy theory of A_∞ algebra developed there. The contents of Chapter 1 is classical and no new points of view is introduced. We include them here since most of the reference the author found requires much background on algebraic geometry etc.

In Chapter 2, we explain systematically how the points of view of homological algebra of A_∞ or L_∞ algebra can be applied to the problem of moduli. (A_∞ and L_∞ algebra are generalization of differential graded algebra and of differential graded Lie algebra, respectively.) In §2.1, we give a definition of them and define A_∞ and L_∞ homomorphisms. Also we developed homotopy theory of them. Namely we define homotopy between two A_∞ or L_∞ homomorphisms and homotopy equivalence between two A_∞ or L_∞ algebras.

We next study a Maurer-Cartan equation from the point of view of "Functor from Artin ring" by [93] which we explained in §1.7.

We then sketch an important theorem which says that the gauge equivalence class of solutions of Maurer-Cartan equation is invariant of homotopy types of the A_∞ or L_∞ algebra. In the case of differential graded algebra and differential graded Lie algebra, this result is due to [37, 38].

We then construct a Kuranishi family of the solutions of Maurer-Cartan equation, as a quotient ring of appropriate formal power series ring. We use a technique sum over trees (calculation of tree amplitude by Feynman diagram) for this purpose. Several basic results postponed from Chapter 1 (together with its generalization to A_∞ or L_∞ algebra) follows. In §2.4, we briefly explain a translation of the story of Chapter 2 into the language of formal super geometry.

The theory developed in Chapter 2, seems to be studied by various people by now. Let us quote here a few related papers [54, 46, 92, 99, 78, 77, 14, 7, 69, 49, 50, 97, 56] which the authors found. (The author is very sorry for the authors of the other papers on the subject which he did not quote. The author does not have enough knowledge to quote all important papers.) (Y. Soibelman informed me that he and M. Kontsevitch is preparing a book which has overlap with this article.)

In Chapter 3, we study the application of the discussion in Chapters 1 and 2, to homological mirror symmetry. We need to introduce a kind of formal power series ring which we call the universal Novikov ring to study instanton (or quantum) effect (in symplectic geometry side of the story). Our A_∞ algebra is a module over universal Novikov ring, and we need a slight modification of the definition of A_∞ algebra which is explained in §3.1.

In §3.2 and §3.3, Floer homology is explained. In this Chapter 3 we do not assume the reader to be familiar with global symplectic geometry. So we include §3.2, which is an introduction of a part of global symplectic geometry related to §3.3. Especially we explain Floer's original construction [22] of Floer homology. In §3.4, we explain the main construction of [33] which associates A_∞ algebra to Lagrangian submanifolds. Our discussion in §3.2 and §3.3, are rather brief especially in the geometric and analytic points we need for the construction, since the main purpose of this article is to explain algebraic formalism rather than basic geometric-analytic construction which is essential to give examples of the algebraic formalism. Detail of the construction is in [33]. [28, 30, 84] are other surveys.

§3.5 is devoted to the definition of the moduli space $\mathcal{M}(L)$. To define it, we explain the modifications of the argument of Chapter 2 which are necessary to apply it to the case when the coefficient ring is not $\mathbb{C}$ but is universal Novikov ring. §3.6 is devoted to the explanation of the complex geometry side of the story. An important point to be explained is what the Novikov ring in the complex geomegry side corresponds. Roughly speaking Novikov ring will become the ring of functions on the disk which parametrize maximal degenerate family of mirror manifolds. We then discuss that a mirror of a Lagrangian submanifold is a family of vector bundles (or more generally of object of the derived category of coherent sheaves) over maximally degenerate family of Calabi-Yau manifolds. A version of homological mirror symmetry conjecture is then stated that two A_∞ algebra over Novikov rings, one for Floer homology the other for sheaf cohomology, coincide up to homotopy equivalence. There are some points which is not so clear for the author yet, which are related to various deep problems in algebraic geometry. We conclude Chapter 3, with giving an example.

The original plan of the author was to include several other deformation theories related to mirror symmetry in this article. For example extended deformation of Calabi-Yau manifold due to [7], deformation quantization due to [69], and contact homology announced in [21] [5]. They all can be treated by using the formalism of Chapter 2. However this article becomes already too thick and the author would like to postpone them to other occasion[6].

Parts of this article are announcement of our joint paper [33] with Oh,Ohta,Ono. (A preliminary version of [33] is completed in December 2000 and is available from author's home page at the time of writing this article. We are adding several new materials to it, some of which is included in this article. The final version of [33] is now being completed.)

The author would like to thank to the organizers of the conference "Geometry and Physics of Branes" to give him an opportunity to communicate with various researchers in a comfortable atmosphere and to write this article.

[5] Probably there is another example related to the Period of Primitive form due to K.Saito [91]. The author is unable to explain it at the time of writing this article. Some explanation from the point of view of mirror symmetry is found in [74].
[6] The reader who speaks Japanese can find them in [29]

Chapter 1

Classical Deformation theory

1.1 Holomorphic structure on vector bundles.

We start with describing deformation theory (that is a local theory of moduli) of holomorphic structures of complex vector bundles on complex manifolds. It is a classical theory, and is a direct analogue of Kodaira-Spencer [61, 63], who studied the case of deformation of complex structures of complex manifold itself. We present it here since it consists a prototype of the discussion which will appear later in less classical situation.

Let M be a complex manifold and $\Lambda^k_{\mathbb{C}} M = \oplus_{p+q=k} \Lambda^{p,q} M$ be the decomposition of the set of the complex valued k forms according to their types. We denote by $\Omega^{p,q}(M)$ the set of all smooth sections of $\Lambda^{p,q} M$. The complex structure of M is characterized by the Dolbault differential $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. (A standard text book of complex manifold written from the transcendtal point of view is [40].) We remark that $(\Omega(M), \wedge, \overline{\partial})$ is a differential graded algebra, which we define below. Hereafter we denote by R a commutative ring with unit.

Definition 1.1.1. A *differential graded algebra* or *DGA* over R is a triple $(A^*, \cdot, d)$ with the following properties. (1) For each $k \in \mathbb{Z}_{\geq 0}$, A^k is an R module. We write $\deg a = k$ if $a \in A^k$. (2) $\cdot : A^k \otimes A^\ell \to A^{k+\ell}$ is an R module homomorphism, which is associative. Namely $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. (3) $d : A^k \to A^{k+1}$ is an R module homomorphism such that $d \circ d = 0$. (4) $d(a \cdot b) = d(a) \cdot b + (-1)^{\deg a} a \cdot d(b)$.

We may take either $(\oplus \Omega^{0,k}(M), \wedge, \overline{\partial})$ where k is the degree, or $(\sum_{p,q} \Omega^{p,q}(M), \wedge, \overline{\partial})$ where the degree is the (total) degree of differential form. Let $\pi_E : E \to M$ be a complex vector bundle. (A standard text book of differential geometry of holomorphic vector bundle is [58].) We put $\Omega^{p,q}(M; E) = \Gamma(M, \Lambda^{p,q}(M; \Lambda^{p,q} M \otimes E))$. (We omit M in case no confusion occur.) We define a wedge product : $\wedge : \Omega^{p,q}(M) \otimes \Omega^{p',q'}(M; E) \to$

$\Omega^{p+p',q+q'}(M;E)$ in an obvious way. We define holomorphic structure on our complex vector bundle as follows.

Definition 1.1.2. A *holomorphic structure* on E is a sequence of operators $\overline{\partial}_{\mathcal{E}} : \Omega^{0,q}(M;E) \to \Omega^{p,q+1}(M;E)$ $(p,q \in \{1,\cdots,n\})$, such that (1) $\overline{\partial}_{\mathcal{E}} \circ \overline{\partial}_{\mathcal{E}} = 0$. (2) $\overline{\partial}_{\mathcal{E}}(u \wedge \alpha) = \overline{\partial}(u) \wedge \alpha + (-1)^{p+q} u \wedge \overline{\partial}_{\mathcal{E}}(\alpha)$, for $u \in \Omega^{p,q}(M)$, $\alpha \in \Omega^*(M;E)$.

In other words, the holomorphic structure on E is a structure on $\Omega^*(M;E)$ of left graded differential graded module over $(\Omega(M), \wedge, \overline{\partial})$, which we define below.

Definition 1.1.3. A *differential graded module* on a differential graded algebra $(A^*, \cdot, d)$ is, by definition, a triple $(C^*, \cdot, d)$ such that : (1) For each $k \in \mathbb{Z}$, M^k is an R module. We write $\deg a = k$ if $a \in M^k$. (2) $\cdot : A^k \otimes M^\ell \to M^{k+\ell}$ is an R module homomorphism, which is associative. Namely $(a \cdot b) \cdot x = a \cdot (b \cdot x)$, for $a, b \in A$, $x \in M$. (3) $d : M^k \to M^{k+1}$ is an R module homomorphism such that $d \circ d = 0$. (4) $d(a \cdot x) = d(a) \cdot x + (-1)^{\deg a} a \cdot d(x)$, for $a \in A$, $x \in M$.

We remark that Definition 1.1.2 coincides with another definition of holomorphic vector bundle, the one which uses local chart (see [58]).

From now on we *choose* one holomorphic structure $\overline{\partial}_{\mathcal{E}}$ on E and write $\mathcal{E} = (E, \overline{\partial}_{\mathcal{E}})$. Once we fix $\overline{\partial}_{\mathcal{E}}$, other holomorphic structures can be identified with elements of an affine space satisfying a differential equations, as we describe below. We consider the vector bundle $End(E)$ whose fiber at p is $\mathrm{Hom}(E_p, E_p)$. (We omit M in case no confusion can occur.) Let $\Omega^{p,q}(M; End(E))$ be the set of all smooth sections of $\Lambda^{p,q} \otimes End(E)$. We define operators

$$\circ : \Omega^{p,q}(M; End(E)) \otimes \Omega^{p',q'}(M;E) \to \Omega^{p+p',q+q'}(M;E)$$
$$\circ : \Omega^{p,q}(M; End(E)) \otimes \Omega^{p',q'}(M; End(E)) \to \Omega^{p+p',q+q'}(M; End(E))$$

by using $End(E) \otimes E \to E$, $End(E) \otimes End(E) \to End(E)$ and the wedge product in an obvious way. A holomorphic structure $\overline{\partial}_{\mathcal{E}}$ on E induces a holomorphic structure, (still denoted by $\overline{\partial}_{\mathcal{E}}$), on $End(E)$ by

$$\overline{\partial}_{\mathcal{E}}(B) = \overline{\partial}_{\mathcal{E}} \circ B - (-1)^{\deg B} B \circ \overline{\partial}_{\mathcal{E}}. \tag{1.1}$$

Theorem 1.1.1. *Let $\overline{\partial}_{\mathcal{E}'}$ be another holomorphic structure on $\pi_E : E \to M$. Then, there exists a section $B \in \Omega^{0,1}(M; End(E))$, such that*

$$\overline{\partial}_{\mathcal{E}'}(\alpha) = \overline{\partial}_{\mathcal{E}}(\alpha) + B \circ \alpha \tag{1.2}$$

B satisfies the differential equation

$$\overline{\partial}_{\mathcal{E}} B + B \circ B = 0. \tag{1.3}$$

On the other hand, let $B \in \Omega^{0,1}(M; End(E))$ be a section satisfying (1.3). We define $\overline{\partial}_{\mathcal{E}'}$ by (1.2). Then $\overline{\partial}_{\mathcal{E}'}$ defines a holomorphic structure on $\pi_E : E \to M$.

Proof. By definition, we find $(\overline{\partial}_{\mathcal{E}'} - \overline{\partial}_{\mathcal{E}})(u \wedge \alpha) = (-1)^{\deg u} u \wedge (\overline{\partial}_{\mathcal{E}'} - \overline{\partial}_{\mathcal{E}})(\alpha)$. It implies that $\overline{\partial}_{\mathcal{E}'} - \overline{\partial}_{\mathcal{E}}$ is induced by a section of $\Omega^{0,1}(M; End(E))$, which iwe denote by B. To show (1.3) we calculate

$$\begin{aligned}
\overline{\partial}_{\mathcal{E}'}(\overline{\partial}_{\mathcal{E}'}(\alpha)) &= \overline{\partial}_{\mathcal{E}'}(\overline{\partial}_{\mathcal{E}}(\alpha) + B \circ \alpha) \\
&= \overline{\partial}_{\mathcal{E}}(\overline{\partial}_{\mathcal{E}}(\alpha) + B \circ \alpha) + B \circ (\overline{\partial}_{\mathcal{E}}(\alpha) + B \circ \alpha) \\
&= \overline{\partial}_{\mathcal{E}} B \circ \alpha - B \circ \overline{\partial}_{\mathcal{E}}(\alpha) + B \circ \overline{\partial}_{\mathcal{E}}(\alpha) + B \circ B \circ \alpha \\
&= (\overline{\partial}_{\mathcal{E}} B + B \circ B) \circ \alpha.
\end{aligned} \tag{1.4}$$

Since (1.4) holds for any α, we have (1.3). The converse can be proved in the same way. □

Equation (1.3) is an example of *Maurer-Cartan equation*, whose study is one of the main theme of this article.

1.2 Family of holomorphic structures on vector bundle.

We study holomorphic vector bundles which is sufficiently close to $\mathcal{E}$. In other words, we are going to discuss a local theory of moduli.

We first define a family of complex structures. Let $\mathcal{U} \subset \mathbb{C}^n$ be an open set. Let $\pi_{\hat{M}} : \hat{M} \to \mathcal{U}$ be a fiber bundle whose fibers are diffeomorphic to M.

Definition 1.2.1. A smooth (complex analytic) family of complex structures on M parametrized by $\mathcal{U}$ is a complex structure $J_{\hat{M}}$ on $\hat{M}$ such that $\pi_{\hat{M}} : (\hat{M}, J_{\hat{M}}) \to \mathcal{U}$ is holomorphic.

We next define a family of holomorphic vector bundles. Let $\hat{E} \to \hat{M}$ be a complex vector bundle. We assume that the restriction of $\hat{E}$ to $\pi_{\hat{M}}^{-1}(x) \cong M$ is isomorphic to E (as complex vector bundles).

Definition 1.2.2. A smooth (complex analytic) *family of the holomorphic structures* on E over $(\hat{M}, J_{\hat{M}})$ is a holomorphic structure $\overline{\partial}_{\hat{\mathcal{E}}}$ of the bundle $\hat{E}$.

One important case is when $(\hat{M}, J_{\hat{M}})$ is trivial, that is the case when $(\hat{M}, J_{\hat{M}})$ is isomorphic to the direct product $(M, J_M) \times \mathcal{U}$. (However the case when the family $(\hat{M}, J_{\hat{M}})$ is nontrivial also appears later in our story. §3.5.) In that case, we can use Theorem 1.1.1 to identify a family of holomorphic structures on E to a map $\mathcal{U} \to \Omega^{0,1}(M; End(E))$ as follows. Let

$\overline{\partial}_{\hat{\mathcal{E}}}$ be a family of holomorphic structures on $\hat{E} = E \times \mathcal{U} \to \hat{M} = M \times \mathcal{U}$. Each $x \in \mathcal{U}$ determines a holomorphic structure $\overline{\partial}_{\mathcal{E}_x}$ on E. Namely $\overline{\partial}_{\mathcal{E}_x}$ is the restriction of $\overline{\partial}_{\hat{\mathcal{E}}}$ to $M \times \{x\}$. We put $B_x = \overline{\partial}_{\mathcal{E}_x} - \overline{\partial}_{\mathcal{E}}$. Theorem 1.1.1 implies

$$\overline{\partial}_{\mathcal{E}} B_x + B_x \circ B_x = 0. \tag{1.5}$$

Using the fact that $\overline{\partial}_{\hat{\mathcal{E}}}$ is a holomorphic structure on $\hat{E}$ we can prove that the map

$$B : \mathcal{U} \to \Omega^{0,1}(M; End(E)), \quad x \mapsto B_x \tag{1.6}$$

is holomorphic. ($\Omega^{0,1}(M; End(E))$ is a complex vector space (of infinite dimension). So it makes sense to say that the map B is holomorphic.) On the contrary, given a holomorphic map (1.6) satisfying (1.5), we can define $\overline{\partial}_{\hat{\mathcal{E}}}$ by $\overline{\partial}_{\hat{\mathcal{E}}} = \overline{\partial}_{\mathcal{E}\times\mathcal{U}} + B$. Here $\overline{\partial}_{\mathcal{E}\times\mathcal{U}}$ is a holomorphic structure on $E \times \mathcal{U}$ (the direct product) and B is regarded as a smooth section of $\Omega^{0,1}(\hat{M}; End(\hat{E}))$.

Our next purpose is to define and study notions of completness, versality and universality of families. We define them only in the case when complex structure on M is fixed. The case of family of complex structures on a manifold M and the case of holomorphic structures of vector bundles over $\hat{M}$ (with moving complex structures) are similar and are omitted.

We first need to define morphism of two families, for this purpose. We first define equivalence of holomorphic vector bundles. Let $\pi_{E_1} : E_1 \to M_1$ and $\pi_{E_2} : E_2 \to M_2$ be complex vector bundles. We consider a bundle homomorphism $\varphi : E_1 \to E_2$ over a holomorphic map $\overline{\varphi} : M_1 \to M_2$. Namely $\pi_{E_2} \circ \varphi = \overline{\varphi} \circ \pi_{E_1}$ and φ is complex linear on each fiber. A bundle homomorphism φ induces $\varphi_* : \Omega^{p,q}(M; E_1) \to \Omega^{p,q}(M; E_2)$.

Definition 1.2.3. We say $\varphi : (E_1, \overline{\partial}_{\mathcal{E}_1}) \to (E_2, \overline{\partial}_{\mathcal{E}_2})$ is *holomorphic* if $\overline{\partial}_{\mathcal{E}_2} \circ \varphi_* = \varphi_* \circ \overline{\partial}_{\mathcal{E}_1}$. We say that $\varphi : (E_1, \overline{\partial}_{\mathcal{E}_1}) \to (E_2, \overline{\partial}_{\mathcal{E}_2})$ is an *isomorphism* if it is holomorphic and is a bundle isomorphism.

Let $\hat{E}_i = E \times \mathcal{U}_i$ and $\overline{\partial}_{\mathcal{E}_i}$ be a holomorphic structure on it, that is a deformation of holomorphic structures on E_i. We put $\hat{M}_i = M \times \mathcal{U}_i$.

Definition 1.2.4. A *morphism* from $(\hat{M}_1, \overline{\partial}_{\hat{\mathcal{E}}_1})$ to $(\hat{M}_2, \overline{\partial}_{\hat{\mathcal{E}}_2})$ is a pair (Φ, ϕ), where $\phi : \mathcal{U}_1 \to \mathcal{U}_2$ is a holomorphic map and Φ is a holomorphic bundle map $\Phi : (\hat{E}_1, \overline{\partial}_{\hat{\mathcal{E}}_1}) \to (\hat{E}_2, \overline{\partial}_{\hat{\mathcal{E}}_2})$ over $\mathrm{id} \times \phi : M \times \mathcal{U}_1 \to M \times \mathcal{U}_2$.

Let us define the notion of deformations of complex structure and of holomorphic vector bundle. It is a germ of a family and is defined as follows.

Definition 1.2.5. A *deformation* of a complex manifold (M, J) is a $\sim$ isomorphism class of a pair $(((\hat{M}, J), \mathcal{U}), i)$ where $(\hat{M}, J) = M \times \mathcal{U} \to \mathcal{U}$ is a family of complex structures and i is a (biholomorphic) isomorphism $\pi^{-1}(0) \cong (M, J)$.

We say $((M \times \mathcal{U}, J), i) \sim ((M \times \mathcal{U}', J'), i')$, if there exists an open neighborhood $\mathcal{V}$ of 0 such that $\mathcal{V} \subset \mathcal{U} \cap \mathcal{U}'$ and if there exists a biholomorphic map $\varphi : (M \times \mathcal{V}, J) \to (M \times \mathcal{V}, J')$ which commutes with projection : $M \times \mathcal{V} \to \mathcal{V}$ and which satisfies $i \circ \varphi = i$.

Let $\mathcal{E}$ be a holomorphic vector bundle on a complex manifold M. A deformation of $\mathcal{E}$ is an equivalence class of maps $B : \mathcal{U} \to \Omega^{0,1}(M, End(E))$ such that $B(0) = 0$ and $\overline{\partial}_{\mathcal{E}} B(x) + B(x) \circ B(x) = 0$. We say that $B : \mathcal{U} \to \Omega^{0,1}(M, End(E))$ is equivalence to $B' : \mathcal{U}' \to \Omega^{0,1}(M, End(E))$ if there exists an open neighborhood $\mathcal{V}$ of 0 with $\mathcal{V} \subset \mathcal{U} \cap \mathcal{U}'$ and a holomorphic map $\Phi : \mathcal{V} \to \Gamma(M, End(E))$ such that $\Phi(0) = \mathrm{id}$ and $\Phi(x) \circ (\overline{\partial}_{\mathcal{E}} + B(x)) = (\overline{\partial}_{\mathcal{E}} + B'(x)) \circ \Phi(x)$.

We can define a deformation of a pair of complex structure and vector bundle on it in a similar way. Hereafter we sometimes say B is a deformation of $\mathcal{E}$ or $(B,\mathcal{U})$ is a deformation of $\mathcal{E}$ by abuse of notation.

Now we define completeness of a deformation. Roughly speaking, it means that all nearby holomorphic structures are contained in the family.

Definition 1.2.6. A deformation $(B,\mathcal{U})$ of $\mathcal{E}$ is said to be *complete* if the following condition holds.

Let $(B',\mathcal{U}')$ be another deformation of $\mathcal{E}$ and $\Phi_0 : E \to E$ is an automorphism of $\mathcal{E}$. Then, there exists a neighborhood $\mathcal{V}$ of 0 in $\mathcal{U}'$ and a morphism

$$(\Phi, \phi) : (E \times \mathcal{V}, \overline{\partial}_{\mathcal{E}+B'}) \to (E \times \mathcal{U}, \overline{\partial}_{\mathcal{E}+B}) \tag{1.7}$$

of families in the sense of Definition 1.2.4 such that

$$\Phi|_{M \times \{0\}} = \Phi_0. \tag{1.8}$$

The other important notion is universality and versality of a deformation, which we define below. Let $(B,\mathcal{U})$ of $\mathcal{E}$ be a complete deformation.

Definition 1.2.7. We say that $(B,\mathcal{U})$ is *universal* if for each $(B',\mathcal{U}')$ as in Definition 1.2.6 the morphism (Φ, ϕ) as in (1.7) satisfying (1.8) is unique.

We say that $(B,\mathcal{U})$ is *versal* if the differential of ϕ at 0 is unique. Namely if (Φ', ϕ') as in (1.7) is another morphism satisfying (1.8) then $d_0\phi' = d_0\phi$. (Note they both are linear maps : $T_0\mathcal{V} \to T_0\mathcal{U}$).

The difference between versality and universality is related to the stability of bundles[1]. We give an example of versal family which is not universal in §1.3.

[1] See [79] for the definition of stability.

1.3 Cohomology and Deformation.

The Maurer-Cartan equation (1.3) is a nonlinear partial differential equation. In this section, we study its linearization. The solution of linearized equation is related to the cohomology group. Let $\mathcal{E}_i = (E_i, \overline{\partial}_{\mathcal{E}_i})$ be holomorphic vector bundles on M. We consider $\Omega^{0,q}(Hom(E_1, E_2)) = \Gamma(M; \Lambda^{0,q} \otimes Hom(E_1, E_2))$, where $Hom(E_1, E_2)$ is a bundle whose fiber at p is $\mathrm{Hom}((E_1)_p, (E_2)_p)$ Operations $\overline{\partial}_{\mathcal{E}_1}$, $\overline{\partial}_{\mathcal{E}_2}$ defines an operation $\overline{\partial}_{\mathcal{E}_1,\mathcal{E}_2}$: $\Omega^{0,q}(Hom(E_1, E_2)) \to \Omega^{0,q+1}(Hom(E_1, E_2))$ in the same way as (1.1). It is easy to see $\overline{\partial}_{\mathcal{E}_1,\mathcal{E}_2} \circ \overline{\partial}_{\mathcal{E}_1,\mathcal{E}_2} = 0$. Namley $(Hom(E_1, E_2), \overline{\partial}_{\mathcal{E}_1,\mathcal{E}_2})$ is a holomorphic vector bundle.

Definition 1.3.1. The extension $\mathrm{Ext}^q(\mathcal{E}, \mathcal{E}')$ is the qth cohomology of the chain complex $(\Omega^{0,*}(Hom(E_1, E_2)), \overline{\partial}_{\mathcal{E}_1,\mathcal{E}_2})$.

Let $(B, \mathcal{U})$ be a deformation of $\mathcal{E}$. We are going to define a Kodaira-Spencer map $T_0\mathcal{U} \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$. By Definition 1.2.5, we have

$$\overline{\partial}_{\mathcal{E}} B(x) + B(x) \circ B(x) = 0. \tag{1.9}$$

We differentiate (1.9) at 0. Then, in view of $B(0) = 0$, we have

$$\overline{\partial}_{\mathcal{E}} \left(\frac{\partial B}{\partial x^i}(0) \right) = 0.$$

Here $x = (x^1, \cdots, x^n)$ is a complex coordinate of $\mathcal{U} \subseteq \mathbb{C}^n$.

Definition 1.3.2. We put

$$\mathrm{KS}\left(\frac{\partial}{\partial x^i} \right) = \left[\frac{\partial B}{\partial x^i}(0) \right] \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}).$$

KS is a linear map : $T_0\mathcal{U} \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$, which we call the *Kodaira-Spencer map* of our deformation.

Kodaira-Spencer map is gauge equivariant. Namely it is independent of the choice of the representative $(B, \mathcal{U})$ of the deformation. In other words, we have the following lemma . Let $(B', \mathcal{U}')$ be another representative of the deformation.

Lemma 1.3.1. *If $(B, \mathcal{U})$ is equivalent to $(B', \mathcal{U}')$ in the sense of Definition* 1.2.5 *then*

$$\left[\frac{\partial B}{\partial x^i}(0) \right] = \left[\frac{\partial B'}{\partial x^i}(0) \right] \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}).$$

Proof. $\mathcal{V} \subset \mathcal{U} \cap \mathcal{U}'$, $\Phi : \mathcal{V} \to \Gamma(M; End(E))$ in Definition 1.2.5. satisfies :

$$\Phi(x) \circ (\overline{\partial}_{\mathcal{E}} + B(x)) = (\overline{\partial}_{\mathcal{E}} + B'(x)) \circ \Phi(x) \tag{1.10}$$

We differentiate (1.10) at 0 and obtain

$$\frac{\partial \Phi}{\partial x}(0) \circ \overline{\partial}_{\mathcal{E}} + \frac{\partial B'}{\partial x^i}(0) = \frac{\partial B}{\partial x^i}(0) + \overline{\partial}_{\mathcal{E}} \circ \frac{\partial \Phi}{\partial x^i}(0).$$

Namely $\frac{\partial B}{\partial x^i}(0) - \frac{\partial B'}{\partial x^i}(0) = \overline{\partial}_{\mathcal{E}}\left(\frac{\partial \Phi}{\partial x^i}(0)\right)$. □

In a similar way, we can prove the following lemma.

Lemma 1.3.2. *Let* $(B_1,\mathcal{U}_1)$, $(B_2,\mathcal{U}_2)$ *be deformations of* $\mathcal{E}_1,\mathcal{E}_2$ *and* $\Phi : (E_1 \times \mathcal{U}_1, \overline{\partial}_{\mathcal{E}_1+B_1}) \to (E_2 \times \mathcal{U}_2, \overline{\partial}_{\mathcal{E}_2+B_2})$, $\phi : \mathcal{U}_1 \to \mathcal{U}_2$ *be a morphism of family of holomorphic structures in the sense of Definition* 1.2.4. *We assume* $\phi(0) = 0$. *Then the following diagram commutes.*

$$\begin{array}{ccc} T_0\mathcal{U}_1 & \xrightarrow{\text{KS}} & \text{Ext}^1(\mathcal{E},\mathcal{E}) \\ {\scriptstyle d_0\phi}\downarrow & & \| \\ T_0\mathcal{U}_2 & \xrightarrow{\text{KS}} & \text{Ext}^1(\mathcal{E},\mathcal{E}) \end{array}$$

Diagram 1

The following result was proved by [62] in the case of deformation theory of complex structure.

Theorem 1.3.1. *If Kodaira-Spencer map is surjective then the deformation is complete.*

We will prove it in §2.3. Another main result of deformation theory is the following theorem which is due to Kodaira-Nirenberg-Spencer [60] in the case of deformation theory of complex structures.

Theorem 1.3.2. *If* $\text{Ext}^2(\mathcal{E},\mathcal{E}) = 0$ *then there exists a deformation of* $\mathcal{E}$ *such that Kodaira-Spencer map is an isomorphism.*

We will prove it in §1.5. We also prove in §2.3 that the family obtained in Theorem 1.3.2 is unique up to isomorphism.

Remark 1.3.1. The smooth family where Kodaira-Spencer map is surjective does not exist in general in the case when $\text{Ext}^2(\mathcal{E},\mathcal{E}) \neq 0$. (However there are cases where such family exists in the case $\text{Ext}^2(\mathcal{E},\mathcal{E}) \neq 0$. See Example 1.3.1 below.) Kuranishi [71] studied the case $\text{Ext}^2(\mathcal{E},\mathcal{E}) \neq 0$. It leads us to the notion Kuranishi map : $\text{Ext}^1(\mathcal{E},\mathcal{E}) \to \text{Ext}^2(\mathcal{E},\mathcal{E})$. We will discuss it in §1.6.

We remark that :

Proposition 1.3.1. *The deformation obtained in Theorem* 1.3.2 *is versal.*

Proof. Completeness follows from Theorem 1.3.1. Let (Φ, ϕ) (Φ', ϕ') be morphisms as in (1.7) satisfying (1.8). By Lemma 1.3.2 we have the following commutative diagram.

$$\begin{array}{ccccc} T_0\mathcal{V} & \xrightarrow{\text{KS}} & \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) & \xleftarrow{\text{KS}} & T_0\mathcal{V} \\ {\scriptstyle d_0\phi}\big\downarrow & & \| & & \big\downarrow{\scriptstyle d_0\phi'} \\ T_0\mathcal{U} & \xrightarrow{\text{KS}} & \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) & \xleftarrow{\text{KS}} & T_0\mathcal{U} \end{array}$$

Diagram 2

$d_0\psi' = d_0\psi$ follows immediately. □

We now give a few examples of deformations of holomorphic vector bundle.

Example 1.3.1. Let M be a complex manifold and let $L \to M$ be a complex *line* bundle. L has a holomorphic structure if its first Chern class is represented by a 1-1 form. (See [40, 58].) We fix a holomorphic structure of L and denote it by $\overline{\partial}_{\mathcal{L}}$. Other holomorphic structure is equal to $\overline{\partial}_B = \overline{\partial}_{\mathcal{L}} + B$ here $B \in \Omega^{0,1}(M; End(\mathcal{L}))$. Let us study the equation (1.3) in this case.

Using the fact that L is a line bundle, it is easy to see that $End(\mathcal{L})$ is isomorphic to the trivial line bundle (as a holomorphic line bundle). Namely $B \in \Omega^{0,1}(M)$. Since B is 1 form we have $B \circ B = 0$. Hence (1.3) reduces to a *linear* equation :

$$\overline{\partial} B_{\mathcal{L}} = 0.$$

We can find a vector subspace

$$\mathcal{U} \subset \mathrm{Ker}\ (\overline{\partial} : \Omega^{0,1}(M; End(\mathcal{L})) \to \Omega^{0,2}(M; End(\mathcal{L})))$$

such that the restriction $\mathcal{U} \to \mathrm{Ext}^1(\mathcal{L},\mathcal{L})$ of the natural projection is an isomorphism. Hence, in the case of line bundle, we have always a family whose Kodaira-Spencer map is an isomorphism. Note that in case $\dim M \geq 2$, the condition $\mathrm{Ext}^2(\mathcal{L},\mathcal{L}) \cong 0$ of Theorem 1.3.2 may not be satisfied.

It is easy to see that our deformation is universal.

Example 1.3.2. Let $\mathcal{L}_1$, $\mathcal{L}_2$ be line bundles on M such that

$$\begin{aligned} &\mathrm{Ext}^1(\mathcal{L}_2,\mathcal{L}_1) \neq 0. \\ &\mathrm{Ext}^0(\mathcal{L}_2,\mathcal{L}_1) \cong \mathrm{Ext}^1(\mathcal{L}_1,\mathcal{L}_1) \cong \mathrm{Ext}^1(\mathcal{L}_2,\mathcal{L}_2) \cong \mathrm{Ext}^1(\mathcal{L}_1,\mathcal{L}_2) \cong 0. \end{aligned} \tag{1.11}$$

(Let $M = \mathbb{C}P^1$ and $c^1(L_1) = k_2[\mathbb{C}P^1], c^1(L_1) = k_2[\mathbb{C}P^1]$, with $k_1 > k_2$. It is easy to check (1.11) in this case.) Let $B(x) \in \Omega^{0,1}(M; Hom(L_2, L_1))$ be a form representing non zero cohomology class x in $\mathrm{Ext}^1(\mathcal{L}_2,\mathcal{L}_1)$. We consider

$$\mathfrak{B}(x) = \begin{pmatrix} 0 & B(x) \\ 0 & 0 \end{pmatrix} \in \Omega^{0,1}(M; End(L_1 \oplus L_2)).$$

Then

$$\overline{\partial}_{\mathfrak{B}(x)} = \overline{\partial}_{\mathcal{L}_1\oplus\mathcal{L}_2} + \mathfrak{B}(x) = \begin{pmatrix} \overline{\partial}_{\mathcal{L}_1} & B(x) \\ 0 & \overline{\partial}_{\mathcal{L}_2} \end{pmatrix}$$

satisfies $\overline{\partial}_{\mathfrak{B}(x)} \circ \overline{\partial}_{\mathfrak{B}(x)} = 0$ and hence have a deformation of $\mathcal{L}_1 \oplus \mathcal{L}_2$. By (1.11) we can find easily $\mathrm{Ext}^1(\mathcal{L}_1 \oplus \mathcal{L}_2, \mathcal{L}_1 \oplus \mathcal{L}_2) \cong \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$. The Kodaira-Spencer map of our deformation is the identity : $\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1) \to \mathrm{Ext}^1(\mathcal{L}_1\oplus\mathcal{L}_2, \mathcal{L}_1\oplus\mathcal{L}_2) \cong \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$. We thus have a versal deformation.

However this deformation is *not* universal.

In fact, let $r : \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1) \to \mathbb{C}\backslash\{0\}$ be a holomorphic function with $r(0) = 1$. We define $\phi : \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1) \to \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$ by $\phi(x) = r(x)x$. We also define $\Psi : \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1) \to \mathrm{End}(L_1 \oplus L_2)$, by $\Psi(x)(v, w) = (r(x)v, w)$. Ψ defines a bundle homomorphism over ϕ. By definition it is easy to see that $\Psi(x) : ((L_1 \oplus L_2), \overline{\partial}_{\mathcal{L}+\mathfrak{B}(x)}) \to ((L_1 \oplus L_2), \overline{\partial}_{\mathcal{L}+\mathfrak{B}(x)})$ is holomorphic. We thus find a morphism of family of holomorphic structures, which is identity on the fiber of 0 but is not identity at the fiber of other points. Hence our deformation is not universal.

We can explain this phenomenon as follows. We consider the group of (holomorphic) automorphisms of the bundles $(L_1 \oplus L_2; \overline{\partial}_{\mathfrak{B}(x)})$. In case $x = 0$ this bundle is a direct product. Hence we have $\mathrm{Aut}(L_1 \oplus L_2; \overline{\partial}_{\mathfrak{B}(0)}) \cong \mathrm{Aut}(\mathcal{L}_1) \times \mathrm{Aut}(\mathcal{L}_2) \times \mathrm{Ext}^0(\mathcal{L}_1, \mathcal{L}_2) \cong \mathbb{C}_*^2 \times \mathrm{Ext}^0(\mathcal{L}_1, \mathcal{L}_2)$. On the other hand, for $x \neq 0$ we have $\mathrm{Aut}(L_1 \oplus L_2; \overline{\partial}_{\mathfrak{B}(x)}) \cong \mathbb{C}_* \times \mathrm{Ext}^0(\mathcal{L}_1, \mathcal{L}_2)$ where $c \in \mathbb{C}_*$ acts on $L_1 \oplus L_2$ by $(v, w) \mapsto (cv, cw)$. The difference $\mathrm{Aut}(L_1 \oplus L_2; \overline{\partial}_{\mathfrak{B}(0)}) / \mathrm{Aut}(L_1 \oplus L_2; \overline{\partial}_{\mathfrak{B}(x)}) \cong \mathbb{C}_*$ acts on $\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$ as a scaler multiplication. We can easily check that $(L_1 \oplus L_2; \overline{\partial}_{\mathfrak{B}(x)})$ is isomorphic to $(L_1 \oplus L_2; \overline{\partial}_{\mathfrak{B}(x')})$ if and only if $x' = cx$, namely in the case when x and x' lie in the same orbit of $\mathbb{C}_*$-action. Hence the part of the automorphism of $(L_1 \oplus L_2, \overline{\partial}_{B(x)}) : x = 0$ which will be "lost" for $x \neq 0$ will act on $\mathcal{U} = \mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$. The set of isomorphism classes of nearby holomorphic structures will be the orbit space of this $\mathbb{C}_*$-action. The quotient space of this action in our case is a union of $\mathbb{C}P^N$ and one point, where $N + 1 = \mathrm{rank}\,\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$. If we put quotient topology then the quotient space is not Hausdorff at the orbit of origin. This is because $\mathbb{C}_*$ is noncompact.

The phenomenon explained above that is the jump of the dimension of the automorphism group is related to versal but not universal family is rather general. Actually we can prove the following result. We recal that the Lie algebra of $\mathrm{Aut}(\mathcal{E})$ is identified with $\mathrm{Ext}^0(\mathcal{E}, \mathcal{E})$

Theorem 1.3.3. *If the deformation* $(B,\mathcal{U})$ *is versal and if the rank of* $\mathrm{Ext}^0((E, \overline{\partial}_{\mathcal{E}+B(x)}), (E, \overline{\partial}_{\mathcal{E}+B(x)}))$ *is independent of* x *in a neighborhood of* 0*, then* $(B,\mathcal{U})$ *is universal.*

We will prove it in §2.3.

1.4 Bundle valued Harmonic forms.

We prove Theorem 1.3.2 in the next section. To prove Theorem 1.3.2, we use Harmonic theory of vector bundle valued forms, which we review in this section (See [58] for detail). Let $\kappa : \Omega^{p,q}(M) \to \Omega^{q,p}(M)$ be the complex anti-linear homomorphism defined by

$$\begin{aligned}\kappa(u_{i_1,\cdots,i_p;j_1,\cdots,j_q} dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots d\overline{z}^{j_q})\\ = \overline{u}_{i_1,\cdots,i_p;J_1,\cdots,j_q} d\overline{z}^{i_1} \wedge \cdots d\overline{z}^{i_p} \wedge dz^{j_1} \wedge \cdots dz^{j_q}.\end{aligned}$$

We next fix a hermitian metric g on M. Then g induces the Hodge $*$ operator $* : \Lambda^k(M) \to \Lambda^{n-k}M$ by $u \wedge \kappa(*v) = g(u,v)\,\mathrm{Vol}_g$ where $\mathrm{Vol}_g \in \Omega^{2n}(M)$ is the volume element, and $g(u,v)$ in the inner product on $\Lambda^*(M)$ induced by g. We remark that $*$ is complex linear, $\kappa \circ * = * \circ \kappa$ and $* : \Lambda^{p,q}(M) \to \Lambda^{n-q,n-p}(M)$. We can also check

$$** = (-1)^{p+q}\,\mathrm{id} \qquad \text{on } \Lambda^{p,q}(M), \tag{1.12}$$

Definition 1.4.1. We define the operator $\overline{\partial}^*$ is defined by $\overline{\partial}^* = -\kappa \circ * \circ \overline{\partial} \circ \kappa \circ * : \Lambda^{p,q}(M) \to \Lambda^{p,q-1}(M)$.

$\overline{\partial}^*$ is complex linear. Let us next include holomorphic vector bundle. Let $\mathcal{E} = (E, \overline{\partial}_{\mathcal{E}})$ be a holomorphic vector bundle over M. We take and fix a hermitian inner product h on E. h induces an anti-complex linear homomorphism $I_h : E \to E^*$. I_h and κ induces an anti-complex linear homomorphism $\kappa_h : \Omega^{p,q}(M;E) \to \Omega^{q,p}(M;E^*)$. We define $\wedge : \Omega^{p,q}(M;E) \otimes \Omega^{p',q'}(M;E^*) \to \Omega^{p+p',q+q'}(M)$ by $(u \otimes a) \wedge (v \otimes \alpha) = \alpha(a) u \wedge v$. Then we define $* : \Omega^{p,q}(M;E) \to \Omega^{n-q,n-p}(M;E^*)$ by $a \wedge \kappa_h(*B) = g_h(A,B)\Omega_g$, $A, B \in \Omega^{p,q}(M;E)$, where g_h is an inner product on $\Lambda^{p,q}(M;E)$ induced by g and h. Formula (1.12) holds. We define

$$\overline{\partial}^*_{\mathcal{E}} = -\kappa \circ * \circ \overline{\partial}_{\mathcal{E}} \circ \kappa \circ * : \Lambda^{p,q}(M;E) \to \Lambda^{p,q-1}(M;E).$$

We define hermitian inner product $\langle \cdot, \cdot \rangle$ on $\Omega^{p,q}(M)$ and $\Omega^{p,q}(E)$ by

$$\langle u, v \rangle = \int_M g(u,v)\ \mathrm{Vol}_g, \quad \langle A, B \rangle = \int_M g_h(A,B)\ \mathrm{Vol}_g.$$

Let $L^2(M;\Lambda^{p,q}(M))$, $L^2(M;\Lambda^{p,q}(M;E))$ be the completion. They are Hilbert spaces. We can prove $\langle \overline{\partial} u, v \rangle = \langle u, \overline{\partial}^* v \rangle$, $\langle \overline{\partial}_{\mathcal{E}} A, B \rangle = \langle A, \overline{\partial}^*_{\mathcal{E}} B \rangle$ by using stokes theorem and (1.12). We now define Laplace-Beltrami operator by $\Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, $\Delta_{\overline{\partial}_{\mathcal{E}}} = \overline{\partial}_{\mathcal{E}} \overline{\partial}^*_{\mathcal{E}} + \overline{\partial}^*_{\mathcal{E}} \overline{\partial}_{\mathcal{E}}$, and space of Harmonic forms and sections by

$$\begin{aligned}\mathcal{H}^{p,q}(M) &= \mathrm{Ker}(\Delta_{\overline{\partial}} : \Omega^{p,q}(M) \to \Omega^{p,q}(M)),\\ \mathcal{H}^{p,q}(M;\mathcal{E}) &= \mathrm{Ker}(\Delta_{\overline{\partial}_{\mathcal{E}}} : \Omega^{p,q}(M;E) \to \Omega^{p,q}(M;E)).\end{aligned}$$

We can show that $\Delta_{\overline{\partial}}$, $\Delta_{\overline{\partial}_{\mathcal{E}}}$ are elliptic and hence $\mathcal{H}^{p,q}(M)$ and $\mathcal{H}^{p,q}(M;\mathcal{E})$ are of finite dimensional if M is compact without boundary.

Let $\Pi_{\mathcal{H}} : L^2(M;\Lambda^{p,q}(M)) \to \mathcal{H}^{p,q}(M)$, $\Pi_{\mathcal{H},\mathcal{E}} : L^2(M;\Lambda^{p,q}(M;E)) \to \mathcal{H}^{p,q}(M;\mathcal{E})$ be orthonormal projections.

Now the basic results due to Hodge-Kodaira is :

Theorem 1.4.1. *There exists an orthonormal decomposition :*

$$L^2(M;\Lambda^{p,q}(M)) \cong \operatorname{Im}\overline{\partial} \oplus \operatorname{Im}\overline{\partial}^* \oplus \mathcal{H}^{p,q}(M),$$
$$L^2(M;\Lambda^{p,q}(M;E)) \cong \operatorname{Im}\overline{\partial}_{\mathcal{E}} \oplus \operatorname{Im}\overline{\partial}_{\mathcal{E}}^* \oplus \mathcal{H}^{p,q}(M;\mathcal{E}).$$

There exist operators $Q : L^2(M;\Lambda^{p,q}(M)) \to L^2(M;\Lambda^{p,q}(M))$, $Q_{\mathcal{E}} : L^2(M;\Lambda^{p,q}(M);\mathcal{E}) \to L^2(M;\Lambda^{p,q}(M);\mathcal{E})$ *such that* $\Delta_{\overline{\partial}} \circ Q = \mathrm{id} - \Pi_{\mathcal{H}}$, $\Delta_{\overline{\partial}_{\mathcal{E}}} \circ Q_{\mathcal{E}} = \mathrm{id} - \Pi_{\mathcal{H},\mathcal{E}}$.

(For the proof see for example [106].) We remark that Q commutes with $\overline{\partial}$, $\overline{\partial}^*$, Δ^*. We put $G = \overline{\partial}^* \circ Q$, $G_{\mathcal{E}} = \overline{\partial}_{\mathcal{E}}^* \circ Q_{\mathcal{E}}$ and call them the *propagators*. We remark that G, $G_{\mathcal{E}}$ are chain homotopy between identity and the orthonormal projections $\Pi_{\mathcal{H}}$, $\Pi_{\mathcal{H},\mathcal{E}}$. Namely, we can prove easily that :

$$\mathrm{id} - \Pi_{\mathcal{H}} = G \circ \overline{\partial} + \overline{\partial} \circ G, \quad \mathrm{id} - \Pi_{\mathcal{H}_{\mathcal{E}}} = G_{\mathcal{E}} \circ \overline{\partial}_{\mathcal{E}} + \overline{\partial}_{\mathcal{E}} \circ G_{\mathcal{E}}. \tag{1.13}$$

1.5 Construction of versal family and Feynman diagram.

We prove Theorem 1.3.2 in this section. We are going to find a neighborhood $\mathcal{U}$ of origin in $\mathcal{H}^{p,q}(M;(End(E),\overline{\partial}_{\mathcal{E}}))$, construct a holomorphic map $B : \mathcal{U} \to \Omega^{0,1}(M;End(E))$ such that

$$\overline{\partial}_{\mathcal{E}} B(b) + B(b) \circ B(b) = 0, \tag{1.14}$$

and that $d_0 B : T_0\mathcal{U} = \mathcal{H}^{0,1}(M;(End(E),\overline{\partial}_{\mathcal{E}})) \to \Omega^{0,1}(M;End(E))$ is the identity. As we discussed already, this is enough to show Theorem 1.3.2. We take a formal parameter T and put

$$B(b) = Tb + \sum_{k=2}^{\infty} T^k B_k(b). \tag{1.15}$$

We solve equation (1.14) inductively on k. Namely we solve

$$\overline{\partial}_{\mathcal{E}} B_k(b) = -\sum_{\ell+m=k} B_\ell(b) \circ B_m(b) \tag{1.16}$$

inductively on k. The solution of (1.16) is given by using the operator $G_{\mathcal{E}}$, the propagator, introduced in the last section. (Here we write $G_{\mathcal{E}}$ in place of $G_{End(\mathcal{E})}$ for simplicity.)

Lemma 1.5.1. *We define $B_1(b) = b$ and*

$$B_k(b) = \sum_{\ell+m=k} G_{\mathcal{E}}(B_\ell(b) \circ B_m(b)) \tag{1.17}$$

inductively on k. Then it satisfies (1.16)*.*

Proof. We remark that the harmonic projection $\Pi_{\mathcal{H},End(\mathcal{E})} : L^2(M; \Lambda^{0,2}(M; End(E))) \to \mathcal{H}^{0,2}(M; End(E))$ is zero because $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$ by assumption. Hence, by (1.4.1), we have :

$$\begin{aligned}
\overline{\partial}_{\mathcal{E}} B_k(b) &= \sum_{\ell+m=k} \overline{\partial}_{\mathcal{E}} G_{\mathcal{E}}(B_\ell(b) \circ B_m(b)) \\
&= -\sum_{\ell+m=k} B_\ell(b) \circ B_m(b) - \sum_{\ell+m=k} G_{\mathcal{E}} \overline{\partial}_{\mathcal{E}}(B_\ell(b) \circ B_m(b)).
\end{aligned}$$

By using induction hypothesis, we have

$$\begin{aligned}
&\sum_{\ell+m=k} \overline{\partial}_{\mathcal{E}}(B_\ell(b) \circ B_m(b)) \\
&\qquad = \sum_{\ell+m=k} \overline{\partial}_{\mathcal{E}} B_\ell(b) \circ B_m(b) - \sum_{\ell+m=k} B_\ell(b) \circ \overline{\partial}_{\mathcal{E}} B_m(b) \\
&\qquad = -\sum_{\ell+m=k} \sum_{\ell_1+\ell_2=\ell} (B_{\ell_1}(b) \circ B_{\ell_2}(b)) \circ B_m(b) \\
&\qquad\qquad + \sum_{\ell+m=k} \sum_{m_1+m_2=m} B_\ell(b) \circ (B_{m_1}(b) \circ B_{m_2}(b)) = 0.
\end{aligned}$$

(We remark that the associativity of $\circ$ plays an important role here.) □

To complete the proof of Theorem 1.3.2, it suffices to show the following :

Lemma 1.5.2. *There exists $\epsilon > 0$ such that if $|T|\|b\| < \epsilon$ then* (1.15) *converges.*

(Here $\|b\|$ is the Sobolev L^2_k norm of b with sufficiently large k, which is introduced below.) The proof of Lemma 1.5.2 is based on the standard result, in geometric analysis, which we briefly recall here. First for $A \in \Omega^{p,q}(End(E))$ we define its Sobelev norm $\|A\|_{L^2_d}$ by

$$\|A\|^2_{L^2_d} = \sum_{i=0}^{k} \langle \nabla^i A, \nabla^i A \rangle$$

here $\nabla^i A$ is a ith covariant derivative of A and $\langle \nabla^i A, \nabla^i A\rangle$ is its appropriate L^2 inner product defined in a way similar to the last section. Let $L^2_d(M; End(E))$ be the completion of $\Omega^{p,q}(End(E))$ with respect to $\|A\|_{L^2_d}$. Then

(A) $Q_{\mathcal{E}}$ defines a bounded operator

$$L^2_d(M; \Lambda^{p,q}(M; End(E))) \to L^2_{d+2}(M; \Lambda^{p,q}((M; End(E))).$$

(B) $\circ : \Omega^{p,q}((M; End(E)) \otimes \Omega^{p',q'}((M; End(E)) \to \Omega^{p+p',q+q'}((M; End(E))$ can be extended to a continuous operator

$$\begin{aligned} L^2_{d+1}(M; \Lambda^{p,q}(M; End(E))) \otimes L^2_{d+1}(M; \Lambda^{p',q'}(M; End(E))) \\ \to L^2_d(M; \Lambda^{p+p',q+q'}(M; End(E))). \end{aligned}$$

if k is sufficiently large compared to $2n = \dim_{\mathbb{R}} M$.

The proof of (A) is given in many of the standard text books of Harmonic theory, for example [106]. The proof of (B) is given in many of the standard text books on Sobolev space, for example [36]. Now (A),(B) implies

$$\|G_{\mathcal{E}}(B_\ell(b) \circ B_m(b))\|_{L^2_k} < C\|B_\ell(b)\|_{L^2_k}\|B_m(b)\|_{L^2_k}$$

if k is large. Here C is independent of k. Therefore, we can show

$$\|B_k(b)\|_{L^2_d} < (C'\|b\|_{L^2_d})^k$$

inductively on k. Lemma 1.5.2 follows immediately. □

We defined $B_k(b)$ inductively on k. We can rewrite it and define $B_k(b)$ as a sum over Feynman diagrams. In order to show the relation of Theorem 1.3.2 to quantum field theory, let us do it here.

Definition 1.5.1. A *finite oriented graph* Γ consists of the following date : (1) A finite set Vertex(Γ), the set of vertices. (2) A finite set Edge(Γ), the set of edges. (3) Maps $\partial_{\text{source}} : \text{Edge}(\Gamma) \to \text{Vertex}(\Gamma)$, $\partial_{\text{target}} : \text{Edge}(\Gamma) \to \text{Vertex}(\Gamma)$.

A *ribbon structure* of an oriented graph Γ is the cyclic ordering of the set $\partial^{-1}_{\text{source}}(v) \cup \partial^{-1}_{\text{target}}(v)$ for each $v \in \text{Vertex}(\Gamma)$. A graph together with ribbon structure is called a *ribbon graph.*

We take copies of intervals $[0,1]_e$ corresponding to each element of $e \in \text{Edge}(\Gamma)$ and copies of points v corresponding to each element $v \in \text{Vertex}(\Gamma)$. We identify $\{0\} \in [0,1]_e$ with $\partial_{\text{source}}(e)$ and $\{1\} \in [0,1]_e$ with $\partial_{\text{target}}(e)$. We thus obtain a one dimensional complex, which we write $|\Gamma|$.

An embedding of $|\Gamma|$ to an oriented surface (real 2 dimensional manifold) Σ induces a ribbon structure on Γ as in Figure 1.

Figure 1

In this article we only consider finite oriented graphs. So we call them graphs. Now we consider ribbon graphs Γ satisfying the following conditions.

Condition 1.5.1. (1) $\mathrm{Vertex}(\Gamma)$ is decomposed into a disjoint union of $\mathrm{Vertex}_{\mathrm{int}}(\Gamma)$ and $\mathrm{Vertex}_{\mathrm{ext}}(\Gamma)$. (2) If $v \in \mathrm{Vertex}_{\mathrm{int}}(\Gamma)$ then $\sharp\partial^{-1}_{\mathrm{target}}(v) = 2$, $\sharp\partial^{-1}_{\mathrm{source}}(v) = 1$. (3) There exists $v_{\mathrm{last}} \in \mathrm{Vertex}_{\mathrm{ext}}(\Gamma)$, the last virtex, such that $\sharp\partial^{-1}_{\mathrm{target}}(v) = 1$, $\sharp\partial^{-1}_{\mathrm{source}}(v) = 0$. (4) If $v \in \mathrm{Vertex}_{\mathrm{ext}}(\Gamma)\backslash\{v_{\mathrm{last}}\}$ then $\sharp\partial^{-1}_{\mathrm{target}}(v) = 0$, $\sharp\partial^{-1}_{\mathrm{source}}(v) = 1$. (5) $|\Gamma|$ is connected and simply connected.

We say an element of $\mathrm{Vertex}_{\mathrm{int}}(\Gamma)$ an *interior virtex* and an element of $\mathrm{Vertex}_{\mathrm{ext}}(\Gamma)$ an *exterior virtex.* An edge e is called an *exterior edge* if $\{\partial_{\mathrm{target}}(e), \partial_{\mathrm{source}}(e)\} \cap \mathrm{Vertex}_{\mathrm{ext}}(\Gamma) \neq \emptyset$. Otherwise it is called an *interior edge.*

Figure 2

Let Γ be a ribbon graph such that $|\Gamma|$ is simply connected. It is then easy to see that there exists an embedding $|\Gamma| \to \mathbb{R}^2$ such that the ribbon structure is compatible with the orientation of $\mathbb{R}^2$.

We denote by $RG_{k,2}$ the set of all ribbon graph Γ which satisfies Condition 1.5.1 and has exactly $k+1$ exterior vertices.

Let $\Gamma \in RG_{k,2}$. Then, using the embedding $|\Gamma| \to \mathbb{R}^2$ compatible with ribbon structure, we obtain a cyclic order on $\mathrm{Vertex}_{\mathrm{ext}}(\Gamma)$. By regarding the last vertex as 0th one, the cyclic order determine an order on $\mathrm{Vertex}_{\mathrm{ext}}(\Gamma)\backslash\{v_{\mathrm{last}}\}$. So we put $\mathrm{Vertex}_{\mathrm{ext}}(\Gamma)\backslash\{v_{\mathrm{last}}\} = \{v_1, \cdots, v_k\}$. We now are going to define $B_\Gamma : \mathcal{H}^{0,1}(M; (End(E), \overline{\partial}_{\mathcal{E}}))^{k\otimes} \to \Omega^{0,1}(M; End(E))$ for each $\Gamma \in Gr_{k,2}$ such that

$$B_k(b) = \sum_{\Gamma \in Gr_{k,2}} B_\Gamma(b, \cdots, b). \tag{1.18}$$

We define B_Γ by induction on k. Let $\Gamma \in RG_{k,2}$ and v_{last} be its last vertex. Let e_{last} be the unique edge such that $\partial_{\mathrm{target}}(e_{\mathrm{last}}) = v_{\mathrm{last}}$. We remove $[0,1]_{e_{\mathrm{last}}}$ together with its two vertices from $|\Gamma|$. Then $|\Gamma|\backslash[0,1]_{e_{\mathrm{last}}}$ is a union of two components which can be regarded as $|\Gamma_1|$ and $|\Gamma_2|$, where $\Gamma_1 \in RG_{2,\ell}$, $\Gamma_2 \in RG_{2,m}$ with $\ell + m = k$. We may number Γ_1, Γ_2 so that $v_1, \cdots, v_\ell \in \Gamma_1$.

Now we put

$$B_\Gamma(b_1, \cdots, b_k) = G_{\mathcal{E}}(B_{\Gamma_1}(b_1, \cdots, b_\ell) \circ B_{\Gamma_2}(b_{\ell+1}, \cdots, b_k)). \tag{1.19}$$

(1.18) is quite obvious from definition.

The way we rewrite the induction process into sum over trees is a straightforward matter. However, rewriting the definition of B_k as in (1.18) leads us naturally to the following questions.

Question 1.5.1. (1) Can we generalize to the case when the interior vertex has more than two edges ?
(2) When we consider tree instead of ribbon tree, are there any corresponding story ?
(3) What happens when we include the graph which is not simply connected ?

Remarkably they all have good answers.

Answere 1.5.1. (1) We then study deformation of A_∞ algebra in place of differential graded algebra.
(2) We then study deformation of differential graded Lie algebra or more generally of L_∞ algebra.
(3) It corresponds to Reidemeister or Analytic torsion [88], (in the case $H_1(|\Gamma|) = \mathbb{Z}$) and to Chern-Simons Perturbation theory [5, 6, 66, 24], or quantum Kodaira-Spencer theory [10], or pseudoholomorphic map from higher genus Riemann surface (with or without boundary).

We will explain the first two answers more in later sections. The detailed study on the third one is left to future, since Mathematical theory, in the case $H_1(|\Gamma|) \neq \mathbb{Z}$, is not yet enough developed.

1.6 Kuranishi family.

In this section, we remove the assumption $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) = 0$ from Theorem 1.3.1. We need to study deformations parametrized by a singular variety for this purpose. Let us start with briefly reviewing the notion of analytic variety. Since we discuss here only local theory of moduli it is enough to consider the case of analytic subspace of $\mathbb{C}^N$. (See [20] for more detail on analytic variety.)

Definition 1.6.1. Let $X \subset \mathbb{C}^N$ be a locally closed subset. We say that X is a (complex) *analytic subset* if the following holds.

For each $p \in X$ there exists its neighborhood U in $\mathbb{C}^N$ and holomorphic functions $f_1, \cdots, f_m$ such that $X \cap U = \{z | f_1(z) = \cdots = f_m(z) = 0\}$.

Definition 1.6.2. For $p \in X$, we put

$$\mathfrak{I}_{X,p} = \{f \in \mathcal{O}_p | f \equiv 0 \text{ on } X\}. \tag{1.20}$$

Here $\mathcal{O}_p$ is a germ of holomorphic functions at p. (Namely the set of convergent power series in a neighborhood of p.)

The germ of holomorphic functions $\mathcal{O}_{X,p}$ of X at p is defined by $\mathcal{O}_{X,p} = \mathcal{O}_{X,p}/\mathfrak{I}_{X,p}$. $\mathcal{O}_{X,p}$ is a local ring and its maximal ideal is $\mathcal{O}_{X,p,+} = \{[f] \in \mathcal{O}_{X,p} \mid f(p) = 0\}$. (Here and hereafter, we denote by $\mathcal{R}_+$ its maximal ideal of a local ring $\mathcal{R}$.)

Let $X \subset \mathbb{C}^N, X' \subset \mathbb{C}^{N'}$ be analytic sets. A map $F : X \to X'$ is said to be a *holomorphic map* if for each $p \in X$ there exists a neighborhood U of p in $\mathbb{C}^N$ such that the restriction of F to $U \cap X$ is extended to a holomorphic map from U to $\mathbb{C}^{N'}$.

A *germ of analytic subset* at $0 \in \mathbb{C}^N$ is an equivalence class of analytic subset X containing 0, where X is equivalent to X' if there exists a neighborhood $\mathcal{V}$ of 0 such that $\mathcal{V} \cap X = \mathcal{V} \cap X'$.

To study the problem of moduli, we need to consider the case when $\mathfrak{I}_{X,p}$ does not satisfy (1.20), that is a complex analytic analogue of scheme. The simplest example is $X = \{0\} \subset \mathbb{C}$ and $\mathfrak{I}_{X,p} = (x^2)$, that is the set of holomorphic functions $f(x)$ such that $f(0) = f'(0) = 0$. Let us define such objects. We need only its germ at 0 so we restrict ourselves to such case.

Definition 1.6.3. A *germ at* 0 *of analytic subspace* $\mathfrak{X}$ of $\mathbb{C}^N$ is a germ of analytic subset X together with an ideals $\mathfrak{I}_{\mathfrak{X},0} \subset \mathcal{O}_{X,0}$ such that the following holds. Let $f_1, \cdots, f_m$ be a generator of $\mathfrak{I}_{\mathfrak{X},0}$. (Since $\mathcal{O}_0$ is Noetherian it follows that we can choose such generator.) Let f_i are defined on $\mathcal{U}$. Then

$$\mathcal{U} \cap X = \{x \mid f_i(x) = 0, i = 1, 2, \cdots, m\}. \tag{1.21}$$

Remark 1.6.1. By Hilbert's Nullstellensatz, (1.21) is equivalent to $\mathfrak{I}_{X,0} = \{f \mid f^n \in \mathfrak{I}_{\mathfrak{X},0}$ for some $n.\}$

One can define analytic variety as a ringed space[2] which is locally isomorphic to $(\mathfrak{X}, \mathfrak{I}_{\mathfrak{X},0})$, in other words as the space obtained by glueing germs of analytic subspaces in $\mathbb{C}^N$ defined in Definition 1.6.3. We do not try to do so since we do not use it. (See [20].)

Example 1.6.1. Let $F = (f^1, \cdots, f^k) : \mathcal{U} \to \mathbb{C}^k$ be a holomorphic map, where $\mathcal{U}$ is an open neighborhood of 0 in $\mathbb{C}^N$. We assume $f^i(0) = 0$. We put $X = F^{-1}(0)$. We let $\mathfrak{I}_{\mathfrak{X},0}$ be the ideal generated by $f^1, \cdots, f^k$. We then obtain a germ of analytic subspace. We write it by $F^{-1}(0)$ by abuse of notation.

We put $\mathcal{O}_{\mathfrak{X},0} = \mathcal{O}_0/\mathfrak{I}_{\mathfrak{X},0}$. $\mathcal{O}_{\mathfrak{X},0,+} = \{[f] \in \mathcal{O}_{\mathfrak{X},0} \mid f(0) = 0\}$.

Definition 1.6.4. Let $\mathfrak{X} = (X, \mathfrak{I}_{\mathfrak{X},0})$, $(\mathfrak{X}', \mathfrak{I}'_0(\mathfrak{X}'))$ be germs of analytic varieties. A morphism $\mathfrak{F}$ from $\mathfrak{X}$ to $\mathfrak{X}'$ is a ring homomorphism $F^* : \mathcal{O}_{\mathfrak{X}',0} \to \mathcal{O}_{\mathfrak{X},0}$.

[2] We do not define this notion here since we do not use it. See [48].

(Here and hereafter ring all homomorphisms between commutative rings are assumed to preserve unit.) We remark that morphism of germ of analytic subspaces induce a map between analytic sets as follows.

Lemma 1.6.1. *If $\mathfrak{F} : \mathfrak{X} \to \mathfrak{X}'$ is a morphism as in Definition* 1.6.4. *Here* $\mathfrak{X} = (X, \mathfrak{I}_{\mathfrak{X},0})$*,* $X \subset \mathbb{C}^N$*,* $\mathfrak{X}' = (X', \mathfrak{I}'_{\mathfrak{X}',0})$*,* $X' \subset \mathbb{C}^{N'}$. *Then there exists a neighborhood* $\mathcal{U}$ *of* 0 *in* $\mathbb{C}^N$ *and a holomorphic map* $\tilde{F} : \mathcal{U} \to \mathbb{C}^{N'}$ *with* $\tilde{F}(0) = 0$*, such that :* (1) $\tilde{F}(X \cap \mathcal{U}) \subset X'$. (2) *If* $f \in \mathfrak{I}'\mathfrak{I}'_{\mathfrak{X}',0}$ *then* $f \circ F \in \mathfrak{I}_{\mathfrak{X},0}$. (3) *By* (2) *we have a ring homomorphism* $\mathcal{O}_{\mathfrak{X}',0} \to \mathcal{O}_{\mathfrak{X},0}$ *induced by* $f \mapsto f \circ \tilde{F}$. *This homomorphism coincides with* F^**.*

Proof. Let x^i, $i = 1, \cdots, N'$ be the coordinate function on $\mathbb{C}^{N'}$. We have $F^* x_i \in \mathcal{O}_{\mathfrak{X},0}$. Let $\tilde{f}^i \in \mathcal{O}_0$ be any elements with represents $F^* x_i$. It is easy to see that $\tilde{F} = (\tilde{f}^1, \cdots, \tilde{f}^{N'})$ has required properties. □

To define a deformation of complex structures parametrized by a germ of analytic subspace of $\mathbb{C}^N$, we need to define a fiber bundle over complex analytic variety etc. It is a straightforward analogue of the case of complex manifold. But since our main purpose is to study the case of vector bundle, we restrict ourselves to the case of deformation of holomorphic structures of complex vector bundle on a complex manifold M with *fixed* complex structure.

Let $\mathcal{E}$ be a holomorphic vector bundle on M. Let $\mathcal{U} \subset \mathbb{C}^N$ be an open neighborhood of origin and let $X \subset \mathcal{U}$ be a germ of complex analytic subset.

Definition 1.6.5. A *deformation* of $\mathcal{E}$ parametrized by X is a germ of a holomorphic map $B : \mathcal{U} \to \Omega^{0,1}(M; End(E))$ such that $B(0) = 0$ and that

$$\overline{\partial}_{\mathcal{E}} B(x) + B(x) \circ B(x) = 0. \tag{1.22}$$

holds for each $x \in X$.

Using (1.20), it is easy to see that (1.22) is equivalent to the following.

There exists an open covering $\cup U_i = M$, and an open neighborhood $\mathcal{U}$ of 0, and there exists a smooth sections $e_i(x, q)$ of $\Lambda^{0,1}(\mathcal{U} \times U_i, End(E))$ such that

$$\overline{\partial}_{\mathcal{E}} B(x) + B(x) \circ B(x) = \sum f_i(x) e_i(x, q). \tag{1.23}$$

where $f_i \in \mathfrak{I}_{X,0}$. Hereafter we say that (1.23) holds locally and do not mention $U_i, \mathcal{U}$.

In view of (1.23), we can generalize Definition 1.6.5 to the case of a germ of analytic subspace as follows.

Definition 1.6.6. Let $\mathfrak{X} = (X, \mathfrak{I}_{\mathfrak{X},0})$ be a germ of analytic subspace and $B : \mathcal{U} \to \Omega^{0,1}(M; End(E))$ be a deformation of holomorphic structures on

E parametrized by X. We say that B is a *deformation* of $\mathcal{E}$ parametrized by $\mathfrak{X}$ if the following holds. Such that

$$\overline{\partial}_{\mathcal{E}} B(x) + B(x) \circ B(x) = \sum f_i(x) e_i(q,x). \tag{1.24}$$

holds locally with $f_i \in \mathfrak{I}_{\mathfrak{X},0}$.

We way that B is the *same* deformation as B' if

$$B(x) - B'(x) = \sum f_i(x) e_i(q,x),$$

holds locally with $f_i \in \mathfrak{I}_{\mathfrak{X},0}$.

Example 1.6.2. Let $X = \{0\} \subset \mathbb{C}$ and $\mathfrak{I}_{\mathfrak{X},0} = (x^2)$. Let us write $B(x) = xB_1 + x^2 B_2 + \cdots$. (1.24), then implies $\overline{\partial}_{\mathcal{E}} B_1 = 0$. Note any choice of $B_2, \cdots$ define the same family by definition. Hence the set of deformations parametrized by $(\{0\}, (x^2))$ is identified with the kernel of $\overline{\partial}_{\mathcal{E}} : \Omega^{0,1}(End(E)) \to \Omega^{0,2}(End(E))$.

To define completeness and universality of a deformation parametrized by a germ of analytic subspace, we define morphism between deformations. Let $\mathfrak{X} = (X, \mathfrak{I}_{\mathfrak{X},0})$, $\mathfrak{X}' = (X', \mathfrak{I}_{\mathfrak{X}',0})$ be germs analytic subspaces and $B : \mathcal{U} \to \Omega^{0,1}(M; End(E))$, $B' : \mathcal{U}' \to \Omega^{0,1}(M; End(E))$ be deformations of $\mathcal{E}$ parametrized by $\mathfrak{X}$, $\mathfrak{X}'$ respectively.

Definition 1.6.7. A morphism from $(\mathfrak{X}, B)$ to $(\mathfrak{X}', B')$ is a pair $(\mathfrak{F}, \Psi)$, where $\mathfrak{F} : \mathfrak{X} \to \mathfrak{X}'$ is a morphism of analytic subspace and $\Psi : \mathcal{U} \to \Gamma(M; End(E))$ is a holomorphic map such that

$$\overline{\partial}_{\mathcal{E}}(\Psi(x)) + B'(\tilde{F}(x)) \circ \Psi(x) - \Psi(x) \circ B(x) = \sum f_i(x) e_i(q,x) \tag{1.25}$$

holds locally with $f_i \in \mathfrak{I}_{\mathfrak{X},0}$. Here $\tilde{F}$ is as in Lemma 1.6.1.

We regard $(\mathfrak{F}, \Psi)$ as the same morphism as $(\mathfrak{F}', \Psi')$ if $\mathfrak{F} = \mathfrak{F}'$ in the sense of Definition 1.6.4 and if

$$\Psi(x) - \Psi'(x) = \sum f_i(x) e_i(q,x)$$

locally with $f_i \in \mathfrak{I}_{\mathfrak{X},0}$. Here $e_i(q,x)$ is a local section of $\Lambda^{0,1} \otimes End(E)$.

Example 1.6.3. Let us consider a family on $\mathfrak{X} = (\{0\}, (x^2))$ as in Example 1.6.2. Let $B = xB_1$ and $B' = xB_1'$ be two such families. Let $(\mathfrak{F}, \Psi)$ be a morphism from $(\mathfrak{X}, B)$ to $(\mathfrak{X}, B')$. We may choose $\tilde{F}$ in Lemma 1.6.1 so that $\tilde{F}(x) = xF_1$, when $F_1 \in \mathbb{C}$. We may write also $\Psi = \Psi_0 + x\Psi_1$. (1.25) then can be written as $\overline{\partial}_{\mathcal{E}} \Psi_0 = 0$, $F_1 \overline{\partial}_{\mathcal{E}} \Psi_1 + B_1' \circ \Psi_0 - \Psi_0 \circ B_1 = 0$. Hence $B_1' - B_1 = F_1 \overline{\partial}_{\mathcal{E}} \Psi_1$. Thus the set of the deformations of $\mathcal{E}$ parametrized by $\mathfrak{X} = (\{0\}, (x^2))$ is identified with $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$.

Morphism of deformations being defined, we can define completeness and universality in exactly the same way as the case of deformation parametrized by complex manifolds. We leave it to the reader.

To define versality and Kodaira-Spencer map we need to define Zariski tangent space of a germ of analytic subspace.

Definition 1.6.8. Let $\mathfrak{X} \subset \mathbb{C}^N$ be a germ of analytic subspace. The *Zariski tangent space* $T_0\mathfrak{X}$ is defied by

$$T_0\mathfrak{X} = \{V \in T_p\mathbb{C}^N \mid V(f) = 0 \text{ if } f \in \mathfrak{I}_{\mathfrak{X},0}.\}$$

Let $\mathfrak{F} : \mathfrak{X} \to \mathfrak{X}'$ be a morphism. Let $\tilde{F} : \mathcal{U} \to \mathbb{C}^{N'}$ be as in Lemma 1.6.1. Then it is easy to see that $d_0\tilde{F} : T_0\mathbb{C}^N \to T_0\mathbb{C}^{N'}$ induces $d_0\mathfrak{F} : T_0\mathfrak{X} \to T_0\mathfrak{X}'$.

Now we can generalize the definition of versality in the same way as §1.2 by using Zariski tangent space in the same way as Definition 1.2.7.

We next generalize Kodaira-Spencer map. Let us consider the situation of Definition 1.6.5. Let f_i, e_i be as in (1.23). Let $V = \sum V^i \frac{\partial}{\partial x^i} \in T_0\mathfrak{X}$. By differentiating (1.23) we have

$$\begin{aligned}\overline{\partial}_{\mathcal{E}} V(B(x)) + V(B(x)) \circ B(x) + B(x) \circ V(B(x))& \\ = \sum V(f_i)(0)e_i(0) + f_i(0)V(e_i)(0) = 0.&\end{aligned}$$

It follows that

$$V(B(x)) \in \mathrm{Ker}\,(\overline{\partial}_{\mathcal{E}} : \Omega^{0,1}(M; \mathrm{End}(\mathcal{E})) \to \Omega^{0,2}(M; \mathrm{End}(\mathcal{E}))).$$

Definition 1.6.9. We put

$$\mathrm{KS}\,(V) = [V(B(x))] \in \mathrm{Ext}^1(\mathcal{E}, \mathcal{E}).$$

KS is a linear map : $T_0\mathfrak{X} \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$, which we call the *Kodaira-Spencer map* of our family.

Lemmata 1.3.1, 1.3.2 hold in our case also. However Theorem 1.3.1 does *not* hold. In fact, we have the following counter example. Let us consider a universal family parametrized by an open neighborhood of 0 in $\mathbb{C}^2$. We restrict it to $X = \{(x, y) | xy = 0\}$. Since $T_0X = \mathbb{C}^2$ it follows that this family still satisfies the assumption of Theorem 1.3.1. It is not complete however.

Now we have the following generalization of Theorem 1.3.2

Theorem 1.6.1. *Let $\mathcal{E}$ be a holomorphic vector bundle on M. There exists a germ of complex analytic variety $\mathfrak{X}$ and a deformation B of $\mathcal{E}$ parametrized by $\mathfrak{X}$ such that :* (1) *B is complete.* (2) *The Kodaira-Spencer map,* $\mathrm{KS} : T_0\mathfrak{X} \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{E})$ *is an isomorphism.*

Moreover there exists an open neighborhood $\mathcal{U}$ *of origin in* $\mathrm{Ext}^1(\mathcal{E},\mathcal{E})$ *and a holomorphic map* $\mathrm{Kura} : \mathcal{U} \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ *such that* $\mathfrak{X} = \mathrm{Kura}^{-1}(0)$. *Here the* $\mathrm{Kura}^{-1}(0)$ *is as in Example* 1.6.1.

Definition 1.6.10. We call the map Kura the *Kuranishi map.*

We will give a proof of Theorem 1.6.1 in §2.3. Theorem 1.6.1 is a vector bundle analogue of [71]. [71] was generalized to various other situations by many people. See for example [45, 93, 105, 85, 4]. Theorem 1.6.1 has also an analogue in the case of the moduli space of gauge equivalence classes of Yang-Mills equations [102, 16].

Proposition 1.3.1 can be generalized directly to our situation by the same proof. Namely the deformation obtained in Theorem 1.6.1 is versal. Theorem 1.3.1 still holds in the case of deformation parametrized by a germ of analytic subspace. (We prove it in §2.3.)

We close this section by giving an example where the Kuranishi map is nonzero.

Example 1.6.4. Let $\mathbb{T}$ be an elliptic curve and $\mathcal{L} \to \mathbb{T}$ be a line bundle with $c^1(\mathcal{L}) \cap [\mathbb{T}] = -m < 0$. We have

$$\mathrm{Ext}^k(\mathbb{T}; \mathbb{C}, \mathcal{L}) \cong \begin{cases} \mathbb{C}^m & k = 1, \\ 0 & k = 0. \end{cases}$$

Let $\mathfrak{b}_i : i = 1, \cdots, m$ be a generator of $\mathrm{Ext}^1(\mathbb{T}; \mathbb{C}, \mathcal{L})$ we may regard it as a $\mathcal{L}$ valued (0,1)-form.

By Serre duality, we have $\mathrm{Ext}^0(\mathbb{T}; \mathcal{L}, \mathbb{C}) \cong \mathrm{Ext}^1(\mathbb{T}; \mathbb{C}, \mathcal{L})^*$. Let $\mathfrak{b}^i$: $i = 1, \cdots, m$ be the dual basis.

We consider $M = \mathbb{T} \times \mathbb{T}$ and let $\mathrm{pr}_i : i = 1, 2$ be projections to first and second factors. We put

$$\mathcal{E} = \mathrm{pr}_1^* \mathcal{L} \oplus \mathrm{pr}_2^* \mathcal{L} = (\mathcal{L} \boxtimes 1) \oplus (1 \boxtimes \mathcal{L}),$$

where $\boxtimes$ is the exterior product and 1 is the trivial line bundle. We consider

$$\begin{aligned} \mathrm{Ext}^1(M; \mathcal{E}, \mathcal{E}) \cong\, & \mathrm{Ext}^1(M; \mathcal{L} \boxtimes 1, \mathcal{L} \boxtimes 1) \oplus \mathrm{Ext}^1(M; 1 \boxtimes \mathcal{L}, 1 \boxtimes \mathcal{L}) \\ & \oplus \mathrm{Ext}^1(M; 1 \boxtimes \mathcal{L}, \mathcal{L} \boxtimes 1) \oplus \mathrm{Ext}^1(M; \mathcal{L} \boxtimes 1, 1 \boxtimes \mathcal{L}). \end{aligned}$$

The first two factors are isomorphic to $H^{0,1}(M) \cong \mathbb{C}^2$. The third and fourth factors are isomorphic to $\mathbb{C}^{m^2}$ and their generators are $\mathrm{pr}_1^* \mathfrak{b}_i \wedge \mathrm{pr}_2^* \mathfrak{b}^j$ and $\mathrm{pr}_1^* \mathfrak{b}^i \wedge \mathrm{pr}_2^* \mathfrak{b}_j$ respectively.

Using these basis we define coordinate a_i, b_i, x_{ij}, y_{ij} of $\mathrm{Ext}^1(M; \mathcal{E}, \mathcal{E})$ as follows. Let $B \in \mathrm{Ext}^1(M; \mathcal{E}, \mathcal{E})$, then we put :

$$\overline{\partial}_{\mathcal{E}+B} = \overline{\partial}_{\mathcal{E}} + \begin{pmatrix} \sum_i a_i d\overline{z}^i & \sum_{i,j} x_{ij} (\mathrm{pr}_1^* \mathfrak{b}_i \wedge \mathrm{pr}_2^* \mathfrak{b}^j) \\ \sum_{i,j} y_{ij} (\mathrm{pr}_1^* \mathfrak{b}^i \wedge \mathrm{pr}_2^* \mathfrak{b}_j) & \sum b_i d\overline{z}^i. \end{pmatrix},$$

where z_1, z_2 are complex coordinate of the first and the second factor $M = \mathbb{T} \times \mathbb{T}$. We also have

$$\mathrm{Ext}^2(M;\mathcal{E},\mathcal{E}) \cong \mathrm{Ext}^2(M;\mathcal{L}\boxtimes 1,\mathcal{L}\boxtimes 1) \oplus \mathrm{Ext}^2(M;1\boxtimes\mathcal{L},1\boxtimes\mathcal{L}) \cong \mathbb{C}^2.$$

We find

$$(\mathrm{pr}_1^*\,\mathfrak{b}_i \wedge \mathrm{pr}_2^*\,\mathfrak{b}^j) \wedge (\mathrm{pr}_1^*\,\mathfrak{b}^k \wedge \mathrm{pr}_2^*\,\mathfrak{b}_\ell) = \begin{cases} (1,0) & (i,j) = (k,\ell), \\ 0 & \text{otherwise.} \end{cases}$$

$$(\mathrm{pr}_1^*\,\mathfrak{b}^i \wedge \mathrm{pr}_2^*\,\mathfrak{b}_j) \wedge (\mathrm{pr}_1^*\,\mathfrak{b}_k \wedge \mathrm{pr}_2^*\,\mathfrak{b}^\ell) = \begin{cases} (-1,0) & (i,j) = (k,\ell), \\ 0 & \text{otherwise.} \end{cases}$$

Other components of the products $\mathrm{Ext}^1(M;\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(M;\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(M;\mathcal{E},\mathcal{E})$ vanish. Therefore, the Kuranishi map is

$$(a_1,a_2;b_1,b_2;(x_{i,j}),(y_{i,j})) \mapsto \left(\sum_{i,j} x_{i,j}y_{i,j}, -\sum_{i,j} x_{i,j}y_{i,j}\right).$$

Hence origin is a singular point of $\mathrm{Kura}^{-1}(0)$.

We remark that origin is not a stable bundle in the sense of [79]. We need to consider 3 dimensional case to obtain a stable bundle whose Kuranishi map is nontrivial. R.Thomas [103] found such an example for Calabi-Yau 3 fold. We will discuss a mirror of Example 1.6.4 in section §3.6.

1.7 Formal deformation.

In §1.4, we *proved* the convergence of the series (1.15). In less classical situation, which we will study in later chapters, the convergence of a similar series is not yet proved. So until the convergence will be proved (in future hopefully) we need to regard a series like (1.18) as a formal power series. This leads us to the theory of formal deformation developed in algebraic geometry. [45, 93, 4] seems to be a standard reference. We review formal deformation in this section.

We need to translate the notion of deformation or family of structures, into more algebraic language. Let R be a commutative ring with unit. Here we consider the case R is a field. We consider a local ring $\mathcal{R}$ such that $R \cong \mathcal{R}/\mathcal{R}_+$. We assume that there exists an embedding $R \to \mathcal{R}$ preserving unit. Hence the composition $R \to \mathcal{R} \to \mathcal{R}/\mathcal{R}_+ \cong R$ is the identity. Let $(A,\cdot,d)$ be a differential graded algebra defined over R. Hereafter we assume A is free as R module. We simply write A in place of $(A,\cdot,d)$ sometimes for simplicity.

Definition 1.7.1. A *deformation* of A over $\mathcal{R}$ is a pair $(A_{\mathcal{R}}, i)$ where $A_{\mathcal{R}}$ is a differential graded $\mathcal{R}$ algebra and $i : A_{\mathcal{R}}/\mathcal{R}_+ A_{\mathcal{R}} \cong A$, is an isomorphism of differential graded R algebras.

In a similar way, we can define deformation of differential graded module as follows. Let $(C, \cdot, d)$ be a differential graded module over $(A, \cdot, d)$. We sometimes write C in place of $(C, \cdot, d)$.

Definition 1.7.2. A *deformation* of C over $A_{\mathcal{R}}$ is a pair of $(C_{\mathcal{R}}, i)$ such that $C_{\mathcal{R}}$ is a differential graded $A_{\mathcal{R}}$ module and $i : C_{\mathcal{R}}/\mathcal{R}_+ C_{\mathcal{R}} \cong C$ is an isomorphism of differential graded A modules.

In case $A_{\mathcal{R}} = A \otimes_R \mathcal{R}$, (that is the case of trivial deformation of A), we say $C_{\mathcal{R}}$ is a deformation of C over $\mathcal{R}$.

Let us explain a relation of Definitions 1.7.1 and 1.7.2 to the definitions in §1.2. Let $\hat{M} \to \mathcal{U}$ be a family of complex structures on M. We consider a vector bundle $\Lambda^{p,q}(\hat{M}/\mathcal{U})$ whose fiber at $\hat{x} \in \hat{M}$ is $\Lambda^{p,q}_{\hat{x}}(M_x)$ where $x = \pi(\hat{x}) \in \mathcal{U}$.

Definition 1.7.3. A section ω of $\Lambda^{p,q}(\hat{M}/\mathcal{U})$ is said to be *holomorphic in $\mathcal{U}$ direction* if the following holds. Let $\hat{x} \in \hat{M}$. We choose a complex coordinate $w^1, \cdots, w^N$ of a neighborhood of $x = \pi(\hat{x}) \in \mathcal{U}$. We choose $z^1, \cdots, z^n$ so that $z^1, \cdots, z^n, w^1, \cdots, w^N$ is a complex coordinate of a neighborhood of $\hat{x}$. (Here we identify w^i with $w^i \circ \pi$). Now we may write

$$\omega = \sum \omega_{i_1,\cdots,i_p,j_1,\cdots,j_q}(z,w) dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}. \quad (1.26)$$

Now we say ω is holomorphic in base direction if $\omega_{i_1,\cdots,i_p,j_1,\cdots,j_q}(z,w)$ is holomorphic with respect to w. We assume ω to be smooth in both directions. We denote by $\Omega^{p,q}(\hat{M}/\mathcal{U})$ the set of fiberwise (p,q) form which is holomorphic in base direction.

We remark that the holomorphicity in base direction is independent of the choice of the coordinate z, w. It is also easy to see that $\Omega^{p,q}(\hat{M}/\mathcal{U})$ is a module over $\mathcal{O}(\mathcal{U})$, the ring of holomorphic functions on $\mathcal{U}$.

We define an operator $\overline{\partial} : \Omega^{p,q}(\hat{M}/\mathcal{U}) \to \Omega^{p,q+1}(\hat{M}/\mathcal{U})$ by

$$\begin{aligned}&\overline{\partial}(\omega_{i_1,\cdots,i_p,j_1,\cdots,j_q}(z,w) dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}) \\ &\quad = \sum_{\ell} (-1)^p \frac{\partial \omega_{i_1,\cdots,i_p,j_1,\cdots,j_q}}{\partial \overline{z}^{\ell}} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{\ell} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}.\end{aligned}$$

We can define a wedge product $\wedge$ between elements of $\Omega^{*,*}(\hat{M}/\mathcal{U})$ in an obvious way. Thus $(\Omega^{0,*}(\hat{M}/\mathcal{U}), \overline{\partial}, \wedge)$ is a differential graded algebra over $\mathcal{O}(\mathcal{U})$. We consider its germ as follows. Let $\mathcal{V} \subset \mathcal{U}$ be open neighborhoods of 0. We put $\hat{M}(\mathcal{V}) = \pi^{-1}(\mathcal{V})$. We consider pairs $(\omega, \mathcal{V})$ where $\omega \in \Omega^{p,q}(\hat{M}(\mathcal{V})/\mathcal{V})$. We say $(\omega, \mathcal{V}) \sim (\omega', \mathcal{V}')$ if $\omega = \omega'$ on $\hat{M}(\mathcal{V} \cap \mathcal{V}')$. The set of $\sim$ equivalence class of all such pairs is denoted by $\Omega^{p,q}(\hat{M}/\mathcal{U})_0$. It is obvious that $\Omega^{p,q}(\hat{M}/\mathcal{U})_0$ is an $\mathcal{O}_0$ module. It is easy to see that

$$\frac{\Omega^{p,q}(\hat{M}/\mathcal{U})_0}{\mathcal{O}_{0,+}\Omega^{p,q}(\hat{M}/\mathcal{U})_0} \cong \Omega^{p,q}(M).$$

We write the isomorphism by i. The following lemma is now obvious.

Lemma 1.7.1. *$((\Omega^{0,*}(\hat{M}/\mathcal{U})_0, \overline{\partial}, \wedge), i)$ is a deformation of $(\Omega^{0,*}(M), \overline{\partial}_0, \wedge)$ over $\mathcal{O}_0$.*

Remark 1.7.1. Actually, to study deformation of complex structures, we need to study the deformation of the differential graded Lie algebra $\Gamma(M; \Lambda^{0,*} \otimes TM)$ (defined Example 2.1.1), as is discussed by [61, 63, 71], instead of deformation of differential graded Lie algebra $\Omega^{p,q}(M)$.

Let $(B,\mathcal{U})$ be a deformation of a holomorphic vector bundle $\mathcal{E}$ on $\hat{M}$. In a similar way, we can define $\Omega^{p,q}(\hat{M}/\mathcal{U}; E)$ the set of all E valued p, q forms which is holomorphic with respect to the base direction. ($\hat{M} = M \times \mathcal{U}$.) We can also define its germ $\Omega^{p,q}(\hat{M}/\mathcal{U}; E)_0$. We define $\overline{\partial}_{\mathcal{E}+B} : \Omega^{p,q}(\hat{M}/\mathcal{U}; E)_0 \to \Omega^{p,q+1}(\hat{M}/\mathcal{U}; E)_0$ and module structure $\wedge : \Omega^{p,q}(\hat{M}/\mathcal{U})_0 \otimes \Omega^{p',q'}(\hat{M}/\mathcal{U}; E)_0 \to \Omega^{p+p',q+q'}(\hat{M}/\mathcal{U}; E)_0$. Thus we have

Lemma 1.7.2. *$((\Omega^{0,*}(\hat{M}/\mathcal{U}; E)_0, \overline{\partial}_{\mathcal{E}+B}, \wedge), i)$ we defined above is a deformation of $(\Omega^{0,*}(M; E), \overline{\partial}_{\mathcal{E}}, \wedge)$ over $((\Omega^{0,*}(\hat{M}/\mathcal{U})_0, \overline{\partial}, \wedge), i)$.*

The generalization to the case of families parametrized by a germ of analytic subspace as in §1.6 is straightforward. In the case when we do not move the complex structure of M, it is described as follows.

Let $\mathfrak{X} = (X, \mathfrak{I}_{\mathfrak{X},0})$, be a germ of analytic subspace ($X \subseteq \mathcal{U}$). We put $\hat{M} = M \times X$. Let $B : \mathcal{U} \to \Omega^{0,1}(M; End(E))$ be as in Definition 1.6.6. We then have $\overline{\partial}_{\mathcal{E}+B} : \Omega^{p,q}(\hat{M}/\mathcal{U}; E)_0 \to \Omega^{p,q+1}(\hat{M}/\mathcal{U}; E)_0$. By Definition 1.6.6 we have $(\overline{\partial}_{\mathcal{E}+B} \circ \overline{\partial}_{\mathcal{E}+B})(\omega) = \sum f_i e_i$ where $f_i \in \mathfrak{I}_{\mathfrak{X},0}$. We now put

$$\Omega^{p,q}(\hat{M}/\mathfrak{X}; E)_0 = \frac{\Omega^{p,q}(\hat{M}/\mathcal{U}; E)_0}{\mathfrak{I}_{\mathfrak{X},0}\Omega^{p,q}(\hat{M}/\mathcal{U}; E)_0}.$$

Then $\overline{\partial}_{\mathcal{E}+B}$ induces a homomorphism $\overline{\partial}_{\mathcal{E}+B} : \Omega^{p,q}(\hat{M}/\mathfrak{X}; E)_0 \to \Omega^{p,q+1}(\hat{M}/\mathfrak{X}; E)_0$ such that $\overline{\partial}_{\mathcal{E}+B} \circ \overline{\partial}_{\mathcal{E}+B} = 0$. We thus have

Lemma 1.7.3. *$((\Omega^{0,*}(\hat{M}/\mathfrak{X}; E)_0, \overline{\partial}_{\mathcal{E}+B}, \wedge), i)$ we defined above is a deformation of $(\Omega^{0,*}(M; E), \overline{\partial}_{\mathcal{E}}, \wedge)$ over $((\Omega^{0,*}(M) \otimes \mathcal{O}_{\mathfrak{X},0}, \overline{\partial}, \wedge), i)$.*

Now we will study formal deformation. This means that we are going to study formal power series ring rather than convergent power series ring. We first review the formal power series ring and projective limit a bit. We consider the convergent power series ring $\mathcal{O}_{\mathfrak{X},0}$. It is a local ring and its maximal ideal is $\mathcal{O}_{\mathfrak{X},0,+}$. We consider the quotient ring $\mathcal{O}_{\mathfrak{X},0}/\mathcal{O}^m_{\mathfrak{X},0,+}$. It is finite dimensional as a vector space over $\mathbb{C}$. We put

$$\hat{\mathcal{O}}_{\mathfrak{X},0} = \varprojlim \mathcal{O}_{\mathfrak{X},0}/\mathcal{O}^m_{\mathfrak{X},0,+}. \tag{1.27}$$

Here the right hand side is the projective limit. Let us recall its definition here for the convenience of the reader. We remark that there exists an obvious homomorphism $\pi_{m,m'} : \mathcal{O}_{\mathfrak{X},0}/\mathcal{O}^m_{\mathfrak{X},0,+} \to \mathcal{O}_{\mathfrak{X},0}/\mathcal{O}^{m'}_{\mathfrak{X},0,+}$ for $m < m'$.

Definition 1.7.4. Let S_m is a set for each $m \in \mathbb{Z}_{>0}$ and $\pi_{m,m'} : S_m \to S_{m'}$ be a map for $m < m'$ such that $\pi_{m',m''} \circ \pi_{m,m'} = \pi_{m,m''}$. We consider the direct product $\prod S_m$. The *projective limit* $\varprojlim S_m$ is a subset of $\prod S_m$ consisting of $(x_1, x_2, \cdots)$ such that $\pi_{m,m'}(x_m) = x_{m'}$.

In case when S_m are groups, rings, modules etc. and $\pi_{m,m'}$ are homomorphisms, the projective limit $\varprojlim S_m$ is also a group, ring, module etc.

Remark 1.7.2. We defined projective limit only for family S_m parametrized by $m \in \mathbb{Z}_{\geq 0}$. Projective limit can be defined for more general family.

Remark 1.7.3. We define $d : \varprojlim S_m \times \varprojlim S_m \to \mathbb{R}_{\geq 0}$ by

$$d((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = \exp\left(-\inf\{m \,|\, x_m \neq y_m\}\right).$$

Then $(\varprojlim S_m, d)$ is a *complete* metric space.

Example 1.7.1. Let us consider the case when $\mathfrak{X} = \mathcal{U}$ is an open neighborhood of 0 in $\mathbb{C}^N$. In this case, $\mathcal{O}_{\mathcal{U},0}$ is a convergent power series ring $\mathbb{C}\{z_1, \cdots, z_N\}$ of N variables. Then $\mathcal{O}_{\mathcal{U},0}/\mathcal{O}^m_{\mathcal{U},0,+} = \mathbb{C}[z_1, \cdots, z_N]/(z_1, \cdots, z_N)^m$ is the set of all polynomials modulo the terms of order $> m$. Now let $(P_1, \cdots, P_m, \cdots) \in \prod \mathcal{O}_{\mathcal{U},P_0,0}/\mathcal{O}^m_{\mathcal{U},P_0,0}$ be an element of projective limit $\varprojlim \mathcal{O}_{\mathfrak{X},0}/\mathcal{O}^m_{\mathfrak{X},0,+}$. It means that P_m coincides with $P_{m'}$ up to order m. Therefore, $(P_1, \cdots, P_m, \cdots)$ determines a formal power series of z_i. Namely we have $\hat{\mathcal{O}}_{\mathcal{U},x_0,0} \cong \mathbb{C}[[z_1, \cdots, z_N]]$. Here $\mathbb{C}[[z_1, \cdots, z_N]]$ is a formal power series ring.

In general, if the ideal $\mathfrak{I}_{\mathfrak{X},0}$ is generated by $f_1, \cdots, f_m \in \mathcal{O}_0$ then

$$\hat{\mathcal{O}}_{\mathfrak{X},0} \cong \frac{\mathbb{C}[[z_1, \cdots, z_N]]}{(f_1, \cdots, f_m)}. \tag{1.28}$$

Here we regard $f_i \in \mathbb{C}[[z_1, \cdots, z_N]]$ by taking its Taylor series at 0 and $(f_1, \cdots, f_m)$ is an ideal generated by them.

We remark that the ring $\mathcal{O}_{\mathfrak{X},0}/\mathcal{O}^m_{\mathfrak{X},+}$ in (1.27) is of finite dimension over $\mathbb{C}$ (as a vector space). (In the case of a general local ring $\mathcal{R}$ over R, the ring $\mathcal{R}/\mathcal{R}^m_+$ is an Artin algebra over R.) In other words, formal power series ring can be regarded as a projective limit of finite dimensional $\mathbb{C}$

algebras (or of Artin R algebras). This is a reason why theory of formal deformation is based on "functor from Artin ring" [93], which we review here.

To make exposition elementary, we only consider the case when R is an algebraically closed field. (We usually take $R = \mathbb{C}$.) In this case, we do not need the notion of Artin R algebra and consider only an R algebra of finite dimension (as a vector space over R).

Definition 1.7.5. We define the category of finite dimensional local R algebra, abbreviated by $\{\text{f. d. Alg.}/R\}$, as follows. (1) Its object is a local R algebra $\mathcal{R}$ which is commutative with unit and is finite dimensional over R as a vector space. (2) The morphism $\mathcal{R} \to \mathcal{R}'$ is an R algebra homomorphism.

Definition 1.7.6. A *formal moduli functor* is a covariant functor from $\{\text{f. d. Alg.}/R\}$ to $\{\text{Sets}\}$, the category of sets.

For the reader who is not familiar with category theory, let us review what Definitions 1.7.5, 1.7.6, mean. (See [20] Exposé 11 for more detail on the relation of category theory to the theory of moduli.) Let $\mathfrak{F}$ be a formal moduli functor in the sense of Definition 1.7.6. It consists of two kinds of data.

One is $\mathfrak{F}_0$ which associates a set $\mathfrak{F}_0(\mathcal{R})$ to any local ring $\mathcal{R}$ which is commutative with unit and is of finite dimensional over R.

Let $\mathcal{R}$, $\mathcal{R}'$ be two such rings and let $\varphi : \mathcal{R} \to \mathcal{R}'$ be an R algebra homomorphism. Then the second data $\mathfrak{F}_1$ associates to φ a map $\mathfrak{F}_1(\varphi) : \mathfrak{F}_0(\mathcal{R}) \to \mathfrak{F}_0(\mathcal{R}')$.

The condition for $\mathfrak{F}_0$, $\mathfrak{F}_1$ to define a covariant functor is $\mathfrak{F}_1(\varphi' \circ \varphi) = \mathfrak{F}_1(\varphi') \circ \mathfrak{F}_1(\varphi)$, where $\varphi' : \mathcal{R}' \to \mathcal{R}''$.

Our main example of formal moduli functor is one such that $\mathfrak{F}_0(\mathcal{R})$ is the set of isomorphism classes of deformation of a given differential graded modules. To define it we first define :

Definition 1.7.7. Let $(C_{\mathcal{R}}, i)$, $(C'_{\mathcal{R}}, i')$ be deformations of differential graded A module C over $\mathcal{R}$. We say that $(C_{\mathcal{R}}, i)$ is isomorphic to $(C'_{\mathcal{R}}, i')$ if there exists an isomorphism $\Phi : C_{\mathcal{R}} \to C'_{\mathcal{R}}$ of differential graded $A_{\mathcal{R}}$ modules such that the induced isomorphism $\overline{\Phi} : C_{\mathcal{R}}/\mathcal{R}_+ C_{\mathcal{R}} \to C'_{\mathcal{R}}/\mathcal{R}_+ C'_{\mathcal{R}}$ satisfies $i' \circ \overline{\Phi} = i$.

We now define a functor $\mathfrak{Def}_C : \{\text{f. d. Alg.}/R\} \to \{\text{Sets}\}$ for each differential graded A module C, where A is a differential graded ring over R. Let $\mathcal{R}$ be an object of $\{\text{f. d. Alg.}/R\}$.

Definition 1.7.8. $\mathfrak{Def}_{C,0}(\mathcal{R})$ is the set of all isomorphism classes of deformations of C over $\mathcal{R}$. Let $\varphi : \mathcal{R} \to \mathcal{R}'$ be a morphism of the category $\{\mathrm{f.\,d.\,Alg.}\,/R\}$. Let $C_{\mathcal{R}}$ be a deformation of C over $\mathcal{R}$. We put $\mathfrak{Def}_{C,1}(\varphi)(C_{\mathcal{R}}) = C_{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{R}'$.

We remark that if $C_{\mathcal{R}}$ is isomorphic to $C'_{\mathcal{R}}$ in the sense of Definition 1.7.7 then $C_{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{R}'$ is isomorphic to $C'_{\mathcal{R}} \otimes_{\mathcal{R}} \mathcal{R}'$. Hence $\mathfrak{Def}_{C,1}(\varphi) : \mathfrak{Def}_{C,0}(\mathcal{R}) \to \mathfrak{Def}_{C,0}(\mathcal{R}')$ is well-defined. It is easy to see that $\mathfrak{Def}_C$ is a covariant functor : $\{\mathrm{f.\,d.\,Alg.}\,/R\} \to \{\mathrm{Sets}\}$. Namely we can check $\mathfrak{Def}_{C,1}(\varphi') \circ \mathfrak{Def}_{C,1}(\varphi) = \mathfrak{Def}_{C,1}(\varphi' \circ \varphi)$.

To show a relation of moduli functor to moduli space, we need to discuss representability of the functor. In our case of formal deformation theory, we need a notion of pro-representability, which we define below.

Definition 1.7.9. Let $\mathfrak{R}$ be a local R algebra (commutative with unit). We say that $\mathfrak{R}$ is a *pro* $\{\mathrm{f.\,d.\,Alg.}\,/R\}$ *object* if the following holds. (1) Let us denote by $\mathfrak{R}_+$ the maximal ideal of $\mathfrak{R}$. Then, for each m, the quotient ring $\mathfrak{R}/\mathfrak{R}_+^m$ is an object of $\{\mathrm{f.\,d.\,Alg.}\,/R\}$. (In other words, $\mathfrak{R}/\mathfrak{R}_+^m$ is of finite dimension over R.) (2) $\mathfrak{R} \cong \varprojlim \mathfrak{R}/\mathfrak{R}_+^m$. In other words, $\mathfrak{R}$ is complete with respect to the $\mathfrak{R}_+$ adic metric, (which is defined as in Remark 1.7.3).

Definition 1.7.10. Let $\mathfrak{R}$ be a pro $\{\mathrm{f.\,d.\,Alg.}\,/R\}$ object. We define a covariant functor $\mathfrak{F}_{\mathfrak{R}} : \{\mathrm{f.\,d.\,Alg.}\,/R\} \to \{\mathrm{Sets}\}$ as follows. (1) For an object $\mathcal{R}$ of $\{\mathrm{f.\,d.\,Alg.}\,/R\}$, $\mathfrak{F}_{\mathfrak{R},0}(\mathcal{R})$ is the set of all R algebra homomorphisms $: \mathfrak{R} \to \mathcal{R}$. (2) Let $\varphi : \mathcal{R} \to \mathcal{R}'$ be a morphism in the category $\{\mathrm{f.\,d.\,Alg.}\,/R\}$. Let $\psi \in \mathfrak{F}_{\mathfrak{R},0}(\mathcal{R})$. Then $(\mathfrak{F}_{\mathfrak{R},1}(\varphi))(\psi) = \varphi \circ \psi$.

In case $\mathfrak{R}$ itself is an object of $\{\mathrm{f.\,d.\,Alg.}\,/R\}$ (namely $\mathfrak{R}$ is of finite dimension over R), then the functor $\mathfrak{F}_{\mathfrak{R}}$ defined in Definition 1.7.10 is the functor represented by $\mathfrak{R}$ in the usual sense of category theory.

Definition 1.7.11. A covariant functor $\{\mathrm{f.\,d.\,Alg.}\,/R\} \to \{\mathrm{Sets}\}$ is said to be *pro representable* if there exists a pro $\{\mathrm{f.\,d.\,Alg.}\,/R\}$ object $\mathfrak{R}$ such that $\mathfrak{F}$ is equivalent to $\mathfrak{F}_{\mathfrak{R}}$ defined in Definition 1.7.10

We recall that two functors $\mathfrak{F}, \mathfrak{F}' : \{\mathrm{f.\,d.\,Alg.}\,/R\} \to \{\mathrm{Sets}\}$ are said to be equivalent to each other if the following holds : For each object $\mathcal{R}$ of $\{\mathrm{f.\,d.\,Alg.}\,/R\}$ there exists a bijection $\mathcal{H}_{\mathcal{R}} : \mathfrak{F}_0(\mathcal{R}) \to \mathfrak{F}'_0(\mathcal{R})$ such that the following diagram commutes for any morphism $\varphi : \mathcal{R} \to \mathcal{R}'$.

$$\begin{array}{ccc}
\mathfrak{F}_0(\mathcal{R}) & \xrightarrow{\mathcal{H}_{\mathcal{R}}} & \mathfrak{F}'_0(\mathcal{R}) \\
{\scriptstyle \mathfrak{F}_1(\varphi)}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathfrak{F}'_1(\varphi)} \\
\mathfrak{F}_0(\mathcal{R}') & \xrightarrow{\mathcal{H}_{\mathcal{R}'}} & \mathfrak{F}'_0(\mathcal{R}')
\end{array}$$

Diagram 3

We now define universal formal moduli space of deformation of differential graded module. Let A be a differential graded algebra over R and C be a differential graded A module.

Definition 1.7.12. A pro $\{\text{f. d. Alg.}/R\}$ object $\mathfrak{R}$ is said to be a *universal formal moduli space* of the deformation of C if the functor $\mathfrak{F}_{\mathfrak{R},0}(\mathcal{R})$ in Definition 1.7.10 is equivalent to the functor $\mathfrak{Def}_C$ in Definition 1.7.8.

Remark 1.7.4. It is more precise to say that $\operatorname{Spec}\mathfrak{R}$ is a universal formal moduli space rather than to say $\mathfrak{R}$ is a universal moduli space. Here $\operatorname{Spec}\mathfrak{R}$ is a formal scheme (see [45]). Since we do not introduce the notion of formal scheme, we say $\mathfrak{R}$ is a moduli space by abuse of notation.

Exercise 1.7.1. Prove that universal formal moduli space in the sense of Definition 1.7.12 is unique if it exists.

Now we consider our geometric situation of deformation of holomorphic vector bundles. Let $\mathfrak{X} = (X, \mathfrak{I}_{\mathfrak{X},0})$ be an analytic subspace and $B : \mathcal{U} \to \Omega^{0,1}(M; End(E))$ be a deformation of $\mathcal{E}$ parametrized by $\mathcal{X}$. We define $\hat{\mathcal{O}}_{\mathfrak{X},0}$ by (1.27). $\hat{\mathcal{O}}_{\mathfrak{X},0}$ is a pro $\{\text{f. d. Alg.}/\mathbb{C}\}$ object.

Proposition 1.7.1. *If the deformation B of $\mathcal{E}$ is universal, then $\hat{\mathcal{O}}_{\mathfrak{X},0}$ is a universal moduli space of the deformation of $(\Omega^*(M; \mathcal{E}|_{M\times\{x_0\}}), d, \wedge)$.*

On the contrary if there exists a universal moduli space $\mathfrak{R}$ of the deformation of $(\Omega^*(M; \mathcal{E}|_{M\times\{x_0\}}), d, \wedge)$, then there exists a universal family of holomorphic structures of $\mathfrak{R}$ on $\mathfrak{X}$ such that $\hat{\mathcal{O}}_{\mathfrak{X},0}$ is isomorphic to $\mathfrak{R}$.

Proof. (sketch) First we assume that our family is universal. Let $\mathfrak{R}$ be an object of $\{\text{f. d. Alg.}/\mathbb{C}\}$. We may write $\mathcal{R} = \mathbb{C}[[z_1, \cdots, z_N]]/(f_1, \cdots, f_k)$. f_i is a priori a formal power series. However using finite dimensionality of $\mathcal{R}$ we may take polynomials for f_i.

We consider a germ of analytic subspace $\mathfrak{Y} = (\{0\}, (f_1, \cdots, f_k))$. It is easy to see that a morphism $\mathfrak{Y} \to \mathfrak{X}$ corresponds one to one to the $\mathbb{C}$ algebra homomorphism $\hat{\mathcal{O}}_{\mathfrak{X},0} \to \mathcal{R}$. It is an immediate consequence of Lemma 1.7.2 (which still holds in the case of deformation parametrized by a germ of analytic subspace) that the deformation of $\mathcal{E}$ parametrized by $\mathfrak{Y}$ corresponds one to one to the deformation of $(\Omega^*(M; \mathcal{E}|_{M\times\{x_0\}}), d, \wedge)$ over $\mathcal{R}$. We can then prove easily that $\hat{\mathcal{O}}_{\mathfrak{X},0}$ is a universal moduli space of the deformation of $(\Omega^*(M; \mathcal{E}|_{M\times\{x_0\}}), d, \wedge)$.

The proof of converse is more involved since we need to see the relation between the deformation in the category of formal power series and of convergent power series. We do not try to do it here. $\square$

Chapter 2

Homological algebra and Deformation theory

2.1 Homotopy theory of A_∞ and L_∞ algebras.

Now we are going to discuss less classical part of the story. We have so far studied the equation

$$\overline{\partial}B + B \circ B = 0. \tag{2.1}$$

The second term is of second order. We mentioned in §1.5 a possibility to consider the equation where there are terms of third or of higher order. To do so while keeping gauge invariance of the equation, we need to consider A_∞ algebra, due to J.Stasheff [98], which we define in this section. In case we consider the deformation of complex manifolds rather than holomorphic vector bundle on it, we need to consider the equation

$$\overline{\partial}B + \frac{1}{2}[B, B] = 0, \tag{2.2}$$

in place of (2.1). Here $B \in \Omega^{0,1}(M; TM)$ and $[B, B]$ is a combination of wedge product in $\Lambda^{0,1}$ factor and the bracket of vector field in TM factor. (See Example 2.1.1.) To generalize (2.2) so that it includes terms of third or of higher order we introduce the notion, L_∞ algebra. [1]

To define A_∞ and L_∞ algebras we need to review coalgebra, coderivation etc.. Let C be a graded R module. (Here grading starts from 0.) We define its suspension $C[1]$ by $C[1]^k = C^{k+1}$. Hereafter we denote $\deg x$ as the degree of element of $x \in C$ and $\deg' x$ is the degree of the same element regarded as an element of $C[1]$. Namely $\deg' x = \deg x - 1$. We define its

[1] We remark that A in A_∞ algebra stands for associative and L in L_∞ algebra stands for Lie.

Bar complex $BC[1]$ by

$$B_kC[1] = C[1]^{k\otimes}, \qquad BC[1] = \bigoplus_{k=0}^{\infty} B_kC[1].$$

We remark $B_0C[1] = R$. We define the action of the group $\mathfrak{S}_k$ of all permutations of k elements on $B_kC[1]$ by

$$\sigma(x_1 \otimes \cdots \otimes x_k) = \pm x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

where

$$\pm = (-1)^{\sum_{i,\,j \text{ with } i<j,\ \sigma(i)>\sigma(j)} \deg' x_i \deg' x_j}. \tag{2.3}$$

We define $E_kC[1]$ to be the submodule consisting of fixed points of the $\mathfrak{S}_k$ action on $B_kC[1]$ and $EC[1] = \oplus E_kC[1]$.

We define $\Delta : BC[1] \to BC[1] \otimes BC[1]$ by

$$\Delta(x_1 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k). \tag{2.4}$$

Note the term of (2.4) in case of $i = 0$ is $1 \otimes (x_1 \otimes \cdots \otimes x_k) \in B_0C[1] \otimes B_kC[1]$. The restriction of Δ induces $\Delta : EC[1] \to EC[1] \otimes EC[1]$.

Definition 2.1.1. A graded coalgebra $(D = \oplus D^k, \Delta, \epsilon)$ is a graded R module together with $\Delta : D \to D \otimes D$, $\epsilon : D^0 \to R$ such that the following diagrams commute.

$$\begin{tikzcd} D \otimes_R D \otimes D & D \otimes D \arrow[l, "\Delta\otimes 1"'] \\ D \otimes D \arrow[u, "1\otimes\Delta"] & D \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

Diagram 4

$$\begin{tikzcd} D & D \otimes D \arrow[l, "\epsilon\otimes 1"'] \arrow[r, "1\otimes\epsilon"] & D \\ D \arrow[u, equal] \arrow[r, equal] & D \arrow[u, "\Delta"'] \arrow[r, equal] & D \arrow[u, equal] \end{tikzcd}$$

Diagram 5

Coalgebra (D, Δ, ϵ) is said to be *graded cocomutative* if $R \circ \Delta = \Delta : D \to D \otimes D$, where $R(x \otimes y) = (-1)^{\deg x \deg y}(y \otimes x)$.

The following lemma is easy to check.

Lemma 2.1.1. $(BC[1], \Delta, \epsilon)$ *is a colagebra where* ϵ *is an obvious isomorphism* $B_0C[1] \cong R$. *We define a degree of elements of* $BC[1]$ *by* $\deg'(x_1 \otimes \cdots \otimes x_k) = \sum \deg' x_i = \sum \deg x_i - k$. $(EC[1], \Delta, \epsilon)$ *is a graded cocomutative coalgebra. (We remark that we need to take* $\deg'$ *and not* $\deg$ *in the definition of* R*.)*

Definition 2.1.2. A graded homomorphism $\delta : D \to D$ of degree 1 from a coalgebra D to itself is said to be a *coderivation* if the following diagram commutes.

$$\begin{tikzcd} D \otimes D & D \arrow[l, "\Delta"'] \\ D \otimes D \arrow[u, "1\hat{\otimes}\delta+\delta\hat{\otimes}1"] & D \arrow[l, "\Delta"] \arrow[u, "\delta"'] \end{tikzcd}$$

Diagram 6 .

Here we define the graded tensor product $A\hat{\otimes}B$ between two graded homomorphisms A,B by $(A\hat{\otimes}B)(x\otimes y) = (-1)^{\deg B \deg' x}(A(x)) \otimes (B(y))$.

Lemma 2.1.2. *For any sequence of homomorphisms $f_k : B_kC[1] \to C[1]$ of degree 1 for $k = 1, 2, \cdots$, there exists uniquely a coderivation $\delta : BC[1] \to BC[1]$ whose $Hom(B_kC[1], B_1C[1])$ component is f_k and whose $Hom(B_0C[1], B_1C[1])$ component is zero. The same holds for $EC[1]$ in place of $BC[1]$.*

Proof. Put

$$\begin{aligned} \hat{f}_k(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n-k+1} & (-1)^{\deg f_k(\deg' x_1+\cdots+\deg' x_{i-1})} \\ & x_1 \otimes \cdots \otimes x_{i-1} \otimes f_k(x_i \otimes \cdots \otimes x_{i+k-1}) \\ & \otimes x_{i+k} \otimes \cdots \otimes x_n, \end{aligned} \tag{2.5}$$

and $\delta = \sum \hat{f}_k$. □

To simplify the formulae like (2.5) we introduce the following notation. Let D be a coalgebra. We define $\Delta^k : D \to D^{k\otimes}$ by

$$\Delta^k = \underbrace{\cdots (\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1) \circ \Delta}_{k-1 \text{ times}}.$$

Then, for an element $\mathbf{x}$ of D, we put

$$\Delta^k(\mathbf{x}) = \sum_a \mathbf{x}_a^{(k;1)} \otimes \cdots \otimes \mathbf{x}_a^{(k;k)}. \tag{2.6}$$

In general, we use bold face letter such as $\mathbf{x}$ for elements of the bar complex $BC[1]$ and roman letter such as x_k for elements of $C[1]$.

Now Formula (2.5) can be written

$$\hat{f}_k(\mathbf{x}) = \sum_a (-1)^{\deg f_k \deg' \mathbf{x}_a^{(3;1)}} \mathbf{x}_a^{(3;1)} \otimes f_k(\mathbf{x}_a^{(3;2)}) \otimes \mathbf{x}_a^{(3;1)}.$$

Now we are ready to define A_∞ algebra and L_∞ algebra.

Definition 2.1.3. A structure of A_∞ *algebra* on $C[1]$ is a series of R module homomorphisms $\mathfrak{m}_k : B_kC[1] \to C[1]$ $(k = 1, 2, \cdots)$ of degree $+1$ such that the coderivation δ obtained by Lemma 2.1.2 satisfies $\delta\delta = 0$. If we replace B by E, then it will be the definition of structure of L_∞ *algebra* on $C[1]$.

We can write the condition $\delta\delta = 0$ more explicitly as follows.

$$\begin{aligned}\sum_{i=1}^{n-k+1} & (-1)^{\deg' x_1+\cdots+\deg' x_{i-1}} \\ & \mathfrak{m}_{n-k+1}(x_1 \otimes \cdots \otimes x_{i-1} \otimes \mathfrak{m}_k(x_i \otimes \cdots \otimes x_{i+k-1}) \\ & \qquad \otimes x_{i+k} \otimes \cdots \otimes x_n) = 0.\end{aligned} \tag{2.7}$$

In particular we have $\mathfrak{m}_1\mathfrak{m}_1 = 0$. Hence we can define $\mathfrak{m}_1$ cohomology $H(C, \mathfrak{m}_1)$.

Let $(C, d, \cdot)$ be a graded differential algebra. We put

$$\mathfrak{m}_1(x) = (-1)^{\deg x} dx, \qquad \mathfrak{m}_2(x, y) = (-1)^{\deg x(\deg y+1)} x \cdot y,$$

and $\mathfrak{m}_k = 0$ for $k \geq 3$.

Lemma 2.1.3. *$\mathfrak{m}_k$ determines a structure of A_∞ algebra on C.*

Proof. It suffices to show that $Hom(B_kC[1], C[1])$ component of $\delta\delta$ is zero for $k = 1, 2, 3$. The case $k = 1$ is obvious. Let us check the case $k = 3$, and leave the case $k = 2$ to the reader. Let $\pi_1 : BC[1] \to B_1C[1] = C[1]$ is the projection. Then we have :

$$\begin{aligned}\pi_1\delta\delta(x \otimes y \otimes z) &= (-1)^{\deg' x}\mathfrak{m}_2(x, \mathfrak{m}_2(y, z)) + \mathfrak{m}_2(\mathfrak{m}_2(x, y), z) \\ &= (-1)^{\deg x+1+\deg y(\deg z+1)+\deg x(\deg y+\deg z+1)} x \cdot (y \cdot z) \\ &\quad + (-1)^{\deg x(\deg y+1)+(\deg x+\deg y)(\deg z+1)} (x \cdot y) \cdot z = 0.\end{aligned}$$

The last equality follows from the associativity of $\cdot$. (Change of degree and sign is important so that the relation $\delta\delta = 0$ becomes associativity relation.) □

We next discuss the L_∞ case.

Definition 2.1.4. A *differential graded Lie algebra* is a graded R module C together with operations $[\ ,\ ] : C \otimes C \to C$ of degree 0 and $d : C \to C$ of degree 1 such that $dd = 0$ and (1) $d[x, y] = [dx, y] + (-1)^{\deg x}[x, dy]$. (2) $[x, y] = (-1)^{\deg x \deg y+1}[y, x]$. (3) $[[x, y], z] + (-1)^{(\deg x+\deg y)\deg z}[[z, x], y] + (-1)^{(\deg y+\deg z)\deg x}[[y, z], x] = 0$.

Example 2.1.1. Let M be a complex manifold and $T_{\mathbb{C}}M$ be a complex tangent bundle. (Namely holomorphic vector bundle whose local frame is $\frac{\partial}{\partial z^i}$ where z^i, $i = 1, \cdots, n$ is a local complex coordinate.) Let $C^k = \Gamma^k(M; T_{\mathbb{C}}M \otimes \Lambda^{0,k})$. We put $d = \overline{\partial}$ and

$$\begin{aligned}&\left[f\ \frac{\partial}{\partial z^i} \otimes d\overline{z}^{i_1} \wedge \cdots \wedge d\overline{z}^{i_k}, g\ \frac{\partial}{\partial z^j} \otimes d\overline{z}^{j_1} \wedge \cdots d\overline{z}^{j_\ell}\right]\\ &= \left(f\frac{\partial g}{\partial z^i}\frac{\partial}{\partial z^j} - g\frac{\partial f}{\partial z^j}\frac{\partial}{\partial z^i}\right) \otimes d\overline{z}^{i_1} \wedge \cdots d\overline{z}^{i_k} \otimes d\overline{z}^{j_1} \wedge \cdots d\overline{z}^{j_\ell}.\end{aligned}$$

We obtain a differential graded Lie algebra.

Let $(C, [\ ,\ ], d)$ be a differential graded Lie algebra. We put

$$\mathfrak{m}_1(x) = (-1)^{\deg x}dx, \qquad \mathfrak{m}_2(x, y) = (-1)^{\deg x(\deg y+1)}[x, y],$$

and $\mathfrak{m}_k = 0$ for $k \geq 3$.

Lemma 2.1.4. *$\mathfrak{m}_k$ determines a structure of L_∞ algebra on C.*

Proof. Let us first remark that $\mathfrak{m}_2$ is a well-defined operator on $E_2C[1]$. Namely we have

$$\begin{aligned}\mathfrak{m}_2(x, y) &= (-1)^{\deg x(\deg y+1)}[x, y] = (-1)^{\deg x(\deg y+1)+\deg x \deg y+1}[y, x]\\ &= (-1)^{\deg x(\deg y+1)+\deg x \deg y+1+\deg y(\deg x+1)}\mathfrak{m}_2(y, x)\\ &= (-1)^{(\deg x+1)(\deg y+1)}\mathfrak{m}_2(y, x).\end{aligned}$$

It then suffices to show that $Hom(E_kC[1], C[1])$ component of $\delta\delta$ is zero for $k = 1, 2, 3$. The case $k = 1$ is obvious. Let us check the case $k = 3$, and leave the case $k = 2$ to the reader. Let $\pi_1 : EC[1] \to E_1C[1] = C[1]$ is the projection.

We put $x_1 \times \cdots \times x_k = \sum_{\sigma\in\mathfrak{S}_k} \pm x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$, were $\pm$ is as in Formula (2.3). Then we have :

$$\begin{aligned}&\frac{1}{2}\pi_1\delta\delta(x \times y \times z)\\ &\quad= \mathfrak{m}_2(\mathfrak{m}_2(x, y), z) + (-1)^{\deg' z(\deg' x+\deg' y)}\mathfrak{m}_2(\mathfrak{m}_2(z, x), y)\\ &\qquad+ (-1)^{\deg' x(\deg' y+\deg' z)}\mathfrak{m}_2(\mathfrak{m}_2(y, z), x)\\ &\quad= (-1)^{\deg x(\deg y+1)+(\deg x+\deg y)(\deg z+1)}[[x, y], z]\\ &\qquad+ (-1)^{\deg z(\deg x+1)+(\deg y+\deg z)(\deg x+1)+(\deg z+1)(\deg x+\deg y)}[[z, x], y]\\ &\qquad+ (-1)^{\deg y(\deg z+1)+(\deg y+\deg z)(\deg x+1)+(\deg x+1)(\deg y+\deg z)}[[y, z], x]\\ &\quad= 0.\end{aligned}$$

(Note the calculation above is a bit problematic in case 2 is not invertible. We do not try to correct it since the case when R is a field of characteristic zero is our main concern). ∎

Our next purpose is to define homotopy equivalence of A_∞ and L_∞ algebras. For this purpose we first define A_∞ and L_∞ homomorphisms.

Definition 2.1.5. Let (D,Δ,ϵ), (D',Δ',ϵ') be a coalgebras. An R module homomorphism $\varphi : D \to D'$ of degree 0 is said to be a *coalgebra homomorphism* if the following diagram commutes.

$$\begin{array}{ccc} D'\otimes D' & \xleftarrow{\Delta'} & D' \\ {\scriptstyle\varphi\otimes\varphi}\big\uparrow & & {\scriptstyle\varphi}\big\uparrow \\ D\otimes D & \xleftarrow{\Delta} & D \end{array} \qquad\qquad \begin{array}{ccc} R & \xleftarrow{\epsilon'} & D' \\ \big\| & & {\scriptstyle\varphi}\big\uparrow \\ R & \xleftarrow{\epsilon} & D \end{array}$$

Diagram 7 Diagram 8

Lemma 2.1.5. *Let $\varphi_k : B_kC[1] \to C'[1]$, $k = 1,2,\cdots$ be a sequence of degree 0 R module homomorphisms. Then there exists a unique colagebra homomorphism $\hat\varphi : BC[1] \to BC'[1]$ such that its $Hom(B_kC[1],B_1C'[1])$ component is φ_k. The same statement holds if we replace B by E.*

Proof. Let $\varphi : BC[1] \to C'[1]$ be a homomorphism whose $Hom(B_kC[1],B_1C'[1])$ component is φ_k. We then set (using notation (2.6))

$$\hat\varphi(\mathbf{x}) = \sum_k\sum_a \varphi(\mathbf{x}_a^{k;1})\otimes\cdots\otimes\varphi(\mathbf{x}_a^{k;k}).$$

It is easy to check that this $\hat\varphi$ has a required property. The L_∞ case is similar. $\square$

Definition 2.1.6. Let $(C[1],\mathfrak{m}_k)$, $(C'[1],\mathfrak{m}'_k)$ be A_∞ algebras. A sequence of homomorphisms $\varphi_k : B_kC[1] \to C'[1]$ is said to be an A_∞ *homomorphism* if the coalgebra homomorphism $\hat\varphi : BC[1] \to BC'[1]$ obtained by Lemma 2.1.5 satisfies $\hat\varphi\circ\delta = \delta\circ\hat\varphi$. The definition of L_∞ *homomorphism* is similar.

We can define a composition of A_∞ and L_∞ homomorphisms by $\widehat{\varphi\circ\psi} = \hat\varphi\circ\hat\psi$.

We next define a homotopy between A_∞ and L_∞ homomorphisms. The definition we give below is an analogy of a similar definition in the case of differential graded algebra given in [41] Chapter X. There is another way to define homotopy which is an analogy of one in [101] (we do not discuss later here).

Let $(C[1],\mathfrak{m}_k)$ be an A_∞ algebra. We define an A_∞ algebra $C[1]\otimes R[t,dt]$ as follows.

Definition 2.1.7. An element of $C[1]\otimes R[t,dt]$ is written as $P(t)+Q(t)dt$, where $P,Q\in C[t]$ are polynomials with coefficient on C. We put $\deg dt = 1$.

The operator $\mathfrak{m}_k$ is defined as follows. Let $x_i = P_i(t) + Q_i(t)dt$

$$\mathfrak{m}_1(P(t) + Q(t)dt) = \mathfrak{m}_1(P(t)) - \mathfrak{m}_1(Q(t))dt - \frac{dP}{dt}dt \tag{2.8}$$

$$\begin{aligned} &\mathfrak{m}_k(x_1, \cdots, x_k) \\ &\quad = \mathfrak{m}_k(P_1, \cdots, P_k) \\ &\qquad + \sum_{i=1}^{k} (-1)^{\deg' P_1 + \cdots \deg' P_{i-1} + 1} \mathfrak{m}_k(P_1, \cdots, Q_i, \cdots, P_k)dt \end{aligned} \tag{2.9}$$

Here we extend $\mathfrak{m}_k$ to $B_k C[t]$ in an obvious way.

In case when $(C[1], \mathfrak{m}_k)$ is an L_∞ algebra, we define $C[1] \otimes R[t, dt]$ in the same way as an R module. The definition of operations $\mathfrak{m}_k$ is also the same by using (2.8),(2.9).

We omit the proof of the A_∞ and L_∞ formulae. (See the final version of [33].) For $t_0 \in \mathbb{R}$, we define an A_∞ homomorphism $\mathrm{Eval}_{t=t_0} : C[1] \otimes R[t, dt] \to C[1]$ by

$$\mathrm{Eval}_{t=t_0}(P(t) + Q(t)dt) = P(t_0). \tag{2.10}$$

More precisely, we define the $B_1(C \otimes R[t, dt])[1] \to C[1]$ component by (2.10) and set all the other components to be 0.

Definition 2.1.8. Two A_∞ homomorphisms $\varphi, \varphi' : C \to C'$ are said to be *homotopic* to each other, if there exists an A_∞ homomorphism $H : C \to C' \otimes R[t, dt]$ such that $\mathrm{Eval}_{t=0} \circ H = \varphi$, $\mathrm{Eval}_{t=1} \circ H = \varphi'$. We define homotoy between L_∞ homomorphisms in the same way.

In Theorems 2.1.1 and 2.1.2 below, we assume R contains $\mathbb{Q}$.

Theorem 2.1.1. *If φ is homotopic to φ' and φ' is homotopic to φ'' then φ is homotopic to φ''.*

In case when C, C' are differential graded algebras, Theorem 2.1.1 is proved in [41]. The general A_∞ algebra case is similar and is proved in detail in [33]. We omit the proof in this article.

Definition 2.1.9. An A_∞ homomorphism $\varphi : C \to C'$ is said to be a *homotopy equivalence* if there exists an A_∞ homomorphism $\psi : C' \to C$ such that the compositions $\psi \circ \varphi$, $\varphi \circ \psi$ are homotopic to identity. Two A_∞ algebra are said to be *homotopy equivalent* if there exists a homotopy equivalence between them. Homotopy equivalence of the L_∞ algebras is defined in the same way.

The following theorem is useful to show that an A_∞ homomorphism to be a homotopy equivalence.

Theorem 2.1.2. *If $\varphi : C \to C'$ is an A_∞ homomorphism which induces an isomorphism on $\mathfrak{m}_1$ cohomology, then φ is a homotopy equivalence. The same holds for L_∞ case.*

Remark 2.1.1. Theorem 2.1.2 does *not* hold in the category of differential graded algebra. Namely if $\varphi : C \to C'$ is a differential graded algebra homomorphism (of degree 0) which induces an isomorphism on cohomology. Then Theorem 2.1.2 implies that we can find a homotopy inverse of it which is an A_∞ homomorphism. However it is in general not possible to find a homotopy inverse which is a differential graded algebra homomorphism.

Theorem 2.1.2 is proved in somewhat weaker version in the 2000 December version of [33]. The proof of the general case will be included in the final version of [33].

2.2 Maurer-Cartan equation and moduli functor.

We now generalize the Maurer-Cartan equation (1.5) to the case of A_∞ and L_∞ algebras. Namely we consider the equation

$$\sum_k \mathfrak{m}_k(b,\cdots,b) = 0 \qquad A_\infty \text{ case}, \tag{2.11a}$$

$$\sum_k \frac{1}{k!}\mathfrak{m}_k(b,\cdots,b) = 0 \qquad L_\infty \text{ case}. \tag{2.11b}$$

where $b \in C[1]^0 = C^1$. Note that (2.11a) coincides with (1.5) in the case of differential graded algebra, and (2.11b) coincides with (2.2) in the case of differential graded Lie algebra.

There is however one trouble to make sense of equations (2.11a), (2.11b). Namely the left hand side is an infinite sum in the case when infinitely many of $\mathfrak{m}_k$ are nonzero. There are two ways to make sense of (2.11a),(2.11b). One is to define a topology on C and consider the case when the left hand side converges. (We may either consider non Archimedean valuation on our coefficient ring ($R = \mathbb{C}[[T]]$ for example) or may consider the convergence in the classical sense ($R = \mathbb{R}$ or $\mathbb{C}$). Both plays a role in the story of mirror symmetry.) The other possibility is to consider b which is nilpotent (namely product of several of them vanishes). This second point is related to the "functor from Artin ring" discussed in §1.7. Let us take this second point of view in this section. (The first point of view also appears later.) Let us again consider the case when R is an algebraically closed field of characteristic zero.

Let $\mathcal{R}$ be a finite dimensional local R algebra (commutative with unit). Let $\mathcal{R}_+$ be the maximal ideal of $\mathcal{R}$. There exists N such that $\mathcal{R}_+^N = 0$. Let C be an A_∞ algebra. $C_{\mathcal{R}} = C \otimes_R \mathcal{R}$ has a structure of A_∞ algebra.

Let $b \in C^1 \otimes_R \mathcal{R}_+$. Obviously $\mathfrak{m}_k(b, \cdots, b) = 0$ if $k > N$. Hence equation (2.11) makes sense.

Definition 2.2.1. b is said to be a *Maurer-Cartan element* of $C_{\mathcal{R}}$ if it satisfies equation (2.11).

To simplify the notation we introduce the following notation.

$$e^b = \sum_{k=0}^{\infty} \underbrace{b \otimes \cdots \otimes b}_{k \text{ times}} \qquad A_\infty \text{ case}, \tag{2.12a}$$

$$e^b = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{b \otimes \cdots \otimes b}_{k \text{ times}} \qquad L_\infty \text{ case}. \tag{2.12b}$$

We remark that $\mathfrak{m} : BC[1] \to C[1]$ is a homomorphism which is $\mathfrak{m}_k$ on $B_kC[1]$. ($\mathfrak{m} : EC[1] \to C[1]$ is similar.)

Then equation (2.11) can be written as $\mathfrak{m}(e^b) = 0$. Before going further let us explain the meaning of equation (2.11). Let us define a deformed boundary operator $\mathfrak{m}_1^b$ by

$$\mathfrak{m}_1^b(x) = \mathfrak{m}(e^b, x, e^b) \qquad A_\infty \text{ case}, \tag{2.13a}$$

$$\mathfrak{m}_1^b(x) = \mathfrak{m}(e^b, x) \qquad L_\infty \text{ case}. \tag{2.13b}$$

Lemma 2.2.1. $\mathfrak{m}_1^b\mathfrak{m}_1^b = 0$ *if and only if* (2.11) *is satisfied.*

The proof is easy and is omitted. We can deform $\mathfrak{m}_k$ also by

$$\mathfrak{m}_k^b(x_1, \cdots, x_k) = \mathfrak{m}(e^b, x_1, e^b, x_1, \cdots, x_{k-1}, e^b, x_k, e^b), \quad A_\infty \text{ case},$$

$$\mathfrak{m}_k^b(x_1, \cdots, x_k) = \mathfrak{m}(e^b, x_1, \cdots, x_k) \qquad L_\infty \text{ case}.$$

Then we obtain A_∞ or L_∞ algebra $(C, \mathfrak{m}_*^b)$. (See [33].)

We now define a functor $\widetilde{\mathcal{MC}}(C) : \{\text{f. d. Alg.}/R\} \to \{\text{Sets}\}$ as follows. $\widetilde{\mathcal{MC}}(C)(\mathcal{R})$ is the set of all Maurer-Cartan elements of $C_{\mathcal{R}}$. If $\psi : \mathcal{R} \to \mathcal{R}'$ is a morphism in $\{\text{f. d. Alg.}/R\}$ and if b is a Maurer-Cartan element of $C_{\mathcal{R}}$, then $(1 \otimes \psi)(b) \in C_{\mathcal{R}'}$ is a Maurer-Cartan elements of $C_{\mathcal{R}'}$. Thus we obtain $\psi_* : \widetilde{\mathcal{MC}}(C)(\mathcal{R}) \to \widetilde{\mathcal{MC}}(C)(\mathcal{R}')$. We thus obtain a covariant functor $\widetilde{\mathcal{MC}}(C) : \{\text{f. d. Alg.}/R\} \to \{\text{Sets}\}$.

However the set $\widetilde{\mathcal{MC}}(C)(\mathcal{R})$ is usually too big. So we divide it by an appropriate gauge equivalence, which we define below.

Definition 2.2.2. Let $b, b' \in \widetilde{\mathcal{MC}}(C)(\mathcal{R})$. We say that b is *gauge equivalent* to b' if there exists an element $\tilde{b} \in \widetilde{\mathcal{MC}}(C \otimes R[t, dt])(\mathcal{R})$ such that $\text{Eval}_{t=0}\, \tilde{b} = b$, $\text{Eval}_{t=1}\, \tilde{b} = b'$. Here $\text{Eval}_{t=t_0} : C_{\mathcal{R}} \otimes R[t, dt] \to C_{\mathcal{R}}$ is an A_∞ homomorphism as in the last section.

In the theorem below, we assume R contains $\mathbb{Q}$.

Theorem 2.2.1. *Let $b, b', b'' \in \widetilde{\mathcal{MC}}(C)(\mathcal{R})$. If b is gauge equivalent to b' and b' is gauge equivalent to b'', then b is gauge equivalent to b''.*

The proof is similar to the proof of Theorem 2.1.1 and will be given in the final version of [33].

Let us rewrite the definition of gauge equivalence in a bit concrete way. Here we only discuss A_∞ case. (L_∞ case is similar.) Let $\tilde b$ is as in Definition 2.2.2. We put $\tilde b = b(t) + c(t)dt$ where $b(t) \in C^1 \otimes \mathcal{R}_+[t]$, $c(t) \in C^0 \otimes \mathcal{R}_+[t]$. The equation (2.11a) for $\tilde b$ becomes

$$\frac{db(t)}{dt} + \sum_{k=1}^{N}\sum_{i=1}^{k} \mathfrak{m}_k(b(t)^{\otimes i-1}, c(t), b(t)^{\otimes k-i}) = 0, \tag{2.14}$$

$$\sum_{k=1}^{N} \mathfrak{m}_k(b(t)^{\otimes k}) = 0. \tag{2.15}$$

The condition $\mathrm{Eval}_{t=0}\,\tilde b = b$, $\mathrm{Eval}_{t=1}\,\tilde b = b'$ is a boundary condition $b(0) = b,\quad b(1) = b'$. We remark that (2.14) and equation (2.11a) for b implies (2.15). In fact $\frac{d}{dt}\sum_{k=1}^{N}\mathfrak{m}_k(b(t)^{\otimes k})$ is the left hand side of (2.14).

We remark that using (2.13a) equation (2.14) can be written as

$$\frac{db(t)}{dt} + \mathfrak{m}_1^{b(t)}(c(t)) = 0. \tag{2.16}$$

Let us consider the case when C is a differential graded algebra and $R = \mathbb{C}$ or $\mathbb{R}$. Then (2.16) is

$$\frac{db(t)}{dt} + d(c(t)) - b(t)\cdot c(t) + c(t)\cdot b(t) = 0. \tag{2.17}$$

Let $g(t)$ be the solution of the differential equation :

$$\frac{dg(t)}{dt} = g(t)\cdot c(t), \qquad g(0) = I \tag{2.18}$$

We remark that we use the fact that R is a ring *with characteristic 0* to solve equation (2.18). We remark $g(t) - 1 \in \mathcal{R}_+$ for each t hence $g(t)$ is invertible in the ring $C^0_{\mathcal{R}}$.

Lemma 2.2.2. $g(t)^{-1}\mathfrak{m}_1^b(g(t)) = b - b(t)$.

Proof. We may assume $b = 0$ by replacing $\mathfrak{m}_1$ by $\mathfrak{m}_1^b$. The lemma is obvious for $t = 0$. By (2.18), we have :

$$\begin{aligned}\frac{d}{dt}\mathfrak{m}_1(g(t)) &= \mathfrak{m}_1(g(t)\cdot c(t)) = d(g(t)\cdot c(t))\\ &= \mathfrak{m}_1(g(t))\cdot c(t) + g(t)\cdot d(c(t)) = -g(t)\cdot b(t)\cdot c(t) + g(t)\cdot d(c(t))\end{aligned}$$

On the other hand, we have, by (2.18)

$$\begin{aligned}&\frac{d}{dt}(g(t)\cdot b(t)) = \frac{d}{dt}g(t)\cdot b(t) + g(t)\cdot\frac{d}{dt}b(t)\\ &= g(t)\cdot c(t)\cdot b(t) - g(t)\cdot d(c(t)) + g(t)\cdot b(t)\cdot c(t) - g(t)\cdot c(t)\cdot b(t).\end{aligned}$$

The lemma follows. $\square$

In case $C = (\Omega^{0,*}(\mathcal{E}), \overline{\partial}_{\mathcal{E}}, \wedge)$ we studied in Chapter 1, we have $g \in \Gamma(End(\mathcal{E}))$, $\mathfrak{m}_1^B = \overline{\partial}_{\mathcal{E}-B}$. Lemma 2.2.2 then implies $\overline{\partial}_{\mathcal{E}+B} = g^{-1}\circ\overline{\partial}_{\mathcal{E}}\circ g$. In other words g is an isomorphism from $(E, \overline{\partial}_{\mathcal{E}+B})$ to $\overline{\partial}_{\mathcal{E}}$. This justifies our terminology, gauge equivalence.

Definition 2.2.3. $\mathcal{MC}(C)(\mathcal{R})$ is the set of all gauge equivalence classes of elements of $\widetilde{\mathcal{MC}}(C)(\mathcal{R})$.

If $\psi : \mathcal{R} \to \mathcal{R}'$ is a morphism in $\{\text{f. d. Alg.}/R\}$, we can construct $\psi_* : \mathcal{MC}(C)(\mathcal{R}) \to \mathcal{MC}(C)(\mathcal{R}')$ in an obvious way. Hence $\mathcal{MC}(C)$ defines a functor : $\{\text{f. d. Alg.}/R\} \to \{\text{Sets}\}$. We call it the *Maurer-Cartan functor* associated to A_∞ or L_∞ algebra C.

Our next goal is to show that Maurer-Cartan functor is homotopy invariant. Let $\varphi_k : B_kC[1] \to C'[1]$ be an A_∞ or L_∞ homomorphism. It induces $\hat{\varphi} : BC[1] \to BC'[1]$, or $EC[1] \to EC'[1]$.

Lemma 2.2.3. *There exists $\varphi_* : C[1]^0 \to C'[1]^0$ such that $\hat{\varphi}(e^b) = e^{\varphi_*(b)}$*

Proof. Put

$$\varphi_*(b) = \varphi(e^b) = \sum_k \varphi_k(b, \cdots, b).$$

Here $\varphi : BC[1] \to C'[1]$ is a homomorphism which is φ_k on $B_kC[1]$. L_∞ case is similar. $\square$

Using the fact that $\hat{\varphi}$ is a chain map we have :

Lemma 2.2.4. *If $b \in \widetilde{\mathcal{MC}}(C)(\mathcal{R})$ then $\varphi_*(b) \in \widetilde{\mathcal{MC}}(C')(\mathcal{R})$.*

We also have the following :

Lemma 2.2.5. *If $b \sim b'$ and if φ_k is homotopic to φ'_k then $\varphi_*(b) \sim \varphi'_*(b')$.*

Proof. Let $H : C \to C' \otimes R[t, dt]$ be as in Definition 2.1.8. It induces $\tilde{H} : C \otimes R[t, dt] \to C' \otimes R[t, dt]$ as follows. We put $H_k(v_1, \cdots, v_k) = H_k^1(v_1, \cdots, v_k) + H_k^2(v_1, \cdots, v_k)dt$, where $H_k^i : B_kC[1] \to C'[1] \otimes R[t]$. We extend H_k^i to $B_k(C[1] \otimes R[t]) \to C'[1] \otimes R[t]$ in an obvious way and denote it by the same symbol. Let $x_i = P_i(t) + Q_i(t)dt$. Then $\tilde H_k(x_1,\cdots,x_k) = P(t) + Q(t)dt$ where

$$\begin{aligned}
P(t) &= H_k^1(P_1(t),\cdots,P_k(t)),\\
Q(t) &= H_k^2(P_1(t),\cdots,P_k(t))\\
&\qquad + \sum_i (-1)^{\deg' P_1+\cdots \deg' P_{i-1}+1} H_k^1(P_1(t),\cdots,Q_i(t),\cdots,P_k(t)).
\end{aligned}$$

It is easy to check that $\tilde H$ is an A_∞ homomorphism. Let $\tilde b \in \widetilde{\mathcal{MC}}(C \otimes R[t,dt])$ such that $\mathrm{Eval}_{t=0}\,\tilde b = b$, $\mathrm{Eval}_{t=1}\,\tilde b = b$. (Definition 2.2.2). Now $\tilde H_*(\tilde b) \in \widetilde{\mathcal{MC}}(C' \otimes R[t,dt])$ and $\mathrm{Eval}_{t=0}\,\tilde H_*(\tilde b) = \varphi_*(b)$, $\mathrm{Eval}_{t=1}\,\tilde H_*(\tilde b) = \varphi'_*(b)$. □

By Lemmata 2.2.4 and 2.2.5 we obtain a map $\varphi_*(\mathcal R) : \mathcal{MC}(C)(\mathcal R) \to \mathcal{MC}(C')(\mathcal R)$. It is easy to see that the following diagram commutes for each morphism $\psi : \mathcal R \to \mathcal R'$ in $\{\mathrm{f.\,d.\,Alg.}/R\}$.

$$\begin{array}{ccc}
\mathcal{MC}(C)(\mathcal R) & \xrightarrow{\psi_*} & \mathcal{MC}(C)(\mathcal R')\\
{\scriptstyle\varphi_*}\downarrow & & \downarrow{\scriptstyle\varphi_*}\\
\mathcal{MC}(C')(\mathcal R) & \xrightarrow{\psi_*} & \mathcal{MC}(C')(\mathcal R')
\end{array}$$

Diagram 9

Commutativity of Diagram 9 implies that φ_* is a natural transformation of Maurer-Cartan functors : $\mathcal{MC}(C) \to \mathcal{MC}(C')$. Moreover Lemma 2.2.5 implies that $\varphi_* : \mathcal{MC}(C) \to \mathcal{MC}(C')$ depends only on homotopy class of φ. The following theorem follows immediately.

Theorem 2.2.2. *If C is homotopy equivalent to C' then Maurer-Cartan functor $\mathcal{MC}(C)$ is equivalent to $\mathcal{MC}(C')$.*

Remark 2.2.1. Theorem 2.2.2 in the cases of differential graded algebra and of differential graded Lie algebra is due to Goldman-Milson [37, 38]. Its generalization to A_∞ or L_∞ algebra seems to have been a folklore and was quoted by several authors (without proof). (For example by Kontsevitch [69].) We give its rigorous proof here, assuming Theorems 2.1.1, 2.1.2, 2.2.1 which will be proved in the final version of [33].

Remark 2.2.2. We can state Theorem 2.2.2 in more functorial way as follows. Let $\{A_\infty\,\mathrm{alg.}/R\}/$ homotopy be the category whose object is the set of all homotopy equivalence classes of A_∞ algebras over R and morphisms are homotopy class of A_∞ homomorphisms [2]. Let

[2] There exists a set theoretical trouble to say "set of all homotopy equivalence classes of A_∞ algebras". We can go around it by introducing universe in the same way as [13].

$\mathfrak{Funk}(\{\mathrm{f.\,d.\,Alg.}\,/R\},\{\mathrm{Sets}\})$ be the category whose objects are the sets of all equivalence classes of covariant functors from $\{\mathrm{f.\,d.\,Alg.}\,/R\}$ to $\{\mathrm{Sets}\}$. Then $\mathcal{MC}$ induces a functor : $\{A_\infty\,\mathrm{alg.}\,/R\}/\,\mathrm{homotopy} \to \mathfrak{Funk}(\{\mathrm{f.\,d.\,Alg.}\,/R\},\{\mathrm{Sets}\})$.

2.3 Canonical model, Kuranishi map, and moduli space.

In this section, we apply Theorem 2.2.2 to construct a versal formal moduli space representing the Maurer-Cartan functor. Theorem 2.2.2 is useful to construct a moduli space because it enables us to replace a given A_∞ algebra to another one which is homotopy equivalent to the original one but is easier to handle. A good representative of each homotopy class for our purpose is a canonical one, which we define below.

Definition 2.3.1. An A_∞ (or L_∞) algebra $(C,\mathfrak{m}_*)$ is said to be *canonical* if $\mathfrak{m}_1 = 0$.

Remark 2.3.1. In [69] Kontsevich called the same notion "minimal" L_∞ algebra. In the case of differential graded algebra the name "minimal" was used by D. Sullivan in [101] for an important notion which is different from Definition 2.3.1. This is the reason we do not call it "minimal" but canonical. The name canonical may be justified by Proposition 2.3.1 below.

We say an A_∞ (or L_∞) homomorphism $\varphi : C \to C'$ is an *isomorphism* if there exists an A_∞ (or L_∞) homomorphism $\varphi' : C' \to C$ such that the compositions $\varphi' \circ \varphi$ and $\varphi \circ \varphi'$ *are equal to* identity. Here the identity A_∞ homomorphism id_* is defined by $\mathrm{id}_1 = \mathrm{id}$ and $\mathrm{id}_k = 0$ for $k \geq 2$.

Proposition 2.3.1. *A homotopy equivalence between canonical A_∞ (or L_∞) algebras is an isomorphism.*

Proof. The condition $\mathfrak{m}_1 = 0$ implies that the $\mathfrak{m}_1$ cohomology of C is isomorphic to C itself. Since homotopy equivalence φ_* induces an isomorphism on $\mathfrak{m}_1$ cohomology it follows that $\varphi_1; C[1] \to C[1]$ is an isomorphism. We can then easily prove that $\hat{\varphi} : BC[1] \to BC[1]$ is an isomorphisms. The converse of it is a cochain map and is a coalgebra map. Hence there exists ψ_k such that $\hat{\psi} = \hat{\varphi}^{-1}$. ψ_k is the inverse of φ, as required. □

Hereafter in this section we assume that R is a field of characteristic 0. We also assume that $H(C,\mathfrak{m}_1)$ is finite dimensional.

Theorem 2.3.1. *There exists a canonical A_∞ algebra C_{can} homotopy equivalent to a given A_∞ algebra C. The same holds for L_∞ algebra.*

Remark 2.3.2. Theorem 2.3.1 was first proved in [54]. See also [46, 78]. There might be some people who found it independently. (For example the author heard of a talk by A. Polishchuk discussing the same theorem in

1998 January at the Winter school held in Harvard University.) The proof below is similar to one by Kontsevich-Soibelman [70], and also to [33] (2000 December version) §A6.

Proof. The argument is similar to one we explained in §1.5. We prove the case of L_∞ algebra to minimize the overlap with the argument of §1.5. Let C be an L_∞ algebra. We put $C^k_{\rm can} = H^k(C, \mathfrak{m}_1)$. We first need an analogue of Theorem 1.4.1. Here we need to use the fact that our coefficient ring is a field. We have

Lemma 2.3.1. *There exists a linear subspaces $H^k \subseteq C^k$ projections Π_{H^k} : $C^k \to H^k$ and R linear maps $G_k : C^k \to C^{k-1}$ such that*

$$G_{k+1} \circ \mathfrak{m}_1 + \mathfrak{m}_1 \circ G_k = 1 - \Pi_{H^k}. \tag{2.19}$$

Proof. We put $Z_k = \operatorname{Ker} \mathfrak{m}_1 : C^k \to C^{k+1}$. Let $H^k \subseteq Z_k$ be a linear subspace such that the restriction of the projection : $Z_k \to C^k_{\rm can} = H^k(C, \mathfrak{m}_1)$ to H^k is an isomorphism. We put $B^k = \operatorname{Im} \mathfrak{m}_1 : C^{k-1} \to C^k$. Then $H^k \oplus B^k = Z_k$. We also choose $I^k \subseteq C^k$ such that $Z_k \oplus I^k = C^k$. It is easy to see that $\mathfrak{m}_1$ induces an isomorphism $I^{k-1} \to B^k$. Let $G^k : B^k \to I^{k-1}$ be an inverse of it. We extend G^k to C^k so that $G^k = 0$ on H^k and on I^k. It is easy to check (2.19). □

We call G_k a *propagator*. We remark that in §1.4 we constructed a similar operator (See (1.13)).

Now we consider the set of trees which satisfies a bit milder condition than Condition 1.5.1. Namely we consider the following :

Condition 2.3.1. Γ satisfies (1),(3),(4),(5) of Condition 1.5.1 and
$(2)'$ If $v \in \mathrm{Vertex}_\int(\Gamma)$ then $\sharp\partial^{-1}_{\rm target}(v) \ge 2$, $\sharp\partial^{-1}_{\rm source}(v) = 1$.

We denote by Gr_k the set of all oriented graphs Γ satisfying Condition 2.3.1. (We do not take ribbon structure here, since we are studying L_∞ algebra and not A_∞ algebra.) We use the notations in §1.4.

Remark 2.3.3. The relation of trees to A_∞ or L_∞ algebra had been known to algebraic topologists for a long time. See for example [11].

We first put $\varphi_1 = \mathrm{id}$ and $\overline{\mathfrak{m}}_1 = 0$, and are going to define

$$\overline{\mathfrak{m}}_\Gamma : E_k C_{\rm can}[1] \to C_{\rm can}[1], \qquad \varphi_\Gamma : E_k C_{\rm can}[1] \to C[1] \tag{2.20}$$

for $\Gamma \in Gr_k$ inductively on k. We will then put : $\overline{\mathfrak{m}}_k = \sum_{\gamma \in Gr_k} \overline{\mathfrak{m}}_\Gamma$, $\varphi_k = \sum_{\gamma \in Gr_k} \varphi_\Gamma$.

Now let us assume that (2.20) is defined for $\Gamma \in Gr_\ell$, $\ell < k$. Let $\Gamma \in Gr_k$. Let $v_{\rm last}$ be its last vertex. Let $e_{\rm last}$ be the unique edge such that $\partial_{\rm target}(e_{\rm last}) = v_{\rm last}$. We remove $[0,1]_{e_{\rm last}}$ together with its two vertices from $|\Gamma|$. Then $|\Gamma|\backslash[0,1]_{e_{\rm last}}$ is a union $\cup|\Gamma_i|$ of several elements $\Gamma_i \in Gr_{k_i}$ with $\sum_{i=1}^{\ell} k_i = k$. We remark that since we are using graph Γ which does not have a particular ribbon structure, there is no canonical way to order $\Gamma_1, \dots, \Gamma_\ell$. So the construction below should be independent of the order.

Let $x_i \in C_{\rm can}[1]$. Let $\mathfrak{S}_k$ be the group of all permutation of k numbers $\{1, \cdots, k\}$. We put

$$y_{1,\sigma} = \overline{\varphi}_{\Gamma_1}(x_{\sigma(1)}, \cdots, x_{\sigma(k_1)}), \cdots, y_{\ell,\sigma} = \overline{\varphi}_{\Gamma_\ell}(x_{\sigma(k-k_\ell+1)}, \cdots, x_{\sigma(k)}),$$

and then

$$\overline{\varphi}_\Gamma(x_1 \otimes \cdots \otimes x_k) = -\sum_{\sigma \in \mathfrak{S}_k} \pm \frac{1}{k_1! \cdots k_\ell!} G\left(\mathfrak{m}_\ell(y_{1,\sigma}, y_{2,\sigma}, \cdots, y_{\ell,\sigma})\right),$$
$$\overline{\mathfrak{m}}_\Gamma(x_1 \otimes \cdots \otimes x_k) = -\sum_{\sigma \in \mathfrak{S}_k} \pm \frac{1}{k_1! \cdots k_\ell!} \Pi_H\left(\mathfrak{m}_\ell(y_{1,\sigma}, y_{2,\sigma}, \cdots, y_{\ell,\sigma})\right),$$

where $\pm$ is as in (2.3) and G is the homomorphism in Lemma 2.3.1.

We now calculate

$$\begin{aligned}
&\sum_{\Gamma \in Gr_k} \mathfrak{m}_1(\varphi_\Gamma(x_1 \otimes \cdots \otimes x_k)) \\
&\qquad = -\sum_{\Gamma \in Gr_k} \sum_{\sigma \in \mathfrak{S}_k} \pm \mathfrak{m}_1(G\left(\mathfrak{m}_\ell(y_{1,\sigma}, y_{2,\sigma}, \cdots, y_{\ell,\sigma})\right)) \\
&\qquad = \sum_{\sigma \in \mathfrak{S}_k} \pm \mathfrak{m}_\ell(y_{1,\sigma}, y_{2,\sigma}, \cdots, y_{\ell,\sigma}) \\
&\qquad\qquad - \sum_{\Gamma \in Gr_k} \sum_{\sigma \in \mathfrak{S}_k} \pm G(\mathfrak{m}_1(\mathfrak{m}_\ell(y_{1,\sigma}, y_{2,\sigma}, \cdots, y_{\ell,\sigma}))).
\end{aligned} \tag{2.21}$$

We are going to calculate the last line of (2.21) using L_∞ relation of $\mathfrak{m}$. We have

$$\begin{aligned}
&-\mathfrak{m}_1(\mathfrak{m}_\ell(y_{1,\sigma}, y_{2,\sigma}, \cdots, y_{\ell,\sigma})) \\
&\quad = \sum_{m>1} \sum_{n_1+\cdots+n_m=\ell} \sum_{\mu \in \mathfrak{S}_m} \pm \mathfrak{m}_m(\mathfrak{m}_{n_1}(y_{\mu(1),\sigma}, \cdots), \cdots, \mathfrak{m}_{n_m}(\cdots, y_{\mu(\ell),\sigma}))).
\end{aligned} \tag{2.22}$$

We let $\hat{\varphi} : EC_{\rm can}[1] \to EC[1]$ be the coalgebra homomorphism induced by φ_k and $\overline{\delta} : EC_{\rm can}[1] \to EC_{\rm can}[1]$ be the coderivation induced by $\overline{\mathfrak{m}}_k$. The coderivation $\delta : EC[1] \to EC[1]$ is induced by $\mathfrak{m}_*$. We now prove the following lemma.

Lemma 2.3.2. *We have $\delta \circ \hat{\varphi} = \hat{\varphi} \circ \overline{\delta}$ and $\overline{\delta} \circ \overline{\delta} = 0$.*

Proof. We prove the equalities on $\mathfrak{G}^k EC_{\rm can}[1] = \oplus_{i \le k} E_i C_{\rm can}[1]$ by induction on k. The case $k = 1$ is obvious. Let $\mathbf{x} \in E_k C_{\rm can}[1]$. By (2.21), (2.22) we have

$$\mathfrak{m}_1(\varphi_k(\mathbf{x})) = -\sum_{\ell>2}\overline{\mathfrak{m}}_\ell(\hat\varphi(\mathbf{x})) + (G\circ(\mathfrak{m}-\mathfrak{m}_1)\circ\delta\circ\hat\varphi)(\mathbf{x}). \tag{2.23}$$

Here $\mathfrak{m}-\mathfrak{m}_1 : EC[1]\to C[1]$ is an operator which is zero on $E_1C[1]$ and is $\mathfrak{m}_\ell$ on $E_\ell C[1]$, $\ell>2$. We want to apply induction hypothesis to calculate $(\delta\circ\hat\varphi)(\mathbf{x})$. We remark the following :

Sulemma 2.3.1. *If Lemma* 2.3.2 *holds on* $\mathfrak{G}^{k-1}EC_{\rm can}[1]$ *then* $\delta\circ\hat\varphi = \hat\varphi\circ\overline{\delta}$ *as an equality of homomorphisms :* $\mathfrak{G}^kEC_{\rm can}[1]/\mathfrak{G}^1EC_{\rm can}[1] \to \mathfrak{G}^kEC[1]/\mathfrak{G}^1EC[1]$.

The proof of sublemma is easy and is omitted.

Since $\mathfrak{m}-\mathfrak{m}_1$ is zero on $\mathfrak{G}^1EC[1]$ it follows from Sublemma and induction hypothesis that

$$(G\circ(\mathfrak{m}-\mathfrak{m}_1)\circ\delta\circ\hat\varphi)(\mathbf{x}) = (G\circ(\mathfrak{m}-\mathfrak{m}_1)\circ\hat\varphi\circ\overline{\delta})(\mathbf{x}).$$

It follows from definition that $G\circ(\mathfrak{m}-\mathfrak{m}_1)\circ\hat\varphi = \hat\varphi$. We thus obtain $\delta\circ\hat\varphi = \hat\varphi\circ\overline{\delta}$ on $\mathfrak{G}^kEC_{\rm can}[1]$. Thus by induction $\delta\circ\hat\varphi = \hat\varphi\circ\overline{\delta}$ holds.

The second formula $\overline{\delta}\circ\overline{\delta}=0$ follows from the first one as follows. We have

$$\hat\varphi\circ\overline{\delta}\circ\overline{\delta} = \delta\circ\delta\circ\hat\varphi = 0. \tag{2.24}$$

Since φ_1 is an isomorphism and $\hat\varphi$ preserves the filtration $\mathfrak{G}$ it follows that $\hat\varphi$ is injective. Hence (2.24) implies $\overline{\delta}\circ\overline{\delta}=0$. □

We thus constructed $\overline{\mathfrak{m}}_k : E_kC_{\rm can}[1]\to C_{\rm can}[1]$, $\varphi_k : E_kC_{\rm can}[1]\to C[1]$. It is immediate from definition that φ_1 induces an isomorphism on $\mathfrak{m}_1$ cohomology. Therefore, by Theorem 2.1.2, φ_k is a homotopy equivalence of L_∞ algebras. The proof of Theorem 2.3.1 is now complete. □

We next use Theorem 2.3.1 (and Theorem 2.2.2) to construct versal family of deformations. We assume that the cohomology group $H^*(C,\mathfrak{m}_1)$ is finite dimensional. (We recall that we assume R to be a field of characteristic zero.) We replace C by a canonical one $C_{\rm can}$ using Theorem 2.3.1. Since the $\mathfrak{m}_1$ cohomology of $C_{\rm can}$ is isomorphic to $C_{\rm can}$ itself, it follows that $C^k_{\rm can}$ is finite dimensional. Let $\mathbf{e}_i$, $i=1,\cdots,b_1$ be a basis of $C^1_{\rm can}[1]$ and $\mathbf{f}_i$, $i=1,\cdots,b_2$ be a basis of $C^2_{\rm can}[1]$. We define elements $P_i\in R[[X_1,\cdots,X_{b_1}]]$, $i=1,\cdots,b_2$ by

$$\sum_{i=1}^{b_2} P_i(X_1,\cdots,X_k)\mathbf{f}_i = \mathfrak{m}\left(\exp(X_1\mathbf{e}_1+\cdots+X_{b_1}\mathbf{e}_{b_1})\right). \tag{2.25}$$

Here $\exp(X_1\mathbf{e}_1+\cdots+X_{b_1}\mathbf{e}_{b_1})$ is as in (2.13). It is easy to see that P_i is well-defined as a formal power series.

Definition 2.3.2. We define a pro $\{\mathrm{f.\,d.\,Alg.}/R\}$ object $\mathfrak{K}_{C_{\mathrm{can}}}$ by

$$\mathfrak{K}_{C_{\mathrm{can}}} \cong \frac{R[[X_1,\cdots,X_{b_1}]]}{(P_1,\cdots,P_{b_2})}.$$

We call $P_1,\cdots,P_{b_2}$ the *formal Kuranishi map.*

Lemma 2.3.3. *If two A_∞ (or L_∞) algebras C_{can} and C'_{can} are homotopy equivalent then $\mathfrak{K}_{C_{\mathrm{can}}}$ is isomorphic to $\mathfrak{K}_{C'_{\mathrm{can}}}$ as R algebras.*

Proof. Let $\mathbf{e}'_i$ and $\mathbf{f}'_j$ be basis of $C'^0_{\mathrm{can}}[1]$, $C'^1_{\mathrm{can}}[1]$ respectively. We define $F_j(X_1,\cdots,X_{b_1})$, $j=1,\cdots,b_1$ by

$$\begin{aligned}F_1(X_1,\cdots,X_{b_1})\mathbf{e}'_1+\cdots+F_{b_1}(X_1,\cdots,X_{b_1})\mathbf{e}'_{b_1}\\ =\varphi_*(X_1\mathbf{e}_1+\cdots+X_{b_1}\mathbf{e}_{b_1}).\end{aligned} \tag{2.26}$$

here φ_* is defined by $e^{\varphi_* x}=\hat\varphi(e^x)$. Since $\varphi_1: C^1_{\mathrm{can}}[1]\to C'^1_{\mathrm{can}}[1]$ is a linear isomorphism we may choose $\mathbf{f}'_j$ so that $\mathbf{f}'_j=\varphi_1(\mathbf{f}_j)$. It is easy to see that F^* induces an R algebra isomorphism : $R[[X'_1,\cdots,X'_{b_1}]]/(P'_1,\cdots,P'_{b_2})\to R[[X_1,\cdots,X_{b_1}]]/(P_1,\cdots,P_{b_2})$. ☐

Therefore by Theorem 2.3.1 we can define $\mathfrak{K}_C=\mathfrak{K}_{C_{\mathrm{can}}}$ for any A_∞ (or L_∞) algebra C with finite dimensional $\mathfrak{m}_1$ cohomology.

We put $(C_{\mathrm{can}})_{\mathfrak{K}_{C_{\mathrm{can}}}}=C^0_{\mathrm{can}}[1]\otimes_R\mathfrak{K}_{C_{\mathrm{can}}}$ and define : $\mathfrak{b}=\sum X_i\mathbf{e}_i\in(C^0_{\mathrm{can}})_{\mathfrak{K}_{C_{\mathrm{can}}}}$. Then we have $\mathfrak{m}(e^{\mathfrak{b}})=0\in(C^1_{\mathrm{can}})_{\mathfrak{K}_{C_{\mathrm{can}}}}$. Here we remark that $\mathfrak{m}(e^{\mathfrak{b}})$ is an infinite sum $\sum_k\mathfrak{m}_k(\mathfrak{b},\cdots,\mathfrak{b})$ (in L_∞ case we divide the terms by $k!$). But it converges in $(C^0_{\mathrm{can}})_{\mathfrak{K}_{C_{\mathrm{can}}}}$ with respect to the topology induced by the non Archimedean valuation on the formal power series ring. (This means nothing but the infinite sum $\sum_k\mathfrak{m}_k(\mathfrak{b},\cdots,\mathfrak{b})$ makes sense as a formal power series.)

Thus we obtain an A_∞ (or L_∞) algebra $((C_{\mathrm{can}})_{\mathfrak{K}_{C_{\mathrm{can}}}},\mathfrak{m}^{\mathfrak{b}})$.[3] The next lemma implies that this deformation is complete.

Lemma 2.3.4. *Let $\mathcal{R}$ be a finite dimensional local R algebra and $b\in\widetilde{\mathcal{MC}}(C_{\mathrm{can}})(\mathcal{R})$. Then there exists an R algebra homomorphism $\psi:\mathfrak{K}_{C_{\mathrm{can}}}\to\mathcal{R}$ such that $\psi(\mathfrak{b})=b$.*

Proof. We have a polynomials $R_1(Y_1,\cdots,Y_m),\cdots,R_N(Y_1,\cdots,Y_m)$ of m variables such that
$\mathcal{R}\cong R[Y_1,\cdots,Y_m]/(R_1,\cdots,R_N)\cong R[[Y_1,\cdots,Y_m]]/(R_1,\cdots,R_N)$. Let us write by $\mathfrak{a}$ the ideal generated by $R_1,\cdots,R_N$. We write

$$b\equiv F_1(Y_1,\cdots,Y_m)\mathbf{e}_1+\cdots+F_{b_1}(Y_1,\cdots,Y_m)\mathbf{e}_{b_1}\mod\mathfrak{a}.$$

[3] In other words we have a deformation of C parametrized by a formal scheme $\mathrm{Spec}\,\mathfrak{K}_{C_{\mathrm{can}}}$.

Here $\mathbf{e}_i$ are basis of C^1_{can}. We define $\tilde\psi : R[[X_1, \cdots, X_{b_1}]] \to R[[Y_1, \cdots, Y_m]]$ by $\tilde\psi(X_i) = F_i(Y_1, \cdots, Y_m)$. Then $\tilde\psi(X_1\mathbf{e}_1 + \cdots + X_{b_1}\mathbf{e}_{b_1}) = b \mod \mathfrak{a}$. Therefore

$$\tilde\psi(\delta(\exp(X_1\mathbf{e}_1 + \cdots + X_{b_1}\mathbf{e}_{b_1}))) = \delta(e^b) = 0.$$

Hence, by definition, $\tilde\psi$ induces a homomorphism $\psi : \mathfrak{K}_{C_{\mathrm{can}}} \to \mathcal{R}$. □

We remark that the formal Kuranishi map P_i has no term of degree ≤ 1. It follows that the Zariski tangent space $T_0 \operatorname{Spec} \mathfrak{K}_C$ can be identified with $C^1_{\mathrm{can}} = H^1(C; \mathfrak{m}_1)$. The Kodaira-Spencer map then is an identity. This fact together with Lemma 2.3.4 is a formal scheme analogue of Theorem 1.6.1.

We now prove Theorem 1.6.1. Let us take $C^* = \Omega^{0,*}(M; End(E))$ and $\mathfrak{m}_k$ is induced by $\overline\partial_{\mathcal{E}}$ and $\circ$ by Lemma 2.1.3. Then by Theorem 2.3.1 we have $\overline{\mathfrak{m}}_k : B_kC_{\mathrm{can}}[1] \to C_{\mathrm{can}}[1]$, $\varphi_k : B_kC_{\mathrm{can}}[1] \to C[1]$, where $C^k_{\mathrm{can}} = \operatorname{Ext}^k(\mathcal{E}, \mathcal{E})$.

Proposition 2.3.2.

$$\|\overline{\mathfrak{m}}_k(x_1, \cdots, x_k)\| \le C^k \|x_1\| \cdots \|x_k\|, \tag{2.27}$$

$$\|\varphi_k(x_1, \cdots, x_k)\| \le C^k \|x_1\| \cdots \|x_k\|, \tag{2.28}$$

where C is independent of k.

Here the norm in the right hand side of (2.28) is L^2_ℓ norm for any fixed ℓ. (The constant C in (2.28) may depend on ℓ.) The norm on C_{can} can be defined uniquely up to equivalence since C_{can} is finite dimensional.

We omit the proof of proposition since it is straight forward check using Properties (A),(B) we stated during the proof of Lemma 1.5.2 and the definition of $\overline{\mathfrak{m}}_k$ and φ_k given during the proof of Theorem 2.3.1. (In other words the proof is a analogue of the proof of Lemma 1.5.2.)

Proposition 2.3.2 implies that the formal Kuranishi map $P : \operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) \to \operatorname{Ext}^2(\mathcal{E}, \mathcal{E})$ actually converges in a neighborhood of the origin. We put $\mathfrak{X}_{\mathrm{Kura}} = P^{-1}(0)$ in the sense of Example 1.6.1. Then $\tilde b$ determines deformations parametrized by $\mathfrak{X}_{\mathrm{Kura}}$ as follows. We define

$$\begin{aligned} B_{\mathrm{Kura}}(X_1, \cdots, X_\ell) &= \varphi\left(\exp(\tilde b)\right) \\ &= \sum_k \varphi_k(X_1\mathbf{e}_1 + \cdots + X_{b_1}\mathbf{e}_{b_1}, \cdots, X_1\mathbf{e}_1 + \cdots + X_{b_1}\mathbf{e}_{b_1}). \end{aligned} \tag{2.29}$$

It is a formal power series of X_i with values in $\Omega^{0,1}(M, End(E))$. By using (2.28) we find that there exists an open neighborhood $\mathcal{U}_\ell$ of 0 in $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})$ where (2.29) converges in L^2_ℓ sense for $(X_1, \cdots, X_{b_1}) \in \mathcal{U}$. It is easy to see that $B_{\mathrm{Kura}} : \mathcal{U} \to \Omega^{0,1}(M; End(E))$ defines a deformation of $\mathcal{E}$.[4] We already showed that Kodaira-Spencer map is an isomorphism for this family. Let us prove the completeness. The proof is similar to the proof of Lemma 2.3.4, except we need to discuss convergence. Let $\psi_k : B_k\Omega^{0,*}(M, End(E))[1] \to \mathrm{Ext}^*(\mathcal{E},\mathcal{E})$ be a homotopy inverse of $\varphi_k : B_k C_{\mathrm{can}}[1] \to C[1]$. We recall that the existence of ψ_k follows from Theorem 2.1.2. We use (2.28), and check a proof of Theorem 2.1.2 carefully and obtain an estimate

$$\|\psi_k(x_1, \cdots, x_k)\| \le C^k \|x_1\| \cdots \|x_k\|. \tag{2.30}$$

Now let $B : \mathcal{U} \to \Omega^{0,1}(M; End(E))$ be a holomorphic map which defines a deformation of $\mathcal{E}$ parametrized by $\mathfrak{X}$. Here $\mathfrak{X}$ is a germ of analytic subspace. We consider $\psi_* \circ B : \mathcal{U} \to C^1_{\mathrm{can}} = \mathrm{Ext}^1(\mathcal{E},\mathcal{E})$. (Here ψ_* is defined by $\hat{\psi}(e^x) = e^{\psi_*(x)}$.) The map $\psi_* \circ B$ is defined first as a formal power series. We then use (2.30) and the fact that B is a convergent power series, to show that $\psi_* \circ B$ converges in a small neighborhood of origin. By replacing $\mathcal{U}$ if necessary we may assume that it converges on $\mathcal{U}$.

Now, in the same way as the proof of Lemma 2.3.4, we find that composition of $\psi_* \circ B$ and Kuranishi map $P : \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ vanishes. Hence we obtain a ring homomorphism $\Phi : \mathcal{O}_{0,+}/(P_1, \cdots, P_{b_2}) \to \mathfrak{I}_{\mathfrak{X},0}$ by $f \mapsto f \circ \psi_* \circ B$. By definition (Definition 1.6.4), Φ is a morphism $\mathfrak{X} \to \mathfrak{X}_{\mathrm{Kura}}$.

To complete the proof of Theorem 1.6.1, we need to show that the pull back of the deformation $B_{\mathrm{Kura}} : \mathcal{U} \to \Omega^{0,1}(M; End(E))$ by Φ is isomorphic to B. The pull back of B_{Kura} by Φ is $\phi_* \circ \psi_* \circ B : \mathcal{U} \to \Omega^{0,1}(M; End(E))$.

We use the fact that $\phi \circ \psi$ is homotopic to identity to show that $\phi_* \circ \psi_* \circ B$ is gauge equivalent to B as follows. By definition of homotopy (Definition 2.1.8), there exists $H : \Omega^{0,*}(M; End(E)) \to \Omega^{0,*}(M; End(E)) \otimes \mathbb{C}[t, dt]$ such that $\mathrm{Eval}_{t=0} \circ H = \mathrm{id}$, $\mathrm{Eval}_{t=1} \circ H = \phi \circ \psi$. We can check the proof of Theorem 2.1.2 carefully again to find that an estimate similar to (2.26) holds for H. Then we have a map $H_* \circ B : \mathcal{U} \to (\Omega^{0,*}(M; End(E)) \otimes \mathbb{C}[t, dt])^1$, after shrinking $\mathcal{U}$ if necessary. We put $H_* \circ B = b(t, x) + c(t, x)dt$, where $x \in \mathcal{U}$, $b(t, x) \in \Omega^{0,1}(M; End(E))$, $c(t, x) \in \Gamma(M; End(E))$. We use Lemma 2.2.2 here. Namely we solve equation (2.18) to obtain $g(t)(x) \in \Gamma(M; End(E))$. We use the estimate of H to show that $g(t)(x)$ converges if x is in a neighborhood of 0. Now $g(1)$ gives an isomorphism from the deformation B to $\phi_* \circ \psi_* \circ B$. The proof of Theorem 1.6.1 is now complete. $\square$

We now turn to the proof of Theorem 1.3.3 and its formal analogue. We remark that the ring homomorphism $\pi : \mathfrak{K}_C \to R$ induces a homomorphism

$$\pi : H^0(C_{\mathfrak{K}_C}, \mathfrak{m}_1^{\tilde{b}}) \to H^0(C, \mathfrak{m}_1). \tag{2.31}$$

[4] We can modify the family so that B converges in C^∞ topology. See [53]

Theorem 2.3.2. *If* (2.31) *is surjective, then the functor* $\mathfrak{F}_{\mathfrak{K}_C}$: $\{\text{f. d. Alg.}/R\} \to \{\text{Sets}\}$ *is equivalent to the Maurer-Cartan functor* $\mathcal{MC}(C)$: $\{\text{f. d. Alg.}/R\} \to \{\text{Sets}\}$.

Before proving Theorem 2.3.2, let us explain why it is a formal version of Theorem 1.3.3. We consider the case when our canonical A_∞ (or L_∞) algebra satisfies (2.27). Then, as in the proof of Theorem 1.6.1, we find a germ of analytic subvarieties $\mathfrak{X}_{\text{Kura}}$ in $\mathcal{U} \subseteq \mathbb{C}^N$ defined by Kuranishi map $P_i \in \mathcal{O}_{0,+}$ and $B_{\text{Kura}} = \tilde{b} : \mathcal{U} \to (C_{\text{can}}[1])^0$ defines a holomorphic family of A_∞ (or L_∞) algebra parametrized by $\mathfrak{X}_{\text{Kura}}$. The operators $\mathfrak{m}_1^{\tilde{b}}$ define a holomorphic family of chain complexes

$$C^0_{\text{can}} \overset{\mathfrak{m}_1^{\tilde{b}}}{\to} C^1_{\text{can}} \overset{\mathfrak{m}_1^{\tilde{b}}}{\to} C^2_{\text{can}} \to \cdots \tag{2.32}$$

We remark that $H^0(C_{\mathfrak{K}_C}, \mathfrak{m}_1^{\tilde{b}})$ is the set of the germs of holomorphic maps $s : \mathcal{U} \to C^0_{\text{can}}$ such that $\mathfrak{m}_1^{\tilde{b}}(s)$ vanishes on $\mathfrak{X}_{\text{Kura}}$. Hence the surjectivity of (2.31) implies that we have a local frame $s_1, \cdots, s_{b_0}$ of the kernel of (2.32) on $\mathfrak{X}_{\text{Kura}}$ in a neighborhood of 0. We remark that the pointwise 0th cohomology of (2.32) is semi continuous. Hence the existence of the sections s_i implies that the pointwise 0th cohomology of (2.32) is of constant rank in a neighborhood of 0 in $\mathfrak{X}_{\text{Kura}}$. This is the assumption of Theorem 1.3.3. It is also easy to see that the assumption of Theorem 1.3.3 implies the surjectivity of (2.31).

Note that the discussion above shows that Theorem 2.3.2 implies Theorem 1.3.3 and its analogue stated in §1.6.

Proof. We now prove Theorem 2.3.2. We have already constructed a natural transformation $\mathfrak{F}_{\mathfrak{K}_C} \to \mathcal{MC}(C)$. Lemma 2.3.4 implies that this transformation is surjective. So it suffices to show that it is injective. In other words, it suffices to show the following :

Lemma 2.3.5. *We assume* (2.31) *is surjective. If* $\varphi, \varphi' : \mathfrak{K}_C \to \mathcal{R}$ *are* R *algebra homomorphism, and if* $\varphi(\tilde{b}) \sim \varphi'(\tilde{b})$ *in* $\tilde{\mathcal{MC}}(C)(\mathcal{R})$*, then* $\varphi = \varphi'$.

Proof. We may assume C is canonical. Then $H^0(C, \mathfrak{m}_1) = C^0$. Let $\mathbf{1}_1, \cdots, \mathbf{1}_{b_0}$ be a generator of C^0. By the surjectivity of (2.31) we have its lift $\tilde{\mathbf{1}}_i$ to $C^0_{\mathfrak{K}_C}$ such that $\mathfrak{m}_1^{\tilde{b}}(\tilde{\mathbf{1}}_i) = 0$ in $C^1_{\mathfrak{K}_C}$. It is easy to see that $\mathfrak{m}_1^{\tilde{b}}$ generate $C^0_{\mathfrak{K}_C}$ as a $\mathfrak{K}_C$ module. Hence $\mathfrak{m}_1^{\tilde{b}} : C^0_{\mathfrak{K}_C} \to C^1_{\mathfrak{K}_C}$ is zero.

Let $\mathbf{e}_i$ be a generator of C^1 and $\mathfrak{a} \subseteq R[[X_1, \cdots, X_{b_1}]]$ be the ideal generated by formal Kuranishi maps. We recall $\tilde{b} \equiv \sum X_i \mathbf{e}_i \mod \mathfrak{a}$

By assumption, we have $\hat{b} \in \tilde{\mathcal{MC}}(C)(\mathcal{R} \otimes R[t, dt])$ such that $\text{Eval}_{t=0}(\hat{b}) = \varphi(\tilde{b})$, $\text{Eval}_{t=1}(\hat{b}) = \varphi'(\tilde{b})$. We put $\hat{b} = x(t) + y(t)dt$. We remark that $x(t) \in (C \otimes \mathcal{R}[t])^1$ and $\mathfrak{m}(e^{x(t)}) = 0$. Let $x(t) = \sum X_i(t)\mathbf{e}_i$. Then $\tilde{\varphi}(X_i) \mapsto X_i(t)$ defines a ring homomorphism $\tilde{\varphi} : \mathfrak{K}_C \to \mathcal{R}[t]$. Its composition with obvious homomorphism $\mathrm{Eval}_{t=0} : \mathcal{R}[t] \to \mathcal{R}$ and with $\mathrm{Eval}_{t=1} : \mathcal{R}[t] \to \mathcal{R}$ are equal to φ and φ' respectively.

The condition that $\hat{b}$ is a Maurer-Cartan element implies

$$\frac{d}{dt}x(t) = -\mathfrak{m}_1^{b(t)}(y(t)). \tag{2.33}$$

Since $b(t) = \tilde{\varphi}(\tilde{b})$ and since $\mathfrak{m}_1^{\tilde{b}} : C^0_{\mathfrak{K}_C} \to C^1_{\mathfrak{K}_C}$ is zero it follows that the right hand side of (2.33) is zero. Namely $x(0) = x(t)$.

We remark that $\tilde{\varphi}\left(\sum X_i \mathbf{e}_i\right) = x(t)$ and X_i generates $\mathfrak{K}_C$. Therefore $\frac{d}{dt}\tilde{\varphi}(t) = 0$. Hence $\varphi = \varphi'$. The proof of Lemma 2.3.5 and Theorem 2.3.2 is now complete. □

Let us prove Theorem 1.3.1 here. Let us take a versal family constructed in the proof of Theorem 1.6.1 above and write it $(\mathfrak{K}_C, B_{\mathrm{Kura}})$. We already know that it is complete. Let $(\mathcal{U}, B)$ be another family such that the Kodaira-Spencer map is surjective. We may take a submanifold $\mathcal{V} \subset \mathcal{U}$ such that the restriction of Kodaira-Spencer map to $T_0\mathcal{V}$ is an isomorphism. Since $(\mathfrak{K}_C, B_{\mathrm{Kura}})$ is complete, we have a morphism $(\mathfrak{F}, \Phi) : (\mathcal{V}, B) \to (\mathfrak{K}_C, B_{\mathrm{Kura}})$. Using Lemma 1.3.2 and the fact the Kodaira-Spencer map of both deformations are isomorphisms, we find that $d_0\mathfrak{F} : T_0\mathcal{V} \to T_0\mathfrak{X}$ is an isomorphism. Then, using the fact that $\mathcal{V}$ is a manifold, we can apply implicit function theorem to prove that $\mathfrak{F}$ is an isomorphism in a neighborhood of zero. (We remark that we do not need to assume that $\mathfrak{X}$ is a manifold in a neighborhood of zero to apply implicit function theorem here.) Hence the $(\mathcal{V}, B)$ is isomorphic to $(\mathfrak{K}_C, B_{\mathrm{Kura}})$ and is complete. Therefore, $(\mathcal{U}, B)$ is complete. □

2.4 Super space and odd vector field - an alternative formulation of L_∞ algebra -

According to [2, 65], we can rewrite the contents of §2.2, 2.3 using the terminology of formal geometry. In the case of L_∞ algebra we need to use supper formal manifold and in the case of A_∞ algebra we need a kind of "noncommutative geometry". [21] uses the formalism of [2, 65]. It is more useful in L_∞ case than A_∞ case, since the case of super manifold is closer to usual geometry than noncommutative geometry. So we discuss only L_∞ case here. Our argument is very brief here.

We start with explaining formal super manifold (only) in the case we concern with. Let us consider $V = \oplus V^k$ a graded vector space. Let V_{ev}, V_{od} be the sum of their even or odd degree parts, respectively. We regard V_{ev} as the "Bosonic" part and V_{od} as the "Fermionic" part. This mean nothing but we regard ring of functions on it as $\prod_{k,\ell} S_k V_{\mathrm{ev}}^* \otimes \Lambda^\ell V_{\mathrm{od}}^*$, where S_k denotes the kth symmetric power. We remark that $\prod_{k,\ell} S_k V_{\mathrm{ev}}^* \otimes \Lambda^\ell V_{\mathrm{od}}^*$

is a dual to EV. (Note EV is a direct sum hence only a finite sum is allowed. Its dual is a direct product.)

Definition 2.4.1. The ring of function on *formal supper manifold* $C[1]$ is the dual vector space $EC[1]^*$ of $EC[1]$.

Let $\mathbf{e}_i$ be a basis of $C[1]^{\mathrm{ev}} = C^{\mathrm{od}}$ and $\mathbf{f}_i$ be a basis of $C[1]^{\mathrm{od}} = C^{\mathrm{ev}}$. An element of C is written as a finite sum : $\sum x^i \mathbf{e}_i + \sum y^j \mathbf{f}_j$. Then x^i and y^j be basis of the dual vector space C^*. Hence element of $EC[1]^*$ can be written uniquely as

$$h = \sum_k \sum_{j_1 < \cdots < j_k} h_{j_1,\cdots,j_k}(x^1, x^2, \cdots) y^{j_1} \wedge \cdots \wedge y^{j_k} \tag{2.34}$$

where $h_{j_1,\cdots,j_k} \in R[[x^1, x^2, \cdots]]$. The ring structure is determined by $y^i \wedge y^j = -y^j \wedge y^i$ and the ring structure on $R[[x^1, x^2, \cdots]]$.

Definition 2.4.2. A formal vector field on the formal super manifold $C[1]$ is an expression

$$\mathfrak{V} = \sum V_i^x \frac{\partial}{\partial x^i} + \sum V_i^y \frac{\partial}{\partial y^i}$$

where $V_i \in EC[1]^*$. For h as in (2.34) and $\mathfrak{V}$ as above, we put :

$$\begin{aligned} \mathfrak{V}(h) =& \sum_{k,\ell} \sum_{j_1 < \cdots < j_k} V_\ell^x \frac{\partial h_{j_1,\cdots,j_k}}{\partial x^i} y^{j_1} \wedge \cdots \wedge y^{j_k} \\ &+ \sum_k \sum_{j_1 < \cdots < j_k} \sum_{i=1}^k (-1)^{i-1} V_{j_i}^y h_{j_1,\cdots,j_k} y^{j_1} \wedge \cdots \wedge \hat{y}^{j_i} \wedge \cdots \wedge y^{j_k}. \end{aligned} \tag{2.35}$$

(2.35) is characterized by $\frac{\partial}{\partial x^i} x^j = \delta_{ij}$, $\frac{\partial}{\partial y^i} x^j = 0$, $\frac{\partial}{\partial y^i} y^j = \delta_{ij}$, $\frac{\partial}{\partial x^i} y^j = 0$, and $\mathfrak{V}(hh') = \mathfrak{V}(h)h' + (-1)^{\deg \mathfrak{V} \deg h} h \mathfrak{V}(h')$. Here we define degree by : $\deg \frac{\partial}{\partial x^i} = \deg x^i = 0$, $-\deg \frac{\partial}{\partial y^i} = \deg y^i = 1$.

It is easy to see the following :

Lemma 2.4.1. *If* $\deg \mathfrak{V}$ *and* $\deg \mathfrak{W}$ *are even then there exists an super vector field* $[\mathfrak{V}, \mathfrak{W}]$ *such that* $[\mathfrak{V}, \mathfrak{W}](h) = (\mathfrak{V}\mathfrak{W} - \mathfrak{W}\mathfrak{V})(h)$. *If* $\deg \Theta$ *and* $\deg \Xi$ *are odd then there exists a super vector field* $\{\Theta, \Xi\}$ *such that* $\{\Theta, \Xi\}(h) = (\Theta\Xi + \Xi\Theta)(h)$.

It is obvious that $[\mathfrak{V}, \mathfrak{V}] = 0$ for super vector field $\mathfrak{V}$ of *even* degree. However $\{\Theta, \Theta\} \neq 0$ in general for super vector field Θ of *odd* degree. Actually we have :

Lemma 2.4.2. *We assume* 2 *is invertible on* R. *Then, supper vector field* Θ *of degree* 1 *satisfying* $\{\Theta, \Theta\} = 0$ *corresponds one to one to the* L_∞ *structure on* $C[1]$.

Proof. Let Θ defines a derivation $\Theta : EC[1]^* \to EC[1]^*$. Its dual defines a coderivation $\delta : EC[1] \to EC[1]$. $\delta\delta = 0$ is equivalent to $\{\Theta, \Theta\} = 2\Theta^2 = 0$. The lemma follows. □

Let $(C, \mathfrak{m})$, $(C', \mathfrak{m}')$ be L_∞ algebras and Θ, Θ' be odd vector fields corresponding to them by Lemma 2.4.2. Let φ_k be an L_∞ homomorphism. It induces a coalgebra homomorphism $\hat{\varphi} : EC[1] \to EC'[1]$. Hence its dual is an algebra homomorphism $\hat{\varphi}^* : EC'[1]^* \to EC[1]^*$. That is a morphism of formal super manifold $C[1] \to C'[1]$. Since $\hat{\varphi} \circ \delta = \delta \circ \hat{\varphi}$ it follows that $\hat{\varphi}^* \circ \Theta' = \Theta \circ \hat{\varphi}^*$. Thus, L_∞ homomorphism will become a morphism of formal super manifold preserving odd vector fields on it. One may continue and translate various other operations of L_∞ algebras to the language of super manifolds. We do not try to do it here.

Chapter 3

Application to Mirror symmetry

3.1 Novikov ring and filtered A_∞, L_∞ algebras.

In this chapter, we explain relations of the discussion of deformation theory in Chapter 1,2 to Mirror symmetry.

We explain a construction of [33] which associates an "A_∞ algebra" to a Lagrangian submanifold of a symplectic manifold (satisfying some conditions we will explain later). (We review various notions of symplectic geometry we need at the beginning of §3.2.) Actually what we will associate to a Lagrangian submanifold is a slight modification of A_∞ algebra, which we call filtered A_∞ algebra. To define it we first need to define universal Novikov ring. We will discuss universal Novikov ring more in §3.5. Novikov introduced a kind of formal power series ring in [81] to study Morse theory of closed one form. It was applied by [51] etc. to infinite dimensional situation of Floer homology, (which may be regarded as a Morse theory of closed one form on loop space). Namely Floer homology which we will discuss in §3.3, is defined as a module over Novikov ring. In order to use the same ring independent of the symplectic manifold (and of its Lagrangian submanifold) we use a ring which we call universal Novikov ring. We define it below.

Let R be a commutative ring. We consider the formal sum $x = \sum_i a_i T^{\lambda_i}$ satisfying the following conditions.

Condition 3.1.1. (1) $a_i \in R$, (2) $\lambda_i \in \mathbb{R}$, (3) $\lambda_i < \lambda_{i+1}$, (4) $\lim_{i\to\infty} = \infty$.

Definition 3.1.1. The set of all formal sums $x = \sum_i a_i T^{\lambda_i}$ satisfying Conditions 3.1.1 is called the *universal Novikov ring* and is written as $\Lambda_{R,\mathrm{nov}}$. $\Lambda_{R,\mathrm{nov}}$ becomes a ring (R algebra) by an obvious definition of sum and multiplication. We replace (2) by $\lambda_i \geq 0$. We then obtain a subring $\Lambda_{R,\mathrm{nov},0}$. We replace (2) by $\lambda_i > 0$. We then obtain an ideal $\Lambda_{R,\mathrm{nov},+}$ of $\Lambda_{R,\mathrm{nov}}$. (We omit R in case no confusion can occur.)

In case R is a field, $\Lambda_{R,\mathrm{nov},0}$ is a local ring with maximal ring $\Lambda_{R,\mathrm{nov},+}$.

We define a filtration $\mathfrak{F}$ on Λ_{nov} by

$$\mathfrak{F}^\lambda\Lambda_{\mathrm{nov}} = \{x \mid x \text{ is as in (3.1) satisfying Condition 3.1.1 and } \lambda_i \geq \lambda.\}$$

$\mathfrak{F}^\lambda\Lambda_{\mathrm{nov}}$ is a filtration. Namely $\mathfrak{F}^\lambda\Lambda_{\mathrm{nov}}$ is a subabelian group (with respect to $+$) and $\mathfrak{F}^\lambda\Lambda_{\mathrm{nov}} \cdot \mathfrak{F}^{\lambda'}\Lambda_{\mathrm{nov}} \subseteq \mathfrak{F}^{\lambda+\lambda'}\Lambda_{\mathrm{nov}}$. We remark that $\mathfrak{F}$ induces a filtration on $\Lambda_{\mathrm{nov},0}$ and $\Lambda_{\mathrm{nov},+}$

Remark 3.1.1. We remark that $\Lambda_{\mathrm{nov},0}$ is not a Noether ring, since the ascending sequence of ideals $\mathfrak{F}^{1/i}\Lambda_{\mathrm{nov}}$ does not stop. This fact makes harder to study algebra or module over it.

A filtered $\Lambda_{\mathrm{nov},0}$ module is a $\Lambda_{\mathrm{nov},0}$ module C together with filtration $\mathfrak{F}^\lambda C$ such that $\mathfrak{F}^\lambda\Lambda_{\mathrm{nov}} \cdot \mathfrak{F}^{\lambda'}C \subseteq \mathfrak{F}^{\lambda+\lambda'}C$. We say that a Λ_{nov} module homomorphism $\varphi : C \to C'$ is a filtered $\Lambda_{\mathrm{nov},0}$ module homomorphism if $\varphi(\mathfrak{F}^\lambda C) \subset \mathfrak{F}^\lambda C'$.

Filtration defines a metric on Λ_{nov} and a module on it by $d(x,y) = \exp\left(-\inf\{\lambda \,|\, x-y \in \mathfrak{F}^\lambda C\}\right)$. Λ_{nov}, $\Lambda_{\mathrm{nov},0}$ and $\Lambda_{\mathrm{nov},+}$ are complete with respect to this metric. From now on we *assume* all filtered $\Lambda_{\mathrm{nov},0}$ modules are complete. We also *assume* that all filtered A_∞ (or L_∞) algebras are completion of free $\Lambda_{\mathrm{nov},0}$ module.

If C, C' are filtered $\Lambda_{\mathrm{nov},0}$ module, we define a filtration on its tensor product $C \otimes_{\Lambda_{\mathrm{nov},0}} C'$ by

$$\mathfrak{F}^\lambda(C \otimes_{\Lambda_{\mathrm{nov},0}} C') = \bigcup_\mu \mathfrak{F}^\mu C \otimes_{\Lambda_{\mathrm{nov},0}} \mathfrak{F}^{\lambda-\mu}C'.$$

The tensor product $C \otimes_{\Lambda_{\mathrm{nov},0}} C'$ is not complete with respect to the metric induced by this filtration. We denote by $C\hat{\otimes}_{\Lambda_{\mathrm{nov},0}}C'$ the completion.

Graded filtered $\Lambda_{\mathrm{nov},0}$ module is defined in an obvious way. Let C be a graded filtered $\Lambda_{\mathrm{nov},0}$ module. We consider

$$B_kC[1] = \underbrace{C[1]\hat{\otimes}_{\Lambda_{\mathrm{nov},0}} \cdots \hat{\otimes}_{\Lambda_{\mathrm{nov},0}}C[1]}_{k \text{ times}}$$

Let $\hat{B}C[1]$ be the completion of the direct sum $\oplus_k B_kC[1]$. $E_kC[1]$ and $\hat{E}C[1]$ are defined by taking its submodule which is invariant under the action of $\mathfrak{S}_k$.

They are formal coalgebra. Here formal coalgebra is defined replacing $\otimes$ by $\hat{\otimes}$ in the definition of coalgebra. We can define coderivation and cohomomorphism for formal coalgebra in the same way. (We assume them to be filtered.) The following analogy of Lemmata 2.1.2 2.1.5 holds.

Lemma 3.1.1. *Let $f_k : B_kC[1] \to C[1]$, $k = 0, 1, \cdots$, be a sequence of filtered homomorphisms of degree 1. Then there exists a unique coderivation $\delta : \hat{B}C[1] \to \hat{B}C[1]$ whose restriction to $B_kC[1]$ is f_k.*

Let $\varphi_k : B_kC[1] \to C'[1]$, $k = 0, 1, \cdots$, be a sequence of filtered homomorphisms of degree 0. *We assume $\varphi_0(\Lambda_{\mathrm{nov},0}) \subseteq \mathfrak{F}^{\lambda_0} B_1C'[1]$ for some positive λ_0. Then there exists a coalgebra homomorphism $\hat\varphi : \hat BC[1] \to \hat BC[1]$ whose $Hom(B_Ck[1], B_1C'[1])$ component is φ_k.*

The same statement holds when we replace B by E.

We remark that we include f_0 and φ_0 here but not in Lemmata 2.1.2 2.1.5.

Proof. We prove A_∞ case only. We put

$$\begin{aligned}&\hat f_0(x_1 \otimes \cdots \otimes x_k)\\ &\quad = \varphi_0(1) \otimes x_1 \otimes \cdots \otimes x_k + (-1)^{\deg' x_1} x_1 \otimes \varphi_0(1) \otimes x_2 \otimes \cdots \otimes x_k\\ &\qquad + \cdots + (-1)^{\deg' x_1 + \cdots + \deg' x_k} x_1 \otimes \cdots \otimes x_k \otimes \varphi_0(1).\end{aligned}$$

We define $\hat f_k$, $k \geq 1$ in the same way as the proof of Lemma 2.1.2. Then we can prove that $\delta = \hat f_0 + \cdots + \hat f_k + \cdots$ converges by using the fact f_i preserves filtration.

We next define $\hat\varphi$. We put

$$e^{\varphi(1)} = \sum_k \varphi(1) \underset{k \text{ times}}{\otimes \cdots \otimes} \varphi(1).$$

By assumption it converges in $\hat BC'[1]$. Now we put

$$\hat\varphi(\mathbf{x}) = \sum_k \sum_a e^{\varphi(1)} \otimes \overline{\varphi}(\mathbf{x}_a^{k;1}) \otimes e^{\varphi(1)} \otimes \cdots \otimes e^{\varphi(1)} \otimes \overline{\varphi}(\mathbf{x}_a^{k;k}) \otimes e^{\varphi(1)}.$$

Here $\overline{\varphi}$ is φ_k on $B_kC[1]$ $k \neq 0$ and is zero on $B_0C[1]$. It is easy to check that $\hat\varphi$ converges and is a coalgebra homomorphism. □

Remark 3.1.2. $\hat BC[1]$ has another filtration different from $\mathfrak{F}^\lambda \hat BC[1]$. Namely we put $\mathfrak{G}^k \hat BC[1] = \oplus_{i \leq k} B_iC[1]$. We call the filtration $\mathfrak{F}$ the *energy filtration* and $\mathfrak{G}$ the *number filtration*. In case $f_0 \neq 0$ or $\varphi_0 \neq 0$, $\hat f$ or $\hat\varphi$ does *not* preserve the number filtration. They preserve energy filtration. We also remark that $\hat BC[1]$ is not complete with respect to the number filtration.

Definition 3.1.2. A structure of *filtered A_∞ algebra* on filtered graded $\Lambda_{\mathrm{nov},0}$ module C is a series of filtered homomorphisms $\mathfrak{m}_k : B_kC[1] \to C[1]$, $k = 0, 1, \cdots$ of degree 1 such that $\delta\delta = 0$ where $\delta = \sum \hat{\mathfrak{m}}_k : \hat B_kC[1] \to C[1]$ is obtained by Lemma 3.1.1. We also assume that $\mathfrak{m}_0(\Lambda_{\mathrm{nov},0}) \subseteq \mathfrak{F}^{\lambda_0} B_1C[1]$.

A sequence of homomorphisms $\varphi_k : B_kC[1] \to C'[1]$ is a *filtered A_∞ homomorphism* between filtered A_∞ algebras if $\varphi_0(\Lambda_{\mathrm{nov},0}) \subseteq \mathfrak{F}^{\lambda_0} B_1C'[1]$ and the homomorphism $\hat\varphi$ obtained by Lemma 3.1.1 satisfies $\delta\hat\varphi = \hat\varphi\delta$.

L_∞ can be defined in a similar way.

We will explain how to modify the argument of the last chapter to our filtered situation later after we introduce our main example.

3.2 Review of a part of global symplectic geometry.

In this section, we review several points on global symplectic geometry, especially those related to pseudoholomorphic curve, which we need for our main construction. [52, 75, 76] are standard references of them.

A *symplectic manifold* is a pair (M, ω) where ω is a closed 2 form on M such that it is nondegenerate as an anti symmetric two form on T_pM for each $p \in M$. M is automatically even dimensional. Let $2n$ be its dimension. Then ω^n is nowhere vanishing $2n$ form on M and hence determines an orientation. A *Lagrangian submanifold* of a symplectic manifold (M, ω) is an n dimensional closed submanifold L such that $\omega|_L = 0$. We remark that if the dimension of submanifold of M is strictly larger than n then the restriction of ω to L cannot vanish.

A typical example of symplectic manifold is a Kähler manifold. Especially if M is a projective variety that is a complex submanifold of $\mathbb{C}P^n$ then it is a symplectic manifold, whose symplectic structure is obtained as a pull back of the Fubini-Study form ω on $\mathbb{C}P^n$ which is defined by $\pi^*\omega = -4\sqrt{-1}\partial\overline{\partial}\log(|z^0|^2 + \cdots + |z^n|^2)$ where $\pi : \mathbb{C}^{n+1} \to \mathbb{C}P^n$ is the projection.

Another important example of symplectic manifold is a cotangent bundle T^*N of any smooth manifold N. The symplectic form on T^*N is given by $d\theta$. Here θ is a one form on T^*N such that $\theta(X) = u(\pi_*(X))$ where $X \in T_uT^*N$ and $\pi : T^*N \to N$ is the natural projection. If $x^1, \cdots, x^n$ is a coordinate of N then element of T^*N is written as $p_1dx^1 + \cdots + p_ndx^n$. Hence p_i, x^j, $i, j = 1, \cdots, n$ is a coordinate of T^*N. Using this coordinate our symplectic form ω on T^*N is $\omega = dp_1 \wedge dx^1 + \cdots + dp_n \wedge dx^n$.

An example of Lagrangian submanifold of a projective variety is a submanifold consisting of real valued points. Namely let $M \subseteq \mathbb{C}P^n$ be a complex submanifold which is preserved by complex conjugataction τ. We assume $L = \{x \in M | \tau(x) = x\}$ is a submanifold of dimension $n = \dim M/2$. Then we can show L is a Lagrangian submanifold.

There are also various examples of Lagrangian submanifolds in T^*N. One is a conormal bundle T^*_KN of a submanifold K of N. Here

$$T^*_KN = \{(x, u) \in T^*M | x \in N, u|_{T_xN} = 0\}.$$

In fact, if K is defined by equations $x^{k+1} = \cdots = x^n = 0$ then T^*_KN is defined by equations $x^{k+1} = \cdots = x^n = 0, p_1 = \cdots = p_k = 0$. Hence ω zero on T^*_KN.

Another example is a graph of a closed one form u which is defined as follows. For a one form u on N, we put $G_u = \{(x, u(x)) | x \in N\}$. Let us define a diffeomorphism $i : N \to G_u$ by $i(x) = (x, u(x))$. We can show that the pull back of θ by i is u itself. It follows that $i^*\omega = du$. Therefore G_u is a Lagrangian submanifold if and only if u is closed.

Symplectic geometry has a long history. There are many interesting results and their applications. However, for a long time, it seems that there had been only few results in symplectic geometry which is really of global nature. For example, the following question was open for a long time. Does there exists a pair of symplectic manifolds (M,ω) and (M,ω') on a same manifold M such that $[\omega] = [\omega'] \in H^2(M;\mathbb{R})$ but there is no diffeomorphism $\varphi : M \to M$ with $\varphi^*\omega' = \omega$. The reason why such results were not known seems to me that there was basically no general technique which can be applied to study global symplectic geometry. Arnold in 1960's formulates a series of conjectures which is related to global problem of symplectic geometry. (See for example [3].) Roughly speaking, those questions ask whether there exists "symplectic topology". Around the beginning of 1980' several works appeared which shows that such "symplectic topology" does exist and is extremely rich. Among those results, Gromov's one in [42] is quite remarkable. In [42], Gromov introduced a new technique, pseudo-holomorphic curve, to study global structure of symplectic manifold. Let us briefly review it here. Let (M,ω) be a symplectic manifold. An almost complex structure $J : TM \to TM$ is a tensor such that $JJ = -1$.

Definition 3.2.1. J is said to be compatible with ω if (1) $\omega(JX,JY) = \omega(X,Y)$, (2) $\omega(X,JX) > 0$ and (3) $\omega(X,JX) = 0$ implies $X = 0$.

It is proved in [42] (see [75, 76]) that compatible almost complex structure always exists and the set of compatible almost complex structures is contractible [1].

The idea by Gromov is to apply techniques of complex geometry to almost complex manifold (M,J) to get information of a symplectic manifold (M,ω).

There are basically two kinds of methods in complex geometry. One is to use holomorphic functions or holomorphic maps defined on M, and the other is to use holomorphic maps to M. We remark here, in case (M,J) is almost complex (namely in case J is not integrable), there are not so many holomorphic functions on M. Hence the first kinds of methods are hard to apply in our case of (M,J).

The important observation by Gromov is that, even in case when (M,J) is not integrable, there are many holomorphic maps to (M,J) from Riemann surface (complex one dimensional manifold). The basic reason of it is that any almost complex structure on real two dimensional manifold is automatically integrable.

The method then initiated by Gromov is to study moduli space of holomorphic maps from Riemann surface to (M,J) to get information of

[1] It follows that the Chern classes of (the tangent bundle of) symplectic manifold is well-defined.

(M, ω). Gromov called holomorphic map from Riemann surface to an almost complex manifold, a *pseudoholomorphic curve*. The contractibility of the set of almost complex structures guarantees that any invariant obtained by using compatible almost complex structure is an invariant of symplectic manifold if it is independent of continuous change of compatible almost complex structures.

Using an existence of symplectic structure compatible to J, Gromov proved various compactness results of the moduli space of pseudoholomorphic maps, hence its fundamental cycle in principle defines such an invariant. Ruan [89] made this construction (which somehow was implicit in [42]) more explicit.

This invariant in turn was found to be an invariant of topological sigma model with target space M, which Witten [107] introduced in an informal way. The invariant obtained in this way is now called *Gromov-Witten invariant*. See [89, 90, 76, 34] more about it.

Our main concern here is its relative version. Namely we consider a Lagrangian submanifold L of M and study a map $\varphi : D^2 \to M$ such that :

Condition 3.2.1. (1) φ is pseudoholomorphic. Namely $J \circ d\varphi = d\varphi \circ j_{D^2}$. Here j_{D^2} is the standard complex structure of D^2. (2) $\varphi(\partial D^2) \subseteq L$.

Gromov [42] already studied moduli space of such φ to get information of Lagrangian submanifolds in $\mathbb{C}^n$. (For example he proved that there does not exist simply connected compact Lagrangian submanifold in $\mathbb{C}^n$.) Floer [22] used a similar idea to study problems of intersection of Lagrangian submanifolds especially to study the following problem due to Arnold [3].

Problem 3.2.1. Let $L \subset M$ be a Lagrangian submanifold, and let $\varphi : M \to M$ be a Hamiltonian diffeomorphism (which we will define below). We assume L is transversal to $\varphi(L)$. Then, under "some" condition, we have an estimate

$$\sharp L \cap \varphi(L) \geq \sum \operatorname{rank} H_k(L; \mathbb{Z}_2). \tag{3.1}$$

Let us define Hamiltonian diffeomorphism. Let (M, ω) be a symplectic manifold and f be a function on it. There exists a vector field X_f such that $\omega(X_f, V) = df(V)$ holds for any vector V. X_f is called the *Hamiltonian vector field*. We now consider $f : M \times [0, 1] \to \mathbb{R}$. Let $f_t(x) = f(x, t)$. It induces a one parameter family of vector fields X_{f_t}. We define a family of diffeomorphisms $\varphi_t : M \to M$ by

$$\frac{d}{dt}\varphi_t(x) = X_{f_t}(\varphi_t(x)), \quad \varphi_0(x) = x. \tag{3.2}$$

Definition 3.2.2. $\varphi : M \to M$ is called a *Hamiltonian diffeomorphism* if there exists $f_t(x) = f(x, t)$ such that $\varphi_1 = \varphi$, and φ_t is defined by (3.2).

One can prove easily that Hamiltonian diffeomorphism is a symplectic diffeomorphism, namely $\varphi^*\omega = \omega$.

Floer [22] proved (3.1) in case $\pi_2(M, L) = 0$. He used new homology theory which is now called *Floer homology* for this purpose. Let us briefly explain it here. Let L_1, L_2 be two Lagrangian submanifolds such that $\pi_2(M, L_i) = 0$. We assume that L_1 is transversal to L_2. We consider $\mathbb{Z}_2$ vector space $CF(L_1, L_2)$ whose basis is identified to the intersection points $p \in L_1 \cap L_2$. Namely we put

$$CF(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2[p]. \tag{3.3}$$

Floer defined a degree of each $[p]$ and defined a boundary operator on it as follows. Let $p, q \in L_1 \cap L_2$. We consider maps $\varphi : D^2 \to M$ satisfying the following conditions. We put $\partial D^2_+ = \{z \in \partial D | \operatorname{Im} z > 0\}$, $\partial D^2_- = \{z \in \partial D | \operatorname{Im} z < 0\}$.

Condition 3.2.2. (1) φ is pseudoholomorphic. Namely $J \circ d\varphi = d\varphi \circ j_{D^2}$. Here j_{D^2} is the standard complex structure of D^2. (2) $\varphi(\partial_+ D^2) \subset L_1$, $\varphi(\partial_- D^2) \subset L_2$. (3) $\varphi(-1) = p$, $\varphi(1) = q$.

Let $\widetilde{\mathcal{M}}(p, q; L_1, L_2)$ be the moduli space of all such maps φ. The group $\operatorname{Aut}(D^2; \{\pm 1\})$ of biholomorphic maps $D^2 \to D^2$ preserving ± 1 acts on $\widetilde{\mathcal{M}}(p, q; L_1, L_2)$. This group is isomorphic to $\mathbb{R}$. We denote by $\mathcal{M}(p, q; L_1, L_2)$ the quotient space. Floer proved the following :

Theorem 3.2.1. *Under the assumption $\pi_2(M, L_i) = 0$, there exists $\mu : L_1 \cap L_2 \to \mathbb{Z}$ such that the following holds after taking a "generic perturbation[2]" of the pseudoholomorphic curve equation $J \circ d\varphi = d\varphi \circ j_{D^2}$.*

(1) *$\mathcal{M}(p, q; L_1, L_2)$ is a smooth manifold of dimension $\mu(q) - \mu(p) - 1$.*
(2) *If $\mu(q) - \mu(p) - 1 = 0$, then $\mathcal{M}(p, q; L_1, L_2)$ consists of finitely many points.*
(3) *If $\mu(q) - \mu(p) - 1 = 1$, then $\mathcal{M}(p, q; L_1, L_2)$ can be compatcified to $\mathcal{CM}(p, q; L_1, L_2)$ which is a 1 dimensional manifold with boundary.*
(4) *In the situation of* (3) *the boundary of $\mathcal{CM}(p, q; L_1, L_2)$ is identified with*

$$\bigcup_{r \in L_1 \cap L_2, \mu(r) = \mu(q)+1} \mathcal{M}(p, r; L_1, L_2) \times \mathcal{M}(r, q; L_1, L_2). \tag{3.4}$$

(Note that (3.4) is of finite order by (2).) We will not discuss the proof of this theorem which is now classical. μ is called the *Maslov-Viterbo index.*

[2] We do not explain precise meaning of it in the article.

(4) can be explained by the following Figure.

Figure 3

Now the definition of Floer homology is as follows. We put $\deg[p] = \mu(p)$ hence $CF(L_1, L_2)$ is a graded $\mathbb{Z}_2$ vector space. We next define

$$\langle \delta[p], [q] \rangle \equiv \sharp \mathcal{M}(p, q; L_1, L_2) \mod 2$$

in case $\mu(q) - \mu(p) - 1 = 0$ and put

$$\delta[p] = \sum \langle \delta[p], [q] \rangle [q].$$

δ is an operator of degree 1. We show

Corollary 3.2.1. $\delta \circ \delta = 0$.

Proof. Let us calculate the coefficient of $[q]$ in $\delta\delta([p])$. We may write it as

$$\sum_r \langle \delta[p], [r] \rangle \langle \delta[r], [q] \rangle.$$

It suffices to consider the case $\mu(q) - \mu(p) - 1 = 1$. Namely we can apply (4) of Theorem 3.2.1. We then have

$$\begin{aligned} &\sum_r \langle \delta[p], [r] \rangle \langle \delta[r], [q] \rangle \\ &= \bigcup_{r \in L_1 \cap L_2, \mu(r) = \mu(q)+1} \sharp (\mathcal{M}(p, r; L_1, L_2) \times \mathcal{M}(r, q; L_1, L_2)) \\ &= \sharp \partial \mathcal{M}(p, q; L_1, L_2) \equiv 0 \mod 2, \end{aligned}$$

since the order of the boundary of one dimensional compact manifold is even. □

We thus define a *Floer cohomology* by

$$HF(L_1, L_2) = H(CF(L_1, L_2), \delta).$$

Floer proved the following two properties of it.

Theorem 3.2.2. *We assume $\pi_2(M, L_i) = 0$. If φ_i are Hamiltonian diffeomorphisms then $HF(L_1, L_2) \cong HF(\varphi_1 L_1, \varphi_2 L_2)$.*

Theorem 3.2.3. *We assume $\pi_2(M, L) = 0$. Then $HF(L, L) \cong H(L; \mathbb{Z}_2)$.*

It is easy from definition to see that

$$\operatorname{rank} HF(L_1, L_2) \geq \sharp L_1 \cap L_2. \tag{3.5}$$

(3.1) follows from Theorems 3.2.2,3.2.2 and (3.2).

We remark that our discussion so far assumed $\pi_2(M, L) = 0$. After Floer, Oh [82] relaxed the condition $\pi_2(M, L) = 0$. His assumption is that Lagrangian submanifold is monotone and its minimal Maslov number is ≥ 3. We do not explain this condition.

Actually there is an example where (3.1) does not hold in general.

Example 3.2.1. Let us consider S^2. Any one dimensional submanifold of it is a Lagrangian submanifold. Let L be a circle which is in a small neighborhood of north pole. We can find easily a Hamiltonian diffeomorphism φ such that $\varphi(L) \cap L = \emptyset$. On the other hand, $H_*(L) = H_*(S^1) \neq 0$.

Therefore, something wrong should happen if we try to generalize the story of Floer to more general Lagrangian submanifold. For example, in the case of Example 3.2.1 it turns out that Floer homology is not defined for such a Lagrangian submanifold.

Thus finding a good condition for Floer homology to be defined is important for its application to global symplectic geometry. Later it was found that the same problem is also closely related to mirror symmetry. This is the main point of our story. So before stating the results (and conjectures) precisely, we first explain the rough story.

The author together with Oh, Ohta, Ono[3] developed an obstruction theory for the well-definedness of the Floer homology of Lagrangian submanifold [33]. (Our projects starts around 1997 just after necessary analytic machinery is completed in [34].) We found that there are a series of obstructions which takes value in the cohomology of L such that if they all vanish then the Floer homology is well-defined.

However, Floer homology thus defined actually *depends* on various choices involved. Namely there exists a moduli space associated to a Lagrangian submanifold and Floer homology is well-defined as a family parametrized by this moduli space. The condition that the obstruction vanish is equivalent to the condition that this moduli space is nonempty. The algebraic machinery we need to establish it is one we developed in the last chapter (its filtered version precisely). Namely the moduli space parametrizing Floer homology is a moduli space representing appropriate Maurer-Cartan functor.

On the other hand, the author proposed to generalize the story of Lagrangian intersection Floer homology to the case when there are three or more Lagrangian submanifolds involved [4]. He then found that there appears an A_∞ structure [23], in early 1990's. Then Kontsevich proposed a conjecture that the A_∞ structure on Floer homology of Lagrangian submanifold should be a "mirror" of a similar A_∞ structure on sheaf cohomol-

[3] Kontsevich gave us an important suggestion to start this research.

[4] The author was inspired by the idea by Donaldson and Segal when he start this project.

ogy of complex manifold (that is the A_∞ algebra related to the complex $\Omega^{0,*}(End(E))$ introduced in Chapters 1 and 2). Namely Kontsevich proposed a *homological mirror symmetry conjecture* in [66, 69]. It roughly states that there are pairs of symplectic manifold M and complex manifold $M^\wedge$ such that Lagrangian submanifold of M corresponds to a coherent sheaf on $M^\wedge$ and Floer homology of Lagrangian submanifolds in M corresponds to the sheaf cohomology on $M^\wedge$. Moreover A_∞ structure on Floer homology on M corresponds to Yoneda and Massey-Yoneda product on sheaf cohomology on $M^\wedge$.

We remark that homological mirror symmetry conjecture was found before "second string theory revolution". Later Brane theory becomes important. Homological Mirror symmetry conjecture can then be regarded naturally as a correspondence of Branes and as a part of various dualities. After that and after Strominger-Yau-Zaslow's important proposal [100] to construct mirror manifold by using dual torus fibration, several people began to be interested in homological mirror symmetry. Among them, Polishchuk-Zaslow [86] proved a part of it in the case of elliptic curve. (Kontsevich [66] discussed the case of elliptic curve earlier.) Just after that the author generalized it partially to the case of complex torus of higher dimension [27]. The main tool used in [27] is a result of [32] which calculates the A_∞ structure of Floer homology in the case of cotangent bundle. Some more papers on mirror symmetry of Abelian variety appeared after that. There are other important works by P. Seidel and his coauthors (such as [96, 95, 94, 57]) which are also related to homological mirror symmetry conjecture.

While studying the case of complex tori, the author found various interesting and delicate phenomena happen in the study of Floer homology of Lagrangian submanifolds, its product (A_∞) structure, and also its family version ([31]). The study of Floer homology thus tied more with homological algebra and with deformation theory, which we explained in Chapters 1 and 2. Homological mirror symmetry conjecture now becomes more precise after being involved much in homological algebra and deformation theory. For example, we conjecture the coincidence of the two moduli spaces, one for deformation of Floer homology of Lagrangian submanifolds and the other for deformation of coherent sheaves or of vector bundles.

3.3 From Lagrangian submanifold to A_∞ algebra.

Now, after explaining rough story and history, let us discuss the construction of filtered A_∞ algebra associated to Lagrangian submanifolds. Here we consider the case we have only one Lagrangian submanifold L rather than a pair of Lagrangian submanifolds as in Floer's case. (The case there are two or more Langrangian submanifolds are mentioned at the end of

§3.4 and discussed in detail in [33, 30].)

The condition $\pi_2(M,L)=0$ we assumed before implies that there exist no maps $\varphi : (D^2, \partial D^2) \to (M,L)$ satisfying Condition 3.2.1. In fact, since φ is zero homotopic it follows that $\int_{D^2} \varphi^*\omega = 0$. We can easily show that if φ is pseudoholomorphic and is non constant then $\int_{D^2} \varphi^*\omega > 0$. This is the basic reason why Theorem 3.2.2 holds. In other words, presence of pseudoholomorphic disk $\varphi : (D^2, \partial D^2) \to (M,L)$ deforms usual homology group $H(L)$ to Floer homology group $HF(L)$. (See §3.4).

Let us discuss moduli space of pseudoholomorphic disks satisfying Condition 3.2.1. Let $\beta \in \pi_2(M,L)$. We use the following moduli space.

Definition 3.3.1. The moduli space $\mathcal{M}_{k+1}(L;\beta)$ is the set of all $\sim$ equivalence classes of pairs $(\varphi, \vec z)$ where :

(1) $\varphi : (D^2, \partial D^2) \to (M,L)$ satisfies Condition 3.2.1. (2) The homotopy class of φ is β. (3) $\vec z = (z_0, \cdots, z_k)$ where $z_i \in \partial D^2$. We assume that $z_0, \cdots, z_k$ respects the cyclic order of ∂D^2.

We say $(\varphi, \vec z) \sim (\varphi', \vec z')$ if there exists a biholomorphic automorphism $u : D^2 \to D^2$ such that $\varphi' = \varphi \circ u$, $z_i = u(z'_i)$.

The basic task we need to carry out in order to use the moduli space $\mathcal{M}_{k+1}(L;\beta)$ to various problems is :

(A) (Compactifiation) Find an appropriate compactification $\mathcal{CM}_{k+1}(L;\beta)$ of it.

(B) (Transversality) Find an appropriate perturbation of the pseudoholomorphic curve equation $J \circ d\varphi = d\varphi \circ j_{D^2}$, so that the moduli space $\mathcal{CM}_{k+1}(L;\beta)$ will become a “smooth manifold” after perturbation.

(C) (Index theory) Calculate the dimension of $\mathcal{CM}_{k+1}(L;\beta)$.

(D) (Orientation) Find a condition on M, L under which $\mathcal{CM}_{k+1}(L;\beta)$ is oriented.

These are package of results one needs to establish topological field theory by nonlinear partial differential equation and moduli space of its solutions. (Donaldson [18] first used such package of results to establish invariants of 4 manifolds (Donaldon invariants). Gromov applied Donaldson's idea to the moduli space of pseudoholomorphic curves (from closed Riemann surface).)

For point (A), we can easily modify Kontsevich's ([67]) notion of stable map so that it can be applied to the case of Riemann surface with boundary (disk). See [33]. For point (B), there is now a general theory developed in [34] which can be applied to various situations in a uniform way. Hence basically there is nothing new to work out but we can just apply [34][5]. The key notion we use to carry out (B) is *space with Kuranishi structure*. We will explain it informally later. Actually it is a smooth analogue of the notion of complex analytic space discussed in §1.6.

Now the package (A),(B),(C),(D) in our situation can be stated as follows.

Theorem 3.3.1. ([33]) *There exists $\mu : \pi_2(M; L) \to \mathbb{Z}$ (the Maslov index) with the following properties.*

(1) *$\mathcal{CM}_{k+1}(L;\beta)$ is a compact space with Kuranishi structure (with corners), of dimension $\mu(\beta) + n + k - 1$.*
(2) *$\mathcal{CM}_{k+1}(L;\beta)$ is oriented if L is relatively spin, in the sense defined later. Relative spin structure determines an orientation of $\mathcal{CM}_{k+1}(L;\beta)$.*
(3) *The boundary of $\mathcal{CM}_{k+1}(L;\beta)$ is described by fiber products of various $\mathcal{CM}_{k'+1}(L;\beta')$ where $k' \le k$ and $\omega \cap \beta' \le \omega \cap \beta$.*

Explanation of the notion used in the statement of Theorem 3.3.1 will follow (after a few remarks). The statement (3) is a bit vague since we do not mention which fiber product appears. We do not explain it since the main focus of this article is on algebraic formalism and we want to minimize the explanation of geometric analysis parts of the story.

Remark 3.3.1. We explain two more properties of the Maslov index μ in Theorem 3.3.1. Let $\pi_2(M) \to \pi_2(M, L) \to \pi_1(L)$ be a part of homotopy exact sequence of the pair M, L. Then the composition $\pi_2(M) \to \pi_2(M, L) \xrightarrow{\mu} \mathbb{Z}$ coincides with $2c^1(M)$. Here $c^1(M) \in H^2(M;\mathbb{Z})$ determines $\pi_2(M) \to \mathbb{Z}$. Also $\pi_2(M) \to \pi_2(L) \xrightarrow{w^1} \mathbb{Z}_2$ coincides with μ modulo 2. Here $w^1 : \pi_1(L) \to \{0, 1\}$ is the first Stiefel Whitney class of L. (Namely $w^1(\gamma) = 0$ if orientation of TL is preserved along the loop γ. In particular μ is even valued if L is oriented.

Let us now explain notions used in Theorem 3.3.1. We will explain two notions, relative spin structure and a space with Kuranishi structure with corners. To define relative spin structure, let us recall some well-known facts on vector bundle on manifolds. First, for any vector bundle E, there exists a characteristic class $w^2(E) \in H^2(L;\mathbb{Z}_2)$, Stiefel-Whitney class such that the structure group of E is reduced to Spinor group if and only if $w^2(L) = 0$. Second, a real vector bundle E on 3 manifold is trivial if it is oriented and spin. (Namely $w_1(E) = w_2(E) = 0$.)

Definition 3.3.2. Let L be a submanifold of M. Then L is said to be relatively spin if L is oriented and if there exists a cohomology class $st \in H^2(M;\mathbb{Z}_2)$ such that $w^2(TL) = i^* st$.

[5] There is one point to clarify about transversality of $\mathcal{CM}_{k+1}(L;\beta)$ which is not included in the general theory. That is the problem of transversality at diagonal $\subset L^{k+1}$. This is point (2) mentioned in the discussion after Theorem 3.3.2, and is handled in [33].

We take an oriented vector bundle E on M such that $w^2(E) = st$. Then the *relative spin structure* of L is a trivialization of $TL \oplus E$ on the two skeleton of L.

In the case when L is spin, we may take $st = 0$. Hence relative spin structure corresponds one to one to the spin structure of L. We refer [33] for more detail on orientation of our moduli space.

We next briefly explain Kuranishi structure with corners. We consider an open neighborhood $\mathcal{U} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_{\ge 0}$ of 0. Let $F = (f^1, \cdots, f^m) : \mathcal{U} \to \mathbb{R}^m$ be a smooth map. We consider the set $F^{-1}(0)$. As in the case of analytic subset and analytic subspace (which we discussed in §1.6), the subset $F^{-1}(0)$ of $\mathcal{U}$ does not contain enough information in case $df^1, \cdots, df^m$ are not linearly independent. For example, in a simplest situation, we need to count the order of such a space (that is a moduli space of pseudoholomorphic curves). Let us suppose that $n_1 = 1, n_2 = 0$, $m = 1$ and $f(x) = x^2$. (Here $x \in \mathbb{R}$.) Set theoretically, $f^{-1}(0)$ consists of one point. But if we perturb the equation $x^2 = 0$ a bit and consider $x^2 = \epsilon$, then the number of the solutions is zero (if we count it with sign). Hence we need to "remember" additional information coming from the equation $F(x) = 0$ itself. So we need something more than a subset of $\mathcal{U}$. In the case when F is a complex analytic function, this is exactly the idea of analytic space explained in §1.6. In the case when F is a polynomial, this is the idea of scheme. What we need here is its C^∞ analogue. (Working in C^∞ category is inevitable in studying moduli space of pseudoholomorphic curves, (especially with Lagrangian boundary condition), since our problem is strictly real one and it is impossible to assume complex analyticity of the equation.) In case we work in C^∞ category, considering the ring of germs at 0 and dividing it by ideals generated by f^i does not seem to work. This is because the ring of smooth functions does not have some nice properties which are enjoyed by the ring of holomorphic functions. So in place of considering rings (or ringed spaces), we regard a pair $(\mathcal{U}, F)$ itself as a *chart* of our "space". We then define appropriate notion of coordinate change (or equivalently define a way to glue charts). We thus obtain a space with Kuranishi structure, roughly.

More precisely, we need to include one more point into the story. Namely, in general, our moduli space is not given as $F^{-1}(0)$ but as $F^{-1}(0)/\Gamma$ locally. Let us explain the notation. Γ is a finite group. We assume that there is a linear actions of Γ on $\mathbb{R}^{n_1}$ and on $\mathbb{R}^m$. We assume that this action preserves $\mathcal{U}$. Moreover we assume that F is Γ equivariant. Hence the zero points set $F^{-1}(0)$ has a Γ action and we may consider the quotient space $F^{-1}(0)/\Gamma$. By Kuranishi structure in Theorem 3.3.1, we mean an object gluing such triples $(\mathcal{U}, F, \Gamma)$ in an appropriate sense. (Hence those objects are smooth analogue of Deligne-Mumford stack[6], see [15].) We do not try

[6] In some other situation like gauge theory, we need to consider the case when Γ is of

to define what we mean by gluing $(\mathcal{U}, F, \Gamma)$. See [34].

We said that our Kuranishi structure is one with corners, since $\mathcal{U}$ is an open subset of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_{\geq 0}$. We can define a boundary of a space with Kuranishi structure with corners.

Once we obtain oriented Kuranishi structure with corners, we can define its fundamental chain. It is a chain and is not a cycle in general since we are studying analogy of manifold with boundary or corners. We remark that, already in the situation of Theorem 3.2.1 (3), the moduli space $\mathcal{CM}(p, q; L_1, L_2)$ was a one dimensional manifold *with boundary.* Studying its boundary is the main part of the proof of the basic equality $\delta\delta = 0$. The same situation will occur in our more general setting. This is the main difference between the case of pseudoholomorphic curve from closed Riemann surface (where everything can be discussed in the level of homology) and our case of pseudoholomorphic curve from disk (where we need to work in the chain level). This causes various technical troubles which are treated in [33].

Now we finished a brief explanation of the statement of Theorem 3.3.1 and apply it to construct an A_∞ algebra.

Let L be a Lagrangian submanifold of M. We assume that L is relatively spin and fix relative spin structure. Then orientation of the moduli spaces $\mathcal{CM}_{k+1}(L; \beta)$ is induced. We next assume that the Maslov index $\mu : \pi_2(M, L) \to \mathbb{Z}$ is zero. Then, the dimension of our moduli space $\mathcal{CM}_{k+1}(L; \beta)$ is $n + k - 1$ and is independent of β.

Remark 3.3.2. We remark that this in particular implies that $c^1(M)$ is zero on $\pi_2(M)$ by Remark 3.3.1. Calabi-Yau manifold has this property. We also remark that if L is a special Lagrangian submanifold (see [100]) of Calabi-Yau manifold then Maslov index $\mu : \pi_2(M, L) \to \mathbb{Z}$ is zero. (See for example [33] for its proof.) The case of special Lagrangian submanifold in Calabi-Yau manifold is believed to be the most important in Mirror symmetry.

Definition 3.3.3. The evaluation map $ev = (ev_0, \cdots, ev_k) : \mathcal{CM}_{k+1}(L; \beta) \to L^{k+1}$ is defined by $ev[\varphi, \vec{z}] = (\varphi(z_0), \cdots, \varphi(z_k))$ where $\vec{z} = (z_0, \cdots, z_k)$.

As we mentioned already Theorem 3.3.1 and general theory of Kuranishi structure developed in [34] imply that there exists a fundamental chain $[\mathcal{CM}_{k+1}(L; \beta)] \in S_{n+k-1}(\mathcal{CM}_{k+1}(L; \beta))$. Here we may regard it as a singular chain. We use it to define an operator $\mathfrak{m}_k$.

We first take a countably generated complex of singular chains of L over $\mathbb{Q}$. (The choice of this subcomplex is based on delicate technical argument which we do not mention in this article.) We write it $S(L)$. We consider the tensor product $C^*(L) = S_{n-*}(L) \hat{\otimes}_{\mathbb{Q}} \Lambda_{\mathbb{Q},\text{nov},0}$. Here $\hat{\otimes}_{\mathbb{Q}}$ is a

positive dimension. It will then be an analogy of Artin stack.

completion with respect to the metric induced by the filtration on $\Lambda_{\mathbb{Q},\mathrm{nov},0}$. We are going to define a structure of filtered A_∞ algebra on it.

Now we define $\mathfrak{m}_k$ as follows. Let us take an element P_i of $C^{d_i}(L)$. It is a chain of degree $n - d_i$. We now take the fiber product

$$\mathcal{CM}_{k+1}(L;\beta) \times_{(ev_1,\cdots,ev_k)} (P_1 \times \cdots \times P_k).$$

It is a $\mathbb{Q}$ chain of dimension $n + k - 2 - \sum d_i$. We use the evaluation map ev_0 to regard it as a chain in L. We thus obtain

$$ev_* \left(\mathcal{CM}_{k+1}(L;\beta) \times_{(ev_1,\cdots,ev_k)} (P_1 \times \cdots \times P_k)\right) \in S_{n+k-2-\sum d_i}(L). \quad (3.6)$$

We regard (3.6) as an element of $C^{\sum d_i+2-k}(L)$ and write it $\mathfrak{m}_{k,\beta}(P_1, \cdots, P_k)$. In case $\beta = 0$ we need to define $\mathfrak{m}_{k,\beta}$ in a bit different way. Roughly speaking we "put"

$$\mathfrak{m}_{1,0}(P) = \partial P, \qquad \mathfrak{m}_{2,0}(P_1, P_2) = P_1 \cap P_2. \quad (3.7)$$

However (3.7) itself is not correct as we will mention later.

Definition 3.3.4. We put :

$$\mathfrak{m}_k(P_1, \cdots, P_k) = \sum_\beta T^{[\omega]\cap\beta}\mathfrak{m}_{k,\beta}(P_1, \cdots, P_k), \quad (3.8)$$

and extends it to a $\Lambda_{\mathbb{Q},\mathrm{nov},0}$ module homomorphism.

Then the main theorem of [33] is :

Theorem 3.3.2. $\mathfrak{m}_k$ *defines a structure of filtered A_∞ algebra on $C^*(L)$.*

Before mentioning various serious and delicate points in the rigorous argument to justify Definition 3.3.4 and proving Theorem 3.3.2, let us explain parts of the arguments which are easier to explain.

(A) We first need to show that (3.8) converges in $C^*(L)$. This is nontrivial since there are infinitely many terms involved. However we can prove it by using Gromov compactness. Namely Gromov compactness implies that, for each E, there is only a finitely many β such that $\int_\beta \omega < E$ and that $\mathcal{CM}_{k+1}(L;\beta)$ is nonempty. This means that modulo $\mathfrak{F}^E\mathcal{CM}_{k+1}(L;\beta)$ there is only finitely many terms in (3.8), where $\mathfrak{F}^E\mathcal{CM}_{k+1}(L;\beta)$ is the energy filtration. This implies the convergence of (3.8).

(B) We next check that the degree is correct. The degree of P_i is $d_i - 1$ after shifted. On the other hand the degree of the right hand side of (3.8) is $\sum d_i - k + 1$ after shifted. Hence the degree of $\mathfrak{m}_k$ (after shifted) is 1 as required.

(C) Next we check the condition that $\mathfrak{m}_0 \equiv 0$ modulo $\Lambda_{\mathbb{Q},\mathrm{nov},0}$. This is immediate from $\mathfrak{m}_{0,\beta} = 0$ if $\beta = 0$.

(D) The proof of the fact that $\mathfrak{m}_k$ satisfies the A_∞ relation is based on Theorem 3.3.1 (3) and is roughly as follows. We study $\mathfrak{m}_{1,0} \circ \mathfrak{m}_{k,\beta}$. Since $\mathfrak{m}_{1,0}$ is the usual boundary operator by (3.7), it follows that this composition is obtained by using the boundary of the moduli space $\mathcal{CM}_{k+1}(L;\beta)$. Theorem 3.3.1 (3) asserts that the boundary of $\mathcal{CM}_{k+1}(L;\beta)$ is described as a fiber product of various similar moduli spaces. Taking fiber product of the moduli spaces corresponds to taking a composition of the operators obtained by it. Hence by looking which kinds of fiber product appears in the compactification, we find A_∞ relation. Roughly the boundary of the moduli space $\mathcal{CM}_{k+1}(L;\beta)$ is described by the following figure.

Figure 4

Now we mention more delicate parts of argument to justify Definition 3.3.4 and proving Theorem 3.3.2. We discuss them only briefly since those points are not our main concern in this article [7].

(1) We need to specify orientation to define the right hand side of (3.6) as a $\mathbb{Q}$ chain. Basically relative spin structure gives a way to define orientation of $\mathcal{CM}_{k+1}(L;\beta)$. Moreover P_i (which is Poincaré dual to a cochain over $\mathbb{Q}$) is cooriented. So we obtain a coorientaion of the fiber product. However it is rather a delicate problem to handle the orientation of the fiber product and check that A_∞ formula is correct with sign. Actually we need more than 60 pages in [33] for this purpose. It is preferable to find a simpler argument.

(2) The formula (3.7) itself actually can not be justified. The reason is that P is not transversal to P so we can not put $\mathfrak{m}_{2,0}(P,P) = P \cap P$. This is the problem of transversality at diagonal. The way to overcome is as follows. We perturb diagonal and define operations $\mathfrak{m}_{k,0}$ inductively on k so that we can define it in the chain level on a countably generated subcompex $S(L)$ of singular chain complex and it is an A_∞ algebra homotopy equivalent to the De-Rham complex. In other words, we need to use A_∞ algebra for $\mathfrak{m}_{k,0}$, which corresponds to the rational homotopy theory [8].

[7] However the author emphasizes that those ‘technical detail’ (which the author together with Oh, Ohta, Ono spend almost 5 years to work out) is the main part of the story and asserting results without working out that kinds of detail is extremely dangerous.
[8] It is realized by several specialists of surgery theory that transversality at diagonal is one of the most essential point in differential topology.

(3) We need to take a complex $S(L)$ carefully so that it is countable (otherwise we can not use Bair's category theorem to achieve transverality) and the right hand side of (3.6) is again contained in the same complex.

After this brief explanation on the proof of Theorem 3.3.2, we continue our story. Our next task is to state that the A_∞ algebra in Theorem 3.3.2 is independent of various choices involved. Precisely it is invariant up to homotopy equivalence. Let us define homotopy equivalence of filtered A_∞ algebra. It is similar to the case of usual A_∞ algebra described in Chapter 2. There is a few points to modify, which we explain below.

Let C be a filtered A_∞ algebra. We first define a filtered A_∞ algebra $C[1]\hat{\otimes}_{\Lambda_{R,\mathrm{nov}}}\Lambda_{R,\mathrm{nov}}[t,dt]$. The definition is almost the same as Definition 2.1.7. One important difference is that we take completion of the tensor product here. Namely the element of $C[1]\hat{\otimes}_{\Lambda_{R,\mathrm{nov}}}\Lambda_{R,\mathrm{nov}}[t,dt]$ is written as $P(t)+Q(t)dt$ where $P(t),Q(t)$ are *infinite* sum $P(t)=\sum t^iP_i$, $Q(t)=\sum t^iQ_i$ where $P_i,Q_i\in\mathfrak{F}^{\lambda_i}C$ with $\lambda_i\to\infty$. The operation $\mathfrak{m}_k$ on $C[1]\hat{\otimes}_{\Lambda_{R,\mathrm{nov}}}\Lambda_{R,\mathrm{nov}}[t,dt]$ is defined in the same way as Definition 2.1.7.

We write $C[1][t,dt]$ in place of $C[1]\hat{\otimes}_{\Lambda_{R,\mathrm{nov}}}\Lambda_{R,\mathrm{nov}}[t,dt]$ hereafter.

We remark that we can define a filtered A_∞ homomorphism Eval_{t+t_0} : $C[1][t,dt]\to C[1]$. In the same way as (2.10) as follows. Let $P(t)+Q(t)dt\in\Lambda_{R,\mathrm{nov}}[t,dt]$. We put :

$$\mathrm{Eval}_{t+t_0}(P(t)+Q(t)dt)=\sum t_0^iP_i \tag{3.9}$$

(3.9) is an infinite sum but it converges in $C[1]$.

Now we can define homotoy between filtered A_∞ (or L_∞) algebras in the same way as Definition 2.1.8. Theorem 2.1.1 holds in the case of filtered A_∞ (or L_∞) algebra. The definition of homotopy equivalence is also the same as Definition 2.1.9.

Now we have :

Theorem 3.3.3. *The A_∞ algebra is independent of the various choices involved (for example to compatible almost complex structures) up to homotopy equivalence. If φ is a symplectic diffeomorphism then $(C^*(L),\mathfrak{m}_*)$ is homotopy equivalent to $(C^*(\varphi(L)),\mathfrak{m}_*)$.*

We omit the proof which is in [33][9].

[9] The definition of homotoy equivalence we gave in the December 2000 version of [33] looks different from one we gave above. We proved Theorem 3.3.3 in [33] using the definition there. We will rewrite the proof of it and prove the homotopy equivalence in the sense defined above in the final version of [33]. The two definitions are actually equivalent to each other.

3.4 Maurer-Cartan equation for filtered A_∞ algebra.

In this section, we explain the way to modify the argument of §2.3 to filtered A_∞ algebra, especially to the filtered A_∞ algebra $(C^*(L), \mathfrak{m}_*)$ in §3.3. (We only discuss the case of filtered A_∞ algebra C to save notation. All the argument are parallel in the case of filtered L_∞ algebra.) The Maurer-Cartan equation is

$$\delta(e^b) = 0, \quad \text{where } b \in \cup_{\lambda>0}\mathfrak{F}^\lambda C^1. \tag{3.10}$$

Since $e^b = \sum_{k=0}^{\infty} \underbrace{b \otimes \cdots \otimes b}_{k \text{ times}}$ converges as an element of $\hat{B}C[1]$, left hand side of (3.10) makes sense. This point is different from §2.3.

In the same way as §2.3, solution of (3.10) defines a filtered A_∞ algebra $(C, \mathfrak{m}^b)$ by

$$\mathfrak{m}_k^b(x_1, \cdots, x_k) = \mathfrak{m}(e^b, x_1, e^b, \cdots, e^b, x_k, e^b).$$

We remark that $(C, \mathfrak{m}^b)$ is a filtered A_∞ algebra if $b \in \cup_{\lambda>0}\mathfrak{F}^\lambda C^1$, without assuming $\delta(e^b) = 0$. The Maurer-Cartan equation $\delta(e^b) = 0$ is equivalent to $\mathfrak{m}_0^b = 0$. Namely it is equivalent to the condition that $(C, \mathfrak{m}^b)$ is an A_∞ algebra.

Let us elaborate the assumption $b \in \cup_{\lambda>0}\mathfrak{F}^\lambda C^1$. Let us reduce the coefficient ring of our filtered A_∞ algebra C to $R = \Lambda_{\mathrm{nov},0}/\Lambda_{\mathrm{nov},+}$ and obtain $\overline{C} = C \otimes_{\Lambda_{\mathrm{nov},0}} R$. Then $\overline{C}$ together with induced operations $\overline{\mathfrak{m}}_k$ is an A_∞ algebra over R. (Note $\overline{\mathfrak{m}}_0 = 0$ by our assumption that $\mathfrak{m}_0 \equiv 0 \mod \Lambda_{\mathrm{nov},+}$.)

Hence $(C, \mathfrak{m})$ is a "deformation" over $\Lambda_{\mathrm{nov},0}$ in the sense similar to Definition 1.7.2. (However since $\mathfrak{m}_0 \neq 0$ it is not strictly so.) The condition $b \in \cup_{\lambda>0}\mathfrak{F}^\lambda C^1$ implies $(C, \mathfrak{m}^b) \otimes_{\Lambda_{\mathrm{nov},0}} R \cong (\overline{C}, \overline{\mathfrak{m}})$. Namely $(C, \mathfrak{m}^b)$ is a deformation of $(\overline{C}, \overline{\mathfrak{m}})$. In case when b satisfies Maurer-Cartan equation, $\mathfrak{m}_0^b = 0$, it is a deformation of $(\overline{C}, \overline{\mathfrak{m}})$ in the sense of Definition 1.7.2 strictly.

Thus when we study the set of solutions of (3.10), we are studying moduli space of deformations, that is deformations of deformations. We will explain in the next section that studying it is natural in mirror symmetry.

We expand equation (3.10) and obtain

$$\mathfrak{m}_0(1) + \mathfrak{m}_1(b) + \mathfrak{m}_2(b, b) + \cdots = 0. \tag{3.11}$$

Another important difference between (3.11) and (2.11) is that (3.11) is inhomogeneous. (Namely there is a term $\mathfrak{m}_0(1)$.) As a consequence $b = 0$ is not a solution of (3.11). Actually there are cases where (3.11) has no solutions.

Definition 3.4.1. We say that filtered A_∞ algebra C is *unobstructed* if (3.11) has solution. We say that b is a *bounding chain* (or Maurer-Cartan element) of C if (3.10) is satisfied.

We now define gauge equivalence of solutions of (3.10) as follows.

Definition 3.4.2. Let b, b' are bounding chains of C. Then b is said to be gauge equivalent to b' (and is written as $b \sim b'$), if there exists $\tilde b$ a bounding chain of $C[t,dt]$ such that $\mathrm{Eval}_{t=0}(\tilde b) = b$, $\mathrm{Eval}_{t=1}(\tilde b) = b'$.

We can prove that $b \sim b'$, $b' \sim b''$ imply $b \sim b''$ in a similar way to the proof of Theorem 2.2.1. (See the final version of [33].) Lemma 2.2.5 can be generalized to our situation in the same way.

Definition 3.4.3. We denote by $\widetilde{\mathcal{MC}}(C)$ the sets of all bounding chains of filtered A_∞ algebra C. The sets of gauge equivalence class of bounding chains is denoted by $\mathcal{MC}(C)$.

The following Theorem 3.4.1 follows from what we already explained. Let consider a category $\{$filtered A_∞ alg.$/R\}$ whose object is a filtered A_∞ algebra over R and whose morphism is a filtered A_∞ homomorphism. We consider its quotient category[10] $\{$filtered A_∞ alg.$/R\}/$homotopy whose object is a homotopy equivalence class of filtered A_∞ algebra over R and whose morphism is a homotopy class of A_∞ homomorphism.

Theorem 3.4.1. *$C \mapsto \mathcal{MC}(C)$ induces a functor*
$: \{$filtered A_∞ alg.$/R\}/$homotopy $\to \{$Sets$\}$.

Theorem 3.4.1 implies that the set $\mathcal{MC}(C)$ is invariant of homotopy type of C. However 'invariant as sets' does not mean so much. This is the reason we state Theorem 3.4.1 by using quotient category as above. A better way to state homotopy invariance of $\mathcal{MC}(C)$ is given later in this section (Proposition 3.4.2).

To study $\mathcal{MC}(C)$ we use canonical model as in §2.3.

Definition 3.4.4. A filtered A_∞ algebra $(C, \mathfrak{m})$ is said to be *canonical* if $\mathfrak{m}_0 = 0$ and $\mathfrak{m}_1 \equiv 0 \mod \Lambda_{\mathrm{nov},+}$.

To generalize Theorem 2.3.1 we assume a kind of finiteness condition for our A_∞ algebra C.

Definition 3.4.5. A filtered A_∞ algebra C is said to be *weakly finite* if it is homotopy equivalent to C' which is finitely generated as $\Lambda_{\mathrm{nov},0}$ module.

We need one more assumption. We recall that we assumed that C is a completion of free $\Lambda_{\mathrm{nov},0}$ module (as a $\Lambda_{\mathrm{nov},0}$ module). So, as $\Lambda_{\mathrm{nov},0}$ module, we have $C \cong \overline{C} \otimes \Lambda_{\mathrm{nov},0}$. Hence, an R module homomorphism $\overline{f}_k : B_k\overline{C}[1] \to \overline{C}[1]$ induces a filtered $\Lambda_{\mathrm{nov},0}$ module homomorphism $f_k : B_kC[1] \to C[1]$.

[10] See for example [47, 55, 35] for its definition

Definition 3.4.6. A filtered A_∞ algebra C is said to be strongly gapped if there exists λ_i and $\overline{\mathfrak{m}}_{k,i} : B_k\overline{C}[1] \to \overline{C}[1]$ such that

$$\mathfrak{m}_k = \sum_i T^{\lambda_i}\mathfrak{m}_{k,i}, \qquad \lim_{i\to\infty} \lambda_i = \infty. \tag{3.12}$$

Our main example $(C(L), \mathfrak{m}_k)$ is strongly gapped by definition. We proved in [33] §A4, that it is weakly finite.

Theorem 3.4.2. *For any weakly finite, strongly gapped, and unobstructed filtered A_∞ algebra, there exists a canonical filtered A_∞ algebra homotopy equivalence to it.*

We remark that if we require $\mathfrak{m}_1 = 0$ in the definition of canonical filtered A_∞ algebra, then we cannot prove Theorem 3.4.2. This is because we need an analogue of Lemma 2.3.1, which does not hold over $\Lambda_{\mathrm{nov},0}$ but only over a field. [11].

Proof. (Sketch) We consider $\mathfrak{m}_{1,0} = \overline{\mathfrak{m}}_0 \otimes 1$ as in (3.12) (here we put $\lambda_0 = 0$). We remark that $\mathfrak{m}_{1,0} \circ \mathfrak{m}_{1,0} = 0$ follows from $\mathfrak{m}_0 \equiv 0 \mod \Lambda_{\mathrm{nov},+}$. We use the assumption that R is a field to obtain a decomposition $\mathrm{Ker}\,\mathfrak{m}_{1,0} = \mathrm{Im}\,\mathfrak{m}_{1,0} \oplus H(C;\mathfrak{m}_{1,0})$. We put $C^*_{\mathrm{can}} = H^*(C;\mathfrak{m}_{1,0})$. We then obtain a propagator G_k such that $G_{k+1} \circ \mathfrak{m}_{1,0} + \mathfrak{m}_{1,0} \circ G_k = 1 - \Pi_{C^k_{\mathrm{can}}}$.

The construction of the structure $\mathfrak{m}^{\mathrm{can}}_k$ of filtered A_∞ algebra on C_{can} and of filtered A_∞ homomorphism $\varphi_k : B_k C_{\mathrm{can}} \to C$ then goes in a similar way to the proof of Theorem 2.3.1.

The are two differences however. One is that we use graphs such that interior vertex may have 1 or 2 edges. (In Condition 2.3.1, we assumed interior vertices have at least 3 edges.) Second, we assign λ_i to each vertex.

Each interior vertex of our graph then corresponds to a term $\mathfrak{m}_{k,\lambda_i}$ in (3.12) with $(k,\lambda_i) \neq (1,0)$. The rest of the construction of $\mathfrak{m}^{\mathrm{can}}_k$, φ_k is similar to the proof of Theorem 2.3.1 and is omitted. (See the final version of [33].)

To complete the proof, we need the following two results. Let $\varphi : C \to C'$ be a filtered A_∞ homomorphism of weakly finite strongly gapped A_∞ algebra. We assume C is unobstructed and let $b \in C$ is a bounding chain.

Theorem 3.4.3. *If $\varphi_1 : H(C,\mathfrak{m}^b_1) \to H(C',\mathfrak{m}^{\varphi_* b}_1)$ is an isomorphism, then φ is a homotopy equivalence.*

Proposition 3.4.1. *If $\overline{\varphi}_1 : H(\overline{C},\overline{\mathfrak{m}}^b_1) \to H(\overline{C}',\overline{\mathfrak{m}}^{\overline{\varphi}_* b}_1)$ is an isomorphism, then $\varphi_1 : H(C,\mathfrak{m}^b_1) \to H(C',\mathfrak{m}^{\varphi_* b}_1)$ is an isomorphism.*

[11] On the other hand, if we use the field Λ_{nov} as a coefficient ring, we can prove a lemma similar to Lemma 2.3.1. However the propagator G obtained over Λ_{nov} coefficient does not preserve energy filtration. As a consequence, if we try to define operators $\overline{\mathfrak{m}}_k$ using G, it does not converge.

Theorem 3.4.3 is an analogue of Theorem 2.2.1 and can be proved in the same way. Proposition 3.4.1 will follow from a spectral sequence which we explain later (Theorem 3.4.5). It is easy to see that $\varphi_k : B_k C_{\rm can} \to C$ satisfies the assumption of Proposition 3.4.1. Hence it is a homotopy equivalence. $\square$

We can show that homotopy equivalence between canonical A_∞ algebras are isomorphism in the same way as Proposition 2.3.1

We now use Theorem 3.4.2 to define formal Kuranishi map as follows. Let C be a canonical A_∞ algebra. Let $\mathbf{e}_i$ and $\mathbf{f}_i$ are basis of C^1 and C^2 respectively. We $b_i = \operatorname{rank}_R \overline{C}^i$. We take formal parameters $X_1, \cdots, X_{b_1}$, $X_1^\epsilon, \cdots, X_{b_1}^\epsilon$ and put

$$\Lambda_{\rm nov,0}[[X_1,\cdots,X_{b_1}]] = \Lambda_{\rm nov,0}\hat\otimes_R R[[X_1,\cdots,X_{b_1}]]$$
$$\Lambda_{\rm nov,0}\langle X_1^\epsilon,\cdots,X_{b_1}^\epsilon\rangle = \Lambda_{\rm nov,0}\hat\otimes_R R[X_1^\epsilon,\cdots,X_{b_1}^\epsilon]$$

We remark that $\Lambda_{\rm nov,0}\langle X_1^\epsilon,\cdots,X_{b_1}^\epsilon\rangle$ consists of elements

$$\sum a_{i_1,\cdots,i_{b_1}} X_1^{i_1}\cdots X_{b_1}^{i_{b_1}}$$

such that $a_{i_1,\cdots,i_{b_1}} \in \mathfrak{F}^{\lambda_{i_1,\cdots,i_{b_1}}}\Lambda_{\rm nov,0}$ with

$$\lim_{\min\{i_1,\cdots,i_{b_1}\}\to\infty} \lambda_{i_1,\cdots,i_{b_1}} = \infty.$$

$\Lambda_{\rm nov,0}\langle X_1^\epsilon,\cdots,X_{b_1}^\epsilon\rangle$ is called *strictly convergent ring* in Rigid analytic geometry. See [12].

We define $P_j \in \Lambda_{\rm nov,0}[[X_1,\cdots,X_{b_1}]]$, $P_j^\epsilon \in \Lambda_{\rm nov,0}\langle X_1^\epsilon,\cdots,X_{b_1}^\epsilon\rangle$ by

$$\sum P_j(X_1,\cdots,X_{b_1})\mathbf{f}_j = \mathfrak{m}\left(\exp(X_1\mathbf{e}_1+\cdots+X_{b_1}\mathbf{e}_{b_1})\right)$$
$$\sum P_j^\epsilon(X_1^\epsilon,\cdots,X_{b_1}^\epsilon)\mathbf{f}_j, = \mathfrak{m}\left(\exp(T^\epsilon X_1^\epsilon\mathbf{e}_1+\cdots+T^\epsilon X_{b_1}^\epsilon\mathbf{e}_{b_1})\right).$$

Definition 3.4.7. We put $\mathfrak{K}_C = \Lambda_{\rm nov,0}[[X_1,\cdots,X_{b_1}]]/(P_1,\cdots,P_{b_2})$.

We also put : $\mathfrak{K}_C^\epsilon = \Lambda_{\rm nov,0}\langle X_1^\epsilon,\cdots,X_{b_1}^\epsilon\rangle/(P_1^\epsilon,\cdots,P_{b_2}^\epsilon)$.

For $\epsilon < \delta$ we define a homomorphism $\pi_{\epsilon,\delta} : \Lambda_{\rm nov,0}\langle X_1^\epsilon,\cdots,X_{b_1}^\epsilon\rangle \to \Lambda_{\rm nov,0}\langle X_1^\delta,\cdots,X_{b_1}^\delta\rangle$ by $\pi_{\epsilon,\delta}(X_i^\epsilon) = T^{\delta-\epsilon}X_i^\delta$. It induces $\pi_{\epsilon,\delta} : \mathfrak{K}_C^\epsilon \to \mathfrak{K}_C^\delta$. We then define $\mathfrak{K}_C^+ = \varprojlim \mathfrak{K}_C^\epsilon$.

We can prove easily the following proposition.

Proposition 3.4.2. *The isomorphism classes (as $\Lambda_{\rm nov,0}$ algebras) of $\mathfrak{K}_C$, $\mathfrak{K}_C^\epsilon$, $\mathfrak{K}_C^+$ are independent of the homotopy equivalence of C.*

We define $\tilde b = \sum X_i\mathbf{e}_i$, $\tilde b^\epsilon = \sum T^\epsilon X_i^\epsilon \mathbf{e}_i$. $\tilde b^\epsilon$ defines $\tilde b^+ \in \mathfrak{K}_C^+$. They satisfy Maurer-Cartan equation (3.10). Therefore $(C_{\mathfrak{K}_C}, \mathfrak{m}^{\tilde b})$, $(C_{\mathfrak{K}_C^\epsilon}, \mathfrak{m}^{\tilde b^\epsilon})$, $(C_{\mathfrak{K}_C^+}, \mathfrak{m}^{\tilde b^+})$ are A_∞ algebras. (Here we put $C_{\mathfrak{K}_C} = C\hat\otimes_{\Lambda_{\rm nov,0}}\mathfrak{K}_C$ etc.)

We remark that $\mathfrak{K}_C$ is a complete local ring whose maximal ideal is generated by T^λ (for all $\lambda > 0$), and X_i. Hence it parametrizes a deformation of $\overline{C}$ which is an A_∞ algebra over R.

On the other hand, $\mathfrak{K}_C^\epsilon$ is not a local ring. Namely its spectrum has many (closed) points. This is equivalent to the fact that we can define $f_{a_1,\cdots,a_{b_1}} : \Lambda_{\mathrm{nov},0}\langle X_1^\epsilon,\cdots,X_{b_1}^\epsilon\rangle \to \Lambda_{\mathrm{nov},0}$ by $f_{a_1,\cdots,a_{b_1}}(P(X_1^\epsilon,\cdots,X_{b_1}^\epsilon)) = P(a_1,\cdots,a_{b_1})$. (In other words $\mathrm{Spec}(\mathfrak{K}_C^\epsilon)$ is infinitesimally small in T direction but is of positive size in X_i direction.) Hence $\mathfrak{K}_C^+$ is not a local ring either. We consider the ideal $\Lambda_{\mathrm{nov},+}\mathfrak{K}_C^+$ of it and put $\overline{\mathfrak{K}}_C^+ = \mathfrak{K}_C^+/(\Lambda_{\mathrm{nov},+}\mathfrak{K}_C^+)$. We then consider a deformation $C_{\overline{\mathfrak{K}}_C^+} = (C_{\mathfrak{K}_C^+},\mathfrak{m}^{\tilde b^+})\otimes_{\mathfrak{K}_C^+}\overline{\mathfrak{K}}_C^+$. $C_{\overline{\mathfrak{K}}_C^+}$ is a deformation of $\overline{C}$ and is the restriction of the family $(C_{\mathfrak{K}_C^+},\mathfrak{m}^{\tilde b^+})$ to its "sub space" defined by $T = 0$. We find easily see that $C_{\overline{\mathfrak{K}}_C^+}$ is a trivial deformation. Namely it is isomorphic $(\overline{C},\overline{\mathfrak{m}})\otimes_R\overline{\mathfrak{K}}_C^+$. Thus, $(C_{\mathfrak{K}_C^+},\mathfrak{m}^{\tilde b^+})$ is a deformation of C whose restriction to $T = 0$ is trivial.

$(C_{\mathfrak{K}_C},\mathfrak{m}^{\tilde b})$, $(C_{\mathfrak{K}_C^+},\mathfrak{m}^{\tilde b^+})$ have the following completeness properties (Lemma 3.4.1) similar to Lemma 2.3.4. To state the lemma, we need some notations.

Let $\{$filtered complete Artin local$/\Lambda_{\mathrm{nov},0}\}$ be the category of filtered complete Artin local $\Lambda_{\mathrm{nov},0}$ algebra. $\{$filtered complete$/\Lambda_{\mathrm{nov},0}\}$ is the category of filtered complete $\Lambda_{\mathrm{nov},0}$ algebra. (They can be defined in the same way as Definition 1.7.5.)

To each a filtered A_∞ algebra C (which may not be canonical), we define functors $\mathcal{MC}(C) : \{$filtered complete Artin local$/\Lambda_{\mathrm{nov},0}\} \to \{$Sets$\}$, $\mathcal{MC}^+(C) : \{$filtered complete$/\Lambda_{\mathrm{nov},0}\} \to \{$Sets$\}$ as follows.

If $\mathfrak{R}$ be a filtered complete Artin local $\Lambda_{\mathrm{nov},0}$ algebra, then $\mathcal{MC}(C)(\mathfrak{R})$ is the set of all gauge equivalence classes of $b^+ \in C^+_{\mathfrak{R}}$ such that $b \equiv 0 \mod \mathfrak{R}_+$ and that b satisfies the Maurer-Cartan equation. (Here $\mathfrak{R}_+$ is a maximal ideal of $\mathfrak{R}$.) If $\mathcal{R}$ is a filtered complete $\Lambda_{\mathrm{nov},0}$ algebra, then $\mathcal{MC}^+(C)(\mathcal{R})$ is a set of all gauge equivalence classes of b such that $b \equiv 0 \mod \Lambda_{\mathrm{nov},+}\mathcal{R}$ and b satisfies the Maurer-Cartan equation.

Lemma 3.4.1. *If $b \in \mathcal{MC}(C)(\mathfrak{R})$, then there exists a $\Lambda_{\mathrm{nov},0}$ algebra homomorphism $\varphi : \mathfrak{K}_C \to \mathfrak{R}$ such that $\varphi(\tilde b) = b$. If $b \in \mathcal{MC}^+(C)(\mathcal{R})$, then there exists a $\Lambda_{\mathrm{nov},0}$ algebra homomorphism $\psi : \mathfrak{K}_C^+ \to \mathcal{R}$ such that $\psi(\tilde b^+) = b$.*

The proof of Lemma 3.4.1 is similar to the proof of Lemma 2.3.4. We next discuss universality. We need a similar condition to one in Theorem 2.3.2. We consider the homomorphisms :

$$\pi : H^0(C_{\mathfrak{K}_C},\mathfrak{m}_1^{\tilde b}) \to H^0(\overline{C},\overline{\mathfrak{m}}_1), \tag{3.13a}$$

$$\pi : H^0(C_{\mathfrak{K}_C^+},\mathfrak{m}_1^{\tilde b^+}) \to H^0(\overline{C},\overline{\mathfrak{m}}_1). \tag{3.13b}$$

Lemma 3.4.2. *If* (3.13) *is surjective then the homomorphisms φ and ψ in Lemma* 3.4.1 *is unique.*

The proof is the same as the proof of Theorem 2.3.2.

These two lemmata immediately imply the following Theorem 3.4.4. We need some more notation to state it. We define another functors $\mathfrak{F}_{\mathfrak{K}_C}$: {filtered complete Artin local/$\Lambda_{\text{nov},0}$} $\to$ {Sets} so that $\mathfrak{F}_{\mathfrak{K}_C}(\mathfrak{R})$ is the set of all $\Lambda_{\text{nov},0}$ algebra homomorphisms $\mathfrak{K}_C \to \mathfrak{R}$. We define $\mathfrak{F}_{\mathfrak{K}_C^+}$: {filtered complete/$\Lambda_{\text{nov},0}$} $\to$ {Sets} so that $\mathfrak{F}_{\mathfrak{K}_C^+}(\mathcal{R})$ is the set of all $\Lambda_{\text{nov},0}$ algebra homomorphisms $\psi : \mathfrak{K}_C^+ \to \mathcal{R}$.

Theorem 3.4.4. *If* (3.13) *is surjective then the functor $\mathcal{MC}(C)$ is equivalent to $\mathfrak{F}_{\mathfrak{K}_C}$ and $\mathcal{MC}^+(C)$ is equivalent to $\mathfrak{F}_{\mathfrak{K}_C^+}$.*

Let us consider the case of the filtered A_∞ algebra $C(L)$ of Lagrangian submanifold defined in the last section. We assume that L is connected and $C(L)$ is unobstructed. The A_∞ algebra $\overline{C}(L) = C(L)/(\Lambda_{\text{nov},+} \cdot C(L))$ is one by rational homotopy type of L. In particular $H^0(\overline{C}(L); \overline{\mathfrak{m}}_1) = R$ since L is connected. Its generator is the fundamental cycle $[L]$. We proved in [33] that $[L]$ is gives nonzero element in $H^0(C_{\mathfrak{K}_C}, \mathfrak{m}_1^{\tilde{b}})$, $H^0(C_{\mathfrak{K}_C^+}, \mathfrak{m}_1^{\tilde{b}})$.[12] Therefore (3.13) is surjective in this case. Hence $\mathfrak{F}_{\mathfrak{K}_C}$, $\mathfrak{F}_{\mathfrak{K}_C^+}$ are universal moduli spaces of appropriate Maurer-Cartan functors. This fact may be related to the stability of the mirror object in complex side. (Compare [104].)

We recall that $\mathfrak{K}_C^+$ represents the moduli functor of deformations of C such that its restriction to $T = 0$ is trivial. This moduli functor is not infinitesimal one, since it is a functor from {filtered complete/$\Lambda_{\text{nov},0}$} whose object is not necessary Artin or local. So it makes sense to say a point of it. (On the other hand, $\mathfrak{K}_C$ represents a moduli functor of *infinitesimal* deformation of C.)

Theorem 3.4.4 implies $\mathcal{MC}(C) = Hom_{\Lambda_{\text{nov},0}}(\mathfrak{K}_C^+, \Lambda_{\text{nov},0})$ if (3.13) is surjective. (Here the right hand side is the set of all $\Lambda_{\text{nov},0}$ algebra homomorphisms which is continuous with respect to the $\Lambda_{\text{nov},+}$ adic topology.) In other words, $\mathcal{MC}(C)$ is the set of R valued points[13] of $\mathfrak{K}_C^+$.

We next explain a spectral sequence which describe a relation of $(C, \mathfrak{m}_1^b)$ to $(\overline{C}, \overline{\mathfrak{m}}_1)$. In the case of filtered A_∞ algebra $C(L)$ of Lagrangian submanifold, it gives a relation between cohomology of L and Floer cohomology $H(C, \mathfrak{m}_1^b)$. In the case when $\pi_2(M, L) = 0$, Theorem 3.2.3 asserts that Floer cohomology is isomorphic to the usual cohomology. The reason for it was there was no holomorphic disk when $\pi_2(M, L) = 0$. If this assumption is not satisfied, then Floer cohomology may not be equal to the usual

[12] We proved in [33] that it is a (homotopy) unit of our A_∞ algebra $C(L)$. In particular, it gives a nontrivial element of the cohomology.

[13] See for example [48] for its definition.

cohomology of Lagrangian submanifold. The spectral sequence we discuss below describe the procedure how they are deformed.

Remark 3.4.1. A translation of this phenomenon to physics language might be, "instanton effect changes the dimension of the moduli space of the vacuum since it change the mass of some particle from zero to positive number".

Let b be a bounding chain of C. In general, if there exists a filtration on chain complex, we obtain a spectral sequence. (See any text book of homological algebra). Filtered A_∞ algebra C has a filtration (energy filtration) and hence $(C, \mathfrak{m}_1^b)$ is a filtered complex. However the filtration is parametrized by real number and not by integer. So we fix sufficiently small $\lambda_0 > 0$ and use the filtration $\mathcal{F}^k C = \mathfrak{F}^{k\lambda_0} C$.

The other trouble is that the ring $\Lambda_{\mathrm{nov},0}$ is not Noetherian. It causes a serious trouble when proving convergence of spectral sequence. This problem is overcome in [33] §A4. We then obtain the following theorem.

Theorem 3.4.5. *We assume that C is weakly finite and strongly gapped. Let b be a bounding chain of it. Then, there exists a spectral sequence $E_r^{p,q}$ with the following properties.*

(1) $E_2^{p,q} \cong H(\overline{C}, \overline{\mathfrak{m}}_1) \otimes_R \mathcal{F}^q \Lambda_{\mathrm{nov},0} / \mathcal{F}^{q+1} \Lambda_{\mathrm{nov},0}$.
(2) *There exists a filtration $F^q H^p(C, \mathfrak{m}_1^b)$ on $H^p(C, \mathfrak{m}_1^b)$ and r_0 such that* $E_{r_0}^{p,q} \cong E_{r_0+1}^{p,q} \cong \cdots \cong E_\infty^{p,q} \cong F^q H^p(C, \mathfrak{m}_1^b) / F^{q+1} H^p(C, \mathfrak{m}_1^b)$.

Theorem 3.4.5 was proved by Y.G.Oh [83] in the case of monotone Lagrangian submanifolds with minimal Maslov number ≥ 3. See [33] §A4 for the proof of Theorem 3.4.5. We remark Proposition 3.4.1 follows from Theorem 3.4.5.

Before going to the next section, we explain very briefly the case when there are more than one Lagrangian submanifolds. See [33] on the case when there are two Lagrangian submanifolds and [30] for the case when there are 3 or more Lagrangian submanifolds.

Let L_1, L_2 be two Lagrangian submanifolds. We *assume* that their Maslov index are zero. We also assume that they are relatively spin, namely we assume that there exists $st \in H^2(M; \mathbb{Z}_2)$ which reduces to the second Stiefel-Whitney class of L_i. We assume that we can take the same st for both of the Lagrangian sumanifolds. We then obtain filtered A_∞ algebras $(C(L_i), \mathfrak{m}_i)$. The Lagrangian intersection Floer homology then is a filtered A_∞ bimodule $CF(L_1, L_2)$. Namely it is a left $(C(L_1), \mathfrak{m}_i)$ and right $(C(L_2), \mathfrak{m}_i)$ module. We do not define A_∞ bimodule here. (See [33] for its definition.)

In case there are three of more Lagrangian submanifolds L_i, then we can define a product operations.

We can use the ring $\mathfrak{K}^+_{C(L_i)}$ defined in Definition 3.4.7 to rewrite Lagrangian intersection Floer cohomology and their product structures as follows.

Theorem 3.4.6. *Let L_i be a countable set of mutually transversal Lagrangian submanifolds. We assume that their Maslov index are all zero. We also assume that there exists $st \in H^2(M;\mathbb{Z}_2)$ which restricts to $w^2(L_i)$ for any i. We fix relative spin structure for each L_i.*

Then for each i, j, there finitely generated exists $\mathfrak{K}^+_{C(L_i)}$ $\mathfrak{K}^+_{C(L_j)}$ differential graded bimodule $(\mathcal{D}(L_i, L_j), \mathfrak{m}_1)$ and operations

$$\mathfrak{m}_k : \mathcal{D}(L_{i_1}, L_{i_2})\hat{\otimes}_{\mathfrak{K}_{C(L_2)}}\mathcal{D}(L_{i_2}, L_{i_3})\hat{\otimes}_{\mathfrak{K}^+_{C(L_3)}} \cdots$$
$$\cdots\hat{\otimes}_{\mathfrak{K}^+_{C(L_{k-1})}}\mathcal{D}(L_{i_{k-1}}, L_{i_k}) \to \mathcal{D}(L_{i_1}, L_{i_k}),$$

which satisfies A_∞ formula.

See [33] ([31]) for the proof of Theorem 3.4.6. As $\mathfrak{K}_C(L_i)\hat{\otimes}_{\Lambda_{\text{nov},0}}\mathfrak{K}_C(L_j)$ module, $\mathcal{D}(L_i, L_j)$ is :

$$\mathcal{D}(L_i, L_j) \cong \bigoplus_{p \in L_1 \cap L_2} \mathfrak{K}_C(L_i)\hat{\otimes}_{\Lambda_{\text{nov},0}}\Lambda_{\text{nov},0}[p]\hat{\otimes}_{\Lambda_{\text{nov},0}}\mathfrak{K}_C(L_j).$$

We remark that $\text{Spec}(\mathfrak{K}^+_{C(L_i)})$ is a moduli space parametrizing a deformation of A_∞ algebra $C(L_i)$. Bimodule over $\mathfrak{K}^+_{C(L_i)}$, $\mathfrak{K}^+_{C(L_j)}$ is regarded as a coherent sheaf over the product $\text{Spec}(\mathfrak{K}^+_{C(L_i)}) \times_{\text{Spec}\,\Lambda_{\text{nov},0}} \text{Spec}(\mathfrak{K}^+_{C(L_j)})$. Hence its cohomology sheaf (which is an object of derived category of coherent sheaves on $\text{Spec}(\mathfrak{K}^+_{C(L_i)}) \times_{\text{Spec}\,\Lambda_{\text{nov},0}} \text{Spec}(\mathfrak{K}^+_{C(L_j)})$) is a family of Floer homologies. (This is only a local family. To study global family we need more. See [31].) We will discuss the mirror object of one constructed in Theorem 3.4.5.

Remark 3.4.2. The bimodule $\mathcal{D}(L_i, L_j)$ and operations in Theorem 3.4.5 is invariant of various choices involved, for example the choice of compatible almost complex structure J and of various perturbations. However it is not independent of Hamiltonian diffeomphism. Namely $\mathcal{D}(L_1, L_2) \neq \mathcal{D}(\varphi_1(L_1), \varphi_2(L_2))$ in general. However Floer homology coincides if we change the coefficient ring to Λ_{nov}. See [33] for the proof.

Remark 3.4.3. We remark that we assumed that Maslov index $\pi_2(M, L) \to \mathbb{Z}$ vanishes in this section. The reason we need it is that otherwise the operator $\mathfrak{m}_k$ does not preserve degree, because the dimension of the moduli space $\mathcal{M}_{k+1}(L;\beta)$ depends on the cohomology class β. This assumption is not used anywhere else. So by considering $\mathbb{Z}_2$ graded Floer homology we may remove this assumption without difficulty. For the application to symplectic geometry, to study the general case is necessary. However the homomorphism (3.13) then may not be surjective. Including the case Maslov index is nonzero leads us the notion of *extended moduli space.* (See [87].) In the side of complex geometry, we may consider the $\mathbb{Z}_2$ graded chain complex of coherent sheaves. Including such objects also leads us extended moduli space.

Remark 3.4.4. Usually in Mirror symmetry we include a flat $U(1)$ bundle on L. Actually such a parameter is already included in our story. Namely our parameter T may be regarded a complex number. Its real part which is related to $H^1(L;\mathbb{R})$ by the spectral sequence in Theorem 3.4.5. $H^1(L;\mathbb{R})$ parametrize a deformation of our Lagrangian submanifold L. Then the imaginary part $H^1(L;\sqrt{-1}\mathbb{R})$ corresponds to the deformation of the trivial bundle on L to a flat (nontrivial) $U(1)$ bunlde.

3.5 Homological mirror symmetry.

We now are going back to the complex geometry side of the story and explain more what is expected to be a mirror of the construction of §§3.2,3.3, and 3.4. Actually the author is not an expert of complex geometry part of the story. There are many deep mathematics involved, some of which the author does not know enough. So he is afraid that there might be some error in this section. He dares to write them here since it seems almost impossible to find a person who has enough knowledge to all of many aspects of the mirror symmetry. For example the person who have enough backgroud of both symplectic part and complex part of the story is rare.

We remark that the author is much influenced by Kontsevich-Soibelman [70] while writing this section.

We first introduce some more Novikov rings. Let $\lozenge$ be a sub semigroup of $\mathbb{R}$ (namely $\lozenge \subseteq \mathbb{R}$ such that $a, b \in \lozenge$ implies $a+b \in \lozenge$). We put $\Lambda^{\lozenge}_{R,\mathrm{nov}} = \left\{\sum a_i T^{\lambda_i} \,\middle|\, \lambda_i \in \lozenge\right\}$. For example $\Lambda^{\mathbb{R}_{\ge 0}}_{R,\mathrm{nov}} = \Lambda_{R,\mathrm{nov},0}$, $\Lambda^{\mathbb{R}_{>0}}_{R,\mathrm{nov}} = \Lambda_{R,\mathrm{nov},+}$. On the other hand, $\Lambda^{\mathbb{Z}_{\ge 0}}_{R,\mathrm{nov}}$ is the formal power series ring $R[[T]]$ and $\Lambda^{\mathbb{Z}}_{R,\mathrm{nov}}$ is the Laurant polynomial ring $R[[T]][T^{-1}]$.

The example which appeared in symplectic geometry of Lagrangian submanifold is the case when $\lozenge$ is the semigroup generated by the symplectic areas (that is the integration of symplectic form) of pseudoholomorphic disks. In the case studied by Novikov himself, that is the case of Morse theory of closed one form θ, the semigroup $\lozenge$ is the set of all $\theta \cap \ell$ for $\ell \in \pi_1(M)$.

In mirror symmetry, we consider the case $\Lambda^{\mathbb{Q}_{\ge 0}}_{R,\mathrm{nov}}$, and its maximal ideal $\Lambda^{\mathbb{Q}_{\ge 0}}_{R,\mathrm{nov},+} = \Lambda^{\mathbb{Q}_{>0}}_{R,\mathrm{nov}}$ and $\Lambda^{\mathbb{Q}}_{R,\mathrm{nov}}$. One may also take : $\Lambda^{\mathbb{Q},0}_{R,\mathrm{nov}} = \bigcup_m \Lambda^{\mathbb{Z}[1/m]}_{R,\mathrm{nov}}$, $\Lambda^{\mathbb{Q}_{\ge 0},0}_{R,\mathrm{nov}} = \Lambda^{\mathbb{Q},0}_{R,\mathrm{nov}} \cap \Lambda^{\mathbb{Q}_{\ge 0}}_{R,\mathrm{nov}}$, $\Lambda^{\mathbb{Q}_{\ge 0},0}_{R,\mathrm{nov},+} = \Lambda^{\mathbb{Q},0}_{R,\mathrm{nov}} \cap \Lambda^{\mathbb{Q}_{>0}}_{R,\mathrm{nov}}$.

Exercise 3.5.1. Prove that $\Lambda^{\mathbb{Q},0}_{R,\mathrm{nov}}$ is the algebraic closure of $R[[T]][T^{-1}]$

if R is a algebraically closed field.

Elements of $\Lambda_{\mathbb{C},\mathrm{nov}}^{\mathbb{Q}_{\ge 0},0}$ is a "formal Puiseux series" $\sum_{k=0}^{\infty} a_k T^{k/n}$. The ring $\Lambda_{R,\mathrm{nov}}^{\mathbb{Q}}$ is a completion of $\Lambda_{R,\mathrm{nov}}^{\mathbb{Q},0}$.

The geometric meaning of these rings are as follows. The ring $\mathbb{C}[[T]]$ is a ring of functions of one variables. Since we are considering *formal* power series we may say it is a "ring of holomorphic functions on $D^2(0)$, disk of radius zero". Elements of $\mathbb{C}[[T]][T^{-1}]$ is meromorphic function that is a function which is defined outside the origin but have only a pole at 0. Hence $\mathbb{C}[[T]][T^{-1}]$ may be regarded as a "ring of holomorphic functions on $D^2(0)\backslash\{0\}$".

Considering $\Lambda_{R,\mathrm{nov}}^{\mathbb{Z}[1/n]}$ corresponds to taking an n fold Galois cover $D^2_n(0) \to D^2(0)$. Hence elements of its sum $\Lambda_{R,\mathrm{nov}}^{\mathbb{Q},0}$ may be regarded as a inductive limit $\varinjlim \mathcal{O}(D^2_n(0)\backslash\{0\})$. In algebraic geometry, it is impossible to consider universal cover in the usual sense, so one considers system of finite covers and take its limit. In that sense $\Lambda_{R,\mathrm{nov}}^{\mathbb{Q},0}$ is a ring of functions of "algebraic universal cover" of $D^2(0)\backslash\{0\}$.

The universal Novikov ring $\Lambda_{\mathrm{nov},0}$ we used in §3.3, 3.4 is of more transcendental nature and may be regarded as a ring of holomorphic functions of the "usual universal cover" of $D^2(0)\backslash\{0\}$.

Now let us explain how they appears in the complex geometry side of the mirror symmetry. Let us start with a symplectic manifold (M,ω). Mirror symmetry predicts that there exists a complex manifold $(M^\vee, J)$ which is a mirror to (M,ω) under some assumption. (It is not conjectured that any symplectic manifold has a mirror, however.)

To be more precise, we have to modify it a bit. First in mirror symmetry, one usually include a closed 2 form B on M, which is called a B field. The sum $\Omega = \omega + \sqrt{-1}B$ is called a complexified symplectic structure and (M,Ω) is expected to correspond to a complex manifold $(M^\vee, J) = (M,\Omega)^\vee$. We need to include B since the moduli space of complex structure is complex analytic object so its mirror (moduli space of symplectic manifold) should be modified so that it will be a complex object.

Furthermore, when we have a symplectic manifold (M,ω) we actually have a family of them that is a family $(M,z\omega)$ where z is a complex number such that $\operatorname{Re} z > 0$. (Then $(\operatorname{Re} z)\omega$ becomes the symplectic form and $(\operatorname{Im} z)\omega$ becomes a B field.) Here I would like to state two points which seems to be widely accepted among researchers of mirror symmetry.

($\star$) The mirror $(M,\omega)^\vee$ exists only if $\omega \in H^2(M;\mathbb{Q})$. Namely ω is rational homology class.
($\star\star$) If $\omega \in H^2(M;\mathbb{Z})$ and $k \in \mathbb{Z}$ then $(M,z\omega)^\vee = (M,(z+k\sqrt{-1})\omega)^\vee$.

The author does not try to explain the reason that why they are believed. Let us assume them. We then may assume $\omega \in H^2(M;\mathbb{Z})$ since we can replace ω by $n\omega$ ($n \in \mathbb{Z}$) if necessary. We put $q = e^{-2\pi z}$.

Remark 3.5.1. We remark that, in Definition 3.3.4, T appeared as $T^{\omega\cap\beta}$. Hence if we put $T = e^{-1}$ then then $T^{z\omega\cap\beta}$ will be $q^{\omega\cap\beta}$ the same form as in Definition 3.3.4. Thus by *redefining* $T = q$, the formal parameter T in the §3.3,3.4 may be regarded as a coordinate of the disk parametrizing the mirror family. (See [31] for more detail about this point.)

($\star\star$) above implies that we have a family of complex manifolds $(M^\vee, J_q) = (M, z\omega)^\vee$ parametrized by $q \in D^2(1)\backslash\{0\}$. Another conjecture (see [44, 70]) predicts that such a family is a maximal degenerate family of Calabi-Yau manifolds. Here we recall the definition of maximal degenerate family briefly. See [73, 70] for more detail.

Let $\pi : \hat M \to D^2\backslash\{0\}$ be a family of complex manifold parametrized by a unit disk D^2 minus origin. We are interested in the case when the fibers are smooth. We assume that is extends to a flat family $\pi^+ : \overline{\hat M} \to D^2$ over D^2 but assume the fiber of the origin $(\pi^+)^{-1}(0)$ is singular.

The story here is related to the theory of variation of Hodge structures. (See [39].) We have a fiber bundle $\mathcal{H}^k(M) \to D^2\backslash\{0\}$ whose fiber at $q \in D^2\backslash\{0\}$ is the cohomology group of $(\pi^+)^{-1}(q) = (M, J_q)$. The bundle $\mathcal{H}^k(M)$ is a flat bundle and the flat connection is the famous Gauss-Manin connection. Let us denote its monodromy on $H^n(M)$ by $\rho : H^n(M) \to H^n(M)$. (Here $n = \dim_{\mathbb{C}} M$.) (Since $\pi_1(D^2\backslash\{0\}) = \mathbb{Z}$ we only need to consider the generator.) It is known that the eigenvalue of ρ are all roots of unity. Namely $\rho^N - 1$ is nilpotent for some N. It is also known that $(\rho^N - 1)^{n+1} = 0$.

Definition 3.5.1. The family $\pi : \hat M \to D^2\backslash\{0\}$ is said to be a maximal degenerate family if $(\rho^N - 1)^n \ne 0$, $(\rho^N - 1)^{n+1} = 0$

In [44, 70], interesting conjecture is proposed about the behavior of Calabi-Yau metric on (M, J_q) when $q \to 0$ for a maximal degenerate family $\hat M \to D^2\backslash\{0\}$, by using Gromov-Hausdorff convergence (see [43]) of Riemannian manifolds.

We remark that maximal degenerate family is the opposite extreme to the case when monodromy is given by a Dehn twist along symplectic sphere, which is studied in detail by [96, 57, 95, 94].

Now let us return to our situation of $\pi : \hat M \to D^2\backslash\{0\}$. We obtain a differential graded algebra $\Omega^{0,*}(\hat M/D^2\backslash\{0\})$ over $\mathcal{O}(D^2\backslash\{0\})$, in the same way as we explained at the beginning of §1.7. The ring $\mathcal{O}(D^2\backslash\{0\})$ is different from the ring of meromorphic functions on D^2. Namely $\mathcal{O}(D^2\backslash\{0\})$ contains a function which has an essential singularity at 0. However using an extension of our family to 0, one can find a differential graded algebra $\hat C$ over $\mathcal{O}(D^2)$ so that $\hat C \otimes_{\mathcal{O}(D^2)} \mathcal{O}(D^2\backslash\{0\})$ is homotopy equivalent to

$\Omega^{0,*}(\hat M/D^2\backslash\{0\})$. Moreover $\hat C \otimes_{\mathcal{O}(D^2)} \mathbb{C}[[T]][T^{-1}]$ is independent to the extension but depends only on the family over $D^2\backslash\{0\}$.

However, the differential graded algebra $\hat C$ over $\mathbb{C}[[T]]$ depends on the choice of the extension of our family at 0. The choice of such extension is called a choice of model sometimes. It seems that no canonical choice of such model is know in the general situation, and seems to be related to a deep problem in algebraic geomegry. A natural choice of a model is known in some case. For example in the case of elliptic curve [59, 80].

The discussion on the Novikov ring at beginning of this section suggests that it is important to consider the family parametrized not only by $\mathbb{C}[[T]]$ but also by $\Lambda_{\mathbb{C},\mathrm{nov}}^{\mathbb{Q}_{\geq 0},0}$ or $\Lambda_{\mathbb{C},\mathrm{nov}}^{\mathbb{Q}_{\geq 0}}$. To obtain such a family, we first take an n fold cover of the base $D^2\backslash\{0\}$, pullback the family, and, consider the "limit" when $n \to \infty$. It seems that what Kontsevich-Soibelman [70] suggested about the relation of Mirror symmetry and Rigid analytic geometry is somehow related to this point. Let me mention just one example to show this relation.

Example 3.5.1. Let us consider a (real) 2 torus T^2. Its symplectic form ω is unique up to constant. Its mirror $(T^2, z\omega)^\vee$ is $\mathbb{C}/(\mathbb{Z}\oplus\sqrt{-1}z\mathbb{Z})$. This gives a standard family of elliptic curve parametrized by $q = e^{-2\pi z} \in D^2\backslash\{0\}$. The monodromy matrix is $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. In this case there is a canonical choice of the model. Namely we put type I singular fiber (in the classification of Kodaira [59]) over origin. Now we replace q by $q^{1/n}$. Then the monodromy matrix will become $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$. Hence the the singular fiber will become of type I_n.

Now, what happens when $n \to \infty$? We will have a dense set of singular points which consists of all rational points of S^1 and its completion (that is S^1) may be regarded a limit. This seems to be the picture of Rigid analytic geometry ([12, 9]).

This S^1 in turn will be the Gromov-Hausdorff limit of the Riemannian manifold $\mathbb{C}/(\mathbb{Z}\oplus\sqrt{-1}z\mathbb{Z})$ equipped with Calabi-Yau metric (which is nothing but the flat metric in this case) with diameter normalized to 1.

This seems to be the simplest case of the picture proposed by Kontsevich-Soibelman [70].

Now let us try to formulate homological mirror symmetry conjecture. Suppose we have a Lagrangian submanifold L of a Calabi-Yau manifold (M,ω) which is relatively spin and Maslov index $\pi_2(M,L) \to \mathbb{Z}$ is zero. We define :

Definition 3.5.2. L is said to be *rational* if $\int_{D^1} \varphi^*\omega \in \mathbb{Q}$ for any $[\varphi] \in \pi_2(M,L)$.

Remark 3.5.2. The definition of rationality here is tentative one. For example let us consider the case of symplectic 2-torus (T^2,ω). (2 is the real dimension.) In $T^2 = \mathbb{R}^2/(\mathbb{Z}\oplus\mathbb{Z})$, we consider Lagrangian submanifolds $\mathbb{R}/\mathbb{Z}\times\{a\}$, where $a\in\mathbb{R}/\mathbb{Z}$. I would rather like to call it rational only when $a\in\mathbb{Q}/\mathbb{Z}$. But in the sense of Definition 3.5.2 it is always rational. Such a trouble might disappear in the case when M is simply connected.

To a rational Lagrangian submanifold L, we can associate an A_∞ algebra $(C(L),\mathfrak{m})$ over $\Lambda^{\mathbb{Q},0}_{\mathrm{nov},0}$. Actually, using the fact that $\pi_2(M,L)$ is finitely generated, we can define it over $\Lambda^{\mathbb{Z}[1/m]}_{\mathrm{nov},0}$ for some $m\in\mathbb{Z}_{>0}$.

We suppose that we have a mirror family $(M,z\omega)^\vee$ parametrized by $q=e^{-z}$.[14] By Remark 3.5.1 we may identify q with T in §3.3, 3.4. We take an m hold cover of our mirror family and get a family $\pi_m:\hat{M}_m\to D^2\backslash\{0\}$. Mirror object is expected to be a family of holomorphic vector bundle over this family. But we need a bit more general object than vector bundle, namely an object of the derived category of coherent sheaves. In our situation, it may be regarded as a complex

$$\hat{\mathcal{E}}_1\xrightarrow{\delta_1}\hat{\mathcal{E}}_2\xrightarrow{\delta_2}\hat{\mathcal{E}}_3\xrightarrow{\delta_3}\cdots\xrightarrow{\delta_{N-1}}\hat{\mathcal{E}}_N. \tag{3.14}$$

here $\hat{\mathcal{E}}_i$ is a family of holomorphic vector bundles over $\pi_m:\hat{M}_m\to D^2\backslash\{0\}$ and δ_i are holomorphic section of $Hom(\hat{\mathcal{E}}_i,\hat{\mathcal{E}}_{i+1})$ over $\hat{M}_m$. We write such an object $\mathbb{E}$.

One can define a differential graded algebra describing a deformation of this objects. For example we can proceed as follows [15]. We put

$$\Omega^k(\mathbb{E},\mathbb{E}) = \bigoplus_{i,j:i\le j\le i+k}\Omega^{0,k+i-j}(\hat{M}_m/\mathcal{O}(D^2\backslash\{0\});Hom(\mathcal{E}_i,\mathcal{E}_j)). \tag{3.15}$$

For $\varphi\in\Omega^k(\mathbb{E},\mathbb{E})$ we denote its $\Omega^{0,p-\ell}(\hat{M}_m/\mathcal{O}(D^2\backslash\{0\});Hom(\mathcal{E}_i,\mathcal{E}_{i+\ell}))$ component by $\varphi_{i,\ell}$. We then put

$$(d\varphi)_{i,j} = \pm\overline{\partial}_{\mathcal{E}_j}\circ\varphi_{i,j}\pm\varphi_{i,j}\circ\overline{\partial}_{\mathcal{E}_i}\pm\delta_j\circ\varphi_{i,j-1}\pm\varphi_{i+1,j}\circ\delta_i$$

We also put

$$(\varphi\circ\phi)_{i,j} = \sum_\ell \pm\varphi_{\ell,j}\circ\phi_{i,\ell}$$

(See [27] Chapter 4 for sign.) We can check $(\Omega^*(\mathbb{E},\mathbb{E}),d,\circ)$ is a differential graded algebra and hence an A_∞ algebra over $\mathcal{O}(D^2\backslash\{0\})$ (where $T^{1/m}$ is the coordinate). If we can extend $\mathbb{E}$ to a family over D^2 then we have a differential graded algebra over $\mathcal{O}(D^2)$. We formalize it as in §1.7 and

[14] Construction of mirror family is now studied extensively by several people. In this paper the author do not try to discuss it.

[15] Compare [27] Chapter 4. There we developed a similar but more general construction in the case of twisted complex.

obtain a differential graded algebra over $\Lambda_{\mathrm{nov},0}^{\mathbb{Z}[1/m]}$ or $\Lambda_{\mathrm{nov}}^{\mathbb{Z}[1/m]}$. We remark that a differential graded algebra over $\Lambda_{\mathrm{nov},0}^{\mathbb{Z}[1/m]}$ depends on choice of the extension (model).

Now a part of homological mirror symmetry conjecture is stated as follows.

Conjecture 3.5.1. *Let L be a rational Lagrangian submanifold which is relatively spin and its Maslov index is zero. We assume that $(C(L), \mathfrak{m})$ is unobstructed.*

Then there exists an object $\mathbb{E}$ as in (3.15) *together with its extension to $0 \in D^2_m$, such that $(C(L), \mathfrak{m})$ is homotopy equivalent to $(\Omega^*(\mathbb{E}, \mathbb{E}), d, \circ)$ as an A_∞ algebra over $\Lambda_{\mathrm{nov},0}^{\mathbb{Z}[1/m]}$.*

We proceed to the case corresponding to Theorem 3.4.5 as follows. Let us consider the case when we have two objects $\mathbb{E}$ and $\mathbb{F}$ of derived category of coherent sheaves on $\hat{M}_m \to D^2\backslash\{0\}$. Then we can define $\Omega^*(\mathbb{E}, \mathbb{F})$ in a way similar to (3.15). It is a differential graded bimodule over $(\Omega^*(\mathbb{E}, \mathbb{E}), d, \circ)$ and $(\Omega^*(\mathbb{F}, \mathbb{F}), d, \circ)$. It induces a $\mathfrak{K}_{\Omega^*(\mathbb{E},\mathbb{E}),}$, $\mathfrak{K}_{\Omega^*(\mathbb{F},\mathbb{F})}$ differential graded bimodule $\mathcal{D}^*(\mathbb{E}, \mathbb{F})$.

Conjecture 3.5.2. *If Lagrangian submanifold L_i corresponds to $\mathbb{E}_i$ by Conjecture* 3.5.1 *then the differential graded bimodule $\mathcal{D}^*(\mathbb{E}_i, \mathbb{E}_j)$ is chain homotopy equivalent to the differential graded bimodule $\mathcal{D}^*(L_i, L_j)$.*

We remark that Conjecture 3.5.1 *implies $\mathfrak{K}_{C(L_i)} \cong \mathfrak{K}_{\Omega(\mathbb{E}_i,\mathbb{E}_i)}$. And hence $\mathcal{D}^*(\mathbb{E}_i, \mathbb{E}_j)$, $\mathcal{D}^*(L_i, L_j)$ are both $\mathfrak{K}_{C(L_i)}$ $\mathfrak{K}_{C(L_j)}$ differential graded bimodule.*

We can continue and can state the coincidence of product structures. In the complex side it is Massey-Yoneda product induced by obvious composition operators $\circ : \Omega^k(\mathbb{E}_1, \mathbb{E}_2) \otimes_{\Lambda_{\mathrm{nov},0}^{\mathbb{Z}[1/m]}} \Omega^\ell(\mathbb{E}_2, \mathbb{E}_3) \to \Omega^{k+\ell}(\mathbb{E}_1, \mathbb{E}_3)$. The most natural way to do so is to use the notion of filtered A_∞ category defined in [30]. We add two remarks.

Remark 3.5.3. In Conjecture 3.5.1 we took $\Lambda_{\mathrm{nov},0}^{\mathbb{Z}[1/m]}$ with large m as a coefficient ring. However the number m depends on $\mathbb{E}$. So to have a better statement it is natural to use $\Lambda_{\mathrm{nov},0}^{\mathbb{Q},0}$ or $\Lambda_{\mathrm{nov}}^{\mathbb{Q},0}$ in place of $\Lambda_{\mathrm{nov},0}^{\mathbb{Z}[1/m]}$ by taking limit. To go to the limit however, we have to clarify the following point. Let us denote the mth branched cover of D^2 by D^2_m. (D^2_m is actually D^2 but its coordinate is $T^{1/m}$.) Let $\hat{M}_m \to D^2_m\backslash\{0\}$. Let $\hat{M}^+_m \to D^2_m$ be a model of $\hat{M}_m$. (that is an extension of it to the origin.) Let us consider $\hat{M}_{mm'} \to D^2_{mm'}\backslash\{0\}$. We want to find $\hat{M}^+_{mm'} \to D^2_{mm'}$ together with $\hat{M}^+_{mm'} \to \hat{M}^+_m$. The naive choice, that is the fiber product $D^2_{mm'} \times_{D^2_m} \hat{M}^+_m$, does not seem to be a good choice. For example this is not correct choice in case of Example 3.5.1. In the case of Abelian variety, [80] seems to give an appropriate choice. The author does not know whether the choice of such systems $\hat M_m^+ \to D_m^2$ together with maps $\hat M_{mm'}^+ \to \hat M_m^+$ are know in the general case of, say, Calabi-Yau manifolds.

We suppose that such a choice is given. Then we can say of the coherent sheaves of such a projective system. In other words, we may consider a derived category of coherent sheaf over $\hat M$ equipped with a kind of Etal topology. Then, A_∞ category we obtain is defined over $\Lambda_{\rm nov}^{\mathbb{Q},0}$. The coherent sheaves on Berkovich spectra [9] (as discussed in [70]) might be related to such objects. The author does not have enough knowledge to discuss them at the time of writing this article.

We finally explain a mirror to Example 1.6.4. Let us first consider $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let x, y be coordinates of $\mathbb{R}^2$. We take a Lagrangian submanifold L_k of T^2 defined by $y = -kx$, where $k \in \mathbb{Z}_{\ge 0}$. We use mirror symmetry of elliptic curve by [86] then L_0 will becomes to the trivial bundle on mirror $\mathbb{T}$ (elliptic curve) and L_k becomes a complex line bundle $\mathcal{L}$.

We remark in this case Floer's condition $\pi_2(T^2, L_k) = 0$ is satisfied. Hence Floer homology is defined as in §3.3. We find that the intersections of L_0 and L_k consists of k points. We can calculate its Maslov-Viterbo index and find that it is 1. Hence $HF^0(L_0, L_k) = 0$, $HF^1(L_0, L_k) = \mathbb{C}^k$. We thus in a situation mirror to Example 1.6.4.

Now we take two Lagrangian submanifolds $L_{(1)} = L_k \times L_0$ and $L_{(2)} = L_0 \times L_k$ on direct product $T^2 \times T^2$. We take symplectic form ω on it such that $(T^2 \times \{0\}) \cap \omega = (\{0\} \times \omega) \cap \omega$. Then the mirror family is a product $\mathbb{T}_q \times \mathbb{T}_q$. When $q = e^{-z}$ converges to 0 the two factors will degenerates as in Example 3.5.1. It is easy to see that this family is maximally degenerate family.

Now it is easy to see that $L_{(1)}$, $L_{(2)}$ are mirror of $\mathrm{pr}_1^* \mathcal{L}$ and $\mathrm{pr}_2^* \mathcal{L}$, respectively. The mirror of $\mathcal{E} = \mathrm{pr}_1^* \mathcal{L} \oplus \mathrm{pr}_2^* \mathcal{L}$ should be the union $L_{(1)} \cup L_{(2)}$. It is immersed however. So the construction of §3.3,3.4 does not apply directly. However we can modify it as follows. Take $C(L_1) \oplus C(L_2)$ and add two generators $[p^{12}]$, $[p^{21}]$ to each intersection points $p \in L_1 \cap L_2$. There are k^2 intersection points $p \in L_1 \cap L_2$ which we write p_{ij} $i, j = 1, \cdots, k$. Now

$$C(L_1) \oplus C(L_2) \oplus \bigoplus_{ij} (\Lambda_{\rm nov,0}[p_{ij}^{12}] \oplus \Lambda_{\rm nov,0}[p_{ij}^{21}])$$

is our complex. (See [1].) The boundary operator $\mathfrak{m}_1$ is nontrivial on $C(L_i)$ and hence $\mathfrak{m}_1$ cohomology is

$$H^*(L_1) \oplus H^*(L_2) \oplus \bigoplus_{ij} (\Lambda_{\rm nov,0}[p_{ij}^{12}] \oplus \Lambda_{\rm nov,0}[p_{ij}^{21}]). \tag{3.16}$$

We can define a structure of A_∞ algebra on it. To calculate formal Kuranishi map we need to calculate the product structure. Since there is no holomorphic disk which bounds L_i, it follows that the operator $\mathfrak{m}_2$ is equal to the usual cup product and $\mathfrak{m}_3$ and higher are zero, on the first two components. We remark also that there is no pseudoholomorphic disk bounding union of two Lagrangian submanifolds L_1, L_2, other than trivial one. The trivial disk contributes

$$\mathfrak{m}_2(p_{ij}^{12}, p_{ij}^{21}) = [p_{ij;1}], \quad \mathfrak{m}_2(p_{ij}^{21}, p_{ij}^{12}) = -[p_{ij;2}] \tag{3.17}$$

Where $p_{ij;1}$ is a singular 0 chain $p_{ij} \in L_{(1)}$ and $p_{ij;2}$ is a singular 0 chain $p_{ij} \in L_{(2)}$. All other products are zero. We remark that $H^2(L_i) = H_0(L_i)$. Hence the degree 2 part of (3.16) is $\mathbb{C}^2$. (3.17) show that $\mathfrak{m}_2(p_{ij}^{12}, p_{ij}^{21})$ $\mathfrak{m}_2(p_{ij}^{21}, p_{ij}^{12})$ will be the first and second factor of $H^2(L_i) \oplus H^2(L_j) = \mathbb{C}^2$, respectively. We thus find :

$$\sum_{ij} x_{ij}[p_{ij;1}] + y_{ij}[p_{ij;2}] \mapsto \left(\sum_{ij} x_{ij}y_{ij}, -\sum_{ij} x_{ij}y_{ij} \right)$$

is the (nonzero part of) Kuranishi map. This map coincides with one in Example 1.6.4.

In this example, the operator is independent of $q = e^{-z}$. We can find an example that the operator actually depends on q (and is a theta function of it) in the case $M^\vee = \mathbb{T}^3$. See Chaper 4 of [27] and Chaper 7 of (2000 December version of) [33]. More example seems to be found in Physics literature.

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