Lectures on deformations of complex manifolds

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Sogliono, el piú delle volte, coloro che desiderano acquistare grazia appresso uno Principe, farsegli incontro con quelle cose che infra le loro abbino piú care, o delle quali vegghino lui piú delettarsi; donde si vede molte volte essere loro presentati cavalli, arme, drappi d'oro, pietre preziose e simili ornamenti degni della grandezza di quelli.
Desiderando io, adunque, offerirmi alla Vostra Magnificenzia con qualche testimone della servitú mia verso di quella, non ho trovato, intra la mia suppellettile, cosa quale io abbi piú cara o tanto esístimi quanto la cognizione delle [deform]azioni degli uomini grandi, imparata da me con una lunga esperienzia delle cose moderne e una continua lezione delle antique; le quali avendo io con gran diligenzia lungamente escogitate ed esaminate, e ora in uno piccolo volume ridotte, mando alla Magnificenzia Vostra.

Nicolaus Maclavellus, De principatibus

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LECTURE I

Smooth families of compact complex manifolds

In this chapter we introduce the notion of a family $f\colon \mathcal{X} \to B$ of compact complex manifolds as a proper holomorphic submersion of complex manifolds. Easy examples (I.4, I.6) will show that in general the fibres $X_t := f^{-1}(t)$ are not biholomorphic each other. Using integration of vector fields we prove that the family is locally trivial if and only if a certain morphism $\mathcal{KS}$ of sheaves over B is trivial, while the restriction of $\mathcal{KS}$ at a point $b \in B$ is a linear map $\mathrm{KS}\colon T_{b,B} \to H^1(X_b, T_{X_b})$, called the *Kodaira-Spencer map*, which can interpreted as the first derivative at the point b of the map

$$B \to \{\text{isomorphism classes of complex manifolds}\}, \quad t \mapsto X_t.$$

Then, according to Kodaira, Nirenberg and Spencer we define a *deformation* of a complex manifolds X as the data of a family $\mathcal{X} \to B$, of a base point $0 \in B$ and of an isomorphism $X \simeq X_0$. The isomorphism class of a deformation involves only the structure of f in a neighbourhood of X_0.

In the last section we state, without proof, the principal pioneer theorems about deformations proved using hard analysis by Kodaira, Nirenberg and Spencer in the period 1956-58.

1. Dictionary

For every complex manifold M we denote by:

- $\mathcal{O}_M(U)$ the $\mathbb{C}$-algebra of holomorphic functions $f\colon U \to \mathbb{C}$ defined on an open subset $U \subset M$.
- $\mathcal{O}_M$ the trivial complex line bundle $\mathbb{C} \times M \to M$.
- T_M the holomorphic tangent bundle to M. The fibre of T_M at a point $x \in M$, i.e. the complex tangent space at x, is denoted by $T_{x,M}$.

If $x \in M$ is a point we denote by $\mathcal{O}_{M,x}$ the $\mathbb{C}$-algebra of germs of holomorphic functions at a point $x \in M$; a choice of local holomorphic coordinates $z_1, \ldots, z_n$, $z_i(x) = 0$, gives an isomorphism $\mathcal{O}_{M,x} = \mathbb{C}\{z_1, \ldots, z_n\}$, being $\mathbb{C}\{z_1, \ldots, z_n\}$ the $\mathbb{C}$-algebra of convergent power series.

In order to avoid a too heavy notation we sometimes omit the subscript M, when the underlying complex manifold is clear from the context.

Definition I.1. A *smooth family of compact complex manifolds* is a proper holomorphic map $f\colon M \to B$ such that:

(1) M, B are nonempty complex manifolds and B is connected.
(2) The differential of f, $f_*\colon T_{p,M} \to T_{f(p),B}$ is surjective at every point $p \in M$.

Two families $f\colon M \to B$, $g\colon N \to B$ over the same base are isomorphic if there exists a holomorphic isomorphism $N \to M$ commuting with f and g.

From now on, when there is no risk of confusion, we shall simply say *smooth family* instead of smooth family of compact complex manifolds.

Note that if $f\colon M \to B$ is a smooth family then f is open, closed and surjective. If $V \subset B$ is an open subset then $f\colon f^{-1}(V) \to V$ is a smooth family; more generally for every

Marco Manetti: *Deformations of complex manifolds* version June 28, 2011 holomorphic map of connected complex manifolds $C \to B$, the pull-back $M \times_B C \to C$ is a smooth family.

For every $b \in B$ we denote $M_b = f^{-1}(b)$: M_b is a regular submanifold of M.

Definition I.2. A smooth family $f\colon M \to B$ is called *trivial* if it is isomorphic to the product $M_b \times B \to B$ for some (and hence all) $b \in B$. It is called *locally trivial* if there exists an open covering $B = \cup U_a$ such that every restriction $f\colon f^{-1}(U_a) \to U_a$ is trivial.

Lemma I.3. *Let $f\colon M \to B$ be a smooth family, $b \in B$. The normal bundle $N_{M_b/M}$ of M_b in M is trivial.*

Proof. Let $E = T_{b,B} \times M_b \to M_b$ be the trivial bundle with fibre $T_{b,B}$. The differential $f_*\colon T_{x,M} \to T_{b,B}$, $x \in M_b$ induces a surjective morphism of vector bundles $(T_M)_{|M_b} \to E$ whose kernel is exactly T_{M_b}.

By definition $N_{M_b/M} = (T_M)_{|M_b}/T_{M_b}$ and then $N_{M_b/M} = T_{b,B} \times M_b$. □

By a classical result (Ehresmann's theorem, [**37**, Thm. 2.4]), if $f\colon M \to B$ is a family, then for every $b \in B$ there exists an open neighbourhood $b \in U \subset B$ and a diffeomorphism $\phi\colon f^{-1}(U) \to M_b \times U$ making the following diagram commutative

$$\begin{tikzcd} & M_b \arrow[dl, "i"'] \arrow[dr, "Id\times\{b\}"] & \\ f^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & M_b \times U \arrow[dl, "p_2"] \\ & U & \end{tikzcd}$$

being $i\colon M_b \to M$ the inclusion. In particular the diffeomorphism type of the fibre M_b is independent from b. Later on (Theorem IV.30) we will prove a result that implies Ehresmann's theorem.

The following examples of families show that, in general, if $a, b \in B$, $a \neq b$, then M_a is not biholomorphic to M_b.

Example I.4. Consider $B = \mathbb{C} - \{0, 1\}$,

$$M = \{([x_0, x_1, x_2], \lambda) \in \mathbb{P}^2 \times B \,|\, x_2^2 x_0 = x_1(x_1 - x_0)(x_1 - \lambda x_0)\},$$

and $f\colon M \to B$ the projection. Then f is a family and the fibre M_λ is a smooth plane cubic with j-invariant

$$j(M_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

(Recall that two elliptic curves are biholomorphic if and only if they have the same j-invariant.)

Example I.5. (Universal family of hypersurfaces)

For fixed integers $n, d > 0$, consider the projective space $\mathbb{P}^N$, $N = \begin{pmatrix} d+n \\ n \end{pmatrix} - 1$, with homogeneous coordinates $a_{i_0,\ldots,i_n}$, $i_j \geq 0$, $\sum_j i_j = d$, and denote

$$X = \left\{ ([x], [a]) \in \mathbb{P}^n \times \mathbb{P}^N \;\middle|\; \sum_{i_0+\ldots+i_n=d} a_{i_0,\ldots,i_n} x_0^{i_0} \ldots x_n^{i_n} = 0 \right\}.$$

X is a smooth hypersurface of $\mathbb{P}^n \times \mathbb{P}^N$, the differential of the projection $X \to \mathbb{P}^N$ is not surjective at a point $([x], [a])$ if and only if $[x]$ is a singular point of X_a.

Let $B = \{[a] \in \mathbb{P}^N \,|\, X_a \text{ is smooth }\}$, $M = f^{-1}(B)$: then B is open (exercise), $f\colon M \to B$ is a family and every smooth hypersurface of degree d of $\mathbb{P}^n$ is isomorphic to a fibre of f.

Example I.6. (Hopf surfaces)
Let $A \in GL(2,\mathbb{C})$ be a matrix with eigenvalues of norm > 1 and let $\langle A\rangle \simeq \mathbb{Z} \subset GL(2,\mathbb{C})$ be the subgroup generated by A. The action of $\langle A\rangle$ on $X = \mathbb{C}^2 - \{0\}$ is free and properly discontinuous: in fact a linear change of coordinates $C\colon \mathbb{C}^2 \to \mathbb{C}^2$ changes the action of $\langle A\rangle$ into the action of $\langle C^{-1}AC\rangle$ and therefore it is not restrictive to assume A is a lower triangular matrix.
Therefore the quotient $S_A = X/\langle A\rangle$ is a compact complex manifold called *Hopf surface*: the holomorphic map $X \to S_A$ is the universal cover and then for every point $x \in S_A$ there exists a natural isomorphism $\pi_1(S_A, x) \simeq \langle A\rangle$. We have already seen that if A, B are conjugated matrix then S_A is biholomorphic to S_B. Conversely if $f\colon S_A \to S_B$ is a biholomorphism then f lifts to a biholomorphism $g\colon X \to X$ such that $gA = B^k g$; since f induces an isomorphism of fundamental groups $k = \pm 1$.
By Hartogs' theorem g extends to a biholomorphism $g\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that $g(0) = 0$; since for every $x \neq 0$ $\lim_{n\to\infty} A^n(x) = +\infty$ and $\lim_{n\to\infty} B^{-n}(x) = 0$ it must be $gA = Bg$. Taking the differential at 0 of $gA = Bg$ we get that A is conjugated to B.

Exercise I.7. If $A = e^{2\pi i\tau} I \in GL(2,\mathbb{C})$, $\tau = a + ib$, $b < 0$, then the Hopf surface S_A is the total space of a holomorphic G-principal bundle $S_A \to \mathbb{P}^1$, where $G = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. △

Example I.8. (Complete family of Hopf surfaces)
Denote $B = \{(a, b, c) \in \mathbb{C}^3 \mid |a| > 1,\ |c| > 1\}$, $X = B \times (\mathbb{C}^2 - \{0\})$ and let $\mathbb{Z} \simeq G \subset \mathrm{Aut}(X)$ be the subgroup generated by

$$(a, b, c, z_1, z_2) \mapsto (a, b, c, az_1, bz_1 + cz_2)$$

The action of G on X is free and properly discontinuous, let $M = X/G$ be its quotient and $f\colon M \to B$ the projection on the first coordinates: f is a family whose fibres are Hopf surfaces. Every Hopf surface is isomorphic to a fibre of f, this motivate the adjective "complete".
In particular all the Hopf surfaces are diffeomorphic to $S^1 \times S^3$ (to see this look at the fibre over $(2, 0, 2)$).

Notation I.9. For every pair of pointed manifolds (M, x), (N, y) we denote by $\mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$ the set of germs of holomorphic maps $f\colon (M, x) \to (N, y)$. Every element of $\mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$ is an equivalence class of pairs (U, f), where $x \in U \subset M$ is an open neighbourhood of x, $f\colon U \to N$ is a holomorphic map such that $f(x) = y$ and $(U, f) \sim (V, g)$ if and only if there exists an open subset $x \in W \subset U \cap V$ such that $f_{|W} = g_{|W}$.
The category $\mathbf{Ger}^{\mathbf{sm}}$ of germs of complex manifolds is the category whose object are the pointed complex manifold (M, x) and the morphisms are the $\mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$ defined above. A germ of complex manifold is nothing else that an object of $\mathbf{Ger}^{\mathbf{sm}}$.

In Lecture III we will consider $\mathbf{Ger}^{\mathbf{sm}}$ as a full subcategory of the category of analytic singularities $\mathbf{Ger}$.

Exercise I.10. $\mathbf{Ger}^{\mathbf{sm}}$ is equivalent to its full subcategory whose objects are $(\mathbb{C}^n, 0)$, $n \in \mathbb{N}$. △

Roughly speaking a deformation is a "framed germ" of family; more precisely

Definition I.11. Let (B, b_0) be a pointed manifold, a *deformation* $M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$ of a compact complex manifold M_0 over (B, b_0) is a pair of holomorphic maps

$$M_0 \xrightarrow{i} M \xrightarrow{f} B$$

such that:

1. $fi(M_0) = b_0$.
2. There exists an open neighbourhood $b_0 \in U \subset B$ such that $f\colon f^{-1}(U) \to U$ is a proper smooth family.
3. $i\colon M_0 \to f^{-1}(b_0)$ is an isomorphism of complex manifolds.

M is called the total space of the deformation and (B, b_0) the base germ space.

Definition I.12. Two deformations of M_0 over the same base

$$M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0), \qquad M_0 \xrightarrow{j} N \xrightarrow{g} (B, b_0)$$

are isomorphic if there exists an open neighbourhood $b_0 \in U \subset B$, and a commutative diagram of holomorphic maps

$$\begin{tikzcd} M_0 \arrow[r, "i"] \arrow[d, "j"'] & f^{-1}(U) \arrow[d, "f"] \arrow[dl] \\ g^{-1}(U) \arrow[r, "g"'] & U \end{tikzcd}$$

with the diagonal arrow a holomorphic isomorphism.

For every pointed complex manifold (B, b_0) we denote by $\mathrm{Def}_{M_0}(B, b_0)$ the set of isomorphism classes of deformations of M_0 with base (B, b_0). It is clear from the definition that if $b_0 \in U \subset B$ is open, then $\mathrm{Def}_{M_0}(B, b_0) = \mathrm{Def}_{M_0}(U, b_0)$.

Exercise I.13. There exists an action of the group $\mathrm{Aut}(M_0)$ of holomorphic isomorphisms of M_0 on the set $\mathrm{Def}_{M_0}(B, b_0)$: if $g \in \mathrm{Aut}(M_0)$ and $\xi : M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$ is a deformation we define

$$\xi^g : M_0 \xrightarrow{ig^{-1}} M \xrightarrow{f} (B, b_0).$$

Prove that $\xi^g = \xi$ if and only if $g\colon f^{-1}(b_0) \to f^{-1}(b_0)$ can be extended to an isomorphism $\hat{g}\colon f^{-1}(V) \to f^{-1}(V)$, $b_0 \in V$ open neighbourhood, such that $f\hat{g} = f$. △

If $\xi : M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$ is a deformation and $g\colon (C, c_0) \to (B, b_0)$ is a holomorphic map of pointed complex manifolds then

$$g^*\xi : M_0 \xrightarrow{(i,c_0)} M \times_B C \xrightarrow{pr} (C, c_0)$$

is a deformation with base point c_0. It is clear that the isomorphism class of $g^*\xi$ depends only by the class of g in $\mathrm{Mor}_{\mathbf{Ger}}((C, c_0), (B, b_0))$.
Therefore every $g \in \mathrm{Mor}_{\mathbf{Ger}}((C, c_0), (B, b_0))$ induces a well defined pull-back morphism

$$g^*\colon \mathrm{Def}_{M_0}(B, b_0) \to \mathrm{Def}_{M_0}(C, c_0).$$

2. Dolbeault cohomology

If M is a complex manifold and E is a holomorphic vector bundle on M, we denote:

- $E^\vee$ the dual bundle of E.
- $\Gamma(U, E)$ the space of holomorphic sections $s\colon U \to E$ on an open subset $U \subset M$.
- $\Omega^1_M = T^\vee_M$ the holomorphic cotangent bundle of M.
- $\Omega^p_M = \bigwedge^p T^\vee_M$ the bundle of holomorphic differential p-forms.

For every open subset $U \subset M$ we denote by $\Gamma(U, \mathcal{A}^{p,q}_M)$ the $\mathbb{C}$-vector space of differential (p, q)-forms on U. If $z_1, \ldots, z_n$ are local holomorphic coordinates, then $\phi \in \Gamma(U, \mathcal{A}^{p,q}_M)$ is written locally as $\phi = \sum \phi_{I,J} dz_I \wedge d\overline{z}_J$, where $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$, $dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p}$, $d\overline{z}_J = d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_q}$ and the $\phi_{I,J}$ are C^∞ functions.
Similarly, if $E \to M$ is a holomorphic vector bundle we denote by $\Gamma(U, \mathcal{A}^{p,q}(E))$ the space of differential (p, q)-forms on U with value in E; locally, if $e_1, \ldots, e_r$ is a local frame for E, an element of $\Gamma(U, \mathcal{A}^{p,q}(E))$ is written as $\sum_{i=1}^r \phi_i e_i$, with $\phi_i \in \Gamma(U, \mathcal{A}^{p,q})$. Note that there exist natural isomorphisms $\Gamma(U, \mathcal{A}^{p,q}(E)) \simeq \Gamma(U, \mathcal{A}^{0,q}(\Omega^p_M \otimes E))$.
We begin recalling the well known

Lemma I.14 (Dolbeault's lemma). *Let*

$$\Delta^n_R = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, |z_1| < R, \ldots, |z_n| < R\}$$

be a polydisk of radius $R \leq +\infty$ ($\Delta^n_{+\infty} = \mathbb{C}^n$) and let $\phi \in \Gamma(\Delta^n_R, \mathcal{A}^{p,q})$, $q > 0$, such that $\overline{\partial}\phi = 0$. Then there exists $\psi \in \Gamma(\Delta^n_R, \mathcal{A}^{p,q-1})$ such that $\overline{\partial}\psi = \phi$.

Proof. [**37**, Thm. 3.3], [**26**, pag. 25]. □

If E is a holomorphic vector bundle, the $\overline{\partial}$ operator extends naturally to the *Dolbeault operator* $\overline{\partial}\colon \Gamma(U,\mathcal{A}^{p,q}(E))\to\Gamma(U,\mathcal{A}^{p,q+1}(E))$ by the rule $\overline{\partial}(\sum_i \phi_i e_i)=\sum_i(\overline{\partial}\phi_i)e_i$. If $h_1,\ldots,h_r$ is another local frame of E then there exists a matrix (a_{ij}) of holomorphic functions such that $h_i=\sum_j a_{ij}e_j$ and then

$$\overline{\partial}\left(\sum_i \phi_i h_i\right)=\overline{\partial}\left(\sum_{i,j}\phi_i a_{ij}e_j\right)=\sum_{i,j}\overline{\partial}(\phi_i a_{ij})e_j=\sum_i(\overline{\partial}\phi_i)a_{ij}e_j=\sum_i(\overline{\partial}\phi_i)h_i.$$

It is obvious that $\overline{\partial}^2=0$.

Definition I.15. The Dolbeault's cohomology of E, $H^{p,*}_{\overline{\partial}}(U,E)$ is the cohomology of the complex

$$0\longrightarrow\Gamma(U,\mathcal{A}^{p,0}(E))\xrightarrow{\overline{\partial}}\Gamma(U,\mathcal{A}^{p,1}(E))\xrightarrow{\overline{\partial}}\ldots\xrightarrow{\overline{\partial}}\Gamma(U,\mathcal{A}^{p,q}(E))\xrightarrow{\overline{\partial}}\ldots$$

Note that $H^{p,0}_{\overline{\partial}}(U,E)=\Gamma(U,\Omega^p_M\otimes E)$ is the space of holomorphic sections.
The Dolbeault cohomology has several functorial properties; the most relevant are:

(1) Every holomorphic morphism of holomorphic vector bundles $E\to F$ induces a morphism of complexes $\Gamma(U,\mathcal{A}^{p,*}(E))\to\Gamma(U,\mathcal{A}^{p,*}(F))$ and then morphisms of cohomology groups $H^{p,*}_{\overline{\partial}}(U,E)\to H^{p,*}_{\overline{\partial}}(U,F)$.
(2) The wedge product

$$\Gamma(U,\mathcal{A}^{p,q}(E))\otimes\Gamma(U,\mathcal{A}^{r,s}(F))\xrightarrow{\wedge}\Gamma(U,\mathcal{A}^{p+r,q+s}(E\otimes F)),$$

$$\left(\sum\phi_i e_i\right)\otimes\left(\sum\psi_j f_j\right)\to\sum\phi_i\wedge\psi_j e_i\otimes e_j.$$

commutes with Dolbeault differentials and then induces a *cup* product

$$\cup\colon H^{p,q}_{\overline{\partial}}(U,E)\otimes H^{r,s}_{\overline{\partial}}(U,F)\to H^{p+r,q+s}_{\overline{\partial}}(U,E\otimes F).$$

(3) The composition of the wedge product with the trace map $E\otimes E^\vee\to\mathcal{O}_M$ gives bilinear morphisms of cohomology groups

$$\cup\colon H^{p,q}_{\overline{\partial}}(U,E)\times H^{r,s}_{\overline{\partial}}(U,E^\vee)\to H^{p+r,q+s}_{\overline{\partial}}(U,\mathcal{O}_M).$$

Theorem I.16. *If M is a compact complex manifold of dimension n and $E\to M$ is a holomorphic vector bundle then for every $p,q\geq 0$:*

(1) $\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(M,E)<\infty$.
(2) (Serre's duality) *The bilinear map* $\Gamma(M,\mathcal{A}^{p,q}(E))\times\Gamma(M,\mathcal{A}^{n-p,n-q}(E^\vee))\to\mathbb{C}$,

$$(\phi,\psi)\mapsto\int_M\phi\wedge\psi$$

induces a perfect pairing $H^{p,q}_{\overline{\partial}}(M,E)\times H^{n-p,n-q}_{\overline{\partial}}(M,E^\vee)\to\mathbb{C}$ *and then an isomorphism* $H^{p,q}_{\overline{\partial}}(M,E)^\vee\simeq H^{n-p,n-q}_{\overline{\partial}}(M,E^\vee)$.

Proof. [**37**]. □

From now on we denote for simplicity $H^q(M,E)=H^{0,q}_{\overline{\partial}}(M,E)$, $h^q(M,E)=\dim_{\mathbb{C}}H^q(M,E)$, $H^q(M,\Omega^p(E))=H^{p,q}_{\overline{\partial}}(M,E)$.

Definition I.17. If M is a complex manifold of dimension n, the holomorphic line bundle $K_M=\bigwedge^n T^\vee_M=\Omega^n_M$ is called the *canonical bundle* of M.

Since $\Omega^p_M=K_M\otimes(\Omega^{n-p}_M)^\vee$, an equivalent statement of the Serre's duality is $H^p(M,E)^\vee\simeq H^{n-p}(M,K_M\otimes E^\vee)$ for every holomorphic vector bundle E and every $p=0,\ldots,n$.

The *Hodge numbers* of a fixed compact complex manifold M are by definition

$$h^{p,q}=\dim_{\mathbb{C}}H^{p,q}_{\overline{\partial}}(M,\mathcal{O})=\dim_{\mathbb{C}}H^{0,q}_{\overline{\partial}}(M,\Omega^p).$$

The *Betti numbers* of M are the dimensions of the spaces of the De Rham cohomology of M, i.e.

$$b_p = \dim_{\mathbb{C}} H^p_d(M,\mathbb{C}), \quad H^p_d(M,\mathbb{C}) = \frac{d\text{-closed } p\text{-forms}}{d\text{-exact } p\text{-forms}}.$$

EXERCISE I.18. Let $p \geq 0$ be a fixed integer and, for every $0 \leq q \leq p$, denote by $F_q \subset H^p_d(M,\mathbb{C})$ the subspace of cohomology classes represented by a d-closed form $\eta \in \oplus_{i\leq q}\Gamma(M,\mathcal{A}^{p-i,i})$. Prove that there exist injective linear morphisms $F_q/F_{q-1} \to H^{p-q,q}_{\overline{\partial}}(M,\mathcal{O})$. Deduce that $b_p \leq \sum_q h^{p-q,q}$. △

EXERCISE I.19. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and assume that $X = f^{-1}(0)$ is a regular smooth submanifold; denote $i\colon X \to \mathbb{C}^n$ the embedding.
Let $\phi \in \Gamma(\mathbb{C}^n,\mathcal{A}^{p,q})$, $q > 0$, be a differential form such that $\overline{\partial}\phi = 0$ in an open neighbourhood of X. Prove that $i^*\phi$ is $\overline{\partial}$-exact in X. (Hint: prove that there exists $\psi \in \Gamma(\mathbb{C}^n,\mathcal{A}^{p,q})$ such that $\overline{\partial}\phi = \overline{\partial}(f\psi)$.) △

EXERCISE I.20. Let $h\colon \mathbb{C}^n \to \mathbb{C}$ be holomorphic and let $U = \{z \in \mathbb{C}^n \mid h(z) \neq 0\}$. Prove that $H^q(U,\mathcal{O}_U) = 0$ for every $q > 0$. (Hint: consider the open disk $\Delta = \{t \in \mathbb{C} \mid |t| < 1\}$ and the holomorphic maps $\phi\colon U \times \Delta \to \mathbb{C}^{n+1}$, $(z,t) \mapsto (z,(1+t)h^{-1}(z))$, $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$, $f(z,u) = h(z)u - 1$; ϕ is a biholomorphism onto the open set $\{(z,u) \in \mathbb{C}^{n+1} \,|, |uh(z) - 1| < 1\}$; use Exercise I.19.) △

EXERCISE I.21. Prove that the following facts are equivalent:

(1) For every holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ there exists a holomorphic function $h\colon \mathbb{C} \to \mathbb{C}$ such that $f(z) = h(z+1) - h(z)$ for every z.
(2) $H^1(\mathbb{C} - \{0\},\mathcal{O}_{\mathbb{C}}) = 0$.

(Hint: Denote $p\colon \mathbb{C} \to \mathbb{C} - \{0\}$ the universal covering $p(z) = e^{2\pi i z}$. Given f, use a partition of unity to find a C^∞ function g such that $f(z) = g(z+1) - g(z)$; then $\overline{\partial}g$ is the pull back of a $\overline{\partial}$-closed form on $\mathbb{C} - \{0\}$.) △

3. Čech cohomology

Let E be a holomorphic vector bundle on a complex manifold M. Let $\mathcal{U} = \{U_a\}$, $a \in \mathcal{I}$, $M = \cup_a U_a$ be an open covering. For every $k \geq 0$ let $C^k(\mathcal{U},E)$ be the set of skewsymmetric sequences $\{f_{a_0,a_1,\ldots,a_k}\}$, $a_0,\ldots,a_k \in \mathcal{I}$, where $f_{a_0,a_1,\ldots,a_k}\colon U_{a_0} \cap \ldots \cap U_{a_k} \to E$ is a holomorphic section. skewsymmetric means that for every permutation $\sigma \in \Sigma_{k+1}$, $f_{a_{\sigma(0)},a_{\sigma(1)},\ldots,a_{\sigma(k)}} = (-1)^\sigma f_{a_0,a_1,\ldots,a_k}$.

The Čech differential $d\colon C^k(\mathcal{U},E) \to C^{k+1}(\mathcal{U},E)$ is defined as

$$(df)_{a_0,\ldots,a_{k+1}} = \sum_{i=0}^{k+1}(-1)^i f_{a_0,\ldots,\widehat{a_i},\ldots,a_{k+1}}.$$

Since $d^2 = 0$ (exercise) we may define cocycles $Z^k(\mathcal{U},E) = \ker d \subset C^k(\mathcal{U},E)$, coboundaries $B^k(\mathcal{U},E) = \operatorname{Im} d \subset Z^k(\mathcal{U},E)$ and cohomology groups $H^k(\mathcal{U},E) = Z^k(\mathcal{U},E)/B^k(\mathcal{U},E)$.

PROPOSITION I.22. *For every holomorphic vector bundle E and every locally finite covering $\mathcal{U} = \{U_a\}$, $a \in \mathcal{I}$, there exists a natural morphism of $\mathbb{C}$-vector spaces $\theta\colon H^k(\mathcal{U},E) \to H^{0,k}_{\overline{\partial}}(M,E)$.*

PROOF. Let $t_a\colon M \to \mathbb{C}$, $a \in \mathcal{I}$, be a partition of unity subordinate to the covering $\{U_a\}$: $supp(t_a) \subset U_a$, $\sum_a t_a = 1$, $\sum \overline{\partial} t_a = 0$.
Given $f \in C^k(\mathcal{U},E)$ and $a \in \mathcal{I}$ we consider

$$\phi_a(f) = \sum_{c_1,\ldots,c_k} f_{a,c_1,\ldots,c_k}\overline{\partial}t_{c_1} \wedge \ldots \wedge \overline{\partial}t_{c_k} \in \Gamma(U_a,\mathcal{A}^{0,k}(E)),$$

$$\phi(f) = \sum_a t_a\phi_a(f) \in \Gamma(M,\mathcal{A}^{0,k}(E)).$$

Since every $f_{a,c_1,\dots,c_k}$ is holomorphic, it is clear that $\overline{\partial}\phi_a = 0$ and then

$$\overline{\partial}\phi(f) = \sum_a \overline{\partial}t_a \wedge \phi_a(f) = \sum_{c_0,\dots,c_k} f_{c_0,\dots,c_k}\overline{\partial}t_{c_0}\wedge\dots\wedge\overline{\partial}t_{c_k}.$$

We claim that ϕ is a morphism of complexes; in fact

$$\phi(df) = \sum_a t_a \sum_{c_0,\dots,c_k} df_{a,c_0,\dots,c_k}\overline{\partial}t_{c_0}\wedge\dots\wedge\overline{\partial}t_{c_k} =$$

$$\sum_a t_a\left(\overline{\partial}\phi(f) - \sum_{i=0}^{k}\sum_{c_i}\overline{\partial}t_{c_i}\wedge\sum_{c_0,\dots,\widehat{c_i},\dots,c_k} f_{a,c_0,\dots,\widehat{c_i},\dots,c_k}\overline{\partial}t_{c_0}\wedge\dots\wedge\widehat{\overline{\partial}t_{c_i}}\wedge\dots\wedge\overline{\partial}t_{c_k}\right) =$$

$$= \sum_a t_a\overline{\partial}\phi(f) = \overline{\partial}\phi(f).$$

Setting θ as the morphism induced by ϕ in cohomology, we need to prove that θ is independent from the choice of the partition of unity. We first note that, if $df = 0$ then, over $U_a\cap U_b$, we have

$$\begin{aligned}\phi_a(f)-\phi_b(f) &= \textstyle\sum_{c_1,\dots,c_k}(f_{a,c_1,\dots,c_k} - f_{b,c_1,\dots,c_k})\overline{\partial}t_{c_1}\wedge\dots\wedge\overline{\partial}t_{c_k}\\ &= \textstyle\sum_{c_1,\dots,c_k}\sum_{i=1}^k(-1)^{i-1}f_{a,b,c_1,\dots,\hat{c}_i,\dots,c_k}\overline{\partial}t_{c_1}\wedge\dots\wedge\overline{\partial}t_{c_k}\\ &= \textstyle\sum_{i=1}^k(-1)^{i-1}\sum_{c_1,\dots,c_k}f_{a,b,c_1,\dots,\hat{c}_i,\dots,c_k}\overline{\partial}t_{c_1}\wedge\dots\wedge\overline{\partial}t_{c_k}\\ &= \sum_{i=1}^{k}\sum_{c_i}\overline{\partial}t_{c_i}\wedge\sum_{c_1,\dots,\widehat{c_i},\dots,c_k} f_{a,b,c_1,\dots,\widehat{c_i},\dots,c_k}\overline{\partial}t_{c_1}\wedge\dots\wedge\widehat{\overline{\partial}t_{c_i}}\wedge\dots\wedge\overline{\partial}t_{c_k}\\ &= 0.\end{aligned}$$

Let v_a be another partition of 1, $\eta_a = t_a - v_a$, and denote, for $f\in Z^k(\mathcal{U},E)$,

$$\tilde{\phi}_a = \sum_{c_1,\dots,c_k} f_{a,c_1,\dots,c_k}\overline{\partial}v_{c_1}\wedge\dots\wedge\overline{\partial}v_{c_k},$$

$$\psi_a^j = \sum_{c_1,\dots,c_k} f_{a,c_1,\dots,c_k}\overline{\partial}t_{c_1}\wedge\dots\wedge\overline{\partial}t_{c_{j-1}}\wedge v_{c_j}\overline{\partial}v_{c_{j+1}}\wedge\dots\wedge\overline{\partial}v_{c_k},\qquad j=1,\dots,k.$$

The same argument as above shows that $\tilde{\phi}_a = \tilde{\phi}_b$ and $\psi_a^j = \psi_b^j$ for every a,b,j. Therefore all the ψ_a^j come from a global section $\psi^j\in\Gamma(M,\mathcal{A}^{0,k-1}(E))$; moreover $\phi - \tilde{\phi} = \sum_j(-1)^{j-1}\overline{\partial}\psi^j$ and then ϕ, $\tilde{\phi}$ determine the same cohomology class. ☐

Exercise I.23. In the same situation of Proposition I.22 define, for every $k\geq 0$, $D^k(\mathcal{U},E)$ as the set of sequences $\{f_{a_0,a_1,\dots,a_k}\}$, $a_0,\dots,a_k\in\mathcal{I}$, where $f_{a_0,a_1,\dots,a_k}\colon U_{a_0}\cap\dots\cap U_{a_k}\to E$ is a holomorphic section. Denote by $i\colon C^k(\mathcal{U},E)\to D^k(\mathcal{U},E)$ the natural inclusion. The same definition of the Čech differential gives a differential $d\colon D^k(\mathcal{U},E)\to D^{k+1}(\mathcal{U},E)$ making i a morphism of complexes. Moreover, it is possible to prove (see e.g. [**73**, p. 214]) that i induce isomorphisms between cohomology groups. Prove:

(1) Given two holomorphic vector bundles E, F consider the linear maps

$$D^k(\mathcal{U},E)\otimes D^{p-k}(\mathcal{U},F)\xrightarrow{\cup}D^p(\mathcal{U},E\otimes F),\qquad (f\cup g)_{a_0,\dots,a_p} = f_{a_0,\dots,a_k}\otimes g_{a_k,\dots,a_p}.$$

Prove that $\cup$ is associative and $d(f\cup g) = df\cup g + (-1)^k f\cup dg$, where $f\in D^k(\mathcal{U},E)$.

(2) The antisymmetrizer $p\colon D^k(\mathcal{U},E)\to C^k(\mathcal{U},E)$,

$$(pf)_{a_0,\dots,a_n} = \frac{1}{(n+1)!}\sum_\sigma(-1)^\sigma f_{a_{\sigma(0)},\dots,a_{\sigma(n)}},\qquad \sigma\in\Sigma_{n+1},$$

is a morphism of complexes and then induce a morphism $p\colon H^k(D^*(\mathcal{U},E))\to H^k(\mathcal{U},E)$ such that $pi = Id$ (Hint: the readers who are frightened by combinatorics may use linearity and compatibility with restriction to open subsets $N\subset M$ of d,p to reduce the verification of $dp(f) = pd(f)$ in the case $\mathcal{U}=\{U_a\}$, $a=1,\dots,m$ finite cover and $f_{a_1,\dots,a_k}\neq 0$ only if $a_i = i$).

(3) The same definition of ϕ given in the proof of I.22 gives a morphism of complexes $\phi_E\colon D^*(\mathcal{U},E)\to\Gamma(M,\mathcal{A}^{0,*}(E))$ which is equal to the composition of ϕ and p. In particular ϕ_E induces $\tilde{\theta}\colon H^k(D^*(\mathcal{U},E))\to H^k(M,E)$ such that $\theta p=\tilde{\theta}$.
(4) Prove that, if $dg=0$ then $\phi_{E\otimes F}(f\cup g)=\phi_E(f)\wedge\phi_F(g)$. (Hint: write $0=\sum_b t_b dg_{b,a_k,\dots,a_p}$.)
(5) If E,F are holomorphic vector bundles on M then there exists a functorial cup product
$$\cup\colon H^p(\mathcal{U},E)\otimes H^q(\mathcal{U},F)\to H^{p+q}(\mathcal{U},E\otimes F)$$
commuting with θ and the wedge product in Dolbeault cohomology.

△

Theorem I.24 (Leray). *Let $\mathcal{U}=\{U_a\}$ be a locally finite covering of a complex manifold M, E a holomorphic vector bundle on M: if $H^{k-q}_{\overline{\partial}}(U_{a_0}\cap\ldots\cap U_{a_q},E)=0$ for every $q<k$ and $a_0,\dots,a_q$, then $\theta\colon H^k(\mathcal{U},E)\to H^k_{\overline{\partial}}(M,E)$ is an isomorphism.*

Proof. The complete proof requires sheaf theory and spectral sequences; here we prove "by hand" only the cases $k=0,1$: this will be sufficient for our applications.
For $k=0$ the theorem is trivial, in fact $H^0_{\overline{\partial}}(M,E)$ and $H^0(\mathcal{U},E)$ are both isomorphic to the space of holomorphic sections of E over M. Consider thus the case $k=1$; by assumption $H^1_{\overline{\partial}}(U_a,E)=0$ for every a.
Let $\phi\in\Gamma(M,\mathcal{A}^{0,1}(E))$ be a $\overline{\partial}$-closed form, then for every a there exists $\psi_a\in\Gamma(U_a,\mathcal{A}^{0,0}(E))$ such that $\overline{\partial}\psi_a=\phi$. The section $f_{a,b}=\psi_a-\psi_b\colon U_a\cap U_b\to E$ is holomorphic and then $f=\{f_{a,b}\}\in C^1(\mathcal{U},E)$; since $f_{a,b}-f_{c,b}+f_{c,a}=0$ for every a,b,c we have $f\in Z^1(\mathcal{U},E)$; define $\sigma(\phi)\in H^1(\mathcal{U},E)$ as the cohomology class of f. It is easy to see that $\sigma(\phi)$ is independent from the choice of the sections ψ_a; we want to prove that $\sigma=\theta^{-1}$. Let t_a be a fixed partition of unity.
Let $f\in Z^1(\mathcal{U},E)$, then $\theta(f)=[\phi]$, $\phi=\sum_b f_{a,b}\overline{\partial}t_b$; we can choose $\psi_a=\sum_b f_{a,b}t_b$ and then
$$\sigma(\phi)_{a,c}=\sum_b(f_{a,b}-f_{c,b})t_b=f_{a,c},\qquad\Rightarrow\sigma\theta=Id.$$
Conversely, if $\phi_{|U_a}=\overline{\partial}\psi_a$ then $\theta\sigma([\phi])$ is the cohomology class of
$$\overline{\partial}\sum_b(\psi_a-\psi_b)t_b=\overline{\partial}\sum_b\psi_a t_b-\overline{\partial}\sum_b\psi_b t_b=\phi-\overline{\partial}\sum_b\psi_b t_b.$$
□

Remark I.25. The theory of Stein manifolds (see e.g. [**28**]) says that the hypotheses of Theorem I.24 are satisfied for every k whenever every U_a is biholomorphic to an open convex subset of $\mathbb{C}^n$.

Example I.26. Let $T\to\mathbb{P}^1$ be the holomorphic tangent bundle, x_0,x_1 homogeneous coordinates on $\mathbb{P}^1$, $U_i=\{x_i\neq 0\}$. Since the tangent bundle of $U_i=\mathbb{C}$ is trivial, by Dolbeault's lemma, $H^1(U_i,T)=0$ and by Leray's theorem $H^i(\mathbb{P}^1,T)=H^i(\{U_0,U_1\},T)$, $i=0,1$.
Consider the affine coordinates $s=x_1/x_0$, $t=x_0/x_1$, then the holomorphic sections of T over U_0,U_1 and $U_{0,1}=U_0\cap U_1$ are given respectively by convergent power series
$$\sum_{i=0}^{+\infty}a_is^i\frac{\partial}{\partial s},\qquad\sum_{i=0}^{+\infty}b_it^i\frac{\partial}{\partial t},\qquad\sum_{i=-\infty}^{+\infty}c_is^i\frac{\partial}{\partial s}.$$
Since, over $U_{0,1}$, $t=s^{-1}$ and $\dfrac{\partial}{\partial t}=-s^2\dfrac{\partial}{\partial s}$, the Cech differential is given by
$$d\left(\sum_{i=0}^{+\infty}a_is^i\frac{\partial}{\partial s},\sum_{i=0}^{+\infty}b_it^i\frac{\partial}{\partial t}\right)=\sum_{i=0}^{+\infty}a_is^i\frac{\partial}{\partial s}+\sum_{i=-\infty}^{2}b_{2-i}s^i\frac{\partial}{\partial s},$$

and then $H^1(\{U_0,U_1\},T)=0$ and

$$H^0(\{U_0,U_1\},T)=\left\langle\left(\frac{\partial}{\partial s},-t^2\frac{\partial}{\partial t}\right),\left(s\frac{\partial}{\partial s},-t\frac{\partial}{\partial t}\right),\left(s^2\frac{\partial}{\partial s},-\frac{\partial}{\partial t}\right)\right\rangle.$$

Example I.27. If $X=\mathbb{P}^1\times\mathbb{C}^n_t$ then $H^1(X,T_X)=0$. If $\mathbb{C}\subset\mathbb{P}^1$ is an affine open subset with affine coordinate s, then $H^0(X,T_X)$ is the free $\mathcal{O}(\mathbb{C}^n)$-module generated by

$$\frac{\partial}{\partial t_1},\ldots,\frac{\partial}{\partial t_n},\frac{\partial}{\partial s},s\frac{\partial}{\partial s},s^2\frac{\partial}{\partial s}.$$

The proof is essentially the same (replacing the constant terms a_i,b_i,c_i with holomorphic functions over $\mathbb{C}^n$) of Example I.26.

4. The Kodaira-Spencer map

Notation I.28. Given a holomorphic map $f\colon X\to Y$ of complex manifolds and complexified vector fields $\eta\in\Gamma(X,\mathcal{A}^{0,0}(T_X))$, $\gamma\in\Gamma(Y,\mathcal{A}^{0,0}(T_Y))$ we write $\gamma=f_*\eta$ if for every $x\in X$ we have $f_*\eta(x)=\gamma(f(x))$, where $f_*\colon T_{x,X}\to T_{f(x),Y}$ is the differential of f.

Let $f\colon M\to B$ be a fixed smooth family of compact complex manifolds, $\dim B=n$, $\dim M=m+n$; for every $b\in B$ we let $M_b=f^{-1}(b)$.

Definition I.29. A holomorphic coordinate chart $(z_1,\ldots,z_m,t_1,\ldots,t_n)\colon U\hookrightarrow\mathbb{C}^{m+n}$, $U\subset M$ open, is called *admissible* if $f(U)$ is contained in a coordinate chart $(v_1,\ldots,v_n)\colon V\hookrightarrow\mathbb{C}^n$, $V\subset B$, such that $t_i=v_i\circ f$ for every $i=1,\ldots,n$.

Since the differential of f has everywhere maximal rank, by the implicit function theorem, M admits a locally finite covering of admissible coordinate charts.

Lemma I.30. *Let $f\colon M\to B$ be a smooth family of compact complex manifolds. For every $\gamma\in\Gamma(B,\mathcal{A}^{0,0}(T_B))$ there exists $\eta\in\Gamma(M,\mathcal{A}^{0,0}(T_M))$ such that $f_*\eta=\gamma$.*

Proof. Let $M=\cup U_a$ be a locally finite covering of admissible charts; on every U_a there exists $\eta_a\in\Gamma(U_a,\mathcal{A}^{0,0}(T_M))$ such that $f_*\eta_a=\gamma$.
It is then sufficient to take $\eta=\sum_a\rho_a\eta_a$, being $\rho_a\colon U_a\to\mathbb{C}$ a partition of unity subordinate to the covering $\{U_a\}$. □

Let $T_f\subset T_M$ be the holomorphic vector subbundle of tangent vectors v such that $f_*v=0$. If $z_1,\ldots,z_m,t_1,\ldots,t_n$ is an admissible system of local coordinates then $\dfrac{\partial}{\partial z_1},\ldots,\dfrac{\partial}{\partial z_m}$ is a local frame of T_f. Note that the restriction of T_f to M_b is equal to T_{M_b}.

For every open subset $V\subset B$ let $\Gamma(V,T_B)$ be the space of holomorphic vector fields on V. For every $\gamma\in\Gamma(V,T_B)$ take $\eta\in\Gamma(f^{-1}(V),\mathcal{A}^{0,0}(T_M))$ such that $f_*\eta=\gamma$. In an admissible system of local coordinates z_i,t_j we have $\eta=\sum_i\eta_i(z,t)\dfrac{\partial}{\partial z_i}+\displaystyle\sum_j\gamma_i(t)\dfrac{\partial}{\partial t_j}$, with $\gamma_i(t)$ holomorphic, $\overline{\partial}\eta=\sum_i\overline{\partial}\eta_i(z,t)\dfrac{\partial}{\partial z_i}$ and then $\overline{\partial}\eta\in\Gamma(f^{-1}(V),\mathcal{A}^{0,1}(T_f))$.
Obviously $\overline{\partial}\eta$ is $\overline{\partial}$-closed and then we can define the *Kodaira-Spencer* map

$$\mathcal{KS}(V)_f\colon\Gamma(V,T_B)\to H^1(f^{-1}(V),T_f),\qquad \mathcal{KS}(V)_f(\gamma)=[\overline{\partial}\eta].$$

Lemma I.31. *The map $\mathcal{KS}(V)_f$ is a well-defined homomorphism of $\mathcal{O}(V)$-modules.*

Proof. If $\tilde{\eta}\in\Gamma(f^{-1}(V),\mathcal{A}^{0,0}(T_M))$, $f_*\tilde{\eta}=\gamma$, then $\eta-\tilde{\eta}\in(f^{-1}(V),\mathcal{A}^{0,0}(T_f))$ and $[\overline{\partial}\tilde{\eta}]=[\overline{\partial}\eta]\in H^1(f^{-1}(V),T_f)$.
If $g\in\mathcal{O}(V)$ then $f_*(f^*g)\eta=g\gamma$, $\overline{\partial}(f^*g)\eta=(f^*g)\overline{\partial}\eta$. □

If $V_1 \subset V_2 \subset B$ then the Kodaira-Spencer maps $\mathcal{KS}(V_i)_f \colon \Gamma(V_i, T_B) \to H^1(f^{-1}(V_i), T_f)$, $i = 1, 2$, commute with the restriction maps $\Gamma(V_2, T_B) \to \Gamma(V_1, T_B)$, $H^1(f^{-1}(V_2), T_f) \to H^1(f^{-1}(V_1), T_f)$. Therefore we get a well defined $\mathcal{O}_{B,b}$-linear map

$$\mathcal{KS}_f \colon \Theta_{B,b} \to (R^1 f_* T_f)_b,$$

where $\Theta_{B,b}$ and $(R^1 f_* T_f)_b$ are by definition the direct limits, over the set of open neighbourhood V of b, of $\Gamma(V, T_B)$ and $H^1(f^{-1}(V), T_f)$ respectively.

If $b \in B$, then there exists a linear map $\mathrm{KS}_f \colon T_{b,B} \to H^1(M_b, T_{M_b})$ such that for every open subset $b \in V \subset B$ there exists a commutative diagram

$$\begin{array}{ccc} \Gamma(V, T_B) & \xrightarrow{\mathcal{KS}(V)_f} & H^1(f^{-1}(V), T_f) \\ \big\downarrow & & \big\downarrow{\scriptstyle r} \\ T_{b,B} & \xrightarrow{\mathrm{KS}_f} & H^1(M_b, T_{M_b}) \end{array}$$

where the vertical arrows are the natural restriction maps.
In fact, if V is a polydisk then $T_{b,B}$ is the quotient of the complex vector space $\Gamma(V, T_B)$ by the subspace $I = \{\gamma \in \Gamma(V, T_B) \mid \gamma(b) = 0\}$; by $\mathcal{O}(V)$-linearity I is contained in the kernel of $r \circ \mathcal{KS}(V)_f$.

The Kodaira-Spencer map has at least two geometric interpretations: obstruction to the holomorphic lifting of vector fields and first-order variation of complex structures (this is a concrete feature of the general philosophy that deformations are a derived construction of automorphisms).

Proposition I.32. *Let $f \colon M \to B$ be a family of compact complex manifolds and $\gamma \in \Gamma(V, T_B)$, then $\mathcal{KS}(V)_f(\gamma) = 0$ if and only if there exists $\eta \in \Gamma(f^{-1}(V), T_M)$ such that $f_*\eta = \gamma$.*

Proof. One implication is trivial; conversely let $\eta \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_M))$ such that $f_*\eta = \gamma$. If $[\overline{\partial}\eta] = 0$ then there exists $\tau \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_f))$ such that $\overline{\partial}(\eta - \tau) = 0$, $\eta - \tau \in \Gamma(f^{-1}(V), T_M)$ and $f_*(\eta - \tau) = \gamma$. ☐

To compute the Kodaira-Spencer map in terms of Cech cocycles we assume that V is a polydisk with coordinates $t_1, \ldots, t_n$ and we fix a locally finite covering $\mathcal{U} = \{U_a\}$ of admissible holomorphic coordinates $z_1^a, \ldots, z_m^a, t_1^a, \ldots, t_n^a \colon U_a \to \mathbb{C}$, $t_i^a = f^* t_i$.
On $U_a \cap U_b$ we have the transition functions

$$\begin{cases} z_i^b = g_{i,a}^b(z^a, t^a), & i = 1, \ldots, m \\ t_i^b = t_i^a, & i = 1, \ldots, n \end{cases}$$

Consider a fixed integer $h = 1, \ldots, n$ and $\eta \in \Gamma(f^{-1}(V), \mathcal{A}^{0,0}(T_M))$ such that $f_*\eta = \dfrac{\partial}{\partial t_h}$; in local coordinates we have

$$\eta = \sum_i \eta_i^a(z^a, t^a) \frac{\partial}{\partial z_i^a} + \frac{\partial}{\partial t_h^a}, \qquad \eta = \sum_i \eta_i^b(z^b, t^b) \frac{\partial}{\partial z_i^b} + \frac{\partial}{\partial t_h^b}.$$

Since, for every a, $\eta - \dfrac{\partial}{\partial t_h^a} \in \Gamma(U_a, \mathcal{A}^{0,0}(T_f))$ and $\overline{\partial}\left(\eta - \dfrac{\partial}{\partial t_h^a}\right) = \overline{\partial}\eta$, $\mathcal{KS}(V)_f\left(\dfrac{\partial}{\partial t_h}\right) \in H^1(\mathcal{U}, T_f)$ is represented by the cocycle

Formula I.33.

$$\mathcal{KS}(V)_f\left(\frac{\partial}{\partial t_h}\right)_{b,a} = \left(\eta - \frac{\partial}{\partial t_h^b}\right) - \left(\eta - \frac{\partial}{\partial t_h^a}\right) = \frac{\partial}{\partial t_h^a} - \frac{\partial}{\partial t_h^b} = \sum_i \frac{\partial g_{i,a}^b}{\partial t_h^a} \frac{\partial}{\partial z_i^b}.$$

The above formula allows to prove easily the invariance of the Kodaira-Spencer maps under base change; more precisely if $f \colon M \to B$ is a smooth family, $\phi \colon C \to B$ a holomorphic map, $\hat{\phi}$, $\hat{f}$ the pullbacks of ϕ and f,

$$\begin{array}{ccc} M \times_B C & \xrightarrow{\hat{\phi}} & M \\ \Big\downarrow{\scriptstyle \hat{f}} & & \Big\downarrow{\scriptstyle f} \\ C & \xrightarrow{\phi} & B \end{array}$$

$c \in C$, $b = f(c)$.

THEOREM I.34. *In the above notation, via the natural isomorphism $M_b = \hat{f}^{-1}(c)$, we have*

$$\mathrm{KS}_{\hat{f}} = \mathrm{KS}_f \phi_* \colon T_{c,C} \to H^1(M_b, T_{M_b}).$$

PROOF. It is not restrictive to assume $B \subset \mathbb{C}^n_t$, $C \subset \mathbb{C}^s_u$ polydisks, $c = \{u_i = 0\}$ and $b = \{t_i = 0\}$, $t_i = \phi_i(u)$.
If $z^a, t^a \colon U_a \to \mathbb{C}$, $z^b, t^b \colon U_b \to \mathbb{C}$ are admissible local coordinate sets with transition functions $z^b_i = g^b_{i,a}(z^a, t^a)$, then $z^a, u^a \colon U_a \times_B C \to \mathbb{C}$, $z^b, t^b \colon U_b \times_B C \to \mathbb{C}$ are admissible with transition functions $z^b_i = g^b_{i,a}(z^a, \phi(u^a))$.
Therefore

$$\mathrm{KS}_{\hat{f}}\left(\frac{\partial}{\partial u_h}\right)_{b,a} = \sum_i \frac{\partial g^b_{i,a}}{\partial u^a_h}\frac{\partial}{\partial z^b_i} = \sum_{i,j} \frac{\partial g^b_{i,a}}{\partial t^a_j}\frac{\partial \phi_j}{\partial u^a_h}\frac{\partial}{\partial z^b_i} = \mathrm{KS}_f\left(\phi_* \frac{\partial}{\partial u_h}\right)_{b,a}.$$

□

It is clear that the Kodaira-Spencer map $\mathrm{KS}_f \colon T_{b_0,B} \to H^1(M_0, T_{M_0})$ is defined for every isomorphism class of deformation $M_0 \to M \xrightarrow{f} (B, b_0)$: The map $\mathcal{K}S_f \colon \Theta_{B,b_0} \to (R^1 f_* T_f)_{b_0}$ is defined up to isomorphisms of the $\mathcal{O}_{B,b_0}$ module $(R^1 f_* T_f)_{b_0}$.

DEFINITION I.35. Consider a deformation $\xi : M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$, $fi(M_0) = b_0$, with Kodaira-Spencer map $\mathrm{KS}_\xi \colon T_{b_0,B} \to H^1(M_0, T_{M_0})$. ξ is called:

(1) *Versal* if KS_ξ is surjective and for every germ of complex manifold (C, c_0) the morphism

$$\mathrm{Mor}_{\mathbf{Ger}}((C, c_0), (B, b_0)) \to \mathrm{Def}_{M_0}(C, c_0), \qquad g \mapsto g^*\xi$$

is surjective.
(2) *Semiuniversal* if it is versal and KS_ξ is bijective.
(3) *Universal* if KS_ξ is bijective and for every pointed complex manifolds (C, c_0) the morphism

$$\mathrm{Mor}_{\mathbf{Ger}}((C, c_0), (B, b_0)) \to \mathrm{Def}_{M_0}(C, c_0), \qquad g \mapsto g^*\xi$$

is bijective.

Versal deformations are also called *complete*; semiuniversal deformations are also called *miniversal* or *Kuranishi* deformations.
Note that if ξ is semiuniversal, $g_1, g_2 \in \mathrm{Mor}_{\mathbf{Ger}}((C, c_0), (B, b_0))$ and $g_1^*\xi = g_2^*\xi$ then, according to Theorem I.34, $dg_1 = dg_2 \colon T_{c_0,C} \to T_{b_0,B}$.

EXERCISE I.36. A universal deformation $\xi : M_0 \xrightarrow{i} M \xrightarrow{f} (B, b_0)$ induces a representation (i.e. a homomorphism of groups)

$$\rho \colon \mathrm{Aut}(M_0) \to \mathrm{Aut}_{\mathbf{Ger}}((B, b_0)), \qquad \rho(g)^*\xi = \xi^g, \quad g \in \mathrm{Aut}(M_0).$$

Every other universal deformation over the germ (B, b_0) gives a conjugate representation.
△

5. Rigid varieties

DEFINITION I.37. A deformation $M_0 \to M \to (B, b_0)$ is called *trivial* if it is isomorphic to

$$M_0 \overset{Id\times\{b_0\}}{\longrightarrow} M_0 \times B \overset{pr}{\longrightarrow} (B, b_0).$$

LEMMA I.38. *Let $f\colon M \to \Delta^n_R$ be a smooth family of compact complex manifolds, $t_1, \ldots, t_n$ coordinates in the polydisk Δ^n_R. If there exist holomorphic vector fields $\chi_1, \ldots, \chi_n$ on M such that $f_*\chi_h = \dfrac{\partial}{\partial t_h}$ then there exists $0 < r \le R$ such that $f\colon f^{-1}(\Delta^n_r) \to \Delta^n_r$ is the trivial family.*

PROOF. For every $r \le R$, $h \le n$ denote

$$\Delta^h_r = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, |z_1| < r, \ldots, |z_h| < r, z_{h+1} = 0, \ldots, z_n = 0\} \subset \Delta^n_R.$$

We prove by induction on h that there exists $R \ge r_h > 0$ such that the restriction of the family f over $\Delta^h_{r_h}$ is trivial. Taking $r_0 = R$ the statement is obvious for $h = 0$. Assume that the family is trivial over $\Delta^h_{r_h}$, $h < n$; shrinking Δ^n_R if necessary it is not restrictive to assume $R = r_h$ and the family trivial over Δ^h_R.
The integration of the vector field χ_{h+1} gives an open neighbourhood $M\times\{0\} \subset U \subset M\times\mathbb{C}$ and a holomorphic map $H\colon U \to M$ with the following properties (see e.g. [**8**, Ch. VII]):

(1) For every $x \in M$, $\{x\} \times \mathbb{C} \cap U = \{x\} \times \Delta(x)$ with $\Delta(x)$ a disk.
(2) For every $x \in M$ the map $H_x = H(x, -)\colon \Delta(x) \to M$ is the solution of the Cauchy problem

$$\begin{cases} \dfrac{dH_x}{dt}(t) = \chi_{h+1}(H_x(t)) \\ H_x(0) = x \end{cases}$$

In particular if $H(x,t)$ is defined then $f(H(x,t)) = f(x) + (0, \ldots, t, \ldots, 0)$ (t in the $(h+1)$-th coordinate).
(3) If $V \subset M$ is open and $V \times \Delta \subset U$ then for every $t \in \Delta$ the map $H(-,t)\colon V \to M$ is an open embedding.

Since f is proper there exists $r \le R$ such that $f^{-1}(\Delta^h_r) \times \Delta_r \subset U$; then the holomorphic map $H\colon f^{-1}(\Delta^h_r) \times \Delta_r \to f^{-1}(\Delta^{h+1}_r)$ is a biholomorphism (exercise) giving a trivialization of the family over Δ^{h+1}_r. $\square$

EXAMPLE I.39. Lemma I.38 is generally false if f is not proper (cf. the exercise in Lecture 1 of [**43**]).
Consider for instance an irreducible polynomial $F \in \mathbb{C}[x_1, \ldots, x_n, t]$; denote by $f\colon \mathbb{C}^n_x \times \mathbb{C}_t \to \mathbb{C}_t$ the projection on the second factor and

$$V = \left\{(x,t) \,\middle|\, F(x,t) = \frac{\partial F}{\partial x_i}(x,t) = 0,\ i = 1, \ldots, n\right\}.$$

Assume that $f(V)$ is a finite set of points and set $B = \mathbb{C} - f(V)$, $X = \{(x,t) \in \mathbb{C}^n \times B \,|\, F(x,t) = 0\}$. Then X is a regular hypersurface, the restriction $f\colon X \to B$ is surjective and its differential is surjective everywhere.
X is closed in the affine variety $\mathbb{C}^n \times B$, by Hilbert's Nullstellensatz there exist regular functions $g_1, \ldots, g_n \in \mathcal{O}(\mathbb{C}^n \times B)$ such that

$$g := \sum_{i=1}^n g_i \frac{\partial F}{\partial x_i} \equiv 1 \qquad (\mathrm{mod}\ F).$$

On the open subset $U = \{g \neq 0\}$ the algebraic vector field

$$\chi = \sum_{i=1}^n \frac{g_i}{g}\left(\frac{\partial F}{\partial x_i}\frac{\partial}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial t} - \sum_{i=1}^n \frac{g_i}{g}\frac{\partial F}{\partial t}\frac{\partial}{\partial x_i}$$

is tangent to X and lifts $\dfrac{\partial}{\partial t}$.
In general the fibres of $f\colon X \to B$ are not biholomorphic: consider for example the case

$F(x, y, \lambda) = y^2 - x(x-1)(x-\lambda)$. Then $B = \mathbb{C} - \{0, 1\}$ and $f\colon X \to B$ is the restriction to the affine subspace $x_0 \neq 0$ of the family $M \to B$ of Example I.4.
The fibre $X_\lambda = f^{-1}(\lambda)$ is $M_\lambda - \{\text{point}\}$, where M_λ is an elliptic curve with j-invariant $j(\lambda) = 2^8(\lambda^2 - \lambda + 1)^3\lambda^{-2}(\lambda - 1)^{-2}$. If X_a is biholomorphic to X_b then, by Riemann's extension theorem, also M_a is biholomorphic to M_b and then $j(a) = j(b)$.

Exercise I.40. Find a holomorphic vector field χ lifting $\dfrac{\partial}{\partial\lambda}$ and tangent to $\{F = 0\} \subset \mathbb{C}^2 \times \mathbb{C}$, where $F(x, y, \lambda) = y^2 - x(x-1)(x-\lambda)$ (Hint: use the Euclidean algorithm to find $a, b \in \mathbb{C}[x]$ such that $ay\dfrac{\partial F}{\partial y} + b\dfrac{\partial F}{\partial x} = 1 + 2aF$). △

Theorem I.41. *A deformation $M_0 \to M \xrightarrow{f} (B, b_0)$ of a compact manifold is trivial if and only if $\mathcal{KS}_f\colon \Theta_{B,b_0} \to (R^1f_*T_f)_{b_0}$ is trivial.*

Proof. One implication is clear; conversely assume $\mathcal{KS}_f = 0$, it is not restrictive to assume B a polydisk with coordinates $t_1, \dots, t_n$ and f a smooth family. After a possible shrinking of B we have $\mathcal{KS}(B)_f\left(\dfrac{\partial}{\partial t_i}\right) = 0$ for every $i = 1, \dots, n$. According to I.32 there exist holomorphic vector fields ξ_i such that $f_*\xi_i = \dfrac{\partial}{\partial t_i}$; by I.38 the family is trivial over a smaller polydisk $\Delta \subset B$. □

Note that if a smooth family $f\colon M \to B$ is locally trivial, then for every $b \in B$ the Kodaira-Spencer map $\mathrm{KS}_f\colon T_{b,B} \to H^1(M_b, T_{M_b})$ is trivial for every $b \in B$.

Theorem I.42. (Semicontinuity and base change)
Let $E \to M$ be a holomorphic vector bundle on the total space of a smooth family $f\colon M \to B$. Then, for every $i \geq 0$:

1. *$b \mapsto h^i(M_b, E)$ is upper semicontinuous.*
2. *If $b \mapsto h^i(M_b, E)$ is constant, then for every $b \in B$ there exists an open neighbourhood $b \in U$ and elements $e_1, \dots, e_r \in H^i(f^{-1}(U), E)$ such that:*
   (a) *$H^i(f^{-1}(U), E)$ is the free $\mathcal{O}(U)$-module generated by $e_1, \dots, e_n$.*
   (b) *$e_1, \dots, e_r$ induce a basis of $H^i(M_c, E)$ for every $c \in U$.*
3. *If $b \mapsto h^{i-1}(M_b, E)$ and $b \mapsto h^{i+1}(M_b, E)$ are constant then also $b \mapsto h^i(M_b, E)$ is constant.*

Proof. [**4**, Ch. 3, Thm. 4.12], [**41**, I, Thm. 2.2], [**37**]. □

Corollary I.43. *Let X be a compact complex manifold. If $H^1(X, T_X) = 0$ then every deformation of X is trivial.*

Definition I.44. A compact complex manifold X is called *rigid* if $H^1(X, T_X) = 0$.

Corollary I.45. *Let $f\colon M \to B$ a smooth family of compact complex manifolds. If $b \mapsto h^1(M_b, T_{M_b})$ is constant and $\mathrm{KS}_f = 0$ at every point $b \in B$ then the family is locally trivial.*

Proof. (cf. Example I.49) Easy consequence of Theorems I.41 and I.42. □

Example I.46. Consider the following family of Hopf surfaces $f\colon M \to \mathbb{C}$, $M = X/G$ where $X = B \times (\mathbb{C}^2 - \{0\})$ and $G \simeq \mathbb{Z}$ is generated by $(b, z_1, z_2) \mapsto (b, 2z_1, b^2z_1 + 2z_2)$.
The fibre M_b is the Hopf surface $S_{A(b)}$, where $A(b) = \begin{pmatrix} 2 & 0 \\ b^2 & 2 \end{pmatrix}$ and then M_0 is not biholomorphic to M_b for every $b \neq 0$.
This family is isomorphic to $N \times_{\mathbb{C}} B$, where $B \to \mathbb{C}$ is the map $b \mapsto b^2$ and N is the quotient of $\mathbb{C} \times (\mathbb{C}^2 - \{0\})$ by the group generated by $(s, z_1, z_2) \mapsto (s, 2z_1, sz_1 + 2z_2)$. By base-change property, the Kodaira-Spencer map $\mathrm{KS}_f\colon T_{0,B} \to H^1(M_0, T_{M_0})$ is trivial.
On the other hand the family is trivial over $B - \{0\}$, in fact the map

$$(B - \{0\}) \times (\mathbb{C}^2 - \{0\}) \to (B - \{0\}) \times (\mathbb{C}^2 - \{0\}), \qquad (b, z_1, z_2) \mapsto (b, b^2z_1, z_2)$$

induces to the quotient an isomorphism $(B - \{0\}) \times M_1 \simeq (M - f^{-1}(0))$. Therefore the Kodaira-Spencer map $\mathrm{KS}_f \colon T_{b,B} \to H^1(M_b, T_{M_b})$ is trivial for every b.
According to the base-change theorem the dimension of $H^1(M_b, T_{M_b})$ cannot be constant: in fact it is proved in [**41**] that $h^1(M_0, T_{M_0}) = 4$ and $h^1(M_b, T_{M_b}) = 2$ for $b \neq 0$.

Example I.47. Let $M \subset \mathbb{C}_b \times \mathbb{P}^3_x \times \mathbb{P}^1_u$ be the subset defined by the equations

$$u_0x_1 = u_1(x_2 - bx_0), \qquad u_0x_2 = u_1x_3,$$

$f \colon M \to \mathbb{C}$ the projection onto the first factor and $f^* \colon M^* = (M - f^{-1}(0)) \to (\mathbb{C} - \{0\})$ its restriction.
Assume already proved that f is a family (this will be done in the next chapter); we want to prove that:

(1) f^* is a trivial family.
(2) f is not locally trivial at $b = 0$.

Proof of 1. After the linear change of coordinates $x_2 - bx_0 \mapsto x_0$ the equations of $M^* \subset \mathbb{C} - \{0\} \times \mathbb{P}^3 \times \mathbb{P}^1$ become

$$u_0x_1 = u_1x_0, \qquad u_0x_2 = u_1x_3$$

and there exists an isomorphism of families $\mathbb{C} - \{0\} \times \mathbb{P}^1_s \times \mathbb{P}^1_u \to M^*$, given by

$$(b, [t_0, t_1], [u_0, u_1]) \mapsto (b, [t_0u_1, t_0u_0, t_1u_1, t_1u_0], [u_0, u_1]).$$

$\square$

Proof of 2. Let $Y \simeq \mathbb{P}^1 \subset M_0$ be the subvariety of equation $b = x_1 = x_2 = x_3 = 0$. Assume f locally trivial, then there exist an open neighbourhood $0 \in U \subset \mathbb{C}$ and a commutative diagram of holomorphic maps

$$\begin{array}{ccc} Y \times U & \xrightarrow{j} & M \\ \Big\downarrow{\scriptstyle pr} & & \Big\downarrow{\scriptstyle f} \\ U & \overset{i}{\hookrightarrow} & \mathbb{C} \end{array}$$

where i is the inclusion, j is injective and extends the identity $Y \times \{0\} \to Y \subset M_0$.
Possibly shrinking U it is not restrictive to assume that the image of j is contained in the open subset $V_0 = \{x_0 \neq 0\}$. For $b \neq 0$ the holomorphic map $\delta \colon V_0 \cap M_b \to \mathbb{C}^3$,

$$\delta(b, [x_0, x_1, x_2, x_3], [u_0, u_1]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right),$$

is injective; therefore for $b \in U$, $b \neq 0$, the holomorphic map $\delta j(-, b) \colon Y \simeq \mathbb{P}^1 \to \mathbb{C}^3$ is injective. This contradicts the maximum principle of holomorphic functions. $\square$

Example I.48. In the notation of Example I.47, the deformation $M_0 \to M \xrightarrow{b} (\mathbb{C}, 0)$ is not universal: in order to see this it is sufficient to prove that M is isomorphic to the deformation g^*M, where $g \colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is the holomorphic map $g(b) = b + b^2$.
The equation of g^*M is

$$u_0x_1 = u_1(x_2 - (b + b^2)x_0), \qquad u_0x_2 = u_1x_3,$$

and the isomorphism of deformations $g^*M \to M$ is given by

$$(b, [x_0, x_1, x_2, x_3], [u_0, u_1]) = (b, [(1 + b)x_0, x_1, x_2, x_3], [u_0, u_1]).$$

Example I.49. Applying the base change $\mathbb{C} \to \mathbb{C}$, $b \mapsto b^2$, to the family $M \to \mathbb{C}$ of Example I.47 we get a family with trivial KS at every point of the base but not locally trivial at 0.
We will prove in II.5 that $H^1(M_b, T_{M_b}) = 0$ for $b \neq 0$ and $H^1(M_0, T_{M_0}) = \mathbb{C}$.

6. Historical survey, I

The deformation theory of complex manifolds began in the years 1957-1960 by a series of papers of Kodaira-Spencer [**39**], [**40**], [**41**] and Kodaira-Nirenberg-Spencer [**38**].
The main results of these papers were the completeness and existence theorem for versal deformations.

THEOREM I.50. (Completeness theorem, [**40**])
A deformation ξ over a smooth germ $(B,0)$ of a compact complex manifold M_0 is versal if and only if the Kodaira-Spencer map $\mathrm{KS}_\xi\colon T_{0,B}\to H^1(M_0,T_{M_0})$ is surjective.

Note that if a deformation $M_0 \longrightarrow M \xrightarrow{f} (B,0)$ is versal then we can take a linear subspace $0\in C\subset B$ making the Kodaira-Spencer map $T_{0,C}\to H^1(M_0,T_{M_0})$ bijective; by completeness theorem $M_0\to M\times_B C\to (C,0)$ is semiuniversal.

In general, a compact complex manifold does not have a versal deformation over a smooth germ. The problem of determining when such a deformation exists is one of the most difficult in deformation theory.
A partial answer is given by

THEOREM I.51. (Existence theorem, [**38**])
Let M_0 be a compact complex manifold. If $H^2(M_0,T_{M_0})=0$ then M_0 admits a semiuniversal deformation over a smooth base.

The condition $H^2(M_0,T_{M_0})=0$ is sufficient but it is quite far from being necessary. The "majority" of manifolds having a versal deformation over a smooth germ has the above cohomology group different from 0.

The next problem is to determine when a semiuniversal deformation is universal: a sufficient (and almost necessary) condition is given by the following theorem.

THEOREM I.52. ([**67**], [**79**]) *Let $\xi : M_0 \longrightarrow M \longrightarrow (B,0)$ be a semiuniversal deformation of a compact complex manifold M_0. If $b\mapsto h^0(M_b,T_{M_b})$ is constant (e.g. if $H^0(M_0,T_{M_0})=0$) then ξ is universal.*

REMARK I.53. If a compact complex manifold M has finite holomorphic automorphisms then $H^0(M,T_M)=0$, while the converse is generally false (take as an example the Fermat quartic surface in $\mathbb{P}^3$, cf. [**71**]).

EXAMPLE I.54. Let $M\to B$ be a smooth family of compact complex tori of dimension n, then $T_{M_b}=\oplus_{i=1}^n\mathcal{O}_{M_b}$ and then $h^0(M_b,T_{M_b})=n$ for every b.

EXAMPLE I.55. If K_{M_0} is ample then, by a theorem of Matsumura [**55**], $H^0(M_0,T_{M_0})=0$.

EXERCISE I.56. The deformation $M_0 \longrightarrow M \xrightarrow{f} \mathbb{C}$, where f is the family of Example I.47, is not universal. △

LECTURE II

Deformations of Segre-Hirzebruch surfaces

In this chapter we compute the Kodaira-Spencer map of some particular deformations and, using the completeness theorem I.50, we give a concrete description of the semiuniversal deformations of the Segre-Hirzebruch surfaces $\mathbb{F}_k$ (Theorem II.28).
As a by-product we get examples of deformation-unstable submanifolds (Definition II.29). A sufficient condition for stability of submanifolds is the well known *Kodaira stability theorem* (Thm. II.30) which is stated without proof in the last section.

1. Segre-Hirzebruch surfaces

We consider the following description of the Segre-Hirzebruch surface $\mathbb{F}_q$, $q \geq 0$.

$$\mathbb{F}_q = (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\})/\sim,$$

where the equivalence relation $\sim$ is given by the $(\mathbb{C}^*)^2$-action

$$(l_0, l_1, t_0, t_1) \mapsto (\lambda l_0, \lambda l_1, \lambda^q \mu t_0, \mu t_1), \qquad \lambda, \mu \in \mathbb{C}^*.$$

The projection $\mathbb{F}_q \to \mathbb{P}^1$, $[l_0, l_1, t_0, t_1] \mapsto [l_0, l_1]$ is well defined and it is a $\mathbb{P}^1$-bundle (cf. Example II.13).
Note that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$; $\mathbb{F}_q$ is covered by four affine planes $\mathbb{C}^2 \simeq U_{i,j} = \{l_i t_j \neq 0\}$. In this affine covering we define local coordinates according to the following table

$U_{0,0}: \quad z = \dfrac{l_1}{l_0}, \quad s = \dfrac{t_1 l_0^q}{t_0}$	$U_{0,1}: \quad z = \dfrac{l_1}{l_0}, \quad s' = \dfrac{t_0}{t_1 l_0^q}$
$U_{1,0}: \quad w = \dfrac{l_0}{l_1}, \quad y' = \dfrac{t_1 l_1^q}{t_0}$	$U_{1,1}: \quad w = \dfrac{l_0}{l_1}, \quad y = \dfrac{t_0}{t_1 l_1^q}$

We also denote

$$V_0 = \{l_0 \neq 0\} = U_{0,0} \cup U_{0,1}, \qquad V_1 = \{l_1 \neq 0\} = U_{1,0} \cup U_{1,1}.$$

We shall call z, s principal affine coordinates and $U_{0,0}$ principal affine subset. Since the changes of coordinates are holomorphic, the above affine covering gives a structure of complex manifold of dimension 2 on $\mathbb{F}_k$.

Exercise II.1. If we consider the analogous construction of $\mathbb{F}_q$ with $\mathbb{R}$ instead of $\mathbb{C}$ we get $\mathbb{F}_q$=*torus* for q even and $\mathbb{F}_q$=*Klein bottle* for q odd. △

Definition II.2. For $q > 0$ we set $\sigma_\infty = \{t_1 = 0\}$. Clearly σ_∞ is isomorphic to $\mathbb{P}^1$.

Proposition II.3. $\mathbb{F}_0$ *is not homeomorphic to* $\mathbb{F}_1$.

Proof. Topologically $\mathbb{F}_0 = S^2 \times S^2$ and therefore $H_2(\mathbb{F}_0, \mathbb{Z}) = \mathbb{Z}[S^2 \times \{p\}] \oplus \mathbb{Z}[\{p\} \times S^2]$, where $p \in S^2$ and $[V] \in H_2$ denotes the homology class of a closed subvariety $V \subset S^2 \times S^2$

of real dimension 2.
The matrix of the intersection form $q\colon H_2 \times H_2 \to H_0 = \mathbb{Z}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and therefore $q(a,a)$ is even for every $a \in H_2(\mathbb{F}_0, \mathbb{Z})$.
Consider the following subvarieties of $\mathbb{F}_1$:

$$\sigma = \{t_0 = 0\}, \qquad \sigma' = \{t_0 = l_0 t_1\}.$$

σ and σ' intersect transversely at the point $t_0 = l_0 = 0$ and therefore their intersection product is equal to $q([\sigma],[\sigma']) = \pm 1$. On the other hand the continuous map

$$r : (\mathbb{F}_1 - \sigma_\infty) \times [0,1] \to (\mathbb{F}_1 - \sigma_\infty), \qquad r((l_0,l_1,t_0,t_1),a) = (l_0,l_1,at_0,t_1)$$

shows that σ is a deformation retract of $(\mathbb{F}_1 - \sigma_\infty)$. Since $r_1\colon \sigma' \to \sigma$ is an isomorphism we have $[\sigma] = [\sigma'] \in H_2(\mathbb{F}_1 - \sigma_\infty, \mathbb{Z})$ and then a fortiori $[\sigma] = [\sigma'] \in H_2(\mathbb{F}_1, \mathbb{Z})$. Therefore $q([\sigma],[\sigma]) = \pm 1$ is not even and $\mathbb{F}_0$ cannot be homeomorphic to $\mathbb{F}_1$. □

It is easy to find projective embeddings of the surfaces $\mathbb{F}_q$;

Example II.4. The Segre-Hirzebruch surface $\mathbb{F}_q$ is isomorphic to the subvariety $X \subset \mathbb{P}^{q+1} \times \mathbb{P}^1$ of equation

$$u_0(x_1, x_2, \ldots, x_q) = u_1(x_2, x_3, \ldots, x_{q+1}),$$

where $x_0, \ldots, x_{q+1}$ and u_0, u_1 are homogeneous coordinates in $\mathbb{P}^{q+1}$ and $\mathbb{P}^1$ respectively. An isomorphism $\mathbb{F}_q \to X$ is given by:

$$u_0 = l_0, \quad u_1 = l_1, \quad x_0 = t_0, \quad x_i = t_1 l_0^{i-1} l_1^{q+1-i},\ i = 1, \ldots q+1.$$

Denote by $T \to \mathbb{F}_q$ the holomorphic tangent bundle, in order to compute the spaces $H^0(\mathbb{F}_q, T)$ and $H^1(\mathbb{F}_q, T)$ we first notice that the open subsets V_0, V_1 are isomorphic to $\mathbb{C} \times \mathbb{P}^1$. Explicit isomorphisms are given by

$$V_0 \to \mathbb{C}_z \times \mathbb{P}^1, \qquad (l_0,l_1,t_0,t_1) \mapsto \left(z = \frac{l_1}{l_0}, [t_0,t_1]\right),$$

$$V_1 \to \mathbb{C}_w \times \mathbb{P}^1, \qquad (l_0,l_1,t_0,t_1) \mapsto \left(w = \frac{l_0}{l_1}, [t_0,t_1]\right).$$

According to Example I.27 $H^1(V_i, T) = 0$, $i = 0, 1$, and then $H^0(\mathbb{F}_q, T)$ and $H^1(\mathbb{F}_q, T)$ are isomorphic, respectively, to the kernel and the cokernel of the Čech differential

$$H^0(V_0, T) \oplus H^0(V_1, T) \xrightarrow{d} H^0(V_0 \cap V_1, T), \qquad d(\chi, \eta) = \chi - \eta.$$

In the affine coordinates (z,s), (w,y) we have that:

(1) $H^0(V_0, T)$ is the free $\mathcal{O}(\mathbb{C}_z)$-module generated by $\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial s}, s\dfrac{\partial}{\partial s}, s^2\dfrac{\partial}{\partial s}$.
(2) $H^0(V_1, T)$ is the free $\mathcal{O}(\mathbb{C}_w)$-module generated by $\dfrac{\partial}{\partial w}, \dfrac{\partial}{\partial y}, y\dfrac{\partial}{\partial y}, y^2\dfrac{\partial}{\partial y}$.
(3) $H^0(V_0 \cap V_1, T)$ is the free $\mathcal{O}(\mathbb{C}_z - \{0\})$-module generated by $\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial s}, s\dfrac{\partial}{\partial s}, s^2\dfrac{\partial}{\partial s}$.

The change of coordinates is given by

$$\begin{cases} z = w^{-1} \\ s = y^{-1} w^q \end{cases} \qquad \begin{cases} w = z^{-1} \\ y = s^{-1} z^{-q} \end{cases}$$

and then

$$\begin{cases} \dfrac{\partial}{\partial w} = -z^2 \dfrac{\partial}{\partial z} + q y^{-1} w^{-q-1} \dfrac{\partial}{\partial s} = -z^2 \dfrac{\partial}{\partial z} + qzs\dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial y} = -y^{-2} w^q \dfrac{\partial}{\partial s} = -z^q s^2 \dfrac{\partial}{\partial s} \end{cases}$$

$$d\left(\sum_{i\geq 0} z^i\left(a_i\frac{\partial}{\partial z}+(b_i+c_is+d_is^2)\frac{\partial}{\partial s}\right),\sum_{i\geq 0} w^i\left(\alpha_i\frac{\partial}{\partial w}+(\beta_i+\gamma_iy+\delta_iy^2)\frac{\partial}{\partial y}\right)\right)=$$

$$=\sum_{i\geq 0} z^i\left(a_i\frac{\partial}{\partial z}+b_i\frac{\partial}{\partial s}+c_is\frac{\partial}{\partial s}+d_is^2\frac{\partial}{\partial s}\right)$$

$$+\sum_{i\geq 0} z^{-i}\left(\alpha_i\left(z^2\frac{\partial}{\partial z}-qzs\frac{\partial}{\partial s}\right)+\beta_is^2z^q\frac{\partial}{\partial s}+\gamma_is\frac{\partial}{\partial s}+\delta_iz^{-q}\frac{\partial}{\partial s}\right)$$

An easy computation gives the following

Lemma II.5.

$$\sum_{i\in\mathbb{Z}} z^i\left(a_i\frac{\partial}{\partial z}+b_i\frac{\partial}{\partial s}+c_is\frac{\partial}{\partial s}+d_is^2\frac{\partial}{\partial s}\right)\in H^0(V_0\cap V_1,T)$$

belongs to the image of the Čech differential if and only if $b_{-1}=b_{-2}=\ldots=b_{-q+1}=0$. *In particular the vector fields*

$$z^{-h}\frac{\partial}{\partial s}\in H^0(V_0\cap V_1,T),\qquad h=1,\ldots,q-1$$

represent a basis of $H^1(\mathbb{F}_q,T)$ *and then* $h^1(\mathbb{F}_q,T)=\max(0,q-1)$.

Exercise II.6. Prove that $h^0(\mathbb{F}_q,T)=\max(6,q+5)$. △

Theorem II.7. *If* $a\neq b$ *then* $\mathbb{F}_a$ *is not biholomorphic to* $\mathbb{F}_b$.

Proof. Assume $a>b$. If $a\geq 2$ then the dimension of $H^1(\mathbb{F}_a,T_{\mathbb{F}_a})$ is bigger than the dimension of $H^1(\mathbb{F}_b,T_{\mathbb{F}_b})$. If $a=1$, $b=0$ we apply Proposition II.3. □

We will show in II.24 that $\mathbb{F}_a$ is diffeomorphic to $\mathbb{F}_b$ if and only if $a-b$ is even.

2. Decomposable bundles on projective spaces

For $n>0$, $a\in\mathbb{Z}$ we define

$$\mathcal{O}_{\mathbb{P}^n}(a)=(\mathbb{C}^{n+1}-0)\times\mathbb{C}/\mathbb{C}^*,$$

where the action of the multiplicative group $\mathbb{C}^*=\mathbb{C}-0$ is

$$\lambda(l_0,\ldots,l_n,t)=(\lambda l_0,\ldots,\lambda l_n,\lambda^a t),\qquad \lambda\in\mathbb{C}^*.$$

The projection $\mathcal{O}_{\mathbb{P}^n}(a)\to\mathbb{P}^n$, $[l_0,\ldots,l_n,t]\mapsto[l_0,\ldots,l_n]$, is a holomorphic line bundle. Notice that $\mathcal{O}_{\mathbb{P}^n}=\mathcal{O}_{\mathbb{P}^n}(0)\to\mathbb{P}^n$ is the trivial vector bundle of rank 1.

The obvious projection maps give a commutative diagram

$$\begin{array}{ccc}(\mathbb{C}^{n+1}-0)\times\mathbb{C} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(a)\\ \downarrow & & \downarrow p\\ (\mathbb{C}^{n+1}-0) & \xrightarrow{\pi} & \mathbb{P}^n\end{array}$$

inducing an isomorphism between $(\mathbb{C}^{n+1}-0)\times\mathbb{C}$ and the fibred product of p and π; in particular for every open subset $U\subset\mathbb{P}^n$ the space $H^0(U,\mathcal{O}_{\mathbb{P}^n}(a))$ is naturally isomorphic to the space of holomorphic maps $f\colon\pi^{-1}(U)\to\mathbb{C}$ such that $f(\lambda x)=\lambda^a f(x)$ for every $x\in\pi^{-1}(U)$, $\lambda\in\mathbb{C}^*$.
If $U=\mathbb{P}^n$ then, by Hartogs' theorem, every holomorphic map $f\colon\pi^{-1}(U)\to\mathbb{C}$ can be extended to a function $f\colon\mathbb{C}^{n+1}\to\mathbb{C}$. Considering the power series expansion of f we get a natural isomorphism between $H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(a))$ and the space of homogeneous polynomials of degree a in the homogeneous coordinates $l_0,\ldots,l_n$.

Exercise II.8. Prove that $h^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(a))=\binom{n+a}{n}$. △

EXERCISE II.9. Under the isomorphism $\sigma_\infty = \mathbb{P}^1$ we have $N_{\sigma_\infty/\mathbb{F}_q} = \mathcal{O}_{\mathbb{P}^1}(-q)$. △

On the open set $U_i = \{l_i \neq 0\}$ the section $l_i^a \in H^0(U_i, \mathcal{O}_{\mathbb{P}^n}(a))$ is nowhere 0 and then gives a trivialization of $\mathcal{O}_{\mathbb{P}^n}(a)$ over U_i. The multiplication maps

$$H^0(U_i, \mathcal{O}_{\mathbb{P}^n}(a)) \otimes H^0(U_i, \mathcal{O}_{\mathbb{P}^n}(b)) \to H^0(U_i, \mathcal{O}_{\mathbb{P}^n}(a+b)), \qquad f \otimes g \mapsto fg,$$

give natural isomorphisms of line bundles

$$\mathcal{O}_{\mathbb{P}^n}(a) \otimes \mathcal{O}_{\mathbb{P}^n}(b) = \mathcal{O}_{\mathbb{P}^n}(a+b), \quad \mathcal{H}om(\mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(b)) = \mathcal{O}_{\mathbb{P}^n}(b-a)$$

(In particular $\mathcal{O}_{\mathbb{P}^n}(a)^\vee = \mathcal{O}_{\mathbb{P}^n}(-a)$.)

DEFINITION II.10. A holomorphic vector bundle $E \to \mathbb{P}^n$ is called *decomposable* if it is isomorphic to a direct sum of line bundles of the form $\mathcal{O}_{\mathbb{P}^n}(a)$.
Equivalently a vector bundle is decomposable if it is isomorphic to

$$(\mathbb{C}^{n+1} - 0) \times \mathbb{C}^r/\mathbb{C}^* \to (\mathbb{C}^{n+1} - 0)/\mathbb{C}^* = \mathbb{P}^n,$$

where the action is $\lambda(l_0, \ldots, l_n, t_1, \ldots, t_r) = (\lambda l_0, \ldots, \lambda l_n, \lambda^{a_1} t_1, \ldots, \lambda^{a_r} t_r)$.

LEMMA II.11. *Two decomposable bundles of rank r, $E = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i)$, $F = \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(b_i)$, $a_1 \leq a_2, \ldots, \leq a_r$, $b_1 \leq b_2, \ldots, \leq b_r$, are isomorphic if and only if $a_i = b_i$ for every $i = 1, \ldots, r$.*

PROOF. Immediate from the formula

$$h^0(\mathbb{P}^n, (\oplus_i \mathcal{O}_{\mathbb{P}^n}(a_i)) \otimes \mathcal{O}_{\mathbb{P}^n}(s)) = \sum_i h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a_i + s)) = \sum_{\{i \,|\, a_i + s \geq 0\}} \binom{a_i + s + n}{n}.$$

□

EXAMPLE II.12. If $n \geq 2$ not every holomorphic vector bundle is decomposable. Consider for example the surjective morphism

$$\phi\colon \oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(1) e_i \to \mathcal{O}_{\mathbb{P}^n}(2), \qquad \sum f_i e_i \mapsto \sum f_i l_i.$$

We leave it as an exercise to show that the kernel of ϕ is not decomposable (Hint: first prove that $\ker \phi$ is generated by the global sections $l_i e_j - l_j e_i$).

For every holomorphic vector bundle $E \to X$ on a complex manifold X we denote by $\mathbb{P}(E) \to X$ the projective bundle whose fibre over $x \in X$ is $\mathbb{P}(E)_x = \mathbb{P}(E_x)$. If $E \to X$ is trivial over an open subset $U \subset X$ then also $\mathbb{P}(E)$ is trivial over U; this proves that $\mathbb{P}(E)$ is a complex manifold and the projection $\mathbb{P}(E) \to X$ is proper.

EXAMPLE II.13. For every $a, b \in \mathbb{Z}$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) = \mathbb{F}_{|a-b|}$.
To see this it is not restrictive to assume $a \geq b$; we have

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) = (\mathbb{C}^2 - 0) \times (\mathbb{C}^2 - 0)/\mathbb{C}^* \times \mathbb{C}^*,$$

where the action is $(\lambda, \eta)(l_0, l_1, t_0, t_1) = (\lambda l_0, \lambda l_1, \lambda^a \eta t_0, \lambda^b \eta t_1)$. Setting $\mu = \lambda^b \eta$ we recover the definition of $\mathbb{F}_{a-b}$.
More generally if $E \to X$ is a vector bundle and $L \to X$ is a line bundle then $\mathbb{P}(E \otimes L) = \mathbb{P}(E)$.

EXAMPLE II.14. The tangent bundle $T_{\mathbb{P}^1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)$. Let l_0, l_1 be homogeneous coordinates on $\mathbb{P}^1$; $s = \dfrac{l_1}{l_0}$, $t = \dfrac{l_0}{l_1}$ are coordinates on $U_0 = \{l_0 \neq 0\}$, $U_1 = \{l_1 \neq 0\}$ respectively. The sections of $T_{\mathbb{P}^1}$ over an open set U correspond to pairs $\left(f_0(s)\dfrac{\partial}{\partial s}, f_1(t)\dfrac{\partial}{\partial t}\right)$, $f_i \in \mathcal{O}(U \cap U_i)$, such that $f_1(t) = -t^2 f_0(t^{-1})$.
The isomorphism $\phi\colon \mathcal{O}_{\mathbb{P}^1}(2) \to T_{\mathbb{P}^1}$ is given by $\phi(l_0^a l_1^{2-a}) = \left(s^{2-a}\dfrac{\partial}{\partial s}, -t^a \dfrac{\partial}{\partial t}\right)$.

THEOREM II.15 (Euler exact sequence). *On the projective space $\mathbb{P}^n$ there exists an exact sequence of vector bundles*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\sum l_i \frac{\partial}{\partial l_i}} \oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(1)\frac{\partial}{\partial l_i} \xrightarrow{\phi} T_{\mathbb{P}^n} \longrightarrow 0,$$

where on the affine open subset $l_h \neq 0$, with coordinates $s_i = \dfrac{l_i}{l_h}$, $i \neq h$,

$$\begin{cases} \phi\left(l_i \dfrac{\partial}{\partial l_j}\right) = s_i \dfrac{\partial}{\partial s_j} & i, j \neq h \\ \phi\left(l_h \dfrac{\partial}{\partial l_j}\right) = \dfrac{\partial}{\partial s_j} & j \neq h \end{cases}, \qquad \begin{cases} \phi\left(l_i \dfrac{\partial}{\partial l_h}\right) = -\displaystyle\sum_{j \neq h} s_i s_j \dfrac{\partial}{\partial s_j} & i \neq h \\ \phi\left(l_h \dfrac{\partial}{\partial l_h}\right) = -\displaystyle\sum_{j \neq h} s_j \dfrac{\partial}{\partial s_j} \end{cases}$$

PROOF. The surjectivity of ϕ is clear. Assume $\phi\left(\sum_{i,j} a_{ij} l_i \dfrac{\partial}{\partial l_j}\right) = 0$, looking at the quadratic terms in the set $l_h \neq 0$ we get $a_{ih} = 0$ for every $i \neq h$. In the open set $l_0 \neq 0$ we have

$$\phi\left(\sum_i a_{ii} l_i \frac{\partial}{\partial l_i}\right) = \sum_{i=1}^n a_{ii} s_i \frac{\partial}{\partial s_i} - \sum_{i=1}^n a_{00} s_i \frac{\partial}{\partial s_i} = 0$$

and then the matrix a_{ij} is a multiple of the identity. □

REMARK II.16. It is possible to prove that the map ϕ in the Euler exact sequence is surjective at the level of global sections, this gives an isomorphism

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) = gl(n+1, \mathbb{C})/\mathbb{C}Id = pgl(n+1, \mathbb{C}) = T_{Id}PGL(n+1, \mathbb{C}).$$

Moreover it is possible to prove that every biholomorphism of $\mathbb{P}^n$ is a projectivity and the integration of holomorphic vector fields corresponds to the exponential map in the complex Lie group $PGL(n+1, \mathbb{C})$.

EXERCISE II.17. Use the Euler exact sequence and the surjectivity of ϕ on global sections to prove that for every $n \geq 2$ the tangent bundle of $\mathbb{P}^n$ is not decomposable. △

COROLLARY II.18. *The canonical bundle of $\mathbb{P}^n$ is $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$.*

PROOF. From the Euler exact sequence we have

$$\bigwedge^n T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n} = \bigwedge^{n+1}(\oplus_{i=0}^n \mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{O}_{\mathbb{P}^n}(n+1)$$

and then $K_{\mathbb{P}^n} = (\bigwedge^n T_{\mathbb{P}^n})^\vee = \mathcal{O}_{\mathbb{P}^n}(-n-1)$. □

EXERCISE II.19. Prove that $h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-a)) = \dbinom{a-1}{n}$. △

LEMMA II.20. *Let $E \to \mathbb{P}^1$ be a holomorphic vector bundle of rank r. If:*

(1) *$H^0(\mathbb{P}^1, E(s)) = 0$ for $s << 0$, and*
(2) *There exists a constant $c \in \mathbb{N}$ such that $h^0(\mathbb{P}^1, E(s)) \geq rs - c$ for $s >> 0$.*

Then E is decomposable.

PROOF. Using the assumptions 1 and 2 we may construct recursively a sequence $a_1, \ldots, a_r \in \mathbb{Z}$ and sections $\alpha_i \in H^0(\mathbb{P}^1, E(a_i))$ such that:

(1) a_{h+1} is the minimum integer s such that the map
$$\oplus_{i=1}^h \alpha_i \colon \bigoplus_{i=1}^h H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(s-a_i)) \to H^0(\mathbb{P}^1, E(s))$$
is not surjective.
(2) α_{h+1} does not belong to the image of
$$\oplus_{i=1}^h \alpha_i \colon \bigoplus_{i=1}^h H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_{h+1}-a_i)) \to H^0(\mathbb{P}^1, E(a_{h+1})).$$

Notice that $a_1 \leq a_2 \leq \ldots \leq a_r$.
We prove now by induction on h that the morphism of vector bundles

$$\oplus_{i=1}^{h}\alpha_i \colon \bigoplus_{i=1}^{h}\mathcal{O}_{\mathbb{P}^1}(-a_i) \to E$$

is injective on every fibre; this implies that $\oplus_{i=1}^{r}\alpha_i \colon \bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^1}(-a_i) \to E$ is an isomorphism.
For $h = 0$ it is trivial. Assume $\oplus_{i=1}^{h}\alpha_i$ injective on fibres and let $p \in \mathbb{P}^1$. Choose homogeneous coordinates l_0, l_1 such that $p = \{l_1 = 0\}$ and set $s = l_1/l_0$.
Assume that there exist $c_1, \ldots, c_h \in \mathbb{C}$ such that $\alpha_{h+1}(p) = \sum c_i(l_0^{a_{h+1}-a_i}\alpha_i)(p) \in E(a_{h+1})_p$.
If $e_1, \ldots, e_r$ is a local frame for E at p we have locally

$$\alpha_{h+1} - \sum_{i=1}^{h} c_i l_0^{a_{h+1}-a_i}\alpha_i = \sum_{j=1}^{r} f_j(s) l_0^{a_{h+1}} e_j$$

with $f_j(s)$ holomorphic functions such that $f_j(0) = 0$.
Therefore $f_j(s)/s$ is still holomorphic and $l_0^{-1}(\alpha_{h+1} - \sum c_i l_0^{a_{h+1}-a_i}\alpha_i) \in H^0(\mathbb{P}^1, E(a_{h+1}-1))$, in contradiction with the minimality of a_{h+1}. □

THEOREM II.21. *Let $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ be an exact sequence of holomorphic vector bundles on $\mathbb{P}^1$.*

1. *If F, G are decomposable then also E is decomposable.*
2. *If $E = \oplus\mathcal{O}_{\mathbb{P}^1}(-a_i)$ then $\min(a_i)$ is the minimum integer s such that $H^0(\mathbb{P}^1, F(s)) \to H^0(\mathbb{P}^1, G(s))$ is not injective.*

PROOF. The kernel of $H^0(\mathbb{P}^1, F(s)) \to H^0(\mathbb{P}^1, G(s))$ is exactly $H^0(\mathbb{P}^1, E(s))$.
If $F = \oplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^1}(b_i)$, $G = \oplus_{i=1}^{p}\mathcal{O}_{\mathbb{P}^1}(c_i)$ then for $s >> 0$ $h^0(\mathbb{P}^1, F(s)) = r(s+1) + \sum b_i$, $h^0(\mathbb{P}^1, G(s)) = p(s+1) + \sum c_i$ and then the rank of E is $r - p$ and $h^0(\mathbb{P}^1, E(s)) \geq (r-p)(s+1) + \sum b_i - \sum c_i$. According to Lemma II.20, the vector bundle E is decomposable. □

We also state, without proof, the following

THEOREM II.22. (1) *Every holomorphic line bundle on $\mathbb{P}^n$ is decomposable.*
(2) *(Serre) Let E be a holomorphic vector bundle on $\mathbb{P}^n$, then:*
 (a) *$H^0(\mathbb{P}^n, E(s)) = 0$ for $s << 0$.*
 (b) *$E(s)$ is generated by global sections and $H^p(\mathbb{P}^n, E(s)) = 0$ for $p > 0$, $s >> 0$.*
(3) *(Bott vanishing theorem) For every $0 < p < n$:*

$$H^p(\mathbb{P}^n, \Omega^q(a)) = \begin{cases} \mathbb{C} & \text{if } p = q, \quad a = 0 \\ 0 & \text{otherwise} \end{cases}$$

Moreover $H^0(\mathbb{P}^n, \Omega^q(a)) = H^n(\mathbb{P}^n, \Omega^{n-q}(-a))^{\vee} = 0$ whenever $a < q$.

PROOF. [**37**] □

3. Semiuniversal families of Segre-Hirzebruch surfaces

Let $q > 0$ be a fixed integer, define $M \subset \mathbb{C}_t^{q-1} \times \mathbb{P}_l^1 \times \mathbb{P}_x^{q+1}$ as the set of points of homogeneous coordinates $(t_2, \ldots, t_q, [l_0, l_1], [x_0, \ldots, x_{q+1}])$ satisfying the vectorial equation

$$l_0(x_1, x_2, \ldots, x_q) = l_1(x_2 - t_2x_0, \ldots, x_q - t_qx_0, x_{q+1}). \tag{1}$$

We denote by $f \colon M \to \mathbb{C}^{q-1}$, $p \colon M \to \mathbb{C}^{q-1} \times \mathbb{P}_l^1$ the projections.

LEMMA II.23. *There exists a holomorphic vector bundle of rank 2, $E \to \mathbb{C}^{q-1} \times \mathbb{P}_l^1$ such that the map $p \colon M \to \mathbb{C}^{q-1} \times \mathbb{P}_l^1$ is a smooth family isomorphic to $\mathbb{P}(E) \to \mathbb{C}^{q-1} \times \mathbb{P}_l^1$.*

PROOF. Let $\pi\colon \mathbb{C}^{q-1}\times\mathbb{P}^1_l \to \mathbb{P}^1_l$ be the projection; define E as the kernel of the morphism of vector bundles over $\mathbb{C}^{q-1}\times\mathbb{P}^1_l$

$$\bigoplus_{i=0}^{q+1}\pi^*\mathcal{O}_{\mathbb{P}^1}\xrightarrow{A}\bigoplus_{i=1}^{q}\pi^*\mathcal{O}_{\mathbb{P}^1}(1),$$

$$A(t_2,\dots,t_q,[l_0,l_1])\begin{pmatrix} x_0\\ x_1\\ \vdots\\ x_{q+1}\end{pmatrix}=\begin{pmatrix} l_0x_1-l_1(x_2-t_2x_0)\\ l_0x_2-l_1(x_3-t_3x_0)\\ \vdots\\ l_0x_q-l_1x_{q+1}\end{pmatrix}.$$

We first note that A is surjective on every fibre, in fact for fixed $t_2,\dots,t_q,l_0,l_1\in\mathbb{C}$, $A(t_i,l_j)$ is represented by the matrix

$$\begin{pmatrix} t_2l_1 & l_0 & -l_1 & \dots & 0 & 0\\ t_3l_1 & 0 & l_0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & l_0 & -l_1\end{pmatrix}.$$

Since either $l_0\neq 0$ or $l_1\neq 0$ the above matrix has maximal rank.
By definition we have that M is the set of points of $x\in\mathbb{P}(\oplus_{i=0}^{q+1}\pi^*\mathcal{O}_{\mathbb{P}^1})$ such that $A(x)=0$ and then $M=\mathbb{P}(E)$. $\square$

For every $k\geq 0$ denote by $T_k\subset\mathbb{C}^{q-1}_t$ the subset of points of coordinates $(t_2,\dots,t_q)$ such that there exists a nonzero $(q+2)$-uple of homogeneous polynomials of degree k

$$(x_0(l_0,l_1),\dots,x_{q+1}(l_0,l_1))$$

which satisfy identically (t being fixed) the Equation 1. Note that $t\in T_k$ if and only if there exists a nontrivial morphism $\mathcal{O}_{\mathbb{P}^1}(-k)\to E_t$ and then $t\in T_k$ if and only if $-k\leq -a$. Therefore $t\in T_k-T_{k-1}$ if and only if $a=k$.

LEMMA II.24. *In the notation above:*
1. $T_0=\{0\}$.
2. $T_k\subset T_{k+1}$.
3. *If* $2k+1\geq q$ *then* $T_k=\mathbb{C}^{q-1}$.
4. *If* $2k\leq q$ *and* $t\in T_k-T_{k-1}$ *then* $M_t=\mathbb{F}_{q-2k}$.

PROOF. 1 and 2 are trivial.
Denoting by $S_k\subset\mathbb{C}[l_0,l_1]$ the space of homogeneous polynomials of degree k, $\dim_{\mathbb{C}}S_k=k+1$; interpreting Equation 1 as a linear map (depending on the parameter t) $A_k(t)\colon S_k^{q+2}\to S_{k+1}^q$, we have that $t\in T_k$ if and only if $\ker A_k(t)\neq 0$.
Since $(q+2)(k+1)>q(k+2)$ whenever $2k>q-2$, item 3 follows immediately.
Let E_t be the restriction of the vector bundle E to $\{t\}\times\mathbb{P}^1$, E_t is the kernel of the surjective morphism $A(t)\colon \oplus_{i=0}^{q+1}\mathcal{O}_{\mathbb{P}^1}\to\oplus_{i=1}^{q}\mathcal{O}_{\mathbb{P}^1}(1)$. According to Theorem II.21, E_t is decomposable. Since $\bigwedge^2 E_t=\mathcal{O}_{\mathbb{P}^1}(-q)$ we have $E_t=\mathcal{O}_{\mathbb{P}^1}(-a)\oplus\mathcal{O}_{\mathbb{P}^1}(a-q)$ with $-a\leq a-q$ and $M_t=\mathbb{P}(E_t)=\mathbb{F}_{q-2a}$.
$\square$

LEMMA II.25. *In the notation above* $(t_2,\dots,t_q)\in T_k$ *if and only if there exists a nonzero triple* $(x_0,x_1,x_{q+1})\in\oplus\mathbb{C}[s]$ *of polynomials of degree* $\leq k$ *such that*

$$x_{q+1}=s^qx_1+x_0\left(\sum_{i=2}^{q}t_is^{q+1-i}\right).$$

PROOF. Setting $s=l_0/l_1$ we have by definition that $(t_2,\dots,t_q)\in T_k$ if and only if there exists a nontrivial sequence $x_0,\dots,x_{q+1}\in\mathbb{C}[s]$ of polynomials of degree $\leq k$ such that $x_{i+1}=sx_i+t_{i+1}x_0$ for every $i=1,\dots,q$ ($t_{q+1}=0$ by convention). Clearly this set of equation is equivalent to $x_{i+1}=s^ix_1+x_0\sum_{j=1}^{i}t_{j+1}s^{i-j}$.
Given x_0,x_1,x_{q+1} as in the statement, we can define recursively $x_i=s^{-1}(x_{i+1}-t_{i+1}x_0)$ and the sequence $x_0,\dots,x_{q-1}$ satisfies the defining equation of T_k. $\square$

Corollary II.26. *$(t_2, \ldots, t_q) \in T_k$ if and only if the $(q-k-1)\times(k+1)$ matrix $B_k(t)_{ij} = (t_{q-k-i+j})$ has rank $\leq k$.*

Proof. If $2k+1 \leq q$ then $T_k = \mathbb{C}^{q-1}$, $q-k-1 \leq k$ and the result is trivial: thus it is not restrictive to assume $k+1 \leq q-k-1$ and then $\mathrm{rank} B_k(t) \leq k$ if and only if $\ker B_k(t) \neq 0$.
We note that if x_0, x_1, x_{q+1} satisfy the equation $x_{q+1} = s^q x_1 + x_0(\sum_{i=2}^q t_i s^{q+1-i})$ then x_1, x_{q+1} are uniquely determined by x_0; conversely a polynomial $x_0(s)$ of degree $\leq k$ can be extended to a solution of the equation if and only if all the coefficients of $s^{k+1}, s^{k+2}, \ldots, s^{q-1}$ in the polynomial $x_0(\sum_{i=2}^q t_i s^{q+1-i})$ vanish. Writing $x_0 = a_0 + a_1 s + \ldots + a_k s^k$, this last condition is equivalent to $(a_0, \ldots, a_k) \in \ker B_k(t)$. $\square$

Therefore T_k is defined by the vanishing of the $\binom{q-k-1}{k+1}$ minors of $B_k(t)$, each one of which is a homogeneous polynomial of degree $k+1$ in $t_2, \ldots, t_q$. In particular T_k is an algebraic cone.

As an immediate consequence of Corollary II.26 we have that for $q \geq 2$, $0 < 2k \leq q$, the subset $\{t_{k+1} \neq 0, t_{k+2} = t_{k+3} = \ldots = t_q = 0\}$ is contained in $T_k - T_{k-1}$. In particular $\mathbb{F}_q$ is diffeomorphic to $\mathbb{F}_{q-2k}$ for every $k \leq q/2$.

Proposition II.27. *If $2k < q$ then T_k is an irreducible affine variety of dimension $2k$.*

Proof. Denote

$$Z_k = \{([v], t) \in \mathbb{P}^k \times \mathbb{C}^{q-1} \,|\, v \in \mathbb{C}^{k+1} - 0,\ B_k(t)v = 0\}$$

and by $p\colon Z_k \to T_k$ the projection on the second factor. p is surjective and if $t_{k+1} = 1$, $t_i = 0$ for $i \neq k+1$, then $B_k(t)$ has rank k and $p^{-1}(t)$ is one point. Therefore it is sufficient to prove that Z_k is an irreducible variety of dimension $2k$.
Let $\pi\colon Z_k \to \mathbb{P}^k$ be the projection. We have $([a_0, \ldots, a_k], (t_2, \ldots, t_q)) \in Z_k$ if and only if for every $i = 1, \ldots, q-k-1$

$$0 = \sum_{j=0}^{k} t_{i+1+j} a_j = \sum_{l=2}^{q} t_l a_{l-i-1},$$

where $a_l = 0$ for $l < 0$, $l > k$ and then the fibre over $[a_0, \ldots, a_k]$ is the kernel of the matrix $A_{ij} = (a_{j-i-1})$ $i = 1, \ldots, q-k-1$, $j = 2, \ldots, q$. Since at least one a_i is $\neq 0$ the rank of A_{ij} is exactly $q-k-1$ and then the fibre is a vector subspace of dimension k. By a general result in algebraic geometry [**72**],[**51**] Z is an irreducible variety of dimension $2k$. $\square$

Theorem II.28. *In the above notation the Kodaira-Spencer map $\mathrm{KS}_f\colon T_{0,\mathbb{C}^{q-1}} \to H^1(M_0, T_{M_0})$ is bijective for every $q \geq 1$ and therefore, by completeness theorem I.50, deformation $\mathbb{F}_q \to M \to (\mathbb{C}^{q-1}, 0)$ is semiuniversal.*

Proof. We have seen that $M_0 = \mathbb{F}_q$. Let $V_0, V_1 \subset \mathbb{F}_q$ be the open subset defined in Section 1. Denote $M_i \subset M$ the open subset $\{l_i \neq 0\}$, $i = 0, 1$.
We have an isomorphism $\phi_0\colon \mathbb{C}^{q-1} \times V_0 \to M_0$, commuting with the projections onto $\mathbb{C}^{q-1}$, given in the affine coordinates (z, s) by:

$$l_0 = 1, \quad l_1 = z, \quad x_0 = 1, \quad x_h = z^{q-h+1}s - \sum_{j=1}^{q-h} t_{h+j} z^j = z(x_{h+1} - t_{h+1}x_0), \quad h > 0.$$

Similarly there exists an isomorphism $\phi_1\colon \mathbb{C}^{q-1} \times V_1 \to M_1$,

$$l_0 = w, \quad l_1 = 1, \quad x_0 = y, \quad x_h = w^{h-1} + y\sum_{j=2}^{h} t_j w^{h-j} = wx_{h-1} + t_h x_0, \quad h > 0.$$

In the intersection $M_0 \cap M_1$ we have:

$$\begin{cases} z = w^{-1} \\ s = \dfrac{x_{q+1}}{x_0} = y^{-1}w^q + \displaystyle\sum_{j=2}^{q} t_j w^{q+1-j}. \end{cases}$$

According to Formula I.33, for every $h = 2, \ldots, q$

$$\mathrm{KS}_f\left(\frac{\partial}{\partial t_h}\right) = \frac{\partial w^{-1}}{\partial t_h}\frac{\partial}{\partial z} + \frac{\partial(y^{-1}w^q + \sum_{j=2}^{q} t_j w^{q+1-j})}{\partial t_h}\frac{\partial}{\partial s} = z^{h-q-1}\frac{\partial}{\partial s}.$$

□

4. Historical survey, II

One of the most famous theorems in deformation theory (at least in algebraic geometry) is the stability theorem of submanifolds proved by Kodaira in 1963.

Definition II.29. Let Y be a closed submanifold of a compact complex manifold X. Y is called *stable* if for every deformation $X \xrightarrow{i} \mathcal{X} \xrightarrow{f} (B, 0)$ there exists a deformation $Y \xrightarrow{j} \mathcal{Y} \xrightarrow{g} (B, 0)$ and a commutative diagram of holomorphic maps

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "i_{|X}"'] & \mathcal{Y} \arrow[d, "g"] \arrow[dl] \\ \mathcal{X} \arrow[r, "f"] & B \end{tikzcd}$$

The same argument used in Example I.46 shows that $\sigma_\infty \subset \mathbb{F}_q$ is not stable for every $q \geq 2$, while $\sigma_\infty \subset \mathbb{F}_1$ is stable because $\mathbb{F}_1$ is rigid.

Theorem II.30. (Kodaira stability theorem for submanifolds, [**36**])
Let Y be a closed submanifold of a compact complex manifold X. If $H^1(Y, N_{Y/X}) = 0$ then Y is stable.

Just to check Theorem II.30 in a concrete case, note that $h^1(\sigma_\infty, N_{\sigma_\infty/\mathbb{F}_q}) = \max(0, q-1)$.

Theorem II.30 has been generalized to arbitrary holomorphic maps of compact complex manifolds in a series of papers by Horikawa [**30**].

Definition II.31. Let $\alpha\colon Y \to X$ be a holomorphic map of compact complex manifolds. A *deformation of* α over a germ $(B, 0)$ is a commutative diagram of holomorphic maps

$$\begin{array}{ccccc} Y & \xrightarrow{i} & \mathcal{Y} & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle Id} \\ X & \xrightarrow{j} & \mathcal{X} & \xrightarrow{g} & B \end{array}$$

where $Y \xrightarrow{i} \mathcal{Y} \xrightarrow{f} (B, 0)$ and $X \xrightarrow{j} \mathcal{X} \xrightarrow{g} (B, 0)$ are deformations of Y and X respectively.

Definition II.32. In the notation of II.31, the map α is called:

(1) *Stable* if every deformation of X can be extended to a deformation of α.
(2) *Costable* if every deformation of Y can be extended to a deformation of α.

Consider two locally finite coverings $\mathcal{U} = \{U_a\}$, $\mathcal{V} = \{V_a\}$, $a \in \mathcal{I}$, $Y = \cup U_a$, $X = \cup V_a$ such that U_a, V_a are biholomorphic to polydisks and $\alpha(U_a) \subset V_a$ for every a (U_a is allowed to be the empty set).

Given $a \in \mathcal{I}$ and local coordinate systems $(z_1, \dots, z_m)\colon U_a \to \mathbb{C}^m$, $(u_1, \dots, u_n)\colon V_a \to \mathbb{C}^n$ we have linear morphisms of vector spaces

$$\alpha^*\colon \Gamma(V_a, T_X) \to \Gamma(U_a, \alpha^* T_X), \qquad \alpha^*\left(\sum_i g_i \frac{\partial}{\partial u_i}\right) = \sum_i \alpha^*(g_i)\frac{\partial}{\partial u_i}$$

$$\alpha_*\colon \Gamma(U_a, T_Y) \to \Gamma(U_a, \alpha^* T_X), \qquad \alpha_*\left(\sum_i h_i \frac{\partial}{\partial z_i}\right) = \sum_{i,j} h_i \frac{\partial u_j}{\partial z_i}\frac{\partial}{\partial u_j}$$

Define $\mathbb{H}^*(\alpha_*)$ as the cohomology of the complex

$$0 \longrightarrow C^0(\mathcal{U}, T_Y) \xrightarrow{d_0} C^1(\mathcal{U}, T_Y) \oplus C^0(\mathcal{U}, \alpha^* T_X) \xrightarrow{d_1} \dots$$

where $d_i(f, g) = (df, dg + (-1)^i \alpha_* f)$, being d the usual Čech differential.

Similarly define $\mathbb{H}^*(\alpha^*)$ as the cohomology of the complex

$$0 \longrightarrow C^0(\mathcal{V}, T_X) \xrightarrow{d_0} C^1(\mathcal{V}, T_X) \oplus C^0(\mathcal{U}, \alpha^* T_X) \xrightarrow{d_1} \dots$$

where $d_i(f, g) = (df, dg + (-1)^i \alpha^* f)$.

Theorem II.33 (Horikawa). *The groups $\mathbb{H}^k(\alpha_*)$ and $\mathbb{H}^k(\alpha^*)$ do not depend on the choice of the coverings $\mathcal{U}, \mathcal{V}$. Moreover:*

(1) *If $\mathbb{H}^2(\alpha_*) = 0$ then α is stable.*
(2) *If $\mathbb{H}^2(\alpha^*) = 0$ then α is costable.*

Exercise II.34. Give a Dolbeault-type definition of the groups $\mathbb{H}^k(\alpha_*)$, $\mathbb{H}^k(\alpha^*)$. △

Exercise II.35. If $\alpha\colon Y \to X$ is a regular embedding then $\mathbb{H}^k(\alpha_*) = H^{k-1}(Y, N_{Y/X})$. (Hint: take $U_a = V_a \cap Y$, and local systems of coordinates $u_1, \dots, u_n$ such that $Y = \{u_{m+1} = \dots = u_n = 0\}$. Then prove that the projection maps $C^{k+1}(\mathcal{U}, T_Y) \oplus C^k(\mathcal{U}, \alpha^* T_X) \to C^k(\mathcal{U}, N_{Y/X})$ give a quasiisomorphism of complexes. △

The following (non trivial) exercise is reserved to experts in algebraic geometry:

Exercise II.36. Let $\alpha\colon Y \to Alb(Y)$ be the Albanese map of a complex projective manifold Y. If $X = \alpha(Y)$ is a curve then $\alpha\colon Y \to X$ is costable. △

LECTURE III

Analytic singularities

Historically, a major step in deformation theory has been the introduction of deformations of complex manifolds over (possibly non reduced) analytic singularities.
This chapter is a short introductory course on analytic algebras and analytic singularities; moreover we give an elementary proof of the Nullstellenstaz for the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of convergent complex power series.
Quite important in deformation theory are the smoothness criterion III.7 and the two dimension bounds III.40 and III.41.

1. Analytic algebras

Let $\mathbb{C}\{z_1, \ldots, z_n\}$ be the ring of convergent power series with complex coefficient. Every $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ defines a holomorphic function in a nonempty open neighbourhood U of $0 \in \mathbb{C}^n$; for notational simplicity we still denote by $f\colon U \to \mathbb{C}$ this function.

If f is a holomorphic function in a neighbourhood of 0 and $f(0) \neq 0$ then $1/f$ is holomorphic in a (possibly smaller) neighbourhood of 0. This implies that f is invertible in $\mathbb{C}\{z_1, \ldots, z_n\}$ if and only if $f(0) \neq 0$ and therefore $\mathbb{C}\{z_1, \ldots, z_n\}$ is a local ring with maximal ideal $\mathfrak{m} = \{f \mid f(0) = 0\}$. The ideal $\mathfrak{m}$ is generated by $z_1, \ldots, z_n$.

Definition III.1. The *multiplicity* of a power series $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ is defined as

$$\mu(f) = \sup\{s \in \mathbb{N} \mid f \in \mathfrak{m}^s\} \in \mathbb{N} \cup \{+\infty\}.$$

The *valuation* $\nu(S)$ of a nonempty subset $S \subset \mathbb{C}\{z_1, \ldots, z_n\}$ is

$$\nu(S) = \sup\{s \in \mathbb{N} \mid S \subset \mathfrak{m}^s\} = \inf\{\mu(f) \mid f \in S\} \in \mathbb{N} \cup \{+\infty\}.$$

We note that $\nu(S) = +\infty$ if and only if $S = \{0\}$ and $\mu(f)$ is the smallest integer d such that the power series expansion of f contains a nontrivial homogeneous part of degree d.
The local ring $\mathbb{C}\{z_1, \ldots, z_n\}$ has the following important properties:

- $\mathbb{C}\{z_1, \ldots, z_n\}$ is Noetherian ([**28**, II.B.9], [**24**]).
- $\mathbb{C}\{z_1, \ldots, z_n\}$ is a unique factorization domain ([**28**, II.B.7], [**24**]).
- $\mathbb{C}\{z_1, \ldots, z_n\}$ is a Henselian ring ([**51**], [**23**], [**24**]).
- $\mathbb{C}\{z_1, \ldots, z_n\}$ is a regular local ring of dimension n (see e.g. [**3**], [**24**], [**56**] for the basics about dimension theory of local Noetherian ring).

We recall, for the reader's convenience, that the dimension of a local Noetherian ring A with maximal ideal $\mathfrak{m}$ is the minimum integer d such that there exist $f_1, \ldots, f_d \in \mathfrak{m}$ with the property $\sqrt{(f_1, \ldots, f_d)} = \mathfrak{m}$. In particular $\dim A = 0$ if and only if $\sqrt{0} = \mathfrak{m}$, i.e. if and only if $\mathfrak{m}$ is nilpotent.
We also recall that a morphism of local rings $f\colon (A, \mathfrak{m}) \to (B, \mathfrak{n})$ is called local if $f(\mathfrak{m}) \subset \mathfrak{n}$.

Definition III.2. A local $\mathbb{C}$-algebra is called an *analytic algebra* if it is isomorphic to $\mathbb{C}\{z_1, \ldots, z_n\}/I$, for some $n \geq 0$ and some ideal $I \subset (z_1, \ldots, z_n)$.
We denote by **An** the category with objects the analytic algebras and morphisms the local morphisms of $\mathbb{C}$-algebras.

Every analytic algebra is a local Noetherian ring. Every local Artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$ is an analytic algebra.

The ring $\mathbb{C}\{z_1,\ldots,z_n\}$ is, in some sense, a free object in the category $\mathbf{An}$ as explained in the following lemma

LEMMA III.3. *Let $(R,\mathfrak{m})$ be an analytic algebra. Then the map*

$$\mathrm{Mor}_{\mathbf{An}}(\mathbb{C}\{z_1,\ldots,z_n\},R)\to\underbrace{\mathfrak{m}\times\ldots\times\mathfrak{m}}_{n\ factors},\qquad f\mapsto(f(z_1),\ldots,f(z_n))$$

is bijective.

PROOF. We first note that, by the lemma of Artin-Rees ([**3**, 10.19]), $\cap_n\mathfrak{m}^n=0$ and then every local homomorphism $f\colon\mathbb{C}\{z_1,\ldots,z_n\}\to R$ is uniquely determined by its factorizations

$$f_s\colon\mathbb{C}\{z_1,\ldots,z_n\}/(z_1,\ldots,z_n)^s\to R/\mathfrak{m}^s.$$

Since $\mathbb{C}\{z_1,\ldots,z_n\}/(z_1,\ldots,z_n)^s$ is a $\mathbb{C}$-algebra generated by $z_1,\ldots,z_n$, every f_s is uniquely determined by $f(z_i)$; this proves the injectivity.
For the surjectivity it is not restrictive to assume $R=\mathbb{C}\{u_1,\ldots,u_m\}$; given $\phi=(\phi_1,\ldots,\phi_n)$, $\phi_i\in\mathfrak{m}$, let U be an open subset $0\in U\subset\mathbb{C}^m_u$ where the $\phi_i=\phi_i(u_1,\ldots,u_m)$ are convergent power series. The map $\phi=(\phi_1,\ldots,\phi_n)\colon U\to\mathbb{C}^n$ is holomorphic, $\phi(0)=0$ and $\phi^*(z_i)=\phi_i$. □

Another important and useful tool is the following

THEOREM III.4 (Rückert's nullstellensatz). *Let $I,J\subset\mathbb{C}\{z_1,\ldots,z_n\}$ be proper ideals, then*

$$\mathrm{Mor}_{\mathbf{An}}(\mathbb{C}\{z_1,\ldots,z_n\}/_I,\mathbb{C}\{t\})=\mathrm{Mor}_{\mathbf{An}}(\mathbb{C}\{z_1,\ldots,z_n\}/_J,\mathbb{C}\{t\})\qquad\iff\qquad\sqrt{I}=\sqrt{J},$$

where the left equality is intended as equality of subsets of $\mathrm{Mor}_{\mathbf{An}}(\mathbb{C}\{z_1,\ldots,z_n\},\mathbb{C}\{t\})$

A proof of Theorem III.4 will be given in Section 4.

LEMMA III.5. *Every analytic algebra is isomorphic to $\mathbb{C}\{z_1,\ldots,z_k\}/_I$ for some $k\geq 0$ and some ideal $I\subset(z_1,\ldots,z_k)^2$.*

PROOF. Let $A=\mathbb{C}\{z_1,\ldots,z_n\}/I$ be an analytic algebra such that I is not contained in $(z_1,\ldots,z_n)^2$; then there exists $u\in I$ and an index i such that $\dfrac{\partial u}{\partial z_i}(0)\neq 0$. Up to permutation of indices we may suppose $i=n$ and then, by inverse function theorem $z_1,\ldots,z_{n-1},u$ is a system of local holomorphic coordinates. Therefore A is isomorphic to $\mathbb{C}\{z_1,\ldots,z_{n-1}\}/I^c$, where I^c is the kernel of the surjective morphism

$$\mathbb{C}\{z_1,\ldots,z_{n-1}\}\to\mathbb{C}\{z_1,\ldots,z_{n-1},u\}/I=A.$$

The conclusion follows by induction on n. □

DEFINITION III.6. An analytic algebra is called *smooth* if it is isomorphic to the power series algebra $\mathbb{C}\{z_1,\ldots,z_k\}$ for some $k\geq 0$.

PROPOSITION III.7. *Let $R=\mathbb{C}\{z_1,\ldots,z_k\}/I$, $I\subset(z_1,\ldots,z_k)^2$, be an analytic algebra. The following conditions are equivalent:*

(1) $I=0$.
(2) *R is smooth.*
(3) *for every surjective morphism of analytic algebras $B\to A$, the morphism*

$$\mathrm{Mor}_{\mathbf{An}}(R,B)\to\mathrm{Mor}_{\mathbf{An}}(R,A)$$

is surjective.

(4) *for every $n\geq 2$ the morphism*

$$\mathrm{Mor}_{\mathbf{An}}(R,\mathbb{C}\{t\}/_{(t^n)})\to\mathrm{Mor}_{\mathbf{An}}(R,\mathbb{C}\{t\}/_{(t^2)})$$

is surjective.

Proof. [$1 \Rightarrow 2$] and [$3 \Rightarrow 4$] are trivial, while [$2 \Rightarrow 3$] is an immediate consequence of the Lemma III.3.
To prove [$4 \Rightarrow 1$], assume $I \neq 0$ and let $s = \nu(I) \geq 2$ be the valuation of I, i.e. the greatest integer s such that $I \subset (z_1, \ldots, z_k)^s$: we claim that $\mathrm{Mor}_{\mathbf{An}}(R, \mathbb{C}[t]/(t^{s+1})) \to \mathrm{Mor}_{\mathbf{An}}(R, \mathbb{C}[t]/(t^2))$ is not surjective.
Choosing $f \in I - (z_1, \ldots, z_k)^{s+1}$, after a possible generic linear change of coordinates of the form $z_i \mapsto z_i + a_i z_1$, $a_2, \ldots, a_k \in \mathbb{C}$, we may assume that f contains the monomial z_1^s with a nonzero coefficient, say $f = cz_1^s + \ldots$; let $\alpha\colon R \to \mathbb{C}[t]/(t^2)$ be the morphism defined by $\alpha(z_1) = t$, $\alpha(z_i) = 0$ for $i > 1$.
Assume that there exists $\beta\colon R \to \mathbb{C}[t]/(t^{s+1})$ that lifts α, then $\beta(z_1) - t, \beta(z_2), \ldots, \beta(z_k) \in (t^2)$ and therefore $\beta(f) \equiv ct^s \pmod{t^{s+1}}$. □

Lemma III.8. *For every analytic algebra R with maximal ideal $\mathfrak{m}$ there exist natural isomorphisms*

$$\mathrm{Hom}_{\mathbb{C}}({}^{\mathfrak{m}}/_{\mathfrak{m}^2}, \mathbb{C}) = \mathrm{Der}_{\mathbb{C}}(R, \mathbb{C}) = \mathrm{Mor}_{\mathbf{An}}(R, {}^{\mathbb{C}[t]}/_{(t^2)}).$$

Proof. Exercise. □

Exercise III.9. The ring of entire holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ is an integral domain but it is not factorial (Hint: consider the sine function $\sin(z)$).
For every connected open subset $U \subset \mathbb{C}^n$, the ring $\mathcal{O}(U)$ is integrally closed in its field of fractions (Hint: Riemann extension theorem). △

2. Analytic singularities and fat points

Let M be a complex manifold, as in Lecture I we denote by $\mathcal{O}_{M,x}$ the ring of germs of holomorphic functions at a point $x \in M$. The elements of $\mathcal{O}_{M,x}$ are the equivalence classes of pairs (U, g), where U is open, $x \in U \subset M$, $g\colon U \to \mathbb{C}$ is holomorphic and $(U, g) \sim (V, h)$ if there exists an open subset W, $x \in W \subset U \cap V$ such that $g_{|W} = h_{|W}$.
By definition of holomorphic function and the identity principle we have that $\mathcal{O}_{\mathbb{C}^n,0}$ is isomorphic to the ring of convergent power series $\mathbb{C}\{z_1, \ldots, z_n\}$.

Let $f\colon M \to N$ be a holomorphic map of complex manifolds, for every open subset $V \subset N$ we have a homomorphism of $\mathbb{C}$-algebras

$$f^*\colon \Gamma(V, \mathcal{O}_N) \to \Gamma(f^{-1}(V), \mathcal{O}_M), \qquad f^*g = g \circ f$$

If $x \in M$ then the limit above maps f^*, for V varying over all the open neighbourhood of $y = f(x)$, gives a local homomorphism of local $\mathbb{C}$-algebras $f^*\colon \mathcal{O}_{N,y} \to \mathcal{O}_{M,x}$.
It is clear that $f^*\colon \mathcal{O}_{N,y} \to \mathcal{O}_{M,x}$ depends only on the behavior of f in a neighbourhood of x and then depends only on the class of f in the space $\mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$.
A choice of local holomorphic coordinates $z_1, \ldots, z_n$ on M such that $z_i(x) = 0$, gives an invertible morphism in $\mathrm{Mor}_{\mathbf{Ger}}((M, x), (\mathbb{C}^n, 0))$ and then an isomorphism $\mathcal{O}_{M,x} = \mathbb{C}\{z_1, \ldots, z_n\}$.

Exercise III.10. Given $f, g \in \mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$, prove that $f = g$ if and only if $f^* = g^*$. △

Definition III.11. An *analytic singularity* is a triple (M, x, I) where M is a complex manifold, $x \in M$ is a point and $I \subset \mathcal{O}_{M,x}$ is a proper ideal.
The germ morphisms $\mathrm{Mor}_{\mathbf{Ger}}((M, x, I), (N, y, J))$ are the equivalence classes of morphisms $f \in \mathrm{Mor}_{\mathbf{Ger}}((M, x), (N, y))$ such that $f^*(J) \subset I$ and $f \sim g$ if and only if $f^* = g^*\colon \mathcal{O}_{N,y}/J \to \mathcal{O}_{M,x}/I$.
We denote by **Ger** the category of analytic singularities (also called germs of complex spaces).

Lemma III.12. *The contravariant functor* **Ger** $\to$ **An***,*

$$Ob(\mathbf{Ger}) \to Ob(\mathbf{An}), \qquad (M, x, I) \mapsto \mathcal{O}_{M,x}/I;$$

$$\mathrm{Mor}_{\mathbf{Ger}}((M, x, I), (N, y, J)) \to \mathrm{Mor}_{\mathbf{An}}\left(\frac{\mathcal{O}_{N,y}}{J}, \frac{\mathcal{O}_{M,x}}{I}\right), \qquad f \mapsto f^*;$$

is an equivalence of categories. Its "inverse" **An** $\to$ **Ger** *(cf.* [**49**, 1.4]*) is called* Spec *(sometimes* Specan*).*

PROOF. Since $\mathbb{C}\{z_1,\dots,z_n\}/I$ is isomorphic to $\mathcal{O}_{\mathbb{C}^n,0}/I$ the above functor is surjective on isomorphism classes.
We only need to prove that $\mathrm{Mor}_{\mathbf{Ger}}((M,x,I),(N,y,J)) \to \mathrm{Mor}_{\mathbf{An}}(\mathcal{O}_{N,y}/J,\mathcal{O}_{M,x}/I)$ is surjective, being injective by definition of $\mathrm{Mor}_{\mathbf{Ger}}$. To see this it is not restrictive to assume $(M,x)=(\mathbb{C}^m_u,0)$, $(N,y)=(\mathbb{C}^n_z,0)$.
Let $g^*\colon \mathbb{C}\{z_1,\dots,z_n\}/J \to \mathbb{C}\{u_1,\dots,u_m\}/I$ be a local homomorphism and choose, for every $i=1,\dots,n$, a convergent power series $f_i\in\mathbb{C}\{u_1,\dots,u_m\}$ such that $f_i\equiv g^*(z_i)$ (mod I). Note that $f_i(0)=0$.
If U is an open set, $0\in U\subset\mathbb{C}^m$, such that f_i are convergent in U, then we may define a holomorphic map $f=(f_1,\dots,f_n)\colon U\to\mathbb{C}^n$. By construction $f^*(z_i)=g^*(z_i)\in \mathbb{C}\{u_1,\dots,u_m\}/_I$ and then by Lemma III.3 $f^*=g^*$. $\square$

DEFINITION III.13. Given an analytic singularity $(X,x)=(M,x,I)$, the analytic algebra $\mathcal{O}_{X,x}:=\mathcal{O}_{M,x}/I$ is called the *algebra of germs of analytic functions* of (X,x).
The *dimension* of (X,x) is by definition the dimension of the analytic algebra $\mathcal{O}_{X,x}$.

DEFINITION III.14. A *fat point* is an analytic singularity of dimension 0.

LEMMA III.15. *Let $X=(M,x,I)$ be an analytic singularity; the following conditions are equivalent.*

1. *The maximal ideal of $\mathcal{O}_{X,x}$ is nilpotent.*
2. *X is a fat point.*
3. *The ideal I contains a power of the maximal ideal of $\mathcal{O}_{M,x}$.*
4. *If V is open, $x\in V\subset M$, and $f_1,\dots,f_h\colon V\to\mathbb{C}$ are holomorphic functions generating the ideal I, then there exists an open neighbourhood $U\subset V$ of x such that*
$$U\cap\{f_1=\ldots=f_h=0\}=\{x\}.$$
5. $\mathrm{Mor}_{\mathbf{An}}(\mathcal{O}_{X,x},\mathbb{C}\{t\})$ *contains only the trivial morphism $f\mapsto f(0)\in\mathbb{C}\subset\mathbb{C}\{t\}$.*

PROOF. $[1\Leftrightarrow 2\Leftrightarrow 3]$ are trivial.
$[3\Rightarrow 4]$ It is not restrictive to assume that V is contained in a coordinate chart; let $z_1,\dots,z_n\colon V\to\mathbb{C}$ be holomorphic coordinates with $z_i(x)=0$ for every i. If 3 holds then there exists $s>0$ such that $z_i^s\in I$ and then there exists an open subset $x\in U\subset V$ and holomorphic functions $a_{ij}\colon U\to\mathbb{C}$ such that $z_i^s=\sum_j a_{ij}f_j$. Therefore $U\cap V\cap\{f_1=\ldots=f_h=0\}\subset U\cap\{z_1^s=\ldots=z_n^s=0\}=\{x\}$.
$[4\Rightarrow 5]$ Let $\phi\colon(\mathbb{C},0)\to(M,x)$ be a germ of holomorphic map such that $\phi^*(I)=0$. If ϕ is defined in an open subset $W\subset\mathbb{C}$ and $\phi(W)\subset U$ then $\phi^*(I)=0$ implies $\phi(W)\subset U\cap\{f_1=\ldots=f_h=0\}$ and therefore $\mathrm{Mor}_{\mathbf{Ger}}((\mathbb{C},0,0),(M,x,I))$ contains only the constant morphism.
$[5\Rightarrow 1]$ is a consequence of Theorem III.4 (with $J=\mathfrak{m}_{M,x}$). $\square$

EXERCISE III.16. If $f\in\mathrm{Mor}_{\mathbf{Ger}}((M,x,I),(N,y,J))$ we define the schematic fibre $f^{-1}(y)$ as the singularity $(M,x,I+f^*\mathfrak{m}_{N,y})$.
Prove that the dimension of a singularity (M,x,I) is the minimum integer d such that there exists a morphism $f\in\mathrm{Mor}_{\mathbf{Ger}}((M,x,I),(\mathbb{C}^d,0,0))$ such that $f^{-1}(0)$ is a fat point. $\triangle$

DEFINITION III.17. The *Zariski tangent space* $T_{x,X}$ of an analytic singularity (X,x) is the $\mathbb{C}$-vector space $\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,x},\mathbb{C})$.

Note that every morphism of singularities $(X,x)\to(Y,y)$ induces a linear morphism of Zariski tangent spaces $T_{x,X}\to T_{y,Y}$.

EXERCISE III.18. (Cartan's Lemma)
Let $(R,\mathfrak{m})$ be an analytic algebra and $G\subset\mathrm{Aut}(R)$ a finite group of automorphisms. Denote $n=\dim_{\mathbb{C}}\mathfrak{m}/\mathfrak{m}^2$.
Prove that there exists an injective homomorphism of groups $G\to GL(\mathbb{C}^n)$ and a G-isomorphism of analytic algebras $R\simeq\mathcal{O}_{\mathbb{C}^n,0}/I$ for some G-stable ideal $I\subset\mathcal{O}_{\mathbb{C}^n,0}$. (Hint: there exists a direct sum decomposition $\mathfrak{m}=V\oplus\mathfrak{m}^2$ such that $gV\subset V$ for every $g\in G$.) $\triangle$

3. The resultant

Let A be a commutative unitary ring and $p \in A[t]$ a monic polynomial of degree d. It is easy to see that $A[t]/(p)$ is a free A-module of rank d with basis $1, t, \dots, t^{d-1}$.
For every $f \in A[t]$ we denote by $R(p, f) \in A$ the determinant of the multiplication map $f\colon A[t]/(p) \to A[t]/(p)$.

Definition III.19. In the notation above, the element $R(p, f)$ is called the *resultant* of p and f.

If $\phi\colon A \to B$ is a morphism of unitary rings then we can extend it to a morphism $\phi\colon A[t] \to B[t]$, $\phi(t) = t$, and it is clear from the definition that $R(\phi(p), \phi(f)) = \phi(R(p, f))$.
By Binet's theorem $R(p, fg) = R(p, f)R(p, g)$.

Lemma III.20. *In the notation above there exist $\alpha, \beta \in A[t]$ with $\deg\alpha < \deg f$, $\deg\beta < \deg p$ such that $R(p, f) = \beta f - \alpha p$. In particular $R(p, f)$ belongs to the ideal generated by p and f.*

Proof. For every $i, j = 0, \dots, d-1$ there exist $h_i \in A[t]$ and $c_{ij} \in A$ such that

$$t^i f = h_i p + \sum_{j=0}^{d-1} c_{ij} t^j, \qquad \deg h_i < \deg f.$$

By definition $R(p, f) = \det(c_{ij})$; if (C^{ij}) is the adjoint matrix of (c_{ij}) we have, by Laplace formula, for every $j = 0, \dots, d-1$

$$\sum_i C^{0i} c_{ij} = \delta_{0j} R(p, f)$$

and then

$$R(p, f) = \sum_{i=0}^{d-1} C^{0i}(t^i f - h_i p) = \beta f - \alpha p.$$

□

Lemma III.21. *In the notation above, if A is an integral domain and p, f have a common factor of positive degree then $R(p, f) = 0$. The converse hold if A is a unique factorization domain.*

Proof. Since A injects into its fraction field, the multiplication $f\colon A[t]/(p) \to A[t]/(p)$ is injective if and only if $R(p, f) \neq 0$.
If $p = qr$ with $\deg r < \deg p$, then the multiplication $q\colon A[t]/(p) \to A[t]/(p)$ is not injective and then its determinant is trivial. If q also divides f then, by the theorem of Binet also $R(p, f) = 0$.
Assume now that A is a unique factorization domain and $R(p, f) = 0$. There exists $q \notin (p)$ such that $fq \in (p)$; by Gauss' lemma $A[t]$ is a UFD and then there exists a irreducible factor p_1 of p dividing f. Since p is a monic polynomial the degree of p_1 is positive. □

Lemma III.22. *Let A be an integral domain and $0 \neq \mathfrak{p} \subset A[t]$ a prime ideal such that $\mathfrak{p} \cap A = 0$. Denote by K the fraction field of A and by $\mathfrak{p}^e \subset K[x]$ the ideal generated by $\mathfrak{p}$. Then:*

(1) *$\mathfrak{p}^e$ is a prime ideal.*
(2) *$\mathfrak{p}^e \cap A[x] = \mathfrak{p}$.*
(3) *There exists $f \in \mathfrak{p}$ such that for every monic polynomial $p \notin \mathfrak{p}$ we have $R(p, f) \neq 0$.*

Proof. [1] We have $\mathfrak{p}^e = \left\{ \dfrac{p}{a} \;\middle|\; p \in \mathfrak{p},\ a \in A - \{0\} \right\}$. If $\dfrac{p_1}{a_1}\dfrac{p_2}{a_2} \in \mathfrak{p}^e$ with $p_i \in A[x]$, $a_i \in A$; then there exists $a \in A - \{0\}$ such that $ap_1p_2 \in \mathfrak{p}$. Since $\mathfrak{p} \cap A = 0$ it must be $p_1 \in \mathfrak{p}$ or $p_2 \in \mathfrak{p}$. This shows that $\mathfrak{p}^e$ is prime.
[2] If $q \in \mathfrak{p}^e \cap A[x]$, then there exists $a \in A$, $a \neq 0$ such that $aq \in \mathfrak{p}$ and therefore $q \in \mathfrak{p}$.
[3] Let $f \in \mathfrak{p} - \{0\}$ be of minimal degree, since $K[t]$ is an Euclidean ring, $\mathfrak{p}^e = fK[t]$ and, since $\mathfrak{p}^e$ is prime, f is irreducible in $K[t]$. If $p \in A[t] \setminus \mathfrak{p}$ is a monic polynomial then $p \notin \mathfrak{p}^e = fK[t]$ and then, according to Lemma III.21, $R(p, f) \neq 0$. □

Theorem III.23. *Let A be a unitary ring, $\mathfrak{p} \subset A[t]$ a prime ideal, $\mathfrak{q} = A \cap \mathfrak{p}$.*
If $\mathfrak{p} \neq \mathfrak{q}[t]$ (e.g. if $\mathfrak{p}$ is proper and contains a monic polynomial) then there exists $f \in \mathfrak{p}$ such that for every monic polynomial $p \notin \mathfrak{p}$ we have $R(p, f) \notin \mathfrak{q}$.
If moreover A is a unique factorization domain we can choose f irreducible.

Proof. $\mathfrak{q}$ is prime and $\mathfrak{q}[t] \subset \mathfrak{p}$, therefore the image of $\mathfrak{p}$ in $(A/\mathfrak{q})[t] = A[t]/\mathfrak{q}[t]$ is still a prime ideal satisfying the hypothesis of Lemma III.22.
It is therefore sufficient to take f as any lifting of the element described in Lemma III.22 and use the functorial properties of the resultant. If A is UFD and f is not irreducible we can write $f = hg$ with $g \in \mathfrak{p}$ irreducible; but $R(p, f) = R(p, h)R(p, g)$ and then also $R(p, g) \notin \mathfrak{q}$. □

Exercise III.24. If $p, q \in A[t]$ are monic polynomials of degrees $d, l > 0$ then for every $f \in A[t]$ we have $R(pq, f) = R(p, f)R(q, f)$. (Hint: write the matrix of the multiplication $f\colon A[t]/(pq) \to A[t]/(pq)$ in the basis $1, t, \ldots, t^{d-1}, p, tp, \ldots, t^{l-1}p$.) △

4. Rückert's Nullstellensatz

The aim of this section is to prove the following theorem, also called *Curve selection lemma*, which is easily seen to be equivalent to Theorem III.4. The proof given here is a particular case of the one sketched in [**51**].

Theorem III.25. *Let $\mathfrak{p} \subset \mathbb{C}\{z_1, \ldots, z_n\}$ be a proper prime ideal and $h \notin \mathfrak{p}$. Then there exists a homomorphism of local $\mathbb{C}$-algebras $\phi\colon \mathbb{C}\{z_1, \ldots, z_n\} \to \mathbb{C}\{t\}$ such that $\phi(\mathfrak{p}) = 0$ and $\phi(h) \neq 0$.*

Corollary III.26. *Let $I \subset \mathbb{C}\{z_1, \ldots, z_n\}$ be a proper ideal and $h \notin \sqrt{I}$. Then there exists a homomorphism of local $\mathbb{C}$-algebras $\phi\colon \mathbb{C}\{z_1, \ldots, z_n\} \to \mathbb{C}\{t\}$ such that $\phi(I) = 0$ and $\phi(h) \neq 0$.*

Proof. If $h \notin \sqrt{I}$ there exists (cf. [**3**]) a prime ideal $\mathfrak{p}$ such that $I \subset \mathfrak{p}$ and $h \notin \mathfrak{p}$. □

Before proving Theorem III.25 we need a series of results that are of independent interest. We recall the following

Definition III.27. A power series $p \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ is called a *Weierstrass polynomial* in t of degree $d \geq 0$ if

$$p = t^d + \sum_{i=0}^{d-1} p_i(z_1, \ldots, z_n)t^i, \qquad p_i(0) = 0.$$

In particular if $p(z_1, \ldots, z_n, t)$ is a Weierstrass polynomial in t of degree d then $p(0, \ldots, 0, t) = t^d$.

Theorem III.28 (Preparation theorem). *Let $f \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ be a power series such that $f(0, \ldots, 0, t) \neq 0$. Then there exists a unique $e \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ such that $e(0) \neq 0$ and ef is a Weierstrass polynomial in t.*

Proof. For the proof we refer to [**23**], [**24**], [**26**], [**37**], [**28**], [**51**]. We note that the condition that the power series $\mu(t) = f(0, \ldots, 0, t)$ is not trivial is also necessary and that the degree of ef in t is equal to the multiplicity at 0 of μ. □

Corollary III.29. *Let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ be a power series of multiplicity d. Then, after a possible generic linear change of coordinates there exists $e \in \mathbb{C}\{z_1, \ldots, z_n\}$ such that $e(0) \neq 0$ and ef is a Weierstrass polynomial of degree d in z_n.*

Proof. After a generic change of coordinates of the form $z_i \mapsto z_i + a_i z_n$, $a_i \in \mathbb{C}$, the series $f(0, \ldots, 0, z_n)$ has multiplicity d. □

Lemma III.30. *Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}[t]$ be polynomials in t with g in Weierstrass' form. if $f = hg$ for some $h \in \mathbb{C}\{x_1, \ldots, x_n, t\}$ then $h \in \mathbb{C}\{x_1, \ldots, x_n\}[t]$.*

We note that if g is not a Weierstrass polynomial then the above result is false; consider for instance the case $n = 0, f = t^3, g = t + t^2$.

PROOF. Write $g = t^s + \sum g_i(x)t^{s-i}$, $g_i(0) = 0$, $f = \sum_{i=0}^{r} f_i(x)t^{r-i}$ $h = \sum_i h_i(x)t^i$, we need to prove that $h_i = 0$ for every $i > r - s$.
Assume the contrary and choose an index $j > r - s$ such that the multiplicity of h_j takes the minimum among all the multiplicities of the power series h_i, $i > r - s$.
From the equality $0 = h_j + \sum_{i>0} g_i h_{j+i}$ we get a contradiction. □

LEMMA III.31. *Let $f \in \mathbb{C}\{x_1, \ldots, x_n\}[t]$ be an irreducible monic polynomial of degree d. Then the polynomial $f_0(t) = f(0, \ldots, 0, t) \in \mathbb{C}[t]$ has a root of multiplicity d.*

PROOF. Let $c \in \mathbb{C}$ be a root of $f_0(t)$. If the multiplicity of c is $l < d$ then the multiplicity of the power series $f_0(t + c) \in \mathbb{C}\{t\}$ is exactly l and therefore $f(x_1, \ldots, x_n, t + c)$ is divided in $\mathbb{C}\{x_1, \ldots, x_n\}[t]$ by a Weierstrass polynomial of degree l. □

LEMMA III.32. *Let $p \in \mathbb{C}\{x\}[y]$ be a monic polynomial of positive degree d in y. Then there exists a homomorphism $\phi\colon \mathbb{C}\{x\}[y] \to \mathbb{C}\{t\}$ such that $\phi(p) = 0$ and $0 \neq \phi(x) \in (t)$.*

PROOF. If $d = 1$ then $p(x, y) = y - p_1(x)$ and we can consider the morphism ϕ given by $\phi(x) = t$, $\phi(y) = p_1(t)$. By induction we can assume the theorem true for monic polynomials of degree $< d$.
If p is reducible we have done, otherwise, writing $p = y^d + p_1(x)y^{d-1} + \ldots + p_d(x)$, after the coordinate change $x \mapsto x$, $y \mapsto y - p_1(x)/d$ we can assume $p_1 = 0$.
For every $i \geq 2$ denote by $\mu(p_i) = \alpha_i > 0$ the multiplicity of p_i (we set $\alpha_i = +\infty$ if $p_i = 0$).
Let $j \geq 2$ be a fixed index such that $\dfrac{\alpha_j}{j} \leq \dfrac{\alpha_i}{i}$ for every i. Setting $m = \alpha_j$, we want to prove that the monic polynomial $p(\xi^j, y)$ is not irreducible.
In fact $p(\xi^j, y) = y^d + \sum_{i\geq 2} h_i(\xi)y^{d-i}$, where $h_i(\xi) = g_i(\xi^j)$.
For every i the multiplicity of h_i is $j\alpha_i \geq im$ and then

$$q(\xi, y) = p(\xi^j, \xi^m y)\xi^{-dm} = t^d + \sum \frac{h_i(\xi)}{\xi^{mi}} y^{d-i} = y^d + \sum \eta_i(\xi)y^{d-i}$$

is a well defined element of $\mathbb{C}\{\xi, y\}$. Since $\eta_1 = 0$ and $\eta_j(0) \neq 0$ the polynomial q is not irreducible and then, by induction there exists a nontrivial morphism $\psi\colon \mathbb{C}\{\xi\}[y] \to \mathbb{C}\{t\}$ such that $\psi(q) = 0$, $0 \neq \psi(\xi) \in (t)$ and we can take $\phi(x) = \psi(\xi^j)$ and $\phi(y) = \psi(\xi^m y)$. □

THEOREM III.33 (Division theorem). *Let $p \in \mathbb{C}\{z_1, \ldots, z_n, t\}$, $p \neq 0$, be a Weierstrass polynomial of degree $d \geq 0$ in t. Then for every $f \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ there exist a unique $h \in \mathbb{C}\{z_1, \ldots, z_n, t\}$ such that $f - hp \in \mathbb{C}\{z_1, \ldots, z_n\}[t]$ is a polynomial of degree $< d$ in t.*

PROOF. For the proof we refer to [**23**], [**24**], [**26**], [**37**], [**28**], [**51**]. □

We note that an equivalent statement for the division theorem is the following:

COROLLARY III.34. *If $p \in \mathbb{C}\{z_1, \ldots, z_n, t\}$, $p \neq 0$, is a Weierstrass polynomial of degree $d \geq 0$ in t, then $\mathbb{C}\{z_1, \ldots, z_n, t\}/(p)$ is a free $\mathbb{C}\{z_1, \ldots, z_n\}$-module with basis $1, t, \ldots, t^{d-1}$.*

PROOF. Clear. □

THEOREM III.35 (Newton-Puiseux). *Let $f \in \mathbb{C}\{x, y\}$ be a power series of positive multiplicity. Then there exists a nontrivial local homomorphism $\phi\colon \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}$ such that $\phi(f) = 0$.*
Moreover if f is irreducible then $\ker \phi = (f)$.

In the above statement nontrivial means that $\phi(x) \neq 0$ or $\phi(y) \neq 0$.

PROOF. After a linear change of coordinates we can assume $f(0, y)$ a non zero power series of multiplicity $d > 0$; by Preparation theorem there exists an invertible power series e such that $p = ef$ is a Weierstrass polynomial of degree d in y.
According to Lemma III.32 there exists a homomorphism $\phi\colon \mathbb{C}\{x\}[y] \to \mathbb{C}\{t\}$ such that $\phi(p) = 0$ and $0 \neq \phi(x) \in (t)$. Therefore $\phi(p(0, y)) \in (t)$ and, being p a Weierstrass

polynomial we have $\phi(y) \in (t)$ and then ϕ extends to a local morphism $\phi\colon \mathbb{C}\{x,y\} \to \mathbb{C}\{t\}$.
Assume now f irreducible, up to a possible change of coordinates and multiplication for an invertible element we may assume that $f \in \mathbb{C}\{x\}[y]$ is an irreducible Weierstrass polynomial of degree $d > 0$.
Let $\phi\colon \mathbb{C}\{x,y\} \to \mathbb{C}\{t\}$ be a nontrivial morphism such that $\phi(f) = 0$, then $\phi(x) \neq 0$ (otherwise $\phi(y)^d = \phi(f) = 0$) and therefore the restricted morphism $\phi\colon \mathbb{C}\{x\} \to \mathbb{C}\{t\}$ is injective.
Let $g \in \ker(\phi)$, by division theorem there exists $r \in \mathbb{C}\{x\}[y]$ such that $g = hf + r$ and then $r \in \ker(\phi)$, $R(f,r) \in \ker(\phi) \cap \mathbb{C}\{x\} = 0$. This implies that f divides r. $\square$

The division theorem allows to extend the definition of the resultant to power series. In fact if $p \in \mathbb{C}\{z_1,\dots,z_n\}[t]$ is a Weierstrass polynomial in t of degree d, for every $f \in \mathbb{C}\{z_1,\dots,z_n,t\}$ we can define the resultant $R(p,f) \in \mathbb{C}\{z_1,\dots,z_n\}$ as the determinant of the morphism of free $\mathbb{C}\{z_1,\dots,z_n\}$-module

$$f\colon \frac{\mathbb{C}\{z_1,\dots,z_n,t\}}{(p)} \to \frac{\mathbb{C}\{z_1,\dots,z_n,t\}}{(p)}$$

induced by the multiplication with f.
It is clear that $R(p,f) = R(p,r)$ whenever $f - r \in (p)$.

Lemma III.36. *Let $p \in \mathbb{C}\{z_1,\dots,z_n,t\}$ be a Weierstrass polynomial of positive degree in t and $V \subset \mathbb{C}\{z_1,\dots,z_n,t\}$ a $\mathbb{C}$-vector subspace.*
Then $R(p,f) = 0$ for every $f \in V$ if and only if there exists a Weierstrass polynomial q of positive degree such that:

(1) *q divides p in $\mathbb{C}\{z_1,\dots,z_n\}[t]$*
(2) *$V \subset q\mathbb{C}\{z_1,\dots,z_n,t\}$*

Proof. One implication is clear, in fact if $p = qr$ then the multiplication by q in not injective in $\mathbb{C}\{z_1,\dots,z_n,t\}/(p)$; therefore $R(p,q) = 0$ and by Binet's theorem $R(p,f) = 0$ for every $f \in (q)$.
For the converse let $p = p_1p_2\dots p_s$ be the irreducible decomposition of p in the UFD $\mathbb{C}\{z_1,\dots,z_n\}[t]$. If $R(p,f) = 0$ and $r = f - hp \in \mathbb{C}\{z_1,\dots,z_n\}[t]$ is the rest of the division then $R(p,r) = 0$ and by Lemma III.21 there exists a factor p_i dividing r and therefore also dividing f.
In particular, setting $V_i = V \cap (p_i)$, we have $V = \cup_i V_i$ and therefore $V = V_i$ for at least one index i and we can take $q = p_i$. $\square$

Proof of III.25. We first consider the easy cases $n = 1$ and $\mathfrak{p} = 0$. If $\mathfrak{p} = 0$ then, after a possible change of coordinates, we may assume $h(0,\dots,0,t) \neq 0$ and therefore we can take $\phi(z_i) = 0$ for $i = 1,\dots,n-1$ and $\phi(z_n) = t$.
If $n = 1$ the only prime nontrivial ideal is (z_1) and therefore the trivial morphism $\phi\colon \mathbb{C}\{z_1\} \to \mathbb{C} \subset \mathbb{C}\{t\}$ satisfies the statement of the theorem.
Assume then $n > 1$, $\mathfrak{p} \neq 0$ and fix a nonzero element $g \in \mathfrak{p}$. After a possible linear change of coordinates and multiplication by invertible elements we may assume both h and g Weierstrass polynomials in the variable z_n. Denoting

$$\mathfrak{r} = \mathfrak{p} \cap \mathbb{C}\{z_1,\dots,z_{n-1}\}[z_n], \qquad \mathfrak{q} = \mathfrak{p} \cap \mathbb{C}\{z_1,\dots,z_{n-1}\} = \mathfrak{r} \cap \mathbb{C}\{z_1,\dots,z_{n-1}\},$$

according to Theorem III.23, there exists $\hat{f} \in \mathfrak{r}$ such that $R(h,\hat{f}) \notin \mathfrak{q}$. On the other hand, by Lemma III.20, $R(g,f) \in \mathfrak{q}$ for every $f \in \mathfrak{p}$.
By induction on n there exists a morphism $\tilde{\psi}\colon \mathbb{C}\{z_1,\dots,z_{n-1}\} \to \mathbb{C}\{x\}$ such that $\tilde{\psi}(\mathfrak{q}) = 0$ and $\tilde{\psi}(R(h,\hat{f})) \neq 0$. Denoting by $\psi\colon \mathbb{C}\{z_1,\dots,z_n\} \to \mathbb{C}\{x,z_n\}$ the natural extension of $\tilde{\psi}$ we have $R(\psi(h),\psi(\hat{f})) \neq 0$ and $R(\psi(g),\psi(f)) = 0$ for every $f \in \mathfrak{p}$. Applying Lemma III.36 to the Weierstrass polynomial $\psi(g)$ and the vector space $V = \psi(\mathfrak{p})$ we prove the existence of an irreducible factor p of $\psi(g)$ such that $\psi(\mathfrak{p}) \subset p\mathbb{C}\{x,z_n\}$.
In particular p divides $\psi(\hat{f})$, therefore $R(\psi(h),p) \neq 0$ and $\psi(h) \notin p\mathbb{C}\{x,z_n\}$.
By Newton-Puiseux' theorem there exists $\eta\colon \mathbb{C}\{x,z_n\} \to \mathbb{C}\{t\}$ such that $\eta(p) = 0$ and $\eta(\psi(h)) \neq 0$. It is therefore sufficient to take ϕ as the composition of ψ and η. $\square$

EXERCISE III.37. Prove that $f, g \in \mathbb{C}\{x, y\}$ have a common factor of positive multiplicity if and only if the $\mathbb{C}$-vector space $\mathbb{C}\{x, y\}/(f, g)$ is infinite dimensional. △

5. Dimension bounds

As an application of Theorem III.25 we give some bounds for the dimension of an analytic algebra; this bounds will be very useful in deformation and moduli theory. The first bound (Lemma III.40) is completely standard and the proof is reproduced here for completeness; the second bound (Theorem III.41, communicated to the author by H. Flenner) finds application in the "T^1-lifting" approach to deformation problems.

We need the following two results of commutative algebra.

LEMMA III.38. *Let $(A, \mathfrak{m})$ be a local Noetherian ring and $J \subset I \subset A$ two ideals. If $J + \mathfrak{m}I = I$ then $J = I$.*

PROOF. This a special case of Nakayama's lemma [**3**], [**51**]. □

LEMMA III.39. *Let $(A, \mathfrak{m})$ be a local Noetherian ring and $f \in \mathfrak{m}$, then $\dim A/(f) \geq \dim A - 1$.*
Moreover, if f is nilpotent then $\dim A/(f) = \dim A$, while if f is not a zerodivisor then $\dim A/(f) = \dim A - 1$.

PROOF. [**3**]. □

LEMMA III.40. *Let R be an analytic algebra with maximal ideal $\mathfrak{m}$, then $\dim R \leq \dim_{\mathbb{C}} \dfrac{\mathfrak{m}}{\mathfrak{m}^2}$ and equality holds if and only if R is smooth.*

PROOF. Let $n = \dim_{\mathbb{C}} \dfrac{\mathfrak{m}}{\mathfrak{m}^2}$ and $f_1, \ldots, f_n \in \mathfrak{m}$ inducing a basis of $\dfrac{\mathfrak{m}}{\mathfrak{m}^2}$. If $J = (f_1, \ldots, f_n)$ by assumption $J + \mathfrak{m}^2 = \mathfrak{m}$ and then by Lemma III.38 $J = \mathfrak{m}$, $R/J = \mathbb{C}$ and $0 = \dim R/J \geq \dim R - n$.
According to Lemma III.5 we can write $R = \mathbb{C}\{z_1, \ldots, z_n\}/I$ for some ideal contained in $(z_1, \ldots, z_n)^2$. Since $\mathbb{C}\{z_1, \ldots, z_n\}$ is an integral domain, according to Lemma III.39 $\dim R = n$ if and only if $I = 0$. □

THEOREM III.41. *Let $R = P/I$ be an analytic algebra, where $P = \mathbb{C}\{z_1, \ldots, z_n\}$, $n > 0$ is a fixed integer, and $I \subset P$ is a proper ideal.*
Denoting by $\mathfrak{m} = (z_1, \ldots, z_n)$ the maximal ideal of P and by $J \subset P$ the ideal

$$J = \left\{ f \in I \;\middle|\; \frac{\partial f}{\partial z_i} \in I, \; \forall i = 1, \ldots, n \right\}$$

we have $\dim R \geq n - \dim_{\mathbb{C}} \dfrac{I}{J + \mathfrak{m}I}$.

PROOF. (taken from [**14**]) We first introduce the *curvilinear obstruction map*

$$\gamma_I \colon \mathrm{Mor}_{\mathbf{An}}(P, \mathbb{C}\{t\}) \to \mathrm{Hom}_{\mathbb{C}}\left(\frac{I}{J + \mathfrak{m}I}, \mathbb{C}\right).$$

Given $\phi\colon P \to \mathbb{C}\{t\}$, if $\phi(I) = 0$ we define $\gamma_I(\phi) = 0$; if $\phi(I) \neq 0$ and s is the biggest integer such that $\phi(I) \subset (t^s)$ we define, for every $f \in I$, $\gamma_I(\phi)f$ as the coefficient of t^s in the power series expansion of $\phi(f)$.
It is clear that $\gamma_I(\phi)(\mathfrak{m}I) = 0$, while if $\phi(I) \subset (t^s)$ and $f \in J$ we have $\phi(f) = f(\phi(z_1), \ldots, \phi(z_n))$,

$$\frac{d\phi(f)}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(\phi(z_1), \ldots, \phi(z_n)) \frac{d\phi(z_i)}{dt} \in (t^s)$$

and therefore $\phi(f) \in (t^{s+1})$ (this is the point where the characteristic of the field plays an essential role).
The ideal I is finitely generated, say $I = (f_1, \ldots, f_d)$, according to Nakayama's lemma we can assume $f_1, \ldots, f_d$ a basis of $I/\mathfrak{m}I$.
By repeated application of Corollary III.26 (and possibly reordering the f_i's) we can assume that there exists an $h \leq d$ such that the following holds:

(1) $f_i\notin\sqrt{(f_1,\dots,f_{i-1})}$ for $i\le h$;
(2) for every $i\le h$ there exists a morphism of analytic algebras $\phi_i : P\to\mathbb{C}\{t\}$ such that $\phi_i(f_i)\neq 0$, $\phi_i(f_j)=0$ if $j<i$ and the multiplicity of $\phi_i(f_j))$ is bigger than or equal to the multiplicity of $\phi_i(f_i))$ for every $j>i$.
(3) $I\subset\sqrt{(f_1,\dots,f_h)}$.

Condition 3) implies that $\dim R=\dim P/(f_1,\dots,f_h)\ge n-h$, hence it is enough to prove that $\gamma_I(\phi_1),\dots,\gamma_I(\phi_h)$ are linearly independent in $\mathrm{Hom}_{\mathbb{C}}\left(\dfrac{I}{J+\mathfrak{m}I},\mathbb{C}\right)$ and this follows immediately from the fact that the matrix $a_{ij}=\gamma_I(\phi_i)f_j$, $i,j=1,\dots,h$, has rank h, being triangular with nonzero elements on the diagonal. $\square$

Exercise III.42. In the notation of Theorem III.41 prove that $I^2\subset J$. Prove moreover that $I=J+\mathfrak{m}I$ if and only if $I=0$. $\triangle$

Exercise III.43. Let $I\subset\mathbb{C}\{x,y\}$ be the ideal generated by the polynomial $f=x^5+y^5+x^3y^3$ and by its partial derivatives $f_x=5x^4+3x^2y^3$, $f_y=5y^4+3x^3y^2$. Prove that J is not contained in $\mathfrak{m}I$, compute the dimension of the analytic algebra $\mathbb{C}\{x,y\}/I$ and of the vector spaces $\dfrac{I}{J+\mathfrak{m}I}$, $\dfrac{I}{\mathfrak{m}I}$. $\triangle$

Exercise III.44. (easy, but for experts) In the notation of III.41, if $I\subset\mathfrak{m}^2$ then

$$\mathrm{Hom}_{\mathbb{C}}\left(\frac{I}{J+\mathfrak{m}I},\mathbb{C}\right)=\mathrm{Ext}^1_R(\Omega_R,\mathbb{C}).$$

(Ω_R is the R-module of separated differentials) $\triangle$

Exercise III.45. In the notation of Theorem III.41, prove that for every short exact sequence $0\to E\to F\to G\to 0$ of R-modules of finite length (i.e. annihilated by some power of the maximal ideal) it is defined a map

$$ob\colon \mathrm{Der}_{\mathbb{C}}(R,G)\to\mathrm{Hom}_R\left(\frac{I}{J},E\right)$$

with the property that $ob(\phi)=0$ if and only if ϕ lifts to a derivation $R\to F$.
Moreover, if $\mathfrak{m}_R E=0$ then $\mathrm{Hom}_R\left(\dfrac{I}{J},E\right)=\mathrm{Hom}_{\mathbb{C}}\left(\dfrac{I}{J+\mathfrak{m}I},E\right)$. $\triangle$

Remark III.46. (T^1-lifting for prorepresentable functors.)
For every morphism of analytic algebras $f\colon R\to A$ and every A-module of finite length M there exists a bijection between $\mathrm{Der}_{\mathbb{C}}(R,M)$ and the liftings of f to morphisms $R\to A\oplus M$.
In the notation of Theorem III.41, if $I\subset\mathfrak{m}^2$, then $\mathrm{Hom}_{\mathbb{C}}\left(\dfrac{I}{J+\mathfrak{m}I},\mathbb{C}\right)$ is the subspace of $\mathrm{Hom}_{\mathbb{C}}\left(\dfrac{I}{\mathfrak{m}I},\mathbb{C}\right)$ of obstructions (see [**13**, Section 5]) of the functor h_R arising from all the small extensions of the form $0\to\mathbb{C}\to A\oplus M\xrightarrow{(Id,p)}A\oplus N\to 0$, where $p\colon M\to N$ is a morphism of A-modules and $A\oplus M\to A$, $A\oplus N\to A$ are the trivial extensions.

6. Historical survey, III

According to [**24**], the preparation theorem was proved by Weierstrass in 1860, while division theorem was proved by Stickelberger in 1887.
The factoriality of $\mathbb{C}\{z_1,\dots,z_n\}$ was proved by E. Lasker in a, long time ignored, paper published in 1905. The same result was rediscovered by W. Rückert (a student of W. Krull) together the Noetherianity in 1931. In the same paper of Rückert it is implicitly contained the Nullstellensatz. The ideas of Rückert's proof are essentially the same used in the proof given in [**28**]. The proof given here is different.

All the algebraic results of this chapter that make sense also for the ring of formal power series $\mathbb{C}[[z_1,\dots,z_n]]$ and their quotients, remain true. In many cases, especially in deformation theory, we seek for solutions of systems of analytic equations but we are able to solve these equation only formally; in this situation a great help comes from the following theorem, proved by M. Artin in 1968.

THEOREM III.47. *Consider two arbitrary morphisms of analytic algebras $f\colon S\to R$, $g\colon S\to\mathbb{C}\{z_1,\dots,z_n\}$ and a positive integer $s>0$. The inclusion $\mathbb{C}\{z_1,\dots,z_n\}\subset\mathbb{C}[[z_1,\dots,z_n]]$ and the projection $\mathbb{C}\{z_1,\dots,z_n\}\to\dfrac{\mathbb{C}\{z_1,\dots,z_n\}}{(z_1,\dots,z_n)^s}$ give structures of S-algebras also on $\mathbb{C}[[z_1,\dots,z_n]]$ and $\dfrac{\mathbb{C}\{z_1,\dots,z_n\}}{(z_1,\dots,z_n)^s}$.*
Assume it is given a morphism of analytic S-algebras

$$\phi\colon R\to\frac{\mathbb{C}\{z_1,\dots,z_n\}}{(z_1,\dots,z_n)^s}=\frac{\mathbb{C}[[z_1,\dots,z_n]]}{(z_1,\dots,z_n)^s}.$$

If ϕ lifts to a S-algebra morphism $R\to\mathbb{C}[[z_1,\dots,z_n]]$ then ϕ lifts also to a S-algebra morphism $R\to\mathbb{C}\{z_1,\dots,z_n\}$.

Beware. Theorem III.47 does not imply that every lifting $R\to\mathbb{C}[[z_1,\dots,z_n]]$ is “convergent”.

PROOF. This is an equivalent statement of the main theorem of [**1**]. We leave as as an exercise to the reader to proof of the equivalence of the two statements. □

EXERCISE III.48. Use Theorem III.47 to prove:

(1) Every irreducible convergent power series $f\in\mathbb{C}\{z_1,\dots,z_n\}$ is also irreducible in $\mathbb{C}[[z_1,\dots,z_n]]$.
(2) $\mathbb{C}\{z_1,\dots,z_n\}$ is integrally closed in $\mathbb{C}[[z_1,\dots,z_n]]$.

△

REMARK III.49. It is possible to give also an elementary proof of item 2 of Exercise III.48 (e.g. [**51**]), while I don't know any proof of item 1 which does not involve Artin's theorem.

LECTURE IV

Infinitesimal deformations of complex manifolds

In this chapter we pass from the classical language of deformation theory to the formalism of differential graded objects. After a brief introduction of dg-vector spaces and dg-algebras, we associate to every deformation $X_0 \hookrightarrow \{X_t\}_{t\in T} \to (T,0)$ its *algebraic data* (Definition IV.27), which is a pair of morphisms of sheaves of dg-algebras on X_0. This algebraic data encodes the Kodaira-Spencer map and also all the "Taylor coefficients" of $t \mapsto X_t$.
We introduce the notion of infinitesimal deformation as an infinitesimal variation of integrable complex structures; this definition will be more useful for our purposes. The infinitesimal Newlander-Nirenberg theorem, i.e. the equivalence of this definition with the more standard definition involving flatness, although not difficult to prove, would require a considerable amount of preliminaries in commutative and homological algebra and it is not given in this notes.
In Section 7 we state without proof the Kuranishi's theorem of existence of semiuniversal deformations of compact complex manifolds. In order to keep this notes short and selfcontained, we avoid the use of complex analytic spaces and we state only the "infinitesimal" version of Kuranishi's theorem. This is not a great gap for us since we are mainly interested in infinitesimal deformations. The interested reader can find sufficient material to filling this gap in the papers [**59**], [**60**] and references therein.

From now on we assume that the reader is familiar with the notion of sheaf, sheaf of algebras, ideal and quotient sheaves, morphisms of sheaves.
If $\mathcal{F}$ is a sheaf on a topological space Y we denote by $\mathcal{F}_y$, $y \in Y$, the stalk at the point y. If $\mathcal{G}$ is another sheaf on Y we denote by $\mathcal{H}om(\mathcal{F},\mathcal{G})$ the sheaf of morphisms from $\mathcal{F}$ to $\mathcal{G}$ and by $\operatorname{Hom}(\mathcal{F},\mathcal{G}) = \Gamma(Y, \mathcal{H}om(\mathcal{F},\mathcal{G}))$.

For every complex manifold X we denote by $\mathcal{A}_X^{p,q}$ the sheaf of differential forms of type (p,q) and $\mathcal{A}_X^{*,*} = \oplus_{p,q}\mathcal{A}_X^{p,q}$. The sheaf of holomorphic functions on X is denoted by $\mathcal{O}_X$; Ω_X^* (resp.: $\overline{\Omega}_X^*$) is the sheaf of holomorphic (resp.: antiholomorphic) differential forms. By definition $\Omega_X^* = \ker(\overline{\partial}\colon \mathcal{A}^{*,0} \to \mathcal{A}^{*,1})$, $\overline{\Omega}_X^* = \ker(\partial\colon \mathcal{A}^{0,*} \to \mathcal{A}^{1,*})$; note that $\phi \in \Omega_X^*$ if and only if $\overline{\phi} \in \overline{\Omega}_X^*$.
If $E \to X$ is a holomorphic vector bundle we denote by $\mathcal{O}_X(E)$ the sheaf of holomorphic sections of E.

1. Differential graded vector spaces

This section is purely algebraic and every vector space is considered over a fixed field $\mathbb{K}$; unless otherwise specified, by the symbol $\otimes$ we mean the tensor product $\otimes_{\mathbb{K}}$ over the field $\mathbb{K}$.

NOTATION IV.1. We denote by $\mathbf{G}$ the category of $\mathbb{Z}$-graded $\mathbb{K}$-vector space. The objects of $\mathbf{G}$ are the $\mathbb{K}$-vector spaces V endowed with a $\mathbb{Z}$-graded direct sum decomposition $V = \oplus_{i\in\mathbb{Z}} V_i$. The elements of V_i are called homogeneous of degree i. The morphisms in $\mathbf{G}$ are the degree-preserving linear maps.

If $V = \oplus_{n\in\mathbb{Z}} V_n \in \mathbf{G}$ we write $\deg(a;V) = i \in \mathbb{Z}$ if $a \in V_i$; if there is no possibility of confusion about V we simply denote $\overline{a} = \deg(a;V)$.

Given two graded vector spaces $V, W \in \mathbf{G}$ we denote by $\mathrm{Hom}^n_{\mathbb{K}}(V,W)$ the vector space of $\mathbb{K}$-linear maps $f\colon V \to W$ such that $f(V_i) \subset W_{i+n}$ for every $i \in \mathbb{Z}$. Observe that $\mathrm{Hom}^0_{\mathbb{K}}(V,W) = \mathrm{Hom}_{\mathbf{G}}(V,W)$ is the space of morphisms in the category $\mathbf{G}$.

The tensor product, $\otimes\colon \mathbf{G}\times\mathbf{G} \to \mathbf{G}$, and the *graded Hom*, $\mathrm{Hom}^*\colon \mathbf{G}\times\mathbf{G}\to\mathbf{G}$, are defined in the following way: given $V, W \in \mathbf{G}$ we set

$$V\otimes W = \bigoplus_{i\in\mathbb{Z}}(V\otimes W)_i, \text{ where } (V\otimes W)_i = \bigoplus_{j\in\mathbb{Z}} V_j \otimes W_{i-j},$$

$$\mathrm{Hom}^*(V,W) = \bigoplus_n \mathrm{Hom}^n_{\mathbb{K}}(V,W).$$

We denote by

$$\langle,\rangle : \mathrm{Hom}^*(V,W)\times V \to W, \qquad \langle f, v\rangle = f(v)$$

the natural pairing.

Definition IV.2. If $V, W \in \mathbf{G}$, the *twisting map* $T\colon V\otimes W \to W\otimes V$ is the linear map defined by the rule $T(v\otimes w) = (-1)^{\overline{v}\,\overline{w}} w\otimes v$, for every pair of homogeneous elements $v\in V$, $w \in W$.

Unless otherwise specified we shall use the **Koszul signs convention**. This means that we choose as *natural isomorphism* between $V\otimes W$ and $W\otimes V$ the twisting map T and we make every commutation rule compatible with T. More informally, to "get the signs right", whenever an "object of degree d passes on the other side of an object of degree h, a sign $(-1)^{dh}$ must be inserted".

As an example, the natural map $\langle,\rangle : V\times \mathrm{Hom}^*(V,W)\to W$ must be defined as $\langle v, f\rangle = (-1)^{\overline{f}\,\overline{v}} f(v)$ for homogeneous f, v. Similarly, if $f, g \in \mathrm{Hom}^*(V,W)$, their tensor product $f\otimes g \in \mathrm{Hom}^*(V\otimes V, W\otimes W)$ must be defined on bihomogeneous tensors as $(f\otimes g)(u\otimes v) = (-1)^{\overline{g}\,\overline{u}} f(u)\otimes g(v)$.

Notation IV.3. We denote by $\mathbf{DG}$ the category of $\mathbb{Z}$-graded differential $\mathbb{K}$-vector spaces (also called complexes of vector spaces). The objects of $\mathbf{DG}$ are the pairs (V,d) where $V = \oplus V_i$ is an object of $\mathbf{G}$ and $d\colon V\to V$ is a linear map, called *differential* such that $d(V_i)\subset V_{i+1}$ and $d^2 = d\circ d = 0$. The morphisms in $\mathbf{DG}$ are the degree-preserving linear maps commuting with the differentials.

For simplicity we will often consider $\mathbf{G}$ as the full subcategory of $\mathbf{DG}$ whose objects are the complexes $(V, 0)$ with trivial differential.

If $(V,d), (W,\delta)\in\mathbf{DG}$ then also $(V\otimes W, d\otimes Id + Id\otimes\delta)\in\mathbf{DG}$; according to Koszul signs convention, since $\delta\in \mathrm{Hom}^1_{\mathbb{K}}(W,W)$, we have $(Id\otimes\delta)(v\otimes w) = (-1)^{\overline{v}} v\otimes\delta(w)$.

There exists also a natural differential ρ on $\mathrm{Hom}^*(V,W)$ given by the formula

$$\delta\langle f, v\rangle = \langle \rho f, v\rangle + (-1)^{\overline{f}}\langle f, dv\rangle.$$

Given (V,d) in $\mathbf{DG}$ we denote as usual by $Z(V) = \ker d$ the space of cycles, by $B(V) = d(V)$ the space of boundaries and by $H(V) = Z(V)/B(V)$ the homology of V. Note that Z, B and H are all functors from $\mathbf{DG}$ to $\mathbf{G}$.

A morphism in $\mathbf{DG}$ is called a quasiisomorphism if it induces an isomorphism in homology.

A differential graded vector space (V,d) is called *acyclic* if $H(V) = 0$.

Definition IV.4. Two morphisms $f, g \in \mathrm{Hom}^n_{\mathbb{K}}(V,W)$ are said to be *homotopic* if their difference $f - g$ is a boundary in the complex $\mathrm{Hom}^*(V,W)$.

Exercise IV.5. Let V, W be differential graded vector spaces, then:

(1) $\mathrm{Hom}_{\mathbf{DG}}(V,W) = Z^0(\mathrm{Hom}^*(V,W))$.
(2) If $f\in B^0(\mathrm{Hom}^*(V,W))\subset \mathrm{Hom}_{\mathbf{DG}}(V,W)$ then the induced morphism $f\colon H(V)\to H(W)$ is trivial.
(3) If $f, g\in \mathrm{Hom}_{\mathbf{DG}}(V,W)$ are homotopic then they induce the same morphism in homology.
(4) V is acyclic if and only if the identity $Id\colon V\to V$ is homotopic to 0. (Hint: if $C\subset V$ is a complement of $Z(V)$, i.e. $V = Z(V)\oplus C$, then V is acyclic if and only if $d\colon C_i \to Z(V)_{i+1}$ is an isomorphism for every i.)

△

The fiber product of two morphisms $B \xrightarrow{f} D$ and $C \xrightarrow{h} D$ in the category **DG** is defined as the complex

$$C \times_D B = \bigoplus_n (C \times_D B)_n, \qquad (C \times_D B)_n = \{(c, b) \in C_n \times B_n \,|\, h(c) = f(b)\}$$

with differential $d(c, b) = (dc, db)$.

A commutative diagram in **DG**

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & B \arrow[d, "f"] \\ C \arrow[r, "h"] & D \end{tikzcd}$$

is called cartesian if the induced morphism $A \to C \times_D B$ is an isomorphism; it is an easy exercise in homological algebra to prove that if f is a surjective (resp.: injective) quasi-isomorphism, then g is a surjective (resp.: injective) quasiisomorphism. (Hint: if f is a surjective quasiisomorphism then $\ker f = \ker g$ is acyclic.)

For every integer $n \in \mathbb{Z}$ let's choose a formal symbol $1[n]$ of degree $-n$ and denote by $\mathbb{K}[n]$ the graded vector space generated by $1[n]$. In other terms, the homogeneous components of $\mathbb{K}[n] \in \mathbf{G} \subset \mathbf{DG}$ are

$$\mathbb{K}[n]_i = \begin{cases} \mathbb{K} & \text{if } i + n = 0 \\ 0 & \text{otherwise} \end{cases}$$

For every pair of integers $n, m \in \mathbb{Z}$ there exists a **canonical** linear isomorphism $S_n^m \in \operatorname{Hom}_{\mathbb{K}}^{n-m}(\mathbb{K}[n], \mathbb{K}[m])$; it is described by the property $S_n^m(1[n]) = 1[m]$.

Given $n \in \mathbb{Z}$, the shift functor $[n]\colon \mathbf{DG} \to \mathbf{DG}$ is defined by setting $V[n] = \mathbb{K}[n] \otimes V$, $V \in \mathbf{DG}$, $f[n] = Id_{\mathbb{K}[n]} \otimes f$, $f \in \operatorname{Mor}_{\mathbf{DG}}$.
More informally, the complex $V[n]$ is the complex V with degrees shifted by n, i.e. $V[n]_i = V_{i+n}$, and differential multiplied by $(-1)^n$. The shift functors preserve the subcategory **G** and commute with the homology functor H. If $v \in V$ we also write $v[n] = 1[n] \otimes v \in V[n]$.

Exercise IV.6. There exist natural isomorphisms

$$\operatorname{Hom}_{\mathbb{K}}^n(V, W) = \operatorname{Hom}_{\mathbf{G}}(V[-n], W) = \operatorname{Hom}_{\mathbf{G}}(V, W[n]).$$

△

Example IV.7. Among the interesting objects in **DG** there are the acyclic complexes $\Omega[n] = \mathbb{K}[n] \otimes \Omega$, where $\Omega = (\Omega_0 \oplus \Omega_1, d)$, $\Omega_0 = \mathbb{K}$, $\Omega_1 = \mathbb{K}[-1]$ and $d\colon \Omega_0 \to \Omega_1$ is the canonical linear isomorphism $d(1[0]) = 1[-1]$. The projection $p\colon \Omega \to \Omega_0 = \mathbb{K}$ and the inclusion $\Omega_1 \to \Omega$ are morphisms in **DG**.

Exercise IV.8. Let V, W be differential graded vector spaces, then:

(1) In the notation of Example IV.7, two morphisms of complexes $f, g\colon V \to W$ are homotopic if and only if there exists $h \in \operatorname{Hom}_{\mathbf{DG}}(V, \Omega \otimes W)$ such that $f - g = (p \otimes Id_{|W}) \circ h$.
(2) If $f, g\colon V \to W$ are homotopic then $f \otimes h$ is homotopic to $g \otimes h$ for every $h\colon V' \to W'$.
(3) (Künneth) If V is acyclic then $V \otimes U$ is acyclic for every $U \in \mathbf{DG}$.

△

2. Review of terminology about algebras

Let R be a commutative ring, by a nonassociative (= not necessarily associative) R-algebra we mean a R-module M endowed with a R-bilinear map $M \times M \to M$.
The nonassociative algebra M is called *unitary* if there exist a "unity" $1 \in M$ such that $1m = m1 = m$ for every $m \in M$.
A *left ideal* (resp.: *right ideal*) of M is a submodule $I \subset M$ such that $MI \subset I$ (resp.: $IM \subset I$). A submodule is called an *ideal* if it is both a left and right ideal.
A homomorphism of R-modules $d\colon M \to M$ is called a *derivation* if satisfies the Leibnitz rule $d(ab) = d(a)b + ad(b)$. A derivation d is called a *differential* if $d^2 = d \circ d = 0$.
A R-algebra is *associative* if $(ab)c = a(bc)$ for every $a, b, c \in M$. Unless otherwise specified, we reserve the simple term *algebra* only to associative algebra (almost all the algebras considered in these notes are either associative or Lie).
If M is unitary, a *left inverse* of $m \in M$ is an element $a \in M$ such that $am = 1$. A *right inverse* of m is an element $b \in M$ such that $mb = 1$.
If M is unitary and associative, an element m is called invertible if has left and right inverses. It is easy to see that if m is invertible then every left inverse of m is equal to every right inverse, in particular there exists a unique $m^{-1} \in M$ such that $mm^{-1} = m^{-1}m = 1$.

Exercise IV.9. Let g be a Riemannian metric on an open connected subset $U \subset \mathbb{R}^n$ and let $\phi\colon U \to \mathbb{R}$ be a differentiable function (called *potential*).
Denote by $R = C^\infty(U, \mathbb{R})$ and by M the (free of rank n) R-module of vector fields on U. If $x_1, \ldots, x_n$ is a system of linear coordinates on $\mathbb{R}^n$ denote by:

(1) $\partial_i = \dfrac{\partial}{\partial x_i} \in M$, $\phi_{ijk} = \partial_i\partial_j\partial_k\phi \in R$.

(2) $g_{ij} = g(\partial_i, \partial_j) \in R$ and g^{ij} the coefficients of the inverse matrix of g_{ij}.

(3) $\partial_i * \partial_j = \displaystyle\sum_{k,l} \phi_{ijl} g^{lk} \partial_k$

Prove that the R-linear extension $M \times M \to M$ of the product $*$ is independent from the choice of the linear coordinates and write down the (differential) equation that ϕ must satisfy in order to have the product $*$ associative. This equation is called WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equation and it is very important in mathematics since 1990.
$\triangle$

3. dg-algebras and dg-modules

Definition IV.10. A graded (associative, $\mathbb{Z}$-commutative) algebra is a graded vector space $A = \oplus A_i \in \mathbf{G}$ endowed with a product $A_i \times A_j \to A_{i+j}$ satisfying the properties:

(1) $a(bc) = (ab)c$.
(2) $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.
(3) (Koszul sign convention) $ab = (-1)^{\bar{a}\,\bar{b}} ba$ for a, b homogeneous.

The algebra A is unitary if there exists $1 \in A_0$ such that $1a = a1 = a$ for every $a \in A$.

Let A be a graded algebra, then A_0 is a commutative $\mathbb{K}$-algebra in the usual sense; conversely every commutative $\mathbb{K}$-algebra can be considered as a graded algebra concentrated in degree 0. If $I \subset A$ is a homogeneous left (resp.: right) ideal then I is also a right (resp.: left) ideal and the quotient A/I has a natural structure of graded algebra.

Example IV.11. The exterior algebra $A = \bigwedge^* V$ of a $\mathbb{K}$-vector space V, endowed with wedge product, is a graded algebra with $A_i = \bigwedge^i V$.

Example IV.12. *(Polynomial algebras.)* Given a set $\{x_i\}$, $i \in I$, of homogeneous indeterminates of integral degree $\overline{x_i} \in \mathbb{Z}$ we can consider the graded algebra $\mathbb{K}[\{x_i\}]$. As a $\mathbb{K}$-vector space $\mathbb{K}[\{x_i\}]$ is generated by monomials in the indeterminates x_i subjected to

the relations $x_i x_j = (-1)^{\overline{x_i}\,\overline{x_j}} x_j x_i$.
In some cases, in order to avoid confusion about terminology, for a monomial $x_{i_1}^{\alpha_1} \dots x_{i_n}^{\alpha_n}$ it is defined:

- The *internal degree* $\sum_h \overline{x_{i_h}} \alpha_h$.
- The *external degree* $\sum_h \alpha_h$.

In a similar way it is defined $A[\{x_i\}]$ for every graded algebra A.

EXERCISE IV.13. Let A be a graded algebra: if every $a \neq 0$ is invertible then $A = A_0$ and therefore A is a field.
Give an example of graded algebra where every homogeneous $a \neq 0$ is invertible but $A \neq A_0$. △

DEFINITION IV.14. A *dg-algebra* (differential graded algebra) is the data of a graded algebra A and a $\mathbb{K}$-linear map $s\colon A \to A$, called *differential*, with the properties:

(1) $s(A_n) \subset A_{n+1}$, $s^2 = 0$.
(2) (graded Leibnitz rule) $s(ab) = s(a)b + (-1)^{\overline{a}} a s(b)$.

A morphism of dg-algebras is a morphism of graded algebras commuting with differentials; the category of dg-algebras is denoted by **DGA**.

EXAMPLE IV.15. Let U be an open subset of a complex variety and denote by $A_i = \oplus_{p+q=i} \Gamma(U, \mathcal{A}_X^{p,q})$. Then $\Gamma(U, \mathcal{A}_X^{*,*}) = \oplus A_i$ admits infinitely many structures of differential graded algebras where the differential of each one of is a linear combination $a\partial + b\overline{\partial}$, $a, b \in \mathbb{C}$.

EXERCISE IV.16. Let (A, s) be a unitary dg-algebra; prove:

(1) $1 \in Z(A)$.
(2) $Z(A)$ is a graded subalgebra of A and $B(A)$ is a homogeneous ideal of $Z(A)$. In particular $1 \in B(A)$ if and only if $H(A) = 0$.

△

A differential ideal of a dg-algebra (A, s) is a homogeneous ideal I of A such that $s(I) \subset I$; there exists an obvious bijection between differential ideals and kernels of morphisms of dg-algebras.
On a polynomial algebra $\mathbb{K}[\{x_i\}]$ a differential s is uniquely determined by the values $s(x_i)$.

EXAMPLE IV.17. Let t, dt be indeterminates of degrees $\overline{t} = 0$, $\overline{dt} = 1$; on the polynomial algebra $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ there exists an obvious differential d such that $d(t) = dt$, $d(dt) = 0$. Since $\mathbb{K}$ has characteristic 0, we have $H(\mathbb{K}[t, dt]) = \mathbb{K}$. More generally if (A, s) is a dg-algebra then $A[t, dt]$ is a dg-algebra with differential $s(a \otimes p(t)) = s(a) \otimes p(t) + (-1)^{\overline{a}} a \otimes p'(t)dt$, $s(a \otimes q(t)dt) = s(a) \otimes q(t)dt$.

DEFINITION IV.18. A morphism of dg-algebras $B \to A$ is called a *quasiisomorphism* if the induced morphism $H(B) \to H(A)$ is an isomorphism.

Given a morphism of dg-algebras $B \to A$ the space $\mathrm{Der}^n_B(A, A)$ of B-derivations of degree n is by definition

$$\mathrm{Der}^n_B(A, A) = \{\phi \in \mathrm{Hom}^n_{\mathbb{K}}(A, A) \mid \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b),\ \phi(B) = 0\}.$$

We also consider the graded vector space

$$\mathrm{Der}^*_B(A, A) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}^n_B(A, A) \in \mathbf{G}.$$

There exist a natural differential d and a natural bracket $[-, -]$ on $\mathrm{Der}^*_B(A, A)$ defined as

$$d\colon \mathrm{Der}^n_B(A, A) \to \mathrm{Der}^{n+1}_B(A, A), \qquad d\phi = d_A\phi - (-1)^n \phi d_A$$

and

$$[f, g] = fg - (-1)^{\overline{f}\,\overline{g}} gf.$$

EXERCISE IV.19. Verify that, if $f \in \mathrm{Der}^p_B(A, A)$ and $g \in \mathrm{Der}^q_B(A, A)$ then $[f, g] \in \mathrm{Der}^{p+q}_B(A, A)$ and $d[f, g] = [df, g] + (-1)^p [f, dg]$. △

Let (A,s) be a fixed dg-algebra, by an A-dg-module we mean a differential graded vector space (M,s) together two associative distributive multiplication maps $A\times M\to M$, $M\times A\to M$ with the properties:

(1) $A_iM_j\subset M_{i+j}$, $M_iA_j\subset M_{i+j}$.
(2) (Koszul) $am=(-1)^{\overline{a}\,\overline{m}}ma$, for homogeneous $a\in A$, $m\in M$.
(3) (Leibnitz) $s(am)=s(a)m+(-1)^{\overline{a}}as(m)$.

If $A=A_0$ we recover the usual notion of complex of A-modules.

Example IV.20. For every morphism of dg-algebras $B\to A$ the space $\operatorname{Der}^*_B(A,A)=\oplus_p\operatorname{Der}^p_B(A,A)$ has a natural structure of A-dg-module, with left multiplication $(af)(b)=a(f(b))$.

If M is an A-dg-module then $M[n]=\mathbb{K}\,[n]\otimes_{\mathbb{K}}M$ has a natural structure of A-dg-module with multiplication maps

$$(e\otimes m)a=e\otimes ma,\qquad a(e\otimes m)=(-1)^{n\overline{a}}e\otimes am,\qquad e\in\mathbb{K}\,[n],\ m\in M,\ a\in A.$$

The tensor product $N\otimes_AM$ is defined as the quotient of $N\otimes_{\mathbb{K}}M$ by the graded submodules generated by all the elements $na\otimes m-n\otimes am$.

Given two A-dg-modules $(M,d_M),(N,d_N)$ we denote by

$$\operatorname{Hom}^n_A(M,N)=\{\,f\in\operatorname{Hom}^n_{\mathbb{K}}(M,N)\mid f(ma)=f(m)a,\ m\in M,a\in A\}$$

$$\operatorname{Hom}^*_A(M,N)=\bigoplus_{n\in\mathbb{Z}}\operatorname{Hom}^n_A(M,N).$$

The graded vector space $\operatorname{Hom}^*_A(M,N)$ has a natural structure of A-dg-module with left multiplication $(af)(m)=af(m)$ and differential

$$d\colon \operatorname{Hom}^n_A(M,N)\to\operatorname{Hom}^{n+1}_A(M,N),\qquad df=[d,f]=d_N\circ f-(-1)^nf\circ d_M.$$

Note that $f\in\operatorname{Hom}^0_A(M,N)$ is a morphism of A-dg-modules if and only if $df=0$. A *homotopy* between two morphism of dg-modules $f,g\colon M\to N$ is a $h\in\operatorname{Hom}^{-1}_A(M,N)$ such that $f-g=dh=d_Nh+hd_M$. Homotopically equivalent morphisms induce the same morphism in homology.

Morphisms of A-dg-modules $f\colon L\to M$, $h\colon N\to P$ induce, by composition, morphisms $f^*\colon\operatorname{Hom}^*_A(M,N)\to\operatorname{Hom}^*_A(L,N)$, $h_*\colon\operatorname{Hom}^*_A(M,N)\to\operatorname{Hom}^*_A(M,P)$;

Lemma IV.21. *In the above notation if f is homotopic to g and h is homotopic to l then f^* is homotopic to g^* and l_* is homotopic to h_*.*

Proof. Let $p\in\operatorname{Hom}^{-1}_A(L,M)$ be a homotopy between f and g, It is a straightforward verification to see that the composition with p is a homotopy between f^* and g^*. Similarly we prove that h_* is homotopic to l_*. □

Lemma IV.22. (Base change) *Let $A\to B$ be a morphism of unitary dg-algebras, M an A-dg-module, N a B-dg-modules. Then there exists a natural isomorphism of B-dg-modules*

$$\operatorname{Hom}^*_A(M,N)\simeq\operatorname{Hom}^*_B(M\otimes_AB,N).$$

Proof. Consider the natural maps:

$$\operatorname{Hom}^*_A(M,N)\underset{R}{\overset{L}{\rightleftarrows}}\operatorname{Hom}^*_B(M\otimes_AB,N)\ ,$$

$$Lf(m\otimes b)=f(m)b,\qquad Rg(m)=g(m\otimes 1).$$

We left as exercise the easy verification that L,R are isomorphisms of B-dg-modules and $R=L^{-1}$. □

Given a morphism of dg-algebras $B\to A$ and an A-dg-module M we set:

$$\operatorname{Der}^n_B(A,M)=\{\phi\in\operatorname{Hom}^n_{\mathbb{K}}(A,M)\,|\,\phi(ab)=\phi(a)b+(-1)^{n\overline{a}}a\phi(b),\ \phi(B)=0\}$$

$$\operatorname{Der}^*_B(A,M)=\bigoplus_{n\in\mathbb{Z}}\operatorname{Der}^n_B(A,M).$$

As in the case of Hom^*, there exists a structure of A-dg-module on $\operatorname{Der}^*_B(A,M)$ with product $(a\phi)(b)=a\phi(b)$ and differential

$$d\colon\operatorname{Der}^n_B(A,M)\to\operatorname{Der}^{n+1}_B(A,M),\qquad d\phi=[d,\phi]=d_M\phi-(-1)^n\phi d_A.$$

Given $\phi\in \operatorname{Der}^n_B(A,M)$ and $f\in \operatorname{Hom}^m_A(M,N)$ their composition $f\phi$ belongs to $\operatorname{Der}^{n+m}_B(A,N)$.

4. Kodaira-Spencer's maps in dg-land

In this section, we define on the central fibre of a deformation a sheaf of differential graded algebras $\mathcal{B}$ which contains (well hidden) the "Taylor coefficients" of the variation of the complex structures given by the deformation (the first derivative being the Kodaira-Spencer map).

Lemma IV.23. *Let U be a differential manifold (not necessarily compact), $\Delta\subset\mathbb{C}^n$ a polydisk with coordinates $t_1,\ldots,t_n$ and $f(x,t)\in C^\infty(U\times\Delta,\mathbb{C})$.*
Then there exist $f_1,\ldots,f_n,f_{\overline{1}},\ldots,f_{\overline{n}}\in C^\infty(U\times\Delta,\mathbb{C})$ such that

$$f_i(x,0)=\frac{\partial f}{\partial t_i}(x,0),\qquad f_{\bar{i}}(x,0)=\frac{\partial f}{\partial \bar{t}_i}(x,0)\qquad \textit{and}$$

$$f(x,t)=f(x,0)+\sum t_if_i(x,t)+\sum \overline{t_i}f_{\bar{i}}(x,t).$$

Proof. The first 2 equalities follow from the third. Writing $t_j=u_j+iv_j$, $\overline{t_j}=u_j-iv_j$, with u_j,v_j real coordinates on $\mathbb{C}^n=\mathbb{R}^{2n}$ we have

$$f(x,u,v)=f(x,0,0)+\int_0^1\frac{d}{ds}f(x,su,sv)ds=$$

$$=f(x,0,0)+\sum_j u_j\int_0^1\frac{d}{du_j}f(x,su,sv)ds+\sum_j v_j\int_0^1\frac{d}{dv_j}f(x,su,sv)ds$$

Rearranging in the coordinates t_j, $\overline{t_j}$ we get the proof. ☐

Let X be a fixed complex manifold; denote by $\mathcal{D}er^*_{\overline{\Omega}^*_X}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X)\subset \mathcal{H}om(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X)$ the sheaf of $\overline{\Omega}^*_X$-derivations of the sheaf of graded algebras $\mathcal{A}^{0,*}_X$; we have the following

Proposition IV.24. *In the notation above there exists a natural isomorphism of sheaves*

$$\theta\colon \mathcal{A}^{0,*}_X(T_X)\xrightarrow{\sim}\mathcal{D}er^*_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X).$$

In local holomorphic coordinates $z_1,\ldots,z_m$,

$$\theta\colon \mathcal{A}^{0,p}_X(T_X)\to \mathcal{D}er^p_{\overline{\Omega}^*_X}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X)\subset \mathcal{D}er^p_{\mathbb{C}}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X)$$

is given by $\theta\left(\phi\dfrac{\partial}{\partial z_i}\right)(fd\overline{z}_I)=\phi\wedge\dfrac{\partial f}{\partial z_i}d\overline{z}_I$.
The Dolbeault differential in $\mathcal{A}^{0,*}_X(T_X)$ corresponds, via the isomorphism θ, to the restriction to $\mathcal{D}er^*_{\overline{\Omega}^*_X}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X)$ of the adjoint operator

$$[\overline{\partial},-]\colon \mathcal{D}er^*_{\mathbb{C}}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X)\to \mathcal{D}er^{*+1}_{\mathbb{C}}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X).$$

Proof. The morphism θ is injective and well defined. Let $U\subset X$ be an open polydisk with coordinates $z_1,\ldots,z_m$. Take $\xi\in\Gamma(U,\mathcal{D}er^p_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_X,\mathcal{A}^{0,*}_X))$ and denote $\phi_i=\xi(z_i)\in\Gamma(U,\mathcal{A}^{0,p}_X)$. We want to prove that $\xi=\theta\left(\sum_i\phi_i\dfrac{\partial}{\partial z_i}\right)$.
Since, over U, $\mathcal{A}^{0,*}_X$ is generated by $\mathcal{A}^{0,0}_X$ and $\overline{\Omega}^*_X$, it is sufficient to prove that for every open subset $V\subset U$, every point $x\in V$ and every C^∞-function $f\in\Gamma(V,\mathcal{A}^{0,0}_X)$ the equality $\xi(f)(x)=\sum_i\phi_i\dfrac{\partial f}{\partial z_i}(x)$ holds.
If $z_i(x)=x_i\in\mathbb{C}$, then by Lemma IV.23 we can write

$$f(z_1,\ldots,z_m)=f(x_1,\ldots,x_m)+\sum_{i=1}^m(z_i-x_i)f_i(z_1,\ldots,z_m)+\sum_{i=1}^m(\overline{z}_i-\overline{x}_i)f_{\bar{i}}(z_1,\ldots,z_m)$$

for suitable C^∞ functions f_i, $f_{\bar{i}}$; therefore

$$\xi(f)(x) = \sum_{i=1}^m \xi(z_i - x_i) f_i(x_1, \ldots, x_m) = \sum_{i=1}^m \phi_i \frac{\partial f}{\partial z_i}(x).$$

In particular, for $\xi, \eta \in \Gamma(U, \mathcal{D}er^p_{\overline{\Omega}^*_X}(\mathcal{A}^{0,*}_X, \mathcal{A}^{0,*}_X))$, we have $\xi = \eta$ if and only if $\xi(z_i) = \eta(z_i)$ for $i = 1, \ldots, m$. Since $\overline{\partial}\,\overline{\Omega}^*_X \subset \overline{\Omega}^*_X$, the adjoint operator $[\overline{\partial}, -]$ preserves $\mathcal{D}er^p_{\overline{\Omega}^*_X}(\mathcal{A}^{0,*}_X, \mathcal{A}^{0,*}_X)$, moreover

$$\theta\left(\overline{\partial}\phi \frac{\partial}{\partial z_i}\right) z_j = (\overline{\partial}\phi)\delta_{ij} = \overline{\partial}(\phi\delta_{ij}) - (-1)^{\overline{\phi}}\left(\phi\frac{\partial}{\partial z_i}\right)(\overline{\partial} z_j) = \left[\overline{\partial}, \theta\left(\phi\frac{\partial}{\partial z_i}\right)\right] z_j,$$

and then $\theta\overline{\partial} = [\overline{\partial}, -]\theta$. □

According to Proposition IV.24, the standard bracket on $\mathcal{D}er^*_{\overline{\Omega}^*_X}(\mathcal{A}^{0,*}_X, \mathcal{A}^{0,*}_X)$ induces a bracket on the sheaf $\mathcal{A}^{0,*}_X(T_X)$ given in local coordinates by

$$\left[f\frac{\partial}{\partial z_i} d\overline{z}_I, g\frac{\partial}{\partial z_j} d\overline{z}_J\right] = \left(f\frac{\partial g}{\partial z_i}\frac{\partial}{\partial z_j} - g\frac{\partial f}{\partial z_j}\frac{\partial}{\partial z_i}\right) d\overline{z}_I \wedge d\overline{z}_J.$$

Note that for $f, g \in \Gamma(U, \mathcal{A}^{0,0}_X(T_X))$, $[f, g]$ is the usual bracket on vector fields on a differentiable manifolds.

Let $B \subset \mathbb{C}^n$ be an open subset, $0 \in B$, and let $M_0 \xrightarrow{i} M \xrightarrow{f} (B, 0)$ be a deformation of a compact complex manifold M_0; let $t_1, \ldots, t_n$ be a set of holomorphic coordinates on B. It is not restrictive to assume $M_0 \subset M$ and i the inclusion map.

Definition IV.25. In the notation above, denote by $I_M \subset \mathcal{A}^{*,*}_M$ the graded ideal sheaf generated by $\overline{t_i}, dt_i, d\overline{t_i}$. Denote by $\mathcal{B}^{*,*}_M$ the quotient sheaf $\mathcal{A}^{*,*}_M / I_M$.[1]

If $z_1, \ldots, z_m, t_1, \ldots, t_n$ are admissible (Defn. I.29) local holomorphic coordinates on an admissible chart $W \subset M$, $W \simeq (W \cap M_0) \times \Delta$, $0 \in \Delta \subset B$ polydisk, then every $\phi \in \Gamma(W, \mathcal{B}^{*,*}_M)$ has a representative in $\Gamma(W, \mathcal{A}^{*,*}_M)$ of the form

$$\phi_0(z) + \sum_i t_i\phi_i(z, t), \qquad \phi_0(z) \in \Gamma(W \cap M_0, \mathcal{A}^{*,*}_{M_0}), \quad \phi_i \in \Gamma(W, \mathcal{A}^{*,*}_M).$$

By a recursive use of Lemma IV.23 we have that, for every $s > 0$, ϕ is represented by

$$\sum_{|I|<s} t^I\phi_I(z) + \sum_{|I|=s} t^I\phi_I(z, t).$$

The ideal sheaf I_M is preserved by the differential operators $d, \partial, \overline{\partial}$ and therefore we have the corresponding induced operators on the sheaf of graded algebras $\mathcal{B}^{*,*}_M$. Denoting by $\mathcal{B}^{0,*}_M \subset \mathcal{B}^{*,*}_M$ the image of $\mathcal{A}^{0,*}_M$ we have that $\mathcal{B}^{0,*}_M$ is a sheaf of dg-algebras with respect to the differential $\overline{\partial}$.

Lemma IV.26. *In the notation above, let $U, V \subset M$ be open subsets; if $U \cap M_0 = V \cap M_0$ then $\Gamma(U, \mathcal{B}^{*,*}_M) = \Gamma(V, \mathcal{B}^{*,*}_M)$ and therefore $\mathcal{B}^{*,*}_M$ is a sheaf of dg-algebras over M_0.*

Proof. It is not restrictive to assume $V \subset U$, then $U = V \cup (U - M_0)$ and by the sheaf properties it is sufficient to show that $\Gamma(U - M_0, \mathcal{B}^{*,*}_M) = \Gamma(V - M_0, \mathcal{B}^{*,*}_M) = 0$. More generally if $U \subset M$ is open and $U \cap M_0 = \emptyset$ then $\Gamma(U, \mathcal{B}^{*,*}_M) = 0$; in fact there exists an open covering $U = \cup U_i$ such that $\overline{t_i}$ is invertible in U_i.
If $W \subset M_0$ is open we define $\Gamma(W, \mathcal{B}^{*,*}_M) = \Gamma(U, \mathcal{B}^{*,*}_M)$, where U is any open subset of M such that $U \cap M_0 = W$. □

The pull-back $i^* \colon \mathcal{A}^{*,*}_M \to \mathcal{A}^{*,*}_{M_0}$ factors to a surjective morphism $i^* \colon \mathcal{B}^{*,*}_M \to \mathcal{A}^{*,*}_{M_0}$ of sheaves of differential graded algebras over M_0.
Note also that the image in $\mathcal{B}^{*,*}_M$ of the sheaf of antiholomorphic differential forms $\overline{\Omega}^*_M$ is

[1] It is also possible to define $\mathcal{B}$ as the quotient of $\mathcal{A}$ by the ideal generated by $\overline{t_i}, dt_i, d\overline{t_i}$ and the C^∞ functions on B with vanishing Taylor series at 0: the results of this chapter will remain essentially unchanged

naturally isomorphic to the sheaf $\overline{\Omega}^*_{M_0}$. In fact if $z_1,\ldots,z_m,t_1,\ldots,t_n$ are local admissible coordinates at a point $p\in M_0$ and $\psi\in\overline{\Omega}^q_M$ then

$$\psi\equiv\sum\psi_{j_1,\ldots,j_q}(z)d\overline{z}_{j_1}\wedge\ldots\wedge d\overline{z}_{j_q}\qquad(\text{mod }\overline{t_i},d\overline{t_i}),\qquad\partial\psi_{j_1,\ldots,j_q}=0.$$

Therefore to every deformation $M_0\xrightarrow{i}M\xrightarrow{f}(B,0)$ we can associate an injective morphism of sheaves of dg-algebras on M_0:

$$\overline{\Omega}^*_{M_0}\xrightarrow{\hat{f}}\mathcal{B}^{0,*}_M\subset\mathcal{B}^{*,*}_M.$$

Definition IV.27. The *algebraic data* of a deformation $M_0\xrightarrow{i}M\xrightarrow{f}(B,0)$ is the pair of morphisms of sheaves of dg-algebras on M_0:

$$\overline{\Omega}^*_{M_0}\xrightarrow{\hat{f}}\mathcal{B}^{*,*}_M\xrightarrow{i^*}\mathcal{A}^{*,*}_{M_0}.$$

We note that $\hat{f}$ injective, i^* surjective and $i^*\hat{f}$ the natural inclusion. Moreover $\hat{f}$ and i^* commute with both differentials $\partial,\overline{\partial}$.

If $M_0\xrightarrow{j}N\xrightarrow{g}(B,0)$ is an isomorphic deformation then there exists an isomorphism of sheaves of dg-algebras $\mathcal{B}^{*,*}_M\to\mathcal{B}^{*,*}_N$ which makes commutative the diagram

$$\begin{tikzcd}
\overline{\Omega}^*_{M_0} \arrow[r, "\hat{f}"] \arrow[d, "\hat{g}"'] & \mathcal{B}^{*,*}_M \arrow[d, "i^*"] \arrow[dl] \\
\mathcal{B}^{*,*}_N \arrow[r, "j^*"'] & \mathcal{A}^{*,*}_{M_0}
\end{tikzcd}$$

Similarly if $(C,0)\to(B,0)$ is a germ of holomorphic map, then the pull-back of differential forms induces a commutative diagram

$$\begin{tikzcd}
\overline{\Omega}^*_{M_0} \arrow[r] \arrow[d] & \mathcal{B}^{*,*}_M \arrow[d] \arrow[dl] \\
\mathcal{B}^{*,*}_{M\times_B C} \arrow[r] & \mathcal{A}^{*,*}_{M_0}
\end{tikzcd}$$

Before going further in the theory, we will show that the Kodaira-Spencer map of a deformation $M_0\xrightarrow{i}M\xrightarrow{f}(B,0)$ of a compact connected manifold M_0 can be recovered from its algebraic data $\overline{\Omega}^*_{M_0}\xrightarrow{\hat{f}}\mathcal{B}^{*,*}_M\xrightarrow{i^*}\mathcal{A}^{*,*}_{M_0}$

Lemma IV.28. *In the notation above, consider $\mathcal{A}^{0,*}_{M_0}$ as a sheaf of $\mathcal{B}^{0,*}_M$-modules with the structure induced by i^* and denote for every $j\geq 0$.*

$$\mathcal{T}^j_M=\frac{\mathcal{D}er^j_{\overline{\Omega}^*}(\mathcal{B}^{0,*}_M,\mathcal{A}^{0,*}_{M_0})}{i^*\,\mathcal{D}er^j_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_{M_0},\mathcal{A}^{0,*}_{M_0})}.$$

Then there exists a natural linear isomorphism

$$T_{0,B}=\ker(\Gamma(M_0,\mathcal{T}^0_M)\to\Gamma(M_0,\mathcal{T}^1_M),\quad h\mapsto\overline{\partial}_{\mathcal{A}}h-h\overline{\partial}_{\mathcal{B}}).$$

Proof. We consider $T_{0,B}$ as the $\mathbb{C}$-vector space of $\mathbb{C}$-derivations $\mathcal{O}_{B,0}\to\mathbb{C}$. Let $h\in\Gamma(M_0,\mathcal{D}er^*_{\overline{\Omega}^*}(\mathcal{B}^{0,*}_M,\mathcal{A}^{0,*}_{M_0}))$ be such that $\overline{\partial}_{\mathcal{A}}h-h\overline{\partial}_{\mathcal{B}}\in i^*\,\mathcal{D}er^1_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_{M_0},\mathcal{A}^{0,*}_{M_0})$; in particular $\overline{\partial}h(t_i)=0$ for every i, the function $h(t_i)$ is holomorphic and then constant. Moreover, $h(t_i)=0$ for every i if and only if $h(\ker i^*)=0$ if and only if $h\in i^*\,\mathcal{D}er^0_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_{M_0},\mathcal{A}^{0,*}_{M_0})$. This gives a linear injective morphism

$$\ker(\Gamma(M_0,\mathcal{T}^0_M)\to\Gamma(M_0,\mathcal{T}^1_M))\to T_{0,B}.$$

To prove the surjectivity, consider a derivation $\delta\colon\mathcal{O}_{B,0}\to\mathbb{C}$ and let $M_0=\cup U_a$, $a\in\mathcal{I}$, be a locally finite covering with every U_a open polydisk with coordinate systems $z^a_1,\ldots,z^a_m\colon U_a\to\mathbb{C}$. Let $t_1,\ldots,t_n$ be coordinates on B.

Over U_a, every $\phi\in\mathcal{B}^{0,*}_M$ can be written as $\phi_0(z)+\sum t_i\phi_i(z)+\sum t_it_j\ldots$, with $\phi_i\in\mathcal{A}^{0,*}_{M_0}$. Setting $h_a(\phi)=\sum_i\delta(t_i)\phi_i$ we see immediately that h_a is a $\overline{\Omega}^*_{U_a}$-derivation lifting δ. Taking a partition of unity ρ_a subordinate to the covering $\{U_a\}$, we can take $h=\sum_a\rho_a h_a$. $\square$

Let $h\in\Gamma(M_0,\mathcal{D}er^*_{\overline{\Omega}^*}(\mathcal{B}^{0,*}_M,\mathcal{A}^{0,*}_{M_0}))$ be such that $\psi=\overline{\partial}_{\mathcal{A}}h-h\overline{\partial}_{\mathcal{B}}\in i^*\,\mathcal{D}er^1_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_{M_0},\mathcal{A}^{0,*}_{M_0})$ and let $\delta\colon\mathcal{O}_{B,0}\to\mathbb{C}$ be the corresponding derivation, $\delta(t_i)=h(t_i)$.
According to the isomorphism (Proposition IV.24) $\mathcal{D}er^j_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_{M_0},\mathcal{A}^{0,*}_{M_0})=\mathcal{A}^{0,j}_{M_0}(T_{M_0})$ we have $\psi\in\Gamma(M_0,\mathcal{A}^{0,1}(T_{M_0}))$.
Moreover, being ψ exact in the complex $\mathcal{D}er^*_{\overline{\Omega}^*}(\mathcal{B}^{0,*}_M,\mathcal{A}^{0,*}_{M_0})$, it is closed in $\mathcal{D}er^0_{\overline{\Omega}^*}(\mathcal{A}^{0,*}_{M_0},\mathcal{A}^{0,*}_{M_0})$, ψ is a $\overline{\partial}$-closed form of $\Gamma(M_0,\mathcal{A}^{0,1}(T_{M_0}))$ and the cohomology class $[\psi]\in H^1(M_0,T_{M_0})$ is depends only on the class of h in $\Gamma(M_0,\mathcal{T}^0_M)$. It is now easy to prove the following

Proposition IV.29. *In the above notation,* $[\psi]=[\overline{\partial}h-h\overline{\partial}]=\mathrm{KS}_f(\delta)$.

Proof. (sketch) Let $\eta\in\Gamma(M,\mathcal{A}^{0,0}_M(T_M))$ be a complexified vector field such that $(f_*\eta)(0)=\delta$. We may interpret η as a $\overline{\Omega}^*_M$-derivation of degree 0 $\eta\colon\mathcal{A}^{0,*}_M\to\mathcal{A}^{0,*}_M$; passing to the quotient we get a $\overline{\Omega}^*_{M_0}$-derivation $h\colon\mathcal{B}^{0,*}_M\to\mathcal{A}^{0,*}_{M_0}$. The condition $(f_*\eta)(0)=\delta$ means that h lifts δ and therefore ψ corresponds to the restriction of $\overline{\partial}\eta$ to the fibre M_0. $\square$

5. Transversely holomorphic trivializations

Definition IV.30. A transversely holomorphic trivialization of a deformation $M_0\xrightarrow{i}M\xrightarrow{f}(B,0)$ is a diffeomorphism $\phi\colon M_0\times\Delta\to f^{-1}(\Delta)$ such that:

(1) $\Delta\subset B$ is an open neighbourhood of the base point $0\in B$
(2) $\phi(x,0)=i(x)$ and $f\phi$ is the projection on the second factor.
(3) For every $x\in M_0$, $\phi\colon\{x\}\times\Delta\to M$ is a holomorphic function.

Theorem IV.31. *Every deformation of a compact complex manifold admits a transversely holomorphic trivialization.*

Proof. (cf. also [**10**], [**78**]) Let $f\colon M\to B$ be a deformation of M_0; it is not restrictive to assume $B\subset\mathbb{C}^n$ a polydisk with coordinates $t_1,\ldots,t_n$ and $0\in B$ the base point of the deformation. We identify M_0 with the central fibre $f^{-1}(0)$.
After a possible shrinking of B there exist a finite open covering $M=\cup W_a$, $a=1,\ldots,r$, and holomorphic projections $p_a\colon W_a\to U_a=W_a\cap M_0$ such that $(p_a,f)\colon W_a\to U_a\times B$ is a biholomorphism for every a and U_a is a local chart with coordinates $z^a_i\colon U_a\to\mathbb{C}$, $i=1,\ldots,m$.
Let $\rho_a\colon M_0\to[0,1]$ be a C^∞ partition of unity subordinate to the covering $\{U_a\}$ and denote $V_a=\rho_a^{-1}(]0,1])$; we note that $\{V_a\}$ is a covering of M_0 and $\overline{V_a}\subset U_a$. After a possible shrinking of B we may assume $p_a^{-1}(\overline{V_a})$ closed in M.
For every subset $C\subset\{1,\ldots,r\}$ and every $x\in M_0$ we denote

$$H_C=\left(\bigcap_{a\in C}W_a-\bigcup_{a\notin C}p_a^{-1}(\overline{V_a})\right)\times\left(\bigcap_{a\in C}U_a-\bigcup_{a\notin C}\overline{V_a}\right)\subset M\times M_0,$$

$$C_x=\{a\,|\,x\in\overline{V_a}\,\},\qquad H=\bigcup_C H_C.$$

Clearly $(x,x)\in H_{C_x}$ and then H is an open subset of $M\times M_0$ containing the graph G of the inclusion $M_0\to M$. Since the projection $pr\colon M\times M_0\to M$ is open and M_0 is compact, after a possible shrinking of B we may assume $pr(H)=M$.
Moreover if $(y,x)\in H$ and $x\in\overline{V_a}$ then $(y,x)\in H_C$ for some C containing a and therefore $y\in W_a$.
For every a consider the C^∞ function $q_a\colon H\cap(M\times U_a)\to\mathbb{C}^m$,

$$q_a(y,x)=\sum_b\rho_b(x)\frac{\partial z^a}{\partial z^b}(x)(z^b(p_b(y))-z^b(x)).$$

By the properties of H, q_a is well defined and separately holomorphic in the variable y. If $(y,x)\in H\cap(M\times(U_a\cap U_c))$ then

$$q_c(y,x)=\frac{\partial z^c}{\partial z^a}(x)q_a(y,x)$$

and then

$$\Gamma=\{(y,x)\in H\,|\,q_a(y,x)=0\ \text{ whenever } x\in U_a\}$$

is a well defined closed subset of H.
If $y\in V_a\subset M_0$ and x is sufficiently near to y then $x\in(\bigcap_{b\in C_y}U_b-\bigcup_{b\notin C}\overline{V_b})$ and, for every $b\in C_y$,

$$z^b(y)=z^b(x)+\frac{\partial z^b}{\partial z^a}(x)(z^a(y)-z^a(x))+o(\|z^a(y)-z^a(x)\|).$$

Therefore

$$q_a(y,x)=z^a(y)-z^a(x)+o(\|z^a(y)-z^a(x)\|).$$

In particular the map $x\mapsto q_a(y,x)$ is a local diffeomorphism at $x=y$.
Denote $K\subset H$ the open subset of points (y,x) such that, if $y\in p_a^{-1}(V_a)$ then $u\mapsto q_a(y,u)$ has maximal rank at $u=x$; note that K contains G.
Let Γ_0 be the connected component of $\Gamma\cap K$ that contains G; Γ_0 is a C^∞-subvariety of K and the projection $pr\colon\Gamma_0\to M$ is a local diffeomorphism. Possibly shrinking B we may assume that $pr\colon\Gamma_0\to M$ is a diffeomorphism.
By implicit function theorem Γ_0 is the graph of a C^∞ projection $\gamma\colon M\to M_0$.
After a possible shrinking of B, the map $(\gamma,f)\colon M\to M_0\times B$ is a diffeomorphism, take $\phi=(\gamma,f)^{-1}$.
To prove that, for every $x\in M_0$, the map $t\mapsto\phi(x,t)$ is holomorphic we note that $f\colon\phi(\{x\}\times B)\to B$ is bijective and therefore $\phi(x,-)=f^{-1}pr\colon\{x\}\times B\to\phi(\{x\}\times B)$.
The map $f^{-1}\colon B\to\phi(\{x\}\times B)$ is holomorphic if and only if $\phi(\{x\}\times B)=\gamma^{-1}(x)$ is a holomorphic subvariety and this is true because for x fixed every map $y\mapsto q_a(y,x)$ is holomorphic. ☐

Let $z_1,\dots,z_m,t_1,\dots,t_n$ be an admissible system of local coordinates at a point $p\in M_0\subset M$. $z_1,\dots,z_m,t_1,\dots,t_n$ is also a system of local coordinates over $M_0\times B$.
In these systems, a transversely holomorphic trivialization $\phi\colon M_0\times B\to M$ is written as

$$\phi(z,t)=(\phi_1(z,t),\dots,\phi_m(z,t),t_1,\dots,t_n),$$

where every ϕ_i, being holomorphic in $t_1,\dots,t_n$, can be written as

$$\phi_i(z,t)=z_i+\sum_{I>0}t^I\phi_{i,I}(z),\qquad I=(i_1,\dots,i_n),\quad \phi_{i,I}\in C^\infty.$$

In a neighbourhood of p,

$$\phi^*dz_i=dz_i+\sum_{I>0}t^I\sum_{j=1}^m\left(\frac{\partial\phi_{i,I}}{\partial z_j}dz_j+\frac{\partial\phi_{i,I}}{\partial\overline{z_j}}d\overline{z_j}\right),\qquad (\text{mod } I_{M_0\times B}),$$

$$\phi^*d\overline{z_i}=d\overline{z_i},\qquad (\text{mod } I_{M_0\times B}).$$

Lemma IV.32. *Every transversely holomorphic trivialization $\phi\colon M_0\times B\to M$ induces isomorphisms of sheaves of graded algebras over M_0*

$$\phi^*\colon\mathcal{B}^{*,*}_M\to\mathcal{B}^{*,*}_{M_0\times B},\qquad \phi^*\colon\mathcal{B}^{0,*}_M\to\mathcal{B}^{0,*}_{M_0\times B}$$

making commutative the diagrams

$$\begin{tikzcd}
\overline{\Omega}^*_{M_0} \arrow[r] \arrow[d] & \mathcal{B}^{*,*}_M \arrow[d] \arrow[dl, "\phi^*"] \\
\mathcal{B}^{*,*}_{M_0\times B} \arrow[r] & \mathcal{A}^{*,*}_{M_0}
\end{tikzcd}\qquad
\begin{tikzcd}
\overline{\Omega}^*_{M_0} \arrow[r] \arrow[d] & \mathcal{B}^{0,*}_M \arrow[d] \arrow[dl, "\phi^*"] \\
\mathcal{B}^{0,*}_{M_0\times B} \arrow[r] & \mathcal{A}^{0,*}_{M_0}
\end{tikzcd}$$

Beware: It is not true in general that, for $p>0$, $\phi^*(\mathcal{B}^{p,q})\subset\mathcal{B}^{p,q}$.

PROOF. For every open subset $U \subset M$, the pull-back

$$\phi^*\colon \Gamma(U, \mathcal{A}_M^{*,*}) \to \Gamma(\phi^{-1}(U), \mathcal{A}_{M_0\times B}^{*,*})$$

is an isomorphism preserving the ideals I_M and $I_{M_0\times B}$. Since $U\cap M_0 = \phi^{-1}(U)\cap M_0$, the pull-back ϕ^* induces to the quotient an isomorphism of sheaves of graded algebras $\phi^*\colon \mathcal{B}_M^{*,*} \to \mathcal{B}_{M_0\times B}^{*,*}$.
From the above formulas follows that $\phi^*(\mathcal{B}_M^{p,k-p}) \subset \oplus_{q\le p}\mathcal{B}_{M_0\times B}^{q,k-q}$ and ϕ^* is the identity on $\overline{\Omega}^*_{M_0}$. This shows that $\phi^*(\mathcal{B}_M^{0,*}) = \mathcal{B}_{M_0\times B}^{0,*}$ and proves the commutativity of the diagrams. □

The $\overline{\partial}$ operator on $\mathcal{A}_M^{*,*}$ factors to $\mathcal{B}_M^{0,*}$ and therefore induces operators

$$\overline{\partial}\colon \mathcal{B}_M^{0,*} \to \mathcal{B}_M^{0,*+1}, \qquad \overline{\partial}_\phi = \phi^*\overline{\partial}(\phi^*)^{-1}\colon \mathcal{B}_{M_0\times B}^{0,*} \to \mathcal{B}_{M_0\times B}^{0,*+1}.$$

If $z_1,\ldots,z_m,t_1,\ldots,t_n$ are admissible local coordinates at $p\in M_0$, we have

$$(\phi^*)^{-1}dz_i = dz_i + \sum_{j=1}^m a_{ij}dz_j + b_{ij}d\overline{z_j}, \qquad (\text{mod } I_M),$$

where a_{ij}, b_{ij} are C^∞ functions vanishing on M_0 and

$$(\phi^*)^{-1}d\overline{z_i} = d\overline{z_i}, \qquad (\text{mod } I_M).$$

Thus we get immediately that $\overline{\partial}_\phi(d\overline{z_i}) = 0$. Let's now f be a C^∞ function in a neighbourhood of $p\in U\subset M_0\times B$ and let $\overline{\pi}\colon \mathcal{A}_M^{*,*} \to \mathcal{A}_M^{0,*}$ be the projection. By definition $\overline{\partial}_\phi f$ is the class in $\mathcal{B}_{M_0\times B}^{0,*}$ of

$$\phi^*\overline{\pi}d(\phi^*)^{-1}f = \phi^*\overline{\pi}(\phi^*)^{-1}df = \sum_{i=1}^m \frac{\partial f}{\partial z_i}\phi^*\overline{\pi}(\phi^*)^{-1}dz_i + \sum_{i=1}^m \frac{\partial f}{\partial \overline{z_i}}\phi^*\overline{\pi}(\phi^*)^{-1}d\overline{z_i}$$

and then

$$\overline{\partial}_\phi f = \overline{\partial} f + \sum_{ij} b_{ij}\frac{\partial f}{\partial z_i}d\overline{z_j}.$$

If $\psi\colon M_0\times B \to M$ is another transversely holomorphic trivialization and $\theta = \phi^*(\psi^*)^{-1}$ then $\overline{\partial}_\psi = \theta\overline{\partial}_\phi\theta^{-1}$.

6. Infinitesimal deformations

Let $M_0 \xrightarrow{i} M \xrightarrow{f} (B,0)$ be a deformation of a compact complex manifold and $J\subset \mathcal{O}_{B,0}$ a proper ideal such that $\sqrt{I} = \mathfrak{m}_{B,0}$; after a possible shrinking of B we can assume that:

(1) $B\subset \mathbb{C}^n$ is a polydisk with coordinates $t_1,\ldots,t_n$ and J is generated by a finite number of holomorphic functions on B.
(2) $f\colon M\to B$ is a family admitting a transversely holomorphic trivialization $\phi\colon M_0\times B\to M$.

Denote by $(X,0)$ the fat point $(B,0,J)$ and by $\mathcal{O}_{X,0} = \mathcal{O}_{B,0}/J$ its associated analytic algebra. If $\mathfrak{m}^s_{B,0}\subset J$ then the holomorphic functions t^I, $I=(i_1,\ldots,i_n)$, $|I|<s$, generate $\mathcal{O}_{X,0}$ as a $\mathbb{C}$-vector space.
Denote by $I_{M,J}\subset \mathcal{A}_M^{*,*}$ the graded ideal sheaf generated by I_M and J, $\mathcal{B}_{M,J}^{*,*} = \mathcal{A}_M^{*,*}/I_{M,J} = \mathcal{B}_M^{*,*}/(J)$, $\mathcal{O}_{M,J} = \mathcal{O}_M/(J)$. The same argument used in Lemma IV.26 shows that $\mathcal{B}_{M,J}^{*,*}$ and $\mathcal{O}_{M,J}$ are sheaves over M_0. In the same manner we define $\mathcal{B}_{M_0\times B,J}^{*,*}$

LEMMA IV.33. *Let $U\subset M_0$ be an open subset, then there exist isomorphisms*

$$\Gamma(U,\mathcal{O}_{M_0\times B,J}) = \Gamma(U,\mathcal{O}_{M_0})\otimes_{\mathbb{C}}\mathcal{O}_{X,0}, \qquad \Gamma(U,\mathcal{B}_{M_0\times B,J}^{*,*}) = \Gamma(U,\mathcal{A}_{M_0}^{*,*})\otimes_{\mathbb{C}}\mathcal{O}_{X,0}.$$

The same holds for M instead of $M_0\times B$ provided that U is contained in an admissible coordinate chart.

Proof. We have seen that every $\phi \in \Gamma(U, \mathcal{B}^{p,q}_{M_0\times B,J})$ is represented by a form $\sum_{|I|<s} t^I\phi_I$, with $\phi_I \in \Gamma(U, \mathcal{A}^{p,q}_{M_0})$. Writing every t^I as a linear combination of the elements of a fixed basis of $\mathcal{O}_{X,0}$ and rearranging the terms we get the desired result. The same argument applies to $\mathcal{O}_{M_0\times B,J}$ and, if U is sufficiently small, to $\mathcal{B}^{*,*}_{M,J}$, $\mathcal{O}_{M,J}$. □

Corollary IV.34. $\mathcal{O}_{M,J} = \ker(\overline{\partial}\colon \mathcal{B}^{0,0}_{M,J} \to \mathcal{B}^{0,1}_{M,J})$.

Proof. If $U \subset M_0$ is a sufficiently small open subset, we have $\Gamma(U, \mathcal{B}^{*,*}_{M,J}) = \Gamma(U, \mathcal{A}^{*,*}_{M_0})\otimes_{\mathbb{C}} \mathcal{O}_{X,0}$ and then

$$\ker\left(\overline{\partial}\colon \Gamma(U, \mathcal{B}^{0,0}_{M,J}) \to \Gamma(U, \mathcal{B}^{0,1}_{M,J})\right) =$$

$$= \ker\left(\overline{\partial}\colon \Gamma(U, \mathcal{A}^{0,0}_{M_0}) \otimes \mathcal{O}_{X,0} \to \Gamma(U, \mathcal{A}^{0,1}_{M_0}) \otimes \mathcal{O}_{X,0}\right) = \Gamma(U, \mathcal{O}_{M,J}).$$

□

The transversely holomorphic trivialization ϕ gives a commutative diagram of morphisms of sheaves of graded algebras

$$\begin{tikzcd} \overline{\Omega}^*_{M_0} \otimes \mathcal{O}_{X,0} \arrow[r] \arrow[d] & \mathcal{B}^{0,*}_{M,J} \arrow[d] \arrow[dl, "\phi^*"] \\ \mathcal{B}^{0,*}_{M_0\times B,J} \arrow[r] & \mathcal{A}^{0,*}_{M_0} \end{tikzcd}$$

with ϕ^* an isomorphism. The operator $\overline{\partial}_\phi = \phi^*\overline{\partial}(\phi^*)^{-1}$ is a $\mathcal{O}_{X,0}$-derivation of degree 1 such that $\overline{\partial}^2_\phi = \frac{1}{2}[\overline{\partial}_\phi, \overline{\partial}_\phi] = 0$ and then $\eta_\phi = \overline{\partial}_\phi - \overline{\partial}\colon \mathcal{B}^{0,*}_{M_0\times B,J} \to \mathcal{B}^{0,*+1}_{M_0\times B,J}$ is a $\overline{\Omega}^*_{M_0} \otimes \mathcal{O}_{X,0}$-derivation.

According to Lemma IV.33 we have $\mathcal{B}^{0,*}_{M_0\times B,J} = \mathcal{A}^{0,*}_{M_0}\otimes\mathcal{O}_{X,0}$; moreover, if $g_0 = 1, g_1(t), \dots, g_r(t)$ is a basis of $\mathcal{O}_{X,0}$ with $g_i \in \mathfrak{m}_{X,0}$ for $i > 0$, then we can write $\eta_\phi = \sum_{i>0} g_i(t)\eta_i$, with every η_i a $\overline{\Omega}^*_{M_0}$-derivation of degree 1 of $\mathcal{A}^{0,*}_{M_0}$. By Proposition IV.24 $\eta_\phi \in \Gamma(M_0, \mathcal{A}^{0,1}(T_{M_0})) \otimes \mathfrak{m}_{X,0}$. In local holomorphic coordinates $z_1, \dots, z_m$ we have $\overline{\partial}_\phi(d\overline{z_i}) = 0$ and

$$\overline{\partial}_\phi f = \overline{\partial} f + \sum_{i,j,k} g_i(t) b^i_{j,k}(z)\frac{\partial f}{\partial z_j} d\overline{z_k}$$

for every C^∞ function f. The $b^i_{j,k}$ are C^∞ functions on M_0.

A different choice of transversely holomorphic trivialization $\psi\colon M_0 \times B \to M$ gives a conjugate operator $\overline{\partial}_\psi = \theta\overline{\partial}_\phi\theta^{-1}$, where $\theta = \phi^*(\psi^*)^{-1}$.

This discussion leads naturally to the definition of deformations of a compact complex manifolds over a fat points.

Definition IV.35. A deformation of M_0 over a fat point $(X, 0)$ is a section

$$\eta \in \Gamma(M_0, \mathcal{A}^{0,1}(T_{M_0})) \otimes \mathfrak{m}_{X,0} = \mathrm{Der}^1_{\overline{\Omega}^*_{M_0}}(\mathcal{A}^{0,*}_{M_0}, \mathcal{A}^{0,*}_{M_0}) \otimes \mathfrak{m}_{X,0}$$

such that the operator $\overline{\partial} + \eta \in \mathrm{Der}^1_{\mathcal{O}_{X,0}}(\mathcal{A}^{0,*}_{M_0} \otimes \mathcal{O}_{X,0}, \mathcal{A}^{0,*}_{M_0} \otimes \mathcal{O}_{X,0})$ is a differential, i.e. $[\overline{\partial} + \eta, \overline{\partial} + \eta] = 0$.

Two deformations $\eta, \mu \in \Gamma(M_0, \mathcal{A}^{0,1}(T_{M_0}))\otimes\mathfrak{m}_{X,0}$ are isomorphic if and only if there exists an automorphism of sheaves of graded algebras $\theta\colon \mathcal{A}^{0,*}_{M_0} \otimes \mathcal{O}_{X,0} \to \mathcal{A}^{0,*}_{M_0} \otimes \mathcal{O}_{X,0}$ commuting with the projection $\mathcal{A}^{0,*}_{M_0} \otimes \mathcal{O}_{X,0} \to \mathcal{A}^{0,*}_{M_0}$ and leaving point fixed the subsheaf $\overline{\Omega}^*_{M_0} \otimes \mathcal{O}_{X,0}$ such that $\overline{\partial} + \mu = \theta(\overline{\partial} + \eta)\theta^{-1}$.

According to IV.24 the adjoint operator $[\overline{\partial}, -]$ corresponds to the Dolbeault differential in the complex $\mathcal{A}^{0,*}(T_{M_0})$ and therefore $(\overline{\partial}+\eta)^2 = 0$ if and only if $\eta \in \Gamma(M_0, \mathcal{A}^{0,1}(T_{M_0}))\otimes\mathfrak{m}_{X,0}$ satisfies the **Maurer-Cartan** equation

$$\overline{\partial}\eta + \frac{1}{2}[\eta, \eta] = 0 \in \Gamma(M_0, \mathcal{A}^{0,2}(T_{M_0})) \otimes \mathfrak{m}_{X,0}.$$

We denote with both $\mathrm{Def}_{M_0}(X,0)$ and $\mathrm{Def}_{M_0}(\mathcal{O}_{X,0})$ the set of isomorphism classes of deformations of M_0 over $(X,0)$. By an *infinitesimal deformation* we mean a deformation over a fat point; by a *first order deformation* we mean a deformation over a fat point $(X,0)$ such that $\mathfrak{m}_{X,0}\neq 0$ and $\mathfrak{m}_{X,0}^2=0$.

The Proposition IV.29 allows to extend naturally the definition of the Kodaira-Spencer map $\mathrm{KS}\colon T_{0,X}\to H^1(M_0,T_{M_0})$ to every infinitesimal deformation over $(X,0)$.
Consider in fact $\delta\in\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{X,0},\mathbb{C})=T_{0,X}$, then

$$h=Id\otimes\delta\colon \mathcal{A}^{0,*}_{M_0}\otimes\mathcal{O}_{X,0}\to\mathcal{A}^{0,*}_{M_0}$$

is a $\overline{\Omega}^*_{M_0}$-derivation lifting δ. Since

$$(\overline{\partial}h-h(\overline{\partial}+\eta))(f\otimes 1)=h(-\eta(f))$$

we may define $\mathrm{KS}(\delta)$ as the cohomology class of the derivation

$$\mathcal{A}^{0,*}_{M_0}\to\mathcal{A}^{0,*+1}_{M_0},\qquad f\mapsto h(-\eta(f)),$$

which corresponds, via the isomorphism of Proposition IV.24, to

$$(Id\otimes\delta)(-\eta)\in\Gamma(M_0,\mathcal{A}^{0,1}(T_{M_0})),$$

where $Id\otimes\delta\colon\Gamma(M_0,\mathcal{A}^{0,1}(T_{M_0}))\otimes\mathfrak{m}_{X,0}$. According to the Maurer-Cartan equation $\overline{\partial}\eta=-\frac{1}{2}[\eta,\eta]\in\Gamma(M_0,\mathcal{A}^{0,2}(T_{M_0}))\otimes\mathfrak{m}^2_{X,0}$ and then

$$\overline{\partial}((Id\otimes\delta)(-\eta))=(Id\otimes\delta)(-\overline{\partial}\eta)=0.$$

A morphism of fat points $(Y,0)\to(X,0)$ is the same of a morphism of local $\mathbb{C}$-algebras $\alpha\colon\mathcal{O}_{X,0}\to\mathcal{O}_{Y,0}$; It is natural to set $Id\otimes\alpha(\eta)\in\Gamma(M_0,\mathcal{A}^{0,1}(T_{M_0}))\otimes\mathfrak{m}_{Y,0}$ as the pull-back of the deformation η. It is immediate to see that the Kodaira-Spencer map of $Id\otimes\alpha(\eta)$ is the composition of the Kodaira-Spencer map of η and the differential $\alpha\colon T_{Y,0}\to T_{X,0}$.

7. Historical survey, IV

The importance of infinitesimal deformations increased considerably after the proof (in the period 1965-1975) of several ineffective existence results of semiuniversal deformations of manifolds, of maps etc.., over singular bases.
The archetype of these results is the well known theorem of Kuranishi (1965) [**45**], asserting the existence of the semiuniversal deformation of a compact complex manifold over a base which is an analytic singularity. An essentially equivalent formulation of Kuranishi theorem is the following

Theorem IV.36. *Let M_0 be a compact complex manifold with $n=h^1(M_0,T_{M_0})$, $r=h^2(M_0,T_{M_0})$.*
Then there exist a polydisk $\Delta\subset\mathbb{C}^n$, a section $\eta\in\Gamma(M,\mathcal{A}^{0,1}(T_f))$, being $M=M_0\times\Delta$ and $f\colon M\to\Delta$ the projection, and $q=(q_1,\dots,q_r)\colon\Delta\to\mathbb{C}^r$ a holomorphic map such that:

(1) *$q(0)=0$ and $\dfrac{\partial q_i}{\partial t_j}(0)=0$ for every i,j, being $t_1,\dots,t_n$ holomorphic coordinates on Δ.*
(2) *η vanishes on M_0 and it is holomorphic in $t_1,\dots,t_n$; this means that it is possible to write*

$$\eta=\sum_{I>0}t^I\eta_I,\qquad I=(i_1,\dots,i_n),\quad \eta_I\in\Gamma(M_0,\mathcal{A}^{0,1}(T_{M_0})).$$

(3) *η satisfies the Maurer-Cartan equation to modulus $q_1,\dots,q_s$, i.e.*

$$\overline{\partial}\eta+\frac{1}{2}[\eta,\eta]\in\sum q_i\Gamma(M,\mathcal{A}^{0,2}(T_f)).$$

(4) *Given a fat point $(X,0)$ the natural map*

$$\eta\colon\mathrm{Mor}_{\mathbf{An}}(\mathcal{O}_{\Delta,0}/(q_1,\dots,q_s),\mathcal{O}_{X,0})\to\mathrm{Def}_{M_0}(X,0),\qquad \alpha\mapsto\alpha(\eta)$$

is surjective for every $(X,0)$ and bijective whenever $\mathcal{O}_{X,0}=\mathbb{C}[t]/(t^2)$.

It is now clear that the study of infinitesimal deformations can be used to deduce the structure of the holomorphic map q and the existence of the semiuniversal deformation over a smooth base. For example we have the following

COROLLARY IV.37. *Let M_0 be a compact complex manifolds such that for every $n \geq 2$ the natural map $\mathrm{Def}_{M_0}(\mathbb{C}[t]/(t^n)) \to \mathrm{Def}_{M_0}(\mathbb{C}[t]/(t^2))$ is surjective. Then M_0 has a semiuniversal deformation $M_0 \longrightarrow M \longrightarrow (H^1(M_0, T_{M_0}), 0)$.*

PROOF. (sketch) In the notation of Theorem IV.36 we have $(q_1, \ldots, q_s) \subset \mathfrak{m}^2_{\Delta,0}$ and then, according to Proposition III.7, $q_1 = \ldots = q_s = 0$. In particular η satisfies the Maurer-Cartan equation and by the Newlander-Nirenberg's theorem (cf. [**9**, 1.4], [**78**]) the small variation of almost complex structure [**9**, 2.1, 2.5], [**78**]

$$-\eta\colon \mathcal{A}^{1,0}_M \to \mathcal{A}^{0,1}_M, \qquad -{}^t\eta\colon T^{0,1}_M \to T^{1,0}_M$$

is integrable and gives a complex structure on M with structure sheaf $\mathcal{O}_{M,\eta} = \ker(\overline{\partial} + \eta\partial\colon \mathcal{A}^{0,0}_M \to \mathcal{A}^{0,1}_M)$.
The projection map $(M, \mathcal{O}_{M,\eta}) \to \Delta$ is a family with bijective Kodaira-Spencer map, by completeness theorem I.50 it is a semiuniversal deformation. $\square$

It is useful to remind here the following result proved by Malgrange [**50**]

THEOREM IV.38. *Let $q_1, \ldots, q_m\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ be germs of holomorphic functions and $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ a germ of C^∞ function. If $\overline{\partial} f \equiv 0, \pmod{q_1, \ldots, q_m}$ then there exists a germ of holomorphic function $g\colon (\mathbb{C}^n, 0) \to \mathbb{C}$ such that $f \equiv g, \pmod{q_1, \ldots, q_m}$.*

LECTURE V

Differential graded Lie algebras (DGLA)

The classical formalism (Grothendieck-Mumford-Schlessinger) of infinitesimal deformation theory is described by the procedure (see e.g [**2**], [**66**])

$$\text{Deformation problem} \quad \rightsquigarrow \quad \text{Deformation functor/groupoid}$$

The above picture is rather easy and suffices for many applications; unfortunately in this way we forget information which can be useful.

It has been suggested by several people (Deligne, Drinfeld, Quillen, Kontsevich [**43**], Schlessinger-Stasheff [**68, 69**], Goldman-Millson [**20, 21**] and many others) that a possible and useful way to preserve information is to consider a factorization

$$\text{Deformation problem} \quad \rightsquigarrow \quad DGLA \quad \rightsquigarrow \quad \text{Deformation functor/groupoid}$$

where by $DGLA$ we mean a differential graded Lie algebra depending from the data of the deformation problem and the construction

$$DGLA \rightsquigarrow \text{Deformation functor}, \qquad L \rightsquigarrow \operatorname{Def}_L,$$

is a well defined, functorial procedure explained in this Lecture.

More precisely, we introduce (as in [**44**]) the *deformation functor* associated to a differential graded Lie algebra and we prove in particular (Corollary V.52) that *quasiisomorphic differential graded Lie algebras give isomorphic deformation functors*: this is done in the framework of Schlessinger's theory and extended deformation functors.

We refer to [**20**] for a similar construction which associate to every DGLA a deformation groupoid.

Some additional comments on this procedure will be done in Section 9; for the moment we only point out that, for most deformation problems, the correct $DGLA$ is only defined up to quasiisomorphism and then the results of this Lecture are the necessary background for the whole theory.

In this chapter $\mathbb{K}$ will be a fixed field of characteristic 0. We assume that the reader is familiar with basic concepts about Lie algebras and their representations [**31**], [**33**]; unless otherwise stated we allow the Lie algebras to be infinite dimensional.

1. Exponential and logarithm

For every associative $\mathbb{K}$-algebra R we denote by R_L the associated Lie algebra with bracket $[a,b] = ad(a)b = ab - ba$; the linear operator $ad(a) \in \operatorname{End}(R)$ is called the *adjoint* of a, the morphism $ad\colon R_L \to \operatorname{End}(R)$ is a morphism of Lie algebras. If $I \subset R$ is an ideal then I is also a Lie ideal of R_L

EXERCISE V.1. Let R be an associative $\mathbb{K}$-algebra, $a, b \in R$, prove:

(1)
$$ad(a)^n b = \sum_{i=0}^{n} (-1)^i \binom{n}{i} a^{n-i} b a^i.$$

(2) If a is nilpotent in R then also $ad(a)$ is nilpotent in $\mathrm{End}(R)$ and

$$e^{ad(a)}b := \sum_{n\geq 0} \frac{ad(a)^n}{n!} b = e^a b e^{-a}.$$

$\triangle$

Let V be a fixed $\mathbb{K}$-vector space and denote

$$P(V) = \left\{ \sum_{n=0}^{\infty} v_n \;\middle|\; v_n \in \bigotimes^n V \right\} \simeq \prod_{n=0}^{\infty} \bigotimes^n V.$$

With the natural notion of sum and Cauchy product $P(V)$ becomes an associative $\mathbb{K}$-algebra; the vector subspace

$$\mathfrak{m}(V) = \left\{ \sum_{n=1}^{\infty} v_n \;\middle|\; v_n \in \bigotimes^n V \right\} \subset P(V)$$

is an ideal, $\mathfrak{m}(V)^s = \{\sum_{n=s}^{\infty} v_n\}$ for every s and $P(V)$ is complete for the $\mathfrak{m}(V)$-adic topology: this means that a series $\displaystyle\sum_{i=0}^{\infty} x_i$ is convergent whenever $x_i \in \mathfrak{m}(V)^i$ for every i.
In particular, it is well defined the *exponential*

$$e\colon \mathfrak{m}(V) \to E(V) := 1 + \mathfrak{m}(V) = \{1 + \sum_{n=1}^{\infty} v_n \,|\, v_n \in \bigotimes^n V\} \subset P(V), \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and and the *logarithm*

$$\log\colon E(V) \to \mathfrak{m}(V), \qquad \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

We note that $E(V)$ is a multiplicative subgroup of the set of invertible elements of $P(V)$ (being $\sum_{n=0}^{\infty} x^n$ the inverse of $1-x$, $x \in \mathfrak{m}(V)$). It is well known that exponential and logarithm are one the inverse of the other. Moreover if $[x,y] = xy - yx = 0$ then $e^{x+y} = e^x e^y$ and $\log((1+x)(1+y)) = \log(1+x) + \log(1+y)$.

Every linear morphism of $\mathbb{K}$-vector spaces $f_1\colon V \to W$ induces a natural, homogeneous and continuous homomorphism of $\mathbb{K}$-algebras $f\colon P(V) \to P(W)$. It is clear that $f(\mathfrak{m}(V)) \subset \mathfrak{m}(W)$, $f\colon E(V) \to E(W)$ is a group homomorphism and f commutes with the exponential and the logarithm.
Consider for instance the three homomorphisms

$$\Delta, p, q\colon P(V) \to P(V \oplus V)$$

induced respectively by the diagonal $\Delta_1(v) = (v,v)$, by $p_1(v) = (v,0)$ and by $q_1(v) = (0,v)$.
We define

$$\widehat{l}(V) = \{x \in P(V) \,|\, \Delta(x) = p(x) + q(x)\}, \qquad \widehat{L}(V) = \{x \in P(V) \,|\, \Delta(x) = p(x)q(x)\}.$$

It is immediate to observe that $V \subset \widehat{l}(V) \subset \mathfrak{m}(V)$ and $1 \in \widehat{L}(V) \subset E(V)$.

THEOREM V.2. *In the above notation we have:*
(1) $\widehat{l}(V)$ *is a Lie subalgebra of* $P(V)_L$.
(2) $\widehat{L}(V)$ *is a multiplicative subgroup of* $E(V)$.
(3) *Let* $f_1\colon V \to W$ *be a linear map and* $f\colon P(V) \to P(W)$ *the induced algebra homomorphism. Then* $f(\widehat{l}(V)) \subset \widehat{l}(W)$ *and* $f(\widehat{L}(V)) \subset \widehat{L}(W)$.
(4) *The exponential gives a bijection between* $\widehat{l}(V)$ *and* $\widehat{L}(V)$.

PROOF. We first note that for every $n \geq 0$ and every pair of vector spaces U, W we have a canonical isomorphism

$$\bigotimes^n (U \oplus W) = \bigoplus_{i=0}^{n} (\bigotimes^i U \oplus \bigotimes^{n-i} W)$$

and therefore

$$P(U\oplus W)=\prod_{i,j=0}^{\infty}\bigotimes^{i}U\oplus\bigotimes^{j}W.$$

In particular for every $x\in P(U)\otimes\mathbb{K}\subset P(U\oplus W)$, $y\in\mathbb{K}\otimes P(W)\subset P(U\oplus W)$ we have $xy=yx$. In our case, i.e. when $U=W=V$ this implies that $p(x)q(y)=q(y)p(x)$ for every $x,y\in P(V)$.
Let $x,y\in\widehat{l}(V)$ then

$$\begin{aligned}\Delta([x,y])&=\Delta(x)\Delta(y)-\Delta(y)\Delta(x)\\&=(p(x)+q(x))(p(y)+q(y))-(p(y)+q(y))(p(x)+q(x))\\&=p([x,y])+q([x,y]).\end{aligned}$$

If $x,y\in\widehat{L}(V)$ then

$$\Delta(yx^{-1})=\Delta(y)\Delta(x)^{-1}=p(y)q(x)q(x)^{-1}p(x)^{-1}=p(yx^{-1})q(yx^{-1})$$

and therefore $yx^{-1}\in\widehat{L}(V)$.
If $g\colon P(V\oplus V)\to P(W\oplus W)$ is the algebra homomorphism induced by $f_1\oplus f_1\colon V\oplus V\to W\oplus W$ it is clear that $\Delta f=g\Delta$, $pf=gp$ and $qf=gq$. This implies immediately item 3.
If $x\in\widehat{l}(V)$ then the equalities

$$\Delta(e^{x})=e^{\Delta(x)}=e^{p(x)+q(x)}=e^{p(x)}e^{q(x)}=p(e^{x})q(e^{x})$$

prove that $e(\widehat{l}(V))\subset\widehat{L(V)}$. Similarly if $y\in\widehat{L}(V)$ then

$$\Delta(\log(y))=\log(\Delta(y))=\log(p(y)q(y))=\log(p(y))+\log(q(y))=p(\log(y))+q(\log(y))$$

and therefore $\log(\widehat{L}(V))\subset\widehat{l}(V)$. □

Corollary V.3. *For every vector space V the binary operation*

$$*\colon\widehat{l}(V)\times\widehat{l}(V)\to\widehat{l}(V),\qquad x*y=\log(e^{x}e^{y})$$

induces a group structure on the Lie algebra $\widehat{l}(V)$.
Moreover for every linear map $f_1\colon V\to W$ the induced morphism of Lie algebras $f\colon\widehat{l}(V)\to\widehat{l}(W)$ is also a homomorphism of groups.

Proof. Clear. □

In the next sections we will give an explicit formula for the product $*$ which involves only the bracket of the Lie algebra $\widehat{l}(V)$.

2. Free Lie algebras and the Baker-Campbell-Hausdorff formula

Let V be a vector space over $\mathbb{K}$, we denote by

$$T(V)=\bigoplus_{n\geq 0}\bigotimes^{n}V,\qquad\overline{T(V)}=\bigoplus_{n\geq 1}\bigotimes^{n}V\subset T(V).$$

The tensor product induce on $T(V)$ a structure of unital associative algebra, the natural embedding $T(V)\subset P(V)$ is a morphism of unitary algebras and $\overline{T(V)}$ is the ideal $T(V)\cap\mathfrak{m}(V)$.
The algebra $T(V)$ is called *tensor algebra* generated by V and $\overline{T(V)}$ is called the *reduced tensor algebra* generated by V.

Lemma V.4. *Let V be a $\mathbb{K}$-vector space and $\imath\colon V=\bigotimes^{1}V\to\overline{T(V)}$ the natural inclusion. For every associative $\mathbb{K}$-algebra R and every linear map $f\colon V\to R$ there exists a unique homomorphism of $\mathbb{K}$-algebras $\phi\colon\overline{T(V)}\to R$ such that $f=\phi\imath$.*

Proof. Clear. □

Definition V.5. Let V be a $\mathbb{K}$-vector space; the *free Lie algebra* generated by V is the smallest Lie subalgebra $l(V)\subset\overline{T(V)}_{L}$ which contains V.

Equivalently $l(V)$ is the intersection of all the Lie subalgebras of $T(V)_L$ containing V. For every integer $n>0$ we denote by $l(V)_n \subset \bigotimes^n V$ the linear subspace generated by all the elements

$$[v_1,[v_2,[\dots,[v_{n-1},v_n]].]], \qquad n>0, \quad v_1,\dots,v_n\in V.$$

By definition $l(V)_n=[V,l(V)_{n-1}]$ and therefore $\oplus_{n>0}l(V)_n\subset l(V)$. On the other hand the Jacobi identity $[[x,y],z]=[x,[y,z]]-[y,[x,z]]$ implies that

$$[l(V)_n,l(V)_m]\subset [V,[l(V)_{n-1},l(V)_m]]+[l(V)_{n-1},[V,l(V)_m]]$$

and therefore, by induction on n, $[l(V)_n,l(V)_m]\subset l(V)_{n+m}$.
As a consequence $\oplus_{n>0}l(V)_n$ is a Lie subalgebra of $l(V)$ and then $\oplus_{n>0}l(V)_n=l(V)$, $l(V)_n=l(V)\cap\bigotimes^n V$.

Every morphism of vector spaces $V\to W$ induce a morphism of algebras $\overline{T(V)}\to\overline{T(W)}$ which restricts to a morphism of Lie algebras $l(V)\to l(W)$.
The name *free Lie algebra* of $l(V)$ is motivated by the following universal property:
Let V be a vector space, H a Lie algebra and $f\colon V\to H$ a linear map. Then there exists a unique homomorphism of Lie algebras $\phi\colon l(V)\to H$ which extends f.
We will prove this property in Theorem V.6.
Let H be a Lie algebra with bracket $[,]$ and $\sigma_1\colon V\to H$ a linear map.
Define recursively, for every $n\ge 2$, the linear map

$$\sigma_n\colon \bigotimes^n V\to H, \qquad \sigma_n(v_1\otimes\ldots\otimes v_n)=[\sigma_1(v_1),\sigma_{n-1}(v_2\otimes\ldots\otimes v_n)].$$

For example, if $V=H$ and σ_1 is the identity then $\sigma_n(v_1\otimes\ldots\otimes v_n)=[v_1,[v_2,[\dots,[v_{n-1},v_n]].]]$.

Theorem V.6 (Dynkyn-Sprecht-Wever). *In the notation above, the linear map*

$$\sigma=\sum_{n=1}^{\infty}\frac{\sigma_n}{n}\colon l(V)\to H$$

is the unique homomorphism of Lie algebras extending σ_1.

Proof. The adjoint representation $\theta\colon V\to \mathrm{End}(H)$, $\theta(v)x=[\sigma_1(v),x]$ extends to a unique morphism of associative algebras $\theta\colon\overline{T(V)}\to\mathrm{End}(H)$ by the composition rule

$$\theta(v_1\otimes\ldots\otimes v_s)x=\theta(v_1)\theta(v_2)\ldots\theta(v_s)x.$$

We note that, if $v_1,\dots,v_n,w_1,\dots,w_m\in V$ then

$$\sigma_{n+m}(v_1\otimes\ldots\otimes v_n\otimes w_1\otimes\ldots\otimes w_m)=\theta(v_1\otimes\ldots\otimes v_n)\sigma_m(w_1\otimes\ldots\otimes w_m).$$

Since every element of $l(V)$ is a linear combination of homogeneous elements it is sufficient to prove, by induction on $n\ge 1$, the following properties

A_n: If $m\le n$, $x\in l(V)_m$ and $y\in l(V)_n$ then $\sigma(xy-yx)=[\sigma(x),\sigma(y)]$.
B_n: If $m\le n$, $y\in l(V)_m$ and $h\in H$ then $\theta(y)h=[\sigma(y),h]$.

The initial step $n=1$ is straightforward, assume therefore $n\ge 2$.
$[A_{n-1}+B_{n-1}\Rightarrow B_n]$ We can consider only the case $m=n$. The element y is a linear combination of elements of the form $ab-ba$, $a\in V$, $b\in l(V)_{n-1}$ and, using B_{n-1} we get

$$\theta(y)h=[\sigma(a),\theta(b)h]-\theta(b)[\sigma(a),h]=[\sigma(a),[\sigma(b),h]]-[\sigma(b),[\sigma(a),h].$$

Using A_{n-1} we get therefore

$$\theta(y)h=[[\sigma(a),\sigma(b)],h]=[\sigma(y),h].$$

$[B_n\Rightarrow A_n]$

$$\begin{aligned}\sigma_{n+m}(xy-yx)&=\theta(x)\sigma_n(y)-\theta(y)\sigma_m(x)=[\sigma(x),\sigma_n(y)]-[\sigma(y),\sigma_m(x)]\\&=n[\sigma(x),\sigma(y)]-m[\sigma(y),\sigma(x)]=(n+m)[\sigma(x),\sigma(y)].\end{aligned}$$

Since $l(V)$ is generated by V as a Lie algebra, the unicity of σ follows. ☐

Corollary V.7. *For every vector space V the linear map*

$$\sigma\colon \overline{T(V)} \to l(V), \qquad \sigma(v_1\otimes\ldots\otimes v_n)=\frac{1}{n}[v_1,[v_2,[\ldots,[v_{n-1},v_n]].]]$$

is a projection.

Proof. The identity on $l(V)$ is the unique Lie homomorphism extending the natural inclusion $V\to l(V)$. $\square$

The linear map σ defined in Corollary V.7 extends naturally to a projector $\sigma\colon P(V)\to P(V)$. We have the following theorem

Theorem V.8 (Friedrichs). *In the notation above*

$$\widehat{l}(V)=\{x\in P(V)\,|\,\sigma(x)=x\} \quad \textit{and} \quad l(V)=T(V)\cap\widehat{l}(V).$$

Proof. The two equalities are equivalent, we will prove the second. We have already seen that $T(V)$ and $\widehat{l}(V)$ are Lie subalgebras of $P(V)_L$ containing V and then $l(V)\subset T(V)\cap\widehat{l}(V)$.
Define the linear map

$$\delta\colon T(V)\to T(V\oplus V), \qquad \delta(x)=\Delta(x)-p(x)-q(x).$$

By definition $T(V)\cap\widehat{l}(V)=\ker\delta$ and we need to prove that if $\delta(x)=0$ for some homogeneous x then $x\in l(V)$. For later computation we point out that, under the identification $T(V\oplus V)=T(V)\otimes T(V)$, for every monomial $\prod_i x_i$ with $x_i\in\ker\delta$ we have

$$\delta(\prod_i x_i)=\prod_i(x_i\otimes 1+1\otimes x_i)-(\prod_i x_i)\otimes 1-1\otimes(\prod_i x_i).$$

In particular if $x\in\overline{T(V)}$ then $\delta(x)$ is the natural projection of $\Delta(x)$ onto the subspace $\bigoplus_{i,j\geq 1}\bigotimes^i V\otimes\bigotimes^j V$.

Let $\{y_i\,|\,i\in\mathcal{I}\}$ be a fixed homogeneous basis of $l(V)$. We can find a total ordering on the set $\mathcal{I}$ such that if $y_i\in l(V)_n$, $y_j\in l(V)_m$ and $n<m$ then $i<j$. For every index $h\in\mathcal{I}$ we denote by $J_h\subset T(V)$ the ideal generated by y_h^2 and the y_i's for every $i>h$, then J_h is a homogeneous ideal and $y_h\notin J_h$.
A *standard monomial* is a monomial of the form $y=y_{i_1}y_{i_2}\ldots y_{i_h}$ with $i_1\leq\ldots\leq i_h$. The *external degree* of the above standard monomial y is by definition the positive integer h.
Since $y_iy_j=y_jy_i+\sum_h a_hy_h$, $a_h\in\mathbb{K}$, the standard monomials generate $\overline{T(V)}$ as a vector space and the standard monomials of external degree 1 are a basis of $l(V)$.

Claim V.9. *For every $n>0$ the following hold:*

(1) *The image under δ of the standard monomials of external degree h with $2\leq h\leq n$ are linearly independent.*
(2) *The standard monomials of external degree $\leq n$ are linearly independent.*

Proof of Claim. Since the standard monomials of external degree 1 are linearly independent and contained in the kernel of δ it is immediate to see the implication $[1\Rightarrow 2]$.
We prove [1] by induction on n, being the statement true for $n=1$.
Consider a nontrivial, finite linear combination $l.c.$ of standard monomials of external degree ≥ 2 and $\leq n$. There exists an index $h\in\mathcal{I}$ such that we can write $l.c.=z+\sum_{i=1}^n y_h^iw_i$, where z, w_i are linear combination of standard monomials in y_j, $j>h$ and at least one of the w_i is non trivial. If we consider the composition ϕ of $\delta\colon T(V)\to T(V\oplus V)=T(V)\otimes T(V)$ with the projection $T(V)\otimes T(V)\to T(V)/J_h\otimes T(V)$ we have

$$\phi(l.c.)=\sum_{i=1}^n iy_h\otimes y_h^{i-1}w_i=y_h\otimes\sum_{i=1}^n iy_h^{i-1}w_i.$$

Since $\sum_{i=1}^n iy_h^{i-1}w_i$ is a nontrivial linear combination of standard monomials of external degrees $\leq n-1$, by inductive assumption, it is different from 0 on $T(V)$. $\square$

From the claim follows immediately that the kernel of δ is generated by the standard monomials of degree 1 and therefore $\ker\delta = l(V)$. □

EXERCISE V.10. Let $x_1, \dots, x_n, y$ be linearly independent vectors in a vector space V. Prove that the $n!$ vectors

$$\sigma_{n+1}(x_{\tau(1)} \dots x_{\tau(n)} y), \qquad \tau \in \Sigma_n,$$

are linearly independent in the free Lie algebra $l(V)$.
(Hint: Let W be a vector space with basis $e_0, \dots, e_n$ and consider the subalgebra $A \subset \mathrm{End}(W)$ generated by the endomorphisms $\phi_1, \dots, \phi_n$, $\phi_i(e_j) = \delta_{ij} e_{i-1}$. Take a suitable morphisms of Lie algebras $l(V) \to A \oplus W$.) △

Our main use of the projection $\sigma\colon P(V) \to \widehat{l}(V)$ consists in the proof of the an explicit description of the product $*\colon \widehat{l}(V) \times \widehat{l}(V) \to \widehat{l}(V)$.

THEOREM V.11 (Baker-Campbell-Hausdorff formula). *For every $a, b \in \widehat{l}(V)$ we have*

$$a * b = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_1+q_1>0 \\ \vdots \\ p_n+q_n>0}} \frac{\left(\sum_{i=1}^{n}(p_i+q_i)\right)^{-1}}{p_1!q_1!\dots p_n!q_n!} ad(a)^{p_1} ad(b)^{q_1} \dots ad(b)^{q_n-1} b.$$

In particular $a * b - a - b$ belongs to the Lie ideal of $\widehat{l}(V)$ generated by $[a, b]$.

PROOF. Use the formula of the statement to define momentarily a binary operator $\bullet$ on $\widehat{l}(V)$; we want to prove that $\bullet = *$.
Consider first the case $a, b \in V$, in this situation

$$a * b = \sigma \log(e^a e^b) = \sigma\left(\sum_{n>0} \frac{(-1)^{n-1}}{n}\left(\sum_{p+q>0} \frac{a^p b^q}{p!q!}\right)^n\right) =$$

$$= \sigma\left(\sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_1+q_1>0 \\ \vdots \\ p_n+q_n>0}} \frac{a^{p_1} b^{q_1} \dots a^{p_n} b^{q_n}}{p_1!q_1!\dots p_n!q_n!}\right)$$

$$= \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_1+q_1>0 \\ \vdots \\ p_n+q_n>0}} \frac{1}{m} \frac{\sigma_m(a^{p_1} b^{q_1} \dots a^{p_n} b^{q_n})}{p_1!q_1!\dots p_n!q_n!}, \qquad m := \sum_{i=1}^{n}(p_i+q_i).$$

The elimination of the operators σ_m gives

$$a * b = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_1+q_1>0 \\ \vdots \\ p_n+q_n>0}} \frac{\left(\sum_{i=1}^{n}(p_i+q_i)\right)^{-1}}{p_1!q_1!\dots p_n!q_n!} ad(a)^{p_1} ad(b)^{q_1} \dots ad(b)^{q_n-1} b.$$

Choose a vector space H and a surjective linear map $H \to \widehat{l}(V)$, its composition with the inclusion $\widehat{l}(V) \subset \mathfrak{m}(V) \subset P(V)$ extends to a continuous morphism of associative algebras $q\colon P(H) \to P(V)$; since $\widehat{l}(V)$ is a Lie subalgebra of $P(V)$ we have $q(l(H)_n) \subset \widehat{l}(V)$ for every n and then $q(\widehat{l}(H)) \subset \widehat{l}(V)$. Being $q\colon \widehat{l}(H) \to \widehat{l}(V)$ a morphism of Lie algebras, we have that q commutes with $\bullet$.
On the other hand q also commutes with exponential and logarithms and therefore q commutes with the product $*$. Since $* = \bullet\colon H \times H \to \widehat{l}(H)$ the proof is done. □

The first terms of the Baker-Campbell-Hausdorff formula are:

$$a * b = a + b + \frac{1}{2}[a,b] + \frac{1}{12}[a,[a,b]] - \frac{1}{12}[b,[b,a]] + \dots$$

3. Nilpotent Lie algebras

We recall that every Lie algebra L has a *universal enveloping algebra* U characterized by the properties [**31**, 17.2], [**33**, Ch. V]:

(1) U is an associative algebra and there exists an injective morphism of Lie algebras $i\colon L \to U_L$.
(2) For every associative algebra R and every morphism $f\colon L \to R_L$ of Lie algebras there exists a unique morphism of associative algebras $g\colon U \to R$ such that $f = gi$.

A concrete exhibition of the universal enveloping algebra is given by $U = \overline{T(L)}/I$, where I is the ideal generated by all the elements $a \otimes b - b \otimes a - [a,b]$, $a,b \in L$. The only non trivial condition to check is the injectivity of the natural map $L \to U$. This is usually proved using the well known Poincaré-Birkhoff-Witt's theorem [**33**, Ch. V].

Exercise V.12. Prove that, for every vector space V, $\overline{T(V)}$ is the universal enveloping algebra of $l(V)$. △

Definition V.13. The *lower central series* of a Lie algebra L is defined recursively by $L^1 = L$, $L^{n+1} = [L, L^n]$.
A Lie algebra L is called *nilpotent* if $L^n = 0$ for $n >> 0$.

It is clear that if L is a nilpotent Lie algebra then the adjoint operator $ad(a) = [a,-]\colon L \to L$ is nilpotent for every $a \in L$. According to Engel's theorem [**31**, 3.2] the converse is true if L is finite dimensional.

Example V.14. It is immediate from the construction that the lower central series of the free Lie algebra $l(V) \subset \overline{T(V)}$ is $l(V)^n = \bigoplus_{i \geq n} l(V)_i = l(V) \cap \bigoplus_{i \geq n} \bigotimes^i V$.

If V is a nilpotent Lie algebra, then the Baker-Campbell-Hausdorff formula defines a product $V \times V \xrightarrow{*} V$,

$$a * b = \sum_{n>0} \frac{(-1)^{n-1}}{n} \sum_{\substack{p_1+q_1>0 \\ \vdots \\ p_n+q_n>0}} \frac{\left(\sum_{i=1}^{n}(p_i+q_i)\right)^{-1}}{p_1!q_1!\dots p_n!q_n!} ad(a)^{p_1} ad(b)^{q_1} \dots ad(a)^{p_n} ad(b)^{q_n-1} b.$$

It is clear from the definition that the product $*$ commutes with every morphism of nilpotent Lie algebra. The identity on V induce a morphism of Lie algebras $\pi\colon l(V) \to V$ such that $\pi(l(V)_n) = 0$ for $n >> 0$; this implies that π can be extended to a morphism of Lie algebras $\pi\colon \widehat{l}(V) \to V$.

Proposition V.15. *The Baker-Campbell-Hausdorff product $*$ induces a group structure on every nilpotent Lie algebras V.*

Proof. The morphism of Lie algebras $\pi\colon \widehat{l}(V) \to V$ is surjective and commutes with $*$. □

It is customary to denote by $exp(V)$ the group $(V,*)$. Equivalently it is possible to define $exp(V)$ as the set of formal symbols e^v, $v \in V$, endowed with the group structure $e^v e^w = e^{v*w}$.

Example V.16. Assume that $V \subset M = M(n,n,\mathbb{K})$ is the Lie subalgebra of strictly upper triangular matrices. Since the product of n matrices of V is always equal to 0, the inclusion $V \to M$ extends to a morphism of associative algebras $\phi\colon P(V) \to M$ and the morphism

$$\phi\colon exp(V) \to GL(n,\mathbb{K}), \qquad \phi(e^A) = \sum_{i=0}^{\infty} \frac{A^i}{i!} \in GL(n,\mathbb{K}).$$

is a homomorphism of groups.

The above example can be generalized in the following way

EXAMPLE V.17. Let R be an associative unitary $\mathbb{K}$-algebra, $R^* \subset R$ the multiplicative group of invertible elements and $N \subset R$ a nilpotent subalgebra (i.e. $N^n = 0$ for $n >> 0$). Let V be a nilpotent Lie algebra and $\xi\colon V \to N \subset R$ a representation. This means that $\xi\colon V \to N_L$ is a morphism of Lie algebras.
Denoting by $\imath\colon V \hookrightarrow U$ the universal enveloping algebra, we have a commutative diagram

$$\begin{array}{ccccc} l(V) & \xrightarrow{\pi} & V & \xrightarrow{\xi} & N_L \\ \downarrow & & \downarrow & & \downarrow \\ \overline{T(V)} & \xrightarrow{\eta} & U & \xrightarrow{\psi} & R \end{array}$$

where π, ξ are morphisms of Lie algebras and η, ψ homomorphisms of algebras. Since the image of the composition $\phi = \psi\eta$ is contained in the nilpotent subalgebra N the above diagram extends to

$$\begin{array}{ccc} \widehat{l(V)} & \longrightarrow & P(V) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \phi} \\ V & \xrightarrow{\xi} & R \end{array}$$

with ϕ homomorphism of associative algebras. If $f \in N$ it makes sense its exponential $e^f \in R$. For every $v \in V$ we have $e^{\xi(v)} = \phi(e^v)$ and for every $x, y \in V$

$$e^{\xi(x)}e^{\xi(y)} = \phi(e^x)\phi(e^y) = \phi(e^x e^y) = \phi(e^{x*y}) = e^{\xi(x*y)}.$$

The same assertion can be stated by saying that the exponential map $e^\xi\colon (V, *) = exp(V) \to R^*$ is a homomorphism of groups.

4. Differential graded Lie algebras

DEFINITION V.18. A *differential graded Lie algebra* (DGLA) $(L, [,], d)$ is the data of a $\mathbb{Z}$-graded vector space $L = \oplus_{i\in\mathbb{Z}} L^i$ together a bilinear bracket $[,]\colon L \times L \to L$ and a linear map $d \in \operatorname{Hom}^1(L, L)$ satisfying the following condition:

(1) $[\ ,\ ]$ is homogeneous skewsymmetric: this means $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{\bar{a}\bar{b}}[b, a] = 0$ for every a, b homogeneous.
(2) Every triple of homogeneous elements a, b, c satisfies the (graded) Jacobi identity
$$[a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}}[b, [a, c]].$$
(3) $d(L^i) \subset L^{i+1}$, $d \circ d = 0$ and $d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$. The map d is called the differential of L.

EXERCISE V.19. Let $L = \oplus L^i$ be a DGLA and $a \in L^i$:

(1) If i is even then $[a, a] = 0$.
(2) If i is odd then $[a, [a, b]] = \dfrac{1}{2}[[a, a], b]$ for every $b \in L$ and $[[a, a], a] = 0$.

△

EXAMPLE V.20. If $L = \oplus L^i$ is a DGLA then L^0 is a Lie algebra in the usual sense. Conversely, every Lie algebra can be considered as a DGLA concentrated in degree 0.

EXAMPLE V.21. Let (A, d_A), $A = \oplus A_i$, be a dg-algebra over $\mathbb{K}$ and (L, d_L), $L = \oplus L^i$, a DGLA.
Then $L \otimes_{\mathbb{K}} A$ has a natural structure of DGLA by setting:

$$(L \otimes_{\mathbb{K}} A)^n = \oplus_i (L^i \otimes_{\mathbb{K}} A_{n-i}),$$

$$d(x \otimes a) = d_L x \otimes a + (-1)^{\bar{x}} x \otimes d_A a, \quad [x \otimes a, y \otimes b] = (-1)^{\bar{a}\,\bar{y}}[x, y] \otimes ab.$$

EXAMPLE V.22. Let E be a holomorphic vector bundle on a complex manifold M. We may define a DGLA $L = \oplus L^p$, $L^p = \Gamma(M, \mathcal{A}^{0,p}(\mathcal{E}nd(E)))$ with the Dolbeault differential and the natural bracket. More precisely if e, g are local holomorphic sections of $\mathcal{E}nd(E)$ and ϕ, ψ differential forms we define $d(\phi e) = (\overline{\partial}\phi)e$, $[\phi e, \psi g] = \phi \wedge \psi[e, g]$.

EXAMPLE V.23. Let $(\mathcal{F}^*, d)$ be a sheaf of dg-algebras on a topological space; the space $\mathrm{Der}^*(\mathcal{F}^*, \mathcal{F}^*)$ is a DGLA with bracket $[f, g] = fg - (-1)^{\overline{f}\,\overline{g}}gf$ and differential $\delta(f) = [d, f]$.

DEFINITION V.24. We shall say that a DGLA L is *ad_0-nilpotent* if for every i the image of the adjoint action $ad : L^0 \to \mathrm{End}(L^i)$ is contained in a nilpotent (associative) subalgebra.
EXERCISE V.25.
1) Every nilpotent DGLA (i.e. a DGLA whose descending central series is definitively trivial) is ad_0-nilpotent.
2) If L is ad_0-nilpotent then L^0 is a nilpotent Lie algebra.
3) The converses of 1) and 2) are generally false. △

DEFINITION V.26. A linear map $f\colon L \to L$ is called a *derivation of degree* n if $f(L^i) \subset L^{i+n}$ and satisfies the graded Leibnitz rule $f([a, b]) = [f(a), b] + (-1)^{n\overline{a}}[a, f(b)]$.

We note that the Jacobi identity is equivalent to the assertion that, if $a \in L^i$ then $ad(a)\colon L \to L$, $ad(a)(b) = [a, b]$, is a derivation of degree i. The differential d is a derivation of degree 1.
By following the standard notation we denote by $Z^i(L) = \ker(d\colon L^i \to L^{i+1})$, $B^i(L) = \mathrm{Im}(d\colon L^{i-1} \to L^i)$, $H^i(L) = Z^i(L)/B^i(L)$.

DEFINITION V.27. The *Maurer-Cartan equation* (also called the deformation equation) of a DGLA L is

$$da + \frac{1}{2}[a, a] = 0, \qquad a \in L^1.$$

The solutions $MC(L) \subset L^1$ of the Maurer-Cartan equation are called the Maurer-Cartan elements of the DGLA L.

There is an obvious notion of morphisms of DGLAs; we denote by **DGLA** the category of differential graded Lie algebras.
Every morphism of DGLAs induces a morphism between cohomology groups. It is moreover clear that morphisms of DGLAs preserve the solutions of the Maurer-Cartan equation.
A *quasiisomorphism* of DGLAs is a morphism inducing isomorphisms in cohomology. Two DGLA's are *quasiisomorphic* if they are equivalent under the equivalence relation generated by quasiisomorphisms.
The cohomology of a DGLA is itself a differential graded Lie algebra with the induced bracket and zero differential:

DEFINITION V.28. A DGLA L is called *Formal* if it is quasiisomorphic to its cohomology DGLA $H^*(L)$.

EXERCISE V.29. Let $D\colon L \to L$ be a derivation, then the kernel of D is a graded Lie subalgebra. △

EXAMPLE V.30. Let (L, d) be a DGLA and denote $\mathrm{Der}^i(L, L)$ the space of derivations $f\colon L \to L$ of degree i. The space $\mathrm{Der}^*(L, L) = \oplus_i \mathrm{Der}^i(L, L)$ is a DGLA with bracket $[f, g] = fg - (-1)^{\overline{f}\,\overline{g}}gf$ and differential $\delta(f) = [d, f]$.

For a better understanding of some of next topics it is useful to consider the following functorial construction. Given a DGLA $(L, [,], d)$ we can construct a new DGLA $(L', [,]', d')$ by setting $(L')^i = L^i$ for every $i \neq 1$, $(L')^1 = L^1 \oplus \mathbb{K}\, d$ (here d is considered as a formal symbol of degree 1) with the bracket and the differential

$$[a + vd, b + wd]' = [a, b] + vd(b) + (-1)^{\overline{a}}wd(a), \qquad d'(a + vd) = [d, a + vd]' = d(a).$$

The natural inclusion $L \subset L'$ is a morphism of DGLA; for a better understanding of the Maurer-Cartan equation it is convenient to consider the affine embedding $\phi\colon L^1 \to (L')^1$,

$\phi(a) = a + d$. For an element $a \in L^1$ we have

$$d(a) + \frac{1}{2}[a,a] = 0 \qquad \Longleftrightarrow \qquad [\phi(a), \phi(a)]' = 0.$$

Let's now introduce the notion of gauge action on the Maurer-Cartan elements of an ad_0-nilpotent DGLA. Note that $[L^0, L^1 \oplus \mathbb{K}\, d] \subset L^1$; in particular if L is ad_0-nilpotent then also L' is ad_0-nilpotent.

Given an ad_0-nilpotent DGLA N, the exponential of the adjoint action gives homomorphisms of groups

$$exp(N^0) = (N^0, *) \to GL(N^i), \quad e^a \mapsto e^{ad(a)}, \qquad i \in \mathbb{Z}$$

where $*$ is the product given by the Baker-Campbell-Hausdorff formula.
These homomorphisms induce actions of the group $exp(N^0)$ onto the vector spaces N^i given by

$$e^a b = e^{ad(a)} b = \sum_{n \geq 0} \frac{1}{n!} ad(a)^n(b).$$

Lemma V.31. *In the above notation, if W is a linear subspace of N^i and $[N^0, N^i] \subset W$ then the exponential adjoint action preserves the affine subspaces $v + W$, $v \in N_i$.*

Proof. Let $a \in N^0$, $v \in N^i$, $w \in W$, then

$$e^a(v+w) = v + \sum_{n\geq 1} \frac{1}{n!} ad(a)^{n-1}([a,v]) + \sum_{n \geq 0} \frac{1}{n!} ad(a)^n(w).$$

$\square$

Lemma V.32. *In the above notation the exponential adjoint action preserves the quadratic cone $Z = \{v \in N^1 \mid [v,v] = 0\}$.*
For every $v \in Z$ and $u \in N^{-1}$ the element $exp([u,v])$ belongs to the stabilizer of v.

Proof. By Jacobi identity $2[v,[a,v]] = -2[v,[v,a]] = [a,[v,v]]$ for every $a \in N^0$, $v \in N^1$.
Let $a \in N^0$ be a fixed element, for every $u \in N^1$ define the polynomial function $F_u \colon \mathbb{K} \to N^2$ by

$$F_u(t) = e^{-ad(ta)}[e^{ad(ta)}u, e^{ad(ta)}u].$$

For every $s, t \in \mathbb{K}$, if $v = e^{ad(sa)}u$ then

$$F_u(t+s) = e^{ad(-sa)} F_v(t), \quad \frac{\partial F_v}{\partial t}(0) = -[a,[v,v]] + 2[v,[a,v]] = 0$$

$$\frac{\partial F_u}{\partial t}(s) = e^{ad(-sa)} u \frac{\partial F_v}{\partial t}(0) = 0.$$

Since the field $\mathbb{K}$ has characteristic 0 every function F_v is constant, proving the invariance of Z.
If $u \in N^{-1}$ and $v \in Z$, then by Jacobi identity $[[u,v],v] = ad([u,v])v = 0$ and then $exp([u,v])v = v$. $\square$

If L is an ad_0-nilpotent DGLA then V.31 and V.32 can be applied to $N = L'$. Via the affine embedding $\phi \colon L^1 \to (L')^1$, the exponential of the adjoint action on L' induces the so called *Gauge action* of $exp(L^0)$ over the set of solution of the Maurer-Cartan equation, given explicitly by

$$\begin{aligned} exp(a)(w) &= \phi^{-1}\left(e^{ad(a)}\phi(w)\right) = \sum_{n\geq 0} \frac{1}{n!} ad(a)^n(w) - \sum_{n \geq 1} \frac{1}{n!} ad(a)^{n-1}(da) \\ &= w + \sum_{n \geq 0} \frac{ad(a)^n}{(n+1)!}([a,w] - da). \end{aligned}$$

REMARK V.33. If w is a solution of the Maurer-Cartan equation and $u \in L^{-1}$ then $[w,u]+du=[w+d,u]\in L^0$ belongs to the stabilizer of w under the gauge action.
For every $a \in L^0$, $w \in L^1$, the polynomial $\gamma(t)=exp(ta)(w)\in L^1\otimes \mathbb{K}[t]$ is the solution of the "Cauchy problem"

$$\begin{cases} \dfrac{d\gamma(t)}{dt} = [a,\gamma(t)]-da \\ \gamma(0)=w \end{cases}$$

5. Functors of Artin rings

5-A. Basic definitions. We denote by:

- **Set** the category of sets in a fixed universe; we also make the choice of a fixed set $\{0\}\in$ **Set** of cardinality 1.
- **Grp** the category of groups.
- $\mathbf{Art}_{\mathbb{K}}$ the category of local Artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ (with as morphisms the local homomorphisms). If $A \in \mathbf{Art}_{\mathbb{K}}$, we will denote by $\mathfrak{m}_A$ its maximal ideal.

A *small extension* e in $\mathbf{Art}_{\mathbb{K}}$ is an exact sequence of abelian groups

$$e: \quad 0\longrightarrow M\xrightarrow{i}B\xrightarrow{p}A\longrightarrow 0$$

such that $B\xrightarrow{p}A$ is a morphism in $\mathbf{Art}_{\mathbb{K}}$ and $\ker p = i(M)$ is annihilated by the maximal ideal of B (that is, as a B-module it is a $\mathbb{K}$-vector space).

Given a surjective morphism $B\to A$ in $\mathbf{Art}_{\mathbb{K}}$ with kernel J, there exists a sequence of small extensions

$$0\longrightarrow \mathfrak{m}_B^n J/\mathfrak{m}_B^{n+1}J\longrightarrow B/\mathfrak{m}_B^{n+1}J\longrightarrow B/\mathfrak{m}_B^{n}J\longrightarrow 0, \qquad n\geq 0.$$

Since, by Nakayama's lemma, there exists $n_0\in\mathbb{N}$ such that $\mathfrak{m}_B^n J=0$ for every $n\geq n_0$ we get that every surjective morphism is $\mathbf{Art}_{\mathbb{K}}$ is the composition of a finite number of small extensions.

DEFINITION V.34. A *Functor of Artin rings* is a covariant functor $F\colon \mathbf{Art}_{\mathbb{K}}\to \mathbf{Set}$ such that $F(\mathbb{K})\simeq\{0\}$.

EXAMPLE V.35. If V is a $\mathbb{K}$-vector space we may interpret V as a functor of Artin rings $V\colon \mathbf{Art}_{\mathbb{K}}\to\mathbf{Set}$, $V(A)=V\otimes_{\mathbb{K}}\mathfrak{m}_A$. If $V=0$ we get the *trivial functor* $0\colon \mathbf{Art}_{\mathbb{K}}\to\mathbf{Set}$.

The functors of Artin rings are the object of a new category whose morphisms are the natural transformation of functors. A natural transformation $\eta\colon F\to G$ is an isomorphism if and only if $\eta(A)\colon F(A)\to G(A)$ is bijective for every $A\in\mathbf{Art}_{\mathbb{K}}$.

DEFINITION V.36. Let $F,G\colon \mathbf{Art}_{\mathbb{K}}\to\mathbf{Set}$ be two functors of Artin rings and $\eta\colon F\to G$ a natural transformation; η is called *smooth* if for every small extension

$$0\longrightarrow M\longrightarrow B\xrightarrow{p}A\longrightarrow 0$$

the map

$$(\eta,p)\colon F(B)\to G(B)\times_{G(A)}F(A)$$

is surjective.
A functor of Artin rings F is called *smooth* if the morphism $F\to 0$ is smooth.

EXERCISE V.37. $F\colon \mathbf{Art}_{\mathbb{K}}\to\mathbf{Set}$ is smooth if and only if for every surjective morphism $B\to A$ is $\mathbf{Art}_{\mathbb{K}}$, the map $F(B)\to F(A)$ is also surjective.
If V is a vector space then V is smooth as a functor of Artin rings (cf. Example V.35). △

EXERCISE V.38. Let R be an analytic algebra and let $h_R\colon \mathbf{Art}_{\mathbb{C}}\to\mathbf{Set}$ be the functor of Artin rings defined by $h_R(A)=\mathrm{Mor}_{\mathbf{An}}(R,A)$.
Prove that h_R is smooth if and only if R is smooth. △

Example V.39. Let M_0 be a compact complex manifold and define for every $A \in \mathbf{Art}_{\mathbb{C}}$

$$\mathrm{Def}_{M_0}(A) = \mathrm{Def}_{M_0}(\mathcal{O}_{X,0}) = \mathrm{Def}_{M_0}(X,0)$$

where $(X,0) = \mathrm{Spec}(A)$ is a fat point such that $\mathcal{O}_{X,0} = A$; since it is always possible to write A as a quotient of $\mathbb{C}\{z_1,\ldots,z_n\}$ for some $n \geq 0$, such a fat point $(X,0)$ always exists. According to III.12 the isomorphism class of $(X,0)$ depends only on A.
Every morphism $\mathcal{O}_{X,0} \to \mathcal{O}_{Y,0}$ in $\mathbf{Art}_{\mathbb{C}}$ is induced by a unique morphism $(Y,0) \to (X,0)$. The pull-back of infinitesimal deformations gives a morphism $\mathrm{Def}_{M_0}(X,0) \to \mathrm{Def}_{M_0}(Y,0)$. Therefore $\mathrm{Def}_{M_0}\colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ is a functor of Artin rings.

Definition V.40. The *tangent space* to a functor of Artin rings $F\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ is by definition

$$t_F = F\left(\frac{\mathbb{K}\,[t]}{(t^2)}\right) = F(\mathbb{K} \oplus \mathbb{K}\,\epsilon), \qquad \epsilon^2 = 0.$$

Exercise V.41. Prove that, for every analytic algebra R there exists a natural isomorphism $t_{h_R} = \mathrm{Der}_{\mathbb{C}}(R,\mathbb{C})$ (see Exercise V.38). △

5-B. Automorphisms functor. In this section every tensor product is intended over $\mathbb{K}$, i.e $\otimes = \otimes_{\mathbb{K}}$. Let $S \xrightarrow{\alpha} R$ be a morphism of graded $\mathbb{K}$-algebras, for every $A \in \mathbf{Art}_{\mathbb{K}}$ we have natural morphisms $S \otimes A \xrightarrow{\alpha} R \otimes A$ and $R \otimes_{\mathbb{K}} A \xrightarrow{p} R$, $p(x \otimes a) = x\overline{a}$, where $\overline{a} \in \mathbb{K}$ is the class of a in the residue field of A.

Lemma V.42. *Given $A \in \mathbf{Art}_{\mathbb{K}}$ and a commutative diagram of morphisms of graded $\mathbb{K}$-algebras*

$$\begin{tikzcd} S \otimes A \arrow[r, "\alpha"] \arrow[d, "\alpha"] & R \otimes A \arrow[d, "p"] \\ R \otimes A \arrow[r, "p"] \arrow[ur, "f"] & R \end{tikzcd}$$

we have that f is an isomorphism and $f(R \otimes J) \subset R \otimes J$ for every ideal $J \subset A$.

Proof. f is a morphism of graded A-algebras, in particular for every ideal $J \subset A$, $f(R \otimes J) \subset Jf(R \otimes A) \subset R \otimes J$. In particular, if $B = A/J$, then f induces a morphism of graded B-algebras $\overline{f}\colon R \otimes B \to R \otimes B$.
We claim that if $\mathfrak{m}_A J = 0$ then f is the identity on $R \otimes J$; in fact for every $x \in R$, $f(x \otimes 1) - x \otimes 1 \in \ker p = R \otimes \mathfrak{m}_A$ and then if $j \in J$, $x \in R$.

$$f(x \otimes j) = jf(x \otimes 1) = x \otimes j + j(f(x \otimes 1) - x \otimes 1) = x \otimes j.$$

Now we prove the lemma by induction on $n = \dim_{\mathbb{K}} A$, being f the identity for $n = 1$. Let

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a small extension with $J \neq 0$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R \otimes J & \longrightarrow & R \otimes A & \longrightarrow & R \otimes B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & & \\ 0 & \longrightarrow & R \otimes J & \longrightarrow & R \otimes A & \longrightarrow & R \otimes B & \longrightarrow & 0 \end{array}$$

By induction $\overline{f}$ is an isomorphism and by snake lemma also f is an isomorphism. □

Definition V.43. For every $A \in \mathbf{Art}_{\mathbb{K}}$ let $\mathrm{Aut}_{R/S}(A)$ be the set of commutative diagrams of graded $\mathbb{K}$-algebra morphisms

$$\begin{tikzcd} S \otimes A \arrow[r] \arrow[d] & R \otimes A \arrow[d] \\ R \otimes A \arrow[r] \arrow[ur, "f"] & R \end{tikzcd}$$

According to Lemma V.42 $\mathrm{Aut}_{R/S}$ is a functor from the category $\mathbf{Art}_{\mathbb{K}}$ to the category of groups $\mathbf{Grp}$. Here we consider $\mathrm{Aut}_{R/S}$ as a functor of Artin rings (just forgetting the group structure).

Let $\mathrm{Der}^0_S(R,R)$ be the space of S-derivations $R \to R$ of degree 0. If $A \in \mathbf{Art}_{\mathbb{K}}$ and $J \subset \mathfrak{m}_A$ is an ideal then, since $\dim_{\mathbb{K}} J < \infty$ there exist natural isomorphisms

$$\mathrm{Der}^0_S(R,R)\otimes J = \mathrm{Der}^0_S(R, R\otimes J) = \mathrm{Der}^0_{S\otimes A}(R\otimes A, R\otimes J),$$

where $d = \sum_i d_i \otimes j_i \in \mathrm{Der}^0_S(R,R)\otimes J$ corresponds to the $S\otimes A$-derivation

$$d\colon R\otimes A \to R\otimes J \subset R\otimes A, \qquad d(x\otimes a) = \sum_i d_i(x)\otimes j_i a.$$

For every $d \in \mathrm{Der}^0_{S\otimes A}(R\otimes A, R\otimes A)$ denote $d^n = d\circ\ldots\circ d$ the iterated composition of d with itself n times. The generalized Leibnitz rule gives

$$d^n(uv) = \sum_{i=0}^n \binom{n}{i} d^i(u) d^{n-1}(v), \qquad u,v\in R\otimes A.$$

Note in particular that if $d \in \mathrm{Der}^0_S(R,R)\otimes\mathfrak{m}_A$ then d is a nilpotent endomorphism of $R\otimes A$ and

$$e^d = \sum_{n\geq 0}\frac{d^n}{n!}$$

is a morphism of $\mathbb{K}$-algebras belonging to $\mathrm{Aut}_{R/S}(A)$.

Proposition V.44. *For every $A \in \mathbf{Art}_{\mathbb{K}}$ the exponential*

$$exp\colon \mathrm{Der}^0_S(R,R)\otimes\mathfrak{m}_A \to \mathrm{Aut}_{R/S}(A)$$

is a bijection.

Proof. This is obvious if $A = \mathbb{K}$; by induction on the dimension of A we may assume that there exists a nontrivial small extension

$$0\longrightarrow J\longrightarrow A\longrightarrow B\longrightarrow 0$$

such that $exp\colon \mathrm{Der}^0_S(R,R)\otimes\mathfrak{m}_B \to \mathrm{Aut}_{R/S}(B)$ is bijective.

We first note that if $d \in \mathrm{Der}^0_S(R,R)\otimes\mathfrak{m}_A$, $h\in \mathrm{Der}^0_S(R,R)\otimes J$ then $d^ih^j = h^jd^i = 0$ whenever $j>0$, $j+i\geq 2$ and then $e^{d+h} = e^d + h$; this easily implies that exp is injective. Conversely take a $f \in \mathrm{Aut}_{R/S}(A)$; by the inductive assumption there exists $d\in \mathrm{Der}^0_S(R,R)\otimes \mathfrak{m}_A$ such that $\overline{f} = \overline{e^d} \in \mathrm{Aut}_{R/S}(B)$; denote $h = f - e^d\colon R\otimes A \to R\otimes J$. Since $h(ab) = f(a)f(b) - e^d(a)e^d(b) = h(a)f(b) + e^d(a)h(b) = h(a)\overline{b} + \overline{a}h(b)$ we have that $h \in \mathrm{Der}^0_S(R,R)\otimes J$ and then $f = e^{d+h}$. □

The same argument works also if $S\to R$ is a morphism of sheaves of graded $\mathbb{K}$-algebras over a topological space and $\mathrm{Der}^0_S(R,R)$, $\mathrm{Aut}_{R/S}(A)$ are respectively the vector space of S-derivations of degree 0 of R and the $S\otimes A$-algebra automorphisms of $R\otimes A$ lifting the identity on R.

Example V.45. Let M be a complex manifold, $R = \mathcal{A}^{0,*}_M$, $S = \overline{\Omega}^*_M$. According to Proposition IV.24 $\mathrm{Der}^0_S(R,R) = \Gamma(M, \mathcal{A}^{0,0}(T_M))$ and then the exponential gives isomorphisms

$$exp\colon \Gamma(M,\mathcal{A}^{0,0}(T_M))\otimes\mathfrak{m}_A \to \mathrm{Aut}_{R/S}(A).$$

Since exp is clearly functorial in A, interpreting the vector space $\Gamma(M,\mathcal{A}^{0,0}(T_M))$ as a functor (Example V.35), we have an isomorphism of functors $exp\colon \Gamma(M,\mathcal{A}^{0,0}(T_M)) \to \mathrm{Aut}_{R/S}$.

5-C. The exponential functor. Let L be a Lie algebra over $\mathbb{K}$, V a $\mathbb{K}$-vector space and $\xi\colon L \to \mathrm{End}(V)$ a representation of L.
For every $A \in \mathbf{Art}_{\mathbb{K}}$ the morphism ξ can be extended naturally to a morphism of Lie algebras $\xi\colon L \otimes A \to \mathrm{End}_A(V \otimes A)$. Taking the exponential we get a functorial map

$$exp(\xi)\colon L \otimes \mathfrak{m}_A \to GL_A(V \otimes A), \qquad exp(\xi)(x) = e^{\xi(x)} = \sum_{i=0}^{\infty} \frac{\xi^n}{n!}x,$$

where GL_A denotes the group of A-linear invertible morphisms.
Note that $exp(\xi)(-x) = (exp(\xi)(x))^{-1}$. If ξ is injective then also $exp(\xi)$ is injective (easy exercise).

Theorem V.46. *In the notation above the image of $exp(\xi)$ is a subgroup. More precisely for every $a, b \in L \otimes \mathfrak{m}_A$ there exists $c \in L \otimes \mathfrak{m}_A$ such that $e^{\xi(a)}e^{\xi(b)} = e^{\xi(c)}$ and $a + b - c$ belong to the Lie ideal of $L \otimes \mathfrak{m}_A$ generated by $[a, b]$.*

Proof. This is an immediate consequence of the Campbell-Baker-Hausdorff formula. $\square$

In the above notation denote $P = \mathrm{End}(V)$ and let $ad(\xi)\colon L \to \mathrm{End}(P)$ be the adjoint representation of ξ,

$$ad(\xi)(x)f = [\xi(x), f] = \xi(x)f - f\xi(x).$$

Then for every $a \in L \otimes \mathfrak{m}_A$, $f \in \mathrm{End}_A(V \otimes A) = P \otimes A$ we have (cf. Exercise V.1, [**31**, 2.3])

$$e^{ad(\xi)(a)}f = e^{\xi(a)}fe^{-\xi(a)}.$$

6. Deformation functors associated to a DGLA

Let $L = \oplus L^i$ be a DGLA over $\mathbb{K}$, we can define the following three functors:

(1) The Gauge functor $G_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grp}$, defined by $G_L(A) = exp(L^0 \otimes \mathfrak{m}_A)$. It is immediate to see that G_L is smooth.
(2) The Maurer-Cartan functor $MC_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ defined by

$$MC_L(A) = MC(L \otimes \mathfrak{m}_A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x, x] = 0 \right\}.$$

(3) The gauge action of the group $exp(L^0 \otimes \mathfrak{m}_A)$ on the set $MC(L \otimes \mathfrak{m}_A)$ is functorial in A and gives an action of the group functor G_L over MC_L. We call $\mathrm{Def}_L = MC_L/G_L$ the corresponding quotient. By definition $\mathrm{Def}_L(A) = MC_L(A)/G_L(A)$ for every $A \in \mathbf{Art}_{\mathbb{K}}$.
The functor Def_L is called the *deformation functor* associated to the DGLA L.

The reader should make attention to the difference between the deformation functor Def_L associated to a DGLA L and the functor of deformations of a DGLA L.

Proposition V.47. *Let $L = \oplus L^i$ be a DGLA. If $[L^1, L^1] \cap Z^2(L) \subset B^2(L)$ (e.g. if $H^2(L) = 0$) then MC_L and Def_L are smooth functors.*

Proof. It is sufficient to prove that for every small extension

$$0 \longrightarrow J \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

the map $MC(L \otimes \mathfrak{m}_A) \xrightarrow{\alpha} MC(L \otimes \mathfrak{m}_B)$ is surjective.
Given $y \in L^1 \otimes \mathfrak{m}_B$ such that $dy + \frac{1}{2}[y, y] = 0$ we first choose $x \in L^1 \otimes \mathfrak{m}_A$ such that $\alpha(x) = y$; we need to prove that there exists $z \in L^1 \otimes J$ such that $x - z \in MC(L \otimes \mathfrak{m}_A)$.
Denote $h = dx + \dfrac{1}{2}[x, x] \in L^2 \otimes J$; we have

$$dh = d^2x + [dx, x] = [h, x] - \frac{1}{2}[[x, x], x].$$

Since $[L^2 \otimes J, L^1 \otimes \mathfrak{m}_A] = 0$ we have $[h, x] = 0$, by Jacobi identity $[[x, x], x] = 0$ and then $dh = 0$, $h \in Z^2(L) \otimes J$.

On the other hand $h \in ([L^1, L^1] + B^2(L)) \otimes \mathfrak{m}_A$, using the assumption of the Proposition $h \in (B^2(L) \otimes \mathfrak{m}_A) \cap L^2 \otimes J$ and then there exist $z \in L^1 \otimes \mathfrak{m}_A$ such that $dz = h$.
Since $Z^1(L) \otimes \mathfrak{m}_A \to Z^1(L) \otimes \mathfrak{m}_B$ is surjective it is possible to take $z \in L^1 \otimes J$: it is now immediate to observe that $x - z \in MC(L \otimes \mathfrak{m}_A)$. $\square$

Exercise V.48. Prove that if MC_L is smooth then $[Z^1, Z^1] \subset B^2$. $\triangle$

Proposition V.49. *If $L \otimes \mathfrak{m}_A$ is abelian then* $\mathrm{Def}_L(A) = H^1(L) \otimes \mathfrak{m}_A$. *In particular* $t_{\mathrm{Def}_L} = H^1(L) \otimes \mathbb{K}\,\epsilon$, $\epsilon^2 = 0$.

Proof. The Maurer-Cartan equation reduces to $dx = 0$ and then $MC_L(A) = Z^1(L) \otimes \mathfrak{m}_A$. If $a \in L^0 \otimes \mathfrak{m}_A$ and $x \in L^1 \otimes \mathfrak{m}_A$ we have

$$exp(a)x = x + \sum_{n \geq 0} \frac{ad(a)^n}{(n+1)!}([a, x] - da) = x - da$$

and then $\mathrm{Def}_L(A) = \dfrac{Z^1(L) \otimes \mathfrak{m}_A}{d(L^0 \otimes \mathfrak{m}_A)} = H^1(L) \otimes \mathfrak{m}_A$. $\square$

Exercise V.50. If $[Z^1, Z^1] = 0$ then $MC_L(A) = Z^1 \otimes \mathfrak{m}_A$ for every A. $\triangle$

It is clear that every morphism $\alpha\colon L \to N$ of DGLA induces morphisms of functors $G_L \to G_N$, $MC_L \to MC_N$. These morphisms are compatible with the gauge actions and therefore induce a morphism between the deformation functors $\mathrm{Def}_\alpha\colon \mathrm{Def}_L \to \mathrm{Def}_N$.
The following Theorem V.51 (together its Corollary V.52) is sometimes called the *basic theorem of deformation theory.* For the clarity of exposition the (nontrivial) proof of V.51 is postponed at the end of Section 8.

Theorem V.51. *Let $\phi\colon L \to N$ be a morphism of differential graded Lie algebras and denote by $H^i(\phi)\colon H^i(L) \to H^i(N)$ the induced maps in cohomology.*

(1) *If $H^1(\phi)$ is surjective and $H^2(\phi)$ injective then the morphism* $\mathrm{Def}_\phi\colon \mathrm{Def}_L \to \mathrm{Def}_N$ *is smooth.*
(2) *If $H^0(\phi)$ is surjective, $H^1(\phi)$ is bijective and $H^2(\phi)$ is injective then* $\mathrm{Def}_\phi\colon \mathrm{Def}_L \to \mathrm{Def}_N$ *is an isomorphism.*

Corollary V.52. *Let $L \to N$ be a quasiisomorphism of DGLA. Then the induced morphism* $\mathrm{Def}_L \to \mathrm{Def}_N$ *is an isomorphism.*

Exercise V.53. Let L be a formal DGLA, then Def_L is smooth if and only if the induced bracket $[\ ,\ ]\colon H^1 \times H^1 \to H^2$ is zero. $\triangle$

Example V.54. Let $L = \oplus L^i$ be a DGLA and choose a vector space decomposition $N^1 \oplus B^1(L) = L^1$.
Consider the DGLA $N = \oplus N^i$ where $N^i = 0$ if $i < 1$ and $N^i = L^i$ if $i > 1$ with the differential and bracket induced by L. The natural inclusion $N \to L$ gives isomorphisms $H^i(N) \to H^i(L)$ for every $i \geq 1$. In particular the morphism $\mathrm{Def}_N \to \mathrm{Def}_L$ is smooth and induce an isomorphism on tangent spaces $t_{\mathrm{Def}_N} = t_{\mathrm{Def}_L}$.

Beware. One of the most frequent wrong interpretations of Corollary V.52 asserts that if $L \to N$ is a quasiisomorphism of nilpotent DGLA then $MC(L)/exp(L^0) \to MC(N)/exp(N^0)$ is a bijection. This is false in general: consider for instance $L = 0$ and $N = \oplus N^i$ with $N^i = \mathbb{C}$ for $i = 1, 2$, $N^i = 0$ for $i \neq 1, 2$, $d\colon N^1 \to N^2$ the identity and $[a, b] = ab$ for $a, b \in N^1 = \mathbb{C}$.

Let T_M be the holomorphic tangent bundle of a complex manifold M. The Kodaira-Spencer DGLA is defined as

$$KS(M) = \oplus KS(M)^p, \qquad KS(M)^p = \Gamma(M, \mathcal{A}^{0,p}(T_M))$$

with the Dolbeault differential and the bracket (cf. Proposition IV.24)

$$[\phi d\bar{z}_I, \psi d\bar{z}_J] = [\phi, \psi] d\bar{z}_I \wedge d\bar{z}_J$$

for $\phi, \psi \in \mathcal{A}^{0,0}(T_M)$, $I, J \subset \{1, ..., n\}$ and $z_1, ..., z_n$ local holomorphic coordinates.

THEOREM V.55. *Let $L = KS(M_0)$ be the Kodaira-Spencer differential graded Lie algebra of a compact complex manifold M_0. Then there exists an isomorphism of functors*

$$\mathrm{Def}_{M_0} = \mathrm{Def}_L.$$

PROOF. Fix $A \in \mathbf{Art}_{\mathbb{C}}$, according to Propositions IV.24 and V.44 the exponential

$$exp\colon L^0 \otimes \mathfrak{m}_A = \Gamma(M_0, \mathcal{A}^{0,0}(T_{M_0})) \otimes \mathfrak{m}_A \to \mathrm{Aut}_{\mathcal{A}^{0,*}/\overline{\Omega}^*}(A)$$

is an isomorphism.
Therefore Def_{M_0} is the quotient of

$$MC_L(A) = \left\{\eta \in \Gamma(M_0, \mathcal{A}^{0,1}(T_{M_0})) \otimes \mathfrak{m}_A \;\middle|\; \overline{\partial}\eta + \frac{1}{2}[\eta,\eta] = 0\right\},$$

by the equivalence relation $\sim$, given by $\eta \sim \mu$ if and only if there exists $a \in L^0 \otimes \mathfrak{m}_A$ such that

$$\overline{\partial} + \mu = e^a(\overline{\partial} + \eta)e^{-a} = e^{ad(a)}(\overline{\partial} + \eta)$$

or, equivalently, if and only if $\phi(\mu) = e^{ad(a)}\phi(\eta)$, where ϕ is the affine embedding defined above.
Keeping in mind the definition of the gauge action on the Maurer-Cartan elements we get immediately that this equivalence relation on $MC_L(A)$ is exactly the one induced by the gauge action of $exp(L^0 \otimes \mathfrak{m}_A)$. ☐

COROLLARY V.56. *Let M_0 be a compact complex manifold. If either $H^2(M_0, T_{M_0}) = 0$ or its Kodaira-Spencer DGLA $KS(M_0)$ is quasiisomorphic to an abelian DGLA, then* Def_{M_0} *is smooth.*

7. Extended deformation functors (EDF)

We will always work over a fixed field $\mathbb{K}$ of characteristic 0. All vector spaces, linear maps, algebras, tensor products etc. are understood of being over $\mathbb{K}$, unless otherwise specified.
We denote by:

- **NA** the category of all differential $\mathbb{Z}$-graded associative (graded)-commutative nilpotent finite dimensional $\mathbb{K}$-algebras.
- By **NA**$\cap$**DG** we denote the full subcategory of $A \in$ **NA** with trivial multiplication, i.e. $A^2 = 0$.

In other words an object in **NA** is a finite dimensional complex $A = \oplus A_i \in$ **DG** endowed with a structure of dg-algebra such that $A^n = AA\ldots A = 0$ for $n >> 0$. Note that if $A = A_0$ is concentrated in degree 0, then $A \in$ **NA** if and only if A is the maximal ideal of a local artinian $\mathbb{K}$-algebra with residue field $\mathbb{K}$.
If $A \in$ **NA** and $I \subset A$ is a differential ideal, then also $I \in$ **NA** and the inclusion $I \to A$ is a morphism of dg-algebras.

DEFINITION V.57. A *small extension* in **NA** is a short exact sequence in **DG**

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

such that α is a morphism in **NA** and I is an ideal of A such that $AI = 0$; in addition it is called *acyclic* if I is an acyclic complex, or equivalently if α is a quasiisomorphism.

EXERCISE V.58.

- Every surjective morphism $A \xrightarrow{\alpha} B$ in the category **NA** is the composition of a finite number of small extensions.
- If $A \xrightarrow{\alpha} B$ is a surjective quasiisomorphism in **NA** and $A_i = 0$ for every $i > 0$ then α is the composition of a finite number of acyclic small extensions. This is generally false if $A_i \neq 0$ for some $i > 0$.

△

Definition V.59. A covariant functor $F\colon \mathbf{NA} \to \mathbf{Set}$ is called a *predeformation functor* if the following conditions are satisfied:

(1) $F(0) = 0$ is the one-point set.
(2) For every pair of morphisms $\alpha\colon A \to C$, $\beta\colon B \to C$ in $\mathbf{NA}$ consider the map
$$\eta\colon F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$
Then:
 (a) η is surjective when α is surjective.
 (b) η is bijective when α is surjective and $C \in \mathbf{NA} \cap \mathbf{DG}$ is an acyclic complex.
(3) For every acyclic small extension
$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$
the induced map : $F(A) \to F(B)$ is surjective.

If we consider the above definition for a functor defined only for algebras concentrated in degree 0, then condition 3 is empty, while conditions 1 and 2 are essentially the classical Schlessinger's conditions [**67**], [**13**], [**52**].

Lemma V.60. *For a covariant functor $F\colon \mathbf{NA} \to \mathbf{Set}$ with $F(0) = 0$ it is sufficient to check condition 2b of definition V.59 when $C = 0$ and when $B = 0$ separately.*

Proof. Follows immediately from the equality
$$A \times_C B = (A \times B) \times_C 0$$
where $A \xrightarrow{\alpha} C$, $B \xrightarrow{\beta} C$ are as in 2b of V.59 and the fibred product on the right comes from the morphism $A \times B \to C$, $(a, b) \mapsto \alpha(a) - \beta(b)$. □

Definition V.61. A predeformation functor $F\colon \mathbf{NA} \to \mathbf{Set}$ is called a *deformation functor* if $F(I) = 0$ for every acyclic complex $I \in \mathbf{NA} \cap \mathbf{DG}$.

The predeformation functors (resp.: deformation functors) together their natural transformations form a category which we denote by **PreDef** (resp.: **Def**).

Lemma V.62. *Let $F\colon \mathbf{NA} \to \mathbf{Set}$ be a deformation functor. Then:*

(1) *For every acyclic small extension*
$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$
the induced map : $F(A) \to F(B)$ is bijective.
(2) *For every pair of complexes $I, J \in \mathbf{NA} \cap \mathbf{DG}$ and every pair of homotopic morphisms $f, g\colon I \to J$, we have $F(f) = F(g)\colon F(I) \to F(J)$.*

Proof. We need to prove that for every acyclic small extension
$$0 \longrightarrow I \longrightarrow A \xrightarrow{\rho} B \longrightarrow 0$$
the diagonal map $F(A) \to F(A) \times_{F(B)} F(A)$ is surjective; in order to prove this it is sufficient to prove that the diagonal map $A \to A \times_B A$ induces a surjective map $F(A) \to F(A \times_B A)$. We have a canonical isomorphism $\theta\colon A \times I \to A \times_B A$, $\theta(a, x) = (a, a + x)$ which sends $A \times \{0\}$ onto the diagonal; since $F(A \times I) = F(A) \times F(I) = F(A)$ the proof of item 1 is concluded.
For item 2, we can write $I = I^0 \times I^1$, $J = J^0 \times J^1$, with $d(I^0) = d(J^0) = 0$ and I^1, J^1 acyclic. Then the inclusion $I^0 \xrightarrow{i} I$ and the projection $J \xrightarrow{p} J^0$ induce bijections $F(I^0) = F(I)$, $F(J^0) = F(J)$. It is now sufficient to note that $pfi = pgi\colon I^0 \to J^0$. □

A standard argument in Schlessinger's theory [**67**, 2.10] shows that for every predeformation functor F and every $A \in \mathbf{NA} \cap \mathbf{DG}$ there exists a natural structure of vector space on $F(A)$, where the sum and the scalar multiplication are described by the maps
$$A \times A \xrightarrow{+} A \quad \Rightarrow \quad F(A \times A) = F(A) \times F(A) \xrightarrow{+} F(A)$$
$$s \in \mathbb{K}, \quad A \xrightarrow{\cdot s} A \quad \Rightarrow \quad F(A) \xrightarrow{\cdot s} F(A)$$

We left as an exercise to check that the vector space axioms are satisfied; if $A \to B$ is a morphism in $\mathbf{NA} \cap \mathbf{DG}$ then the commutativity of the diagrams

$$\begin{array}{ccc} A\times A & \xrightarrow{+} & A \\ \downarrow & & \downarrow \\ B\times B & \xrightarrow{+} & B \end{array}, \qquad \begin{array}{ccc} A & \xrightarrow{\cdot s} & A \\ \downarrow & & \downarrow \\ B & \xrightarrow{\cdot s} & B \end{array}, \qquad s \in \mathbb{K}$$

shows that $F(A) \to F(B)$ is $\mathbb{K}$-linear. Similarly if $F \to G$ is a natural transformations of predeformation functors, the map $F(A) \to G(A)$ is $\mathbb{K}$-linear for every $A \in \mathbf{NA} \cap \mathbf{DG}$.

In particular, for every predeformation functor F and for every integer n the sets $F(\Omega[n])$ (see Example IV.7) and $F(\mathbb{K}[n])$ are vector spaces and the projection $p\colon \Omega[n] \to \mathbb{K}[n]$ induce a linear map $F(\Omega[n]) \to F(\mathbb{K}[n])$

Definition V.63. Let F be a predeformation functor, the *tangent space* of F is the graded vector space $TF[1]$, where

$$TF = \bigoplus_{n\in\mathbb{Z}} T^nF, \qquad T^{n+1}F = TF[1]^n = \operatorname{coker}(F(\Omega[n]) \xrightarrow{p} F(\mathbb{K}[n])), \quad n \in \mathbb{Z}.$$

A natural transformation $F \to G$ of predeformation functors is called a *quasiisomorphism* if induces an isomorphism on tangent spaces, i.e. if $T^nF \simeq T^nG$ for every n.

We note that if F is a deformation functor then $F(\Omega[n]) = 0$ for every n and therefore $TF[1]^n = T^{n+1}F = F(\mathbb{K}\epsilon)$, where ϵ is an indeterminate of degree $-n \in \mathbb{Z}$ such that $\epsilon^2 = 0$. In particular $T^1F = t_{F^0}$, where $F^0\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$, $F^0(A) = F(\mathfrak{m}_A)$, is the *truncation* of F in degree 0.

One of the most important examples of deformation functors is the deformation functor associated to a differential graded Lie algebra.

Given a DGLA L and $A \in \mathbf{NA}$, the tensor product $L \otimes A$ has a natural structure of nilpotent DGLA with

$$(L\otimes A)^i = \bigoplus_{j\in\mathbb{Z}} L^j \otimes A_{i-j}$$
$$d(x\otimes a) = dx\otimes a + (-1)^{\overline{x}} x\otimes da$$
$$[x\otimes a, y\otimes b] = (-1)^{\overline{a}\,\overline{y}}[x,y]\otimes ab$$

Every morphism of DGLA, $L \to N$ and every morphism $A \to B$ in $\mathbf{NA}$ give a natural commutative diagram of morphisms of differential graded Lie algebras

$$\begin{array}{ccc} L\otimes A & \longrightarrow & N\otimes A \\ \downarrow & & \downarrow \\ L\otimes B & \longrightarrow & N\otimes B \end{array}$$

The Maurer-Cartan functor $MC_L\colon \mathbf{NA} \to \mathbf{Set}$ of a DGLA L is by definition

$$MC_L(A) = MC(L\otimes A) = \left\{ x \in (L\otimes A)^1 \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\}.$$

Lemma V.64. *For every differential graded Lie algebra L, MC_L is a predeformation functor.*

Proof. It is evident that $MC_L(0) = 0$ and for every pair of morphisms $\alpha\colon A \to C$, $\beta\colon B \to C$ in $\mathbf{NA}$ we have

$$MC_L(A \times_C B) = MC_L(A) \times_{MC_L(C)} MC_L(B)$$

Let $0 \longrightarrow I \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ be an acyclic small extension and $x \in MC_L(B)$. Since α is surjective there exists $y \in (L\otimes A)^1$ such that $\alpha(y) = x$. Setting

$$h = dy + \frac{1}{2}[y,y] \in (L\otimes I)^2$$

we have

$$dh = \frac{1}{2}d[y,y] = [dy,y] = [h,y] - \frac{1}{2}[[y,y],y].$$

By Jacobi identity $[[y,y],y]=0$ and, since $AI=0$ also $[h,y]=0$; thus $dh=0$ and, being $L\otimes I$ acyclic by Künneth formula, there exists $s\in(L\otimes I)^1$ such that $ds=h$. The element $y-s$ lifts x and satisfies the Maurer-Cartan equation. We have therefore proved that MC_L is a predeformation functor. ☐

Exercise V.65. Prove that $MC\colon \mathbf{DGLA}\to\mathbf{PreDef}$ is a faithful functor and every differential graded Lie algebra can be recovered, up to isomorphism, from its Maurer-Cartan functor. △

It is interesting to point out that, if $A\to B$ is a surjective quasiisomorphism in $\mathbf{NA}$, then in general $MC_L(A)\to MC_L(B)$ is not surjective. As an example take L a finite-dimensional non-nilpotent complex Lie algebra, considered as a DGLA concentrated in degree 0 and fix $a\in L$ such that $ad(a)\colon L\to L$ has an eigenvalue $\lambda\neq 0$. Up to multiplication of a by $-\lambda^{-1}$ we can assume $\lambda=-1$. Let $V\subset L$ be the image of $ad(a)$, the linear map $Id+ad(a)\colon V\to V$ is not surjective and then there exists $b\in L$ such that the equation $x+[a,x]+[a,b]=0$ has no solution in L.
Let u,v,w be indeterminates of degree 1 and consider the dg-algebras

$$B=\mathbb{C}u\oplus\mathbb{C}v,\qquad B^2=0,\ d=0$$

$$A=\mathbb{C}u\oplus\mathbb{C}v\oplus\mathbb{C}w\oplus\mathbb{C}dw,\qquad uv=uw=dw,\ vw=0$$

The projection $A\to B$ is a quasiisomorphism but the element $a\otimes u+b\otimes v\in MC_L(B)$ cannot lifted to $MC_L(A)$. In fact if there exists $\xi=a\otimes u+b\otimes v+x\otimes w\in MC_L(A)$ then

$$0=d\xi+\frac{1}{2}[\xi,\xi]=(x+[a,x]+[a,b])\otimes dw$$

in contradiction with the previous choice of a,b.

For every DGLA L and every $A\in\mathbf{NA}$ we define $\mathrm{Def}_L(A)$ as the quotient of $MC(L\otimes A)$ by the gauge action of the group $exp((L\otimes A)^0)$. The gauge action commutes with morphisms in $\mathbf{NA}$ and with morphisms of differential graded Lie algebras; therefore the above definition gives a functor $\mathrm{Def}_L\colon\mathbf{NA}\to\mathbf{Set}$.

Theorem V.66. *For every DGLA L, $\mathrm{Def}_L\colon\mathbf{NA}\to\mathbf{Set}$ is a deformation functor with $T^i\mathrm{Def}_L=H^i(L)$.*

Proof. If $C\in\mathbf{NA}\cap\mathbf{DG}$ is a complex then $L\otimes C$ is an abelian DGLA and according to Proposition V.49, $MC_L(C)=Z^1(L\otimes C)$ and $\mathrm{Def}_L(C)=H^1(L\otimes C)$. In particular $T^i\mathrm{Def}_L=H^1(L\otimes\mathbb{K}[i-1])=H^i(L)$ and, by Künneth formula, $\mathrm{Def}_L(C)=0$ if C is acyclic.

Since Def_L is the quotient of a predeformation functor, conditions 1 and 3 of V.59 are trivially verified and then it is sufficient to verify condition 2.
Let $\alpha\colon A\to C$, $\beta\colon B\to C$ morphism in $\mathbf{NA}$ with α surjective. Assume there are given $a\in MC_L(A)$, $b\in MC_L(B)$ such that $\alpha(a)$ and $\beta(b)$ give the same element in $\mathrm{Def}_L(C)$; then there exists $u\in(L\otimes C)^0$ such that $\beta(b)=e^u\alpha(a)$. Let $v\in(L\otimes A)^0$ be a lifting of u, changing if necessary a with its gauge equivalent element e^va, we may suppose $\alpha(a)=\beta(b)$ and then the pair (a,b) lifts to $MC_L(A\times_C B)$: this proves that the map

$$\mathrm{Def}_L(A\times_C B)\to\mathrm{Def}_L(A)\times_{\mathrm{Def}_L(C)}\mathrm{Def}_L(B)$$

is surjective.
If $C=0$ then the gauge action $exp((L\otimes(A\times B))^0)\times MC_L(A\times B)\to MC_L(A\times B)$ is the direct product of the gauge actions $exp((L\otimes A)^0)\times MC_L(A)\to MC_L(A)$, $exp((L\otimes B)^0)\times MC_L(B)\to MC_L(B)$ and therefore $\mathrm{Def}_L(A\times B)=\mathrm{Def}_L(A)\times\mathrm{Def}_L(B)$.
Finally assume $B=0$, C acyclic complex and denote $D=\ker\alpha\simeq A\times_C B$. Let $a_1,a_2\in MC_L(D)$, $u\in(L\otimes A)^0$ be such that $a_2=e^ua_1$; we need to prove that there exists $v\in(L\otimes D)^0$ such that $a_2=e^va_1$.
Since $\alpha(a_1)=\alpha(a_2)=0$ and $L\otimes C$ is an abelian DGLA we have $0=e^{\alpha(u)}0=0-d\alpha(u)$ and then $d\alpha(u)=0$. $L\otimes C$ is acyclic and then there exists $h\in(L\otimes A)^{-1}$ such that

$d\alpha(h) = -\alpha(u)$ and $u + dh \in (L \otimes D)^0$. Setting $w = [a_1, h] + dh$, then, according to Remark V.33, $e^w a_1 = a_1$ and $e^u e^w a_1 = e^v a_1 = a_2$, where $v = u * w$ is determined by Baker-Campbell-Hausdorff formula. We claim that $v \in L \otimes D$: in fact $v = u * w \equiv u + w \equiv u + dh \pmod{[L \otimes A, L \otimes A]}$ and since $A^2 \subset D$ we have $v = u * w \equiv u + dh \equiv 0 \pmod{L \otimes D}$. $\square$

LEMMA V.67. *For every DGLA L, the projection $\pi\colon MC_L \to \mathrm{Def}_L$ is a quasiisomorphism.*

PROOF. Let $i \in \mathbb{Z}$ be fixed; in the notation of V.63 we can write $\Omega[i-1] = \mathbb{K}\,\epsilon \oplus \mathbb{K}\,d\epsilon$, where $\epsilon^2 = \epsilon d\epsilon = (d\epsilon)^2 = 0$ and $\overline{\epsilon} = 1 - i$, $\overline{d\epsilon} = 2 - i$. We have

$$MC_L(\mathbb{K}\,\epsilon) = \{x\epsilon \in (L \otimes \mathbb{K}\,\epsilon)^1 | d(x\epsilon) = 0\} = Z^i(L) \otimes \mathbb{K}\,\epsilon$$

$$\begin{aligned} MC_L(\mathbb{K}\,\epsilon \oplus \mathbb{K}\,d\epsilon) &= \{x\epsilon + yd\epsilon \in (L \otimes \Omega[i-1])^1 \,|\, dx\epsilon + (-1)^{1-i} x d\epsilon + dy d\epsilon = 0\} \\ &= \{(-1)^i dy\epsilon + yd\epsilon |\, y \in L^{i-1}\}. \end{aligned}$$

Therefore the image of $p\colon MC_L(\mathbb{K}\,\epsilon \oplus \mathbb{K}\,d\epsilon) \to MC_L(\mathbb{K}\,\epsilon)$ is exactly $B^i(L) \otimes \mathbb{K}\,\epsilon$ and then

$$MC_L(\Omega[i-1]) \xrightarrow{p} MC_L(\mathbb{K}\,[i-1]) \xrightarrow{\pi} \mathrm{Def}_L(\mathbb{K}\,[i-1]) \longrightarrow 0$$

is exact. $\square$

8. Obstruction theory and the inverse function theorem for deformation functors

LEMMA V.68. *Let $F\colon \mathbf{NA} \to \mathbf{Set}$ be a deformation functor; for every complex $I \in \mathbf{NA} \cap \mathbf{DG}$ there exists a natural isomorphism*

$$F(I) = \bigoplus_{i \in \mathbb{Z}} TF[1]^i \otimes H_{-i}(I) = \bigoplus_{i \in \mathbb{Z}} T^{i+1}F \otimes H_{-i}(I) = H^1(TF \otimes I).$$

PROOF. Let $s\colon H_*(I) \to Z_*(I)$ be a linear section of the natural projection, then the composition of s with the natural embedding $Z_*(I) \to I$ is unique up to homotopy and its cokernel is an acyclic complex, therefore it gives a well defined isomorphism $F(H_*(I)) \to F(I)$. This says that it is not restrictive to prove the lemma for complexes with zero differential. Moreover since F commutes with direct sum of complexes we can reduce to consider the case when $I = \mathbb{K}^s[n]$ is a vector space concentrated in degree $-n$. Every $v \in I$ gives a morphism $TF[1]^n = F(\mathbb{K}\,[n]) \xrightarrow{v} F(I)$ and we can define a natural map $TF[1]^n \otimes I \to F(I)$, $x \otimes v \mapsto v(x)$. It is easy to verify that this map is an isomorphism of vector spaces. $\square$

THEOREM V.69. *Let $0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\alpha} B \longrightarrow 0$ be an exact sequence of morphisms in $\mathbf{NA}$ and let $F\colon \mathbf{NA} \to \mathbf{Set}$ be a deformation functor.*

1. *If $AI = 0$ then there exist natural transitive actions of the abelian group $F(I)$ on the nonempty fibres of $F(A) \to F(B)$.*
2. *If $AI = 0$ then there exists a natural "obstruction map" $F(B) \xrightarrow{ob} F(I[1])$ with the property that $ob(b) = 0$ if and only if b belongs to the image of $F(A) \to F(B)$.*
3. *If B is a complex, i.e. $A^2 \subset I$, then there exist natural transitive actions of the abelian group $F(B[-1])$ on the nonempty fibres of $F(I) \to F(A)$.*

Here natural means in particular that commutes with natural transformation of functors.

PROOF. [1] There exists an isomorphism of dg-algebras

$$A \times I \longrightarrow A \times_B A; \qquad (a, t) \mapsto (a, a + t)$$

and then there exists a natural surjective map

$$\vartheta_F : F(A) \times F(I) = F(A \times I) \to F(A) \times_{F(B)} F(A)$$

The commutativity of the diagram

$$\begin{array}{ccccccc} A \times I \times I & \longrightarrow & A \times I & & (a, t, s) & \mapsto & (a, t + s) \\ \downarrow & & \downarrow & , & \downarrow & & \downarrow \\ A \times I & \longrightarrow & A & & (a + t, s) & \mapsto & (a + t + s) \end{array}$$

implies in particular that the composition of ϑ_F with the projection in the second factor give a natural transitive action of the abelian group $F(I)$ on the fibres of the map $F(A) \to F(B)$.

[2] We introduce the mapping cone of ι as the dg-algebra $C = A \oplus I[1]$ with the product $(a, m)(b, n) = (ab, 0)$ (note that, as a graded algebra, C is the trivial extension of A by $I[1]$) and differential

$$d_C = \begin{pmatrix} d_A & \iota \\ 0 & d_{I[1]} \end{pmatrix} : A \oplus I[1] \to A[1] \oplus I[2]$$

We left as exercise the easy verification that $C \in \mathbf{NA}$, the inclusion $A \to C$ and the projections $C \to I[1]$, $C \to B$ are morphisms in $\mathbf{NA}$.
The kernel of $C \to B$ is isomorphic to $I \oplus I[1]$ with differential

$$\begin{pmatrix} d_I & Id_{I[1]} \\ 0 & d_{I[1]} \end{pmatrix}.$$

Therefore $0 \longrightarrow I \oplus I[1] \longrightarrow C \longrightarrow B \longrightarrow 0$ is an acyclic small extension and then $F(C) = F(B)$. On the other hand $A = C \times_{I[1]} 0$ and then the map

$$F(A) \to F(C) \times_{F(I[1])} 0$$

is surjective. It is sufficient to define *ob* as the composition of the inverse of $F(C) \to F(B)$ with $F(C) \to F(I[1])$.

3) The derived inverse mapping cone is the dg-algebra $D = A \oplus B[-1]$ with product $(x, m)(x, n) = (xy, 0)$ and differential

$$d_D = \begin{pmatrix} d_A & 0 \\ \alpha & d_{B[-1]} \end{pmatrix} : A \oplus B[-1] \to A[1] \oplus B$$

Here the projection $D \to A$ and the inclusions inclusion $I \to D$, $B[-1] \to D$ are morphisms in $\mathbf{NA}$.
Since $0 \longrightarrow B[-1] \longrightarrow D \longrightarrow A \longrightarrow 0$ is a small extension, by Item 1, there exist natural actions of $F(B[-1])$ on the nonempty fibres of $F(D) \to F(A)$. The quotient of $I \to D$ is the acyclic complex $B \oplus B[-1]$, and then, according to 2b of V.59, $F(I) \to F(D)$ is an isomorphism. $\square$

EXERCISE V.70. Prove that the stabilizers of the actions described in Theorem V.69 are vector subspaces. $\triangle$

Given two integers $p \leq q$ we denote by $\mathbf{NA}_p^q$ the full subcategory of $\mathbf{NA}$ whose objects are algebras $A = \oplus A_i$ such that $A_i \neq 0$ only if $p \leq i \leq q$.

THEOREM V.71. *Let $\theta\colon F \to G$ be a morphism of deformation functors. Assume that $\theta\colon TF[1]^i \to TG[1]^i$ is surjective for $p - 1 \leq i \leq q$ and injective for $p \leq i \leq q + 1$. Then:*

(1) *for every surjective morphism $\alpha\colon A \to B$ in the category $\mathbf{NA}_{p-1}^q$ the morphism*
$$(\alpha, \theta)\colon F(A) \to F(B) \times_{G(B)} G(A)$$
is surjective.
(2) *$\theta\colon F(A) \to G(A)$ is surjective for every $A \in \mathbf{NA}_{p-1}^q$.*
(3) *$\theta\colon F(A) \to G(A)$ is a bijection for every $A \in \mathbf{NA}_p^q$.*

PROOF. The proof uses the natural generalization to the differential graded case of some standard techniques in Schlessinger's theory, cf. [**13**].
We first note that, according to Lemma V.68, for every complex $I \in \mathbf{NA}_p^q \cap \mathbf{DG}$ we have that $\theta\colon F(I) \to G(I)$ is bijective, $\theta\colon F(I[1]) \to G(I[1])$ is injective and $\theta\colon F(I[-1]) \to G(I[-1])$ is surjective.
Moreover, since $F(0) = G(0) = 0$, we have $F(0) \times_{G(0)} G(A) = G(A)$ and then Item 2 is an immediate consequence of Item 1.
STEP 1: For every small extension in $\mathbf{NA}_{p-1}^q$,

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

and every $b\in F(B)$ we have either $\alpha^{-1}(b)=\emptyset$ or $\theta(\alpha^{-1}(b))=\alpha^{-1}(\theta(b))$.

In fact we have a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha} & F(B) \\ \Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\theta} \\ G(A) & \xrightarrow{\alpha} & G(B) \end{array}$$

and compatible transitive actions of the abelian groups $F(I)$, $G(I)$ on the fibres of the horizontal maps. Since $F(I)\to G(I)$ is surjective this proves Step 1.

Step 2: Let

$$0{\longrightarrow}I\xrightarrow{\iota}A\xrightarrow{\alpha}B{\longrightarrow}0$$

be a small extension in $\mathbf{NA}^q_{p-1}$ and $b\in F(B)$. Then b lifts to $F(A)$ if and only if $\theta(b)$ lifts to $G(A)$.

The *only if* part is trivial, let's prove the *if* part. If $\theta(b)$ lifts to $G(A)$ then $ob(\theta(b))=0$ in $G(I[1])$; since the obstruction maps commute with natural transformation of functors and $F(I[1])\to G(I[1])$ is injective, also $ob(b)=0$ in $F(I[1])$ and then b lifts to $F(A)$.

Step 3: For every surjective morphism $\beta\colon A\to C$ in the category $\mathbf{NA}^q_{p-1}$, the morphism

$$(\alpha,\theta)\colon F(A)\to F(C)\times_{G(C)}G(A)$$

is surjective.

Let J be the kernel of β and consider the sequence of homogeneous differential ideals $J=J_0\supset J_1=AJ_0\supset J_2=AJ_1\cdots$. Since A is nilpotent we have $J_n\neq 0$ and $J_{n+1}=0$ for some $n\geq 0$. Denoting by $I=J_n$ and $B=A/I$ we have a small extension

$$0{\longrightarrow}I{\longrightarrow}A\xrightarrow{\alpha}B{\longrightarrow}0$$

By induction on $\dim_{\mathbb{K}}A$ we can assume that the natural morphism $F(B)\to F(C)\times_{G(C)}G(B)$ is surjective and therefore it is sufficient to prove that $F(A)\to F(B)\times_{G(B)}G(A)$ is surjective.

Let $\tilde{a}\in G(A)$ be fixed element and let $b\in F(B)$ such that $\theta(b)=\alpha(\tilde{a})$. By Step 2 $\alpha^{-1}(b)$ is not empty and then by Step 1 $\tilde{a}\in\theta(F(A))$.

Step 4: For every surjective morphism $f\colon A\to B$ in the category $\mathbf{NA}^q_p$ and every $a\in F(A)$ we define

$$S_F(a,f)=\{\xi\in F(A\times_B A)\,|\,\xi\mapsto(a,a)\in F(A)\times_{F(B)}F(A)\subset F(A)\times F(A)\}.$$

By definition, if f is a small extension and $I=\ker f$ then $S_F(a,f)$ is naturally isomorphic to the stabilizer of a under the action of $F(I)$ on the fibre $f^{-1}(f(a))$. It is also clear that:

1. $\theta(S_F(a,f))\subset S_G(\theta(a),f)$.
2. If $\alpha\colon B\to C$ is a surjective morphism if $\mathbf{NA}$, then $S_F(a,f)=h^{-1}(S_F(a,\alpha f))$, where $h\colon F(A\times_B A)\to F(A\times_C A)$ is induced by the natural inclusions $A\times_B A\subset A\times_C A$.

Step 5: For every surjective morphism $f\colon A\to B$ in $\mathbf{NA}^q_p$ and every $a\in F(A)$ the map $\theta\colon S_F(a,f)\to S_G(\theta(a),f)$ is surjective.

This is trivially true if $B=0$, we prove the general assertion by induction on $\dim_{\mathbb{K}}B$. Let

$$0{\longrightarrow}I{\longrightarrow}B\xrightarrow{\alpha}C{\longrightarrow}0$$

be a small extension with $I\neq 0$, set $g=\alpha f$ and denote by $h\colon A\times_C A\to I$ the surjective morphism in $\mathbf{NA}^q_p$ defined by $h(a_1,a_2)=f(a_1)-f(a_2)$; we have an exact sequence

$$0{\longrightarrow}A\times_B A\xrightarrow{\iota}A\times_C A\xrightarrow{h}I{\longrightarrow}0.$$

According to 2a of V.59 the maps

$$F(A\times_B A)\to F(A\times_C A)\cap h^{-1}(0);\qquad S_F(a,f)\to S_F(a,g)\cap h^{-1}(0)$$

are surjective.

Let $\tilde{\xi} \in S_G(\theta(a), f)$ and let $\eta \in S_F(a, g)$ such that $\theta(\eta) = \iota(\tilde{\xi})$. Since $F(I) = G(I)$ we have $h(\eta) = 0$ and then η lifts to some $\xi_1 \in S_F(a, f)$. According to Theorem V.69 there exist surjective maps commuting with θ

$$F(A \times_B A) \times F(I[-1]) \xrightarrow{\varrho} F(A \times_B A) \times_{F(A \times_C A)} F(A \times_B A)$$

$$G(A \times_B A) \times G(I[-1]) \xrightarrow{\varrho} G(A \times_B A) \times_{G(A \times_C A)} G(A \times_B A)$$

Since $F(I[-1]) \to G(I[-1])$ is surjective there exists $v \in F(I[-1])$ such that $\varrho(\theta(\xi_1), \theta(v)) = (\theta(\xi_1), \tilde{\xi})$; defining $\xi \in F(A \times_B A)$ by the formula $\varrho(\xi_1, v) = (\xi_1, \xi)$ we get $\theta(\xi) = \tilde{\xi}$ and then $\xi \in S_F(a, f)$.

Step 6: For every $A \in \mathbf{NA}_p^q$ the map $\theta\colon F(A) \to G(A)$ is injective.

According to Lemma V.68 this is true if $A^2 = 0$; if $A^2 \neq 0$ we can suppose by induction that there exists a small extension

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\alpha} B \longrightarrow 0$$

with $I \neq 0$ and $\theta\colon F(B) \to G(B)$ injective.

Let $a_1, a_2 \in F(A)$ be two elements such that $\theta(a_1) = \theta(a_2)$; by assumption $f(a_1) = f(a_2)$ and then there exists $t \in F(I)$ such that $\vartheta_F(a_1, t) = (a_1, a_2)$. Since ϑ is a natural transformation $\vartheta_G(\theta(a_1), \theta(t)) = (\theta(a_1), \theta(a_2))$ and then $\theta(t) \in S_G(\theta(a_1), \alpha)$. By Step 5 there exists $s \in S_F(a_1, \alpha)$ such that $\theta(s) = \theta(t)$ and by injectivity of $\theta\colon F(I) \to G(I)$ we get $s = t$ and then $a_1 = a_2$.

□

As an immediate consequence we have:

Corollary V.72. *A morphism of deformation functors $\theta\colon F \to G$ is an isomorphism if and only if it is a quasiisomorphism.*

Proof of Theorem V.51. We apply Theorem V.71 to the morphism of deformation functors $\theta = \mathrm{Def}_\phi\colon \mathrm{Def}_L \to \mathrm{Def}_N$.

According to Theorem V.66, the first item of V.51 is exactly the first item of V.71 for $p = 1, q = 0$, while the second item of V.51 is exactly the third item of V.71 for $p = q = 0$. □

9. Historical survey, V

The material Sections 1, 2 and 3 is standard and well exposed in the literature about Lie algebras; in Sections 4, 5 and 6 we follows the approach of [**52**], while the material of Sections 7 and 8 comes from [**53**].

Some remarks on the introduction of this Lecture:

A) Given a deformation problem, in general it is not an easy task to find a factorization as in the introduction and in some cases it is still unknown.

B) Even in the simplest examples, the governing DGLA is only defined up to (non canonical) quasiisomorphism and then the Theorem V.51 is a necessary background for the whole theory.

For example, there are very good reasons to consider, for the study of deformations of a compact complex manifold M, the DGLA $L = \oplus L^i$, where L^i is the completion of $\Gamma(M, \mathcal{A}^{0,i}(T_M))$ is a suitable Sobolev's norm. According to elliptic regularity the inclusion $\mathrm{KS}(M) \subset L$ is a quasiisomorphism of DGLA.

In general a correct procedure gives, for every deformation problem P with associated deformation functor Def_P, a connected subcategory $\mathbf{P} \subset \mathbf{DGLA}$ with the following properties:

(1) If L is an object of $\mathbf{P}$ then $\mathrm{Def}_L = \mathrm{Def}_P$.
(2) Every morphism in $\mathbf{P}$ is a quasiisomorphism of DGLA.
(3) If $\mathrm{Mor}_{\mathbf{P}}(L, N) \neq \emptyset$ then the induced isomorphism $\mathrm{Def}_\alpha\colon \mathrm{Def}_L \to \mathrm{Def}_N$ is independent from the choice of $\alpha \in \mathrm{Mor}_{\mathbf{P}}(L, N)$.

C) It may happen that two people, say Circino and Olibri, starting from the same deformation problem, get two non-quasiisomorphic DGLA governing the problem. This is possible because the DGLA governs an extended (or derived) deformation problem. If Circino and Olibri have in mind two different extensions of the problem then they get different DGLA.
D) Although the interpretation of deformation problems in terms of solutions of Maurer-Cartan equation is very useful on its own, in many situation it is unavoidable to recognize that the category of DGLA is too rigid for a "good" theory. The appropriate way of extending this category will be the introduction of L_∞-algebras; these new objects will be described in Lecture IX.

LECTURE VI

Kähler manifolds

This chapter provides a basic introduction to Kähler manifolds. We first study the local theory, following essentially Weil's book [**80**] and then, assuming harmonic and elliptic theory, we give a proof of the $\partial\overline{\partial}$-lemma which is presented both in the classical version (Theorem VI.37, Item 2) and in the "homological" version (Theorem VI.37, Item 1).
The material of this Lecture is widely present in the literature, with the possible exception of the homological version of $\partial\overline{\partial}$-lemma; I only tried to simplify the presentation and some proofs. The main references are [**80**], [**81**] and [**11**]

1. Covectors on complex vector spaces

Given a complex vector space E of dimension n we denote by:

- $E^\vee = \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ its dual.
- $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$, with the structure of $\mathbb{C}$-vector space induced by the scalar multiplication $a(v \otimes b) = v \otimes ab$.
- $\overline{E}$ its complex conjugate.

The conjugate $\overline{E}$ is defined as the set of formal symbols $\overline{v}$, $v \in E$ with the vector space structure given by

$$\overline{v} + \overline{w} = \overline{v + w}, \qquad a\overline{v} = \overline{\overline{a}v}.$$

The conjugation $\overline{}\colon E \to \overline{E}$, $v \mapsto \overline{v}$ is a $\mathbb{R}$-linear isomorphism.
There exists a list of natural isomorphisms (details left as exercise)

(1) $(E_{\mathbb{C}})^\vee = (E^\vee)_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})$
(2) $\overline{E^\vee} = \overline{E}^\vee$ given by $\overline{f}(\overline{v}) = \overline{f(v)}$, $f \in E^\vee$, $v \in E$.

(3) $E \oplus \overline{E} \to E_{\mathbb{C}}$, $\quad (v, \overline{w}) \mapsto v \otimes 1 - iv \otimes i + w \otimes 1 + iw \otimes i$, being i a square root of -1.
(4) $E^\vee \oplus \overline{E^\vee} \to E^\vee_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{C})$, $\quad (f, \overline{g})(v) = f(v) + \overline{g(v)}$.

Under these isomorphisms, the image of $E^\vee$ (resp.: $\overline{E}^\vee$) inside $E^\vee_{\mathbb{C}}$ is the subspace of f such that $f(iv) = if(v)$ (resp.: $f(iv) = -if(v)$). Moreover $E^\vee = \overline{E}^\perp$, $\overline{E}^\vee = E^\perp$.

For $0 \le p, q \le n$ we set $\mathcal{A}^{p,q} = \bigwedge^p E^\vee \otimes \bigwedge^q \overline{E}^\vee$: this is called the space of (p,q)-covectors of E. We also set $\mathcal{A}^p = \oplus_{a+b=p}\mathcal{A}^{a,b}$ (the space of p-covectors) and $\mathcal{A} = \oplus_{a,b}\mathcal{A}^{a,b}$. Denote by $P_{a,b}\colon \mathcal{A} \to \mathcal{A}^{a,b}$, $P_p\colon \mathcal{A} \to \mathcal{A}^p$ the projections.
If $z_1, \ldots, z_n$ is a basis of $E^\vee$ then $\overline{z_1}, \ldots, \overline{z_n}$ is a basis of $\overline{E}^\vee$ and therefore

$$z_{i_1} \wedge \ldots \wedge z_{i_p} \wedge \overline{z_{j_1}} \wedge \ldots \wedge \overline{z_{j_q}}, \qquad i_1 < \ldots < i_p,\, j_1 < \ldots < j_q$$

is a basis of $\mathcal{A}^{p,q}$. Since $E^\vee_{\mathbb{C}} = E^\vee \oplus \overline{E}^\vee$, we have $\bigwedge^* E^\vee_{\mathbb{C}} = \mathcal{A}$.
The complex conjugation is defined in $\mathcal{A}$ and gives a $\mathbb{R}$-linear isomorphism $\overline{}\colon \mathcal{A} \to \mathcal{A}$. On the above basis, the conjugation acts as

$$\overline{z_{i_1} \wedge \ldots \wedge z_{i_p} \wedge \overline{z_{j_1}} \wedge \ldots \wedge \overline{z_{j_q}}} = (-1)^{pq} z_{j_1} \wedge \ldots \wedge z_{j_q} \wedge \overline{z_{i_1}} \wedge \ldots \wedge \overline{z_{i_p}}.$$

Since $\overline{\mathcal{A}^{a,b}} = \mathcal{A}^{b,a}$, we have $P_{a,b}(\overline{\eta}) = \overline{P_{b,a}(\eta)}$.

Definition VI.1. The operator $C\colon \mathcal{A} \to \mathcal{A}$ is defined by the formula

$$C = \sum_{a,b} i^{a-b} P_{a,b}.$$

Note that $\overline{C(u)} = C(\overline{u})$ (i.e. C is a real operator) and $C^2 = \sum_p (-1)^p P_p$.

2. The exterior algebra of a Hermitian space

Let E be a complex vector space of dimension n. A Hermitian form on E is a $\mathbb{R}$-bilinear map $h\colon E \times E \to \mathbb{C}$ satisfying the conditions

(1) $h(av, w) = ah(v, w)$, $h(v, aw) = \overline{a}h(v, w)$, $a \in \mathbb{C}$, $v, w \in E$.
(2) $h(w, v) = \overline{h(v, w)}$, $v, w \in E$.

Note that $h(v, v) \in \mathbb{R}$ for every v. h is called positive definite if $h(v, v) > 0$ for every $v \neq 0$.

Definition VI.2. A *Hermitian space* is a pair (E, h) where h is a positive definite Hermitian form on E.

It is well known that a Hermitian form h on a finite dimensional vector space E is positive definite if and only if it admits a unitary basis, i.e. a basis $e_1, \ldots, e_n$ of E such that $h(e_i, e_j) = \delta_{ij}$.
Every Hermitian space (E, h) induces canonical Hermitian structures on the associated vector spaces. For example

$$\overline{h}\colon \overline{E} \times \overline{E} \to \mathbb{C}, \qquad \overline{h}(\overline{v}, \overline{w}) = \overline{h(v, w)}$$

and

$$h^p\colon \textstyle\bigwedge^p E \times \bigwedge^p E \to \mathbb{C}, \qquad h^p(v_1 \wedge \ldots \wedge v_p, w_1 \wedge \ldots \wedge w_p) = \det(h(v_i, w_j))$$

are Hermitian forms. If $e_1, \ldots, e_n$ is a unitary basis of E then $\overline{e_1}, \ldots, \overline{e_n}$ is a unitary basis for $\overline{h}$ and $e_{i_1} \wedge \ldots \wedge e_{i_p}$, $i_1 < \ldots < i_p$, is a unitary basis for h^p.

Similarly, if (F, k) is another Hermitian space then we have natural Hermitian structures on $E \otimes F$ and $\mathrm{Hom}_{\mathbb{C}}(E, F)$ given by

$$hk\colon E \otimes F \to \mathbb{C}, \qquad hk(v \otimes f, w \otimes g) = h(v, w)k(f, g)$$

$$h^\vee k\colon \mathrm{Hom}_{\mathbb{C}}(E, F) \to \mathbb{C}, \qquad h^\vee k(f, g) = \sum_{i=1}^n k(f(e_i), g(e_i))$$

where e_i is a unitary basis of E. It is an easy exercise (left to the reader) to prove that $h^\vee k$ is well defined and positive definite.
In particular the complex dual $E^\vee$ is a Hermitian space and the dual basis of a unitary basis for h is a unitary basis for $h^\vee$.

Let $e_1, \ldots, e_n$ be a basis of E, $z_1, \ldots, z_n \in E^\vee$ its dual basis; then

$$h(v, w) = \sum_{i,j} h_{ij} z_i(v)\overline{z_j(w)}$$

where $h_{ij} = h(e_i, e_j)$. We have $h_{ji} = \overline{h_{ij}}$ and the basis is unitary if and only if $h_{ij} = \delta_{ij}$. We then write $h = \sum_{ij} h_{ij} z_i \otimes \overline{z_j}$; in doing this we also consider h as an element of $E^\vee \otimes \overline{E}^\vee = (E \otimes \overline{E})^\vee$.
Taking the real and the imaginary part of h we have $h = \rho - i\omega$, with $\rho, \omega\colon E \times E \to \mathbb{R}$. It is immediate to observe that ρ is symmetric, ω is skewsymmetric and

$$\rho(iv, iw) = \rho(v, w), \quad \omega(iv, iw) = \omega(v, w), \quad \rho(iv, w) = \omega(v, w).$$

Since $z_i \wedge \overline{z_j} = z_i \otimes \overline{z_j} - \overline{z_j} \otimes z_i$, we can write

$$\omega = \frac{i}{2}(h - \overline{h}) = \frac{i}{2} \sum_{ij} h_{ij} z_i \wedge \overline{z_j} \in \mathcal{A}^{1,1}.$$

Note that ω is real, i.e. $\overline{\omega} = \omega$, and the Hermitian form is positive definite if and only if for every $v \neq 0$, $h(v,v) = \rho(v,v) = \omega(v, iv) > 0$. The basis $e_1, \dots, e_n$ is unitary if and only if $\omega = \dfrac{i}{2}\sum_i z_i \wedge \overline{z_i}$.

Let now $e_1, \dots, e_n$ be a fixed unitary basis of a Hermitian space (E, h) with dual basis $z_1, \dots, z_n$ and denote $u_j = \dfrac{i}{2} z_j \wedge \overline{z_j}$; if $z_j = x_j + iy_j$ then $u_j = x_j \wedge y_j$ and

$$\frac{\omega^{\wedge n}}{n!} = u_1 \wedge \dots \wedge u_n = x_1 \wedge y_1 \wedge \dots \wedge x_n \wedge y_n.$$

Since $x_1, y_1, \dots, x_n, y_n$ is a system of coordinates on E, considered as a real oriented vector space of dimension $2n$ and the quadratic form ρ is written in this coordinates

$$\rho(v,v) = \sum_{i=1}^{n} (x_i(v)^2 + y_i(v)^2),$$

we get from the above formula that $\omega^{\wedge n}/n! \in \bigwedge_{\mathbb{R}}^{2n} \mathrm{Hom}_{\mathbb{R}}(E, \mathbb{R})$ is the volume element associated to the scalar product ρ on E.

For notational simplicity, if $A = \{a_1, \dots, a_p\} \subset \{1, \dots, n\}$ and $a_1 < a_2 < \dots < a_p$, we denote $|A| = p$ and

$$z_A = z_{a_1} \wedge \dots \wedge z_{a_p}, \quad \overline{z}_A = \overline{z}_{a_1} \wedge \dots \wedge \overline{z}_{a_p}, \quad u_A = u_{a_1} \wedge \dots \wedge u_{a_p}.$$

For every decomposition of $\{1, \dots, n\} = A \cup B \cup M \cup N$ into four disjoint subsets, we denote

$$z_{A,B,M,N} = \frac{1}{\sqrt{2^{|A|+|B|}}} z_A \wedge \overline{z}_B \wedge u_M \in \mathcal{A}^{|A|+|M|, |B|+|M|}.$$

These elements give a basis of $\mathcal{A}$ which we call *standard basis*.
Note that $\overline{z_{A,B,M,N}} = (-1)^{|A|\,|B|} z_{B,A,M,N}$.

Definition VI.3. The $\mathbb{C}$-linear operator $*\colon \mathcal{A}^{p,q} \to \mathcal{A}^{n-q,n-p}$ is defined as

$$*z_{A,B,M,N} = sgn(A,B) i^{|A|+|B|} z_{A,B,N,M},$$

where $sgn(A,B) = \pm 1$ is the sign compatible with the formulas

$$z_{A,B,M,N} \wedge *\overline{z_{A,B,M,N}} = z_{A,B,M,N} \wedge \overline{*z_{A,B,M,N}} = u_1 \wedge \dots \wedge u_n. \tag{2}$$

$$C^{-1} * z_{A,B,M,N} = (-1)^{\frac{(|A|+|B|)(|A|+|B|+1)}{2}} z_{A,B,N,M} = (-1)^{\frac{(p+q)(p+q+1)}{2} + |M|} z_{A,B,N,M}. \tag{3}$$

Exercise VI.4. Verify that Definition VI.3 is well posed. △

In particular

$$*^2 z_{A,B,M,N} = (-1)^{|A|+|B|} z_{A,B,M,N} = (-1)^{|A|+|B|+2|M|} z_{A,B,M,N}$$

and then

$$(C^{-1}*)^2 = Id, \qquad *^2 = C^2 = \sum_p (-1)^p P_p.$$

If we denote $vol\colon \mathbb{C} \to \mathcal{A}^{n,n}$ the multiplication for the "volume element" $\omega^{\wedge n}/n!$, then vol is an isomorphism and we can consider the $\mathbb{R}$-bilinear maps

$$(\ ,\ )\colon \mathcal{A}^{a,b} \times \mathcal{A}^{a,b} \to \mathbb{C}, \qquad (v,w) = vol^{-1}(v \wedge \overline{*w}) = vol^{-1}(v \wedge *\overline{w}).$$

Clearly $(\ ,\ )$ is $\mathbb{C}$-linear on the first member and $\mathbb{C}$-antilinear in the second; since

$$(z_{A,B,M,N}, z_{A',B',M',N'}) = \begin{cases} 1 & \text{if } A = A', B = B', M = M', N = N' \\ 0 & \text{otherwise} \end{cases}$$

we have that $(\ ,\ )$ is a positive definite Hermitian form with the $z_{A,B,M,N}$'s, $|A| + |M| = a$, $|B| + |M| = b$, a unitary basis; since $*$ sends unitary basis into unitary basis we also get that $*\colon \mathcal{A}^{a,b} \to \mathcal{A}^{n-b,n-a}$ is an isometry.

LEMMA VI.5. *The Hermitian form* $(\, ,\,)$ *is the Hermitian form associated to the Hermitian space* $(E, h/2)$. *In particular* $(\, ,\,)$ *and* $*$ *are independent from the choice of the unitary basis* $e_1, \dots, e_n$.

PROOF. The basis $\sqrt{2}e_1, \dots, \sqrt{2}e_n$ is a unitary basis for $h/2$ and then the standard basis is a unitary basis for the associated Hermitian structures on $\mathcal{A}$.
From the formula $(v, w)\omega^{\wedge n} = n!(v \wedge \overline{*w})$ and from the fact that the wedge product is nondegenerate follows that $*$ depends only by ω and $(\, ,\,)$. □

Consider now, for every $j = 1, \dots, n$, the linear operators

$$L_j \colon \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q+1}, \quad L_j(\eta) = \eta \wedge u_j,$$

$$\Lambda_j \colon \mathcal{A}^{p,q} \to \mathcal{A}^{p-1,q-1}, \quad \Lambda_j(\eta) = \eta \dashv \left(\frac{2}{i}\overline{e_j} \wedge e_j\right),$$

where $\dashv$ denotes the contraction on the right. More concretely, in the standard basis

$$L_i z_{A,B,M,N} = \begin{cases} z_{A,B,M\cup\{i\},N-\{i\}} & \text{if } i \in N \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda_i z_{A,B,M,N} = \begin{cases} z_{A,B,M-\{i\},N\cup\{i\}} & \text{if } i \in M \\ 0 & \text{otherwise} \end{cases}$$

It is therefore immediate to observe that $L_i * = *\Lambda_i$ and $*L_i = \Lambda_i *$. Setting $L = \sum_i L_i$, $\Lambda = \sum_i \Lambda_i$ we have therefore

$$L(\eta) = \eta \wedge \omega, \qquad \Lambda = *^{-1} L * = * L *^{-1}.$$

LEMMA VI.6. *The operators* L *and* Λ *do not depend from the choice of the unitary basis.*

PROOF. ω and $*$ do not depend. □

PROPOSITION VI.7. *The following commuting relations hold:*

$$[L, C] = 0, \quad [\Lambda, C] = 0, \quad [*, C] = 0, \quad [\Lambda, L] = \sum_{p=0}^{2n} (n-p) P_p.$$

PROOF. Only the last is nontrivial, we have:

$$L z_{A,B,M,N} = \sum_{i\in N} z_{A,B,M\cup\{i\},N-\{i\}}, \qquad \Lambda z_{A,B,M,N} = \sum_{i\in M} z_{A,B,M-\{i\},N\cup\{i\}},$$

$$\Lambda L z_{A,B,M,N} = \sum_{i\in N} z_{A,B,M,N} + \sum_{j\in M}\sum_{i\in N} z_{A,B,M\cup\{i\}-\{j\},N\cup\{j\}-\{i\}},$$

$$L\Lambda z_{A,B,M,N} = \sum_{i\in M} z_{A,B,M,N} + \sum_{j\in M}\sum_{i\in N} z_{A,B,M\cup\{i\}-\{j\},N\cup\{j\}-\{i\}}.$$

Therefore we get

$$(\Lambda L - L\Lambda) z_{A,B,M,N} = (|N| - |M|) z_{A,B,M,N} = (n - |A| - |B| - 2|M|) z_{A,B,M,N}.$$

and then

$$[\Lambda, L] = \sum_{p=0}^{2n} (n-p) P_p.$$

□

3. The Lefschetz decomposition

The aim of this section is to study the structure of $\bigwedge^{*,*} E^\vee$ as a module over the algebra Φ generated by the linear operators $C^{-1}*, L, \Lambda$.

In the notation of the previous section, it is immediate to see that there exists a direct sum decomposition of Φ-modules $\bigwedge^{*,*} E^\vee = \bigoplus V_{A,B}$, where $V_{A,B}$ is the subspace generated by the $2^{n-|A|-|B|}$ elements $z_{A,B,M,N}$, being A, B fixed.

It is also clear that every $V_{A,B}$ is isomorphic to one of the Φ-modules $V(h,\tau)$, $h \in \mathbb{N}$, $\tau = \pm 1$, defined in the following way:

(1) $V(h,\tau)$ is the $\mathbb{C}$-vector space with basis u_M, $M \subset \{1, \dots, h\}$.
(2) The linear operators L, Λ and $C^{-1}*$ act on $V(h,\tau)$ as

$$Lu_M = \sum_{i \notin M} u_{M \cup \{i\}}, \quad \Lambda u_M = \sum_{i \in M} u_{M - \{i\}}, \quad C^{-1} * u_M = \tau u_{M^c},$$

where $M^c = \{1, \dots, h\} - M$ denotes the complement of M.

We have a direct sum decomposition

$$V(h,\tau) = \bigoplus_{\alpha \equiv h \pmod 2} V_\alpha,$$

where V_α is the subspace generated by the u_M with $|M^c| - |M| = \alpha$. An element of V_α is called homogeneous of weight α. Set $P_\alpha \colon V(h,\tau) \to V_\alpha$ the projection.

Note that $L\colon V_\alpha \to V_{\alpha-2}$, $\Lambda\colon V_\alpha \to V_{\alpha+2}$ and $C^{-1}*\colon V_\alpha \to V_{-\alpha}$.

We have already seen that

$$[\Lambda, L] = \sum_{\alpha \in \mathbb{Z}} \alpha P_\alpha, \qquad LC^{-1}* = C^{-1} * \Lambda, \qquad C^{-1} * L = \Lambda C^{-1} * .$$

A simple combinatorial argument shows that for every $r \geq 0$,

$$L^r u_M = r! \sum_{M \subset N, |N| = |M| + r} u_N.$$

Lemma VI.8. *For every $r \geq 1$ we have*

$$[\Lambda, L^r] = \sum_\alpha r(\alpha - r + 1) L^{r-1} P_\alpha.$$

Proof. This has already done for $r = 1$, we prove the general statement for induction on r. We have

$$[\Lambda, L^{r+1}] = [\Lambda, L^r]L + L^r[\Lambda, L] = \sum_\alpha r(\alpha - r + 1)L^{r-1}P_\alpha L + \sum_\alpha \alpha P_\alpha.$$

Since $P_\alpha L = LP_{\alpha+2}$ we have

$$[\Lambda, L^{r+1}] = \sum_\alpha r(\alpha - r + 1)L^r P_{\alpha+2} + \sum_\alpha \alpha P_\alpha = \sum_\alpha (r(\alpha - r - 1) + \alpha) L^r P_\alpha.$$

□

Definition VI.9. A homogeneous vector $v \in V_\alpha$ is called *primitive* if $\Lambda v = 0$.

Proposition VI.10. *Let $v \in V_\alpha$ be a primitive element, then:*

(1) $L^q v = 0$ *for every* $q \geq \max(\alpha + 1, 0)$. *In particular if* $\alpha < 0$ *then* $v = L^0 v = 0$.
(2) *If* $\alpha \geq 0$, *then for every* $p > q \geq 0$

$$\Lambda^{p-q} L^p v = \prod_{r=q+1}^{p} r(\alpha - r + 1) L^q v;$$

in particular $\Lambda^\alpha L^\alpha v = \alpha!^2 v$.

Proof. We first note that for $s, r \geq 1$

$$\Lambda^s L^r v = \Lambda^{s-1}[\Lambda, L^r]v = r(\alpha - r + 1)\Lambda^{s-1}L^{r-1}v.$$

and then for every $p > q \geq 0$

$$\Lambda^{p-q}L^p v = \prod_{r=q+1}^{p} r(\alpha - r + 1)L^q v.$$

If $p > q > \alpha$ then $r(\alpha - r + 1) \neq 0$ for every $r > q$ and then $L^q v = 0$ if and only if $\Lambda^{p-q}L^p v = 0$: taking $p >> 0$ we get the required vanishing. $\square$

Lemma VI.11. *Let $\alpha \geq 0$, $m = (h-\alpha)/2$ and $v = \sum_{|M|=m} a_M u_M \in V_\alpha$, $a_M \in \mathbb{C}$. If v is primitive, then for every M*

$$a_M = (-1)^m \sum_{N \subset M^c, |N|=m} a_N.$$

Proof. For $m = 0$ the above equality becomes $a_\emptyset = a_\emptyset$ and therefore we can assume $m > 0$. Let $M \subset \{1, \ldots, h\}$ be a fixed subset of cardinality m, since

$$0 = \Lambda v = \sum_{|H|=m} a_H \sum_{i \in H} u_{H-\{i\}} = \sum_{|N|=m-1} u_N \sum_{i \notin N} a_{N\cup\{i\}}$$

we get for every $N \subset \{1, \ldots, h\}$ of cardinality $m-1$ the equality

$$R_N: \qquad \sum_{i \in M-N} a_{N\cup\{i\}} = -\sum_{i \notin M\cup N} a_{N\cup\{i\}}.$$

For every $0 \leq r \leq m$ denote by

$$S_r = \sum_{|H|=m, |H\cap M|=r} a_H.$$

Fixing an integer $1 \leq r \leq m$ and taking the sum of the equalities R_N, for all N such that $|N \cap M| = r-1$ we get

$$rS_r = -(m-r+1)S_{r-1}$$

and then

$$a_M = S_m = -\frac{S_{m-1}}{m} = \frac{2S_{m-2}}{m(m-1)} = \ldots = (-1)^m \frac{m!}{m!} S_0 = (-1)^m \sum_{N\subset M^c, |N|=m} a_N.$$

$\square$

Lemma VI.12. *If $v \in V_\alpha$, $\alpha \geq 0$, is primitive, then for every $0 \leq r \leq \alpha$*

$$C^{-1} * L^r v = \tau(-1)^m \frac{r!}{(\alpha - r)!} L^{\alpha - r} v,$$

where $m = (h-\alpha)/2$.

Proof. Consider first the case $r = 0$; writing $v = \sum a_N u_N$ with $|N| = m$, $a_N \in \mathbb{C}$, we have:

$$\frac{L^\alpha v}{\alpha!} = \sum_{|N|=m} a_N \sum_{\substack{N\subset M\\ |M|=m+\alpha}} u_M = \sum_{|N|=m} a_N \sum_{\substack{M\subset N^c\\ |M|=m}} u_{M^c} = \sum_{|M|=m} u_{M^c} \sum_{\substack{N\subset M^c\\ |N|=m}} a_N.$$

$$C^{-1} * v = \tau \sum_{|M|=m} a_M u_{M^c}.$$

The equality $C^{-1} * v = \tau(\alpha!)^{-1}L^\alpha v$ follows immediately from Lemma VI.11. If $r \geq 1$ then

$$C^{-1} * L^r v = \Lambda^r C^{-1} * v = \frac{\tau(-1)^m}{\alpha!}\Lambda^r L^\alpha v.$$

Using the formula of VI.10 we get

$$C^{-1} * L^r v = \frac{\tau(-1)^m}{\alpha!} \prod_{j=\alpha-r+1}^{\alpha} j(\alpha - j + 1) L^{\alpha - r} v = \tau(-1)^m \frac{r!}{(\alpha - r)!} L^{\alpha - r} v.$$

□

THEOREM VI.13. (Lefschetz decomposition)

(1) *Every $v \in V_\alpha$ can be written in a unique way as*
$$v = \sum_{r \geq \max(-\alpha,0)} L^r v_r$$
with every $v_r \in V_{\alpha+2r}$ primitive.

(2) *For a fixed $q \geq h$ there exist noncommutative polynomials $G^q_{\alpha,r}(\Lambda, L)$ with rational coefficients such that $v_r = G^q_{\alpha,r}(\Lambda, L)v$ for every $v \in V_\alpha$.*

PROOF. Assume first $\alpha \geq 0$, we prove the existence of the decomposition $v = \sum_{r\geq 0} L^r v_r$ as above by induction on the minimum q such that $\Lambda^q v = 0$. If $q = 1$ then v is already primitive. If $\Lambda^{q+1} v = 0$ then $w = \Lambda^q v \in V_{\alpha+2q}$ is primitive and then, setting $\gamma = \prod_{r=1}^q r(\alpha + 2q - r + 1)$, we have $\gamma > 0$ and
$$\Lambda^q \left(v - L^q \frac{w}{\gamma} \right) = w - \Lambda^q L^q \frac{w}{\gamma} = 0.$$
This prove the existence when $\alpha \geq 0$. If $\alpha < 0$ then $C^{-1} * v \in V_{-\alpha}$ and we can write:
$$C^{-1} * v = \sum_{r\geq 0} L^r v_r, \qquad v = \sum_{r\geq 0} C^{-1} * L^r v_r, \quad v_r \in V_{-\alpha+2r}.$$
According to Lemma VI.12
$$v = \sum_{r\geq 0} \gamma_r L^{-\alpha+r} v_r = \sum_{r\geq -\alpha} \gamma_{r+\alpha} L^r v_r$$
for suitable rational coefficients γ_r.

The unicity of the decomposition and item 2 are proved at the same time. If
$$v = \sum_{r=\max(-\alpha,0)}^{q} L^r v_r$$
is a decomposition with every $v_r \in V_{\alpha+2r}$ primitive, then $L^{\alpha+q} v = L^{\alpha+2q} v_q$ and
$$v_q = \frac{1}{(\alpha + 2q)!^2} \Lambda^{\alpha+2q} L^{\alpha+2q} v_q = \frac{1}{(\alpha + 2q)!^2} \Lambda^{\alpha+2q} L^{\alpha+q} v.$$
Therefore v_q is uniquely determined by v and we can take $G^q_{\alpha,q} = (\alpha + 2q)!^{-2} \Lambda^{\alpha+2q} L^{\alpha+q}$. Since $v - L^q v_q = (1 - L^q G^q_{\alpha,q})v = \sum_{r=\max(-\alpha,0)}^{q-1} L^r v_r$ we can proceed by decreasing induction on q. □

COROLLARY VI.14. *$v \in V_\alpha$, $\alpha \geq 0$, is primitive if and only if $L^{\alpha+1} v = 0$.*

PROOF. Let $v = \sum_{r\geq 0} L^r v_r$ be the Lefschetz decomposition of v, then $\sum_{r>0} L^{\alpha+r+1} v_r$ is the Lefschetz decomposition of $L^{\alpha+1} v$. Therefore $L^{\alpha+1} v = 0$ if and only if $v = v_0$. □

It is clear that Theorem VI.13 and Corollary VI.14 hold also for every finite direct sum of Φ-modules of type $V(h, \tau)$.

For later use we reinterpret Lemma VI.12 for the Φ-module $\mathcal{A}$: we have
$$\mathcal{A} = \bigoplus_{A,B} V_{A,B}, \qquad V_{A,B} = V\left(n - |A| - |B|, (-1)^{\frac{(|A|+|B|)(|A|+|B|+1)}{2}}\right)$$
where the sum is taken over all pairs of disjoint subsets A, B of $\{1, \ldots, n\}$. The space $\mathcal{A}_\alpha = \bigoplus (V_{A,B})_\alpha$ is precisely the space $\bigoplus_a \mathcal{A}^{a,n-\alpha-a}$ of $(n - \alpha)$-covectors. We then get the following

LEMMA VI.15. *If $v \in \mathcal{A}$ is a primitive p-covector, $p \leq n$, then*
$$C^{-1} * L^r v = \begin{cases} (-1)^{\frac{p(p+1)}{2}} \dfrac{r!}{(n-p-r)!} L^{n-p-r} v & \textit{if } r \leq n - p \\ 0 & \textit{if } r > n - p \end{cases}$$

4. Kähler identities

Let M be a complex manifold of dimension n and denote by $\mathcal{A}^{*,*}$ the sheaf of differential forms on M. By definition $\mathcal{A}^{a,b}$ is the sheaf of sections of the complex vector bundle $\bigwedge^a T_M^\vee \otimes \bigwedge^b \overline{T_M}^\vee$. The operators $P_{a,b}$, P_p and C, defined on the fibres of the above bundles, extend in the obvious way to operators in the sheaf $\mathcal{A}^{*,*}$.

If $d\colon \mathcal{A}^{*,*} \to \mathcal{A}^{*,*}$ is the De Rham differential we denote:

$$d^C = C^{-1}dC, \qquad \partial = \frac{d + id^C}{2}, \qquad \overline{\partial} = \frac{d - id^C}{2},$$
$$d = Cd^C C^{-1}, \qquad d = \partial + \overline{\partial}, \qquad d^C = i(\overline{\partial} - \partial).$$

If η is a (p,q)-form then we can write $d\eta = \eta' + \eta''$ with $\eta' \in \mathcal{A}^{p+1,q}$, $\eta'' \in \mathcal{A}^{p,q+1}$ and then

$$d^C(\eta) = C^{-1}d(i^{p-q}\eta) = \frac{i^{p-q}}{i^{p-q+1}}\eta' + \frac{i^{p-q}}{i^{p-q-1}}\eta'' = i^{-1}\eta' + i\eta'', \qquad \partial\eta = \eta', \quad \overline{\partial}\eta = \eta''.$$

Since $0 = d^2 = \partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2$ we get $0 = \partial^2 = \partial\overline{\partial} + \overline{\partial}\partial = \overline{\partial}^2$ and then $(d^C)^2 = 0$, $dd^C = 2i\partial\overline{\partial} = -d^C d$.

Using the structure of graded Lie algebra on the space of $\mathbb{C}$-linear operators of the sheaf of graded algebras $\mathcal{A}^{*,*}$ (with the total degree $\overline{v} = a + b$ if $v \in \mathcal{A}^{a,b}$), the above relation can be rewritten as

$$[d,d] = dd + dd = 2d^2 = 0, \qquad [d^C, d^C] = [d, d^C] = [\partial, \partial] = [\overline{\partial}, \overline{\partial}] = [\partial, \overline{\partial}] = 0.$$

Note finally that d and C are real operators and then also d^C is; moreover $\overline{\partial}\overline{\eta} = \overline{\partial\eta}$.

A Hermitian metric on M is a positive definite Hermitian form h on the tangent vector bundle T_M. If $z_1, \dots, z_n$ are local holomorphic coordinates then $h_{ij} = h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right)$ is a smooth function and the matrix (h_{ij}) is Hermitian and positive definite. The local expression of h is then $h = \sum_{ij} h_{ij} dz_i \otimes d\overline{z}_j$ and the differential form

$$\omega = \frac{i}{2}\sum_{i,j} h_{ij} dz_i \wedge d\overline{z}_j \in \Gamma(M, \mathcal{A}^{1,1})$$

is globally definite and gives the imaginary part of $-h$; ω is called the (real, $(1,1)$) associated form to h.

The choice of a Hermitian metric on M induces, for every open subset $U \subset M$, linear operators

$$L\colon \Gamma(U, \mathcal{A}^{a,b}) \to \Gamma(U, \mathcal{A}^{a+1,b+1}), \qquad Lv = v \wedge \omega,$$
$$*\colon \Gamma(U, \mathcal{A}^{a,b}) \to \Gamma(U, \mathcal{A}^{n-b,n-a}),$$
$$\Lambda\colon \Gamma(U, \mathcal{A}^{a,b}) \to \Gamma(U, \mathcal{A}^{a-1,b-1}), \qquad \Lambda = *^{-1}L* = (C^{-1}*)^{-1}LC^{-1}*.$$

The commuting relations between them

$$[L, C] = [\Lambda, C] = [*, C] = [L, *^2] = 0, \qquad [\Lambda, L^r] = \sum_p r(n-p-r+1)P_p$$

are still valid.

A differential form v is primitive if $\Lambda v = 0$; the existence of the polynomials $G^n_{n-p,r}(\Lambda, L)$ (cf. Theorem VI.13) gives the existence and unicity of Lefschetz decomposition for every differential p-form

$$v = \sum_{r \geq \max(p-n,0)} L^r v_r, \qquad \Lambda v_r = 0.$$

We set:

$$\delta = -*d*, \qquad \delta^C = -*d^C* = C^{-1}\delta C,$$
$$\partial^* = -*\overline{\partial}* = \frac{\delta - i\delta^C}{2}, \qquad \overline{\partial}^* = -*\partial* = \frac{\delta + i\delta^C}{2}.$$

Definition VI.16. The Hermitian metric h is called a *Kähler metric* if $d\omega = 0$.

Almost all the good properties of Kähler metrics come from the following

THEOREM VI.17. (Kähler identities) *Let h be a Kähler metric on a complex manifold, then:*

$[L,d]=0$	$[L,d^C]=0$	$[L,\partial]=0$	$[L,\overline{\partial}]=0$
$[\Lambda,d]=-\delta^C$	$[\Lambda,d^C]=\delta$	$[\Lambda,\partial]=i\overline{\partial}^*$	$[\Lambda,\overline{\partial}]=-i\partial^*$
$[L,\delta]=d^C$	$[L,\delta^C]=-d$	$[L,\partial^*]=i\overline{\partial}$	$[L,\overline{\partial}^*]=-i\partial$
$[\Lambda,\delta]=0$	$[\Lambda,\delta^C]=0$	$[\Lambda,\partial^*]=0$	$[\Lambda,\overline{\partial}^*]=0$

PROOF. It is sufficient to prove that $[L,d]=0$ and $[\Lambda,d]=-\delta^C$. In fact, since $\Lambda = *^{-1}L* = *L*^{-1}$ we have $[\Lambda,\delta]+*[L,d]*=0$ and $[L,\delta]+*[\Lambda,d]*=0$: this will prove the first column. The second column follows from the first using the fact that C commutes with L and Λ. The last two columns are linear combinations of the first two.
If v is a p-form then, since $d\omega=0$,

$$[L,d]v = dv\wedge\omega - d(v\wedge\omega) = -(-1)^p v\wedge d\omega = 0.$$

According to the Lefschetz decomposition it is sufficient to prove that $[\Lambda,d]L^r u = -\delta^C L^r u$ for every $r\geq 0$ and every primitive p-form u ($p\leq n$). We first note that, being u primitive, $L^{n-p+1}u=0$ and then $L^{n-p+1}du = dL^{n-p+1}u = 0$. This implies that the Lefschetz decomposition of du is $du = u_0 + Lu_1$.
Setting $\alpha = n-p$, we have $u\in V_\alpha$, $u_0\in V_{\alpha-1}$, $u_1\in V_{\alpha+1}$:

$$[\Lambda,d]L^r u = \Lambda L^r du - d\Lambda L^r u = \Lambda L^r u_0 + \Lambda L^{r+1}u_1 - r(\alpha-r+1)dL^{r-1}u =$$
$$= r(\alpha-r)L^{r-1}u_0 + (r+1)(\alpha-r+1)L^r u_1 - r(\alpha-r+1)L^{r-1}u_0 - r(\alpha-r+1)L^r u_1 =$$
$$= -rL^{r-1}u_0 + (\alpha-r+1)L^r u_1.$$

On the other hand we have by VI.15

$$-\delta^C L^r u = C^{-1}*d*CL^r u = C^{-1}*dC^2C^{-1}*L^r u$$
$$= C^{-1}*dC^2(-1)^{p(p+1)/2}\frac{r!}{(\alpha-r)!}L^{\alpha-r}u$$

and then

$$-\delta^C L^r u = (-1)^{p(p-1)/2}\frac{r!}{(\alpha-r)!}C^{-1}*L^{\alpha-r}(u_0+Lu_1).$$

Again by VI.15,

$$C^{-1}*L^{\alpha-r}u_0 = (-1)^{(p+1)(p+2)/2}\frac{(\alpha-r)!}{(r-1)!}L^{r-1}u_0,$$
$$C^{-1}*L^{\alpha-r+1}u_1 = (-1)^{(p-1)p/2}\frac{(\alpha-r+1)!}{r!}L^r u_1.$$

Putting all the terms together we obtain the result. $\square$

COROLLARY VI.18. *If ω is the associated form of a Kähler metric h then $d\omega^{\wedge p} = \delta\omega^{\wedge p} = 0$ for every $p\geq 0$.*

PROOF. The equality $d\omega^{\wedge p}=0$ follows immediately from the Leibnitz rule. Since $\omega^{\wedge p}$ is a (p,p) form, we have $C\omega^{\wedge p} = \omega^{\wedge p}$ and then also $d^C\omega^{\wedge p}=0$.
We prove $\delta\omega^{\wedge p}=0$ by induction on p, being the result trivial when $p=0$. If $p>0$ we have

$$0 = d^C\omega^{\wedge p-1} = L\delta\omega^{\wedge p-1} - \delta L\omega^{\wedge p-1} = -\delta\omega^{\wedge p}.$$

$\square$

The gang of Laplacians is composed by:

(1) $\Delta_d = \Delta = [d,\delta] = d\delta+\delta d$.
(2) $\Delta_{d^C} = \Delta^C = C^{-1}\Delta C = [d^C,\delta^C] = d^C\delta^C + \delta^C d^C$.
(3) $\Delta_\partial = \square = [\partial,\partial^*] = \partial\partial^* + \partial^*\partial$.
(4) $\Delta_{\overline{\partial}} = \overline{\square} = [\overline{\partial},\overline{\partial}^*] = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$.

A straightforward computation shows that $\Delta + \Delta^C = 2\square + 2\overline{\square}$.

Corollary VI.19. *In the above notation, if h is a Kähler metric then:*

$$[d, \delta^C] = [d^C, \delta] = [\partial, \overline{\partial}^*] = [\overline{\partial}, \partial^*] = 0, \qquad \frac{1}{2}\Delta = \frac{1}{2}\Delta^C = \Box = \overline{\Box}.$$

In particular Δ is bihomogeneous of degree $(0,0)$.

Proof. According to Theorem VI.17 and the Jacobi identity we have

$$[d, \delta^C] = [d, [d, \Lambda]] = \frac{1}{2}[[d, d], \Lambda] = 0.$$

The proof of $[d^C, \delta] = [\partial, \overline{\partial}^*] = [\overline{\partial}, \partial^*] = 0$ is similar and left as exercise. For the equalities among Laplacians it is sufficient to shows that $\Delta = \Delta^C$ and $\Box = \overline{\Box}$. According to the Kähler identities

$$\Delta = [d, \delta] = [d, [\Lambda, d^C]] = [[d, \Lambda], d^C] + [\Lambda, [d, d^C]].$$

Since $[d, d^C] = dd^C + d^C d = 0$ we have

$$\Delta = [d, \delta] = [[d, \Lambda], d^C] = [\delta^C, d^C] = \Delta^C.$$

The proof of $\Box = \overline{\Box}$ is similar and it is left to the reader. $\Box$

Corollary VI.20. *In the above notation, if h is a Kähler metric, then Δ commutes with all the operators $P_{a,b}$, $*$, d, L, C, Λ, d^C, ∂, $\overline{\partial}$, δ, δ^C, ∂^*, $\overline{\partial}^*$.*

Proof. Since Δ is of type $(0,0)$ it is clear that commutes with the projections $P_{a,b}$. Recalling that $\delta = - * d *$ we get $d = * \delta *$ and then

$$*\Delta = *d\delta + *\delta d = - * d * d * + * \delta * \delta * = \delta d * + d\delta * = \Delta * .$$

$$[L, \Delta] = [L, [d, \delta]] = [[L, d], \delta] + [[L, \delta], d] = [d^C, d] = 0.$$

$$[d, \Delta] = [d, [d, \delta]] = \frac{1}{2}[[d, d], \delta] = 0.$$

Now it is sufficient to observe that all the operators in the statement belong to the $\mathbb{C}$-algebra generated by $P_{a,b}$, $*$, d and L. $\Box$

Definition VI.21. A p-form v is called *harmonic* if $\Delta v = 0$.

Corollary VI.22. *Let h be a Kähler metric and let $v = \sum_r L^r v_r$ be the Lefschetz decomposition of a p-form.*
Then v is harmonic if and only if v_r is harmonic for every r.

Proof. Since Δ commutes with L, if $\Delta v_r = 0$ for every r then also $\Delta v = 0$. Conversely, since $v_r = G^n_{p,r}(\Lambda, L)v$ for suitable noncommutative polynomials with rational coefficients $G^n_{p,r}$, and Δ commutes with Λ, L then v harmonic implies $\Delta v_r = 0$ for every r. $\Box$

Corollary VI.23. *In the above notation, if h is a Kähler metric and v is a closed primitive (p,q)-form then v is harmonic.*

Note that if either $p = 0$ or $q = 0$ then v is always primitive.

Proof. It is sufficient to prove that $\delta v = 0$, we have

$$\delta v = C\delta^C C^{-1} v = i^{q-p} C\delta^C v = i^{q-p} C[d, \Lambda] v = 0.$$

$\Box$

5. Kähler metrics on compact manifolds

In this section we assume M compact complex manifold of dimension n. We denote by $L^{a,b} = \Gamma(M, \mathcal{A}^{a,b})$, $L^p = \bigoplus_{a+b=p} L^{a,b}$, $L = \bigoplus_p L^p$.
Every Hermitian metric h on M induces a structure of pre-Hilbert space on $L^{a,b}$ for every a, b (and then also on L) given by:

$$(\phi, \psi) = \int_M \phi \wedge \overline{*\psi}.$$

We have already seen that the operator $*\colon L^{a,b} \to L^{n-a,n-b}$ is an isometry commuting with the complex conjugation and then we also have:

$$(\phi, \psi) = \int_M \phi \wedge \overline{*\psi} = \int_M \phi \wedge *\overline{\psi} = (-1)^{a+b} \int_M *\phi \wedge \overline{\psi} = \int_M \overline{\psi} \wedge *\phi = \overline{(\psi, \phi)}.$$

Proposition VI.24. *With respect to the above pre-Hilbert structures we have the following pairs (written in columns) of formally adjoint operator:*

operator	d	d^C	∂	$\overline{\partial}$	L
formal adjoint	δ	δ^C	∂^*	$\overline{\partial}^*$	Λ

In particular, all the four Laplacians are formally self-adjoint operators.

Proof. We show here only that δ is the formal adjoint of d. The proof of the remaining assertions is essentially the same and it is left as exercise.
Let ϕ be a p-form and ψ a $p+1$-form. By Stokes theorem

$$0 = \int_M d(\phi \wedge \overline{*\psi}) = \int_M d\phi \wedge \overline{*\psi} + (-1)^p \int_M \phi \wedge d\overline{*\psi}.$$

Since $d\overline{*\psi} = \overline{d * \psi}$ and $d * \psi = (-1)^{2n-p} *^2 d * \psi = -(-1)^p * \delta\psi$ we get

$$0 = \int_M d\phi \wedge \overline{*\psi} - \int_M \phi \wedge \overline{*\delta\psi} = (d\phi, \psi) - (\phi, \delta\psi).$$

□

Let D be any of the operator $d, d^C, \partial, \overline{\partial}$; denote D^* its formal adjoint and by $\Delta_D = DD^* + D^*D$ its Laplacian (i.e. $\Delta_d = \Delta$, $\Delta_{\overline{\partial}} = \overline{\Box}$ etc...). The space of D-harmonic p-forms is denoted by $\mathcal{H}^p_D = \ker \Delta_D \cap L^p$.

Lemma VI.25. *We have* $\ker \Delta_D = \ker D \cap \ker D^*$.

Proof. The inclusion $\supset$ is immediate from the definitions of the Laplacian. The inclusion $\subset$ comes from

$$(\Delta_D\phi, \phi) = (DD^*\phi, \phi) + (D^*D\phi, \phi) = (D^*\phi, D^*\phi) + (D\phi, D\phi) = \|D^*\phi\|^2 + \|D\phi\|^2.$$

□

The theory of elliptic self-adjoint operators on compact manifolds gives:

Theorem VI.26. *In the notation above the spaces of D-harmonic forms $\mathcal{H}^p_D$ are finite dimensional and there exist orthogonal decompositions*

$$L^p = \mathcal{H}^p_D \overset{\perp}{\bigoplus} \operatorname{Im} \Delta_D.$$

Proof. See e.g. [**78**]. □

Corollary VI.27. *The natural projection maps*

$$\mathcal{H}^p_d \to H^p(M, \mathbb{C}), \qquad \mathcal{H}^{p,q}_{\overline{\partial}} \to H^q_{\overline{\partial}}(M, \Omega^p)$$

are isomorphism.

PROOF. We first note that, according to Lemma VI.25, every harmonic form is closed and then the above projection maps makes sense. It is evident that $\operatorname{Im}\Delta\subset\operatorname{Im}d+\operatorname{Im}\delta$. On the other hand, since d,δ are formally adjoint and $d^2=\delta^2=0$ we have $\ker d\perp\operatorname{Im}\delta$, $\ker\delta\perp\operatorname{Im}d$: this implies that $\operatorname{Im}d$, $\operatorname{Im}\delta$ and $\mathcal{H}^p_d$ are pairwise orthogonal. Therefore $\operatorname{Im}\Delta=\operatorname{Im}d\oplus\operatorname{Im}\delta$ and $\ker d=\mathcal{H}^p_d\oplus\operatorname{Im}d$; the conclusion follows by De Rham theorem. The isomorphism $\mathcal{H}^{p,q}_{\overline{\partial}}\to H^q_{\overline{\partial}}(M,\Omega^p)$ is proved in the same way (with Dolbeault's theorem instead of De Rham) and it is left as exercise. □

COROLLARY VI.28. *The map* $\Delta_D\colon\operatorname{Im}\Delta_D\to\operatorname{Im}\Delta_D$ *is bijective.*

PROOF. Trivial consequence of Theorem VI.26. □

We define the *harmonic projection* $H_D\colon L^p\to\mathcal{H}^p_D$ as the orthogonal projection and the *Green operator* $G_D\colon L^p\to\operatorname{Im}\Delta_D$ as the composition of

$$G_D\colon L^p\xrightarrow{Id-H_D}\operatorname{Im}\Delta_D\xrightarrow{\Delta_D^{-1}}\operatorname{Im}\Delta_D.$$

Note that $\Delta_DG_D=G_D\Delta_D=Id-H_D$ and $G_DH_D=H_DG_D=0$.

LEMMA VI.29. *If K is an operator commuting with Δ_D then K commutes with G_D.*

PROOF. Exercise (Hint: K preserves image and kernel of Δ_D). □

If h is a Kähler metric, then the equality $\Delta=2\overline{\Box}$ implies that

$$H_d=H_{d^C}=H_\partial=H_{\overline{\partial}},\qquad G_d=G_{d^C}=\frac{1}{2}G_\partial=\frac{1}{2}G_{\overline{\partial}}.$$

In particular, according to Lemma VI.29 and Corollary VI.20, $G_d=G_{d^C}$ commutes with d,d^C.

COROLLARY VI.30. *If h is a Kähler metric on a compact manifold then: Every holomorphic p-form on M is harmonic.*

PROOF. According to Corollary VI.27 the inclusion $\mathcal{H}^{p,0}_{\overline{\partial}}\subset\Gamma(M,\Omega^p)$ is an isomorphism and then if η is a holomorphic p-form we have $\Delta(\eta)=2\overline{\Box}(\eta)=0$. □

EXERCISE VI.31. Let $v\neq0$ be a primitive (p,q)-form on a compact manifold M with Kähler form ω. Prove that

$$\int_M v\wedge\overline{v}\wedge\omega^{\wedge n-p-q}\neq0.$$

△

6. Compact Kähler manifolds

In this section we will prove that certain good properties concerning the topology and the complex structure of compact complex manifolds are true whenever we assume the existence of a Kähler metric. This motivates the following definition:

DEFINITION VI.32. A complex manifold M s called a *Kähler manifolds* if there exists a Kähler metric on M.

We note that, while every complex manifold admits a Hermitian metric (this is an easy application of partitions of unity, cf. [**37**, Thm. 3.14]), not every complex manifold is Kählerian. We recall the following

THEOREM VI.33. (1) *$\mathbb{C}^n$, $\mathbb{P}^n$ and the complex tori are Kähler manifolds.*
(2) *If M is a Kähler manifold and $N\subset M$ is a regular submanifold then also N is a Kähler manifolds.*

For a proof of Theorem VI.33 we refer to [**26**].

From now on M is a fixed compact Kähler manifold on dimension n.
For every $m \leq 2n$ we denote by $H^m(M,\mathbb{C}) = H^m(M,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ the De Rham cohomology $\mathbb{C}$-vector spaces. We note that a differential m-form η is d-closed if and only if its conjugate $\overline{\eta}$ is. In particular the complex conjugation induce an isomorphism of vector spaces $H^m(M,\mathbb{C}) = \overline{H^m(M,\mathbb{C})}$.
If $p+q=m$ we denote by $F^{p,q} \subset H^m(M,\mathbb{C})$ the subspace of cohomology classes represented by d-closed form of type (p,q) (note that a (p,q)-form η is d-closed if and only if it is $\partial\eta = \overline{\partial}\eta = 0$). It is clear that $\overline{F^{p,q}} = F^{q,p}$.

Theorem VI.34 (Hodge decomposition). *In the notation above we have*

$$H^m(M,\mathbb{C}) = \bigoplus_{p+q=m} F^{p,q}$$

and the natural morphisms $F^{p,q} \to H^{p,q}_{\partial}(M)$, $F^{p,q} \to H^{p,q}_{\overline{\partial}}(M)$ *are isomorphisms.*

Proof. Take a Kähler metric on M and use it to define the four Laplacians, the harmonic projectors and the Green operators. According to Corollary VI.19 the Laplacian Δ is bihomogeneous of bidegree $(0,0)$ and we have

$$\ker\Delta \cap L^q = \bigoplus_{a+b=q} \ker\Delta \cap L^{a,b}.$$

The isomorphism $\ker\Delta \cap L^q \to H^q(M,\mathbb{C})$ induces injective maps $\ker\Delta\cap L^{a,b} \to F^{a,b}$; this maps are also surjective because every closed form α is cohomologically equivalent to its harmonic projection $H\alpha$ and H is bihomogeneous of bidegree $(0,0)$.
The last equalities follow from the isomorphisms

$$\ker\Delta\cap L^{a,b} = \ker\Box\cap L^{a,b} = H^{a,b}_{\partial}(M), \quad \ker\Delta\cap L^{a,b} = \ker\overline{\Box}\cap L^{a,b} = H^{a,b}_{\overline{\partial}}(M).$$

$\square$

Corollary VI.35. *If M is a compact Kähler manifold then:*
1. $b_i = \sum_{a+b=i} h^{a,b}$.
2. $h^{p,q} = h^{q,p}$*, in particular* b_i *is even if* i *is odd.*
3. $h^{p,p} > 0$*, in particular* $b_i > 0$ *if* i *is even.*
4. *Every holomorphic* p*-form on* M *is* d*-closed.*

($b_i = \dim_{\mathbb{C}} H^i(M,\mathbb{C})$ *are the Betti numbers,* $h^{p,q} = \dim_{\mathbb{C}} H^q(M,\Omega^p)$ *the Hodge numbers.)*

Proof. Items 1 and 2 are immediate consequence of the Hodge decomposition. Take a Kähler metric on M and use it to define the four Laplacians, the harmonic projectors and the Green operators. Let ω be the associated form of the Kähler metric on M. According to Corollary VI.18, $\omega^{\wedge p}$ is harmonic and then $\ker\overline{\Box}\cap L^{p,p} = \ker\Delta\cap L^{p,p} \neq 0$.
Finally, by Corollary VI.30 the holomorphic forms are Δ-harmonic and therefore d-closed.

$\square$

Example VI.36. The Hopf surfaces (Example I.6) have $b_1 = b_3 = 1$, $b_2 = 0$ and then are not Kähler.

Finally we are in a position to prove the following

Theorem VI.37. ($\partial\overline{\partial}$-Lemma) *Let M be a compact Kähler manifold. Then*
1. *There exists a linear operator* $\sigma\colon L \to L$ *of bidegree* $(0,-1)$ *such that*

$$[\partial,\sigma] = 0, \qquad [\overline{\partial},\sigma]\partial = [\overline{\partial},\sigma\partial] = \partial.$$

2. $\operatorname{Im}\partial\overline{\partial} = \ker\partial\cap\operatorname{Im}\overline{\partial} = \ker\overline{\partial}\cap\operatorname{Im}\partial.$

Proof. [1] Choose a Kähler metric and define $\sigma = G_{\overline{\partial}}\overline{\partial}^*$. According to VI.19, VI.20 and VI.29 we have $\sigma = \overline{\partial}^* G_{\overline{\partial}}$, $[\partial,\sigma] = 0$ and, denoting by H the harmonic projection,

$$[\overline{\partial},\sigma]\partial = G_{\overline{\partial}}\Delta_{\overline{\partial}}\partial = (Id - H)\partial = \partial.$$

[2] (cf. Exercise VI.39) We prove only $\operatorname{Im}\partial\overline{\partial} = \ker\overline{\partial}\cap\operatorname{Im}\partial$, being the other equality the conjugate of this one. The inclusion $\subset$ is evident, conversely let $x=\partial\alpha$ be a $\overline{\partial}$-closed differential form; we can write

$$x=\partial\alpha=[\overline{\partial},\sigma]\partial\alpha=\overline{\partial}\sigma\partial\alpha+\sigma\overline{\partial}\partial\alpha=-\overline{\partial}\partial\sigma\alpha-\sigma\overline{\partial}x=\partial\overline{\partial}(\sigma\alpha).$$

□

Corollary VI.38. *Let M be a compact Kähler manifold. Then for every p,q the natural maps*

$$\frac{\ker\partial\cap\ker\overline{\partial}\cap L^{p,q}}{\partial\overline{\partial}L^{p-1,q-1}}\to\frac{\ker\partial\cap\ker\overline{\partial}\cap L^{p,q}}{\overline{\partial}(\ker\partial\cap L^{p,q-1})}\to\frac{\ker\overline{\partial}\cap L^{p,q}}{\overline{\partial}L^{p,q-1}}=H^q(M,\Omega^p)$$

$$\frac{\ker\partial\cap\ker\overline{\partial}\cap L^{p,q}}{\partial\overline{\partial}L^{p-1,q-1}}\to\frac{\ker\partial\cap\ker\overline{\partial}\cap L^{p,q}}{\partial(\ker\overline{\partial}\cap L^{p-1,q})}\to\frac{\ker\partial\cap L^{p,q}}{\partial L^{p-1,q}}$$

are isomophisms.

Proof. The two lines are conjugates each other and then it is sufficient to prove that the maps on the first row are isomorphisms.
Choose a Kähler metric, every $\overline{\partial}$-closed form ϕ can be written as $\phi=\alpha+\overline{\partial}\psi$ with $\overline{\Box}\alpha=0$. Since $\Box=\overline{\Box}$ we have $\partial\alpha=0$ and then the above maps are surjective.
According to Theorem VI.37 we have

$$\partial\overline{\partial}(L^{p-1,q-1})\subset\overline{\partial}(\ker\partial\cap L^{p,q-1})\subset\ker\partial\cap\overline{\partial}(L^{p,q-1})\subset\partial\overline{\partial}(L^{p-1,q-1})$$

and then all the maps are injective. □

Exercise VI.39. Prove that for a double complex $(L^{*,*},d,\delta)$ of vector spaces (with d,δ differentials of respective bidegrees $(1,0)$ and $(0,1)$) the following conditions are equivalent:

(1) There exists a linear operator $\sigma\colon L^{*,*}\to L^{*,*-1}$ of bidegree $(0,-1)$ such that
$$[d,\sigma]=0,\qquad [\delta,\sigma]d=[\delta,\sigma d]=d.$$
(2) $\operatorname{Im} d\delta=\ker\delta\cap\operatorname{Im} d$.

(Hint: The implication $[1\Rightarrow 2]$ is the same as in Theorem VI.37. In order to prove $[2\Rightarrow 1]$ write $L^{a,b}=F^{a,b}\oplus C^{a,b}$ with $F^{a,b}=dL^{a-1,b}$ and observe that the complexes $(F^{a,*},\delta)$ are acyclic. Define first $\sigma\colon F^{a,b}\to F^{a,b-1}$ such that $[\delta,\sigma]d=d$ and then $\sigma\colon C^{a,b}\to C^{a,b-1}$ such that $[d,\sigma]=0$.) △

7. Historical survey, VI

Most of the properties of Kähler manifolds are stable under deformation. For example:

Theorem VI.40. *Let $f\colon M\to B$ be a family of compact complex manifolds and assume that M_b is Kählerian for some $b\in B$.*
Then there exists an open neighbourhood $b\in U\subset B$ such the functions $h^{p,q}\colon U\to\mathbb{N}$, $h^{p,q}(u)=\dim_{\mathbb{C}}H^{p,q}(M_u)$ are constant and $\sum_{p+q=m}h^{p,q}(u)=\dim_{\mathbb{C}}H^m(M_u,\mathbb{C})$ for every $u\in U$.

Proof. (Idea) Exercise I.18 implies $\sum_{p+q=m}h^{p,q}(u)\geq\dim_{\mathbb{C}}H^m(M_u,\mathbb{C})$ and the equality holds whenever M_u is Kählerian. On the other side, by semicontinuity theorem I.42 the functions $h^{p,q}$ are semicontinuous and by Ehresmann's theorem the function $u\mapsto\dim_{\mathbb{C}}H^m(M_u,\mathbb{C})$ is locally constant. □

Theorem VI.40 is one of the main ingredients for the proof of the following theorem, proved by Kodaira (cf. [**37**], [**78**])

Theorem VI.41. *Let $f\colon M\to B$ be a family of compact complex manifolds. Then the subset $\{b\in B\mid M_b$ is Kählerian $\}$ is open in B.*

The proof of VI.41 requires hard functional and harmonic analysis.

It seems that the name *Kähler manifolds* comes from the fact that they were defined in a note of Erich Kähler (1906-2000) of 1933 but all their (first) good properties were estabilished by W.V.D. Hodge some years later.

LECTURE VII

Deformations of manifolds with trivial canonical bundle

In the first part of this chapter we prove, following [**21**] and assuming Kuranishi theorem IV.36, the following

THEOREM VII.1 (Bogomolov-Tian-Todorov). *Let M be a compact Kähler manifold with trivial canonical bundle $K_M = \mathcal{O}_M$. Then M admits a semiuniversal deformation with smooth base $(H^1(M, T_M), 0)$.*

According to Corollary IV.37, it is sufficient to to show that the natural map

$$\mathrm{Def}_M\left(\frac{\mathbb{C}[t]}{(t^{n+1})}\right) \to \mathrm{Def}_M\left(\frac{\mathbb{C}[t]}{(t^2)}\right)$$

is surjective for every $n \geq 1$. This will be done using Corollary V.52 and the so called Tian-Todorov's lemma.
A generalization of this theorem has been given recently by H. Clemens [**10**]. We will prove of Clemens' theorem in Chapter IX.

In the second part we introduce some interesting classes of dg-algebras which arise naturally both in mathematics and in physics: in particular we introduce the notion of differential Gerstenhaber algebra and differential Gerstenhaber-Batalin-Vilkovisky algebra. Then we show (Example VII.30) that the algebra of polyvector fields on a manifold with trivial canonical bundle carries the structure of differential Gerstenhaber-Batalin-Vilkovisky algebra.

1. Contraction on exterior algebras

Let $\mathbb{K}$ be a fixed field and E a vector space over $\mathbb{K}$ of dimension n; denote by $E^\vee$ its dual and by $\langle,\rangle\colon E \times E^\vee \to \mathbb{C}$ the natural pairing. Given $v \in E$, the (left) contraction by v is the linear operator $v\vdash\colon \bigwedge^b E^\vee \to \bigwedge^{b-1} E^\vee$ defined by the formula

$$v \vdash (z_1 \wedge \ldots \wedge z_b) = \sum_{i=1}^{b} (-1)^{i-1} \langle v, z_i\rangle z_1 \wedge \ldots \wedge \widehat{z_i} \wedge \ldots \wedge z_b.$$

For every $a \leq b$ the contraction

$$\bigwedge^a E \times \bigwedge^b E^\vee \xrightarrow{\vdash} \bigwedge^{b-a} E^\vee$$

is the bilinear extension of

$$\begin{aligned}(v_a \wedge \ldots \wedge v_1) \vdash (z_1 \wedge \ldots \wedge z_b) &= v_a \vdash ((v_{a-1} \wedge \ldots \wedge v_1) \vdash (z_1 \wedge \ldots \wedge z_b)) \\ &= \sum_{\sigma \in G} (-1)^\sigma \left(\prod_{i=1}^{a} \langle v_i, z_{\sigma(i)}\rangle\right) z_{\sigma(a+1)} \wedge \ldots \wedge z_{\sigma(b)}\end{aligned}$$

where $G \subset \Sigma_b$ is the subset of permutations σ such that $\sigma(a+1) < \sigma(a+2) < \ldots < \sigma(b)$.
We note that if $a = b$ then the contraction is a nondegenerate pairing giving a natural isomorphism $(\bigwedge^a E)^\vee = \bigwedge^a E^\vee$. This isomorphism is, up to sign, the same considered is Section VI.2.
If $a > b$ we use the convention that $\vdash = 0$.

LEMMA VII.2. (1) *For every $v \in E$ the operator $v\vdash$ is a derivation of degree -1 of the graded algebra $\bigwedge^* E^\vee$.*

(2) *For every* $v \in \bigwedge^a E$, $w \in \bigwedge^b E$, $z \in \bigwedge^c E^\vee$, *we have*

$$(v \wedge w) \vdash z = v \vdash (w \vdash z).$$

In particular the operator $w \vdash \colon \bigwedge^c E^\vee \to \bigwedge^{c-b} E^\vee$ *is the adjoint of* $\wedge w \colon \bigwedge^{c-b} E \to \bigwedge^c E$.

(3) *If* $v \in \bigwedge^a E^\vee$, $w \in \bigwedge^b E$, $\Omega \in \bigwedge^n E^\vee$, *where* $\dim E = n$, $a \leq b$, *then:*

$$v \wedge (w \vdash \Omega) = (v \vdash w) \vdash \Omega.$$

PROOF. [1] Complete v to a basis $v = e_1, \ldots, e_n$ of E and let $z_1, \ldots, z_n$ be its dual basis. Every $w \in \bigwedge^* E^\vee$ can be written in a unique way as $w = z_1 \wedge w_1 + w_2$ with $w_1, w_2 \in \bigwedge^* v^\perp$. According to the definition of $\vdash$ we have $v \vdash w = w_1$.
If $w = z_1 \wedge w_1 + w_2$, $u = z_1 \wedge u_1 + u_2$ are decompositions as above then

$$\begin{aligned}
(v \vdash w) \wedge u + (-1)^{\overline{w}} w \wedge (v \vdash u) &= w_1 \wedge (z_1 \wedge u_1 + u_2) + (-1)^{\overline{w_2}} (z_1 \wedge w_1 + w_2) \wedge u_1 \\
&= w_1 \wedge u_2 + (-1)^{\overline{w_2}} w_2 \wedge u_1. \\
v \vdash (w \wedge u) &= v \vdash ((z_1 \wedge w_1 + w_2) \wedge (z_1 \wedge u_1 + u_2)) \\
&= v \vdash (z_1 \wedge w_1 \wedge u_2 + w_2 \wedge z_1 \wedge u_1 + w_2 \wedge u_2) \\
&= w_1 \wedge u_2 + (-1)^{\overline{w_2}} w_2 \wedge u_1.
\end{aligned}$$

[2] Immediate from the definition.
[3] Induction on a; if $a = 1$ then complete v to a basis $v = z_1, \ldots, z_n$ of $E^\vee$ and denote $e_1, \ldots, e_n \in E$ its dual basis. Writing

$$w = e_1 \wedge w_1 + w_2, \qquad w_i \in \textstyle\bigwedge^* v^\perp, \qquad w_i \vdash \Omega = v \wedge \eta_i, \qquad \eta_i \in \bigwedge^* e_1^\perp,$$

we have by Item 2

$$w \vdash \Omega = (e_1 \wedge w_1) \vdash \Omega + (w_2 \vdash \Omega) = e_1 \vdash (w_1 \vdash \Omega) + (w_2 \vdash \Omega) = \eta_1 + v \wedge \eta_2,$$

and then

$$v \wedge (w \vdash \Omega) = v \wedge \eta_1 = w_1 \vdash \Omega = (v \vdash w) \vdash \Omega.$$

If $a > 1$ and $v = v_1 \wedge v_2$, with $v_1 \in E^\vee$, $v_2 \in \bigwedge^{a-1} E^\vee$ then by item 2 and inductive assumption

$$v_1 \wedge v_2 \wedge (w \vdash \Omega) = v_1 \wedge ((v_2 \vdash w) \vdash \Omega) = (v_1 \vdash (v_2 \vdash w)) \vdash \Omega = ((v_1 \wedge v_2) \vdash w) \vdash \Omega.$$

□

LEMMA VII.3. *For every vector space E of dimension n and every integer $a = 0, \ldots, n$, the contraction operator defines a natural isomorphism*

$$\textstyle\bigwedge^a E \xrightarrow{i} \bigwedge^n E \otimes \bigwedge^{n-a} E^\vee, \qquad i(v) = Z \otimes (v \vdash \Omega)$$

where $(Z, \Omega) \in \bigwedge^n E \times \bigwedge^n E^\vee$ *is any pair satisfying* $Z \vdash \Omega = 1$.

PROOF. Trivial. □

EXERCISE VII.4. Let $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ be an exact sequence of vector spaces with $\dim G = n < \infty$. Use the contraction operator to define, for every $a \leq \dim E$, a natural surjective linear map $\bigwedge^{a+n} F \to \bigwedge^a E \otimes \bigwedge^n G$. △

2. The Tian-Todorov's lemma

The isomorphism i of Lemma VII.3 can be extended fiberwise to vector bundles; in particular, if M is a complex manifold of dimension n and T_M is its holomorphic tangent bundle, we have holomorphic isomorphisms

$$i\colon \bigwedge^a T_M \longrightarrow \bigwedge^n T_M \otimes \bigwedge^{n-a} T_M^\vee = \Omega_M^{n-a}(K_M^\vee)$$

which extend to isomorphisms between their Dolbeault's sheaf resolutions

$$i\colon (\mathcal{A}^{0,*}(\bigwedge^a T_M), \overline{\partial}) \longrightarrow (\mathcal{A}^{0,*}(\bigwedge^n T_M \otimes \bigwedge^{n-a} T_M^\vee), \overline{\partial}) = (\mathcal{A}^{n-a,*}(K_M^\vee), \overline{\partial}).$$

If $z_1, \ldots, z_n$ are local holomorphic coordinates then a local set of generators of $\bigwedge^a T_M$ is given by the polyvector fields $\dfrac{\partial}{\partial z_I} = \dfrac{\partial}{\partial z_{i_1}} \wedge \ldots \wedge \dfrac{\partial}{\partial z_{i_a}}$, being $I = (i_1, \ldots, i_a)$ a multiindex. If Ω is a local frame of K_M and Z a local frame of $K_M^\vee$ such that $Z \vdash \Omega = 1$, then

$$i\left(\frac{\partial}{\partial z_I} d\overline{z}_J\right) = Z \otimes \left(\frac{\partial}{\partial z_I} d\overline{z}_J \vdash \Omega\right) = Z \otimes \left(\frac{\partial}{\partial z_I} \vdash \Omega\right) d\overline{z}_J.$$

Given a fixed Hermitian metric h on the line bundle $K_M^\vee$ we denote by $D = D' + \overline{\partial}$ the unique hermitian connection on $K_M^\vee$ compatible with the complex structure.
We recall (cf. [**35**]) that $D'\colon \mathcal{A}^{0,b}(K_M^\vee \otimes \Omega_M^a) \to \mathcal{A}^{0,b}(K_M^\vee \otimes \Omega_M^{a+1})$ is defined in local coordinates as

$$D'(Z \otimes \phi) = Z \otimes (\theta \wedge \phi + \partial\phi), \qquad \phi \in \mathcal{A}^{a,b},$$

where $\theta = \partial \log(|Z|^2) = \partial \log(h(Z, Z))$ is the connection form of the frame Z.
We have moreover $(D')^2 = 0$ and $D'\overline{\partial} + \overline{\partial}D' = \Theta$ is the curvature of the metric.

We can now define a $\mathbb{C}$-linear operator (depending on h)[1]

$$\Delta\colon \mathcal{A}^{0,b}(\bigwedge^a T_M) \to \mathcal{A}^{0,b}(\bigwedge^{a-1} T_M), \qquad \Delta(\phi) = i^{-1}D'(i(\phi)).$$

Lemma VII.5. *Locally on M, with Ω, Z and θ as above we have*

$$\Delta(\phi) \vdash \Omega = \theta \wedge (\phi \vdash \Omega) + \partial(\phi \vdash \Omega)$$

for every $\phi \in \mathcal{A}^{0,b}(\bigwedge^* T_M)$.

Proof. By definition

$$i\Delta(\phi) = Z \otimes (\Delta(\phi) \vdash \Omega),$$
$$i\Delta(\phi) = D'(i(\phi)) = D'(Z \otimes (\phi \vdash \Omega)) = Z \otimes (\theta \wedge (\phi \vdash \Omega) + \partial(\phi \vdash \Omega)).$$

□

Lemma VII.6. *In local holomorphic coordinates $z_1, \ldots, z_n$ we have*

$$\Delta\left(f\frac{\partial}{\partial z_I} d\overline{z}_J\right) = \left((\theta f + \partial f) \vdash \frac{\partial}{\partial z_I}\right) d\overline{z}_J, \qquad f \in \mathcal{A}^{0,0},$$

where θ is the connection form of the frame $Z = \dfrac{\partial}{\partial z_1} \wedge \ldots \wedge \dfrac{\partial}{\partial z_n}$ and the right hand side is considered $= 0$ when $I = \emptyset$.

Proof. We first note that if $\phi \in \mathcal{A}^{0,0}(\bigwedge^a T_M)$ then $i(\phi d\overline{z}_J) = i(\phi) d\overline{z}_J$ and

$$D' i(\phi d\overline{z}_J) = D'(Z \otimes (\phi \vdash \Omega) \otimes d\overline{z}_J) = D'(Z \otimes (\phi \vdash \Omega)) \otimes d\overline{z}_J :$$

this implies that $\Delta(\phi d\overline{z}_J) = \Delta(\phi) d\overline{z}_J$. According to Lemma VII.5

$$\Delta\left(f\frac{\partial}{\partial z_I}\right) \vdash \Omega = \theta \wedge \left(f\frac{\partial}{\partial z_I} \vdash \Omega\right) + \partial\left(f\frac{\partial}{\partial z_I} \vdash \Omega\right)$$

Since $\Omega = dz_n \wedge \ldots \wedge dz_1$ we have $\partial\left(\dfrac{\partial}{\partial z_I} \vdash \Omega\right) = 0$ and then, by Item 3 of Lemma VII.2,

$$\Delta\left(f\frac{\partial}{\partial z_I}\right) \vdash \Omega = (\theta f + \partial f) \wedge \left(\frac{\partial}{\partial z_I} \vdash \Omega\right) = \left((\theta f + \partial f) \vdash \frac{\partial}{\partial z_I}\right) \vdash \Omega.$$

[1]don't confuse this Δ with the Laplacian

□

Setting $\mathcal{P}^{a,b} = \mathcal{A}^{0,b}(\bigwedge^{-a} T_M)$ for every $a \leq 0$, $b \geq 0$, the direct sum $\mathcal{P} = (\bigoplus_{a,b} \mathcal{P}^{a,b}, \overline{\partial})$ is a sheaf of dg-algebras, where the sections of $\mathcal{A}^{0,b}(\bigwedge^{a} T_M)$ have total degree $b-a$ and $\overline{\partial}\colon \mathcal{A}^{0,b}(\bigwedge^{a} T_M) \to \mathcal{A}^{0,b+1}(\bigwedge^{a} T_M)$ is the Dolbeault differential. The product on $\mathcal{P}$ is the 'obvious' one:

$$(\xi \otimes \phi) \wedge (\eta \otimes \psi) = (-1)^{\overline{\phi}\,\overline{\eta}}(\xi \wedge \eta) \otimes (\phi \wedge \psi).$$

Lemma VII.7. *The $\mathbb{C}$-linear operator $\Delta\colon \mathcal{P} \to \mathcal{P}$ has degree $+1$; moreover $\Delta^2 = 0$ and $[\Delta, \overline{\partial}] = \Delta\overline{\partial} + \overline{\partial}\Delta = i^{-1}\Theta i$.*

Proof. Evident. □

Consider the bilinear symmetric map of degree 1, $Q\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$

$$Q(\alpha, \beta) = \Delta(\alpha \wedge \beta) - \Delta(\alpha) \wedge \beta - (-1)^{\overline{\alpha}} \alpha \wedge \Delta(\beta).$$

A brutal computation in local coordinates shows that Q is independent of the metric. In fact, for every pair of C^∞ functions f, g

$$Q\left(f\frac{\partial}{\partial z_I} d\overline{z}_J, g\frac{\partial}{\partial z_H} d\overline{z}_K\right) = (-1)^{|J|\,|H|} Q\left(f\frac{\partial}{\partial z_I}, g\frac{\partial}{\partial z_H}\right) d\overline{z}_J \wedge d\overline{z}_K$$

and

$$Q\left(f\frac{\partial}{\partial z_I}, g\frac{\partial}{\partial z_H}\right) = (\theta f g + \partial(fg)) \vdash \left(\frac{\partial}{\partial z_I} \wedge \frac{\partial}{\partial z_H}\right) -$$

$$-g\left((\theta f + \partial f) \vdash \frac{\partial}{\partial z_I}\right) \wedge \frac{\partial}{\partial z_H} - (-1)^{|I|} f \frac{\partial}{\partial z_I} \wedge \left((\theta g + \partial g) \vdash \frac{\partial}{\partial z_H}\right).$$

According to Lemma VII.2, Item 1:

$$Q\left(f\frac{\partial}{\partial z_I}, g\frac{\partial}{\partial z_H}\right) = f\left(\partial g \vdash \frac{\partial}{\partial z_I}\right) \wedge \frac{\partial}{\partial z_H} + (-1)^{|I|} g \frac{\partial}{\partial z_I} \wedge \left(\partial f \vdash \frac{\partial}{\partial z_H}\right).$$

In particular if $|I| = 0$, $|H| = 1$ then

$$Q\left(f d\overline{z}_J, g\frac{\partial}{\partial z_h} d\overline{z}_K\right) = (-1)^{|J|} g \frac{\partial f}{\partial z_h} d\overline{z}_J \wedge d\overline{z}_K,$$

while, if $|I| = |H| = 1$ then

$$Q\left(f\frac{\partial}{\partial z_i} d\overline{z}_J, g\frac{\partial}{\partial z_h} d\overline{z}_K\right) = (-1)^{|J|}\left(f\frac{\partial g}{\partial z_i}\frac{\partial}{\partial z_h} - g\frac{\partial f}{\partial z_h}\frac{\partial}{\partial z_i}\right) d\overline{z}_J \wedge d\overline{z}_K.$$

Recalling the definition of the bracket $[\ ,\ ]$ in the Kodaira-Spencer algebra $KS_M = \bigoplus_b \mathcal{A}^{0,b}(T_M)$ we have:

Lemma VII.8 (Tian-Todorov). *If $\alpha \in \mathcal{A}^{0,a}(T_M)$, $\beta \in \mathcal{A}^{0,b}(T_M)$ then*

$$(-1)^a[\alpha, \beta] = \Delta(\alpha \wedge \beta) - \Delta(\alpha) \wedge \beta - (-1)^{a-1} \alpha \wedge \Delta(\beta).$$

In particular the bracket of two Δ-closed forms is Δ-exact.

Example VII.9. If M is compact Kähler and $c_1(M) = 0$ in $H^2(M, \mathbb{C})$ then by [**35**, 2.23] there exists a Hermitian metric on $K_M^\vee$ such that $\Theta = 0$; in this case $[\Delta, \overline{\partial}] = 0$ and $\ker \Delta$ is a differential graded subalgebra of KS_M.

Example VII.10. If M has a nowhere vanishing holomorphic n-form Ω ($n = \dim M$) we can set on $K_M^\vee$ the trivial Hermitian metric induced by the isomorphism $\Omega\colon K_M^\vee \to \mathcal{O}_M$. In this case, according to Lemma VII.5, the operator Δ is defined by the rule

$$(\Delta\alpha) \vdash \Omega = \partial(\alpha \vdash \Omega).$$

3. A formality theorem

THEOREM VII.11. *Let M be a compact Kähler manifold with trivial canonical bundle $K_M = \mathcal{O}_M$. Then the Kodaira-Spencer DGLA*

$$KS_M = \bigoplus_p \Gamma(M, \mathcal{A}^{0,p}(T_M))$$

is quasiisomorphic to an abelian DGLA.

PROOF. Let $\Omega \in \Gamma(M, K_M)$ be a nowhere vanishing holomorphic n-form ($n = \dim M$); via the isomorphism $\Omega\colon K_M^\vee \to \mathcal{O}_M$, the isomorphism of complexes

$$i\colon (\mathcal{A}^{0,*}(T_M), \overline{\partial}) \to (\mathcal{A}^{n-1,*}, \overline{\partial})$$

is given in local holomorphic coordinates by

$$i\left(f\frac{\partial}{\partial z_i}d\overline{z}_I\right) = f\left(\frac{\partial}{\partial z_i} \vdash \Omega\right)d\overline{z}_I$$

and induces a structure of DGLA, isomorphic to KS_M on

$$L^{n-1,*} = \bigoplus_p \Gamma(M, \mathcal{A}^{n-1,p}).$$

Taking on $K_M^\vee$ the trivial metric induced by $\Omega\colon K_M^\vee \to \mathcal{O}_M$, the connection D is equal to the De Rham differential and then the Tian-Todorov's lemma implies that the bracket of two ∂-closed form of $L^{n-1,*}$ is ∂-exact; in particular

$$Q^* = \ker\partial \cap L^{n-1,*}$$

is a DGL subalgebra of $L^{n-1,*}$.
Consider the complex $(R^*, \overline{\partial})$, where

$$R^p = \frac{\ker\partial \cap L^{n-1,p}}{\partial L^{n-2,p}}$$

endowed with the trivial bracket, again by Lemma VII.8 the projection $Q^* \to R^*$ is a morphism of DGLA.
It is therefore sufficient to prove that the DGLA morphisms

$$L^{n-1,*} \longleftarrow Q^* \longrightarrow R^*$$

are quasiisomorphisms.
According to the $\partial\overline{\partial}$-lemma VI.37, $\overline{\partial}(\ker\partial) \subset \operatorname{Im}\partial$ and then the operator $\overline{\partial}$ is trivial on R^*: therefore

$$H^p(R^*) = \frac{\ker\partial \cap L^{n-1,p}}{\partial L^{n-2,p}}, \qquad H^p(L^{n-1,*}) = \frac{\ker\overline{\partial} \cap L^{n-1,p}}{\overline{\partial}L^{n-1,p-1}},$$

$$H^p(Q^*) = \frac{\ker\partial \cap \ker\overline{\partial} \cap L^{n-1,p}}{\overline{\partial}(\ker\partial \cap L^{n-1,p-1})}.$$

The conclusion now follows immediately from Corollary VI.38. $\square$

COROLLARY VII.12. *Let M be a compact Kähler manifold with trivial canonical bundle $K_M = \mathcal{O}_M$. For every local Artinian $\mathbb{C}$-algebra $(A, \mathfrak{m}_A)$ we have*

$$\operatorname{Def}_M(A) = H^1(M, T_M) \otimes \mathfrak{m}_A.$$

In particular

$$\operatorname{Def}_M\left(\frac{\mathbb{C}[t]}{(t^{n+1})}\right) \to \operatorname{Def}_M\left(\frac{\mathbb{C}[t]}{(t^2)}\right)$$

is surjective for every $n \geq 2$.

PROOF. According to Theorem V.55 and Corollary V.52 we have $\operatorname{Def}_M = \operatorname{Def}_{R^*}$. Since R^* is an abelian DGLA we have by Proposition V.49

$$\operatorname{Def}_{R^*}(A) = H^1(R^*) \otimes \mathfrak{m}_A = H^1(KS_M) \otimes \mathfrak{m}_A = H^1(M, T_M) \otimes \mathfrak{m}_A.$$

$\square$

4. Gerstenhaber algebras and Schouten brackets

LEMMA VII.13. *Let $(G,\wedge)$ be a graded $\mathbb{Z}$-commutative algebra and let $[,]\colon G[-1]\times G[-1]\to G[-1]$ be a skewsymmetric bilinear map of degree 0 such that*

$$ad_a=[a,-]\in \operatorname{Der}^{\deg(a,G[-1])}(G,G),\qquad \forall a\in G[-1].$$

(Note that this last condition is equivalent to the so-called Odd Poisson identity

$$[a,b\wedge c]=[a,b]\wedge c+(-1)^{\overline{a}(\overline{b}-1)}b\wedge[a,c],$$

$$[a\wedge b,c]=a\wedge[b,c]+(-1)^{\overline{c}(\overline{b}-1)}[a,c]\wedge b,$$

for every $a,b,c\in G[-1]$, $\overline{x}=\deg(x,G[-1])$.)
Let $\mathcal{G}\subset G$ be a set of homogeneous generators of the algebra G, then:

(1) $[,]$ *is uniquely determined by the values $[a,b]$, $a,b\in\mathcal{G}$.*
(2) *A derivation $d\in\operatorname{Der}^n(G,G)$ satisfies $[d,ad_a]=ad_{d(a)}$ for every $a\in G[-1]$ if and only if*
$$d[a,b]=[da,b]+(-1)^{n\overline{a}}[a,db]$$
for every $a,b\in\mathcal{G}$.
(3) $[,]$ *satisfies the Jacobi condition $ad_{[a,b]}=[ad_a,ad_b]$ if and only if*
$$[[a,b],c]=[a,[b,c]]-(-1)^{\overline{a}\,\overline{b}}[b,[a,c]].$$
for every $a,b,c\in\mathcal{G}$.

PROOF. 1) is clear.
If $a\in\mathcal{G}$ then by 2) the derivations $[d,ad_a]$ and $ad_{d(a)}$ take the same values in $\mathcal{G}$ and then $[d,ad_a]=ad_{d(a)}$. The skewsymmetry of $[,]$ implies that for every $b\in G[-1]$ the derivations $[d,ad_b]$ and $ad_{d(b)}$ take the same values in $\mathcal{G}$.
The proof of 3) is made by applying twice 2), first with $d=ad_a$, $a\in\mathcal{G}$, and then with $d=ad_b$, $b\in G[-1]$. □

DEFINITION VII.14. A *Gerstenhaber algebra* is the data of a graded $\mathbb{Z}$-commutative algebra $(G,\wedge)$ and a morphism of graded vector spaces $ad\colon G[-1]\to\operatorname{Der}^*(G,G)$ such that the bracket

$$[,]\colon G[-1]_i\times G[-1]_j\to G[-1]_{i+j},\qquad [a,b]=ad_a(b)$$

induce a structure of graded Lie algebra on $G[-1]$ (cf. [**17**, p. 267]).
A morphism of Gerstenhaber algebras is a morphism of graded algebras commuting with the bracket $[,]$.

For every graded vector space G there exists an isomorphism from the space of bilinear skewsymmetric maps $[,]\colon G[-1]\times G[-1]\to G[-1]$ of degree 0 and the space of bilinear symmetric maps $Q\colon G\times G\to G$ of degree 1; this isomorphism, called *décalage*, is given by the formula[2]

$$Q(a,b)=(-1)^{\deg(a,G[-1])}[a,b].$$

Therefore a Gerstenhaber algebra can be equivalently defined as a graded algebra $(G,\wedge)$ endowed with a bilinear symmetric map $Q\colon G\times G\to G$ of degree 1 satisfying the identities

Odd Poisson $\qquad Q(a,b\wedge c)=Q(a,b)\wedge c+(-1)^{(\overline{a}+1)\overline{b}}b\wedge Q(a,c),$

Jacobi $\qquad Q(a,Q(b,c))=(-1)^{\overline{a}}Q(Q(a,b),c)+(-1)^{\overline{a}\,\overline{b}}Q(b,Q(a,c)),$

where $\overline{a}=\deg(a,G)$, $\overline{b}=\deg(b,G)$.

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[2]The décalage isomorphism is natural up to sign; the choice of $\deg(a,G[-1])$ instead of $\deg(a,G)$ is purely conventional.

EXAMPLE VII.15. (Schouten algebras) A particular class of Gerstenhaber algebras are the so called *Schouten algebras*: here the bracket is usually called *Schouten bracket*.
Consider a commutative $\mathbb{K}$-algebra A_0 and let $A_{-1} \subset \mathrm{Der}_{\mathbb{K}}(A_0, A_0)$ be an A_0-submodule such that $[A_{-1}, A_{-1}] \subset A_{-1}$. Define

$$A = \bigoplus_{i \geq 0} A_{-i}, \qquad A_{-i} = \bigwedge\nolimits_{A_0}^{i} A_{-1}.$$

With the wedge product, A is a graded algebra of nonpositive degrees.
There exists a unique structure of Gerstenhaber algebra $(A, \wedge, [,])$ such that for every $a, b \in A[-1]_1 = A_0$, $f, g \in A[-1]_0 = A_{-1}$

$$ad_a(b) = 0, \quad ad_f(a) = f(a), \quad ad_f(g) = [f, g].$$

In fact A is generated by $A_0 \cup A_{-1}$ and, according to Lemma VII.13, the skew-symmetric bilinear map

$$[\xi_0 \wedge \ldots \wedge \xi_n, h] = \sum_{i=0}^{n} (-1)^{n-i} \xi_i(h) \xi_0 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \xi_n$$

$$[\xi_0 \wedge \ldots \wedge \xi_n, \zeta_0 \wedge \ldots \wedge \zeta_m] =$$
$$= \sum_{i=0}^{n} \sum_{j=0}^{m} (-1)^{i+j} [\xi_i, \zeta_j] \wedge \xi_0 \wedge \ldots \wedge \widehat{\xi_i} \wedge \ldots \wedge \xi_n \wedge \zeta_0 \wedge \ldots \wedge \widehat{\zeta_j} \wedge \ldots \wedge \zeta_m$$

where $h \in A_0$, $\xi_0, \ldots, \xi_n, \zeta_0, \ldots, \zeta_m \in A_{-1}$ is well defined and it is the unique extension of the natural bracket such that $ad(A[-1]) \subset \mathrm{Der}^*(A, A)$.
We need to show that $[,]$ satisfies the Jacobi identity

$$[[a, b], c] = [a, [b, c]] - (-1)^{\bar{a}\bar{b}} [b, [a, c]].$$

Again by Lemma VII.13 we may assume that $0 \leq \bar{a} \leq \bar{b} \leq \bar{c}$. There are 5 possible cases, where the Jacobi identity is satisfied for trivial reasons, as summarized in the following table:

$\bar{a}$	$\bar{b}$	$\bar{c}$	Jacobi is true because..
1	1	1	all terms are $= 0$
0	1	1	all terms are $= 0$
0	0	1	definition of $[,]$ on A_{-1}
0	0	0	Jacobi identity on A_{-1}

EXAMPLE VII.16. Let M be a complex manifold of dimension n, the sheaf of graded algebras $\mathcal{T} = \oplus_{i \leq 0} \mathcal{T}_i$, $\mathcal{T}_i = \mathcal{A}^{0,0}(\bigwedge^{-i} T_M)$, admits naturally a Schouten bracket.
In local holomorphic coordinates $z_1, \ldots, z_n$, since

$$\left[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right] = 0, \qquad \left[\frac{\partial}{\partial z_I}, g\right]_{Sch} = (-1)^{|I|-1} \left(\partial g \vdash \frac{\partial}{\partial z_I}\right),$$

the Odd Poisson identity implies that the Schouten bracket takes the simple form

$$\left[f\frac{\partial}{\partial z_I}, g\frac{\partial}{\partial z_H}\right]_{Sch} = (-1)^{|I|-1} f \left(\partial g \vdash \frac{\partial}{\partial z_I}\right) \wedge \frac{\partial}{\partial z_H} - g\frac{\partial}{\partial z_I} \wedge \left(\partial f \vdash \frac{\partial}{\partial z_H}\right).$$

DEFINITION VII.17. A *differential Gertstenhaber algebra* is a Gerstenhaber algebra $(G, \wedge, [,])$ endowed with a differential $d \in \mathrm{Der}^1(G, G)$ making $(G, d, [,])$ a differential graded Lie algebra.

EXAMPLE VII.18. Given any Gertstenhaber algebra G and an element $a \in G_0 = G[-1]_1$ such that $[a, a] = 0$ we have that $d = ad_a$ gives a structure of differential Gerstenhaber algebra.

EXERCISE VII.19. For every $f \in \mathbb{K}[x_1, \ldots, x_n]$ the Koszul complex of the sequence $\dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_n}$ carries a structure of differential Gerstenhaber algebra. △

5. d-Gerstenhaber structure on polyvector fields

Let M be a fixed complex manifold, then the sheaf of dg-algebras $\mathcal{P}$ defined in Section 2, endowed with the Schouten bracket

$$\left[f\frac{\partial}{\partial z_I}d\overline{z}_J, g\frac{\partial}{\partial z_H}d\overline{z}_K\right]_{Sch} = (-1)^{|J|(|H|-1)}\left[f\frac{\partial}{\partial z_I}, g\frac{\partial}{\partial z_H}\right]_{Sch} d\overline{z}_J\wedge d\overline{z}_K$$

is a sheaf of differential Gerstenhaber algebras.
We have only to verify that locally $\overline{\partial}$ is a derivation of the graded Lie algebra $(\mathcal{P},[,])$: this follows immediately from Lemma VII.13 and from the fact that locally the Kodaira-Spencer DGLA generates $\mathcal{P}$ as a graded algebra.
Via the décalage isomorphism, the Schouten bracket corresponds to the symmetric bilinear map of degree 1 $Q\colon \mathcal{P}\times\mathcal{P}\to\mathcal{P}$ given in local holomorphic coordinates by the formulas

$$Q\left(fd\overline{z}_J\frac{\partial}{\partial z_I}, gd\overline{z}_K\frac{\partial}{\partial z_H}\right) = (-1)^{|K|(|I|-1)+|J|}d\overline{z}_J\wedge d\overline{z}_K Q\left(f\frac{\partial}{\partial z_I}, g\frac{\partial}{\partial z_H}\right),$$

where

$$Q\left(f\frac{\partial}{\partial z_I}, g\frac{\partial}{\partial z_H}\right) = f\left(\partial g\vdash\frac{\partial}{\partial z_I}\right)\wedge\frac{\partial}{\partial z_H} + (-1)^{|I|}g\frac{\partial}{\partial z_I}\wedge\left(\partial f\vdash\frac{\partial}{\partial z_H}\right).$$

Notice that, in the notation of Section 2,

$$Q(\alpha,\beta) = \Delta(\alpha\wedge\beta) - \Delta(\alpha)\wedge\beta - (-1)^{\overline{\alpha}}\alpha\wedge\Delta(\beta)$$

and therefore we also have the following

Lemma VII.20 (Tian-Todorov). *for every $\alpha,\beta\in\mathcal{P}[-1]$,*

$$[\alpha,\beta]_{Sch} = \alpha\wedge\Delta\beta + (-1)^{\deg(\alpha,\mathcal{P}[-1])}(\Delta(\alpha\wedge\beta) - \Delta\alpha\wedge\beta).$$

There exists a natural morphism $\widehat{\ }\colon \mathcal{P}\to \mathcal{H}om(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$ of sheaves of bigraded vector spaces on M given in local coordinates by

$$\widehat{\phi\frac{\partial}{\partial z_I}}(\eta) = \phi\wedge\left(\frac{\partial}{\partial z_I}\vdash\eta\right).$$

Since, for every $\phi\in\mathcal{P}^{0,p} = \mathcal{A}^{0,p}$, $\eta\in\mathcal{A}^{*,*}$, we have

$$\frac{\partial}{\partial z_I}\vdash(\phi\wedge\eta) = (-1)^{p|I|}\phi\wedge\left(\frac{\partial}{\partial z_I}\vdash\eta\right)$$

the hat morphism $\widehat{\ }$ is a morphism of algebras, being the product in $\mathcal{H}om(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$ the composition product. We observe that the composition product is associative and therefore $\mathcal{H}om(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$ has also a natural structure of sheaf of graded Lie algebras. Since $\mathcal{P}$ is graded commutative, $[\widehat{a},\widehat{b}]=0$ for every $a,b\in\mathcal{P}$.

Lemma VII.21. *For every $a,b\in\mathcal{P}$ homogeneous,*

(1) $\widehat{\overline{\partial}a} = [\overline{\partial},\widehat{a}]$.
(2) $\widehat{Q(a,b)} = [[\partial,\widehat{a}],\widehat{b}] = -(-1)^{\overline{a}}\widehat{a}\partial\widehat{b} - (-1)^{\overline{a}\,\overline{b}+\overline{b}}\widehat{b}\partial\widehat{a} \pm \partial\widehat{a}\widehat{b} \pm \widehat{b}\widehat{a}\partial$

Proof. The proof of the first identity is straightforward and left to the reader.
By Jacobi identity,

$$0 = [\partial,[\widehat{a},\widehat{b}]] = [[\partial,\widehat{a}],\widehat{b}] - (-1)^{\overline{a}\,\overline{b}}[[\partial,\widehat{b}],\widehat{a}]$$

and therefore both sides of the equality VII.21 are graded symmetric.
Moreover, since $\widehat{b\wedge c} = \widehat{b}\widehat{c}$ and

$$Q(a,b\wedge c) = Q(a,b)\wedge c + (-1)^{(\overline{a}+1)\overline{b}}b\wedge Q(a,c),$$

$$[[\partial,\widehat{a}],\widehat{b}\widehat{c}] = [[\partial,\widehat{a}],\widehat{b}]\widehat{c} + (-1)^{(\overline{a}+1)\overline{b}}\widehat{b}[[\partial,\widehat{a}],\widehat{c}],$$

it is sufficient to check the equality only when $a,b = f, d\overline{z}_j, \dfrac{\partial}{\partial z_i}$, $f\in\mathcal{P}^{0,0} = \mathcal{A}^{0,0}$.
i) If $\phi\in\mathcal{P}^{0,*}$ then

$$[\partial,\widehat{\phi}]\eta = \partial(\phi\wedge\eta) - (-1)^{\overline{\phi}}\phi\wedge\partial\eta = \partial\phi\wedge\eta.$$

In particular $[\partial, \widehat{d\bar{z}_j}] = 0$, $Q(d\bar{z}_j, b) = 0$ for every b.
ii) If $f, g \in \mathcal{P}^{0,0}$ then $Q(f,g) \in \mathcal{P}^{1,0} = 0$ and

$$[[\partial, \widehat{f}], \widehat{g}]\eta = \partial f \wedge g\eta - g(\partial f \wedge \eta) = 0.$$

If $f \in \mathcal{P}^{0,0}$ then $Q\left(f, \dfrac{\partial}{\partial z_i}\right) = \dfrac{\partial}{\partial z_i} \vdash \partial f = \dfrac{\partial f}{\partial z_i}$ and

$$\left[[\partial, \widehat{f}], \widehat{\frac{\partial}{\partial z_i}}\right]\eta = \partial f \wedge \left(\frac{\partial}{\partial z_i} \vdash \eta\right) + \frac{\partial}{\partial z_i} \vdash (\partial f \wedge \eta) = \left(\frac{\partial}{\partial z_i} \vdash \partial f\right) \wedge \eta$$

where the last equality follows from the Leibniz rule applied to the derivation $\dfrac{\partial}{\partial z_i} \vdash$.
Finally $Q\left(\dfrac{\partial}{\partial z_i}, \dfrac{\partial}{\partial z_j}\right) = 0$; since $\partial, \widehat{\dfrac{\partial}{\partial z_i}}, \widehat{\dfrac{\partial}{\partial z_j}}$ are derivations of $\mathcal{A}^{*,*}$, also $\left[\left[\partial, \widehat{\dfrac{\partial}{\partial z_i}}\right], \widehat{\dfrac{\partial}{\partial z_j}}\right]$ is a derivation of bidegree $(-1,0)$ and then it is sufficient to check the equality for $\eta = dz_i$. This last verification is completely straightforward and it is left to the reader. □

Exercise VII.22. Prove that $\overline{\Omega}^* = \{a \in \mathcal{P} \mid [\partial, \widehat{a}] = 0\}$. △

6. GBV-algebras

In this section $\mathbb{K}$ is a fixed field of characteristic 0.

Definition VII.23. A GBV *(Gerstenhaber-Batalin-Vilkovisky)* algebra is the data of a graded algebra $(G, \wedge)$ and a linear map $\Delta\colon G \to G$ of degree 1 such that:

(1) $\Delta^2 = 0$
(2) The symmetric bilinear map of degree 1

$$Q(a,b) = \Delta(a \wedge b) - \Delta(a) \wedge b - (-1)^{\overline{a}} a \wedge \Delta(b)$$

satisfies the odd Poisson identity

$$Q(a, b \wedge c) = Q(a,b) \wedge c + (-1)^{(\overline{a}+1)\overline{b}} b \wedge Q(a,c).$$

Note that the second condition on the above definition means that for every homogeneous $a \in G$, the linear map $Q(a, -)$ is a derivation of degree $\overline{a} + 1$.
The map Q corresponds, via the décalage isomorphism, to a skewsymmetric bilinear map of degree 0, $[,]\colon G[-1] \times G[-1] \to G[-1]$; the expression of $[,]$ in terms of Δ is

$$[a,b] = a \wedge \Delta(b) + (-1)^{\deg(a, G[-1])}(\Delta(a \wedge b) - \Delta(a) \wedge b).$$

Example VII.24. If Δ is a differential of a graded algebra $(G, \wedge)$, then $Q = 0$ and $(G, \wedge, \Delta)$ is a GBV algebra called *abelian*.

Example VII.25. The sheaf $\mathcal{P}$ of polyvector fields on a complex manifold, endowed with the operator Δ described in Section 2 is a sheaf of GBV algebra.

Exercise VII.26. Let $(G, \wedge, \Delta)$ be a GBV algebra. If G has a unit 1, then $\Delta(1) = 0$. △

Lemma VII.27. *For every $a, b \in G$ homogeneous*

$$\Delta Q(a,b) + Q(\Delta(a), b) + (-1)^{\overline{a}} Q(a, \Delta(b)) = 0.$$

Proof. It is sufficient to write Q in terms of Δ and use $\Delta^2 = 0$. □

Theorem VII.28. *If $(G, \wedge, \Delta)$ is a GBV algebra then $(G[-1], [,], \Delta)$ is a DGLA and therefore $(G, \wedge, Q)$ is a Gerstenhaber algebra.*

Proof. Working in $G[-1]$ (i.e. $\overline{a} = \deg(a, G[-1])$) we have from Lemma VII.27

$$\Delta[a,b] = [\Delta(a), b] + (-1)^{\overline{a}}[a, \Delta(b)]$$

and then we only need to prove the Jacobi identity.
Replacing $a = \alpha$, $b = \beta \wedge \gamma$ in the above formula we have

$$[\alpha, \Delta(\beta \wedge \gamma)] = (-1)^{\overline{\alpha}}(\Delta[\alpha, \beta \wedge \gamma] - [\Delta\alpha, \beta \wedge \gamma])$$

and then $[\alpha, \Delta(\beta\wedge\gamma)]$ is equal to

$$(-1)^{\overline{\alpha}}\Delta([\alpha,\beta]\wedge\gamma)+(-1)^{\overline{\alpha}\,\overline{\beta}}\Delta(\beta\wedge[\alpha,\gamma])-(-1)^{\overline{\alpha}}[\Delta\alpha,\beta]\wedge\gamma+(-1)^{(\overline{\alpha}+1)\overline{\beta}}\beta\wedge[\Delta\alpha,\gamma].$$

Writing

$$[\alpha,[\beta,\gamma]]=[\alpha,\beta\wedge\Delta\gamma]+(-1)^{\overline{\beta}}([\alpha,\Delta(\beta\wedge\gamma)]-[\alpha,\Delta\beta\wedge\gamma]),$$
$$[[\alpha,\beta],\gamma]=[\alpha,\beta]\wedge\Delta\gamma+(-1)^{\overline{\alpha}+\overline{\beta}}(\Delta([\alpha,\beta]\wedge\gamma)-\Delta[\alpha,\beta]\wedge\gamma),$$
$$[\beta,[\alpha,\gamma]]=\beta\wedge\Delta[\alpha,\gamma]+(-1)^{\overline{\beta}}(\Delta(\beta\wedge[\alpha,\gamma])-\Delta\beta\wedge[\alpha,\gamma])$$

we get

$$[\alpha,[\beta,\gamma]]=[[\alpha,\beta],\gamma]+(-1)^{\overline{\alpha}\,\overline{\beta}}[\beta,[\alpha,\gamma]].$$

□

Definition VII.29. Let $(G,\wedge,\Delta)$ be a GBV-algebra and d a differential of degree 1 of $(G,\wedge)$. If $d\Delta+\Delta d=0$ then the gadget $(G,\wedge,\Delta,d)$ is called a *differential GBV algebra.*

Example VII.30. Let $\mathcal{P}$ be the algebra of polyvector fields on a complex manifold M. In the notation of Section 2, $(\mathcal{P},\wedge,\Delta,\overline{\partial})$ is a sheaf of differential GBV algebras if and only if the connection D is integrable.
This happen in particular when M has trivial canonical bundle and D is the trivial connection.

Exercise VII.31. If $(G,\wedge,\Delta,d)$ is a differential GBV-algebra then $(G[-1],[,],d+\hbar\Delta)$ is a DGLA for every $\hbar\in\mathbb{K}$. △

7. Historical survey, VII

The Schouten bracket was introduced by Schouten in [**70**] while the Jacobi identity was proved 15 years later by Nijenhuis [**58**].
The now called Gerstenhaber algebras have been first studied in [**17**] as a structure on the cohomology of an associative ring.
Concrete examples of GBV algebra arising from string theory were studied in 1981 by Batalin and Vilkovisky, while the abstract definition of GBV algebra given in this notes was proposed in [**48**] (cf. also [**75**]).

LECTURE VIII

Graded coalgebras

This Lecture is a basic course on graded coalgebra, with particular emphasis on symmetric graded coalgebra. The aim is give the main definitions and to give all the preliminaries for a satisfactory theory of L_∞-algebras.
Through all the chapter we work over a fixed field $\mathbb{K}$ of characteristic 0. Unless otherwise specified all the tensor products are made over $\mathbb{K}$.
The main references for this Lecture are [**61**, Appendix B] [**22**], [**6**].

1. Koszul sign and unshuffles

Let $V, W \in \mathbf{G}$ be graded vector spaces over $\mathbb{K}$. We recall (Definition IV.2) that the *twisting map* $T\colon V\otimes W\to W\otimes V$ is defined by the rule $T(v\otimes w)=(-1)^{\overline{v}\,\overline{w}}w\otimes v$, for every pair of homogeneous elements $v\in V$, $w\in W$.

The *tensor algebra* generated by $V\in\mathbf{G}$ is by definition the graded vector space

$$T(V)=\bigoplus_{n\geq 0}\bigotimes^n V$$

endowed with the associative product $(v_1\otimes\ldots\otimes v_p)(v_{p+1}\otimes\ldots\otimes v_n)=v_1\otimes\ldots\otimes v_n$.

Let $I\subset T(V)$ be the homogeneous ideal generated by the elements $x\otimes y-T(x\otimes y)$, $x,y\in V$; the *symmetric algebra* generated by V is defined as the quotient

$$S(V)=T(V)/I=\bigoplus_{n\geq 0}\bigodot^n V,\qquad \bigodot^n V=\bigotimes^n V/(\bigotimes^n V\cap I)$$

The product in $S(V)$ is denoted by $\odot$. In particular if $\pi\colon T(V)\to S(V)$ is the projection to the quotient then for every $v_1,\ldots,v_n\in V$, $v_1\odot\ldots\odot v_n=\pi(v_1\otimes\ldots\otimes v_n)$.

The *exterior algebra* generated by V is the quotient of $T(V)$ by the homogeneous ideal J generated by the elements $x\otimes y+T(x\otimes y)$.

$$\bigwedge V=T(V)/J=\bigoplus_{n\geq 0}\bigwedge^n V,\qquad \bigwedge^n V=\bigotimes^n V/(\bigotimes^n V\cap J).$$

Every morphism of graded vector spaces $f\colon V\to W$ induces canonically three homomorphisms of graded algebras

$$T(f)\colon T(V)\to T(W),\qquad S(f)\colon S(V)\to S(W),\qquad \bigwedge(f)\colon\bigwedge V\to\bigwedge W.$$

The following convention is adopted in force: let V,W be graded vector spaces and $F\colon T(V)\to T(W)$ a linear map. We denote by

$$F^i\colon T(V)\to\bigotimes^i W,\quad F_j\colon\bigotimes^j V\to T(W),\quad F^i_j\colon\bigotimes^j V\to\bigotimes^i W$$

the compositions of F with the inclusion $\bigotimes^j V\to T(V)$ and/or the projection $T(W)\to\bigotimes^i W$.
Similar terminology is adopted for linear maps $S(V)\to S(W)$.

If $v_1,\ldots,v_n$ is an ordered tuple of homogeneous elements of V and $\sigma\colon\{1,\ldots,s\}\to\{1,\ldots,n\}$ is any map, we denote $v_\sigma=v_{\sigma 1}\odot v_{\sigma 2}\odot\ldots\odot v_{\sigma s}\in\bigodot^s V$.
If $I\subset\{1,\ldots,n\}$ is a subset of cardinality s we define v_I as above, considering I as a strictly increasing map $I\colon\{1,\ldots,s\}\to\{1,\ldots,n\}$.

If $I_1 \cup \ldots \cup I_a = J_1 \cup \ldots \cup J_b = \{1, \ldots, n\}$ are decompositions of $\{1, \ldots, n\}$ into disjoint subsets, we define the *Koszul sign* $\epsilon\left(V, \begin{matrix} I_1, \ldots, I_a \\ J_1, \ldots, J_b \end{matrix}; \{v_h\}\right) = \pm 1$ by the relation

$$\epsilon\left(V, \begin{matrix} I_1, \ldots, I_a \\ J_1, \ldots, J_b \end{matrix}; \{v_h\}\right) v_{I_1} \odot \ldots \odot v_{I_a} = v_{J_1} \odot \ldots \odot v_{J_b}.$$

Similarly, if σ is a permutation of $\{1, \ldots, n\}$, $\epsilon(V, \sigma; v_1, \ldots, v_n) = \pm 1$ is defined by

$$v_1 \odot \ldots \odot v_n = \epsilon(V, \sigma; v_1, \ldots, v_n)(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(n)}),$$

or more explicitly

$$\epsilon(V, \sigma; v_1, \ldots, v_n) = \prod_{i<j} \left(\frac{\sigma_i - \sigma_j}{|\sigma_i - \sigma_j|}\right)^{\overline{v_i}\,\overline{v_j}}, \qquad \overline{v} = \deg(v; V).$$

For notational simplicity we shall write $\epsilon(\sigma; v_1, \ldots, v_n)$ or $\epsilon(\sigma)$ when there is no possible confusion about V and $v_1, \ldots, v_n$.

The action of the twisting map on $\bigotimes^2 V$ extends naturally, for every $n \geq 0$, to an action of the symmetric group Σ_n on the graded vector space $\bigotimes^n V$. This action can be described by the use of Koszul sign, more precisely

$$\sigma(v_1 \otimes \ldots \otimes v_n) = \epsilon(\sigma; v_1, \ldots, v_n)(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)})$$

Denote by $N\colon S(V) \to T(V)$ the linear map

$$\begin{aligned} N(v_1 \odot \ldots \odot v_n) &= \sum_{\sigma \in \Sigma_n} \epsilon(\sigma; v_1, \ldots, v_n)(v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}) \\ &= \sum_{\sigma \in \Sigma_n} \sigma(v_1 \otimes \ldots \otimes v_n), \quad v_1, \ldots, v_n \in V. \end{aligned}$$

Since $\mathbb{K}$ has characteristic 0, a left inverse of $\pi\colon T(V) \to S(V)$ is given by $\sum_n \dfrac{Id^n}{n!} N$, where, according to our convention, $Id^n\colon T(V) \to \bigotimes^n V$ is the projection.
For every homomorphism of graded vector spaces $f\colon V \to W$, we have

$$N \circ S(f) = T(f) \circ N\colon S(V) \to T(W).$$

The image of $N\colon \bigodot^n V \to \bigotimes^n V$ is contained in the subspace $(\bigotimes^n V)^{\Sigma_n}$ of Σ_n-invariant vectors.

Lemma VIII.1. *In the notation above, let $W \subset \bigotimes^n V$ be the subspace generated by all the vectors $v - \sigma(v)$, $\sigma \in \Sigma_n$, $v \in \bigotimes^n V$.*
Then $\bigotimes^n V = (\bigotimes^n V)^{\Sigma_n} \oplus W$ and $N\colon \bigodot^n V \to (\bigotimes^n V)^{\Sigma_n}$ is an isomorphism with inverse $\dfrac{\pi}{n!}$.

Proof. It is clear from the definition of W that $\pi(W) = 0$; moreover $v - N\dfrac{\pi}{n!}v \in W$ for every $v \in \bigotimes^n V$, and therefore $\mathrm{Im}(N) + W = \bigotimes^n V$.
On the other side if v is Σ_n-invariant then

$$v = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma(v) = \frac{1}{n!} N\pi(v)$$

and therefore $\mathrm{Im}(N) = (\bigotimes^n V)^{\Sigma_n}$, $\mathrm{Im}(N) \cap W \subset \mathrm{Im}(N) \cap \ker(\pi) = 0$. $\square$

For every $0 \leq a \leq n$, the multiplication map $V^{\otimes a} \bigotimes V^{\otimes n-a} \to V^{\otimes n}$ is an isomorphism of graded vector spaces; we denote its inverse by

$$\mathfrak{a}_{a,n-a}\colon V^{\otimes n} \to V^{\otimes a} \bigotimes V^{\otimes n-a},$$

$$\mathfrak{a}_{a,n-a}(v_1 \otimes \ldots \otimes v_n) = (v_1 \otimes \ldots \otimes v_a) \otimes (v_{a+1} \otimes \ldots \otimes v_n).$$

The multiplication $\mu\colon (\bigodot^a V)\otimes(\bigodot^{n-a} V)\to\bigodot^n V$ is surjective but not injective; a left inverse is given by $\mathfrak{l}_{a,n-a}\binom{n}{a}^{-1}$, where

$$\mathfrak{l}_{a,n-a}(v_1\odot\ldots\odot v_n)=\sum\epsilon\left(\begin{matrix} I,I^c\\ \{1,\ldots,n\}\end{matrix};v_1,\ldots,v_n\right)v_I\otimes v_{I^c},$$

the sum is taken over all subsets $I\subset\{1,\ldots,n\}$ of cardinality a and I^c is the complement of I to $\{1,\ldots,n\}$.

Definition VIII.2. The set of *unshuffles* of type (p,q) is the subset $S(p,q)\subset\Sigma_{p+q}$ of permutations σ such that $\sigma(i)<\sigma(i+1)$ for every $i\neq p$.

Since $\sigma\in S(p,q)$ if and only if the restrictions $\sigma\colon\{1,\ldots,p\}\to\{1,\ldots,p+q\}$, $\sigma\colon\{p+1,\ldots,p+q\}\to\{1,\ldots,p+q\}$, are increasing maps, it follows easily that the unshuffles are a set of representatives for the cosets of the canonical embedding of $\Sigma_p\times\Sigma_q$ inside Σ_{p+q}. More precisely for every $\sigma\in\Sigma_{p+q}$ there exists a unique decomposition $\sigma=\tau\rho$ with $\tau\in S(p,q)$ and $\rho\in\Sigma_p\times\Sigma_q$.

Exercise VIII.3. Prove the formula

$$\mathfrak{l}_{a,n-a}(v_1\odot\ldots\odot v_n)=\sum_{\sigma\in S(a,n-a)}\epsilon(\sigma)(v_{\sigma(1)}\odot\ldots\odot v_{\sigma(a)})\otimes(v_{\sigma(a+1)}\odot\ldots\odot v_{\sigma(n)})$$

$\triangle$

Lemma VIII.4. *In the above notation, for every* $0\le a\le n$

$$\mathfrak{a}_{a,n-a}N=(N\otimes N)\mathfrak{l}_{a,n-a}\colon\bigodot^n V\to\bigotimes^a V\otimes\bigotimes^{n-a}V.$$

Proof. Easy exercise. $\square$

Consider two graded vector spaces V,M and a homogeneous linear map $f\colon\bigotimes^m V\to M$. The symmetrization $\widetilde{f}\colon\bigodot^m V\to M$ of f is given by the formula

$$\widetilde{f}(a_1\odot a_2\odot\ldots\odot a_m)=\sum_{\sigma\in\Sigma_m}\epsilon(V,\sigma;a_1,\ldots,a_m)f(a_{\sigma_1}\otimes\ldots\otimes a_{\sigma_m}).$$

If $g\colon\bigotimes^l V\to V$ is a homogeneous linear map of degree k, the (non associative) Gerstenhaber composition product $f\bullet g\colon\bigotimes^{m+l-1}V\to M$ is defined as

$$f\bullet g(a_1\otimes\ldots\otimes a_{m+l-1})=$$
$$=\sum_{i=0}^{m-1}(-1)^{k(\overline{a_1}+\ldots+\overline{a_i})}f(a_1\otimes\ldots\otimes a_i\otimes g(a_{i+1}\otimes\ldots\otimes a_{i+l})\otimes\ldots\otimes a_{m+l-1}).$$

The behavior of $\bullet$ with respect to symmetrization is given in the following lemma.

Lemma VIII.5. (Symmetrization lemma) *In the notation above*

$$\widetilde{f\bullet g}(a_1\odot\ldots\odot a_{m+l-1})=$$
$$=\sum_{\sigma\in S(l,m-1)}\epsilon(V,\sigma;a_1,\ldots,a_m)\widetilde{f}(\widetilde{g}(a_{\sigma_1}\odot\ldots\odot a_{\sigma_l})\odot a_{\sigma_{l+1}}\odot\ldots\odot a_{\sigma_{l+m-1}}).$$

Proof. We give only some suggestion, leaving the details of the proof as exercise. First, it is sufficient to prove the formula in the "universal" graded vector space U with homogeneous basis $a_1,\ldots,a_{m+l-1}$ and b_I, where I ranges over all injective maps $\{1,\ldots,l\}\to\{1,\ldots,m+l-1\}$, b_I is homogeneous of degree $k+\overline{a_{I(1)}}+\ldots+\overline{a_{I(l)}}$ and $g(a_I)=b_I$.
Second, by linearity we may assume that $M=\mathbb{K}$ and f an element of the dual basis of the standard basis of $\bigotimes^m U$.
With these assumption the calculation becomes easy. $\square$

2. Graded coalgebras

Definition VIII.6. A coassociative $\mathbb{Z}$-graded coalgebra is the data of a graded vector space $C=\oplus_{n\in\mathbb{Z}}C^n\in\mathbf{G}$ and of a coproduct $\Delta\colon C\to C\otimes C$ such that:

- Δ is a morphism of graded vector spaces.
- (coassociativity) $(\Delta\otimes Id_C)\Delta=(Id_C\otimes\Delta)\Delta\colon C\to C\otimes C\otimes C$.

The coalgebra is called *cocommutative* if $T\Delta=\Delta$.

For simplicity of notation, from now on with the term *graded coalgebra* we intend a $\mathbb{Z}$-graded coassociative coalgebra.

Definition VIII.7. Let (C,Δ) and (B,Γ) be graded coalgebras. A *morphism* of graded coalgebras $f\colon C\to B$ is a morphism of graded vector spaces that commutes with coproducts, i.e. $\Gamma f=(f\otimes f)\Delta$.
The category of graded coalgebras is denoted by $\mathbf{GC}$.

Exercise VIII.8. A *counity* of a graded coalgebra is a morphism of graded vector spaces $\epsilon\colon C\to\mathbb{K}$ such that $(\epsilon\otimes Id_C)\Delta=(Id_C\otimes\epsilon)\Delta=Id_C$.
Prove that if a counity exists, then it is unique (Hint: $(\epsilon\otimes\epsilon')\Delta=?$). $\triangle$

Example VIII.9. Let $C=\mathbb{K}[t]$ be the polynomial ring in one variable t of even degree. A coalgebra structure is given by

$$\Delta(t^n)=\sum_{i=0}^{n}t^i\otimes t^{n-i}.$$

We left to the reader the verification of the coassociativity, of the commutativity and the existence of the counity.
If the degree of t is equal to 0, then for every sequence $\{f_n\}_{n>0}\subset\mathbb{K}$ it is associated a morphism of coalgebras $f\colon C\to C$ defined as

$$f(1)=1,\qquad f(t^n)=\sum_{s=1}^{n}\sum_{\substack{(i_1,\dots,i_s)\in\mathbb{N}^s\\ i_1+\dots+i_s=n}}f_{i_1}f_{i_2}\dots a_{i_s}t^s.$$

The verification that $\Delta f=(f\otimes f)\Delta$ can be done in the following way: Let $\{x^n\}\subset C^\vee=\mathbb{K}[[x]]$ be the dual basis of $\{t^n\}$. Then for every $a,b,n\in N$ we have:

$$\langle x^a\otimes x^b,\Delta f(t^n)\rangle=\sum_{i_1+\dots+i_a+j_1+\dots+j_b=n}f_{i_1}\dots f_{i_a}f_{j_1}\dots f_{j_b},$$

$$\langle x^a\otimes x^b,f\otimes f\Delta(t^n)\rangle=\sum_s\sum_{i_1+\dots+i_a=s}\sum_{j_1+\dots+j_b=n-s}f_{i_1}\dots f_{i_a}f_{j_1}\dots f_{j_b}.$$

Note that the sequence $\{f_n\}$, $n\geq1$, can be recovered from f by the formula $f_n=\langle x,f(t^n)\rangle$. We shall prove later that every coalgebra endomorphism of $\mathbb{K}[t]$ has this form for some sequence $\{f_n\}$, $n\geq1$.

Lemma-Definition VIII.10. *Let (C,Δ) be a graded coassociative coalgebra, we define recursively $\Delta^0=Id_C$ and, for $n>0$, $\Delta^n=(Id_C\otimes\Delta^{n-1})\Delta\colon C\to\bigotimes^{n+1}C$. Then:*

(1) *For every $0\leq a\leq n-1$ we have*

$$\Delta^n=(\Delta^a\otimes\Delta^{n-1-a})\Delta\colon C\to\bigotimes^{n+1}C,$$

$$\mathfrak{a}_{a+1,n-a}\Delta^n=(\Delta^a\otimes\Delta^{n-1-a})\Delta$$

(2) *For every $s\geq1$ and every $a_0,\dots,a_s\geq0$ we have*

$$(\Delta^{a_0}\otimes\Delta^{a_1}\otimes\dots\otimes\Delta^{a_s})\Delta^s=\Delta^{s+\sum a_i}.$$

In particular, if C is cocommutative then the image of Δ^{n-1} is contained in the set of Σ_n-invariant elements of $\bigotimes^n C$.

(3) *If* $f\colon (C,\Delta)\to(B,\Gamma)$ *is a morphism of graded coalgebras then, for every* $n\ge 1$ *we have*
$$\Gamma^n f=(\otimes^{n+1}f)\Delta^n\colon C\to\bigotimes^{n+1}B.$$

Proof. [1] If $a=0$ or $n=1$ there is nothing to prove, thus we can assume $a>0$ and use induction on n. we have:
$$(\Delta^a\otimes\Delta^{n-1-a})\Delta=((Id_C\otimes\Delta^{a-1})\Delta\otimes\Delta^{n-1-a})\Delta=$$
$$=(Id_C\otimes\Delta^{a-1}\otimes\Delta^{n-1-a})(\Delta\otimes Id_C)\Delta=$$
$$=(Id_C\otimes\Delta^{a-1}\otimes\Delta^{n-1-a})(Id_C\otimes\Delta)\Delta=(Id_C\otimes(\Delta^{a-1}\otimes\Delta^{n-1-a})\Delta)\Delta=\Delta^n.$$
[2] Induction on s, being the case $s=1$ proved in item 1. If $s\ge 2$ we can write
$$(\Delta^{a_0}\otimes\Delta^{a_1}\otimes\ldots\otimes\Delta^{a_s})\Delta^s=(\Delta^{a_0}\otimes\Delta^{a_1}\otimes\ldots\otimes\Delta^{a_s})(Id\otimes\Delta^{s-1})\Delta=$$
$$(\Delta^{a_0}\otimes(\Delta^{a_1}\otimes\ldots\otimes\Delta^{a_s})\Delta^{s-1})\Delta=(\Delta^{a_0}\otimes\Delta^{s-1+\sum_{i>0}a_i})\Delta=\Delta^{s+\sum a_i}.$$
The action of Σ_n on $\bigotimes^n C$ is generated by the operators $T_a=Id_{\bigotimes^a C}\otimes T\otimes Id_{\bigotimes^{n-a-2}C}$, $0\le a\le n-2$, and, if $T\Delta=\Delta$ then
$$T_a\Delta^{n-1}=T_a(Id_{\bigotimes^a C}\otimes\Delta\otimes Id_{\bigotimes^{n-a-2}C})\Delta^{n-2}=$$
$$=(Id_{\bigotimes^a C}\otimes\Delta\otimes Id_{\bigotimes^{n-a-2}C})\Delta^{n-2}=\Delta^{n-1}.$$
[3] By induction on n,
$$\Gamma^n f=(Id_B\otimes\Gamma^{n-1})\Gamma f=(f\otimes\Gamma^{n-1}f)\Delta=(f\otimes(\otimes^n f)\Delta^{n-1})\Delta=(\otimes^{n+1}f)\Delta^n.$$
□

Example VIII.11. Let A be a graded associative algebra with product $\mu\colon A\otimes A\to A$ and C a graded coassociative coalgebra with coproduct $\Delta\colon C\to C\otimes C$.
Then $\mathrm{Hom}^*(C,A)$ is a graded associative algebra with product
$$fg=\mu(f\otimes g)\Delta.$$
We left as an exercise the verification that the product in $\mathrm{Hom}^*(C,A)$ is associative.
In particular $\mathrm{Hom}_{\mathbf{G}}(C,A)=\mathrm{Hom}^0(C,A)$ is an associative algebra and $C^\vee=\mathrm{Hom}^*(C,\mathbb{K})$ is a graded associative algebra. (Notice that in general $A^\vee$ is not a coalgebra.)

Example VIII.12. The dual of the coalgebra $C=\mathbb{K}[t]$ (Example VIII.9) is exactly the algebra of formal power series $A=\mathbb{K}[[x]]=C^\vee$. Every coalgebra morphism $f\colon C\to C$ induces a local homomorphism of $\mathbb{K}$-algebras $f^t\colon A\to A$. Clearly $f^t=0$ only if $f=0$, f^t is uniquely determined by $f^t(x)=\sum_{n>0}f_nx^n$ and then every morphism of coalgebras $f\colon C\to C$ is uniquely determined by the sequence $f_n=\langle f^t(x),t^n\rangle=\langle x,f(t^n)\rangle$.
The map $f\mapsto f^t$ is functorial and then preserves the composition laws.

Definition VIII.13. A graded coassociative coalgebra (C,Δ) is called *nilpotent* if $\Delta^n=0$ for $n>>0$.
It is called *locally nilpotent* if it is the direct limit of nilpotent graded coalgebras or equivalently if $C=\cup_n\ker\Delta^n$.

Example VIII.14. The coalgebra $\mathbb{K}[t]$ of Example VIII.9 is locally nilpotent.

Example VIII.15. Let $A=\oplus A_i$ be a finite dimensional graded associative commutative $\mathbb{K}$-algebra and let $C=A^\vee=\mathrm{Hom}^*(A,\mathbb{K})$ be its graded dual.
Since A and C are finite dimensional, the pairing $\langle c_1\otimes c_2,a_1\otimes a_2\rangle=(-1)^{\overline{a_1}\,\overline{c_2}}\langle c_1,a_1\rangle\langle c_2,a_2\rangle$ gives a natural isomorphism $C\otimes C=(A\otimes A)^\vee$ commuting with the twisting maps T; we may define Δ as the transpose of the multiplication map $\mu\colon A\otimes A\to A$.
Then (C,Δ) is a coassociative cocommutative coalgebra. Note that C is nilpotent if and only if A is nilpotent.

Exercise VIII.16. Let (C,Δ) be a graded coalgebra and $p\colon C\to V$ a morphism of graded vector spaces. We shall say that p *cogenerates* C if for every $c\in C$ there exists $n\geq 0$ such that $(\otimes^{n+1}p)\Delta^n(c)\neq 0$ in $\bigotimes^{n+1}V$.
Prove that every morphism of graded coalgebras $B\to C$ is uniquely determined by its composition $B\to C\to V$ with a cogenerator p. △

2-A. The reduced tensor coalgebra. Given a graded vector space V, we denote $\overline{T(V)}=\bigoplus_{n>0}\bigotimes^n V$. When considered as a subset of $T(V)$ it becomes an ideal of the tensor algebra generated by V.
The *reduced tensor coalgebra* generated by V is the graded vector space $\overline{T(V)}$ endowed with the coproduct $\mathfrak{a}\colon \overline{T(V)}\to\overline{T(V)}\otimes\overline{T(V)}$,

$$\mathfrak{a}=\sum_{n=1}^{\infty}\sum_{a=1}^{n-1}\mathfrak{a}_{a,n-a},\qquad \mathfrak{a}(v_1\otimes\ldots\otimes v_n)=\sum_{r=1}^{n-1}(v_1\otimes\ldots\otimes v_r)\otimes(v_{r+1}\otimes\ldots\otimes v_n)$$

The coalgebra $(\overline{T(V)},\mathfrak{a})$ is coassociative (but not cocommutative) and locally nilpotent; in fact, for every $s>0$,

$$\mathfrak{a}^{s-1}(v_1\otimes\ldots\otimes v_n)=\sum_{1\leq i_1<i_2<\ldots<i_s=n}(v_1\otimes\ldots\otimes v_{i_1})\otimes\ldots\otimes(v_{i_{s-1}+1}\otimes\ldots\otimes v_{i_s})$$

and then $\ker\mathfrak{a}^{s-1}=\bigoplus_{n=1}^{s-1}\bigotimes^n V$.
If $\mu\colon\bigotimes^s\overline{T(V)}\to\overline{T(V)}$ denotes the multiplication map then, for every $v_1,\ldots,v_n\in V$, we have

$$\mu\mathfrak{a}^{s-1}(v_1\otimes\ldots\otimes v_n)=\binom{n-1}{s-1}v_1\otimes\ldots\otimes v_n.$$

For every morphism of graded vector spaces $f\colon V\to W$ the induced morphism of graded algebras

$$T(f)\colon\overline{T(V)}\to\overline{T(W)},\qquad T(f)(v_1\otimes\ldots\otimes v_n)=f(v_1)\otimes\ldots\otimes f(v_n)$$

is also a morphism of graded coalgebras.

Exercise VIII.17. Let $p\colon T(V)\to\overline{T(V)}$ be the projection with kernel $\mathbb{K}=\bigotimes^0 V$ and $\phi\colon T(V)\to T(V)\otimes T(V)$ the unique homomorphism of graded algebras such that $\phi(v)=v\otimes 1+1\otimes v$ for every $v\in V$. Prove that $p\phi=\mathfrak{a}p$. △

If (C,Δ) is locally nilpotent then, for every $c\in C$, there exists $n>0$ such that $\Delta^n(c)=0$ and then it is defined a morphism of graded vector spaces

$$\frac{1}{1-\Delta}=\sum_{n=0}^{\infty}\Delta^n\colon C\to\overline{T(C)}.$$

Proposition VIII.18. *Let (C,Δ) be a locally nilpotent graded coalgebra, then:*

(1) *The map* $\dfrac{1}{1-\Delta}=\sum_{n\geq 0}\Delta^n\colon C\to\overline{T(C)}$ *is a morphism of graded coalgebras.*
(2) *For every graded vector space V and every morphism of graded coalgebras $\phi\colon C\to\overline{T(V)}$, there exists a unique morphism of graded vector spaces $f\colon C\to V$ such that ϕ factors as*

$$\phi=T(f)\frac{1}{1-\Delta}=\sum_{n=1}^{\infty}(\otimes^n f)\Delta^{n-1}\colon C\to\overline{T(C)}\to\overline{T(V)}.$$

Proof. [1] We have

$$\left(\left(\sum_{n\geq 0}\Delta^n\right)\otimes\left(\sum_{n\geq 0}\Delta^n\right)\right)\Delta=\sum_{n\geq 0}\sum_{a=0}^{n}(\Delta^a\otimes\Delta^{n-a})\Delta$$

$$=\sum_{n\geq 0}\sum_{a=0}^{n}\mathfrak{a}_{a+1,n+1-a}\Delta^{n+1}=\mathfrak{a}\left(\sum_{n\geq 0}\Delta^n\right)$$

where in the last equality we have used the relation $\mathfrak{a}\Delta^0 = 0$.
[2] The unicity of f is clear, since by the formula $\phi = T(f)(\sum_{n\geq 0}\Delta^n)$ it follows that f is the composition of ϕ and the projection $\overline{T(V)} \to V$.
To prove the existence of the factorization, take any morphism of graded coalgebras $\phi\colon C \to \overline{T(V)}$ and denote by $\phi^i\colon C \to \bigotimes^i V$ the composition of ϕ with the projection. It is sufficient to show that for every $n > 1$, ϕ^n is uniquely determined by ϕ^1. Now, the morphism condition $\mathfrak{a}\phi = (\phi\otimes\phi)\Delta$ composed with the projection $\overline{T(V)}\otimes\overline{T(V)} \to \bigoplus_{i=1}^{n-1}(\bigotimes^i V \otimes \bigotimes^{n-1} V)$ gives the equality

$$\mathfrak{a}\phi^n = \sum_{i=1}^{n-1}(\phi^i\otimes\phi^{n-i})\Delta, \qquad n\geq 2.$$

Using induction on n, it is enough to observe that the restriction of $\mathfrak{a}$ to $\bigotimes^n V$ is injective for every $n\geq 2$.

□

It is useful to restate part of the Proposition VIII.18 in the following form

COROLLARY VIII.19. *Let V be a fixed graded vector space; for every locally nilpotent graded coalgebra C the composition with the projection $\overline{T(V)} \to V$ induces a bijection*

$$\mathrm{Hom}_{\mathbf{GC}}(C, \overline{T(V)}) = \mathrm{Hom}_{\mathbf{G}}(C, V).$$

When C is a reduced tensor coalgebra, Proposition VIII.18 takes the following more explicit form

COROLLARY VIII.20. *Let U, V be graded vector spaces and $p\colon \overline{T(V)} \to V$ the projection. Given $f\colon \overline{T(U)} \to V$, the linear map $F\colon \overline{T(U)} \to \overline{T(V)}$*

$$F(v_1\otimes\ldots\otimes v_n) = \sum_{s=1}^{n}\sum_{1\leq i_1<i_2<\ldots<i_s=n} f(v_1\otimes\ldots\otimes v_{i_1})\otimes\ldots\otimes f(v_{i_{s-1}+1}\otimes\ldots\otimes v_{i_s})$$

is the unique morphism of graded coalgebras such that $pF = f$.

EXAMPLE VIII.21. Let A be an associative graded algebra. Consider the projection $p\colon \overline{T(A)} \to A$, the multiplication map $\mu\colon \overline{T(A)} \to A$ and its conjugate

$$\mu^* = -\mu T(-1), \qquad \mu^*(a_1\otimes\ldots\otimes a_n) = (-1)^{n-1}\mu(a_1\otimes\ldots\otimes a_n) = (-1)^{n-1}a_1a_2\ldots a_n.$$

The two coalgebra morphisms $\overline{T(A)} \to \overline{T(A)}$ induced by μ and μ^* are isomorphisms, the one inverse of the other.
In fact, the coalgebra morphism $F\colon \overline{T(A)} \to \overline{T(A)}$

$$F(a_1\otimes\ldots\otimes a_n) = \sum_{s=1}^{n}\sum_{1\leq i_1<i_2<\ldots<i_s=n}(a_1a_2\ldots a_{i_1})\otimes\ldots\otimes(a_{i_{s-1}+1}\ldots a_{i_s})$$

is induced by μ (i.e. $pF = \mu$), $\mu^*F(a) = a$ for every $a\in A$ and for every $n\geq 2$

$$\mu^*F(a_1\otimes\ldots\otimes a_n) = \sum_{s=1}^{n}(-1)^{s-1}\sum_{1\leq i_1<i_2<\ldots<i_s=n} a_1a_2\ldots a_n =$$

$$= \sum_{s=1}^{n}(-1)^{s-1}\binom{n-1}{s-1}a_1a_2\ldots a_n = \left(\sum_{s=0}^{n-1}(-1)^s\binom{n-1}{s}\right)a_1a_2\ldots a_n = 0.$$

This implies that $\mu^*F = p$ and therefore, if $F^*\colon \overline{T(A)} \to \overline{T(A)}$ is induced by μ^* then $pF^*F = \mu^*F = p$ and by Corollary VIII.19 F^*F is the identity.

EXERCISE VIII.22. Let A be an associative graded algebra over the field $\mathbb{K}$, for every local homomorphism of $\mathbb{K}$-algebras $\gamma\colon \mathbb{K}[[x]] \to \mathbb{K}[[x]]$, $\gamma(x) = \sum\gamma_n x^n$, we can associate a coalgebra morphism $F_\gamma\colon \overline{T(A)} \to \overline{T(A)}$ induced by the linear map

$$f_\gamma\colon \overline{T(A)} \to A, \qquad f(a_1\otimes\ldots\otimes a_n) = \gamma_n a_1\ldots a_n.$$

Prove the composition formula $F_{\gamma\delta} = F_\delta F_\gamma$. (Hint: Example VIII.12.) △

Exercise VIII.23. A graded coalgebra morphism $F\colon \overline{T(U)} \to \overline{T(V)}$ is surjective (resp.: injective, bijective) if and only if $F_1^1\colon U \to V$ is surjective (resp.: injective, bijective). △

2-B. The reduced symmetric coalgebra.

Definition VIII.24. The reduced symmetric coalgebra is by definition $\overline{S(V)} = \bigoplus_{n>0} \bigodot^n V$, with the coproduct $\mathfrak{l} = \sum_n \sum_{i=0}^{n-1} \mathfrak{l}_{i+1}^{n+1}$,

$$\mathfrak{l}(v_1 \odot \ldots \odot v_n) = \sum_{r=1}^{n-1} \sum_{I\subset\{1,\ldots,n\};|I|=r} \epsilon\left(\begin{matrix} I, I^c \\ \{1,\ldots,n\}\end{matrix}; v_1,\ldots,v_n\right) v_I \otimes v_{I^c}.$$

The verification that $\mathfrak{l}$ is a coproduct is an easy consequence of Lemma VIII.4. In fact, the injective map $N\colon \overline{S(V)} \to \overline{T(V)}$ satisfies the equality $\mathfrak{a}N = (N\otimes N)\mathfrak{l}$ and then N is an isomorphism between $(\overline{S(V)}, \mathfrak{l})$ and the subcoalgebra of symmetric tensors of $(\overline{T(V)}, \mathfrak{a})$.

Remark VIII.25. It is often convenient to think the symmetric algebra as a quotient of the tensor algebra and the symmetric coalgebra as a subset of the tensor coalgebra.

The coalgebra $\overline{S(V)}$ is coassociative without counity. It follows from the definition of $\mathfrak{l}$ that $V = \ker \mathfrak{l}$ and $T\mathfrak{l} = \mathfrak{l}$, where T is the twisting map; in particular $(\overline{S(V)}, \mathfrak{l})$ is cocommutative. For every morphism of graded vector spaces $f\colon V \to W$, the morphism $S(f)\colon \overline{S(V)} \to \overline{S(W)}$ is a morphism of graded coalgebras.

If (C, Δ) is any cocommutative graded coalgebra, then the image of Δ^n is contained in the subspace of symmetric tensors and therefore

$$\frac{1}{1-\Delta} = N \circ \frac{e^{\Delta}-1}{\Delta},$$

where

$$\frac{e^{\Delta}-1}{\Delta} = \sum_{n=1}^{\infty} \frac{\pi}{n!}\Delta^{n-1}\colon C \to \overline{S(C)}.$$

Proposition VIII.26. *Let (C, Δ) be a cocommutative locally nilpotent graded coalgebra, then:*

1. *The map $\dfrac{e^{\Delta}-1}{\Delta}\colon C \to \overline{S(C)}$ is a morphism of graded coalgebras.*
2. *For every graded vector space V and every morphism of graded coalgebras $\phi\colon C \to \overline{S(V)}$, there exists a unique factorization*
$$\phi = S(\phi^1)\frac{e^{\Delta}-1}{\Delta} = \sum_{n=1}^{\infty} \frac{\bigodot^n \phi^1}{n!}\Delta^{n-1}\colon C \to \overline{S(C)} \to \overline{S(V)},$$
where $\phi^1\colon C \to V$ is a morphism of graded vector spaces $f\colon C \to V$. (Note that ϕ^1 is the composition of ϕ and the projection $\overline{S(V)} \to V$.)

Proof. Since N is an injective morphism of coalgebras and $\dfrac{1}{1-\Delta} = N \circ \dfrac{e^{\Delta}-1}{\Delta}$, the proof follows immediately from Proposition VIII.18. □

Corollary VIII.27. *Let C be a locally nilpotent cocommutative graded coalgebra, and V a graded vector space. A morphism $\theta \in \mathrm{Hom}_{\mathbf{G}}(C, \overline{S(V)})$ is a morphism of graded coalgebras if and only if there exists $m \in \mathrm{Hom}_{\mathbf{G}}(C, V) \subset \mathrm{Hom}_{\mathbf{G}}(C, \overline{S(V)})$ such that*

$$\theta = \exp(m) - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} m^n,$$

being the n-th power of m is considered with respect to the algebra structure on $\mathrm{Hom}_{\mathbf{G}}(C, \overline{S(V)})$ *(Example VIII.11).*

Proof. An easy computation gives the formula $m^n = (\bigodot^n m)\Delta^{n-1}$ for the product defined in Example VIII.11. $\square$

Exercise VIII.28. Let V be a graded vector space. Prove that the formula

$$\mathfrak{c}(v_1 \wedge \ldots \wedge v_n) = \sum_{r=1}^{n-1} \sum_{\sigma \in S(r,n-r)} (-1)^{\sigma}\epsilon(\sigma)(v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(r)}) \otimes (v_{\sigma(r+1)} \wedge \ldots \wedge v_{\sigma(n)}),$$

where $(-1)^{\sigma}$ is the signature of the permutation σ, defines a coproduct on $\overline{\bigwedge(V)} = \bigoplus_{n>0} \bigwedge^n V$. The resulting coalgebra is called *reduced exterior coalgebra* generated by V. $\triangle$

3. Coderivations

Definition VIII.29. Let (C, Δ) be a graded coalgebra. A linear map $d \in \operatorname{Hom}^n(C, C)$ is called a *coderivation of degree* n if it satisfies the *coLeibnitz rule*

$$\Delta d = (d \otimes Id_C + Id_C \otimes d)\Delta.$$

A coderivation d is called a *codifferential* if $d^2 = d \circ d = 0$.
More generally, if $\theta\colon C \to D$ is a morphism of graded coalgebras, a morphism of graded vector spaces $d \in \operatorname{Hom}^n(C, D)$ is called a coderivation of degree n (with respect to θ) if

$$\Delta_D d = (d \otimes \theta + \theta \otimes d)\Delta_C.$$

In the above definition we have adopted the Koszul sign convention: i.e. if $x, y \in C$, $f, g \in \operatorname{Hom}^*(C, D)$, $h, k \in \operatorname{Hom}^*(B, C)$ are homogeneous then $(f \otimes g)(x \otimes y) = (-1)^{\overline{g}\,\overline{x}} f(x) \otimes g(y)$ and $(f \otimes g)(h \otimes k) = (-1)^{\overline{g}\,\overline{h}} fh \otimes gk$.

The coderivations of degree n with respect a coalgebra morphism $\theta\colon C \to D$ form a vector space denoted $\operatorname{Coder}^n(C, D; \theta)$.
For simplicity of notation we denote $\operatorname{Coder}^n(C, C) = \operatorname{Coder}^n(C, C; Id)$.

Lemma VIII.30. *Let $C \xrightarrow{\theta} D \xrightarrow{\rho} E$ be morphisms of graded coalgebras. The compositions with θ and ρ induce linear maps*

$$\rho_*\colon \operatorname{Coder}^n(C, D; \theta) \to \operatorname{Coder}^n(C, E; \rho\theta), \qquad f \mapsto \rho f;$$

$$\theta^*\colon \operatorname{Coder}^n(D, E; \rho) \to \operatorname{Coder}^n(C, E; \rho\theta), \qquad f \mapsto f\theta.$$

Proof. Immediate consequence of the equalities

$$\Delta_E \rho = (\rho \otimes \rho)\Delta_D, \qquad \Delta_D \theta = (\theta \otimes \theta)\Delta_C.$$

$\square$

Exercise VIII.31. Let C be a graded coalgebra and $d \in \operatorname{Coder}^1(C, C)$ a codifferential of degree 1. Prove that the triple $(L, \delta, [,])$, where:

$$L = \bigoplus_{n \in \mathbb{Z}} \operatorname{Coder}^n(C, C), \quad [f, g] = fg - (-1)^{\overline{g}\,\overline{f}} gf, \quad \delta(f) = [d, f]$$

is a differential graded Lie algebra. $\triangle$

Lemma VIII.32. *Let V, W be graded vector spaces, $f \in \operatorname{Hom}_{\mathbf{G}}(V, W)$ and $g \in \operatorname{Hom}^m(\overline{S(V)}, W)$. Then the morphism $d \in \operatorname{Hom}^m(\overline{S(V)}, \overline{S(W)})$ defined by the rule*

$$d(v_1 \odot \ldots \odot v_n) = \sum_{\emptyset \neq I \subset \{1,\ldots,n\}} \epsilon\begin{pmatrix} I, I^c \\ \{1,\ldots,n\} \end{pmatrix}; v_1, \ldots, v_n\Big)\, g(v_I) \odot S(f)(v_{I^c})$$

is a coderivation of degree m with respect to the morphism of graded coalgebras $S(f)\colon \overline{S(V)} \to \overline{S(W)}$.

PROOF. Let $v_1, v_2, \dots, v_n$ be fixed homogeneous elements of V, we need to prove that

$$\mathfrak{l}d(v_1 \odot \dots \odot v_n) = (d \otimes S(f) + S(f) \otimes d)\mathfrak{l}(v_1 \odot \dots \odot v_n).$$

If $A \subset W$ is the image of f and $B \subset W$ is the image of g, it is not restrictive to assume that $W = A \oplus B$: in fact we can always factorize

$$\begin{tikzcd}
V \arrow[dr, "(f,0)"'] \arrow[drr, "f"] & & \\
& A \oplus B \arrow[r, "+"] & W \\
\overline{S(V)} \arrow[ur, "(0,g)"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

and apply Lemma VIII.30 to the coalgebra morphism $S(+)\colon \overline{S(A \oplus B)} \to \overline{S(W)}$.
Under this assumption we have $(S(A)B \otimes \overline{S(A)}) \cap (\overline{S(A)} \otimes S(A)B) = \emptyset$ and the image of d is contained in $S(A)B \subset \overline{S(A \oplus B)}$. Therefore the images of $\mathfrak{l}d$ and $(d \otimes S(f) + S(f) \otimes d)\mathfrak{l}$ are both contained in $(S(A)B \otimes \overline{S(A)}) \oplus (\overline{S(A)} \otimes S(A)B)$.
Denoting by $p\colon \overline{S(W)} \otimes \overline{S(W)} \to S(A)B \otimes \overline{S(A)}$ the natural projection induced by the decomposition $W = A \oplus B$, since both the operators $\mathfrak{l}d$ and $(d \otimes S(f) + S(f) \otimes d)\mathfrak{l}$ are invariant under the twisting map, it is sufficient to prove that

$$p\mathfrak{l}d(v_1 \odot \dots \odot v_n) = p(d \otimes S(f))\mathfrak{l}(v_1 \odot \dots \odot v_n).$$

We have (all Koszul signs are referred to $v_1, \dots, v_n$)

$$p\mathfrak{l}d(v_1 \odot \dots \odot v_n) = p\mathfrak{l}\left(\sum_{\emptyset \neq J \subset \{1,\dots,n\}} \epsilon\begin{pmatrix} J, J^c \\ \{1,\dots,n\}\end{pmatrix} g(v_J) \odot S(f)(v_{J^c})\right) =$$

$$= \sum_{\emptyset \neq J \subset I \subset \{1,\dots,n\}} \epsilon\begin{pmatrix} J, J^c \\ \{1,\dots,n\}\end{pmatrix} \epsilon\begin{pmatrix} J, I-J, I^c \\ J, J^c\end{pmatrix} g(v_J) \odot S(f)(v_{I-J}) \otimes S(f)(v_{I^c}) =$$

$$= \sum_{\emptyset \neq J \subset I \subset \{1,\dots,n\}} \epsilon\begin{pmatrix} J, I-J, I^c \\ \{1,\dots,n\}\end{pmatrix} g(v_J) \odot S(f)(v_{I-J}) \otimes S(f)(v_{I^c}).$$

On the other hand

$$p(d \otimes S(f))\mathfrak{l}(v_1 \odot \dots \odot v_n) = p(d \otimes S(f))\left(\sum_I \epsilon\begin{pmatrix} I, I^c \\ \{1,\dots,n\}\end{pmatrix} v_I \otimes v_{I^c}\right) =$$

$$= \sum_{J \subset I} \epsilon\begin{pmatrix} I, I^c \\ \{1,\dots,n\}\end{pmatrix} \epsilon\begin{pmatrix} J, I-J, I^c \\ I, I^c\end{pmatrix} g(v_J) \odot S(f)(v_{I-J}) \otimes S(f)(v_{I^c}) =$$

$$= \sum_{J \subset I} \epsilon\begin{pmatrix} J, I-J, I^c \\ \{1,\dots,n\}\end{pmatrix} g(v_J) \odot S(f)(v_{I-J}) \otimes S(f)(v_{I^c}).$$

□

PROPOSITION VIII.33. *Let V be a graded vector space and C a locally nilpotent cocommutative coalgebra. Then for every coalgebra morphism $\theta\colon C \to \overline{S(V)}$ and every integer n, the composition with the projection $\overline{S(V)} \to V$ gives an isomorphism*

$$\mathrm{Coder}^n(C, \overline{S(V)}; \theta) \to \mathrm{Hom}^n(C, V) = \mathrm{Hom}_{\mathbf{G}}(C, V[n]).$$

PROOF. The injectivity is proved essentially in the same way as in Proposition VIII.18: if $d \in \mathrm{Coder}^n(C, \overline{S(V)}; \theta)$ we denote by $\theta^i, d^i\colon C \to \bigodot^i V$ the composition of θ and d with the projection $\overline{S(V)} \to \bigodot^i V$. The coLeibnitz rule is equivalent to the countable set of equalities

$$\mathfrak{l}^i_a d^i = d^a \otimes \theta^{i-a} + \theta^a \otimes d^{i-a}, \qquad 0 < a < i.$$

Induction on i and the injectivity of

$$\mathfrak{l}\colon \bigoplus_{m=2}^{n} \bigodot^m V \to \bigotimes^2 \Big(\bigoplus_{m=1}^{n-1} \bigodot^m V\Big)$$

show that d is uniquely determined by d^1.
For the surjectivity, consider $g \in \operatorname{Hom}^n(C,V)$; according to Proposition VIII.26 we can write $\theta = S(\theta^1)\dfrac{e^\Delta - 1}{\Delta}$ and, by Lemma VIII.32, the map $d = \delta\dfrac{e^\Delta - 1}{\Delta}$, where $\delta\colon \overline{S(C)} \to \overline{S(V)}$ is given by

$$\delta(c_1 \odot \ldots \odot c_n) = \sum_{i\in\{1,\ldots,n\}} \epsilon\left(\begin{matrix}\{i\},\{i\}^c\\ \{1,\ldots,n\}\end{matrix}; c_1,\ldots,c_n\right) g(c_i) \odot S(\theta^1)(c_{\{i\}^c})$$

is a coderivation of degree n with respect to θ that lifts g. □

Corollary VIII.34. *Let V be a graded vector space, $\overline{S(V)}$ its reduced symmetric coalgebra. The application $Q \mapsto Q^1$ gives an isomorphism of vector spaces*

$$\operatorname{Coder}^n(\overline{S(V)},\overline{S(V)}) = \operatorname{Hom}^n(\overline{S(V)},V)$$

whose inverse is given by the formula

$$Q(v_1 \odot \ldots \odot v_n) = \sum_{k=1}^{n} \sum_{\sigma\in S(k,n-k)} \epsilon(\sigma) Q^1_k(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)}.$$

In particular for every coderivation Q we have $Q^i_j = 0$ for every $i > j$ and then the subcoalgebras $\bigoplus_{i=1}^r \bigodot^i V$ are preserved by Q.

Proof. The isomorphism follows from Proposition VIII.33, while the inverse formula comes from Lemma VIII.32. □

LECTURE IX

L_∞ and EDF tools

In this chapter we introduce the category $\mathbf{L}_\infty$ of L_∞-algebras and we define a sequence of natural transformations

$$\mathbf{DGLA} \to \mathbf{L}_\infty \to \mathbf{PreDef} \to \mathbf{Def}$$

whose composition is the functor $L \mapsto \mathrm{Def}_L$ (cf. V.66).
In all the four categories there is a notion of quasi-isomorphism which is preserved by the above natural transformations: we recall that in the category **Def** quasi-isomorphism means isomorphism in tangent spaces and then by Corollary V.72 every quasi-isomorphism is an isomorphism.
Through all the chapter we work over a fixed field $\mathbb{K}$ of characteristic 0. Unless otherwise specified all the tensor products are made over $\mathbb{K}$.

1. Displacing (Décalage)

For every n and every graded vector space V, the twisting map gives a natural isomorphism

$$\mathrm{dp}_n\colon \bigotimes^n(V[1]) \to (\bigotimes^n)V[n], \qquad V[a] = \mathbb{K}[a] \otimes V$$

$$\mathrm{dp}_n(v_1[1] \otimes \ldots \otimes v_n[1]) = (-1)^{\sum_{i=1}^n (n-i)\deg(v_i;V)}(v_1 \otimes \ldots \otimes v_n)[n], \qquad v[a] = 1[a] \otimes v.$$

It is easy to verify that dp_n, called the *displacing*[1] isomorphism, changes symmetric into skewsymmetric tensors and therefore it induces an isomorphism

$$\mathrm{dp}_n\colon \bigodot^n(V[1]) \to (\bigwedge^n V)[n],$$

$$\mathrm{dp}_n(v_1[1] \odot \ldots \odot v_n[1]) = (-1)^{\sum_{i=1}^n (n-i)\deg(v_i;V)}(v_1 \wedge \ldots \wedge v_n)[n].$$

2. DG-coalgebras and L_∞-algebras

Definition IX.1. By a dg-coalgebra we intend a triple (C, Δ, d), where (C, Δ) is a graded coassociative cocommutative coalgebra and $d \in \mathrm{Coder}^1(C, C)$ is a codifferential. If C has a counit $\epsilon\colon C \to \mathbb{K}$, we assume that $\epsilon d = 0$. The category of dg-coalgebras, where morphisms are morphisms of coalgebras commuting with codifferentials, is denoted by **DGC**.

Example IX.2. If A is a finite dimensional dg-algebra with differential $d\colon A \to A[1]$, then $A^\vee$ (Example VIII.15) is a dg-coalgebra with codifferential the transpose of d.

Lemma IX.3. *Let V be a graded vector space and $Q \in \mathrm{Coder}^1(\overline{S(V)}, \overline{S(V)})$. Then Q is a codifferential, i.e. $Q \circ Q = 0$, if and only if for every $n > 0$ and every $v_1, \ldots, v_n \in V$*

$$\sum_{k+l=n+1} \sum_{\sigma \in S(k,n-k)} \epsilon(\sigma; v_1, \ldots, v_n) Q_l^1(Q_k^1(v_{\sigma(1)} \odot \ldots \odot v_{\sigma(k)}) \odot v_{\sigma(k+1)} \odot \ldots \odot v_{\sigma(n)}) = 0.$$

[1] It is often used the french name *décalage*.

PROOF. Denote $P = Q\circ Q = \frac{1}{2}[Q,Q]$: since P is a coderivation we have that $P=0$ if and only if $P^1 = Q^1\circ Q = 0$. According to Corollary VIII.34

$$Q(v_1\odot\ldots\odot v_n) = \sum_{I\subset\{1,\ldots,n\}} \epsilon\begin{pmatrix} I, I^c \\ \{1,\ldots,n\}\end{pmatrix} Q^1(v_I)\odot v_{I^c}$$

and then

$$P^1(v_1\odot\ldots\odot v_n) = \sum_{I\subset\{1,\ldots,n\}} \epsilon\begin{pmatrix} I, I^c \\ \{1,\ldots,n\}\end{pmatrix} Q^1(Q^1(v_I)\odot v_{I^c}).$$

□

Note that $P^1_n = 0$ whenever $Q^1_m = 0$ for every $m \geq \dfrac{n+1}{2}$ and, if Q is a codifferential in $\overline{S(V)}$ then Q^1_1 is a differential in the graded vector space V.

DEFINITION IX.4. Let V be a graded vector space; a codifferential of degree 1 on the symmetric coalgebra $C(V) = \overline{S(V[1])}$ is called an L_∞-*structure* on V. The dg-coalgebra $(C(V), Q)$ is called an L_∞-*algebra.*
An L_∞-algebra $(C(V), Q)$ is called *minimal* if $Q^1_1 = 0$.

DEFINITION IX.5. A *weak morphism* $F\colon (C(V),Q)\to(C(W),R)$ of L_∞-algebras is a morphism of dg-coalgebras. By an L_∞-morphism we always intend a weak morphism of L_∞-algebras.
A weak morphism F is called a *strong morphism* if there exists a morphism of graded vector spaces $F^1_1\colon V\to W$ such that $F = S(F^1_1)$.
We denote by $\mathbf{L}_\infty$ the category having L_∞-algebras as objects and (weak) L_∞-morphisms as arrows.

Consider now two L_∞-algebras $(C(V),Q)$, $(C(W),R)$ and a morphism of graded coalgebras $F\colon C(V)\to C(W)$. Since $FQ-RF\in \mathrm{Coder}^1(C(V),C(W);F)$, we have that F is an L_∞-morphism if and only if $F^1Q = R^1F$.

LEMMA IX.6. *Consider two L_∞-algebras $(C(V),Q)$, $(C(W),R)$ and a morphism of graded vector spaces $F^1\colon C(V)\to W[1]$. Then*

$$F = S(F^1)\frac{e^{\mathfrak{l}}-1}{\mathfrak{l}}\colon (C(V),Q)\to(C(W),R)$$

is an L_∞-morphism if and only if

$$\sum_{i=1}^n R^1_i F^i_n = \sum_{i=1}^n F^1_i Q^i_n \tag{4}$$

for every $n>0$.

PROOF. According to Proposition VIII.26 F is a morphism of coalgebras. Since $FQ-RF\in\mathrm{Coder}^1(C(V),C(W);F)$, we have that F is an L_∞-morphism if and only if $F^1Q = R^1F$.

□

EXERCISE IX.7. An L_∞-morphism F is strong if and only if $F^1_n = 0$ for every $n\geq 2$. △

If $F\colon (C(V),Q)\to(C(W),R)$ is an L_∞-morphism, then by Lemma IX.6 $R^1_1F^1_1 = F^1_1Q^1_1$ and therefore we have a morphism in cohomology $H(F^1_1)\colon H^*(V[1],Q^1_1)\to H^*(W[1],Q^1_1)$.

DEFINITION IX.8. An L_∞-morphism $F\colon (C(V),Q)\to(C(W),R)$ is a *quasiisomorphism* if $H(F^1_1)\colon H^*(V[1],Q^1_1)\to H^*(W[1],Q^1_1)$ is an isomorphism.

The following exercise shows that the above definition is not ambiguous.

EXERCISE IX.9. An L_∞-morphism $F\colon (C(V),Q)\to(C(W),R)$ is a quasiisomorphism if and only if $H(F)\colon H^*(C(V),Q)\to H^*(C(W),R)$ is an isomorphism. △

Given a coderivation $Q\colon \overline{S(V[1])} \to \overline{S(V[1])}[1]$, their components $Q^1_j\colon \bigodot^n(V[1]) \to V[2]$, composed with the inverse of the displacement isomorphism, give linear maps

$$l_j = (Q^1_j \circ \mathrm{dp}_n^{-1})[-n]\colon \bigwedge^n V \to V[2-n].$$

More explicitly

$$l_j(v_1 \wedge \ldots \wedge v_n) = (-1)^{-n}(-1)^{\sum_{i=1}^n (n-i)\deg(v_i;V)} Q^1_j(v_1[1] \odot \ldots \odot v_n[1])$$

The conditions of Lemma IX.3 become

$$\sum_{\substack{k+i=n+1 \\ \sigma\in S(k,n-k)}} (-1)^{k(i-1)} (-1)^\sigma \epsilon(\sigma) l_i(l_k(v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(k)}) \wedge v_{\sigma(k+1)} \wedge \ldots \wedge v_{\sigma(n)}) = 0.$$

Setting $l_1(v) = d(v)$ and $l_2(v_1 \wedge v_2) = [v_1, v_2]$, the first three conditions ($n = 1, 2, 3$) becomes:

$$\begin{aligned}
1:&\quad d^2 = 0 \\
2:&\quad d[x,y] = [dx,y] + (-1)^{\overline{x}}[x,dy] \\
3:&\quad (-1)^{\overline{x}\,\overline{z}}[[x,y],z] + (-1)^{\overline{y}\,\overline{z}}[[z,x],y] + (-1)^{\overline{x}\,\overline{y}}[[y,z],x] = \\
&\quad = (-1)^{\overline{x}\,\overline{z}+1}(dl_3(x,y,z) + l_3(dx,y,z) + (-1)^{\overline{x}} l_3(x,dy,z) + (-1)^{\overline{x}+\overline{y}} l_3(x,y,dz))
\end{aligned}$$

If $l_3 = 0$ we recognize, in the three formulas above, the axioms defining a differential graded Lie algebra structure on V.

Exercise IX.10. Let $(C(V), Q)$ be an L_∞-algebra. Then the bracket

$$[w_1, w_2] = (-1)^{\deg(w_1;V)} Q^1_2(w_1[1] \odot w_2[1]) = l_2(w_1 \wedge w_2)$$

gives a structure of graded Lie algebra on the cohomology of the complex (V, Q^1_1). △

3. From DGLA to L_∞-algebras

In this section we show that to every DGLA structure on a graded vector space V it is associated naturally a L_∞ structure on the same space V, i.e. a codifferential Q on $C(V) = \overline{S(V[1])}$.
The coderivation Q is determined by its components $Q^1_j\colon \bigodot^j V[1] \to V[2]$.

Proposition IX.11. *Let $(V, d, [\ ,\ ])$ be a differential graded Lie algebra. Then the coderivation Q of components*

(1) $Q^1_1(v[1]) = -d(v)$.
(2) $Q^1_2(w_1[1] \odot w_2[1]) = (-1)^{\deg(w_1;V)}[w_1, w_2]$
(3) $Q^1_j = 0$ *for every* $j \geq 3$.

is a codifferential and then gives an L_∞-structure on V.

Proof. The conditions of Lemma IX.3 are trivially satisfied for every $n > 3$. For $n \leq 3$ they becomes (where $\widehat{x} = x[1]$ and $\overline{x} = \deg(x;V)$):

$$\begin{aligned}
n = 1:&\quad Q^1_1 Q^1_1(\widehat{v}) = d^2(v) = 0 \\
n = 2:&\quad Q^1_1 Q^1_2(\widehat{x} \odot \widehat{y}) + Q^1_2(Q^1_1(\widehat{x}) \odot \widehat{y}) + (-1)^{(\overline{x}-1)(\overline{y}-1)} Q^1_2(Q^1_1(\widehat{y}) \odot \widehat{x}) = \\
&\quad = -(-1)^{\overline{x}}(d[x,y] - [dx,y]) + [x,dy] = 0 \\
n = 3:&\quad Q^1_2(Q^1_2(\widehat{x} \odot \widehat{y}) \odot \widehat{z}) + (-1)^{\overline{x}-1} Q^1_2(\widehat{x} \odot Q^1_2(\widehat{y} \odot \widehat{z})) + \\
&\quad +(-1)^{\overline{x}(\overline{y}-1)} Q^1_2(\widehat{y} \odot Q^1_2(\widehat{x} \odot \widehat{z})) = \\
&\quad = (-1)^{\overline{y}}[[x,y],z] + (-1)^{\overline{y}-1}[x,[y,z]] + (-1)^{(\overline{x}-1)\overline{y}}[x,[y,z]] = 0
\end{aligned}$$

□

It is also clear that every morphism of $DGLA$ $f\colon V\to W$ induces a strong morphism of the corresponding L_∞-algebras $S(f[1])\colon C(V)\to C(W)$. Therefore we get in this way a functor

$$\mathbf{DGLA}\to\mathbf{L}_\infty$$

that preserves quasiisomorphisms.
This functor is faithful; the following example, concerning differential graded Lie algebras arising from Gerstenhaber-Batalin-Vilkovisky algebras, shows that it is not fully faithful.

Let (A,Δ) be a GBV-algebra (Section VII.6); we have seen that $(G[-1],[\,,\,],\Delta)$, where

$$[a,b]=a\Delta(b)+(-1)^{\deg(a,G[-1])}(\Delta(ab)-\Delta(a)b)$$

is a differential graded Lie algebra and then it makes sense to consider the associated L_∞-algebra $(C(G[-1]),\delta)=(\overline{S(G)},\delta)$. The codifferential δ is induced by the linear map of degree 1 $\delta^1=\Delta+Q\in\operatorname{Hom}^1_{\mathbb{K}}(\overline{S(G)},G)$, where $\delta^1_1=\Delta$ and

$$\delta^1_2=Q\colon\bigodot^2 G\to G,\qquad Q(a\odot b)=\Delta(ab)-\Delta(a)b-(-1)^{\overline{a}}a\Delta(b)$$

Lemma IX.12. *In the notation above,*

$$\Delta(a_1a_2\dots a_m)=\sum_{\sigma\in S(1,m-1)}\epsilon(\sigma;a_1,\dots,a_m)\Delta(a_{\sigma_1})a_{\sigma_2}\dots a_{\sigma_m}+$$
$$+\sum_{\sigma\in S(2,m-2)}\epsilon(\sigma;a_1,\dots,a_m)Q(a_{\sigma_1},a_{\sigma_2})a_{\sigma_3}\dots a_{\sigma_m}$$

for every $m\geq 2$ and every $a_1,\dots,a_m\in G$.

Proof. For $m=2$ the above equality becomes

$$\Delta(ab)=\Delta(a)b+(-1)^{\overline{a}}a\Delta(b)+Q(a\odot b)$$

which is exactly the definition of Q.
By induction on m we may assume the Lemma true for all integers $<m$ and then

$$\Delta((a_1a_2)a_3\dots a_m)=\sum_{i=1}^{m}(-1)^{\overline{a_1}+\ldots+\overline{a_{i-1}}}a_1\dots\Delta(a_i)a_{i+1}\dots a_m+$$
$$+\sum_{i\geq 3}\epsilon\,Q(a_1a_2\odot a_i)a_3\dots\widehat{a_i}\dots a_m+\sum_{2<i<j}\epsilon\,Q(a_i\odot a_j)a_1a_2\dots\widehat{a_i}\dots\widehat{a_j}\dots a_m.$$

Replacing the odd Poisson identity

$$Q(a_1a_2\odot a_i)=(-1)^{\overline{a_1}}a_1Q(a_2\odot a_i)+(-1)^{(\overline{a_1}+1)\overline{a_2}}a_2Q(a_1\odot a_i)$$

in the above formula, we obtain the desired equality. □

As an immediate consequence we have

Theorem IX.13. *In the notation above, let $(C(G[-1]),\tau)$ be the (abelian) L_∞-algebra whose codifferential is induced by $\Delta\colon G\to G$. Then the morphism of graded vector spaces $f\colon\overline{S(G)}\to G$,*

$$f(a_1\odot\ldots\odot a_m)=a_1a_2\dots a_m$$

induces an isomorphism of L_∞-algebras $F\colon(C(G[-1]),\delta)\to(C(G[-1]),\tau)$.

Proof. According to Lemmas IX.6 and IX.12 the morphism of graded coalgebras induced by f is an L_∞-morphism.
Moreover, according to Example VIII.21 F is an isomorphism of graded coalgebras whose inverse is induced by

$$g\colon\overline{S(G)}\to G,\qquad g(a_1\odot\ldots\odot a_m)=(-1)^{m-1}a_1a_2\dots a_m.$$

□

4. From L_∞-algebras to predeformation functors

Let $Q \in \mathrm{Coder}^1(C(V), C(V))$ be a L_∞ structure on a graded vector space V, we define the Maurer-Cartan functor $MC_V \colon \mathbf{NA} \to \mathbf{Set}$ by setting:

$$MC_V(A) = \mathrm{Hom}_{\mathbf{DGC}}(A^\vee, C(V)).$$

We first note that the natural isomorphism

$$(C(V) \otimes A)^0 = \mathrm{Hom}_{\mathbf{G}}(A^\vee, C(V)), \qquad (v \otimes a)c = c(a)v$$

is an isomorphism of algebras and then, according to Corollary VIII.27, every coalgebra morphism $\theta \colon A^\vee \to C(V)$ is written uniquely as $\theta = \exp(m) - 1$ for some $m \in (V[1] \otimes A)^0 = \mathrm{Hom}_{\mathbf{G}}(A^\vee, V[1])$. As in Lemma IX.6, θ is a morphism of dg-coalgebras if and only if $md_{A^\vee} = Q^1\theta$; considering m as an element of the algebra $(C(V) \otimes A)^0$ this equality becomes the *Maurer-Cartan equation* of an L_∞-structure:

$$(Id_{V[1]} \otimes d_A)m = \sum_{n=1}^{\infty} \frac{1}{n!}(Q^1_n \otimes Id_A)m^n, \qquad m \in (V[1] \otimes A)^0.$$

Via the décalage isomorphism the Maurer-Cartan equation becomes

$$Id_V \otimes d_A(m) = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{\frac{n(n+1)}{2}}(l_n \otimes Id_A)m \wedge \ldots \wedge m, \qquad m \in (V \otimes A)^1.$$

It is then clear that if the L_∞ structure comes from a DGLA V (i.e. $l_n = 0$ for every $n \geq 3$) then the Maurer-Cartan equation reduces to the classical one.

It is evident that MC_V is a covariant functor and $MC_V(0) = 0$. Let $\alpha \colon A \to C$, $\beta \colon B \to C$ be morphisms in $\mathbf{NA}$, then

$$MC_V(A \times_C B) = MC_V(A) \times_{MC_V(C)} MC_V(B)$$

and therefore MC_V satisfies condition 2) of Definition V.59; in particular it makes sense the tangent space TMC_V.

Proposition IX.14. *The functor MC_V is a predeformation functor with $T^iMC_V = H^{i-1}(V[1], Q^1_1)$.*

Proof. If $A \in \mathbf{NA} \cap \mathbf{DG}$ then

$$MC_V(A) = \{m \in (V \otimes A)^1 \mid Id_V \otimes d_A(m) = -l_1 \otimes Id_A(m)\} = Z^1(V \otimes A)$$

the same computation of V.66 shows that there exists a natural isomorphism $T^iMC_V = H^i(V, l_1) = H^{i-1}(V[1], Q^1_1)$.
Let $0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$ be a small acyclic extension in $\mathbf{NA}$, we want to prove that $MC_V(A) \to MC_V(B)$ is surjective.
We have a dual exact sequence

$$0 \longrightarrow B^\vee \longrightarrow A^\vee \longrightarrow I^\vee \longrightarrow 0, \qquad B^\vee = I^\perp.$$

Since $IA = 0$ we have $\Delta_{A^\vee}(A^\vee) \subset B^\vee \otimes B^\vee$.
Let $\phi \in MC_V(B)$ be a fixed element and $\phi^1 \colon B^\vee \to V[1]$; by Proposition VIII.26 ϕ is uniquely determined by ϕ^1. Let $\psi^1 \colon A^\vee \to V[1]$ be an extension of ϕ^1, then, again by VIII.26, ψ^1 is induced by a unique morphism of coalgebras $\psi \colon A^\vee \to C(V)$.
The map $\psi d_{A^\vee} - Q\psi \colon A^\vee \to C(V)[1]$ is a coderivation and then, setting $h = (\psi d_{I^\vee} - Q\psi)^1 \in \mathrm{Hom}_{\mathbf{G}}(I^\vee, V[2])$, we have that ψ is a morphism of dg-coalgebras if and only if $h = 0$.
Note that ψ^1 is defined up to elements of $\mathrm{Hom}_{\mathbf{G}}(I^\vee, V[1]) = (V[1] \otimes I)^0$ and, since $\Delta_{A^\vee}(A^\vee) \subset B^\vee \otimes B^\vee$, ψ^i depends only by ϕ for every $i > 1$. Since I is acyclic and $hd_{I^\vee} + Q^1_1 h = 0$ there exists $\xi \in \mathrm{Hom}_{\mathbf{G}}(I^\vee, V[1])$ such that $h = \xi d_{I^\vee} - Q^1_1\xi$ and then $\theta^1 = \psi^1 - \xi$ induces a dg-coalgebra morphism $\theta \colon A^\vee \to C(V)$ extending ϕ. $\square$

Therefore the Maurer-Cartan functor can be considered as a functor $\mathbf{L_\infty} \to \mathbf{PreDef}$ that preserves quasiisomorphisms. We have already noted that the composition $\mathbf{DGLA} \to \mathbf{L_\infty} \to \mathbf{PreDef}$ is the Maurer-Cartan functor of DGLAs.

5. From predeformation to deformation functors

We first recall the basics of homotopy theory of dg-algebras.

We denote by $\mathbb{K}[t_1,\ldots,t_n,dt_1,\ldots,dt_n]$ the dg-algebra of polynomial differential forms on the affine space $\mathbb{A}^n$ with the de Rham differential. We have $\mathbb{K}[t,dt]=\mathbb{K}[t]\oplus\mathbb{K}[t]dt$ and

$$\mathbb{K}[t_1,\ldots,t_n,dt_1,\ldots,dt_n]=\bigotimes_{i=1}^{n}\mathbb{K}[t_i,dt_i].$$

Since $\mathbb{K}$ has characteristic 0, it is immediate to see that $H_*(\mathbb{K}[t,dt])=\mathbb{K}[0]$ and then by Künneth formula $H_*(\mathbb{K}[t_1,\ldots,t_n,dt_1,\ldots,dt_n])=\mathbb{K}[0]$. Note that for every dg-algebras A and every $s=(s_1,\ldots,s_n)\in\mathbb{K}^n$ we have an evaluation morphism

$$e_s\colon A\otimes\mathbb{K}[t_1,\ldots,t_n,dt_1,\ldots,dt_n]\to A$$

defined by

$$e_s(a\otimes p(t_1,\ldots,t_n,dt_1,\ldots,dt_n))=p(s_1,\ldots,s_n,0,\ldots,0)a$$

For every dg-algebra A we denote $A[t,dt]=A\otimes\mathbb{K}[t,dt]$; if A is nilpotent then $A[t,dt]$ is still nilpotent. If $A\in\mathbf{NA}$, then $A[t,dt]$ is the direct limit of objects in $\mathbf{NA}$. To see this it is sufficient to consider, for every positive real number $\epsilon>0$, the dg-subalgebra

$$A[t,dt]_\epsilon=A\oplus\oplus_{n>0}(A^{\lceil n\epsilon\rceil}t^n\oplus A^{\lceil n\epsilon\rceil}t^{n-1}dt)\subset A[t,dt],$$

where $A^{\lceil n\epsilon\rceil}$ is the subalgebra generated by all the products $a_1a_2\ldots a_s$, $s\geq n\epsilon$, $a_i\in A$.

It is clear that if $A\in\mathbf{NA}$ then $A[t,dt]_\epsilon\in\mathbf{NA}$ for every $\epsilon>0$ and $A[t,dt]$ is the union of all $A[t,dt]_\epsilon$, $\epsilon>0$.

Lemma IX.15. *For every dg-algebra A the evaluation map $e_h\colon A[t,dt]\to A$ induces an isomorphism $H(A[t,dt])\to H(A)$ independent from $h\in\mathbb{K}$.*

Proof. Let $\imath\colon A\to A[t,dt]$ be the inclusion, since $e_h\imath=Id_A$ it is sufficient to prove that $\imath\colon H(A)\to H(A[t,dt])$ is bijective.
For every $n>0$ denote $B_n=At^n\oplus At^{n-1}dt$; since $d(B_n)\subset B_n$ and $A[t,dt]=\imath(A)\bigoplus_{n>0}B_n$ it is sufficient to prove that $H(B_n)=0$ for every n. Let $z\in Z_i(B_n)$, $z=at^n+nbt^{n-1}dt$, then $0=dz=dat^n+((-1)^ia+db)nt^{n-1}dt$ which implies $a=(-1)^{i-1}db$ and then $z=(-1)^{i-1}d(bt^n)$. □

Definition IX.16. Given two morphisms of dg-algebras $f,g\colon A\to B$, a homotopy between f and g is a morphism $H\colon A\to B[t,dt]$ such that $H_0:=e_0\circ H=f$, $H_1:=e_1\circ H=g$ (cf. [**27**, p. 120]).
We denote by $[A,B]$ the quotient of $\mathrm{Hom}_{\mathbf{DGA}}(A,B)$ by the equivalence relation $\sim$ generated by homotopies.

According to Lemma IX.15, homotopic morphisms induce the same morphism in homology.

Lemma IX.17. *Given morphisms of dg-algebras,*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, bend left, "h"] \arrow[r, bend right, "l"'] & C \end{tikzcd},$$

if $f\sim g$ and $h\sim l$ then $hf\sim lg$.

Proof. It is obvious from the definitions that $hg\sim lg$. For every $a\in\mathbb{K}$ there exists a commutative diagram

$$\begin{tikzcd} B\otimes\mathbb{K}[t,dt] \arrow[r, "h\otimes Id"] \arrow[d, "e_a"] & C\otimes\mathbb{K}[t,dt] \arrow[d, "e_a"] \\ B \arrow[r, "h"] & C \end{tikzcd}.$$

If $F\colon A\to B[t,dt]$ is a homotopy between f and g, then, considering the composition of F with $h\otimes Id$, we get a homotopy between hf and hg. □

Since composition respects homotopy equivalence we can also consider the homotopy categories $K(\mathbf{DGA})$ and $K(\mathbf{NA})$. By definition, the objects of $K(\mathbf{DGA})$ (resp.: $K(\mathbf{NA})$) are the same of $\mathbf{DGA}$ (resp.: $\mathbf{NA}$), while the morphisms are $Mor(A,B)=[A,B]$.
If $A,B\in\mathbf{DG}\cap\mathbf{NA}$, then two morphisms $f,g\colon A\to B$ are homotopic in the sense of IX.16 if and only if f is homotopic to g as morphism of complexes. In particular every acyclic complex is contractible as a dg-algebra.

Lemma IX.18. *A predeformation functor $F\colon\mathbf{NA}\to\mathbf{Set}$ is a deformation functor if and only if F induces a functor $[F]\colon K(\mathbf{NA})\to\mathbf{Set}$.*

Proof. One implication is trivial, since every acyclic $I\in\mathbf{NA}\cap\mathbf{DG}$ is isomorphic to 0 in $K(\mathbf{NA})$.
Conversely, let $H\colon A\to B[t,dt]$ be a homotopy, we need to prove that H_0 and H_1 induce the same morphism from $F(A)$ to $F(B)$. Since A is finite-dimensional there exists $\epsilon>0$ sufficiently small such that $H\colon A\to B[t,dt]_\epsilon$; now the evaluation map $e_0\colon B[t,dt]_\epsilon\to B$ is a finite composition of acyclic small extensions and then, since F is a deformation functor $F(B[t,dt]_\epsilon)=F(B)$. For every $a\in F(A)$ we have $H(a)=iH_0(a)$, where $i\colon B\to B[t,dt]_\epsilon$ is the inclusion and then $H_1(a)=e_1H(a)=e_1iH_0(a)=H_0(a)$.

$\square$

Theorem IX.19. *Let F be a predeformation functor, then there exists a deformation functor F^+ and a natural transformation $\eta\colon F\to F^+$ such that:*

1. *η is a quasiisomorphism.*
2. *For every deformation functor G and every natural transformation $\phi\colon F\to G$ there exists a unique natural transformation $\psi\colon F^+\to G$ such that $\phi=\psi\eta$.*

Proof. We first define a functorial relation $\sim$ on the sets $F(A)$, $A\in\mathbf{NA}$; we set $a\sim b$ if and only if there exists $\epsilon>0$ and $x\in F(A[t,dt]_\epsilon)$ such that $e_0(x)=a$, $e_1(x)=b$. By IX.18 if F is a deformation functor then $a\sim b$ if and only if $a=b$. Therefore if we define F^+ as the quotient of F by the equivalence relation generated by $\sim$ and η as the natural projection, then there exists a unique ψ as in the statement of the theorem. We only need to prove that F^+ is a deformation functor.
Step 1: If $C\in\mathbf{DG}\cap\mathbf{NA}$ is acyclic then $F^+(C)=\{0\}$.
Since C is acyclic there exists a homotopy $H\colon C\to C[t,dt]_\epsilon$, $\epsilon\le 1$, such that $H_0=0$, $H_1=Id$; it is then clear that for every $x\in F(C)$ we have $x=H_1(x)\sim H_0(x)=0$.
Step 2: $\sim$ is an equivalence relation on $F(A)$ for every $A\in\mathbf{NA}$.
This is essentially standard (see e.g. [**27**]). In view of the inclusion $A\to A[t,dt]_\epsilon$ the relation $\sim$ is reflexive. The symmetry is proved by remarking that the automorphism of dg-algebras

$$A[t,dt]\to A[t,dt];\qquad a\otimes p(t,dt)\mapsto a\otimes p(1-t,-dt)$$

preserves the subalgebras $A[t,dt]_\epsilon$ for every $\epsilon>0$.
Consider now $\epsilon>0$ and $x\in F(A[t,dt]_\epsilon)$, $y\in F(A[s,ds]_\epsilon)$ such that $e_0(x)=e_0(y)$; we need to prove that $e_1(x)\sim e_1(y)$.
Write $\mathbb{K}[t,s,dt,ds]=\oplus_{n\ge 0}S^n$, where S^n is the n-th symmetric power of the acyclic complex $\mathbb{K}\,t\oplus\mathbb{K}\,s\xrightarrow{d}\mathbb{K}\,dt\oplus\mathbb{K}\,ds$ and define $A[t,s,dt,ds]_\epsilon=A\oplus\oplus_{n>0}(A^{\lceil n\epsilon\rceil}\otimes S^n)$. There exists a commutative diagram

$$\begin{array}{ccc} A[t,s,dt,ds]_\epsilon & \xrightarrow{t\mapsto 0} & A[s,ds]_\epsilon\\ \Big\downarrow{\scriptstyle s\mapsto 0} & & \Big\downarrow{\scriptstyle s\mapsto 0}\\ A[t,dt]_\epsilon & \xrightarrow{t\mapsto 0} & A\end{array}$$

The kernel of the surjective morphism

$$A[t,s,dt,ds]_\epsilon\xrightarrow{\eta}A[t,dt]_\epsilon\times_A A[t,dt]_\epsilon$$

is equal to $\oplus_{n>0}(A^{\lceil n\epsilon\rceil}\otimes(S^n\cap I))$, where $I\subset\mathbb{K}[t,s,dt,ds]$ is the homogeneous differential ideal generated by $st, sdt, tds, dtds$. Since $I\cap S^n$ is acyclic for every $n>0$, the morphism η is a finite composition of acyclic small extensions.

Let $\xi\in F(A[t,s,dt,ds]_\epsilon)$ be a lifting of (x,y) and let $z\in F(A[u,du]_\epsilon)$ be the image of ξ under the morphism

$$A[t,s,dt,ds]_\epsilon\to A[u,du]_\epsilon,\qquad t\mapsto 1-u,\quad s\mapsto u$$

The evaluation of z gives $e_0(z)=e_1(x)$, $e_1(z)=e_1(y)$.

Step 3: If $\alpha\colon A\to B$ is surjective then

$$F(A[t,dt]_\epsilon)\xrightarrow{(e_0,\alpha)}F(A)\times_{F(B)}F(B[t,dt]_\epsilon)$$

is surjective.

It is not restrictive to assume α a small extension with kernel I. The kernel of (e_0,α) is equal to $\oplus_{n>0}(A^{\lceil n\epsilon\rceil}\cap I)\otimes(\mathbb{K}\,t^n\oplus\mathbb{K}\,t^{n-1}dt)$ and therefore (e_0,α) is an acyclic small extension.

Step 4: The functor F^+ satisfies 2a of V.59.

Let $a\in F(A)$, $b\in F(B)$ be such that $\alpha(a)\sim\beta(b)$; by Step 3 there exists $a'\sim a$, $a'\in F(A)$ such that $\alpha(a')=\beta(b)$ and then the pair (a',b) lifts to $F(A\times_C B)$.

Step 5: The functor F^+ satisfies 2b of V.59.

By V.60 it is sufficient to verify the condition separately for the cases $C=0$ and $B=0$. When $C=0$ the situation is easy: in fact $(A\times B)[t,dt]_\epsilon=A[t,dt]_\epsilon\times B[t,dt]_\epsilon$, $F((A\times B)[t,dt]_\epsilon)=F(A[t,dt]_\epsilon)\times F(B[t,dt]_\epsilon)$ and the relation $\sim$ over $F(A\times B)$ is the product of the relations $\sim$ over $F(A)$ and $F(B)$; this implies that $F^+(A\times B)=F^+(A)\times F^+(B)$.

Assume now $B=0$, then the fibred product $D:=A\times_C B$ is equal to the kernel of α. We need to prove that the map $F^+(D)\to F^+(A)$ is injective. Let $a_0,a_1\in F(D)\subset F(A)$ and let $x\in F(A[t,dt]_\epsilon)$ be an element such that $e_i(x)=a_i$, $i=0,1$. Denote by $\overline{x}\in F(C[t,dt]_\epsilon)$ the image of x by α.

Since C is acyclic there exists a morphism of graded vector spaces $\sigma\colon C\to C[-1]$ such that $d\sigma+\sigma d=Id$ and we can define a morphism of complexes

$$h\colon C\to(\mathbb{K}\,s\oplus\mathbb{K}\,ds)\otimes C\subset C[s,ds]_1;\qquad h(v)=s\otimes v+ds\otimes\sigma(v)$$

The morphism h extends in a natural way to a morphism

$$h\colon C[t,dt]_\epsilon\to(\mathbb{K}\,s\oplus\mathbb{K}\,ds)\otimes C[t,dt]_\epsilon$$

such that for every scalar $\zeta\in\mathbb{K}$ there exists a commutative diagram

$$\begin{array}{ccc}
C[t,dt]_\epsilon & \xrightarrow{h} & (\mathbb{K}\,s\oplus\mathbb{K}\,ds)\otimes C[t,dt]_\epsilon\\
\downarrow{\scriptstyle e_\zeta} & & \downarrow{\scriptstyle Id\otimes e_\zeta}\\
C & \xrightarrow{h} & (\mathbb{K}\,s\oplus\mathbb{K}\,ds)\otimes C
\end{array}$$

Setting $\overline{z}=h(\overline{x})$ we have $\overline{z}_{|s=1}=\overline{x}$, $\overline{z}_{|s=0}=\overline{z}_{|t=0}=\overline{z}_{|t=1}=0$. By step 3 $\overline{z}$ lifts to an element $z\in F(A[t,dt]_\epsilon[s,ds]_1)$ such that $z_{|s=1}=x$; Now the specializations $z_{|s=0}$, $z_{|t=0}$, $z_{|t=1}$ are annihilated by α and therefore give a chain of equivalences in $F(D)$

$$a_0=z_{|s=1,t=0}\sim z_{|s=0,t=0}\sim z_{|s=0,t=1}\sim z_{|s=1,t=1}=a_1$$

proving that $a_0\sim a_1$ inside $F(D)$.

The combination of Steps 1, 4 and 5 tell us that F^+ is a deformation functor.

Step 6: The morphism $\eta\colon F\to F^+$ is a quasiisomorphism.
Let ϵ be of degree $1-i$, $\epsilon^2=0$, then $\mathbb{K}\,\epsilon\oplus I_i$ is isomorphic to the dg-subalgebra

$$\mathbb{K}\,\epsilon\oplus\mathbb{K}\,\epsilon t\oplus\mathbb{K}\,\epsilon dt\subset\mathbb{K}\,\epsilon[t,dt]$$

and the map $p\colon F(I_i)\to F(\mathbb{K}\,\epsilon)$ factors as

$$p\colon F(I_i)\hookrightarrow F(I_i)\oplus F(\mathbb{K}\,\epsilon)=F(\mathbb{K}\,\epsilon\oplus\mathbb{K}\,\epsilon t\oplus\mathbb{K}\,\epsilon dt)\xrightarrow{e_1-e_0}F(\mathbb{K}\,\epsilon).$$

On the other hand the evaluation maps e_0, e_1 factor as

$$e_i\colon \mathbb{K}\,\epsilon[t,dt]\xrightarrow{h}\mathbb{K}\,\epsilon\oplus\mathbb{K}\,\epsilon t\oplus\mathbb{K}\,\epsilon dt\xrightarrow{e_i}\mathbb{K}\,\epsilon,\qquad i=0,1$$

where h is the morphism of dg-vector spaces

$$h(\epsilon t^{n+1})=\epsilon t,\quad h(\epsilon t^n dt)=\frac{\epsilon dt}{n+1},\quad \forall n\geq 0.$$

□

Corollary IX.20. *Let L be a differential graded Lie algebra, then there exists a natural isomorphism $MC_L^+=\mathrm{Def}_L$.*

Proof. According to Theorem IX.19 there exists a natural morphism of functors $\psi\colon MC_L^+\to\mathrm{Def}_L$; by V.66 ψ is a quasiisomorphism and then, by Corollary V.72 ψ is an isomorphism. □

Definition IX.21. Let $(C(V),Q)$ be a L_∞-algebra and let $\mathrm{Def}_V=MC_V^+$ be the deformation functor associated to the predeformation functor MC_V. We shall call Def_V the *deformation functor associated to the L_∞-algebra $(C(V),Q)$*.

A morphism of L_∞-algebras $C(V)\to C(W)$ induces in the obvious way a natural transformation $MC_V\to MC_W$ and then, according to IX.19, a morphism $\mathrm{Def}_V\to\mathrm{Def}_W$. Finally, since $MC_V\to\mathrm{Def}_V$ is a quasiisomorphism we have $T^i\mathrm{Def}_V=H^i(V,Q_1^1)$.

The following result is clear.

Corollary IX.22. *Let $\theta\colon C(V)\to C(W)$ be a morphism of L_∞-algebras. The induced morphism $\mathrm{Def}_V\to\mathrm{Def}_W$ is an isomorphism if and only if $\theta_1^1\colon V\to W$ is a quasiisomorphism of complexes.*

6. Cohomological constraint to deformations of Kähler manifolds

Theorem IX.13 shows that the category of L_∞-algebras is more flexible than the category of differential graded Lie algebras. Another example in this direction is given by the main theorem of [**54**].

Let X be a fixed compact Kähler manifold of dimension n and consider the graded vector space $M_X=\mathrm{Hom}^*_{\mathbb{C}}(H^*(X,\mathbb{C}),H^*(X,\mathbb{C}))$ of linear endomorphisms of the singular cohomology of X. The Hodge decomposition gives natural isomorphisms

$$M_X=\bigoplus_i M_X^i,\qquad M_X^i=\bigoplus_{r+s=p+q+i}\mathrm{Hom}_{\mathbb{C}}(H^p(\Omega_X^q),H^r(\Omega_X^s))$$

and the composition of the cup product and the contraction operator $T_X\otimes\Omega_X^p\xrightarrow{\vdash}\Omega_X^{p-1}$ gives natural linear maps

$$\theta_p\colon H^p(X,T_X)\to\bigoplus_{r,s}\mathrm{Hom}^*_{\mathbb{C}}(H^r(\Omega_X^s),H^{r+p}(\Omega_X^{s-1}))\subset M[-1]_X^p=M_X^{p-1}.$$

By Dolbeault's theorem $H^*(KS_X)=H^*(X,T_X)$ and then the maps θ_p give a morphism of graded vector spaces $\theta\colon H^*(KS_X)\to M[-1]_X$. This morphism is generally nontrivial: consider for instance a Calabi-Yau manifold where the map θ_p induces an isomorphism $H^p(X,T_X)=\mathrm{Hom}_{\mathbb{C}}(H^0(\Omega_X^n),H^p(\Omega_X^{n-1}))$.

Theorem IX.23. *In the above notation, consider $M[-1]_X$ as a differential graded Lie algebra with trivial differential and trivial bracket.*
Every choice of a Kähler metric on X induces a canonical lifting of θ to an L_∞-morphism from KS_X to $M[-1]_X$.

The application of Theorem IX.23 to deformation theory, see [**54**], are based on the idea that L_∞-morphisms induce natural transformations of (extended) deformation functors commuting with tangential actions and obstruction maps (cf. Theorem V.69). Being the deformation functor of the DGLA $M[-1]$ essentially trivial, the lifting of θ impose several constraint on deformations of X.

Denote by:

- $A^{*,*}=\bigoplus_{p,q}A^{p,q}$, where $A^{p,q}=\Gamma(X,\mathcal{A}^{p,q})$ the vector space of global (p,q)-forms.
- $N^{*,*}=\operatorname{Hom}^*_{\mathbb{C}}(A^{*,*},A^{*,*})=\bigoplus_{p,q}N^{p,q}$, where $N^{p,q}=\bigoplus_{i,j}\operatorname{Hom}^*_{\mathbb{C}}(A^{i,j},A^{i+p,j+q})$ is the space of homogeneous endomorphisms of $A^{*,*}$ of bidegree (p,q).

The space $N^{*,*}$, endowed with the composition product and total degree $\deg(\phi)=p+q$ whenever $\phi\in N^{p,q}$, is a graded associative algebra and therefore, with the standard bracket

$$[\phi,\psi]=\phi\psi-(-1)^{\deg(\phi)\deg(\psi)}\psi\phi$$

becomes a graded Lie algebra. We note that the adjoint operator $[\overline{\partial},\ ]\colon N^{*,*}\to N^{*,*+1}$ is a differential inducing a structure of DGLA.

Lemma IX.24. *Let X be a compact Kähler manifold, then there exists $\tau\in N^{1,-1}$ such that:*

(1) *τ factors to a linear map $A^{*,*}/\ker\partial\to\operatorname{Im}\partial$.*
(2) *$[\overline{\partial},\tau]=\partial$.*

In particular $\partial\in N^{1,0}$ is a coboundary in the DGLA $(N^{*,*},[\ ,\ ],[\overline{\partial},\ ])$.

Proof. In the notation of Theorem VI.37 it is sufficient to consider $\tau=\sigma\partial=-\partial\sigma$. Note that the above τ is defined canonically from the choice of the Kähler metric. ☐

We fix a Kähler metric on X and denote by: $\mathcal{H}\subset A^{*,*}$ the graded vector space of harmonic forms, $i\colon\mathcal{H}\to A^{*,*}$ the inclusion and $h\colon A^{*,*}\to\mathcal{H}$ the harmonic projector.
We identify the graded vector space M_X with the space of endomorphisms of harmonic forms $\operatorname{Hom}^*_{\mathbb{C}}(\mathcal{H},\mathcal{H})$. We also we identify $\operatorname{Der}^*(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$ with its image into $N=\operatorname{Hom}^*_{\mathbb{C}}(A^{*,*},A^{*,*})$.

According to Lemma IX.24 there exists $\tau\in N^0$ such that

$$h\partial=\partial h=\tau h=h\tau=\partial\tau=\tau\partial=0,\qquad[\overline{\partial},\tau]=\partial.$$

For simplicity of notation we denote by $L=\oplus L^p$ the $\mathbb{Z}$-graded vector space $KS[1]_X$, this means that $L^p=\Gamma(X,\mathcal{A}^{0,p+1}(T_X))$, $-1\le p\le n-1$. The local description of the two linear maps of degree $+1$, $d\colon L\to L$, $Q\colon\odot^2L\to L$ introduced, up to décalage, in Proposition IX.11 is: if $z_1,\dots,z_n$ are local holomorphic coordinates, then

$$d\left(\phi\frac{\partial}{\partial z_i}\right)=(\overline{\partial}\phi)\frac{\partial}{\partial z_i},\qquad\phi\in\mathcal{A}^{0,*}.$$

If I,J are ordered subsets of $\{1,\dots,n\}$, $a=fd\overline{z}_I\dfrac{\partial}{\partial z_i}$, $b=gd\overline{z}_J\dfrac{\partial}{\partial z_j}$, $f,g\in\mathcal{A}^{0,0}$ then

$$Q(a\odot b)=(-1)^{\overline{a}}d\overline{z}_I\wedge d\overline{z}_J\left(f\frac{\partial g}{\partial z_i}\frac{\partial}{\partial z_j}-g\frac{\partial f}{\partial z_j}\frac{\partial}{\partial z_i}\right),\qquad\overline{a}=\deg(a,L).$$

The formula

$$\begin{aligned}\delta(a_1\odot\ldots\odot a_m)=&\sum_{\sigma\in S(1,m-1)}\epsilon(L,\sigma;a_1,\dots,a_m)da_{\sigma_1}\odot a_{\sigma_2}\odot\ldots\odot a_{\sigma_m}+\\&+\sum_{\sigma\in S(2,m-2)}\epsilon(L,\sigma;a_1,\dots,a_m)Q(a_{\sigma_1}\odot a_{\sigma_2})\odot a_{\sigma_3}\odot\ldots\odot a_{\sigma_m}\end{aligned}\tag{5}$$

gives a codifferential δ of degree 1 on $\overline{S}(L)$ and the differential graded coalgebra $(\overline{S}(L),\delta)$ is exactly the L_∞-algebra associated to the Kodaira-Spencer DGLA KS_X.

If $\operatorname{Der}^p(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$ denotes the vector space of $\mathbb{C}$-derivations of degree p of the sheaf of graded algebras $(\mathcal{A}^{*,*},\wedge)$, where the degree of a (p,q)-form is $p+q$ (note that $\partial,\overline{\partial}\in\operatorname{Der}^1(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$), then we have a morphism of graded vector spaces

$$L\xrightarrow{\widehat{\ }}\operatorname{Der}^*(\mathcal{A}^{*,*},\mathcal{A}^{*,*})=\bigoplus_p\operatorname{Der}^p(\mathcal{A}^{*,*},\mathcal{A}^{*,*}),\qquad a\mapsto\widehat{a}$$

given in local coordinates by

$$\widehat{\phi\frac{\partial}{\partial z_i}}(\eta)=\phi\wedge\left(\frac{\partial}{\partial z_i}\vdash\eta\right).$$

LEMMA IX.25. *If* $[\ ,\ ]$ *denotes the standard bracket on* $\mathrm{Der}^*(\mathcal{A}^{*,*},\mathcal{A}^{*,*})$, *then for every pair of homogeneous* $a,b\in L$ *we have:*

(1) $\widehat{da}=[\overline{\partial},\widehat{a}]=\overline{\partial}\widehat{a}-(-1)^{\overline{a}}\widehat{a}\overline{\partial}$.
(2) $\widehat{Q(a\odot b)}=-[[\partial,\widehat{a}],\widehat{b}]=(-1)^{\overline{a}}\widehat{a}\partial\widehat{b}+(-1)^{\overline{a}\,\overline{b}+\overline{b}}\widehat{b}\partial\widehat{a}\pm\partial\widehat{a}\widehat{b}\pm\widehat{b}\widehat{a}\partial$.

PROOF. This is a special case of Lemma VII.21. $\square$

Consider the morphism

$$F_1\colon L\to M_X,\qquad F_1(a)=h\widehat{a}i.$$

We note that F_1 is a morphism of complexes, in fact $F_1(da)=h\widehat{da}i=h(\overline{\partial}\widehat{a}\pm\widehat{a}\overline{\partial})i=0$. By construction F_1 induces the morphism θ in cohomology and therefore the theorem is proved whenever we lift F_1 to a morphism of graded vector spaces $F\colon\overline{S}(L)\to M_X$ such that $F\circ\delta=0$.

Define, for every $m\geq 2$, the following morphisms of graded vector spaces

$$f_m\colon\bigotimes^m L\to M_X,\qquad F_m\colon\bigodot^m L\to M_X,\qquad F=\sum_{m=1}^{\infty}F_m\colon\overline{S}(L)\to M_X,$$

$$f_m(a_1\otimes a_2\otimes\ldots\otimes a_m)=h\widehat{a}_1\tau\widehat{a}_2\tau\widehat{a}_3\ldots\tau\widehat{a_m}i.$$

$$F_m(a_1\odot a_2\odot\ldots\odot a_m)=\sum_{\sigma\in\Sigma_m}\epsilon(L,\sigma;a_1,\ldots,a_m)f_m(a_{\sigma_1}\otimes\ldots\otimes a_{\sigma_m}).$$

THEOREM IX.26. *In the above notation* $F\circ\delta=0$ *and therefore*

$$\Theta=\sum_{m=1}^{\infty}\frac{1}{m!}F^{\odot m}\circ\Delta^{m-1}_{C(KS_X)}\colon(C(KS_X),\delta)\to(C(M[-1]_X),0)$$

is an L_∞*-morphism with linear term* F_1.

PROOF. We need to prove that for every $m\geq 2$ and $a_1,\ldots,a_m\in L$ we have

$$F_m\left(\sum_{\sigma\in S(1,m-1)}\epsilon(L,\sigma)da_{\sigma_1}\odot a_{\sigma_2}\odot\ldots\odot a_{\sigma_m}\right)=$$

$$=-F_{m-1}\left(\sum_{\sigma\in S(2,m-2)}\epsilon(L,\sigma)Q(a_{\sigma_1}\odot a_{\sigma_2})\odot a_{\sigma_3}\odot\ldots\odot a_{\sigma_m}\right),$$

where $\epsilon(L,\sigma)=\epsilon(L,\sigma;a_1,\ldots,a_m)$.
It is convenient to introduce the auxiliary operators $q\colon\bigotimes^2 L\to N[1]$, $q(a\otimes b)=(-1)^{\overline{a}}\widehat{a}\partial\widehat{b}$ and $g_m\colon\bigotimes^m L\to M[1]_X$,

$$g_m(a_1\otimes\ldots\otimes a_m)=-\sum_{i=0}^{m-2}(-1)^{\overline{a_1}+\overline{a_2}+\ldots+\overline{a_i}}h\widehat{a}_1\tau\ldots\widehat{a}_i\tau q(a_{i+1}\otimes a_{i+2})\tau\widehat{a_{i+3}}\ldots\tau\widehat{a_m}i.$$

Since for every choice of operators $\alpha=h,\tau$ and $\beta=\tau,i$ and every $a,b\in L$ we have

$$\alpha\widehat{Q(a\odot b)}\beta=\alpha((-1)^{\overline{a}}\widehat{a}\partial\widehat{b}+(-1)^{\overline{a}\,\overline{b}+\overline{b}}\widehat{b}\partial\widehat{a})\beta=\alpha(q(a\otimes b)+(-1)^{\overline{a}\,\overline{b}}q(b\otimes a))\beta,$$

the symmetrization lemma VIII.5 gives

$$\sum_{\sigma\in\Sigma_m}\epsilon(L,\sigma)g_m(a_{\sigma_1}\otimes\ldots\otimes a_{\sigma_m})=-F_{m-1}\left(\sum_{\sigma\in S(2,m-2)}\epsilon(L,\sigma)Q(a_{\sigma_1}\odot a_{\sigma_2})\odot a_{\sigma_3}\odot\ldots\odot a_{\sigma_m}\right).$$

On the other hand

$$f_m\left(\sum_{i=0}^{m-1}(-1)^{\overline{a_1}+\ldots+\overline{a_i}}a_1\otimes\ldots\otimes a_i\otimes da_{i+1}\otimes\ldots\otimes a_m\right)=$$

$$=\sum_{i=0}^{m-1}(-1)^{\overline{a_1}+\ldots+\overline{a_i}}h\widehat{a_1}\ldots\widehat{a_i}\tau(\overline{\partial}\widehat{a_{i+1}}-(-1)^{\overline{a_{i+1}}}\widehat{a_{i+1}}\overline{\partial})\tau\ldots\tau\widehat{a_m}i$$

$$=\sum_{i=0}^{m-2}(-1)^{\overline{a_1}+\ldots+\overline{a_i}}h\widehat{a_1}\ldots\widehat{a_i}\tau(-(-1)^{\overline{a_{i+1}}}\widehat{a_{i+1}}\overline{\partial}\tau\widehat{a_{i+2}}+(-1)^{\overline{a_{i+1}}}\widehat{a_{i+1}}\tau\overline{\partial}\widehat{a_{i+2}})\tau\ldots\tau\widehat{a_m}i$$

$$=-\sum_{i=0}^{m-2}(-1)^{\overline{a_1}+\ldots+\overline{a_i}}h\widehat{a_1}\ldots\widehat{a_i}\tau((-1)^{\overline{a_{i+1}}}\widehat{a_{i+1}}[\overline{\partial},\tau]\widehat{a_{i+2}})\tau\ldots\tau\widehat{a_m}i$$

$$=-\sum_{i=0}^{m-2}(-1)^{\overline{a_1}+\ldots+\overline{a_i}}h\widehat{a_1}\ldots\widehat{a_i}\tau q(a_{i+1}\otimes a_{i+2})\tau\ldots\tau\widehat{a_m}i$$

$$=g_m(a_1\otimes\ldots\otimes a_m).$$

Using again Lemma VIII.5 we have

$$\sum_{\sigma\in\Sigma_m}\epsilon(L,\sigma)g_m(a_{\sigma_1}\otimes\ldots\otimes a_{\sigma_m})=F_m\left(\sum_{\sigma\in S(1,m-1)}\epsilon(L,\sigma)da_{\sigma_1}\odot a_{\sigma_2}\odot\ldots\odot a_{\sigma_m}\right).$$

□ □

Remark. If X is a Calabi-Yau manifold with holomorphic volume form Ω, then the composition of F with the evaluation at Ω induces an L_∞-morphism $C(KS_X)\to C(\mathcal{H}[n-1])$. For every $m\geq 2$, $\mathrm{ev}_\Omega\circ F_m\colon \bigodot^m L\to\mathcal{H}[n]$ vanishes on $\bigodot^m\{a\in L\,|\,\partial(a\vdash\Omega)=0\}$.

7. Historical survey, IX

L_∞-algebras, also called *strongly homotopy Lie algebras*, are the Lie analogue of the A_∞ (strongly homotopy associative algebras), introduced by Stasheff [**74**] in the context of algebraic topology.

The popularity of L_∞-algebras has been increased recently by their application in deformation theory (after [**68**]), in deformation quantization (after [**44**]) and in string theory (after [**82**], cf. also [**47**]).

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