THE THOM-WHITNEY-SULLIVAN CONSTRUCTION

1. Simplicial objects

Let $\mathbf{\Delta}$ be the category of finite ordinals: the objects are objects are $[0] = \{0\}$, $[1] = \{0, 1\}$, $[2] = \{0, 1, 2\}$ ecc. and morphisms are the non decreasing maps.

Finally $\mathbf{\Delta}_{\rm mon}$ is the category with the same objects as above and whose morphisms are order-preserving injective maps among them.

In order to avoid heavy notations it is convenient to denote also $[n] = \emptyset$ for every $n < 0$ and write

$$M(n, m) = \mathrm{Mor}_{\mathbf{\Delta}}([n], [m]) = \{f : \{0, 1, \dots, n\} \to \{0, 1, \dots, m\} \mid f(i) \le f(i + 1)\},$$

$$I(n, m) = \mathrm{Mor}_{\mathbf{\Delta}_{\rm mon}}([n], [m]) = \{f : \{0, 1, \dots, n\} \to \{0, 1, \dots, m\} \mid f(i) < f(i + 1)\}.$$

Every morphism in $\mathbf{\Delta}_{\rm mon}$, different from the identity, is a finite composition of *face* morphisms:

$$\partial_k : [i - 1] \to [i], \qquad \partial_k(p) = \begin{cases} p & \text{if } p < k \\ p + 1 & \text{if } k \le p \end{cases}, \qquad k = 0, \dots, i.$$

Equivalently ∂_k is the unique strictly monotone map whose image misses k.

The relations about compositions of them are generated by

$$\partial_l \partial_k = \partial_{k+1} \partial_l\,, \qquad \text{for every } l \le k.$$

Definition 1.1 ([We94])**.** Let $\mathbf{C}$ be a category:

1. A *cosimplicial* object in $\mathbf{C}$ is a covariant functor $A^\Delta : \mathbf{\Delta} \to \mathbf{C}$.
2. A *semicosimplicial* object in $\mathbf{C}$ is a covariant functor $A^\Delta : \mathbf{\Delta}_{mon} \to \mathbf{C}$.
3. A *simplicial* object in $\mathbf{C}$ is a contravariant functor $A_\Delta : \mathbf{\Delta} \to \mathbf{C}$.
4. A *semisimplicial* object in $\mathbf{C}$ is a contravariant functor $A_\Delta : \mathbf{\Delta}_{mon} \to \mathbf{C}$.

Notice that a semicosimplicial object A^Δ is a diagram in $\mathbf{C}$:

$$A_0 \rightrightarrows A_1 \; \substack{\rightarrow\\\rightarrow\\\rightarrow} \; A_2 \; \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots,$$

where each A_i is in $\mathbf{C}$, and, for each $i > 0$, there are $i + 1$ morphisms

$$\partial_k : A_{i-1} \to A_i, \qquad k = 0, \dots, i,$$

such that $\partial_l \partial_k = \partial_{k+1} \partial_l$, for any $l \le k$.

Example 1.2. Let $\mathbb{K}$ be a field. Define the standard n-simplex over $\mathbb{K}$ as the affine space

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{K}^{n+1} \mid t_0 + t_1 + \cdots + t_n = 1\}.$$

The vertices of Δ^n are the points

$$e_0 = (1, 0, \dots, 0), \quad e_1 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1).$$

Then the family $\{\Delta^n\}$, $n \ge 0$, is a cosimplicial affine space, where for every monotone map $f : [n] \to [m]$ we set $f : \Delta^n \to \Delta^m$ as the affine map such that $f(e_i) = e_{f(i)}$. Equivalently $f(t_0, \dots, t_n) = \sum t_i e_{f(i)} = (u_0, \dots, u_m)$, where

$$u_i = \sum_{\{j \mid f(j) = i\}} t_j \qquad (\text{we intend that } \sum_{\emptyset} t_j = 0).$$

In particular, for $m = n + 1$ we have

$$\partial_k(t_0, \dots, t_n) = (t_0, \dots, t_{k-1}, 0, t_k, \dots, t_n),$$

and this explain why ∂_k is called face map.

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Example 1.3 ([FHT01])**.** For every $0 \le p \le n$, let A_n^p be the vector space of polynomial differential p-forms on the standard n-simplex Δ^n. Then, the space of polynomial differential forms on the standard n-simplex

$$A_n = \bigoplus_{p=0}^n A_n^p = \frac{\mathbb{K}[t_0,\ldots,t_n,dt_0,\ldots,dt_n]}{(1-\sum t_i, \sum dt_i)}$$

is a differential graded algebra. Notice that there exists a natural isomorphism of differential graded algebras

$$\mathbb{K}[t_1,\ldots,t_n,dt_1,\ldots,dt_n] \to A_n.$$

Since every affine map $f\colon \Delta^n \to \Delta^m$ induce by pull-back a morphism of differential graded algebra $f^*\colon A_m \to_n$ we have that the sequence $\{A_n\}$ is a simplicial differential graded algebra.

In particular the face maps $\partial_k^*\colon A_n^p \to A_{n-1}^p$, $k = 0,\ldots,n$, are given by pull-back of forms under the inclusion of standard simplices

$$(t_0,\ldots,t_{n-1}) \mapsto (t_0,\ldots,t_{k-1},0,t_k,\ldots,t_{n-1}).$$

2. Integration and Stokes formula

Lemma 2.1. *Let $\mathbb{K}$ be a field of characteristic 0, then there exists a unique sequence of linear maps*

$$\int_{\Delta^n} \colon A_n \to \mathbb{K}, \qquad n \ge 0,$$

such that:

(1) $\displaystyle\int_{\Delta^n} \eta = 0$ *if* $\eta \in A_n^p$ *and* $p \neq n$.

(2) $\displaystyle\int_{\Delta^0} \colon A_0^0 = \frac{\mathbb{K}[t_0]}{(t_0-1)} \to \mathbb{K}, \qquad \int_0 p(t_0) = p(1).$

(3) $\displaystyle\int_{\Delta^n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{k_0!k_1!\cdots k_n!}{(k_0+k_1+\cdots+k_n+n)!}.$

(4) *(Stokes formula) For every $n > 0$ and $\omega \in A_n^{n-1}$, we have*

$$\int_{\Delta^n} d\omega = \sum_{k=0}^n (-1)^k \int_{\Delta^{n-1}} \partial_k^* \omega.$$

Proof. The unicity follows from the first two conditions. To prove the existence, define

$$\int_{\Delta^n} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{k_1!\cdots k_n!}{(k_1+\cdots+k_n+n)!}$$

and extend by $\mathbb{K}$ linearity to a map $\int_n \colon A_n^n \to \mathbb{K}$. We first prove by induction on k_0 the formula

$$\int_{\Delta^n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{k_0!k_1!\cdots k_n!}{(k_0+k_1+\cdots+k_n+n)!}.$$

Assume $k_0 > 0$ and denote $a = (k_0-1)!k_1!\cdots k_n!$, $b = k_0+k_1+\cdots+k_n+n$. Since

$$t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} = t_0^{k_0-1} t_1^{k_1} \cdots t_n^{k_n} (1 - \sum_{i=1}^n t_i),$$

by induction hypothesis, we have

$$\int_{\Delta^n} t_0^{k_0} t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge dt_n = \frac{a}{(b-1)!} - \sum_{i=1}^n \frac{a}{b!}(k_i+1)$$

$$= \frac{a}{(b-1)!} - \frac{a}{b!}(b-k_0) = \frac{ab - a(b-k_0)}{b!} = \frac{k_0 a}{b!}.$$

Notice that the symmetric group $\mathfrak{S}_{n+1}$ acts on $(A_{PL})_n$ by permutation of indices and, for every $\sigma \in \mathfrak{S}_{n+1}$, we have

$$\int_{\Delta^n} \sigma(\omega) = (-1)^\sigma \int_{\Delta^n} \omega.$$

(It is sufficient to check the above identity for transpositions).

By linearity, it is sufficient to prove Stokes formula for ω of type

$$\omega = t_1^{k_1} \cdots t_n^{k_n} dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n.$$

Up to permutation of indices, we may assume $i = n$. Assume first $k_n = 0$, i.e.,

$$\omega = t_1^{k_1} \cdots t_{n-1}^{k_{n-1}} dt_1 \wedge \cdots \wedge dt_{n-1}.$$

In this case, $d\omega = 0$, $\partial_k^* \omega = 0$ for every $k \neq 0, n$, and

$$\partial_0^* \omega = t_0^{k_1} \cdots t_{n-2}^{k_{n-1}} dt_0 \wedge \cdots \wedge dt_{n-2} = (-1)^{n-1} t_0^{k_1} \cdots t_{n-2}^{k_{n-1}} dt_1 \wedge \cdots \wedge dt_{n-1},$$

$$\partial_n^* \omega = t_1^{k_1} \cdots t_{n-1}^{k_{n-1}} dt_1 \wedge \cdots \wedge dt_{n-1};$$

therefore

$$\int_{\Delta^{n-1}} \partial_0^* \omega + (-1)^n \int_{\Delta^{n-1}} \partial_n^* \omega = 0.$$

Next, assume $k_n > 0$, then $\partial_k^* \omega = 0$ for every $k \neq 0$, and

$$\int_{\Delta^n} d\omega = \int_{\Delta^n} (-1)^{n-1} k_n t_1^{k_1} \cdots t_n^{k_n - 1} dt_1 \wedge \cdots \wedge dt_n = \frac{(-1)^{n-1} k_1! \cdots k_n!}{(k_1 + \cdots + k_n + n - 1)!},$$

$$\int_{n-1} \partial_0^* \omega = \int_{\Delta^{n-1}} t_0^{k_1} \cdots t_{n-1}^{k_n} dt_0 \wedge \cdots \wedge dt_{n-2}$$

$$= (-1)^{n-1} \int_{\Delta^{n-1}} t_0^{k_1} \cdots t_{n-1}^{k_n} dt_1 \wedge \cdots \wedge dt_{n-1} = \frac{(-1)^{n-1} k_1! \cdots k_n!}{(k_1 + \cdots + k_n + n - 1)!}.$$

□

Exercise Prove that for $\mathbb{K} = \mathbb{R}$ the operator $\int_{\Delta^n}$ is equal to the usual integration on the topological simplex $\Delta^n \cap \{t_i \geq 0 \ \forall i\}$.

3. The Thom-Whitney-Sullivan construction

Here we consider only the semicosimplicial case; the sare results holds, with minor modification also in the cosimplicial case.

Let

$$V^\Delta: \quad V_0 \rightrightarrows V_1 \ \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} \ V_2 \ \substack{\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow\\[-1em]\longrightarrow} \cdots,$$

be a semicosimplicial vector space. Then the graded vector space $\bigoplus_{n\geq 0} V_n[-n]$ has two differentials

$$d = \sum_n (-1)^n d_n, \qquad \text{where} \quad d_n \text{ is the differential of } V_n,$$

and

$$\partial = \sum_i (-1)^i \partial_i, \qquad \text{where} \quad \partial_i \text{ are the face maps.}$$

More explicitly, if $v \in V_n^i$, then the degree of v is $i + n$ and

$$d(v) = (-1)^n d_n(v) \in V_n^{i+1}, \qquad \partial(v) = \partial_0(v) - \partial_1(v) + \cdots + (-1)^{n+1} \partial_{n+1}(v) \in V_{n+1}^i.$$

Since $d^2 = \partial^2 = d\partial + \partial d = 0$ the following definition makes sense:

Definition 3.1. The *normal complex* of V^Δ is the differentiao graded vector space

$$N(V^\Delta) = (\bigoplus_{n\geq 0} V_n[-n], d + \partial).$$

The above definition of normal complex is valid only in the semicosimplicial case. In the cosimplicial case we have $N(V^\Delta) = (\bigoplus_{n\geq 0} K_n[-n], d + \partial)$ where $K_0 = V_0$ and

$$K_n = \bigcap_{f \in M(n, n-1)} \ker(f \colon V_n \to V_{n-1}), \qquad n > 0.$$

Definition 3.2. The Thom-Whitney-Sullivan differential graded vector space of V^Δ is

$$TW(V^\Delta) = \operatorname{Tot}(\bigoplus_{p,q} TW(V^\Delta)^{p,q}, d, \partial)$$

where

$$TW(V^\Delta)^{p,q} = \{(x_n) \in \prod_{n\geq 0} A_n^p \otimes V_n^q \mid (\partial_k^* \otimes Id) x_n = (Id \otimes \partial_k) x_{n-1}, \quad \text{for every} \ \ 0 \leq k \leq n\}.$$

It is immediate to see that $TW(V^\Delta)$ is a differential graded subspace of the total complex of the double complex $\bigoplus_{p,q} \prod_{n\geq 0} A_n^p \otimes V_n^q$.

Theorem 3.3 (Whitney). *The map*

$$I\colon TW(V^{\Delta}) \to N(V^{\Delta})$$

induced by

$$TW(V^{\Delta})^{p,q} \xrightarrow{\textit{inclusion}} \prod_{n\geq 0} A_n^p \otimes V_n^q \xrightarrow{\textit{projection}} A_p^p \otimes V_p^q \xrightarrow{\int_{\Delta^p}\otimes Id} V_p[-p]^{p+q}$$

is a quasiisomorphism of differential graded vector spaces.

We will prove this theorem later on, after a series of preliminary results.

Example 3.4. Let $\mathcal{L}$ be a sheaf of differential graded vector spaces over an algebraic variety X and $\mathcal{U} = \{U_i\}$ an open cover of X; assume that the set of indices i is totally ordered. Then, we can define the semicosimplicial DG vector space of Čech cochains of $\mathcal{L}$ with respect to he cover $\mathcal{U}$:

$$\mathcal{L}(\mathcal{U}): \quad \prod_i \mathcal{L}(U_i) \rightrightarrows \prod_{i<j} \mathcal{L}(U_{ij}) \Rrightarrow \prod_{i<j<k} \mathcal{L}(U_{ijk}) \cdots .$$

Clearly, in this case, the total complex $\mathrm{Tot}(\mathcal{L}(\mathcal{U}))$ is the associated Čech complex $C^*(\mathcal{U}, \mathcal{L})$. We will denote by $TW(\mathcal{U}, \mathcal{L})$ the associated Thom-Whitney complex. The integration map $TW(\mathcal{U}, \mathcal{L}) \to C^*(\mathcal{U}, \mathcal{L})$ is a surjective quasiisomorphism. If $\mathcal{L}$ is a quasicoherent DG-sheaf and every U_i is affine, then the cohomology of $TW(\mathcal{U}, \mathcal{L})$ is the same of the cohomology of $\mathcal{L}$.

Example 3.5. Let

$$\mathfrak{g}^{\Delta}: \quad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \Rrightarrow \mathfrak{g}_2 \cdots ,$$

be a semicosimplicial differential graded Lie algebra, i.e., each $\mathfrak{g}_i$ is a DGLA each ∂_k is a morphism of DGLAs. Then, in this case too, we can apply the Thom-Whitney construction: it is evident $TW(\mathfrak{g}^{\Delta})$ has a structure of a differential graded lie algebra.

Example 3.6. Let $\chi : L \to M$ be a morphism of differential graded Lie algebras. Then, we can consider the semicosimplicial DGLA

$$\chi^{\Delta}: \quad L \rightrightarrows M \Rrightarrow 0 \cdots , \qquad \text{with} \quad \partial_0 = \chi \text{ and } \partial_1 = 0.$$

It turns out that the normal complex $N(\chi^{\Delta})$ coincides with the mapping cone of χ, i.e.,

$$N(\chi^{\Delta})^i = L^i \oplus M^{i-1}, \qquad d(l, m) = (dl, \chi(l) - dm),$$

and the Thom-Whitney-Sullivan construction coincides with the homotopy fiber of χ:

$$TW(\chi^{\Delta}) \simeq \{(l, m(t, dt)) \in L \times M[t, dt] \mid m(0, 0) = 0,\, m(1, 0) = \chi(l)\}.$$

Lemma 3.7. *Let $\mathfrak{g}^{\Delta}$ be a semicosimplicial DGLA, L a DGLA and $\varphi : L \to \mathfrak{g}_0$ a morphism of DGLA, such that $\partial_0 \circ \varphi = \partial_1 \circ \varphi$. Define $h : L \to TW(\mathfrak{g}^{\Delta})$ as*

$$h(l) = (\varphi(l) \otimes 1, \partial_0(\varphi(l)) \otimes 1, \partial_0^2(\varphi(l)) \otimes 1, \ldots, \partial_0^n(\varphi(l)) \otimes 1, \ldots).$$

Then, h is a well defined morphism of DGLAs giving a commutative diagram

$$\begin{array}{ccc} & & TW(\mathfrak{g}^{\Delta}) \\ & \overset{h}{\nearrow} & \downarrow I \\ L & \xrightarrow{\psi} & \mathrm{Tot}(\mathfrak{g}^{\Delta}), \end{array}$$

where $\psi : L \to \mathrm{Tot}(\mathfrak{g}^{\Delta})$ is the composition of φ with the inclusion $\mathfrak{g}_0 \subset \mathrm{Tot}(\mathfrak{g}^{\Delta})$.

Proof. Since $\partial_0\partial_k = \partial_{k+1}\partial_0$, for all k, we have that

$$\delta^k(\partial_0^n(\varphi(l)) \otimes 1) = \partial_0^n(\varphi(l)) \otimes \delta^k(1) = \partial_0^n(\varphi(l)) \otimes 1 =$$
$$\partial_k(\partial_0^{n-1}(\varphi(l))) \otimes 1 = \partial_k(\partial_0^{n-1}(\varphi(l)) \otimes 1),$$

i.e., for every $l \in L$, $h(l) \in TW(\mathfrak{g}^{\Delta})$. Moreover, h commutes with the differentials; in fact, by hypothesis, $d_{\mathfrak{g}_0}(\varphi(l)) = \varphi(d_L(l))$, and so

$$h(d_L(l)) = (d_{\mathfrak{g}_0}(\varphi(l)) \otimes 1, \partial_0(d_{\mathfrak{g}_0}(\varphi(l))) \otimes 1, \partial_0^2(d_{\mathfrak{g}_0}(\varphi(l))) \otimes 1, \ldots, \partial_0^n(d_{\mathfrak{g}_0}(\varphi(l))) \otimes 1, \ldots)$$

is equal to

$$(d_{\mathfrak{g}_0}(\varphi(l)) \otimes 1, d_{\mathfrak{g}_1}(\partial_0(\varphi(l))) \otimes 1, d_{\mathfrak{g}_2}(\partial_0^2(\varphi(l))) \otimes 1, \ldots, d_{\mathfrak{g}_n}(\partial_0^n(\varphi(l))) \otimes 1, \ldots) =$$
$$d_{TW}(\varphi(l) \otimes 1, \partial_0(\varphi(l)) \otimes 1, \partial_0^2(\varphi(l)) \otimes 1, \ldots, \partial_0^n(\varphi(l)) \otimes 1, \ldots)$$

(since all ∂_0 are DGLA morphisms). Analogously, since δ_0 and φ commutes with brackets, h commutes with the brackets, i.e., h is a DGLAs morphism.

Finally, since I contracts the polynomial differential forms in A_n by integrating over the standard simplex Δ_n, we have that, $I(h(l)) = \varphi(l) \in \mathfrak{g}_0^i$, for every $l \in L^i$. $\square$

4. Homotopy operators

For every $n \geq -1$, consider the affine space

$$C^n = \{(s, t_0, t_1, \dots, t_n) \in \mathbb{K}^{n+2} \mid s + \sum t_i = 1\}.$$

The identity on $\mathbb{K}^{n+2}$ induces an isomorphism $c\colon \Delta^{n+1} \to C^n$ and therefore an integration operator

$$\int_{C^n} : \frac{\mathbb{K}[s, t_0, \dots, t_n, ds, dt_0, \dots, dt_n]}{(s + \sum t_i - 1, ds + \sum dt_i)} \to \mathbb{K}, \qquad \int_{C^n} \eta = \int_{\Delta^n} c^*\eta.$$

We have affine maps

$$i\colon \Delta^n \to C^n, \qquad i(t_0, \dots, t_n) = (0, t_0, \dots, t_n)$$

and for every $f \in M(n, m)$ we also denote

$$f\colon C^n \to C^m, \qquad f(1, 0, \dots, 0) = (1, 0, \dots, 0), \quad f(e_i) = e_{f(i)},\ i \geq 0.$$

$$\widehat{f}\colon C^n \times \Delta^m \to \Delta^m, \quad \widehat{f}((s, t_0, \dots, t_n), v) = sv + \sum t_i e_{f(i)},$$
$$\widetilde{f}\colon \Delta^n \times \Delta^m \to \Delta^m, \quad \widetilde{f}(u, v) = \widehat{f}(i(u), v).$$

Finally define for every $k = 0, \dots, n$

$$\widehat{f_k}\colon C^{n-1} \times \Delta^m \to \Delta^m, \quad \widehat{f_k}(u, v) = \widehat{f}(\partial_k u, v).$$

Lemma 4.1. *In the notation above:*
1. $\widehat{f_k} = \widehat{f\partial_k}$,
2. $\widetilde{f}$ *is the composition of the projection* $\Delta^n \times \Delta^m \to \Delta^n$ *and* $f\colon \Delta^n \to \Delta^m$.

Proof. Trivial. $\square$

Lemma 4.2. *In the notation above, for every* $g \in M(m, p)$ *we have a commutative diagram*

$$\begin{array}{ccc} C^n \times \Delta^m & \xrightarrow{\widehat{f}} & \Delta^m \\ \downarrow{\scriptstyle Id \times g} & & \downarrow{\scriptstyle g} \\ C^n \times \Delta^p & \xrightarrow{\widehat{gf}} & \Delta^p \end{array}$$

Proof. Trivial. $\square$

Passing to differential forms we have morphisms for differential graded alebras

$$\widehat{f}^*\colon A_m \to B_n \otimes A_m,$$

where

$$B_m = \frac{\mathbb{K}[s, t_0, \dots, t_n, ds, dt_0, \dots, dt_n]}{(s + \sum t_i - 1, ds + \sum dt_i)}$$

is the de Rham algebra of C^n.

Definition 4.3. For every $n \geq -1$, $m \geq 0$ and $f \in M(n, m)$ define the operator $h_f \in \operatorname{Hom}^{-n-1}(A_m, A_m)$ as the composition

$$h_f\colon A_m \xrightarrow{\widehat{f}^*} B_n \otimes A_m \xrightarrow{\int_{C^n} \otimes Id} A_m.$$

Notice that for $n = -1$ the above operator equals the identity.

Lemma 4.4. *For every* $n \geq 0$, $m \geq 0$, $f \in M(n, m)$ *and* $\eta \in A_m$ *we have*

$$[h_f, d](\eta) = h_f(d\eta) + (-1)^n d h_f(\eta) = \int_{\Delta^n} f^*\eta - \sum_{k=0}^{n} (-1)^k h_{f\partial_k}(\eta).$$

In particular, for $n = 0$ *we have* $h_f(d\eta) + dh_f(\eta) = \eta(e_{f(0)}) - \eta$ *and then the evaluation at a vertex is homotopic to the identity.*

Proof. For every $\beta \in B_n$ we have by Stokes formula

$$\int_{C^n} d\beta = \int_{\Delta^n} i^*\beta - \sum_{k=0}^{n} (-1)^k \int_{C^{n-1}} \partial_k^*\beta.$$

Writing

$$\widehat{f}^*\eta = \sum_i \beta_i \otimes \alpha_i, \qquad \beta_i \in B_n,\ \alpha_i \in A_m$$

we have

$$dh_f(\eta) = d\sum_i \left(\int_{C^n} \beta_i\right) \alpha_i = \sum_i \left(\int_{C^n} \beta_i\right) d\alpha_i\ ,$$

$$\widehat{f}^*(d\eta) = d\widehat{f}^*(\eta) = \sum_i d\beta_i \otimes \alpha_i + \sum_i (-1)^{\overline{\beta_i}} \beta_i \otimes d\alpha_i,$$

$$h_f(d\eta) = \sum_i \left(\int_{C^n} d\beta_i\right) \otimes \alpha_i + (-1)^{n+1} \sum_i \left(\int_{C^n} \beta_i\right) \otimes d\alpha_i,$$

Therefore

$$\begin{aligned}
h_f(d\eta) + (-1)^n dh_f(\eta) &= \sum_i \left(\int_{C^n} d\beta_i\right) \otimes \alpha_i \\
&= \sum_i \left(\int_{\Delta^n} i^*\beta_i\right) \otimes \alpha_i - \sum_{k=0}^{n} (-1)^k \sum_i \left(\int_{C^{n-1}} \partial_k^*\beta_i\right) \otimes \alpha_i \\
&= \left(\int_{\Delta^n} \otimes Id\right) (i^* \otimes Id)\widehat{f}^*(\eta) - \sum_{k=0}^{n} (-1)^k \left(\int_{C^{n-1}} \otimes Id\right) (\partial_k^* \otimes Id)\widehat{f}^*(\eta) \\
&= \left(\int_{\Delta^n} \otimes Id\right) \widetilde{f}^*(\eta) - \sum_{k=0}^{n} (-1)^k \left(\int_{C^{n-1}} \otimes Id\right) \widehat{f\partial_k}^*(\eta) \\
&= \int_{\Delta^n} f^*\eta - \sum_{k=0}^{n} (-1)^k h_{f\partial_k}(\eta).
\end{aligned}$$

□

Lemma 4.5. *Given $f \in M(n, m)$, $g \in M(m, p)$ and $\eta \in A_p$ we have:*

$$g^* h_{gf}(\eta) = h_f(g^*\eta).$$

Proof. Immediate consequence of the commutative diagram

$$\begin{array}{ccccc}
A_p & \xrightarrow{\widehat{gf}^*} & B_n \otimes A_p & \xrightarrow{\int_{C^n} \otimes Id} & A_p \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle Id \otimes g^*} & & \downarrow{\scriptstyle g^*} \\
A_m & \xrightarrow{\widehat{f}^*} & B_n \otimes A_m & \xrightarrow{\int_{C^n} \otimes Id} & A_m
\end{array}$$

□

5. Whitney elementary forms

Definition 5.1. For every $f \in M(n, m)$ define the *elementary form*

$$\omega_f = n! \sum_{i=0}^{n} (-1)^i t_{f(i)} dt_{f(0)} \wedge \cdots \wedge \widehat{dt_{f(i)}} \wedge \cdots \wedge dt_{f(n)} \in A_m^n.$$

Denote by $W_m \subset A_m$ the graded subspace generated by the elementary forms.

Notice that $\omega_f \neq 0$ if and only if f is injective.

Lemma 5.2. *We have:*

(1) *For every $f \in M(n, m)$ and every $g \in M(p, m)$ we have*

$$g^*\omega_f = \sum_{\{h \in M(n,p) \mid f = gh\}} \omega_h.$$

In particular for $n = p$ we have $g^*\omega_f \neq 0$ if and only if $f = g$.

(2) *For every* $f \in M(n,m)$

$$d\omega_f = \sum_k (-1)^k \sum_{\{g \mid g\partial_k = f\}} \omega_g.$$

(3) *For every* $f \in I(n,m)$ *we have*

$$\int_{\Delta^n} f^*\omega_f = 1.$$

In particular $\{W_m\}$ *is a simplicial differential graded subspace of* $\{A_m\}$

Proof. The first item is easy and left as an exercise. More generally, for every finite sequence $0 \le i_0, i_1, \dots, i_n \le m$ denote

$$\omega_{i_0,\dots,i_n} = n! \sum_{k=0}^{n} (-1)^k t_{i_k} dt_{i_0} \wedge \cdots \wedge \widehat{dt_{i_k}} \wedge \cdots \wedge dt_{i_n},$$

then

$$d\omega_{i_0,\dots,i_n} = \sum_{i=0}^{m} \omega_{i,i_0,\dots,i_n}.$$

In fact

$$d\omega_{i_0,\dots,i_n} = n! \sum_{k=0}^{n} dt_{i_0} \wedge \cdots \wedge dt_{i_k} \wedge \cdots \wedge dt_{i_n} = (n+1)! dt_{i_0} \wedge \cdots \wedge dt_{i_k} \wedge \cdots \wedge dt_{i_n}.$$

and

$$\begin{aligned}\sum_{i=0}^{m} \omega_{i,i_0,\dots,i_n} &= (n+1)! \sum_{i=0}^{m} t_i dt_{i_0} \wedge \cdots \wedge dt_{i_k} \wedge \cdots \wedge dt_{i_n} - (n+1) \sum_{i=0}^{m} dt_i \wedge \omega_{i_0,\dots,i_n} \\ &= (n+1)! dt_{i_0} \wedge \cdots \wedge dt_{i_k} \wedge \cdots \wedge dt_{i_n}\end{aligned}$$

It is now sufficient to observe that for $f \in M(n,m)$ we have

$$\sum_{i=0}^{m} \omega_{i,f(0),\dots,f(n)} = \sum_{k=0}^{n} (-1)^k \sum_{f(k-1)<i<f(k)} \omega_{f(0),\dots,f(k-1),i,f(k),\dots,f(n)} = \sum_k (-1)^k \sum_{\{g \mid g\partial_k = f\}} \omega_g.$$

Since

$$f^*\omega_f = n! \sum_{k=0}^{n} (-1)^k t_k dt_0 \wedge \cdots \wedge \widehat{dt_k} \wedge \cdots \wedge dt_n,$$

using the equalities $dt_0 = -\sum_{i>0} dt_i$, $\sum_i t_i = 1$ we obtain

$$\begin{aligned}f^*\omega_f &= n! \left(t_0 dt_1 \wedge \cdots \wedge dt_n - \sum_{k=1}^{n} (-1)^k t_k dt_k \wedge \cdots \wedge \widehat{dt_k} \wedge \cdots \wedge dt_n \right) \\ &= n!(t_0 + \cdots + t_n) dt_1 \wedge \cdots \wedge dt_n = n!\, dt_1 \wedge \cdots \wedge dt_n\end{aligned}$$

and then

$$\int_{\Delta^n} f^*\omega_f = n! \int_{\Delta^n} dt_1 \wedge \cdots \wedge dt_n = 1.$$

□

Remark 5.3. For later use we point out that

$$\bigcap_{k=0}^{m} \ker(\partial_k^* \colon W_m \to W_{m-1}) = W_m^m.$$

Definition 5.4. For every $m \ge 0$ define the operators

$$\pi_m \colon A_m \to W_m, \qquad \pi_m(\eta) = \sum_{n=0}^{m} \sum_{f \in I(n,m)} \left(\int_{\Delta_n} f^*\eta \right) \omega_f$$

$$K_m \colon A_m \to A_m, \qquad K_m(\eta) = \sum_{n=0}^{m} \sum_{f \in I(n,m)} \omega_f \wedge h_f(\eta).$$

Theorem 5.5. *In the above notation we have:*

(1) π_m *is a projector, i.e.* $\pi_m^2 = \pi_m$*;*

(2)
$$K_m d + dK_m = \pi_m - Id;$$
(3)
$$K_p g^* = g^* K_m, \qquad \pi_p g^* = g^* \pi_m, \qquad \textit{for every } g \in M(p,m).$$

Proof. The first item is trivial. For the second we have

$$K_m(d\eta) + dK_m(\eta) =$$
$$\sum_{n=0}^{m} \sum_{f \in I(n,m)} d\omega_f \wedge h_f(\eta) + \sum_{n=0}^{m} \sum_{f \in I(n,m)} \omega_f \wedge ((-1)^n dh_f(\eta) + h_f(d\eta))$$
$$= \sum_{n=0}^{m} \sum_{f \in I(n,m)} d\omega_f \wedge h_f(\eta) + \sum_{n=0}^{m} \sum_{f \in I(n,m)} \omega_f \wedge \left(\int_{\Delta^n} f^*\eta - \sum_{k=0}^{n} (-1)^k h_{f\partial_k}(\eta) \right)$$

Since $h_\emptyset = Id$ and $\sum_{f \in I(0,m)} \omega_f = \sum_{i=0}^m t_i = 1$ we have

$$K_m(d\eta) + dK_m(\eta) - \pi_m(\eta) + \eta = \sum_{n=0}^{m} \sum_{f \in I(n,m)} d\omega_f \wedge h_f(\eta) - \sum_{n=1}^{m} \sum_{f \in I(n,m)} \omega_f \wedge \sum_{k=0}^{n} (-1)^k h_{f\partial_k}(\eta).$$

The vanishing of the right side follows from the equations

$$\sum_{n=0}^{m} \sum_{f \in I(n,m)} d\omega_f \wedge h_f(\eta) = \sum_{n=0}^{m-1} \sum_{f \in I(n,m)} d\omega_f \wedge h_f(\eta) =$$
$$= \sum_{n=0}^{m-1} \sum_{f \in I(n,m)} \sum_{k=0}^{n} (-1)^k \sum_{\{g | f = g\partial_k\}} \omega_g \wedge h_{g\partial_k}(\eta) = \sum_{n=1}^{m} \sum_{g \in I(n,m)} \sum_{k=0}^{n} (-1)^k \omega_g \wedge h_{g\partial_k}(\eta).$$

For the last item it is sufficient to prove that $K_p g^* = g^* K_m$;

$$g^* K_m(\eta) = \sum_{n=0}^{m} \sum_{f \in I(n,m)} g^*(\omega_f) \wedge g^* h_f(\eta) = \sum_{n=0}^{m} \sum_{f \in I(n,m)} \sum_{\{h \in M(n,p) | f = gh\}} \omega_h \wedge g^* h_f(\eta) =$$
$$= \sum_{n=0}^{m} \sum_{h \in I(n,p)} \omega_h \wedge g^* h_{gh}(\eta) = \sum_{n=0}^{m} \sum_{h \in I(n,p)} \omega_h \wedge h_h(g^*\eta) = K_p(g^*\eta).$$

□

6. Proof of Whitney's theorem

Let

$$V^\Delta : \quad V_0 \rightrightarrows V_1 \rightrightarrows V_2 \rightrightarrows \cdots,$$

be a fixed semicosimplicial vector space.

For every p, q we will denote

$$A^{p,q} = \prod_{n \geq 0} A_n^p \otimes V_n^q, \qquad W^{p,q} = \prod_{n \geq 0} W_n^p \otimes V_n^q,$$
$$K \colon A^{p,q} \to A^{p-1,q}, \qquad K(x_0, x_1, \ldots) = (K_0(x_0), K_1(x_1), \ldots),$$
$$\pi \colon A^{p,q} \to W^{p,q}, \qquad \pi(x_0, x_1, \ldots) = (\pi_0(x_0), \pi_1(x_1), \ldots),$$
$$TW(V^\Delta)^{p,q} = \{(x_n) \in A^{p,q} \mid (\partial_k^* \otimes Id)x_n = (Id \otimes \partial_k)x_{n-1} \ \forall 0 \leq k \leq n\},$$
$$W(V^\Delta)^{p,q} = TW(V^\Delta)^{p,q} \cap W^{p,q}, \qquad W(V^\Delta) = \bigoplus_{p,q} W(V^\Delta)^{p,q}.$$

Since the homotopy operator K_m are simplicial we have clearly that K preserves $TW(V^\Delta)$ and π is a projection of $TW(V^\Delta)$ onto $W(V^\Delta)$.

Lemma 6.1. *The inclusion $W(V^\Delta) \to TW(V^\Delta)$ and the map $\pi \colon TW(V^\Delta) \to W(V^\Delta)$ are homotopy equivalences.*

Proof. Immediate from formula $dK + Kd = \pi - Id$. □

Lemma 6.2. *For every p, q the map*

$$\phi\colon W(V^{\Delta})^{p,q} \xrightarrow{inclusion} \prod_{n\geq 0} W_n^p \otimes V_n^q \xrightarrow{projection} W_p^p \otimes V_p^q \xrightarrow{\int_{\Delta^p}\otimes Id} V_p^q$$

is an isomorphism whose components of its inverse E are

$$E_n\colon V_p^q \to W_n^p \otimes V_n^q, \qquad E_n(v) = \sum_{f\in I(p,n)} \omega_f \otimes f(v).$$

Proof. Let's first prove that for every $v \in V_p^q$ the sequence $E_n(v)$ belongs to $W^{p,q}$. For every $g \in I(n,m)$ we have

$$(g^* \otimes Id)E_m(v) = \sum_{f\in I(p,m)} g^*\omega_f \otimes f(v) = \sum_{f\in I(p,m)} \sum_{\{h|f=gh\}} \omega_h \otimes gh(v) =$$

$$= \sum_{h\in I(p,n)} \omega_h \otimes gh(v) = (Id \otimes g)E_n(v).$$

It is obvious that $\phi \circ E = Id$ and if $\phi(x_n) = 0$ then $x_p = 0$ and if $x_n = \sum_{f\in I(p,n)} \omega_f \otimes v_f$ then $(f^* \otimes Id)(x_n) = f^*\omega_f \otimes v_f = (Id \otimes f)(x_p) = 0$ and then $v_f = 0$. This proves that ϕ is injective. □

Lemma 6.3. *The map $\phi\colon W(V^{\Delta}) \to N(V^{\Delta})$ is a isomorphism of complexes and $I = \phi \circ \pi$.*

Proof. We have already proved that it is bijective. As easy application od Stokes formula show that $\partial\phi = \phi d$. □

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