THE FUNCTOR L OF QUILLEN

DEFORMATION THEORY 2011-12; M. M.

Let R be a commutative ring, by a nonassociative (= not necessarily associative) graded R-algebra we mean a graded R-module $M = \oplus M^i$ endowed with a R-bilinear map $M^i \times M^j \to M^{i+j}$.

The nonassociative algebra M is called *unitary* if there exist a "unity" $1 \in M^0$ such that $1m = m1 = m$ for every $m \in M$.

A *left ideal* (resp.: *right ideal*) of M is a graded submodule $I \subset M$ such that $MI \subset I$ (resp.: $IM \subset I$). A graded submodule is called an *ideal* if it is both a left and right ideal.

A homomorphism of R-modules $d\colon M \to M$ is called a *derivation of degree k* if $d(M^i) \subset M^{i+k}$ and satisfies the graded Leibniz rule $d(ab) = d(a)b + (-1)^{k\,\overline{a}} a d(b)$.

If M is as associative graded algebra we denote by M_L the associated graded Lie algebra, with bracket equal to the graded commutator $[a,b] = ab - (-1)^{\overline{a}\,\overline{b}} ba$.

It is easy to see that if $f\colon M \to M$ is a derivation, then also $f\colon M_L \to M_L$ is a derivation.

Notation: For a graded Lie algebra H we denote $[a,b,c] = [a,[b,c]]$ and more generally

$$[a_1,\dots,a_n] = [a_1,[a_2,\dots,a_n]] = [a_1,[a_2,[a_3,\dots,[a_{n-1},a_n]\dots] \,.$$

Notice that the descending central series $H^{[n]}$ may be defined as

$$H^{[n]} = Span\{[a_1,\dots,a_n]\}, \qquad a_1,\dots,a_n \in H.$$

Jacoby identity becomes

$$[[a,b],c] = [a,b,c] - (-1)^{\overline{a}\,\overline{b}}[b,a,c].$$

1. Free graded Lie algebras

Let V be a graded vector space over $\mathbb{K}$, we denote by

$$T(V) = \bigoplus_{n\ge 0} \bigotimes^n V, \qquad \overline{T(V)} = \bigoplus_{n\ge 1} \bigotimes^n V.$$

The tensor product induce on $T(V)$ a structure of unital associative graded algebra and $\overline{T(V)}$ is an ideal of $T(V)$. The algebra $T(V)$ is called **tensor algebra** generated by V and $\overline{T(V)}$ is called the **reduced tensor algebra** generated by V.

Lemma 1.1. *Let V be a $\mathbb{K}$-vector space and $\imath\colon V \to \overline{T(V)}$ the natural inclusion. For every graded associative $\mathbb{K}$-algebra R and every linear map $f \in \operatorname{Hom}^0_{\mathbb{K}}(V,R)$ there exists a unique homomorphism of $\mathbb{K}$-algebras $\phi\colon \overline{T(V)} \to R$ such that $f = \phi\imath$.*

Proof. Clear. □

Lemma 1.2. *Let V be a $\mathbb{K}$-vector space and $\imath\colon V \to \overline{T(V)}$ the natural inclusion. For every $f \in \operatorname{Hom}^k_{\mathbb{K}}(V,\overline{T(V)})$ there exists a unique derivation $\phi\colon \overline{T(V)} \to \overline{T(V)}$ such that $f = \phi\imath$.*

Proof. Leibniz rule forces to define ϕ as

$$\phi(v_1\otimes\cdots\otimes v_n) = \sum_{i=1}^n (-1)^{k(\overline{v_1}+\cdots+\overline{v_{i-1}})}\, v_1\otimes\cdots\otimes f(v_i)\otimes\cdots\otimes v_n \,.$$

□

Definition 1.3. Let V be a graded $\mathbb{K}$-vector space; the **free Lie algebra** generated by V is the smallest graded Lie subalgebra $\mathbb{L}(V)\subset\overline{T(V)}_L$ which contains V.

Equivalently $\mathbb{L}(V)$ is the intersection of all the Lie subalgebras of $T(V)_L$ containing V.

For every integer $n>0$ we denote by $\mathbb{L}(V)_n\subset\mathbb{L}(V)\cap\bigotimes^n V$ the linear subspace generated by all the elements

$$[v_1,v_2,\ldots,v_n],\qquad n>0,\quad v_1,\ldots,v_n\in V.$$

By definition $\mathbb{L}(V)_1=V$, $\mathbb{L}(V)_n=[V,\mathbb{L}(V)_{n-1}]$ and therefore $\oplus_{n>0}\mathbb{L}(V)_n\subset\mathbb{L}(V)$. On the other hand the Jacobi identity $[[x,y],z]=[x,[y,z]]-[y,[x,z]]$ implies that

$$[\mathbb{L}(V)_n,\mathbb{L}(V)_m]\subset[V,[\mathbb{L}(V)_{n-1},\mathbb{L}(V)_m]]+[\mathbb{L}(V)_{n-1},[V,\mathbb{L}(V)_m]]$$

and therefore, by induction on n, $[\mathbb{L}(V)_n,\mathbb{L}(V)_m]\subset\mathbb{L}(V)_{n+m}$.

This implies that the direct sum $\oplus_{n>0}\mathbb{L}(V)_n$ is a graded Lie subalgebra of $\mathbb{L}(V)$; therefore $\oplus_{n>0}\mathbb{L}(V)_n=\mathbb{L}(V)$ and for every n

$$\mathbb{L}(V)_n=\mathbb{L}(V)\cap\bigotimes^n V,\qquad \mathbb{L}(V)^{[n]}=\bigoplus_{i\geq n}\mathbb{L}(V)_i.$$

The construction $V\mapsto\mathbb{L}(V)$ is a functor from the category of graded vector spaces to the category of graded Lie algebras, since every morphism of vector spaces $V\to W$ induce a morphism of algebras $\overline{T(V)}\to\overline{T(W)}$ which restricts to a morphism of Lie algebras $\mathbb{L}(V)\to\mathbb{L}(W)$.

Theorem 1.4 (Dynkyn-Sprecht-Wever). *Assume that V is a vector space and H a graded Lie algebra with bracket $[\,,]$. Let $\sigma_1\in\operatorname{Hom}^0(V,H)$ be a linear map and define, for every $n\geq 2$, the maps*

$$\sigma_n\colon\bigotimes^n V\to H,\qquad \sigma_n(v_1\otimes\cdots\otimes v_n)=[\sigma_1(v_1),\sigma_{n-1}(v_2\otimes\cdots\otimes v_n)]=[\sigma_1(v_1),\sigma_1(v_2),\ldots,\sigma_1(v_n)].$$

Then the linear map

$$\sigma=\sum_{n=1}^{\infty}\frac{\sigma_n}{n}\colon\mathbb{L}(V)\to H,\qquad \sigma(v_1\otimes\cdots\otimes v_n)=\frac{1}{n}[\sigma_1(v_1),\sigma_1(v_2),\ldots,\sigma_1(v_n)],$$

is the unique homomorphism of graded Lie algebras extending σ_1.

Proof. The adjoint representation

$$\theta\colon V\to\operatorname{Hom}^*(H,H),\qquad \theta(v)x=[\sigma_1(v),x],$$

extends to a morphism of graded associative algebras $\theta\colon\overline{T(V)}\to\operatorname{Hom}^*(H,H)$ by the composition rule

$$\theta(v_1\otimes\cdots\otimes v_s)x=\theta(v_1)\theta(v_2)\cdots\theta(v_s)x.$$

By definition

$$\sigma_n(v_1\otimes\cdots\otimes v_n)=\theta(v_1\otimes\cdots\otimes v_{n-1})\sigma_1(v_n)$$

and more generally, for every $v_1,\ldots,v_n,w_1,\ldots,w_m\in V$ we have

$$\sigma_{n+m}(v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\ldots\otimes w_m)=\theta(v_1\otimes\cdots\otimes v_n)\sigma_m(w_1\otimes\cdots\otimes w_m).$$

Since every element of $\mathbb{L}(V)$ is a linear combination of homogeneous elements it is sufficient to prove, by induction on $n\geq 1$, the following properties

- A_n**:** If $m\leq n$, $x\in\mathbb{L}(V)_m$ and $y\in\mathbb{L}(V)_n$ then $\sigma([x,y])=[\sigma(x),\sigma(y)]$.
- B_n**:** If $m\leq n$, $y\in\mathbb{L}(V)_m$ and $h\in H$ then $\theta(y)h=[\sigma(y),h]$.

The initial step $n=1$ is straightforward, assume therefore $n\geq 2$.

$[A_{n-1}+B_{n-1}\Rightarrow B_n]$ We have to consider only the case $m=n$. The element y is a linear combination of elements of the form $[a,b]$, $a\in V$, $b\in\mathbb{L}(V)_{n-1}$ and, using B_{n-1} we get

$$\theta(y)h=[\sigma(a),\theta(b)h]-(-1)^{\overline{a}\,\overline{b}}\theta(b)[\sigma(a),h]=[\sigma(a),[\sigma(b),h]]-(-1)^{\overline{a}\,\overline{b}}[\sigma(b),[\sigma(a),h]].$$

Using A_{n-1} we get therefore

$$\theta(y)h = [[\sigma(a), \sigma(b)], h] = [\sigma(y), h].$$

$[B_n \Rightarrow A_n]$

$$\begin{aligned}\sigma_{n+m}([x,y]) &= \theta(x)\sigma_n(y) - (-1)^{\overline{x}\,\overline{y}}\theta(y)\sigma_m(x) = [\sigma(x), \sigma_n(y)] - (-1)^{\overline{x}\,\overline{y}}[\sigma(y), \sigma_m(x)] \\ &= n[\sigma(x), \sigma(y)] - m(-1)^{\overline{x}\,\overline{y}}[\sigma(y), \sigma(x)] = (n+m)[\sigma(x), \sigma(y)].\end{aligned}$$

Since $\mathbb{L}(V)$ is generated by V as a Lie algebra, the unicity of σ follows. □

Corollary 1.5. *For every vector space V the linear map*

$$\pi\colon \overline{T(V)} \to \mathbb{L}(V), \qquad \pi(v_1 \otimes \cdots \otimes v_n) = \frac{1}{n}[v_1, v_2, \ldots, v_n]$$

is a projection. In particular for every $n > 0$

$$\mathbb{L}(V)_n = \{x \in V^{\otimes n} \mid \pi(x) = x\}.$$

Proof. The identity on $\mathbb{L}(V)$ is the unique Lie homomorphism extending the natural inclusion $V \to \mathbb{L}(V)$. □

Lemma 1.6. *Every $f \in \mathrm{Hom}^k(V, \mathbb{L}(V))$ extends to a unique derivation $\mathbb{L}(V) \to \mathbb{L}(V)$.*

Proof. The composition $f\colon V \to \mathbb{L}(V) \hookrightarrow \overline{T(V)}$ extends to a derivation $F\colon \overline{T(V)} \to \overline{T(V)}$. Leibniz rule gives the unicity and $F(\mathbb{L}(V)) \subset \mathbb{L}(V)$. □

2. The functor L of Quillen

Let s be a formal symbol of degree $+1$. For every graded vector space V denote

$$sV = \{sv \mid v \in V\}, \quad \text{and} \quad s \in \mathrm{Hom}^1(V, sV), \qquad s(v) = sv.$$

Lemma 2.1. *For every $x \in V \otimes V$ we have*

$$\pi((s \otimes s)x) = (s \otimes s)\left(\frac{x + \mathtt{tw}(x)}{2}\right).$$

In particular $(s \otimes s)x \in \mathbb{L}(sV)_2$ if and only if $x = \mathtt{tw}(x)$.

Proof. By linearity may assume $x = u \otimes v$. Then $\mathtt{tw}(x) = (-1)^{\overline{u}\,\overline{v}} v \otimes u$ and

$$\begin{aligned}2\pi((s\otimes s)x) &= 2\pi((-1)^{\overline{u}} su \otimes sv) = (-1)^{\overline{u}} su \otimes sv - (-1)^{\overline{u} + (\overline{u}+1)(\overline{v}+1)} sv \otimes su = \\ &= (-1)^{\overline{u}} su \otimes sv + (-1)^{\overline{v} + \overline{u}\,\overline{v}} sv \otimes su = (s \otimes s)(x + \mathtt{tw}(x)).\end{aligned}$$

□

Let (C, Δ, δ) be a differential graded cocommutative coalgebra. Denote by:

(1) $d_1 \in \mathrm{Der}^1(\mathbb{L}(sC), \mathbb{L}(sC))$ the derivation induced by the map

$$d_1\colon sC \to \mathbb{L}(sC), \qquad d_1(sv) = -s\delta(v).$$

(2) $d_2 \in \mathrm{Der}^1(\mathbb{L}(sC), \mathbb{L}(sC))$ the derivation induced by the map

$$d_2\colon sC \to \mathbb{L}(sC), \qquad d_2(sv) = -(s \otimes s)\Delta(v)$$

(this makes sense since $\mathtt{tw} \circ \Delta = \Delta$).

Theorem 2.2. *In the above notation $d_1^2 = d_2^2 = [d_1, d_2] = 0$ and then $(\mathbb{L}(sC), [,], d_1 + d_2)$ is a DGLA.*

Proof. $d_1^2(sv) = s\delta^2(v) = 0$.

$$d_2d_1(sv) = d_2(-s\delta(v)) = (s\otimes s)\Delta(\delta(v)) = (s\otimes s)(\delta\otimes Id + Id\otimes\delta)\Delta(v) =$$
$$= (-s\delta\otimes s + s\otimes s\delta)\Delta(v) = (d_1\otimes Id + Id\otimes d_1)(s\otimes s)\Delta(v) = d_1d_2(sv).$$

Remains to prove that $[d_2, d_2] = 2d_2^2 = 0$. Given $x \in C\otimes C$ we have the straighforward identities

$$(d_2\otimes Id)(s\otimes s)(x) = -(s\otimes s\otimes s)(\Delta\otimes Id)(x),$$
$$(Id\otimes d_2)(s\otimes s)(x) = (s\otimes s\otimes s)(Id\otimes\Delta)(x),$$

and then for $v \in C$ we have

$$d_2^2(sv) = -(d_2\otimes Id + Id\otimes d_2)(s\otimes s)\Delta(v) = -(s\otimes s\otimes s)(\Delta\otimes Id - Id\otimes\Delta)\Delta(v) = 0.$$

□

Definition 2.3. For every differential graded cocommutative coalgebra C denote $L(C)$ the differential graded Lie algebra $(\mathbb{L}(sC), [,], d_1 + d_2)$.

It is quite obvious that L is a functor.

Proposition 2.4 (Quillen)**.** *The functor L preserves quasiisomorphisms.*

Proof. Omitted. □

3. Twisting morphisms

Let (C, Δ, δ) be a differential graded cocommutative coalgebra and $(H, [,], \partial)$ a DGLA. Then the space $\operatorname{Hom}^*(C, H)$ has a natural structure of DGLA with differential

$$d(f) = \partial f - (-1)^{\overline{f}} f\delta$$

and bracket $[f, g]$ equal to the composition

$$C \xrightarrow{\Delta} C\otimes C \xrightarrow{f\otimes g} H\otimes H \xrightarrow{[,]} H.$$

Exercise Verify that this is a DGLA.

Definition 3.1. A **twisting morphism** is a map $\alpha \in \operatorname{Hom}^1(C, H)$ satisfying the Maurer-Cartan equation.

The composition with $s\colon C \to sC$ give an isomorphism $\operatorname{Hom}^1(C, H) = \operatorname{Hom}^0(sC, H)$ and then every element $\alpha_1 \in \operatorname{Hom}^1(C, H)$ gives a morphims of graded Lie algebras $\alpha\colon L(C) = \mathbb{L}(sC) \to H$.

Lemma 3.2. *$\alpha_1 \in \operatorname{Hom}^1(C, H)$ is a twisting morphism if and only if $\alpha\colon L(C) \to H$ is a morphism of DGLA:*

$$MC(\operatorname{Hom}^*(C, H)) = \operatorname{Hom}_{DGLA}(L(C), H).$$

Proof. α is a morphims of DGLA if and only if

$$d\alpha = \alpha(d_1 + d_2).$$

Being the above maps two α-derivations, by Leibniz rule it is sufficient to prove that they coincide on sC, i.e. that for every $v \in C$

$$d\alpha_1(sv) = \alpha_1d_1(sv) + \frac{1}{2}\alpha_2(d_2(sv))$$

We have $d\alpha_1(sv) = d\alpha_1(v)$, $\alpha_1d_1(sv) = -\alpha_1(\delta(v))$ and then

$$d\alpha_1(sv) - \alpha_1d_1(sv) = (d\alpha_1 + \alpha_1\delta)v = (d\alpha_1)v.$$

Similarly

$$\alpha_2(d_2(sv)) = -\alpha_2((s\otimes s)\Delta(v)) = -[\alpha_1, \alpha_1](v).$$

□