DIFFERENTIAL GRADED LIE ALGEBRAS AND FORMAL DEFORMATION THEORY

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Introduction

This paper aims to do two things: (1) to give a tutorial introduction to differential graded Lie algebras, functors of Artin rings and obstructions; (2) to explain ideas and techniques underlying some recent papers [29, 31, 32, 11, 17] concerning vanishing theorems for obstructions to deformations of complex Kähler manifolds.

We assume that the reader has a basic knowledge of algebraic geometry, homological algebra and deformation theory; for this topic, the young person may read the excellent expository article of Arcata's proceedings [39].

The common denominator is the following *guiding principle*, proposed by Quillen, Deligne and Drinfeld: *in characteristic 0 every deformation problem is governed by a differential graded Lie algebra*. After the necessary background we will restate such principle in a less vague form (Principle 1.9).

The guiding principle has been confined in the realm of abstract ideas and personal communications until the appearance of [37, 13, 24, 25][1] where a clever use of it has permitted interesting applications in concrete deformation problems. In particular the lecture notes [24] give serious and convincing motivations for the validity of the guiding principle (called there meta-theorem).

In this paper we apply these ideas in order to prove vanishing theorems for obstruction spaces. Just to explain the subject of our investigation, consider the example of deformations of a compact complex manifold X with holomorphic tangent bundle Θ_X. The well known Kuranishi's theorem [26, 39, 5, 14] asserts that there exists a deformation $\mathcal{X}\xrightarrow{f}\operatorname{Def}(X)$ of X over a germ of complex space $\operatorname{Def}(X)$ with the property that the Kodaira-Spencer map $T_{\operatorname{Def}(X)} \to H^1(X,\Theta_X)$ is bijective and every deformation of X over an analytic germ S is isomorphic to the pull-back of f by a holomorphic map $S \to \operatorname{Def}(X)$.

From Kuranishi's proof follows moreover that:

1. $\operatorname{Def}(X) \simeq q^{-1}(0)$, where $q\colon H^1(X,\Theta_X) \to H^2(X,\Theta_X)$ is a germ of holomorphic map such that $q(0) = 0$.
2. The differential of q at 0 is trivial.
3. The quadratic part of the Mac-Laurin series of q is isomorphic to the quadratic map

$$H^1(X,\Theta_X) \to H^2(X,\Theta_X), \qquad x \mapsto \frac{1}{2}[x,x],$$

   where $[\ ,\ ]$ is the natural bracket in the graded Lie algebra $H^*(X,\Theta_X)$.

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[1] This list is not intended to be exhaustive

Kuranishi's proof also says that the higher homogeneous components of the Mac-Laurin series of q depend by the Green's operator of the elliptic complex associated to the Dolbeault resolution of Θ_X and therefore are extremely difficult to handle (with very few exceptions, e.g. [42]).
A way to overcome, at least partially, this difficulty is by using infinitesimal methods, i.e. a deeper study of deformations of X over fat points, and obstruction theory. It is possible to prove that it is well defined a vector subspace $O \subset H^2(X, \Theta_X)$ which is minimal for the property that the image of q is contained in a germ of smooth subvariety $Y \subset H^2(X, \Theta_X)$ with tangent space $T_0Y = O$. In particular $O = 0$ if and only if $q = 0$. The advantage of this notion is that the space O, called obstruction space, can be completely computed by infinitesimal methods.
In this paper we will show how the differential graded Lie algebras can be conveniently used to construct non trivial morphisms of vector spaces $H^2(X, \Theta_X) \to W$ such that $w(O) = 0$.

The content of this paper follows closely my talk at the AMS Summer Institute on Algebraic Geometry, Seattle (WA) 2005. The main differences concern Section 5, where we answer to some technical questions arised during the talk of B. Fantechi, and Section 6 where we give a new and simpler proof of the Kodaira's principle "ambient cohomology annihilates obstructions".

1. Differential graded Lie algebras and toy examples of deformation problems

Let $\mathbb{K}$ be a fixed field of characteristic 0: unless otherwise specified the bifunctors Hom and $\otimes$ are intended over $\mathbb{K}$. By a graded vector space we intend a $\mathbb{Z}$-graded vector space over $\mathbb{K}$.
A differential graded vector space is a pair (V, d) where $V = \oplus V^i$ is a graded vector space and d is a differential of degree $+1$. By following the standard notation, for every differential graded vector space (V, d) we denote by $Z^i(V) = \ker(d\colon V^i \to V^{i+1})$, $B^i(V) = \operatorname{Im}(d\colon V^{i-1} \to V^i)$ and $H^i(V) = Z^i(V)/B^i(V)$.

Definition 1.1. A *differential graded Lie algebra* (DGLA for short) is the data of a differential graded vector space (L, d) together a with bilinear map $[-,-]\colon L \times L \to L$ (called bracket) of degree 0 such that:

1. (graded skewsymmetry) $[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]$.
2. (graded Jacobi identity) $[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]]$.
3. (graded Leibniz rule) $d[a, b] = [da, b] + (-1)^{\deg(a)}[a, db]$.

The Leibniz rule implies in particular that the bracket of a DGLA L induces a structure of graded Lie algebra on its cohomology $H^*(L) = \oplus_i H^i(L)$.

Example 1.2. Consider a differential graded vector space $(V, \overline{\partial})$ and denote

$$\operatorname{Hom}^*(V, V) = \bigoplus_i \operatorname{Hom}^i(V, V), \qquad \text{where}$$

$$\operatorname{Hom}^i(V, V) = \{f\colon V \to V \text{ linear } \mid f(V^n) \subset f(V^{n+i}) \text{ for every } n\}.$$

The bracket

$$[f, g] = fg - (-1)^{\deg(f)\deg(g)} gf$$

and the differential

$$df = [\overline{\partial}, f] = \overline{\partial} f - (-1)^{\deg(f)} f\overline{\partial}$$

make $\mathrm{Hom}^*(V, V)$ a differential graded Lie algebra.
Moreover there exists a natural isomorphism of graded Lie algebras

$$H^*(\mathrm{Hom}^*(V, V)) \xrightarrow{\simeq} \mathrm{Hom}^*(H^*(V), H^*(V)).$$

Example 1.3. Given a differential graded Lie algebra L and a commutative $\mathbb{K}$-algebra $\mathfrak{m}$ there exists a natural structure of DGLA in the tensor product $L \otimes \mathfrak{m}$ given by

$$d(x \otimes r) = dx \otimes r, \qquad [x \otimes r, y \otimes s] = [x, y] \otimes rs.$$

If $\mathfrak{m}$ is nilpotent (for example if $\mathfrak{m}$ is the maximal ideal of a local artinian $\mathbb{K}$-algebra), then the DGLA $L \otimes \mathfrak{m}$ is nilpotent; under this assumption, for every $a \in L^0 \otimes \mathfrak{m}$ the operator

$$[a, -]\colon L \otimes \mathfrak{m} \to L \otimes \mathfrak{m}, \qquad [a, -](b) = [a, b],$$

is a nilpotent derivation and

$$e^{[a,-]} = \sum_{n=0}^{+\infty} \frac{[a, -]^n}{n!}\colon L \otimes \mathfrak{m} \to L \otimes \mathfrak{m}$$

is an automorphism of the DGLA $L \otimes \mathfrak{m}$.

In order to introduce the basic ideas of the use of DGLA in deformation theory we begin with an example where technical difficulties are reduced at minimum. Consider a finite complex of vector spaces

$$(V, \overline{\partial}): \qquad 0 \longrightarrow V^0 \xrightarrow{\overline{\partial}} V^1 \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} V^n \longrightarrow 0.$$

Given a local artinian $\mathbb{K}$-algebra A with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbb{K}$, we define a deformation of $(V, \overline{\partial})$ over A as a complex of A-modules of the form

$$0 \longrightarrow V^0 \otimes A \xrightarrow{\overline{\partial}_A} V^1 \otimes A \xrightarrow{\overline{\partial}_A} \cdots \xrightarrow{\overline{\partial}_A} V^n \otimes A \longrightarrow 0$$

such that its residue modulo $\mathfrak{m}_A$ gives the complex $(V, \overline{\partial})$. Since, as a $\mathbb{K}$ vector space, $A = \mathbb{K} \oplus \mathfrak{m}_A$, this last condition is equivalent to say that

$$\overline{\partial}_A = \overline{\partial} + \xi, \qquad \text{where} \quad \xi \in \mathrm{Hom}^1(V, V) \otimes \mathfrak{m}_A.$$

The "integrability" condition $\overline{\partial}_A^2 = 0$ becomes

$$0 = (\overline{\partial} + \xi)^2 = \overline{\partial}\xi + \xi\overline{\partial} + \xi^2 = d\xi + \frac{1}{2}[\xi, \xi],$$

where d and [,] are the differential and the bracket on the differential graded Lie algebra $\mathrm{Hom}^*(V, V) \otimes \mathfrak{m}_A$.

Two deformations $\overline{\partial}_A, \overline{\partial}'_A$ are isomorphic if there exists a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & V^0\otimes A & \xrightarrow{\overline{\partial}_A} & V^1\otimes A & \xrightarrow{\overline{\partial}_A} & \cdots & \xrightarrow{\overline{\partial}_A} & V^n\otimes A & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\phi_0} & & \downarrow{\scriptstyle\phi_1} & & & & \downarrow{\scriptstyle\phi_n} & & \\
0 & \longrightarrow & V^0\otimes A & \xrightarrow{\overline{\partial}'_A} & V^1\otimes A & \xrightarrow{\overline{\partial}'_A} & \cdots & \xrightarrow{\overline{\partial}'_A} & V^n\otimes A & \longrightarrow & 0
\end{array}$$

such that every ϕ_i is an isomorphism of A-modules whose specialization to the residue field is the identity.
Therefore we can write $\phi := \sum_i \phi_i = Id + \eta$, where $\eta \in \mathrm{Hom}^0(V, V) \otimes \mathfrak{m}_A$ and, since $\mathbb{K}$ is assumed of characteristic 0 we can take the logarithm and write $\phi = e^a$ for some $a \in \mathrm{Hom}^0(V, V) \otimes \mathfrak{m}_A$. The commutativity of the diagram is therefore given by the equation $\overline{\partial}'_A = e^a \circ \overline{\partial}_A \circ e^{-a}$. Writing $\overline{\partial}_A = \overline{\partial} + \xi$, $\overline{\partial}'_A = \overline{\partial} + \xi'$ and using the relation $e^a \circ b \circ e^{-a} = e^{[a,-]}(b)$ we get

$$\xi' = e^{[a,-]}(\overline{\partial} + \xi) - \overline{\partial} = \xi + \frac{e^{[a,-]} - 1}{[a,-]}([a,\xi] + [a,\overline{\partial}]) = \xi + \sum_{n=0}^{\infty} \frac{([a,-])^n}{(n+1)!}([a,\xi] - da).$$

In particular both the integrability condition and isomorphism are entirely written in terms of the DGLA structure of $\mathrm{Hom}^*(V, V) \otimes \mathfrak{m}_A$. This leads to the following general construction.

Denote by **Art** the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ and by **Set** the category of sets (we ignore all the set-theoretic problems, for example by restricting to some universe). Unless otherwise specified, for every objects $A \in \mathbf{Art}$ we denote by $\mathfrak{m}_A$ its maximal ideal.
Given a differential graded Lie algebra L we define a covariant functor $\mathrm{MC}_L \colon \mathbf{Art} \to \mathbf{Set}$,

$$\mathrm{MC}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid dx + \frac{1}{2}[x,x] = 0 \right\}$$

The equation $dx + [x,x]/2 = 0$ is called the *Maurer-Cartan* equation and MC_L is called the Maurer-Cartan functor associated to L.
Two elements $x, y \in L \otimes \mathfrak{m}_A$ are said to be *gauge equivalent* if there exists $a \in L^0 \otimes \mathfrak{m}_A$ such that

$$y = e^a * x := x + \sum_{n=0}^{\infty} \frac{([a,-])^n}{(n+1)!}([a,x] - da).$$

The operator $*$ is called *gauge action*; in fact we have $e^a * (e^b * x) = e^{a \bullet b} * x$, where $\bullet$ is the Baker-Campbell-Hausdorff product [18, 40] in the nilpotent Lie algebra $L^0 \otimes \mathfrak{m}_A$, and then $*$ is an action of the exponential group $\exp(L^0 \otimes \mathfrak{m}_A)$ on the graded vector space $L \otimes \mathfrak{m}_A$.
It is not difficult to see that the set of solutions of the Maurer-Cartan equation is stable under the gauge action and then it makes sense to consider the functor $\mathrm{Def}_L \colon \mathbf{Art} \to \mathbf{Set}$ defined as

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{\text{gauge equivalence}}.$$

Remark 1.4. Given a surjective morphism $A \xrightarrow{\alpha} B$ in the category **Art**, an element $x \in \mathrm{MC}_L(B)$ can be lifted to $\mathrm{MC}_L(A)$ if and only if its equivalence class $[x] \in \mathrm{Def}_L(B)$ can

be lifted to $\mathrm{Def}_L(A)$.
In fact if $[x]$ lifts to $\mathrm{Def}_L(A)$ then there exists $y \in \mathrm{MC}_L(A)$ and $b \in L^0 \otimes \mathfrak{m}_B$ such that $\alpha(y) = e^b * x$. It is therefore sufficient to lift b to an element $a \in L^0 \otimes \mathfrak{m}_A$ and consider $x' = e^{-a} * y$.

The above computation shows that the functor of infinitesimal deformations of a complex $(V, \overline{\partial})$ is isomorphic to Def_L, where L is the differential graded Lie algebra $\mathrm{Hom}^*(V, V)$.

The utility of this approach relies on the following result, sometimes called *basic theorem of deformation theory.*

Theorem 1.5 (Schlessinger-Stasheff, Deligne, Goldman-Millson)**.** *Let $f\colon L \to M$ be a morphism of differential graded Lie algebras (i.e. f commutes with differential and brackets). Then f induces a natural transformation of functors $\mathrm{Def}_L \to \mathrm{Def}_M$. Moreover, if:*

1. *$f\colon H^0(L) \to H^0(M)$ is surjective;*
2. *$f\colon H^1(L) \to H^1(M)$ is bijective;*
3. *$f\colon H^2(L) \to H^2(M)$ is injective;*

then $\mathrm{Def}_L \to \mathrm{Def}_M$ is an isomorphism.

Where to find a proof. We do not give a proof here and we refer to the existing literature. The first published proof is contained in [13]; Goldman and Millson assume that both algebras have nonnegative degrees (i.e. $L^i = M^i = 0$ for every $i < 0$) but their proof can be easily extended to the general case (as in [27]) by using the remark about stabilizers given in [24, pag. 19].
Other proofs of (generalizations of) this theorem are in [24, pag. 24], [25] (via homotopy classification of L_∞ algebras) and [28], [30] (via extended deformation functors).
An earlier and essentially equivalent result was given in [37] (Theorem 6.2 in version October 3, 2000). □

Definition 1.6. On the category of differential graded Lie algebras consider the equivalence relation generated by: $L \sim M$ *if there exists a quasiisomorphism* $L \to M$. We shall say that two DGLA are *quasiisomorphic* if they are equivalent under this relation.

Example 1.7. Denote by $\mathbb{K}[t, dt]$ the differential graded algebra of polynomial differential forms over the affine line and for every DGLA L denote $L[t, dt] = L \otimes \mathbb{K}[t, dt]$. As a graded vector space $L[t, dt]$ is generated by elements of the form $aq(t) + bp(t)dt$, for $p, q \in \mathbb{K}[t]$ and $a, b \in L$. The differential and the bracket on $L[t, dt]$ are

$$d(aq(t) + bp(t)dt) = (da)q(t) + (-1)^{\deg(a)} aq(t)'dt + (db)p(t)dt,$$

$$[aq(t), ch(t)] = [a, c]q(t)h(t), \quad [aq(t), ch(t)dt] = [a, c]q(t)h(t)dt.$$

For every $s \in \mathbb{K}$, the evaluation morphism

$$e_s\colon L[t, dt] \to L, \quad e_s(aq(t) + bp(t)dt) = q(s)a$$

is a quasiisomorphism of differential graded Lie algebras.

Corollary 1.8. *If L, M are quasiisomorphic DGLA, then there exists an isomorphism of functors* $\mathrm{Def}_L \simeq \mathrm{Def}_M$.

It is now possible to state a more concrete interpretation of the guiding principle. Recall that an infinitesimal deformation is a deformation over a base $A \in \mathbf{Art}$.

Principle 1.9. *Let $F\colon \mathbf{Art} \to \mathbf{Set}$ be the functor of infinitesimal deformation of some algebro-geometric object defined over $\mathbb{K}$. Then there exists a differential graded Lie algebra L, defined up to quasiisomorphism, such that $F \simeq \mathrm{Def}_L$.*

For a slightly stronger version of Principle 1.9 we refer to the discussion in Section 5.9 of [30].

Definition 1.10. A differential graded Lie algebra L is called *formal* if it is quasiisomorphic, to its cohomology graded Lie algebra $H^*(L)$.

Lemma 1.11. *For every differential graded vector space $(V,\overline{\partial})$, the differential graded Lie algebra $\mathrm{Hom}^*(V,V)$ is formal.*

Proof. More generally, for every pair $(V,\overline{\partial}_V)$, $(W,\overline{\partial}_W)$ of differential graded vector spaces we consider the differential graded vector space

$$\mathrm{Hom}^*(V,W) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(V,W),$$

where

$$\mathrm{Hom}^i(V,W) = \{f\colon V \to W \mid f(V^j) \subset W^{i+j}\} = \prod_j \mathrm{Hom}(V^j, W^{i+j}).$$

and the differential is

$$\delta\colon \mathrm{Hom}^i(V,W) \to \mathrm{Hom}^{i+1}(V,W), \qquad \delta(f) = \overline{\partial}_W f - (-1)^{\deg(f)} f\overline{\partial}_V.$$

For every index i we choose a vector subspace $H^i \subset Z^i(V)$ such that the projection $H^i \to H^i(V)$ is bijective. The graded vector space $H = \oplus H^i$ is a quasiisomorphic subcomplex of V.

The subspace $K = \{f \in \mathrm{Hom}^*(V,V) \mid f(H) \subset H\}$ is a differential graded Lie subalgebra and there exists a commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\alpha} & \mathrm{Hom}^*(V,V) & \longrightarrow & \mathrm{Hom}^*(H,V/H) & \longrightarrow & 0\\
 & & \big\downarrow \beta & & \big\downarrow \gamma & & \big\downarrow Id & & \\
0 & \longrightarrow & \mathrm{Hom}^*(H,H) & \longrightarrow & \mathrm{Hom}^*(H,V) & \longrightarrow & \mathrm{Hom}^*(H,V/H) & \longrightarrow & 0
\end{array}$$

The maps α and β are morphisms of differential graded Lie algebras. Since $\mathrm{Hom}^*(H,V/H)$ is acyclic and γ is a quasiisomorphism, it follows that also α and β are quasiisomorphisms. $\square$

A generic deformation of $(V,\overline{\partial})$ over $\mathbb{K}[[t]]$ is a differential of the form $\tilde{\partial} = \overline{\partial} + tx_1 + t^2x_2 + \cdots$, where $x_i \in \mathrm{Hom}^1(V,V)$ for every i. Taking the series expansion of the integrability condition $[\tilde{\partial},\tilde{\partial}] = 0$ we get an infinite number of equations

$$\begin{array}{ll}
1) & [\overline{\partial}, x_1] = dx_1 = 0\\
2) & [x_1,x_1] = -2[\overline{\partial},x_2] = -2dx_2\\
\vdots & \quad\vdots\\
n) & \sum_{i=1}^{n-1}[x_i,x_{n-i}] = -2[\overline{\partial},x_n] = -2dx_n
\end{array}$$

The first equation tell us that $\overline{\partial} + tx_1$ is a deformation over $\mathbb{K}[t]/(t^2)$ of $\overline{\partial}$ if and only if $\overline{\partial}x_1 + x_1\overline{\partial} = 0$. The second equation tell us that $\overline{\partial} + tx_1$ extends to a deformation over $\mathbb{K}[[t]]$ only if the morphism of complexes $x_1 \circ x_1$ is homotopically equivalent to 0.

Vice versa, the existence of x_1, x_2 satisfying equations 1) and 2) is also sufficient to ensure that $\overline{\partial}+tx_1$ extends to a deformation over $\mathbb{K}[[t]]$. According to Lemma 1.11, the proof of this fact follows immediately from the following proposition.

Proposition 1.12. *If a differential graded Lie algebra L is formal, then the two maps*

$$\mathrm{Def}_L(\mathbb{K}[t]/(t^3)) \to \mathrm{Def}_L(\mathbb{K}[t]/(t^2))$$

$$\mathrm{Def}_L(\mathbb{K}[[t]]) := \lim_{\leftarrow n} \mathrm{Def}_L(\mathbb{K}[t]/(t^n)) \to \mathrm{Def}_L(\mathbb{K}[t]/(t^2))$$

have the same image.

Proof. According to Corollary 1.8 we may assume that L is a graded Lie algebra and therefore its Maurer-Cartan equation becomes $[x,x]=0$, $x \in L^1$.
Therefore $tx_1 \in \mathrm{Def}_L(\mathbb{K}[t]/(t^2))$ lifts to $\mathrm{Def}_L(\mathbb{K}[t]/(t^3))$ if and only if there exists $x_2 \in L^1$ such that

$$t^2[x_1,x_1] \equiv [tx_1+t^2x_2, tx_1+t^2x_2] \equiv 0 \pmod{t^3} \iff [x_1,x_1]=0$$

and $[x_1,x_1]=0$ implies that $tx_1 \in \mathrm{Def}_H(\mathbb{K}[t]/(t^n))$ for every $n \geq 3$. $\square$

Beware: the formality of L does not imply that $\mathrm{Def}_L(\mathbb{K}[[t]]) \to \mathrm{Def}_L(\mathbb{K}[t]/(t^3))$ is surjective.

2. Three examples of differential graded Lie algebras and related deformation problems

In this section we leave our toy example we consider three more important examples of deformation functors, namely deformations of holomorphic bundles, deformations of complex manifolds and embedded deformations of submanifolds. We work in the complex analytic category and then $\mathbb{K}=\mathbb{C}$.

Unless otherwise specified, every complex manifold is assumed compact and connected. For every complex manifold X we denote by:

- Θ_X the holomorphic tangent sheaf of X.
- $\mathcal{A}_X^{p,q}$ the sheaf of differentiable (p,q)-forms of X. More generally if $\mathcal{E}$ is locally free sheaf of $\mathcal{O}_X$-modules we denote by $\mathcal{A}_X^{p,q}(\mathcal{E}) \simeq \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{E}$ the sheaf of (p,q)-forms of X with values in $\mathcal{E}$ and by $A_X^{p,q}(\mathcal{E}) = \Gamma(X, \mathcal{A}_X^{p,q}(\mathcal{E}))$ the space of its global sections.
- For every submanifold $Z \subset X$, we denote by $N_{Z|X}$ the normal sheaf of Z in X.

Example 2.1 (Deformations of locally free sheaves)**.** Let $\mathcal{E}$ be a holomorphic locally free sheaf on a complex manifold X. The functor of isomorphism classes of deformations of $\mathcal{E}$ is denoted by $\mathrm{Def}_{\mathcal{E}}\colon \mathbf{Art} \to \mathbf{Set}$. A deformation of $\mathcal{E}$ over $A \in \mathbf{Art}$ is the data of a locally free sheaf $\mathcal{E}_A$ of $\mathcal{O}_X \otimes A$-modules and a morphism $\pi_A\colon \mathcal{E}_A \to \mathcal{E}$ inducing an isomorphism $\mathcal{E}_A \otimes_A \mathbb{C} \xrightarrow{\simeq} \mathcal{E}$.
Two deformations $\pi_A\colon \mathcal{E}_A \to \mathcal{E}$ and $\pi'_A\colon \mathcal{E}'_A \to \mathcal{E}$ are isomorphic if there exists an isomorphism $\theta\colon \mathcal{E}_A \to \mathcal{E}'_A$ of $\mathcal{O}_X \otimes A$-modules such that $\pi_A = \pi'_A\theta$.
The graded vector space

$$K = \oplus_{i\geq 0} K^i, \qquad \text{where} \quad K^i = A_X^{0,i}(\mathcal{H}om(\mathcal{E},\mathcal{E})),$$

endowed with the Dolbeault differential and the natural bracket is a differential graded Lie algebra such that $H^i(K) = \mathrm{Ext}^i(\mathcal{E},\mathcal{E})$.

Theorem 2.2. *In the notation above there exists an isomorphism of functors*

$$\mathrm{Def}_K \simeq \mathrm{Def}_{\mathcal{E}}\,.$$

Proof. This is well known and we refer to [12], [21, Chap. VII], [13, Sec. 9.4], [8, Pag. 238] for a proof (some of these references deals with the equivalent problem of deformations of a holomorphic vector bundle).
Here we only note that, for every $A \in \mathbf{Art}$ and every $x \in \mathrm{MC}_K(A)$, the associated deformation of $\mathcal{E}$ over A is the kernel of

$$\overline{\partial} + x\colon \mathcal{A}^{0,0}_X(\mathcal{E}) \otimes A \to \mathcal{A}^{0,1}_X(\mathcal{E}) \otimes A.$$

□

Example 2.3 (Deformations of complex manifolds)**.** Recall that, given a complex manifold X, a deformation of X over a local artinian $\mathbb{C}$-algebra A can be interpreted as a morphism of sheaves of algebras $\mathcal{O}_A \to \mathcal{O}_X$ such that $\mathcal{O}_A$ is flat over A and $\mathcal{O}_A \otimes_A \mathbb{C} \to \mathcal{O}_X$ is an isomorphism. Define the functor $\mathrm{Def}_X\colon \mathbf{Art} \to \mathbf{Set}$ of infinitesimal deformations of X setting $\mathrm{Def}_X(A)$ as the set of isomorphism classes of deformations of X over A.
This functor is isomorphic to the deformation functor associated to the *Kodaira-Spencer* differential graded Lie algebra of X, that is

$$KS_X = A^{0,*}_X(\Theta_X) = \oplus_i A^{0,i}_X(\Theta_X).$$

The differential on KS_X is the Dolbeault differential, while the bracket is defined in local coordinates as the $\overline{\Omega}^*$-bilinear extension of the standard bracket on $\mathcal{A}^{0,0}_X(\Theta_X)$ ($\overline{\Omega}^*$ is the sheaf of antiholomorphic differential forms). By Dolbeault theorem we have $H^i(A^{0,*}_X(\Theta_X)) = H^i(X,\Theta_X)$ for every i.
The isomorphism $\mathrm{Def}_{KS_X} \to \mathrm{Def}_X$ is obtained by thinking, via Lie derivation, the elements of $A^{0,i}_X(\Theta_X)$ as derivations of degree i of the sheaf of graded algebras $\oplus_i \mathcal{A}^{0,i}_X$. More precisely, to every $x \in \mathrm{MC}_{KS_X}(A)$ we associate the deformation

$$\mathcal{O}_A(x) = \ker(\mathcal{A}^{0,0}_X \otimes A \xrightarrow{\overline{\partial}+\boldsymbol{l}_x} \mathcal{A}^{0,1}_X \otimes A),$$

where in local holomorphic coordinates $z_1,\dots,z_n$

$$x = \sum_{i,j} x_{ij} d\overline{z}_i \frac{\partial}{\partial z_j}, \qquad \boldsymbol{l}_x(f) = \sum_{i,j} x_{ij} \frac{\partial f}{\partial z_j} d\overline{z}_i.$$

Equivalently we can interpret every element of $A^{0,1}_X(\Theta_X)$ as a morphism of vector bundles $T^{0,1}_X \to T^{1,0}_X$ and then also as a variation of the almost complex structure of X. The Maurer-Cartan equation becomes exactly the integrability condition of the Newlander-Nirenberg theorem (see e.g. [5], [14]).
If we are interested only to infinitesimal deformations, the proof of the isomorphism $\mathrm{Def}_{KS_X} \to \mathrm{Def}_X$ can be done without using almost complex structures and therefore without Newlander-Nirenberg theorem; full details will appear in [17].

Example 2.4 (Embedded deformations of submanifolds)**.** Let X be a complex manifold and let $Z \subset X$ be an analytic subvariety defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. The embedded deformations of Z in X are described by the functor $\mathrm{Hilb}_X^Z\colon \mathbf{Art} \to \mathbf{Set}$,

$$\mathrm{Hilb}_X^Z(A) = \{\text{ideal sheaves } \mathcal{I}_A \subset \mathcal{O}_X \otimes_{\mathbb{C}} A,\ \text{flat over } A \text{ such that } \mathcal{I}_A \otimes_A \mathbb{C} = \mathcal{I}\}.$$

Theorem 2.5. *Assume that Z is smooth: denote by*

$$\pi\colon A_X^{0,*}(\Theta_X) \to A_Z^{0,*}(N_{Z|X})$$

the natural restriction map and by (see Example 1.7)

$$L = \{a \in A_X^{0,*}(\Theta_X)[t, dt] \mid e_0(a) = 0,\ \pi e_1(a) = 0\}.$$

Then L is a differential graded Lie algebra, $H^i(L) = H^{i-1}(Z, N_{Z|X})$ for every $i \in \mathbb{Z}$ and there exists an isomorphism of functors $\mathrm{Def}_L \simeq \mathrm{Hilb}_X^Z$.

Proof. This is proved in [31]. Later on this paper (see Remark 8.2) we will give an explicit description of the isomorphism $\mathrm{Def}_L \simeq \mathrm{Hilb}_X^Z$. □

3. Functors of Artin rings

The philosophical implication of Theorem 1.5 and its Corollary 1.8 is that it is often useful to consider quasiisomorphisms of DGLA where the domain and/or the target are differential graded Lie algebras whose elements may have no geometrical meaning. Therefore it is also useful to have an abstract theory of functors $\mathbf{Art} \to \mathbf{Set}$ that fits with this mathematical setting. Such theory already exists and was introduced, with different motivations, by M. Schlessinger in his PhD thesis and published in the paper [36]. In this section we sketch the main definitions for the benefit of non expert reader; more details and the original motivation can be found both in the original paper and in every introductory book of deformation theory (such as [1], [38]).

Definition 3.1. A *functor of Artin rings* is a covariant functor $F\colon \mathbf{Art} \to \mathbf{Set}$ such that $F(\mathbb{K}) = \{\text{one point}\}$.

The functors of Artin rings are the objects of a category whose morphisms are the natural transformations. For simplicity of notation, if $\phi\colon F \to G$ is a natural transformation we denote by $\phi\colon F(A) \to G(A)$ the corresponding morphism of sets for every $A \in \mathbf{Art}$.

Example 3.2. Let R be a local complete $\mathbb{K}$-algebra with residue field $\mathbb{K}$. The functor

$$\mathrm{h}_R\colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{h}_R(A) = \mathrm{Hom}_{\mathbb{K}-\mathrm{alg}}(R, A),$$

is a functor of Artin rings.

Definition 3.3. A functor $F\colon \mathbf{Art} \to \mathbf{Set}$ is *prorepresentable* if it is isomorphic to h_R for some R as in Example 3.2.

The category $\mathbf{Art}$ is closed under fiber products, i.e. every pair of morphisms $C \to A$, $B \to A$ may be extended to a commutative diagram

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & A \end{array} \tag{1}$$

such that the natural map

$$\mathrm{h}_R(B \times_A C) \to \mathrm{h}_R(B) \times_{\mathrm{h}_R(A)} \mathrm{h}_R(C)$$

is bijective for every R.

Definition 3.4. Let $F\colon \mathbf{Art} \to \mathbf{Set}$ be a functor of Artin rings; for every fiber product

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \xrightarrow{\beta} & A \end{array}$$

in **Art** consider the induced map $\eta\colon F(B \times_A C) \to F(B) \times_{F(A)} F(C)$.
We shall say that F is *homogeneous* if η is bijective whenever β is surjective [34, Def. 2.5].
We shall say that F is a *deformation functor* if:

1. η is surjective whenever β is surjective,
2. η is bijective whenever $A = \mathbb{K}$.

The name deformation functor comes from the fact that almost all functors arising in deformation theory satisfy the condition of Definition 3.4. Every prorepresentable functor is a homogeneous deformation functor.

Remark 3.5. Our definition of deformation functors involves conditions that are slightly more restrictive than the classical Schlessinger conditions H1, H2 of [36] and the semi-homogeneity condition of [34]. The main motivations of this change are:

1. Functors of Artin rings satisfying Schlessinger condition H1, H2 and H3 do not necessarily have a "good" obstruction theory (see [9, Example 6.8]).
2. The definition of deformation functor extends naturally in the framework of derived deformation theory and extended moduli spaces [28].

Finally it is not difficult to prove that a deformation functor is homogeneous if and only if satisfies Schlessinger's condition H4.

The functors MC_L (characteristic $\neq 2$) and Hilb^X_Z (Example 2.4) are homogeneous deformation functors. The functors Def_L (characteristic 0), $\mathrm{Def}_{\mathcal{E}}$ (Example 2.1) and Def_X (Example 2.3) are deformation functors but not homogeneous in general. The verification of 3.4 for MC_L is clear, while the proof that Def_L is a deformation functor follows from Remark 1.4.
For the proof that Hilb^X_Z, $\mathrm{Def}_{\mathcal{E}}$ and Def_X are deformation functors we refer to [36] (especially Examples 3.1 and 3.7), [38], [39] and [43].

Definition 3.6. Let $F\colon \mathbf{Art} \to \mathbf{Set}$ be a deformation functor. The set

$$T^1F = F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$

is called the *tangent space* of F.

Proposition 3.7. *The tangent space of a deformation functor has a natural structure of vector space over $\mathbb{K}$. For every natural transformation of deformation functors $F \to G$, the induced map $T^1F \to T^1G$ is linear.*

Proof. See [36, Lemma 2.10]. □

It is notationally convenient to reserve the letter ϵ to denote elements of $A \in \mathbf{Art}$ annihilated by the maximal ideal $\mathfrak{m}_A$, and in particular of square zero.

Example 3.8. For the functor h_R defined in Example 3.2, its tangent space is

$$T^1 \mathrm{h}_R = \mathrm{Hom}_{\mathbb{K}-\mathrm{alg}}(R, \mathbb{K}[\varepsilon]) = \mathrm{Hom}_{\mathbb{K}}\left(\frac{\mathfrak{m}_R}{\mathfrak{m}_R^2}, \mathbb{K}\right).$$

Therefore $T^1 \mathrm{h}_R$ is isomorphic to the Zariski tangent space of $Spec(R)$ at its closed point.

The formal smoothness of $Spec(R)$ is equivalent to the property that $A \to B$ surjective implies $\mathrm{h}_R(A) \to \mathrm{h}_R(B)$ surjective. This motivate the following definition.

Definition 3.9. A functor of Artin rings F is called *smooth* if $F(A) \to F(B)$ is surjective for every surjective morphism $A \to B$ in $\mathbf{Art}$.
A natural transformation $\phi\colon F \to G$ of functors of Artin rings is called *smooth* if for every surjective morphism $A \to B$ in $\mathbf{Art}$, the map $F(A) \to G(A) \times_{G(B)} F(B)$ is also surjective.

Note that if $\phi\colon F \to G$ is a smooth natural transformation, then $\phi\colon F(A) \to G(A)$ is surjective for every A (take $B = \mathbb{K}$).
In characteristic 0, according to Remark 1.4, for every differential graded Lie algebra L the natural projection $\mathrm{MC}_L \to \mathrm{Def}_L$ is smooth.
The majority of deformation functors arising in concrete cases are not proprepresentable; a weaker version, that correspond to the notion of semiuniversal deformation, is given in next Theorem.

Theorem 3.10 (Schlessinger, [36])**.** *Let F be a deformation functor with finite dimensional vector space. Then there exists a local complete noetherian $\mathbb{K}$-algebra R with residue field $\mathbb{K}$ and a smooth natural transformation $\mathrm{h}_R \to F$ inducing an isomorphism on tangent spaces $T^1 \mathrm{h}_R = T^1 F$. Moreover R is unique up to non-canonical isomorphism.*

In the situation of Theorem 3.10 the geometric properties of $Spec(R)$ can be determined in terms of properties of F. Concerning smoothness we have the following:

Lemma 3.11. *Let R be a local complete noetherian $\mathbb{K}$-algebra with residue field $\mathbb{K}$, F a deformation functor and $\mathrm{h}_R \to F$ as in Theorem 3.10. The following conditions are equivalent:*

1. *R is isomorphic to a power series ring $\mathbb{K}[[x_1, \dots, x_n]]$.*
2. *The functor F is smooth.*
3. *For every $s \geq 2$ the morphism*
$$F\left(\frac{\mathbb{K}[t]}{(t^{s+1})}\right) \to F\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$
is surjective.

Proof. The only nontrivial implication is $[3 \Rightarrow 1]$. We assume for simplicity that $\mathbb{K}$ is an infinite field; the useless case of $\mathbb{K}$ finite would require a different proof.
We first observe that for every $s \geq 1$ the map $\mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^{s+1})}\right) \to \mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$ is surjective.

Let n be the embedding dimension of R, then we can write $R = \mathbb{K}[[x_1, \dots, x_n]]/I$ for some ideal $I \subset (x_1, \dots, x_n)^2$; we want to prove that $I = 0$. Assume therefore $I \neq 0$ and denote by $s \geq 2$ the greatest integer such that $I \subset (x_1, \dots, x_n)^s$: we claim that

$$\mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^{s+1})}\right) \to \mathrm{h}_R\left(\frac{\mathbb{K}[t]}{(t^2)}\right)$$

is not surjective. Choosing $f \in I - (x_1, \dots, x_n)^{s+1}$, after a possible generic linear change of coordinates of the form $x_i \mapsto x_i + a_i x_1$, with $a_2, \dots, a_k \in \mathbb{K}$, we may assume that f contains the monomial x_1^s with a nonzero coefficient, say $f = cx_1^s + \dots$; let $\alpha\colon R \to \mathbb{K}[t]/(t^2)$ be the morphism defined by $\alpha(x_1) = t$, $\alpha(x_i) = 0$ for $i > 1$. Assume that there exists $\tilde{\alpha}\colon R \to \mathbb{K}[t]/(t^{s+1})$ that lifts α and denote by $\beta\colon \mathbb{K}[[x_1, \dots, x_n]] \to \mathbb{K}[t]/(t^{s+1})$ the composition of $\tilde{\alpha}$ with the projection $\mathbb{K}[[x_1, \dots, x_n]] \to R$. Then $\beta(x_1) - t, \beta(x_2), \dots, \beta(x_n) \in (t^2)$ and therefore $\beta(f) \equiv ct^s \not\equiv 0 \pmod{t^{s+1}}$. □

4. Obstructions

In the set-up of functors of Artin rings, with the term obstructions we intend *obstructions for a deformation functor to be smooth.*

We shall say that a morphism $\alpha\colon B \to A$ in **Art** is a *small surjection* if α is surjective and its kernel is annihilated by the maximal ideal $\mathfrak{m}_B$. The artinian property implies that every surjective morphism in **Art** can be decomposed in a finite sequence of small surjection and then a functor F is smooth if and only if $F(B) \to F(A)$ is surjective for every small surjection $B \to A$.

A *small extension* is a small surjection together a framing of its kernel. More precisely a small extension e in **Art** is an exact sequence of abelian groups

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \xrightarrow{\alpha} A \longrightarrow 0,$$

such that α is a morphism in the category **Art** and M is an ideal of B annihilated by the maximal ideal $\mathfrak{m}_B$. In particular M is a finite dimensional vector space over $B/\mathfrak{m}_B = \mathbb{K}$. A small extension as above is called *principal* if $M = \mathbb{K}$.

Definition 4.1. Let F be a functor of Artin rings. An *obstruction theory* (V, v_e) for F is the data of a $\mathbb{K}$-vector space V and for every small extension in **Art**

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

of an *obstruction map* $v_e\colon F(A) \to V \otimes M$ satisfying the following properties:

1. If $a \in F(A)$ can be lifted to $F(B)$, then $v_e(a) = 0$.
2. (base change) For every commutative diagram

$$\begin{array}{ccccccccc}
e_1\colon & 0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
& & & \downarrow{\scriptstyle\alpha_M} & & \downarrow{\scriptstyle\alpha_B} & & \downarrow{\scriptstyle\alpha_A} & & \\
e_2\colon & 0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0.
\end{array}$$

with e_1, e_2 small extensions and α_A, α_B morphisms in **Art**, we have

$$v_{e_2}(\alpha_A(a)) = (Id_V \otimes \alpha_M)(v_{e_1}(a)) \qquad \text{for every} \quad a \in F(A_1).$$

Definition 4.2. An obstruction theory (V, v_e) for F is called *complete* if the converse of item 1 in 4.1 holds; i.e. the lifting exists if and only if the obstruction vanishes.

Clearly if F admits a complete obstruction theory then it admits infinitely ones; it is in fact sufficient to embed V in a bigger vector space. One of the main interest is to look for the "smallest" complete obstruction theory.

Remark 4.3. Let $e\colon\ 0\longrightarrow M\longrightarrow B\longrightarrow A\longrightarrow 0$ be a small extension and $a \in F(A)$; the obstruction $v_e(a) \in V \otimes M$ is uniquely determined by the values $(Id_V \otimes f)v_e(a) \in V$, where f varies along a basis of $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$. On the other hand, by base change we have $(Id_V \otimes f)v_e(a) = v_\epsilon(a)$, where ϵ is the small extension

$$\epsilon:\quad 0\longrightarrow\mathbb{K}\longrightarrow \frac{B\oplus\mathbb{K}}{\{(m,-f(m))\mid m\in M\}}\longrightarrow A\longrightarrow 0.$$

This implies that every obstruction theory is uniquely determined by its behavior on principal small extensions.

Example 4.4. Assume $\mathbb{K}$ of characteristic 0 and let L be a differential graded Lie algebra. We want to show that MC_L has a "natural" obstruction theory $(H^2(L), v_e)$.
Let's consider a small extension in **Art**

$$e\colon\qquad 0\longrightarrow M\longrightarrow A\longrightarrow B\longrightarrow 0$$

and let $x \in \mathrm{MC}_L(B) = \{x \in L^1 \otimes \mathfrak{m}_B \mid dx + \frac{1}{2}[x,x] = 0\}$; we define an obstruction $v_e(x) \in H^2(L \otimes M) = H^2(L) \otimes M$ in the following way:
First take a lifting $\tilde{x} \in L^1 \otimes \mathfrak{m}_A$ of x and consider $h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes M$; we have

$$dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h, \tilde{x}] - \frac{1}{2}[[\tilde{x}, \tilde{x}], \tilde{x}].$$

Since $[L^2 \otimes M, L^1 \otimes \mathfrak{m}_A] = 0$ we have $[h, \tilde{x}] = 0$, by Jacobi identity $[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$ and then $dh = 0$.
Define $v_e(x)$ as the class of h in $H^2(L \otimes M) = H^2(L) \otimes M$; the first thing to prove is that $v_e(x)$ is independent from the choice of the lifting $\tilde{x}$; every other lifting is of the form $y = \tilde{x} + z$, $z \in L^1 \otimes M$ and then

$$d\tilde{y} + \frac{1}{2}[y, y] = h + dz.$$

It is evident from the above computation that $(H^2(L), v_e)$ is a complete obstruction theory for the functor MC_L.

Definition 4.5. A morphism of obstruction theories $(V, v_e) \to (W, w_e)$ is a linear map $\theta\colon V \to W$ such that $w_e = (\theta \otimes Id)v_e$ for every small extension e.
An obstruction theory (O_F, ob_e) for F is called *universal* if for every obstruction theory (V, v_e) there exists an unique morphism $(O_F, ob_e) \to (V, v_e)$.

Theorem 4.6. *Let F be a deformation functor, then:*

1. *There exists an universal obstruction theory (O_F, ob_e) for F which is complete.*
2. *Every element of the universal obstruction target O_F is of the form $ob_e(a)$ for some principal extension*

$$e\colon\quad 0\longrightarrow\mathbb{K}\longrightarrow B\longrightarrow A\longrightarrow 0$$

and some $a \in F(A)$.

Proof. For the proof we refer to [9]. □

It is clear that the universal obstruction theory (O_F, ob_e) is unique up to isomorphism and depends only by F and not by any additional data.

Definition 4.7. The *obstruction space* of a deformation functor F is the universal obstruction target O_F.

Corollary 4.8. *Let (V, v_e) be a complete obstruction theory for a deformation functor F. Then the obstruction space O_F is isomorphic to the vector subspace of V generated by all the obstructions arising from principal extensions.*

Proof. Denote by $\theta\colon O_F \to V$ the morphism of obstruction theories. Every principal obstruction is contained in the image of θ and, since V is complete, the morphism θ is injective. □

Remark 4.9. Most authors use Corollary 4.8 as a definition of obstruction space.

Example 4.10. Let R be a local complete $\mathbb{K}$-algebra with residue field $\mathbb{K}$ and let $n = \dim T^1 \mathrm{h}_R = \dim \mathfrak{m}_R/\mathfrak{m}_R^2$ its embedding dimension. Then we can write $R = P/I$, where $P = \mathbb{K}[[x_1, \dots, x_n]]$ and $I \subset \mathfrak{m}_P^2$. We claim that

$$T^2 \mathrm{h}_R := \mathrm{Hom}_P(I, \mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(I/\mathfrak{m}_P I, \mathbb{K})$$

is the obstruction space of h_R. In fact for every small extension

$$e\colon \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

and every $\alpha \in \mathrm{h}_R(A)$ we can lift α to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & P & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ob_a(\alpha)} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with β a morphism of $\mathbb{K}$-algebras. It is easy to verify that

$$ob_e(\alpha) = \beta_{|I} \in \mathrm{Hom}_{\mathbb{K}}(I/\mathfrak{m}_P I, M) = T^2 \mathrm{h}_R \otimes M$$

is well defined, it is a complete obstruction and that $(T^2 \mathrm{h}_R, ob_e)$ is the universal obstruction theory for the functor h_R (see [9, Prop. 5.3]).

Let $\phi\colon F \to G$ be a natural transformation of deformation functors. Then $(O_G, ob_e \circ \phi)$ is an obstruction theory for F and then there exists an unique linear map $ob_\phi\colon O_F \to O_G$ which is compatible with ϕ in the obvious sense.

Theorem 4.11 (Standard smoothness criterion)**.** *Let $\phi\colon F \to G$ be a morphism of deformation functors. The following conditions are equivalent:*

1. *ϕ is smooth.*
2. *$T^1\phi\colon T^1F \to T^1G$ is surjective and $ob_\phi\colon O_F \to O_G$ is bijective.*
3. *$T^1\phi\colon T^1F \to T^1G$ is surjective and $ob_\phi\colon O_F \to O_G$ is injective.*

Proof. In order to avoid confusion we denote by ob_e^F and ob_e^G the obstruction maps for F and G respectively.

[1 ⇒ 2] Every smooth morphism is in particular surjective; therefore if ϕ is smooth then the induced morphisms $T^1F \to T^1G$, $O_F \to O_G$ are both surjective.
Assume that $ob_\phi(\xi) = 0$ and write $\xi = ob_e^F(x)$ for some $x \in F(A)$ and some small extension $e\colon 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$. Since $ob_e^G(\phi(x)) = 0$ the element x lifts to a pair

$(x, y') \in F(A) \times_{G(A)} G(B)$ and then the smoothness of ϕ implies that x lifts to $F(B)$.
[3 $\Rightarrow$ 1] We need to prove that for every small extension $e\colon 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$ the map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. Fix $(x, y') \in F(A) \times_{G(A)} G(B)$ and let $y \in G(A)$ be the common image of x and y'. Then $ob_e^G(y) = 0$ because y lifts to $G(B)$, hence $ob_e^F(x) = 0$ by injectivity of ob_ϕ. Therefore x lifts to some $x'' \in F(B)$. In general $y'' = \phi(x'')$ is not equal to y'. However, $(y'', y') \in G(B) \times_{G(A)} G(B)$ and therefore there exists $v \in T^1G$ such that $\theta(y'', v) = (y'', y')$ where

$$\theta\colon G(B) \times T^1G = F(B \times_{\mathbb{K}} \mathbb{K}[\epsilon]) \to G(B) \times_{G(A)} G(B)$$

is induced by the isomorphism

$$B \times_{\mathbb{K}} \mathbb{K}[\epsilon] \to B \times_A B, \qquad (b, \overline{b} + \alpha\epsilon) \mapsto (b, b + \alpha\epsilon).$$

By assumption $T^1F \to T^1G$ is surjective, v lifts to a $w \in T^1F$ and setting $\theta(x'', w) = (x'', x')$ we have that x' is a lifting of x which maps to y', as required. □

Remark 4.12. In most concrete cases, given a natural transformation $F \to G$ it is very difficult to calculate the map $O_F \to O_G$, while it is generally easy to describe complete obstruction theories for F and G and compatible morphism between them. In this situation only the implication [3 $\Rightarrow$ 1] of the standard smoothness criterion holds.

Corollary 4.13. *Let L be a differential graded Lie algebra. Then the projection $\mathrm{MC}_L \to \mathrm{Def}_L$ induces an isomorphism on obstruction spaces. Therefore every obstruction theory for MC_L is invariant under the gauge action and factors to an obstruction theory for Def_L.*
In particular, according to Example 4.4, the obstruction space of Def_L is contained in $H^2(L)$.

Proof. The projection $\mathrm{MC}_L \to \mathrm{Def}_L$ is smooth. □

Corollary 4.14. *Let F be a deformation functor and $\mathrm{h}_R \to F$ a smooth natural transformation. Then the dimension of O_F is equal to the minimum number of generators of an ideal I defining R as a quotient of a power series ring, i.e. $R = \mathbb{K}[[x_1, \dots, x_n]]/I$.*

Proof. Apply Nakayama's lemma to the $\mathbb{K}[[x_1, \dots, x_n]]$-module I and use Example 4.10. □

5. Special obstructions

Let F be a deformation functor with a complete obstruction theory (V, v_e). In concrete cases, the difficulty of computation of the map v_e reflects the structure of the small extension e. It is therefore useful to consider obstructions arising from "special" small extensions that are easier to compute. The interesting fact is that, under some additional condition, the knowledge of this "special obstructions" gives lots of information.

The *primary obstruction* is the obstruction map arising from the small extension

$$0 \longrightarrow \mathbb{K} \xrightarrow{xy} \frac{\mathbb{K}[x,y]}{(x^2, y^2)} \longrightarrow \frac{\mathbb{K}[x,y]}{(x^2, xy, y^2)} \longrightarrow 0.$$

Note that

$$\frac{\mathbb{K}[x,y]}{(x^2,xy,y^2)}=\frac{\mathbb{K}[x]}{(x^2)}\times_{\mathbb{K}}\frac{\mathbb{K}[y]}{(y^2)},\qquad \frac{\mathbb{K}[x,y]}{(x^2,y^2)}=\frac{\mathbb{K}[x]}{(x^2)}\otimes\frac{\mathbb{K}[y]}{(y^2)}$$

and then

$$F\left(\frac{\mathbb{K}[x,y]}{(x^2,xy,y^2)}\right)=F\left(\frac{\mathbb{K}[x]}{(x^2)}\right)\times F\left(\frac{\mathbb{K}[y]}{(y^2)}\right)=T^1F\times T^1F.$$

It is a formal consequence of base change property that the associated obstruction map

$$b\colon T^1F\times T^1F\to V$$

is bilinear symmetric.
If F is smooth then the primary obstruction vanishes but the converse is generally false. If $F=\mathrm{Def}_L$ for some differential graded Lie algebra and we consider the natural obstruction theory of Example 4.4, the primary obstruction map $T^1F\times T^1F=H^1(L)\times H^1(L)\to H^2(L)$ is equal to the induced bracket in cohomology. Therefore, according to Corollary 1.8, if L is a formal DGLA, then Def_L is smooth if and only if its primary obstruction is trivial.

In characteristic $\neq 2$, the substitution $\alpha(t)=x+y$ gives a morphism of small extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \xrightarrow{t^2} & \mathbb{K}[t]/(t^3) & \longrightarrow & \mathbb{K}[t]/(t^2) & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle 2} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{K} & \xrightarrow{xy} & \mathbb{K}[x,y]/(x^2,y^2) & \longrightarrow & \mathbb{K}[x,y]/(x^2,xy,y^2) & \longrightarrow & 0.
\end{array}$$

From this and base change axiom it follows that the obstruction of lifting $x\in T^1F$ to $F(\mathbb{K}[t]/(t^3))$ is equal to $\dfrac{1}{2}b(x,x)$. Therefore the vanishing of the primary obstruction map is equivalent to the surjectivity of $F(\mathbb{K}[t]/(t^3))\to F(\mathbb{K}[t]/(t^2))$.

The *curvilinear* obstructions are the obstructions arising from the curvilinear extensions

$$0\longrightarrow\mathbb{K}\xrightarrow{t^n}\mathbb{K}[t]/(t^{n+1})\longrightarrow\mathbb{K}[t]/(t^n)\longrightarrow 0.$$

We have seen that the first curvilinear obstruction ($n=2$) is essentially the primary obstruction map. Lemma 3.11 suggests the validity of Item 1 of the following proposition, while Item 2 is very surprising.

Proposition 5.1. *In the above setup, let $O^c_F\subset V$ be the vector subspace generated by the curvilinear obstructions. Then:*

1. *If $O^c_F=0$ then F is smooth.*
2. *In general O^c_F is a proper subspace of the obstruction space O_F.*
3. *If $\mathbb{K}$ is algebraically closed and $F=\mathrm{h}_R$, then $\dim R\geq\dim_{\mathbb{K}}T^1F-\dim_{\mathbb{K}}O^c_F$.*

Proof. If F has finite dimensional tangent space then Item 1 follows from Theorem 3.10 and Lemma 3.11; the general case is proved in [9, Cor. 6.4]. Item 3 is proved in [20] (a simplified proof is in [10]).
Consider the functor $F=\mathrm{h}_R$, where $R=P/I$, $P=\mathbb{K}[x,y]$ and $I=(x^3,y^3,x^2y^2)$.

Then $O_F = \mathrm{Hom}_P(I, \mathbb{K})$ has dimension 3, while $x^2y^2 \in I$ belongs to the kernel of every curvilinear obstruction. $\square$

We consider now another class of small extensions that, as far as I know, was first used in [15]; such class is quite useful because in concrete cases the induced obstructions are usually much easier to understand (see next Remark 8.5).
For every $A \in \mathbf{Art}$ and every A-module M we denote by $A \oplus M$ the trivial extension (with multiplication rule $(a, m)(b, n) = (ab, an + bm)$). We define a *semitrivial small extension* as an extension of the form

$$0 \to K \to A \oplus M \to A \oplus N \to 0$$

for some $A \in \mathbf{Art}$ and some short exact sequence $0 \to K \to M \to N \to 0$ of finitely generated A-modules with $\mathfrak{m}_A K = 0$.
Unfortunately, in general the semitrivial obstructions do not generate O_F; consider for instance $F = \mathrm{h}_R$, where

$$R = \frac{P}{I} = \frac{\mathbb{C}[x, y]}{(f, f_x, f_y)}, \qquad f = x^3 + xy^5 + y^7,\ f_x = \frac{\partial f}{\partial x},\ f_y = \frac{\partial f}{\partial y}.$$

Then O_F has dimension 3 [35, p. 103], while every semitrivial obstruction $\phi \in \mathrm{Hom}_P(I, \mathbb{C})$ verifies $\phi(f) = 0$.
On the positive side we have the following result which is the basic trick of the abstract T^1-lifting theorem [19], [10].

Theorem 5.2. *If $\mathbb{K}$ is a field of characteristic 0, then every curvilinear obstruction of a deformation functor F is semitrivial. In particular F is smooth if and only if every semitrivial obstruction vanishes.*

Proof. Denote by $\dfrac{\mathbb{K}[t]s}{(t^n)}$ the free $\dfrac{\mathbb{K}[t]}{(t^n)}$-module generated by s. To prove the theorem it is sufficient to consider the morphism

$$\alpha \colon \frac{\mathbb{K}[t]}{(t^{n+1})} \to \frac{\mathbb{K}[t]}{(t^n)} \oplus \frac{\mathbb{K}[t]s}{(t^n)}, \qquad \alpha(t) = t + s$$

and apply the base change axiom to the morphism of small extension

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \xrightarrow{t^n} & \mathbb{K}[t]/(t^{n+1}) & \longrightarrow & \mathbb{K}[t]/(t^n) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle n} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & \mathbb{K} & \xrightarrow{t^{n-1}s} & \dfrac{\mathbb{K}[t]}{(t^n)} \oplus \dfrac{\mathbb{K}[t]s}{(t^n)} & \longrightarrow & \dfrac{\mathbb{K}[t]}{(t^n)} \oplus \dfrac{\mathbb{K}[t]s}{(t^{n-1})} & \longrightarrow & 0.
\end{array}$$

$\square$

Remark 5.3. It is not difficult to prove [4, Appendix] that the space of semitrivial obstructions of h_R is isomorphic to $\mathrm{Ext}^1_R(\Omega^1_{R/\mathbb{K}}, \mathbb{K})$ and then, for $\mathbb{K}$ algebraically closed of characteristic 0, we get the dimension bound

$$\dim R \geq \dim_{\mathbb{K}} \mathrm{Ext}^0_R(\Omega^1_{R/\mathbb{K}}, \mathbb{K}) - \dim_{\mathbb{K}} \mathrm{Ext}^1_R(\Omega^1_{R/\mathbb{K}}, \mathbb{K})$$

proved first in [35].

6. Annihilation of obstructions

Let $F\colon \mathbf{Art}\to\mathbf{Set}$ be the functor of infinitesimal deformations of some geometric object. If such object is "reasonable", then F is a deformation functor and it is provided of a natural and geometrically defined complete obstruction theory (V,v_e). As an example if $F=\mathrm{Def}_X$ is the functor of deformations of a complex manifold, then it has a natural obstruction theory with $V=H^2(\Theta_X)$.
Our goal is to determine the obstruction space O_F as a subset of V. In general the inclusion $O_F\subset V$ is proper: if X is a complex torus of dimension n, then $H^2(\Theta_X)$ has dimension $n^2(n-1)/2$, while it is known since the very first works of Kodaira and Spencer that X has unobstructed deformations [22, p. 408] and then $O_F=0$.
A way to obtain information about O_F is by annihilation maps: we shall say that a linear map $\omega\colon V\to W$ annihilates the obstructions of F if $\omega(O_F)=0$ or equivalently if $\omega v_e=0$ for every principal small extension e. Analogous definitions can be done for annihilations of curvilinear and semitrivial obstructions.
The possibility to describe a deformation functor in the form Def_L for some differential graded Lie algebra L gives a simple and powerful way to construct maps $\omega\colon H^2(L)\to W$ that annihilate obstructions. The idea is easy: assume it is given a morphism $\chi\colon L\to M$ of DGLA, then, according to Example 4.4, the morphism in cohomology $\chi\colon H^2(L)\to H^2(M)$ is compatible with the natural transformation $\chi\colon \mathrm{Def}_L\to\mathrm{Def}_M$ and with obstruction maps. Therefore if $\omega\colon H^2(M)\to W$ annihilates the obstructions of Def_M, the composition $\omega\chi$ annihilates the obstructions of Def_L. The best situation is when Def_M is unobstructed (e.g. if M is abelian) and therefore χ itself annihilates the obstructions of Def_L.
This procedure is purely formal and it is not necessary for the functor Def_M to have any geometrical meaning. In the rest of this section we apply these ideas to deformations of Kähler manifolds.

Let X be a fixed complex manifold. We denote by $(A_X,d)=(\oplus_{p,q}A_X^{p,q},d=\partial+\overline{\partial})$ its De Rham complex, by $(\ker(\partial),\overline{\partial})$ the subcomplex of ∂-closed forms and by $\left(\dfrac{A_X}{\partial A_X},\overline{\partial}\right)$ the quotient complex of ∂-coexact forms. The contraction maps are denoted by

$$\mathcal{A}_X^{0,i}(\Theta_X)\times\mathcal{A}_X^{p,q}\xrightarrow{\lrcorner}\mathcal{A}_X^{p-1,q+i},$$

while the internal product is denoted by

$$\boldsymbol{i}\colon A_X^{0,*}(\Theta_X)\to\mathrm{Hom}^*(A_X,A_X),\quad \boldsymbol{i}_a(\omega)=a\lrcorner\omega,\qquad a\in A_X^{0,*}(\Theta_X),\quad \omega\in A_X.$$

Denoting by $[\ ,\ ]$ the standard bracket (Example 1.2) in the differential graded Lie algebra $\mathrm{Hom}^*(A_X,A_X)$, it is straightforward to check the validity of the *Cartan homotopy formulas* (see e.g. [29])

$$\boldsymbol{i}_{da}=[\overline{\partial},\boldsymbol{i}_a],\qquad \boldsymbol{i}_{[a,b]}=[\boldsymbol{i}_a,[\partial,\boldsymbol{i}_b]]=[[\boldsymbol{i}_a,\partial],\boldsymbol{i}_b].$$

Now we want to introduce a new differential graded Lie algebra which depends by the De Rham complex of X (the same construction can be made for every double complex of vector spaces). Define

$$\mathrm{Htp}\left(\ker(\partial),\frac{A_X}{\partial A_X}\right)=\oplus\,\mathrm{Htp}^i\left(\ker(\partial),\frac{A_X}{\partial A_X}\right),$$

where

$$\mathrm{Htp}^i\left(\ker(\partial), \frac{A_X}{\partial A_X}\right) = \mathrm{Hom}^{i-1}\left(\ker(\partial), \frac{A_X}{\partial A_X}\right),$$

the differential is

$$\mathrm{Htp}^i\left(\ker(\partial), \frac{A_X}{\partial A_X}\right) \ni f \mapsto \delta(f) = \overline{\partial} f + (-1)^i f\overline{\partial} \in \mathrm{Htp}^{i+1}\left(\ker(\partial), \frac{A_X}{\partial A_X}\right)$$

and the bracket is

$$\{f, g\} = f\partial g - (-1)^{\deg(f)\deg(g)} g\partial f.$$

Proposition 6.1. *The linear map*

$$\boldsymbol{i}\colon A_X^{0,*}(\Theta_X) \to \mathrm{Htp}\left(\ker(\partial), \frac{A_X}{\partial A_X}\right)$$

is a morphism of differential graded Lie algebras.

Proof. Immediate consequence of Cartan formulas. □

It is now possible to give an easy proof of the following result known as Kodaira's principle (for more complicated proofs see [6], [33] and [29]).

Theorem 6.2. *Let X be a compact Kähler manifold. Then the obstruction space of* Def_X *is contained in the kernel of the map*

$$\boldsymbol{i}\colon H^2(X, \Theta_X) \to \oplus_{p,q} \mathrm{Hom}(H^p(X, \Omega_X^q), H^{p+2}(X, \Omega_X^{q-1})).$$

Proof. For notational simplicity denote by L the DGLA $\mathrm{Htp}\left(\ker(\partial), \dfrac{A_X}{\partial A_X}\right)$. Since X is Kähler, according to $\partial\overline{\partial}$-lemma (see e.g. [7]) we have

$$H_{\overline{\partial}}^{q,p}(\ker(\partial)) = H_{\overline{\partial}}^{q,p}\left(\frac{A_X}{\partial A_X}\right) = H^p(X, \Omega_X^q)$$

and therefore

$$H^2(L) = \oplus_{p+q=r+s-1} \mathrm{Hom}(H^p(X, \Omega_X^q), H^r(X, \Omega_X^s)).$$

On the other hand, L is quasiisomorphic to an abelian differential graded Lie algebra: to see this is sufficient to consider the abelian subalgebra

$$K = \left\{ f \in L \mid f(\ker(\partial)) \subset \frac{\ker(\partial)}{\partial A_X},\ f(\partial A_X) = 0 \right\} \simeq \mathrm{Htp}\left(\frac{\ker(\partial)}{\partial A_X}, \frac{\ker(\partial)}{\partial A_X}\right)$$

and observe that the inclusion $K \subset L$ is a quasiisomorphism.
According to Corollary 1.8 the functor Def_L is isomorphic to Def_K and therefore it is smooth and its obstruction space O_{Def_L} is trivial. By Proposition 6.1 and Example 2.3, the morphism $\boldsymbol{i}$ induces a natural transformation of functors $\mathrm{Def}_X \to \mathrm{Def}_L$ and a compatible morphism of obstruction theories $O_{\mathrm{Def}_X} \to O_{\mathrm{Def}_L} = 0$. In other words $\boldsymbol{i}$ annihilates the obstruction space of Def_X. □

Corollary 6.3 (Bogomolov-Tian-Todorov [3, 41, 42])**.** *Let X be a compact Kähler manifold with trivial canonical bundle. Then X has unobstructed deformations, i.e.* Def_X *is a smooth functor.*

Proof. Let $n = \dim X$ and $\omega \in H^0(X, \Omega^n_X) \simeq \mathbb{C}$ be a holomorphic volume form. Then the contraction operator induces isomorphisms $H^i(X, \Theta_X) \simeq \operatorname{Hom}(H^0(X, \Omega^n_X), H^i(X, \Omega^{n-1}_X))$ and therefore the morphism

$$\boldsymbol{i}\colon H^2(X, \Theta_X) \to \oplus_{p,q} \operatorname{Hom}(H^p(X, \Omega^q_X), H^{p+2}(X, \Omega^{q-1}_X))$$

is injective. □

7. An approach to "semitrivialized" deformations

Sometimes a deformation of a geometric object is described by a set of deformations of specific parts of it, plus some compatibility condition. For example a deformation of a variety X can be described by deformations of the open subsets of an affine covering $\{U_i\}$, plus the condition that the deformations of U_i, U_j are isomorphic on $U_i \cap U_j$ and such isomorphisms must satisfy the cocycle condition on triple intersections.
We shall talk about semitrivialized deformations when we consider deformations of a geometric object together a trivialization of the deformation of a specific part of it.
The most important example concerns embedded deformations of a subvariety Z of a complex manifold X. Such deformations (over a base B) can be considered as deformations $\mathcal{Z} \subset \mathcal{X}$ of the inclusion map $Z \subset X$ together a trivialization $\mathcal{X} \simeq X \times B$.

Kereping in mind that *deformations=solution on Maurer-Cartan* and *trivializations=actions of the gauge group*, the guiding principle tell us that in characteristic 0 every semitrivialized deformation problem is governed by a morphism of differential graded Lie algebras $\chi\colon L \to M$, according to the following definition.

Definition 7.1. Let $\chi\colon L \to M$ be a morphism of differential graded Lie algebras. For every $A \in \mathbf{Art}$ denote

$$\mathrm{MC}_\chi(A) = \left\{(x, e^a) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A) \mid dx + \frac{1}{2}[x,x] = 0,\ e^a * \chi(x) = 0\right\},$$

$$\mathrm{Def}_\chi(A) = \frac{\mathrm{MC}_\chi(A)}{\exp(L^0 \otimes \mathfrak{m}_A) \times \exp(dM^{-1} \otimes \mathfrak{m}_A)},$$

where the gauge action is given by the formula

$$(e^l, e^{dm}) * (x, e^a) = (e^l * x, e^{dm}e^a e^{-\chi(l)}) = (e^l * x, e^{dm \bullet a \bullet (-\chi(l))}).$$

The above definition gives two functors of Artin rings

$$\mathrm{MC}_\chi, \mathrm{Def}_\chi\colon \mathbf{Art} \to \mathbf{Set}$$

that are deformation functors in the sense of Definition 3.4. A proof of this fact, that involves Baker-Campbell-Hausdorff formula, can be found in [31]. The same argument of Remark 1.4 shows that the projection $\mathrm{MC}_\chi \to \mathrm{Def}_\chi$ is smooth and therefore, according to Theorem 4.11 $\mathrm{MC}_\chi, \mathrm{Def}_\chi$ have the same obstruction theories.

The construction of Def_χ is also functorial in χ; more precisely, every commutative diagram of morphisms of differential graded Lie algebras

$$\begin{array}{ccc} L & \xrightarrow{f} & H \\ \Big\downarrow{\scriptstyle\chi} & & \Big\downarrow{\scriptstyle\eta} \\ M & \xrightarrow{f'} & I \end{array}$$

induces a natural transformation of functors $\mathrm{Def}_\chi \to \mathrm{Def}_\eta$.

In order to compute tangent space and obstruction maps of Def_χ we need to introduce the *suspension of the mapping cone* of χ; it is the differential graded vector space (C_χ, δ), where $C^i_\chi = L^i \oplus M^{i-1}$ and the differential δ is defined as

$$\delta(l, m) = (dl, \chi(l) - dm).$$

The tangent space of Def_χ is isomorphic to $H^1(C_\chi)$. In fact

$$\mathrm{MC}_\chi(\mathbb{K}[\epsilon]) = \left\{(x, e^a) \in (L^1 \otimes \mathbb{K}\epsilon) \times \exp(M^0 \otimes \mathbb{K}\epsilon) \mid dx = 0,\ e^a * \chi(x) = \chi(x) - da = 0\right\}$$

$$\simeq \left\{(x, a) \in L^1 \times M^0 \mid dx = 0,\ \chi(x) - da = 0\right\} = \ker(\delta\colon C^1_\chi \to C^2_\chi).$$

Two elements $(x, a), (y, b) \in \ker\delta$ are gauge equivalent if and only if there exists $(c, z) \in L^0 \times M^{-1}$ such that

$$y = x - dc, \qquad b = dz + a - \chi(c), \qquad \text{or equivalently} \quad (x, a) - (y, b) = \delta(c, z).$$

The obstruction space of Def_χ is naturally contained in $H^2(C_\chi)$. Since the two functors $\mathrm{MC}_\chi, \mathrm{Def}_\chi$ have the same obstruction theories it is sufficient to show that the functor MC_χ has a complete obstruction theory $(H^2(C_\chi), v_e)$. Let

$$\tau\colon \qquad 0 \longrightarrow E \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

be a small extension and $(x, e^q) \in \mathrm{MC}_\chi(B)$.
Since α is surjective there exists a pair $(y, e^p) \in L^1 \otimes \mathfrak{m}_A \times \exp(M^0 \otimes \mathfrak{m}_A)$ such that $\alpha(y) = x$ and $\alpha(p) = q$.
Setting

$$h = dy + \frac{1}{2}[y, y] \in L^2 \otimes E, \qquad r = e^p * \chi(y) \in M^1 \otimes E$$

we have $\delta(h, r) = 0$. In fact,

$$dh = \frac{1}{2}d[y, y] = [dy, y] = [h, y] - \frac{1}{2}[[y, y], y].$$

By Jacobi identity $[[y, y], y] = 0$, while $[h, y] = 0$ because $\mathfrak{m}_A$ annihilates E; therefore $dh = 0$.
Since $\chi(y) = e^{-p} * r = r + e^{-p} * 0$, we have

$$\chi(h) = d(r + e^{-p} * 0) + \frac{[r + e^{-p} * 0, r + e^{-p} * 0]}{2} =$$

$$= dr + d(e^{-p} * 0) + \frac{[e^{-p} * 0, e^{-p} * 0]}{2} = dr,$$

where the last equality follows from the fact that $e^{-p} * 0$ satisfies the Maurer-Cartan equation in $M \otimes \mathfrak{m}_A$.
We define $v_\tau(x, e^q) \in H^2(C_\chi) \otimes E$ as the cohomology class of (h, r). It is clear from

definition that such class is well defined and is exactly the obstruction of lifting (x, e^q) to $\mathrm{MC}_\chi(A)$.

It is an interesting exercise to show that the primary obstruction map is equal to

$$H^1(C_\chi) \to H^2(C_\chi), \qquad (x,a) \mapsto \frac{1}{2}([x,x],[a,\chi(x)]).$$

There exists an analog of Theorem 1.5.

Theorem 7.2. *Consider a commutative diagram of of morphisms of differential graded Lie algebras*

$$\begin{array}{ccc} L & \xrightarrow{f} & H \\ \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\eta} \\ M & \xrightarrow{f'} & I \end{array}$$

and assume that

1. $(f, f')\colon H^0(C_\chi) \to H^0(C_\eta)$ *is surjective.*
2. $(f, f')\colon H^1(C_\chi) \to H^1(C_\eta)$ *is bijective.*
3. $(f, f')\colon H^2(C_\chi) \to H^2(C_\eta)$ *is injective.*

Then the natural transformation $\mathrm{Def}_\chi \to \mathrm{Def}_\eta$ *is an isomorphism.*

Proof. For a proof that use extended deformation functors put together [31, Thm. 7.4] and [30, Thm. 5.71]. Alternatively use [31, Thm. 2.1] to prove next Corollary 7.3 and then apply Theorem 1.5.
Another proof involving L_∞-algebras follows from the results of [11]. □

The functors Def_χ respect the general Principle 1.9; more precisely:

Corollary 7.3. *Let* $\chi\colon L \to M$ *be a morphism of differential graded Lie algebras and consider the DGLA (see Example 1.7)*

$$H = \{(l,m) \in L \times M[t,dt] \mid e_0(m) = 0,\ e_1(m) = \chi(l)\}$$

Then there exists an isomorphism $\mathrm{Def}_\chi \simeq \mathrm{Def}_H$.

Proof. Denote

$$K = \{(l,m) \in L \times M[t,dt] \mid e_1(m) = \chi(l)\}$$

and apply Theorem 7.2 or [31, Thm. 2.1] to the commutative diagram of morphisms of differential graded Lie algebras

$$\begin{array}{ccccc} L & \xrightarrow{f} & K & \hookleftarrow & H \\ \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle e_0} & & \downarrow \\ M & \xrightarrow{Id} & M & \leftarrow & 0 \end{array} \qquad f(l) = (l, \chi(l)).$$

□

Remark 7.4. In most concrete cases the interpretation of a deformations functor as Def_χ is more geometrical than the interpretation as Def_H and is more useful in computations. In other words, Corollary 7.3 is important philosophically but at the moment do not seems very useful in concrete examples; we refer to [11] for a deeper discussion of this.

8. Semiregularity annihilates obstructions

One of the most important applications of the formalism introduced in Section 7 is the description of embedded deformations of a complex submanifold $Z\subset X$ in the form Def_χ.

As in Example 2.4, we work over the field $\mathbb{K}=\mathbb{C}$ and we denote by $L_{Z|X}$ the kernel of the restriction map

$$\pi\colon A^{0,*}_X(\Theta_X)\to A^{0,*}_Z(N_{Z|X}).$$

The natural inclusion $\chi\colon L_{Z|X}\to A^{0,*}_X(\Theta_X)$ is a morphism of differential graded Lie algebras and its cokernel is isomorphic to the Dolbeault complex of $N_{Z|X}$; in particular for every $i\geq 0$

$$H^i(Z,N_{Z|X})\simeq H^i(A^{0,*}_Z(N_{Z|X}))\simeq H^{i+1}(C_\chi).$$

The DGLA $A^{0,*}_X(\Theta_X)$ can be interpreted, via Lie derivation, as a subalgebra of the DGLA of derivations of the graded sheaf $\mathcal{A}^{0,*}_X$. In particular for every $A\in\mathbf{Art}$ and every $a\in A^{0,*}_X(\Theta_X)\otimes\mathfrak{m}_A$, its exponential e^a is an automorphism of the graded sheaf of A-modules $\mathcal{A}^{0,0}_X\otimes A$.

Consider now the associated functor Def_χ. Since χ is injective we have, for every $A\in\mathbf{Art}$

$$\mathrm{MC}_\chi(A)=\{e^\eta\in\mathrm{Aut}_A(\mathcal{A}^{0,0}_X\otimes A)\mid \eta\in A^{0,0}_X(T_X)\otimes\mathfrak{m}_A,\ e^{-\eta}*0\in L^1_{Z|X}\otimes\mathfrak{m}_A\}.$$

Under this identification the gauge action becomes

$$\exp(L^0_{Z|X}\otimes\mathfrak{m}_A)\times\mathrm{MC}_\chi(A)\to\mathrm{MC}_\chi(A),\qquad (e^\mu,e^\eta)\mapsto e^\eta\circ e^{-\mu},$$

and then

$$\mathrm{Def}_\chi(A)=\frac{\mathrm{MC}_\chi(A)}{\exp(L^0_{Z|X}\otimes\mathfrak{m}_A)}.$$

Denote by $I\subset\mathcal{A}^{0,0}_X$ the ideal sheaf of differentiable functions vanishing on Z and by $\mathcal{I}=\mathcal{O}_X\cap I$ the holomorphic ideal sheaf of Z. Define then

$$\theta\colon\ \mathrm{Def}_\chi(A)\to\{\text{ideal sheaves of }\mathcal{O}_X\otimes A\},\qquad \theta(e^\eta)=(\mathcal{O}_X\otimes A)\cap e^\eta(I\otimes A).$$

Theorem 8.1. *The above map θ is well defined and gives an isomorphism of functors $\theta\colon\ \mathrm{Def}_\chi\xrightarrow{\simeq}\mathrm{Hilb}^Z_X$.*

Proof. See [31]. $\square$

Remark 8.2. The Theorem 2.5 is an immediate consequence of Theorem 8.1 and Corollary 7.3.

The computation of Section 6, applied to this situation gives a commutative diagram of morphisms of DGLA

$$\begin{array}{ccc}
L_{Z|X} & \xrightarrow{\ i\ } & \left\{f\in\mathrm{Htp}\left(\ker(\partial),\dfrac{A_X}{\partial A_X}\right)\mid f(I\cap\ker(\partial))\subset\dfrac{I}{I\cap\partial A_X}\right\}\\
\Big\downarrow\chi & & \Big\downarrow\eta\\
A^{0,*}_X(T_X) & \xrightarrow{\ i\ } & \mathrm{Htp}\left(\ker(\partial),\dfrac{A_X}{\partial A_X}\right)
\end{array}$$

inducing a map of complexes

$$A^{0,*}_Z(N_{Z|X}) = \mathrm{Coker}(\chi) \xrightarrow{\boldsymbol{i}} \mathrm{Coker}(\eta),$$

and therefore a morphism in cohomology. The analog of Theorem 6.2 becomes:

Theorem 8.3. *If X is compact Kähler, then the obstructions of* Hilb^Z_X *are contained in the kernel of*

$$H^1(N_{Z|X}) \xrightarrow{\boldsymbol{i}} H^1(\mathrm{Coker}(\eta)) = \bigoplus_i \mathrm{Hom}\left(H^i(I\cap \ker(\partial)), H^i\left(\frac{A_Z}{\partial A_Z}\right)\right).$$

Proof. (Sketch, for more details see [31]) Since $\mathrm{Def}_\chi = \mathrm{Hilb}^Z_X$ and $H^1(\mathrm{Coker}(\eta)) = H^2(C_\eta)$, it is sufficient to prove that the functor Def_η is smooth.
By $\partial\overline{\partial}$-lemma we have that $0\longrightarrow I\cap\partial A_X\longrightarrow \partial A_X\longrightarrow \partial A_Z\longrightarrow 0$ is an exact sequence of acyclic complexes. Denoting

$$K = \left\{ f \in \mathrm{Htp}\left(\ker(\partial), \frac{A_X}{\partial A_X}\right) \mid f(I\cap\ker(\partial)) \subset \frac{I}{I\cap \partial A_X}\right\},$$

the projection $\ker(\partial)\to\ker(\partial)/\partial A_X$ induces a commutative diagram

$$\begin{array}{ccc} \{f\in K\mid f(\partial A_X)=0\} & \xrightarrow{\alpha} & K \\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\eta} \\ \mathrm{Htp}\left(\dfrac{\ker(\partial)}{\partial A_X}, \dfrac{A_X}{\partial A_X}\right) & \xrightarrow{\beta} & \mathrm{Htp}\left(\ker(\partial), \dfrac{A_X}{\partial A_X}\right) \end{array}$$

Since ∂A_X is acyclic, β is a quasiisomorphism of DGLA. Moreover, there exists an exact sequence

$$0\to \mathrm{Htp}\left(\frac{\partial A_X}{I\cap\partial A_X}, \frac{A_X}{\partial A_X}\right) \to \mathrm{Coker}(\alpha) \to \mathrm{Htp}\left(I\cap\partial A_X, \frac{I}{I\cap\partial A_X}\right)\to 0.$$

Since the complexes $\dfrac{\partial A_X}{I\cap\partial A_X} = \partial A_Z$ and $I_Z\cap\partial A_X$ are both acyclic, also $\mathrm{Coker}(\alpha)$ is acyclic and then α is a quasiisomorphism. According to Theorem 7.2 there exists an isomorphism of functors $\mathrm{Def}_\eta = \mathrm{Def}_\mu$.
On the other side, both algebras on the first column are abelian and then the functor Def_μ is smooth. □

Always assuming X Kähler, the semiregularity map, introduced by Kodaira and Spencer [23] for divisors and generalized by S. Bloch [2] to subvarieties, can be defined in the following way: let n be the dimension of X and denote by $\mathcal{H}$ the space of harmonic forms on X of type $(n-p+1, n-p-1)$. By Dolbeault theorem and Serre duality, the dual of $\mathcal{H}$ is isomorphic to $H^{p+1}(X,\Omega^{p-1}_X)$.
The composition of the contraction map and integration on Z gives a bilinear map

$$H^1(Z, N_{Z|X})\times\mathcal{H}\to\mathbb{C}, \qquad (\eta,\omega)\mapsto \int_Z \eta\lrcorner\omega$$

which induces the linear morphism

$$\pi\colon H^1(Z,N_{Z|X})\to\mathcal{H}^\vee = H^{p+1}(X,\Omega^{p-1}_X)$$

called *semiregularity map*.
Since $\mathcal{H}\subset I\cap\ker(\partial)\cap\ker(\overline{\partial})$, the following corollary follows immediately from Theorem 8.3.

Corollary 8.4. *Let Z be a smooth closed submanifold of codimension p of a compact Kähler manifold X. Then the obstruction space of* Hilb_X^Z *is contained in the kernel of the semiregularity map*

$$\pi\colon H^1(Z, N_{Z|X}) \to H^{p+1}(X, \Omega_X^{p-1}).$$

Remark 8.5. Corollary 8.4 was almost proved by S. Bloch in the paper [2]. More precisely he proved that if the semiregularity map is injective then Hilb_X^Z is smooth; although not explicitly stated in [2], the same proof shows that the semiregularity map annihilates semitrivial obstructions.
The annihilation of semitrivial obstructions by semiregularity map has been recently generalized to deformations of coherent modules by Buchweitz and Flenner in [4].

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