Differential graded Lie algebras and Deformation theory

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1 Differential graded Lie algebras

Let L be a lie algebra over a fixed field $\mathbb{K}$ of characteristic 0.

Definition 1. A $\mathbb{K}$-linear map $d : L \to L$ is called a *derivation* if it satisfies the Leibniz rule $d[a,b] = [da,b] + [a,db]$.

Lemma. *If d is nilpotent (i.e. $d^n = 0$ for n sufficiently large) then $e^d = \sum_{n \geq 0} \frac{d^n}{n!} : L \to L$ is an isomorphism of Lie algebras.*

Proof. Exercise. □

Definition 2. L is called *nilpotent* if the *descending central series* $L \supset [L,L] \supset [L,[L,L]] \supset \cdots$ stabilizes at 0. (I.e., if we write $[L]^2 = [L,L]$, $[L]^3 = [L,[L,L]]$, etc., then $[L]^n = 0$ for n sufficiently large.) Note that in the finite-dimensional case, this is equivalent to the other common definition that for every $x \in L$, $\mathrm{ad}x$ is nilpotent.

We have the Baker-Campbell-Hausdorff formula (BCH).

Theorem. *For every nilpotent Lie algebra L there is an associative product $\bullet : L \times L \to L$ satisfying*

1. *functoriality in L; i.e. if $f : L \to M$ is a morphism of nilpotent Lie algebras then $f(a \bullet b) = f(a) \bullet f(b)$.*

2. *If $I \subset R$ is a nilpotent ideal of the associative unitary $\mathbb{K}$-algebra R and for $a \in I$ we define $e^a = \sum_{n \geq 0} \frac{a^n}{n!} \in R$, then $e^{a \bullet b} = e^a \bullet e^b$.*

Heuristically, we can write $a \bullet b = \log(e^a \bullet e^b)$.

Proof. Exercise (use the computation of www.mat.uniroma1.it/people/manetti/dispense/BCHfjords.pdf and the existence of free Lie algebras).

□

This bullet is defined by the BCH formula, which is universal: $a \bullet b = a + b + \frac{1}{2}[a,b] + \ldots$.

For any nilpotent Lie algebra L, we define the group $\exp(L) = \{e^a | a \in L\}$ with $e^a \cdot e^b = e^{a \bullet b}$. Of course we write $1 = e^0$, and then we have $e^a \cdot 1 = e^a = 1 \cdot e^a$ and $e^a e^{-a} = 1$. Associativity comes from the associativity of $\bullet$.

Definition 3. Define DG to be the category of differential graded vector spaces over $\mathbb{K}$. Objects $V \in DG$ look like $V = \oplus_{n \in \mathbb{Z}} V^n$ with differential $d : V^i \to V^{i+1}$.

This category is equipped with a tensor product $V \otimes W$ given by $(V \otimes W)_n = \sum_i V^i \otimes W^{n-i}$, $d(v \otimes w) = dv \otimes w + (-1)^{\deg(v)} v \otimes dw$.

The term $(-1)^{\deg(v)} = (-1)^{\deg(v)\deg(d)}$ (since $\deg(d) = 1$) comes from what is known as the *Koszul rule*: When two goats a and b pass through each other, we pick up a factor of $(-1)^{\deg a \deg b}$.

This category is also equipped with an *internal hom*. Let $V, W \in DG$. For any $n \in \mathbb{Z}$, we define

$$\mathrm{Hom}^n(V,W) = \{f : V \to W \mid f \ \mathbb{K}\text{-linear}, \ f(V^i) \subset W^{i+n} \ \forall i\}.$$

This is a $\mathbb{K}$-vector space. Then we set $\mathrm{Hom}^*(V,W) = \oplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(V,W)$. For any $f \in \mathrm{Hom}^n(V,W)$, we set $(df)(v) = d(f(v)) - (-1)^{\overline{f}} f(dv)$, where $\overline{f} = \deg(f)$. This is an element of DG.

Example. Let $f \in \mathrm{Hom}^n(V,W)$, $g \in \mathrm{Hom}^m(H,K)$. Then we get $f \otimes g \in \mathrm{Hom}^{n+m}(V \otimes H, W \otimes K)$ by $f \otimes g(v \otimes h) = f(v) \otimes g(h) \cdot (-1)^{\overline{g} \cdot \overline{v}}$.

We now come to the main definition of the talk.

Definition 4. A *differential graded Lie algebra* (DGLA) is a DG vector space (L, d) with a bracket $[\cdot, \cdot] : L^i \times L^j \to L^{i+j}$ such that:

1. $[a, b] = -(-1)^{\bar{a}\bar{b}}[b, a]$;

2. graded Leibniz rule: $d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$

3. graded Jacobi rule: $[[a, b], c] = [a, [b, c]] - (-1)^{\bar{a}\bar{b}}[b, [a, c]]$.

Exercise 1. Let $x \in L^i$. Then $[x, [x, x]] = 0$.

Definition 5. A DGLA is called *nilpotent* if its central series $L \supset [L, L] \supset [L, [L, L]] \supset \cdots$ stabilizes to 0.

There are a number of important examples of DGLAs. For more details on this part see also DefHolmaps.pdf and ArXiv:math/0507284

Example. Let $V \in DG$. Then $\mathrm{Hom}^*(V, V)$ is a DGLA with bracket $[f, g] = f \circ g - (-1)^{\bar{f}\bar{g}}g \circ f$.

Example. Let X be a complex manifold and let $T_X \to X$ be its holomorphic tangent bundle. Write $A^{0,k}(T_X)$ for the differential forms of type $(0, k)$ with values in T_X, which are locally given by $\varphi = \sum \varphi_I \frac{\partial}{\partial z_i} d\bar{z}_{i_1} \wedge \ldots \wedge d\bar{z}_{i_k}$. We have the Dolbeault differential $\bar{\partial} : A^{0,k}(T_X) \to A^{0,k+1}(T_X)$, and we obtain the Dolbeault complex $KS_X = \oplus_{k \geq 0} A^{0,k}(T_X)$. This has a DGLA structure. If $\varphi, \psi \in A^{0,0}(T_X)$, we set $[\varphi, \psi]$ to be the usual bracket of vector fields. We extend to KS_x by assuming that the bracket is antiholomorphic bilinear.

Example. Let X be a differentiable manifold with tangent sheaf $T_X \to X$. We have the usual bracket of vector fields, which can be uniquely extended to a bracket $[\cdot, \cdot] : \Lambda^i T_X \times \Lambda^j T_X \to \Lambda^{i+j-1} T_X$ such that

1. $[\cdot, \cdot] : \Lambda^1 T_X \times \Lambda^0 T_X \to \Lambda^0 T_X$ is given by $[\psi, f] = \psi(f)$

2. for $\eta \in \Lambda^i T_X$, $\psi \in \Lambda^j T_X$, $\phi \in \Lambda^k T_X$, $[\eta, \psi \wedge \phi] = [\eta, \psi] \wedge \phi + (-1)^{(i-1)j}\psi \wedge [\eta, \phi]$

3. $\Lambda^* T_X$ is a sheaf of DGLAs with $d = 0$.

We call this bracket the *Schouten bracket.* (if fact it is a sheaf of Gerstenhaber algebras, see e.g. arxiv:math/0507286v1)

Example (Hochschild DGLA)**.** Consider an associative algebra A with multiplication $m : A \otimes A \to A$. Write $g^i = \mathrm{Hom}_{\mathbb{K}}(A^{\otimes i+1}, A)$ for every $i \geq -1$. Then, define $* : g^i \times g^j \to g^{i+j}$ by

$$\phi * \psi : (a_0, \ldots, a_{i+j}) = \sum_s (-1)^{sj} \phi(a_0, \ldots, a_{s-1}, \psi(a_s, \ldots, a_{s+j}), \ldots, a_{i+j}).$$

We can now obtain a DGLA structure on $g = \oplus_i g^i$ by setting $[\phi, \psi] = \phi * \psi - (-1)^{\overline{\phi\psi}}\psi * \phi$ and $d\phi = [m, \phi]$.

Now let L be a nilpotent DGLA, and suppose $a \in L^0$. Then $[a, -] : L \to L$ is a derivation. By the same proof as in the classical case, $e^{[a,-]} : L \to L$ is an isomorphism of GLAs. (The differential is not preserved.)

As a consequence, if we take $Z = \{x \in L^1 \mid [x, x] = 0\}$, then $e^{[a,-]}(Z) \subset Z$ (since $[e^{[a,-]}x, e^{[a,-]}x] = e^{[a,-]}[x, x] = 0$). We thus obtain that Z is stable under the *adjoint action* $\exp(L^0) \to GL(L^1)$.

2 Deformation theory

The beginning of the story of (this aspect of) deformation theory is with Deligne, Kontsevich, et al., who realized that every deformation problem (over fields of characteristic 0) is governed by a DGLA via Maurer-Cartan equation modulo gauge action.

The simplest example[1] for us will be the deformations of a differential. Consider a finite cochain complex $V = 0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$ over $\mathbb{K}$ with differential d such that $d^2 = 0$. What is a deformation of d? If A is a commutative local artinian $\mathbb{K}$-algebra with a map $A \to \mathbb{K} = A/\mathfrak{m}_A$, we can define a new complex $V \otimes_{\mathbb{K}} A = 0 \to V^0 \otimes_{\mathbb{K}} A \to \cdots \to V^n \otimes_{\mathbb{K}} A \to 0$ with differential $\tilde{d}$, where we require that $\tilde{d}|_V = d$. We want to write $A = \mathbb{K} \oplus \mathfrak{m}_A$ with $\tilde{d} = d + \xi$ for some $\xi \in \mathrm{Hom}^1(V, V) \otimes \mathfrak{m}_A$. (Since A is artinian, $\mathfrak{m}_A$ is nilpotent.) The integration equation is that $\tilde{d}^2 = 0$, which expands to $0 = d\xi + \xi d + \xi \circ \xi$, an equality in the associative graded algebra $\mathrm{Hom}^*(V, V) \otimes \mathfrak{m}_A$. With the obvious Lie algebra structure, we can write this as $0 = [d, \xi] + \frac{1}{2}[\xi, \xi]$. Writing δ for the differential in $\mathrm{Hom}^*(V, V)$, this is $0 = \delta\xi + \frac{1}{2}[\xi, \xi]$. This is called the *Maurer-Cartan equation*.

Definition 6. Given a DGLA $L = \oplus L^i$, the *Maurer-Cartan elements* of L are the set $MC(L) = \{\xi \in L^1 | d\xi + \frac{1}{2}[\xi, \xi] = 0\} \subset L^1$.

But when do two such elements give the same deformation? Given two deformations of d and a collection of maps $\phi_n : V^n \otimes A \to V^n \otimes A$ entertwining the complexes $V^* \otimes A$ with differentials $d + \xi$ and $d + \eta$, we say that $\xi \sim \eta$ if $\phi : V \otimes A \to V \otimes A$ is an isomorphism of graded A-modules which restricts to the identity on $V \subset V \otimes A$.

We can then write $\phi = \mathrm{Id} + \tilde{\phi}$ for $\tilde{\phi} : V \to V \otimes \mathfrak{m}_A$, and (since we are in characteristic 0) we can write $\phi = e^a$ for some $a \in \mathrm{Hom}^0(V, V) \otimes \mathfrak{m}_A$. Then $\xi \sim \eta$ iff $e^a(d + \eta)e^{-a} = d + \xi$. The left side can be written as $e^{[a,-]}(d + \eta)$, which can be expanded further as

$$d + \eta + \frac{e^{[a,-]} - \mathrm{Id}}{[a, -]}([a, d + \eta]).$$

At the end, we get that $\xi \sim \eta$ iff there is some $a \in \mathrm{Hom}^0(V, V) \otimes \mathfrak{m}_A$ such that

$$\xi = e^a * \eta := \eta + \frac{e^{[a,-]} - \mathrm{Id}}{[a, -]}([a, \eta] - \delta a).$$

This is called the *gauge action*.

Going back to an arbitrary DGLA, we get the following: if L is nilpotent and $\xi, \eta \in MC(L)$ then $\xi \sim \eta$ iff $\xi = e^a * \eta$ for some $a \in L^0$. (Here, $*$ is the action of $\exp(L^0) = \{e^a | a \in L^0\}$.)

Definition 7. Let **Art** be the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$. Given $A \in$ **Art** and a DGLA L, we define two functors $MC_L :$ **Art** $\to$ **Set** and $Def_L :$ **Art** $\to$ **Set** by $MC_L(A) = MC(L \otimes \mathfrak{m}_A)$ and $Def_L(A) = MC_L(A)/$gauge action of $\exp(L^0 \otimes \mathfrak{m}_A)$. These give deformations and isomorphism classes of deformations.

Exercise 2. There is a natural isomorphism $Def_L(\mathbb{K}[\varepsilon]/(\varepsilon^2)) \cong H^1(L)$.

These functors have several very nice properties.

Theorem (main theorem).   *1. If $f : L \to M$ is a morphism of DGLAs, this gives natural transformations $f : MC_L \to MC_M$ and $f : Def_L \to Def_M$. Thus we can consider this as a bifunctor $Def :$ **DGLA** $\times$ **Art** $\to$ **Set**.*

2. *Assume that $f : L \to M$ is a morphism of DGLAs. If $f : H^1(L) \to H^1(M)$ is surjective and $f : H^2(L) \to H^2(M)$ is injective, then $f : Def_L \to Def_M$ is surjective. That is, any deformation parametrized by M can be lifted to a deformation parametrized by L.*

3. *If $f : H^0(L) \to H^0(M)$ is surjective, $f : H^1(L) \to H^1(M)$ is an isomorphism, and $f : H^2(L) \to H^2(M)$ is injective, then $f : Def_L \to Def_M$ is an isomorphism.*

Example. Assume that L is quasi-isomorphic to an abelian DGLA (i.e. one with $[\cdot, \cdot] = 0$). Then Def_L is a *smooth* functor: if $A \to B$ in **Art** is surjective then $Def_L(A) \to Def_L(B)$ is surjective.

This is great, because given some DGLA L, if we can find a quasi-isomorphic DGLA M with nicer properties (even if it has no geometric meaning), we can use Def_M in place of Def_L.

Example. Suppose X is a complex manifold. The *Kodaira-Spencer complex* $KS = \oplus A^{0,i}(T_X)$ is a DGLA, and Def_{KS} exactly gives deformation of X. If X is Calabi-Yau, then KS ends up being quasi-isomorphic to an abelian DGLA, and hence X has unobstructed deformations.

3   Generalizations

Definition 8. If L is a DGLA, then $H^*(L)$ is a graded Lie algebra, which we can consider a DGLA with $d = 0$. Then, L is called *formal* if it is quasi-isomorphic to its cohomology (as a DGLA).

In other words, if L is formal then we have a "zigzag" of hats in the category **DGLA** beginning at L and ending at $H^*(L)$. (This ends up implying that we can find a single map between L and $H^*(L)$ which is a quasi-isomorphism.) Then, formality implies that $\xi \in Def_L(\mathbb{K}[t]/(t^2))$ can be lifted to $Def_L(\mathbb{K}[[t]])$ iff ξ can be lifted to $Def_L(\mathbb{K}[t]/(t^3))$.

One can embed this theory into the world of L_∞-algebras. Here there is no notion of gauge action, but there is still the Maurer-Cartan equation.

Theorem. *The bifunctor $MC : \mathbf{DGLA} \times \mathbf{Art} \to \mathbf{Set}$, $(L, A) \mapsto MC_L(A)$ determines Def_L for every L.*

Definition 9. Let $x, y \in MC_L(A)$. We say that x is *homotopy equivalent* to y (and write $x \sim y$ if there is some $\xi \in MC_{L[t,dt]}(A)$ such that $e_0(\xi) = x$ and $e_1(\xi) = y$.

Here, $L[t, dt] = L \otimes \mathbb{K}[t, dt]$; the second tensor factor is essentially A_{PL} of the disk. This has two natural evaluation maps $e_0, e_1 : L[t, dt] \to L$.

Note that this is entirely in terms of the Maurer-Cartan equation, so it can be generalized greatly.

Theorem. *Homotopy equivalence is indeed an equivalence relation (even in the L_∞-algebra setting). Moreover If L is a DGLA, then homotopy equivalence is exactly gauge equivalence.*

Proof of second statement. Let L be be DGLA, $A \in \mathbf{Art}$. Suppose $x, y \in MC(L \otimes \mathfrak{m}_A)$. Recall that $x \sim_g y$ if there is some $a \in L^0 \otimes \mathfrak{m}_A$ such that $e^a * x = y$, whereas $x \sim_h y$ if there is some $z(t, dt) \in MC(L[t, dt] \otimes \mathfrak{m}_A)$ such that $z(0, 0) = x$ and $z(1, 0) = y$. So first, assume $x \sim_g y$ via $a \in L^0$. Then we can consider $x \in MC(L \otimes \mathfrak{m}_A) \subset MC(L[t, dt] \otimes \mathfrak{m}_A)$. But in fact, the inclusion $i : L \to L[t, dt]$ is a quasi-isomorphism. Since $\exp(L^0[t])$ acts by gauge on $MC(L[t, dt] \otimes \mathfrak{m}_A)$, then for every t we can set $z(t, dt) = e^{ta} * x$. Then $z(0) = x$ and $z(1) = y$. On the other hand, suppose that $x \sim_h y$. Note that $MC_L = MC_{L^{\geq 1}}$ (in fact Maurer-Cartan only depends on L^1 and L^2), so $Def_L = Def_{L^{\geq 0}}$. Hence $MC_{L[t,dt]} = MC_{L^{\geq 0}[t,dt]}$. So it is not restrictive to assume that $L = \oplus_{n \geq 0} L^n$. In this case, $L[t, dt]^0 = L^0[t]$. Now let $z(t) \in MC(L[t, dt] \otimes \mathfrak{m}_A)$. Then as we have an isomorphism $i : Def_L(A) \to Def_{L[t,dt]}(A)$, we must have some $x \in MC_L(A)$ such that $x \sim_g z(t)$ in $L[t, dt] \otimes \mathfrak{m}_A$. So we have $a(t) \in L[t, dt]^0 \otimes \mathfrak{m}_A = L^0[t] \otimes \mathfrak{m}_A$ such that $e^{a(t)} * x = z(t)$. Now $z(0) = e^{a(0)} * x$ and $z(1) = e^{a(1)} * x$, and these imply that $z(0) \sim_g z(1)$. $\square$

One can prove that every $z(t) \in MC_{L[t,dt]}(A)$ can be written uniquely as $e^{a(t)} * x$ with $x \in MC_L(A)$ and $a(0) = 0$. The proof is similar to the proof of the main theorem (that a quasi-isomorphism of DGLAs induces an isomorphism of deformation functors).

Proof of main theorem. [2] Let L be a DGLA, and consider $MC_L : \mathbf{Art} \to \mathbf{Set}$. Then there exists an extension $MC_L : DG\text{-}\mathbf{Art} \to \mathbf{Set}$. (An object $C \in DG\text{-}\mathbf{Art}$ is a nilpotent DG-commutative $\mathbb{K}$-algebra with $\dim_{\mathbb{K}} C < \infty$. The natural map $\mathbf{Art} \to DG\text{-}\mathbf{Art}$ is given by considering the maximal ideal. Then as before we have $MC_L(\mathfrak{m}_A) = MC(L \otimes_{\mathbb{K}} \mathfrak{m}_A)$ and we obtain $Def_L : DG\text{-}\mathbf{Art} \to \mathbf{Set}$ as $Def_L = MC_L/\text{gauge}$. Now, $Def_L : DG\text{-}\mathbf{Art} \to \mathbf{Set}$ depends uniquely on $MC_L : DG\text{-}\mathbf{Art} \to \mathbf{Set}$, and $MC_{L[t,dt]}(A) = \lim_{n \to} MC_L(\mathfrak{m}_A[t, dt]/(t^n, dt^n))$. We have the following axiomatic properties of $F = Def_L$:

1. $F(0) = *$

2. If $\alpha : A \to C$ and $\beta : B \to C$ in $DG\text{-}\mathbf{Art}$, then we have $\eta : F(A \times_C B) \to F(A) \times_{F(C)} F(B)$. If α is surjective then η is surjective. If $C^2 = 0$ and $H^*(C) = 0$ then η is bijective.

3. If $\alpha : A \to B$ is surjective $A \ker \alpha = 0$ and $H^*(\ker \alpha) = 0$, then $F(A) \to F(B)$ is surjective.

And so suppose we have $F : DG\text{-}\mathbf{Art} \to \mathbf{Set}$ satisfying these axioms. Then $F(\mathbb{K}[-i]) = T_F^i$ (where $\mathbb{K}[-i]$ is a shift of the complex $\mathbb{K}$ to be supported in degree i) is a vector space. So $T^i Def_L = H^i(L)$. $\square$

We have the following generalization.

Theorem. *If $\eta : F \to G$ is a morphism of functors satisfying the above axioms, then η is an isomorphism iff $\eta : T^i F \to T^i G$ are isomorphisms. (These objects T_F^i should be thought of as something like the homotopy groups of the functor F, more precisely $T^i F = \pi_{1-i}(F)$.)*

Exercise 3. if L is a DGLA such that $H^2(L) = 0$ then MC_L is smooth; that is, for all surjections $\alpha : A \to B$ in $\mathbf{Art}$, $MC_L(A) \to MC_L(B)$ is a surjection.

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[2]There are many different proofs: here we follow M. Manetti: Extended deformation functors. Int. Math. Res. Not. 14 (2002) 719-756; ArXiv:math.AG/9910071.