Preface

The notion of stacks grew out of attempts to construct moduli spaces in contexts where ordinary varieties or schemes were deficient. They may be regarded as the next step in the development from affine and projective varieties to abstract varieties to schemes to algebraic spaces. However, even when moduli spaces have been constructed as projective varieties, whenever some objects being parametrized have nontrivial automorphisms the usual (coarse) moduli space is not a faithful reflection of the moduli problem; in such cases, the corresponding stack is more useful.

It is not easy to point to one moment when stacks appeared. Many of the ideas that led to stacks can be found in work of Grothendieck, especially [**38**]. These ideas were developed by Giraud [**31**], and in the work of Artin [**4**] and Knutson [**56**] on algebraic spaces, which were an earlier extension of the notion of schemes. More ideas toward stacks can be found in Mumford's article [**71**], which remains an excellent source of inspiration.

Deligne and Mumford defined a notion of algebraic stack in [**20**] — the concept now known as *Deligne–Mumford stack* — and used moduli stacks of curves to give an alternative and conceptually simpler proof of the main theorem of that paper: the moduli space of curves of genus g over any field is irreducible. Concise definitions were given in [**20**], but the basic theorems were only stated, with details promised for another publication that unfortunately has never appeared. Since then the language of stacks has been found to be useful in a variety of moduli and other problems. Except for a few fragments, such as the appendix of [**89**] and section I.4 of [**25**], there had been little foundational exposition of these ideas until the book [**61**] was published.

Artin [**5**] defined a more general notion of stack, which can be used for moduli problems of objects with infinite automorphism groups. These have become increasingly important in recent years, such as for moduli of vector bundles ([**61**] and Chapter 19) and the construction of intrinsic normal cones to singular varieties ([**10**] and Chapter 17). The book of Laumont and Moret-Bailly [**61**] provides a systematic treatment of stacks, with proofs of many of the main theorems in the subject. However, it presumes quite a sophisticated background, and proceeds at a high level of difficulty.

The aim of these notes is to give a leisurely and elementary introduction to stacks and some of their uses. Although the language is necessarily abstract, by proceeding from examples to theory, and keeping the examples as simple as possible, we hope to make the notes accessible to those uncomfortable with fancy terminology. One of our goals, in fact, is to prepare a reader for [**61**].

We do, however, assume the reader is familiar with basic notions about schemes, including definitions and basic properties of smooth, étale, and flat morphisms. Many of these notions are defined briefly in a glossary at the back of this book. Good references for most of this material are [**74**] and [**47**], perhaps beginning with [**24**] or [**64**] — with [**EGA**] as the definitive source.

In Part I we make several simplifying assumptions, which suffice to describe Deligne–Mumford stacks and algebraic spaces. Many interesting examples, such as moduli spaces of curves, can be worked out in this language. We give a more direct translation between the categorical and the atlas notions of stack than had been available in the literature before. We include descriptions of stacks of dimensions 0 and 1. We compute the Picard group of the stack of elliptic curves, over any field and over the integers (the latter a new result). We do not try to prove all general theorems about Deligne–Mumford stacks, but we do include enough so that we can carry out proofs of the assertions made in [**20**]. Algebraic spaces are defined and studied as special cases of Deligne–Mumford stacks; the reader need not know about them in advance.

In Part II we begin again, without the special assumptions, to define general algebraic (Artin) stacks. In particular, we collect here in one place the various properties that are scattered throughout Part I. In Part II we proceed in a more concise manner, hoping that the reader who has spent time in Part I will be prepared for this. Here we discuss some important examples of algebraic stacks, such as moduli stacks of vector bundles, and cone stacks. We conclude with chapters on sheaves and cohomology.

In the literature one finds many assertions that something is a stack. What is rare is to find justification for such an assertion, such as a verification that a proposed stack satisfies all or even some of the axioms to be a stack. (The most common reason given for why something is a stack is to point out why it is *not* a scheme!) We hope that giving a few such proofs, together with examples, will improve this situation.

At the foundations of the subject there are quite a few categorical constructions, and verifications involve checking that many diagrams commute. Although we have left many of these verifications as exercises, we do include a section of Answers at the end, where the reader can find many of these worked out.

We are all familiar with the idea that a geometric object can have an intrinsic definition as a ringed space, and that it can also be constructed from an atlas by gluing together simple local models. Stacks also have a dual nature. Their intrinsic nature, however, is not a space with some structure; rather, it is a *category*, together with a functor from this category to a base category (usually of schemes or other familiar geometric objects). Stacks can also be realized from a kind of atlas, called a *groupoid*, that consists of a pair of objects together with five morphisms in the base category. (In the familiar case of a manifold obtained by gluing open sets $\{U_\alpha\}$, the two objects are $\coprod U_\alpha$ and $\coprod U_\alpha \cap U_\beta$, and the morphisms express the usual compatibilities among the gluing data.) For stacks, however, the distance between their categorical nature and an atlas is considerably greater than that between a ringed space and an atlas in classical geometry — roughly speaking, the role of topology for ringed spaces has to be replaced by Grothendieck's descent theory — and it will take us a few chapters to work this out thoroughly. We begin in Chapter 1, however, with a brief description of these categorical and groupoid notions, and discuss a collection of examples, which we hope will prepare the reader for this journey.

One feature of this text, especially in Part I, is an emphasis on these atlases, or groupoids. We believe that working explicitly with many of them leads to a better feeling for stacks. Of course, just as with manifolds, describing a particular stack by an atlas is often not the most revealing way to study it. However, the existence of such an atlas is crucial, since it allows the extention of basic notions from algebraic geometry to stacks by putting appropriate geometric conditions on the schemes and the maps among them.

In addition to the introductory Chapter 1, we have included an Appendix C which studies groupoids of sets. These can be thought of as descrete groupoids (or stacks). They provide a model for the general theory, in which many of the constructions appear in a simple setting, without geometric complications. Beginners may want to spend some time with this appendix before reading the main text.

The text include many Examples and Exercises, which are used in similar ways; Exercises may require more work, and Examples are more likely to be referred to later. In both, we frequently omit phrases like "Show that", especially for routine verifications.

Note on terminology. The notion of groupoid and the word "groupoid" originated in the 1920's in algebra, and in the 1950's in geometry, mainly in the work of Ehresmann and Haefliger; they have continued to appear in many areas of mathematics, and today they play a role in noncommutative geometry, cf. [**18**]. A brief history can be found in [**16**]. When the spaces involved are discrete, this notion coincides with that of a (small) category in which all morphisms are isomorphisms. Category theorists have taken the word groupoid to *mean* a category with this property, and many others have followed this practice.[1] As groupoids in the original sense play a fundamental role here, we will use the word in its original meaning, so we will have algebraic groupoids in algebraic geometry, topological groupoids in topology, etc. If this is not confusing enough, however, categories in which all maps are isomorphisms also appear prominently in the development of stacks. We will say that a category *is* a groupoid if it has this property.

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Comments, corrections, suggestions, or contributions (say for other elementary topics that could be included) are eagerly solicited!

[1] The reader has probably heard the joke that a group is a category with one object in which all maps are isomorphisms. Few take this seriously — at least, no one mistakes an abelian group for some kind of abelian category. Unfortunately for the term groupoid, this joke has been taken seriously!

[Tentative] **CONTENTS**

CHAPTER 1

Introduction

Our aim in this chapter is to describe informally a variety of concrete examples that show why stacks are needed, and to illustrate some of the key ingredients of stacks. We start with a brief discussion of the two natures of a stack: as categories, and as atlases/groupoids. In practice it is usually easy to define the appropriate category, but it requires some work, requiring knowledge of the geometry involved, to construct an atlas. Then we look at examples, where these and other "stacky" features can be seen. Many of these examples should be familiar to the reader in some setting. Some of them were important in the early history of stacks, so reading about them will also give a glimpse of this history. Most of these examples will reappear later in the book, and most of the ideas seen here will be developed systematically later. Depending on a reader's background, statements made without proof can be accepted as facts to be used for motivation, or proofs can be worked out as exercises.

Making the notion of stack precise requires a fair amount of rather abstract language, including such mouthfuls as "categories fibered in groupoids". Starting in the next chapter we will develop this language slowly and carefully, with precise versions of most of these and many other examples. We hope that seeing several examples will help the reader digest what is to follow. However, we emphasize that nothing that is done here is logically necessary for reading the rest of the book.

1. Stacks as categories

Stacks are defined with respect to some fixed category $\mathcal{S}$, called the *base category.* For example, $\mathcal{S}$ can be the category (Sch) of schemes (or schemes over some fixed base), or $(\mathbb{C}_{\mathrm{an}})$ of complex analytic spaces, or (Diff) of differentiable manifolds, or (Top) of topological spaces, or even the category (Set) of sets. A *stack* over $\mathcal{S}$ will be a category $\mathfrak{X}$ together with a functor $\mathfrak{X} \to \mathcal{S}$, satisfying some properties — most of which will be left until later to discuss. These properties will depend, in part, on a "topology" on $\mathcal{S}$. A *morphism* from one $\mathfrak{X} \to \mathcal{S}$ to another $\mathfrak{Y} \to \mathcal{S}$ is defined to be a functor from $\mathfrak{X}$ to $\mathfrak{Y}$ that commutes with the projections to $\mathcal{S}$.

We start with some examples of this.

EXAMPLE 1.1A. **Objects (Schemes).** An object X in $\mathcal{S}$ determines a category $\mathfrak{X}$, whose objects are pairs (S, f), where S is an object in $\mathcal{S}$ and $f\colon S \to X$ is a morphism. A morphism from (S', f') to (S, f) in $\mathfrak{X}$ is given by a morphism $g\colon S' \to S$ such that $f \circ g = f'$. The functor $\mathfrak{X} \to \mathcal{S}$ takes an object (S, f) to S, and takes a morphism from (S', f') to (S, f) to the underlying morphism from S' to S. It is a basic fact of Grothendieck/Yoneda that this category $\mathfrak{X}$ determines X up to canonical isomorphism.

This category will be denoted by $\underline{X}$; when the idea of stacks has been thoroughly digested, it can be denoted simply by X, but we will avoid doing this in Part I. We will be primarily interested in the case when X is a scheme and $\mathcal{S}$ is a category of schemes, but the notion is valid for any base category $\mathcal{S}$.

This example is a variation of Grothendieck's idea of replacing a scheme X by its functor of points. This is the contravariant functor h_X from $\mathcal{S}$ to (Set) with $h_X(S) = \mathrm{Hom}_{\mathcal{S}}(S, X)$, called the set of *$S$-valued points* of X. A morphism $q\colon X \to Y$ in $\mathcal{S}$ determines a natural transformation from h_X to h_Y, taking $f\colon S \to X$ to $q \circ f\colon S \to Y$. Just as X can be recovered from h_X, the q can be recovered from the natural transformation.

EXAMPLE 1.1B. **Torsors.** We start with $\mathcal{S} =$ (Top), the category of topological spaces, and let G be a topological group. A *G-torsor*, or *principal G-bundle*, is a (continuous) map $E \to S$, with a (continuous) action of G on E, which we take to be a right action; one requires that it be locally trivial, in the sense that S has an open covering $\{U_\alpha\}$ such that the restriction $E|_{U_\alpha}$ is isomorphic to the trivial bundle $U_\alpha \times G \to U_\alpha$. One has a category, which we denote by BG, whose objects are the G-torsors $E \to S$. (We will explain this notation later.) A morphism from $E' \to S'$ to $E \to S$ is given by a pair of maps $E' \to E$ and $S' \to S$ with the map $E' \to E$ being equivariant (commuting with the action of G) and the induced diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

being cartesian (this means that it commutes and the induced map from E' to the fibered product $S' \times_S E$ is a homeomorphism). The functor from BG to $\mathcal{S}$ is the obvious one that forgets the torsors, i.e., it takes an object $E \to S$ of BG to the object S of $\mathcal{S}$, and a morphism from $E' \to S'$ to $E \to S$ to the underlying map $S' \to S$.

There is an important generalization of this example. If G acts on the right on a space X, one defines a category, denoted $[X/G]$, whose objects are G-torsors $E \to S$, together with an equivariant map from E to X. A morphism from $E' \to S', E' \to X$ to $E \to S, E \to X$ is given by a pair of maps $E' \to E$ and $S' \to S$ giving a map of torsors as above, but, in addition, the composite $E' \to E \to X$ is required to be equal to the given map from E' to X. This may look rather arbitrary now, but we will soon see examples where these categories arise naturally. In this language, the category BG is the same as the category $[\bullet/G]$, where $\bullet$ is a point; and $[X/\{1\}]$ (where $\{1\}$ denotes the group with one element) is the same as $\underline{X}$. If G acts on the left on X, and we consider left G-torsors, we have similarly a category denoted $[G\backslash X]$.

This example (and its generalization) extend to the setting where G is a complex Lie group, and $\mathcal{S}$ is a category of smooth manifolds, or complex analytic spaces. In algebraic geometry, we can take $\mathcal{S}$ to be a category of schemes (all schemes, or schemes over a fixed base), and work with algebraic actions of algebraic groups. The major difference in the algebraic setting is that the notion of local triviality for a torsor is usually taken, not in the Zariski topology, but in the étale topology.

Example 1.1C. **Moduli of curves.** Let $\mathcal{S}$ be the category of all schemes. A family of curves of genus g is a morphism $C \to S$ of schemes which is smooth and proper, whose geometric fibers are connected curves of genus g. The moduli stack $\mathcal{M}_g$ of curves of genus g has for its objects such families. A morphism from a family $C' \to S'$ to $C \to S$ is pair of morphisms $C' \to C$ and $S' \to S$ such that the induced diagram is cartesian, as in the case of torsors. The functor from $\mathcal{M}_g$ to $\mathcal{S}$ is the obvious one that forgets the families of curves.

In the case when $S' = S = \mathrm{Spec}(k)$, where k is a field, and $C' = C$, a curve over k, any automorphism of C over k will determine a morphism in $\mathcal{M}_g$ lying over the identity morphism of $\mathrm{Spec}(k)$. This illustrates the important point that the morphism from S' to S does *not* determine the morphism from C' gto C. Everything about the algebraic geometry of curves and their automorphisms is encoded in $\mathcal{M}_g \to \mathcal{S}$. It is precisely the existence of nontrivial automorphisms that prevents $\mathcal{M}_g$ from "being" a scheme.

There is a more classical object, the "coarse" moduli space M_g, which is a scheme (over $\mathrm{Spec}(\mathbb{Z})$); see [**75**]. Its geometric points correspond to isomorphism classes of curves, and it has the property that for any family of curves $C \to S$, there is a canonical morphism from S to M_g taking a geometric point s to the isomorphism class of the fiber C_s. These morphisms determine a functor from $\mathcal{M}_g$ to the category $\underline{M_g}$ determined by M_g. Moreover, M_g is characterized by a universal property that can be described as follows. Any morphism from $\mathcal{M}_g$ to the category $\underline{N}$ of a scheme N must factor uniquely through $\underline{M_g}$ (cf. [**75**], §5.2); note that the morphism from $\underline{M_g}$ to $\underline{N}$ is given by a morphism from M_g to N. In the language of stacks, the space M_g will be a "coarse moduli space" for the stack $\mathcal{M}_g$.

Many important examples of stacks will be variations of this example. For example, there is a stack $\mathcal{M}_{g,n}$ whose objects are families of curves $C \to S$ together with n pairwise disjoint sections $\sigma_1, \ldots, \sigma_n$ from S to C; the morphisms in $\mathcal{M}_{g,n}$ must be compatible with these sections. There are also "compactifications", which allow fibers to be nodal curves, with an appropriate notion of stability. One can also replace curves by other varieties.

The use of stacks in [**20**] to prove the irreducibility of the variety $M_g(k)$ of curves of genus g over any algebraically closed field k can be sketched as follows. Take $\mathcal{S}$ to be all schemes. Suppose for the moment that $\mathcal{M}_g$ were represented by a scheme M_g that is smooth over $\mathrm{Spec}(\mathbb{Z})$, and that $\mathcal{M}_g$ had a compactification $\overline{\mathcal{M}_g}$ (using stable curves) that is represented by a scheme $\overline{M}_g$ that contains M_g as an open subscheme, with $\overline{M}_g$ smooth and projective over $\mathrm{Spec}(\mathbb{Z})$. The classical fact that $\overline{M}_g(\mathbb{C})$ is connected would imply, by a connectedness theorem of Enriques and Zariski, that all geometric fibers $\overline{M}_g(k)$ of $\overline{M}_g$ over $\mathrm{Spec}(\mathbb{Z})$ are connected. Since a nonsingular connected variety is irreducible, the open subvariety $M_g(k)$ would also be irreducible. Although these assertions are all false for the coarse moduli spaces — even $M_g(\mathbb{C})$ is singular — they are true, suitably interpreted, for the corresponding stacks, and the irreducibility of the coarse varieties $M_g(k)$ follows.

The notion of S-valued points, discussed in Example 1.1A, can be used to make casual set-theoretic notation rigorous. For example, if $\mathcal{S}$ is the category schemes over a base Λ, and a group scheme G over Λ acts on the right on a scheme X in $\mathcal{S}$, the associativity condition "$(x \cdot g) \cdot h = x \cdot (g \cdot h)$" is not strong enough if applied pointwise, but it is if applied to S-valued points for all S in $\mathcal{S}$. Here x, g, and h are taken to be in $h_X(S)$, $h_G(S)$, and $h_G(S)$, and $x \cdot g$ denotes the composite $S \xrightarrow{(x,g)} X \times_\Lambda G \xrightarrow{\sigma} X$, where σ is the action. The equation "$(x \cdot g) \cdot h = x \cdot (g \cdot h)$" for all such x, g, and h is equivalent to the commutativity of the diagram

$$\begin{tikzcd} X \times_\Lambda G \times_\Lambda G \arrow[r, "\sigma \times 1_G"] \arrow[d, "1_X \times m"'] & X \times_\Lambda G \arrow[d, "\sigma"] \\ X \times_\Lambda G \arrow[r, "\sigma"] & X \end{tikzcd}$$

where $m\colon G \times_\Lambda G \to G$ is the product in G; one sees this by considering the universal case where S is $X \times_\Lambda G \times_\Lambda G$, with x, g, and h the three projections. We will often use such abbreviations in this text.

The idea of stacks as categories leads to some complications. Morphisms should then be functors, specifically, morphism from one stack to another will be functors that commute with the given functors to the base category. But in the world of categories, the natural notion of isomorphism is not a strict isomorphism (bijection on the level of objects and morphisms), but rather an *equivalence* of categories. So, quite different looking categories can give rise to isomorphic stacks. While stacks are important tools in algebraic geometry, it is not easy to do algebraic geometry on a category! For example, we would like to say that $\mathcal{M}_g$ is smooth, and that it is an open substack of a smooth compactification, whose complement is a divisor with normal crossing. And we would like to describe line bundles and vector bundles on these stacks, and do intersection theory on them. Only after a considerable amount of preparation will we see how to do these things.

2. Stacks from groupoids

An algebraic stack will come from a kind of atlas, which is called a groupoid. If $\mathcal{S}$ is the base category, a *groupoid in* $\mathcal{S}$, or an $\mathcal{S}$*-groupoid*, consists of a pair of objects U and R in $\mathcal{S}$, together with five morphisms: s (the "source") and t (the "target") from R to U, e (the "identity") from U to R, a morphism m (the "multiplication") from the fibered product[1] $R\ {}_t\!\times_{U,s} R$ to R, and a morphism i (the "inverse") from R to R, which satisfy some natural axioms.

In fact, you already know how to write down these axioms, as follows. Take a category in which all morphisms are isomorphisms and let U be the objects of this category, and R the morphisms or arrows, with s and t be the usual source and target (sending $f \in R$ to its source and target respectively), e the identity (taking an object to the identity map on it), m the composition (taking a pair $f \times g$ to $g \circ f$), and i

[1]For this to make sense, we assume, at least for now, that the fibered product of R with itself over U, using the two projections t and s, must exist in the category $\mathcal{S}$; we usually abbreviate this to $R\ {}_t\!\times_s R$. Similarly whenever we write cartesian products such as $U \times U$, we are assuming they exist as well.

the inverse. The axioms for a category amount to certain compatibilities among these morphisms, such as $s \circ e = \mathrm{id}_U$. If you write down these compatibilities then you get exactly the axioms for a groupoid.

EXERCISE 1.1. Do this now and try to obtain axioms for a groupoid. (You can check against the list at the beginning of Chapter 3 to see if you have missed any.)

A notation like (U, R, s, t, e, m, i) for a groupoid is too unwieldy to be practical. We will often use the notation $R \rightrightarrows U$ for a groupoid with spaces U and R indicated, with the two arrows (for s and t) standing as an abbreviation for all five maps. In fact, e and i are uniquely determined by s, t, and m, so we often leave their construction to the reader. When $\mathcal{S} = (\mathrm{Top})$, we call this a *topological groupoid*, and when $\mathcal{S} = (\mathbb{C}_{\mathrm{an}})$, we call it an *analytic groupoid*. When $\mathcal{S}$ is a category of schemes, it will be called an *algebraic groupoid* or a *groupoid scheme*.

The *isotropy group* $\mathrm{Aut}(x)$ of a point x in U is the set $s^{-1}(x) \cap t^{-1}(x) \subset R$, which is a group with product determined by m.

A morphism from a groupoid $R' \rightrightarrows U'$ to a groupoid $R \rightrightarrows U$ is given by a pair (ϕ, Φ) of morphisms $\phi \colon U' \to U$ and $\Phi \colon R' \to R$ commuting with all the morphisms of the groupoid structure.

One geometric example of a groupoid, called the *fundamental groupoid* of a topological space, is probably familiar to you. Although it will not play much of a role in this book, it shows clearly the not-everywhere-defined grouplike structure of a groupoid. If X is a topological space, its fundamental groupoid can be denoted $\Pi(X) \rightrightarrows X$. The elements of $\Pi(X)$ are triples (x, y, σ), with x and y points of X and σ a homotopy class of paths in X starting at x and ending at y; s and t take this triple to x and y respectively, and $m((x, y, \sigma), (y, z, \tau)) = (x, z, \sigma * \tau)$, where $\sigma * \tau$ is the usual product coming from tracing first a path representing σ and then a path representing τ.[2] This groupoid has advantages over the usual fundamental group (which requires an arbitrary choice of base point), particularly in the study of the Van Kampen theorem when the intersection of the open sets involved is not connected (cf. [**16**]). There are also useful variants of the fundamental groupoid, such as the groupoid $\Pi(X, A) \rightrightarrows A$, where A is a subset of X, and the paths connect points of A. If X is a foliated manifold, one can require the paths and homotopy equivalences to lie within leaves of the foliation; if one replaces homotopy equivalence by holonomy equivalence, one arrives at the *holonomy groupoid* of the foliation [**43**].

A continuous mapping $f \colon X \to Y$ determines a morphism (f, F) from the groupoid $\Pi(X) \rightrightarrows X$ to the groupoid $\Pi(Y) \rightrightarrows Y$, with $F(\sigma) = f \circ \sigma$. Then a homotopy $H \colon X \times [0, 1] \to Y$ from f to g determines a mapping $\theta \colon X \to \Pi(Y)$, taking x in X to the path $t \mapsto H(x, t)$. If likewise (g, G) denotes the morphism of groupoids determined by g, this mapping θ satisfies the identities

$$s(\theta(x)) = f(x), \quad t(\theta(x)) = g(x), \quad \text{and} \quad \theta(s(\sigma)) \cdot G(\sigma) = F(\sigma) \cdot \theta(t(\sigma))$$

[2]For a general space, its fundamental groupoid is a groupoid of sets. If X has a universal covering space, i.e., X is semilocally simply connected, then $\Pi(X)$ has a natural topology so that s and t are local homeomorphisms, and the fundamental groupoid is a topological groupoid.

for x in X and σ in $\Pi(X)$. Maps θ satisfying these identities are called 2-*isomorphisms*; they will be the analogues of homotopies for groupoids.

EXAMPLE 1.2A. **Classical atlases.** If X is a scheme, or manifold, or topological space, and $\{U_\alpha\}$ is an open covering of X (with α varying in some index set), let $U = \coprod U_\alpha$ be the disjoint union, and let $R = \coprod U_\alpha \cap U_\beta$, the disjoint union of all intersections over all ordered pairs (α, β); equivalently, $R = U \times_X U$. The five maps are the obvious ones: s takes a point in $U_\alpha \cap U_\beta$ to the same point in U_α, and t takes it to the same point in U_β; e takes a point in U_α to the same point in $U_\alpha \cap U_\alpha$; i takes a point in $U_\alpha \cap U_\beta$ to the same point in $U_\beta \cap U_\alpha$; for m, if u is in $U_\alpha \cap U_\beta$ and v is in $U_\delta \cap U_\gamma$, requiring $t(u)$ to equal $s(v)$ says that $\beta = \delta$ and $u = v$, so we can set $m(u,v) = u = v$ in $U_\alpha \cap U_\gamma$.

The basic construction of algebraic geometry of *recollement* (gluing) amounts to constructing X from a compatible collection of schemes $\{U_\alpha\}$, with isomorphisms from an open set $U_{\alpha\beta}$ of each U_α to an open set $U_{\beta\alpha}$ of U_β, satisfying axioms of compatibility. These axioms are the same as those for constructing a manifold by gluing open subsets of Euclidean spaces.

There is a similar atlas (groupoid) constructed from an étale covering $\{U_\alpha \to X\}$, but taking R to be $\coprod U_\alpha \times_X U_\beta$. In fact, for any morphism $U \to X$, one can construct a groupoid, with $R = U \times_X U$, with s and t the two projections, e the diagonal, i the map reversing the two factors, and m the composite

$$(U \times_X U) \times_U (U \times_X U) \cong U \times_X U \times_X U \to U \times_X U,$$

where the second map is the projection $p_{1,3}$ to the outside factors. Applying this to the case of an open covering $U = \coprod U_\alpha \to X$ recovers the "gluing" atlas.

A trivial but important special case of this construction takes, for any object X of our category $\mathcal{S}$, the groupoid arising from the identity map from X to X. Here $U = X$, $R = X$, and all the maps of the groupoid are identity maps. When $\mathcal{S}$ is the category of sets, so a groupoid is identified with a category, a set is exactly a category in which the only maps are identity maps. In this sense, one may say that schemes (or spaces) are to stacks as sets are to (groupoid) categories.

In this collection of examples, the canonical map $(s,t)\colon R \to U \times U$ is an embedding (a monomorphism), so that R defines an equivalence relation on U, and X may be thought of as the quotient of U by this equivalent relation. In fact, *algebraic spaces* are constructed from equivalence relations $R \to U \times U$ with projections s and t étale. (Any equivalence relation on a set U, in fact, determines a groupoid of sets.) One major difference between a scheme or algebraic space and a general stack is that, for an atlas for a stack, the morphism from R to $U \times U$ need not be one-to-one (on geometric points).

EXAMPLE 1.2B. **Group actions.** Suppose an algebraic (resp. topological) group G acts on a scheme (resp. topological space) U, say on the right. There is a natural equivalence relation on U: two points u and v are equivalent if they are in the same orbit: $v = u \cdot g$ for some $g \in G$. There is a better groupoid to construct from this action: take $R = U \times G$, and think of a point (u, g) in R as being a point u together with an arrow g from u to $u \cdot g$. This indicates that we look at the atlas

$$U \times G \rightrightarrows U$$

where $s\colon U \times G \to U$ is the first projection and $t\colon U \times G \to U$ is the action (so $s(u, g) = u$ and $t(u, g) = u \cdot g$). For the remaining maps, e is the identity $(e(u) = (u, e_G))$,

$$m\big((u, g), (u \cdot g, h)\big) = (u, g \cdot h),$$

and $i(u, g) = (u \cdot g, g^{-1})$.

This groupoid is sometimes denoted by a semi-direct product notation $U \rtimes G$, and it is called a *transformation groupoid*. This groupoid will, in fact, be an atlas for the stack $[U/G]$ discussed in Example 1.1B. Note that for x in U, the isotropy group $\mathrm{Aut}(x)$ of the groupoid is the same as the isotropy or stabilizer group G_x for the group action. Whenever there are fixed points, the mapping $(s, t)\colon R \to U \times U$ is not an embedding: if $u \in U$ and $g \in G$, with $g \neq e_G$ and $u \cdot g = u$, then (u, g) and (u, e_G) have the same image. The stack determined by this groupoid will capture the action better than the naive quotient U/G, when this latter quotient exists. An extreme example is the action of G on a point $\bullet$; the groupoid $G \rightrightarrows \bullet$ carries the information of the group G (and the stack BG from Example 1.1B), but the quotient space is just the point $\bullet$.

An analogous groupoid $G \times U \rightrightarrows U$ arises from a left action of a group G on U. This groupoid, also denoted $G \ltimes U$, is defined by setting $s(g, u) = u$, $t(g, u) = g \cdot u$, and $m((g, u), (g', g \cdot u)) = (g' \cdot g, u)$. More generally, if G acts on the left on U, and H acts on the right on U, and the actions commute in the sense that $(g \cdot u) \cdot h = g \cdot (u \cdot h)$ for all $g \in G$, $u \in U$, and $h \in H$, there is a groupoid

$$G \times U \times H \rightrightarrows U,$$

with $s(g, u, h) = u$, $t(g, u, h) = g \cdot u \cdot h$, and $m((g, u, h), (g', g \cdot u \cdot h, h') = (g' \cdot g, u, h \cdot h')$. This groupoid may be denoted $G \ltimes U \rtimes H$.

Example 1.2C. **Curves in projective space.** Fix an integer $g \geq 2$. An important fact about moduli of curves is that curves of genus g can be uniformly embedded in projective space. This is based on the canonical sheaf (which, for a curve, is just the sheaf of differentials), an ample sheaf whose third tensor power is very ample. From the classical Riemann–Roch formula, it is computed that this gives an embedding of the curve into $\mathbb{P}^{5g-6}$. For any family of genus g curves $C \to S$, the sheaf $\omega_{C/S}^{\otimes 3}$ gives rise to an embedding of C in a projective bundle over S.

Inside the Hilbert scheme of $\mathbb{P}^{5g-6}$ there is a locus $\mathrm{Hilb}_{g,3}$, smooth of dimension $25g^2 - 47g + 21$, of tricanonically embedded curves of genus g. The canonical sheaf is preserved by automorphisms of curves, and all isomorphisms are given by projective linear transformations. The action of the projective linear group makes

$$PGL_{5g-5} \times \mathrm{Hilb}_{g,3} \rightrightarrows \mathrm{Hilb}_{g,3}$$

an atlas for $\mathcal{M}_g$. More classically, the moduli space M_g (a variety of dimension $3g-3$) is a quotient variety for this action of PGL_{5g-5}.

The proof of irreducibility of the moduli spaces $M_g(k)$ using stacks [**20**] makes use of the existence of another atlas $R \rightrightarrows U$ for $\mathcal{M}_g$, such that U and R are both smooth and have the same dimension $3g-3$ as M_g. Such an atlas exists; in fact, U can be taken to be the disjoint union of finitely many locally closed subvarieties of $\mathrm{Hilb}_{g,3}$ and R a corresponding disjoint union of subvarieties of $PGL_{5g-5} \times \mathrm{Hilb}_{g,3}$. The existence of such an atlas is an important, nontrivial fact which is linked to properties (coming from deformation theory) of curves of genus g.

For a group action on a variety, there might exist a classical quotient variety. But for some purposes the groupoid $U \rtimes G$ is better. In Example 1.2C, we saw that the groupoid has nice properties (e.g., smoothness) which do not hold for the quotient variety.

Let us compare the stack quotient with a more classical quotient. Here for simplicity the base category is taken to be $(\mathbb{C}_{\mathrm{an}})$. If a complex Lie group G acts on a complex space X, a *categorical quotient* is a complex space X/G, with a G-invariant surjective morphism $q\colon X \to X/G$ that satisfies a universal property: for any complex analytic space Y and any G-invariant morphism $f\colon X \to Y$, there is a unique morphism $\bar{f}\colon X/G \to Y$ such that $f = \bar{f} \circ q$.

We compare the quotients in the following example. Let $G = \mathbb{C}^\times$ act by $(x,y)\cdot t = (xt, yt)$ on $\mathbb{C}^2$, and also on $U := \mathbb{C}^2 \smallsetminus \{(0,0)\}$. Then:

(1) The map from U to $\mathbb{P}^1$ that sends (x,y) to $[x:y]$ identifies $\mathbb{P}^1$ as the categorical quotient U/G.

(1′) For any analytic space S, morphisms $S \to \mathbb{P}^1$ are in bijective correspondence with G-torsors over S equipped with a G-equivariant morphism to U, up to G-equivariant isomorphism commuting with the morphisms to U.

(2) The categorical quotient $\mathbb{C}^2/G$ is a point.

(2′) An analytic space S admits infinitely many G-torsors with equivariant maps to $\mathbb{C}^2$, up to G-equivariant isomorphism commuting with the maps to $\mathbb{C}^2$, while possessing always a unique map to a point.

(We just state these as facts for now; the techniques to give complete justifications will come later. Precisely analogous assertions hold as well in the topological and algebraic settings.) On U, where G acts freely, the classical quotient U/G represents the stack quotient $[U/G]$. In the world of stacks, $[\mathbb{A}^2/G]$ will contain $[U/G]$ as a dense open substack, as contrasted with the classical notion of categorical quotient, in which $\mathbb{A}^2/G$ is a point.

As one would expect from the case of manifolds, many different groupoids can be atlases for the same stack. Example 1.2C made reference to two different groupoids for $\mathcal{M}_g$: there were maps $U \to \mathrm{Hilb}_{g,3}$ and $R \to PGL_{5g-5}\times\mathrm{Hilb}_{g,3}$ (componentwise inclusion maps), giving rise to a map of groupoids from $R \rightrightarrows U$ to $PGL_{5g-5} \times \mathrm{Hilb}_{g,3} \rightrightarrows \mathrm{Hilb}_{g,3}$. Of course, an arbitrary map of groupoids (ϕ, Φ) from $R' \rightrightarrows U'$ to $R \to U$ will not determine an isomorphism of their corresponding stacks. There are two properties that will guarantee this, the first corresponding to injectivity, the second to surjectivity. The properties are:

Condition 1.3(i). The diagram

$$\begin{array}{ccc} R' & \xrightarrow{(s,t)} & U' \times U' \\ \Phi\downarrow & & \downarrow \phi\times\phi \\ R & \xrightarrow{(s,t)} & U \times U \end{array}$$

must be cartesian.

Condition 1.3(ii). For every $u \in U$ there is a $u' \in U'$ and an $a \in R$ such that $s(a) = \phi(u')$ and $t(a) = u$; or in other words, the morphism

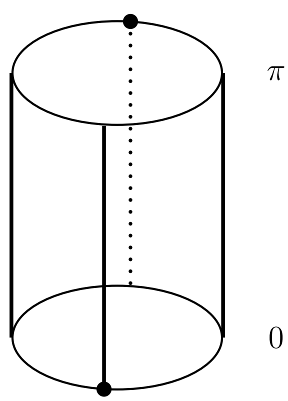
$$V \ = U' \,{}_{\phi}\!\times_{U,s} R \longrightarrow U$$

determined by t must be surjective.

Condition (i) can be expressed in terms of S-valued points: the map $h_{R'}(S) \to h_R(S) \times_{h_{U\times U}(S)} h_{U'\times U'}(S)$ is a bijection for all S. In condition (ii), "surjective" must be interpreted correctly. Requiring surjectivity on the naive point level is too weak, and requiring that $h_V(S) \to h_U(S)$ be surjective for all S is too strong, since that is equivalent to the existence of a splitting morphism from U to V. (For example, a fiber bundle projection should be surjective, but it may have no global section.) What works is to require that the map must be locally surjective, using the topology on $\mathcal{S}$. That is, we require U to have a covering $\{U_\alpha \to U\}$ such that each $U_\alpha \to U$ factors through V.

In the case where $\mathcal{S}$ is the category of sets (with the discrete topology), so the groupoids are categories and maps between them are functors, condition (i) says that this functor is fully faithful, and condition (ii) says that it is essentially surjective; together they say that the functor is an equivalence of categories.

Example 1.4. We conclude this discussion with a geometric example. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, and let X be the cylinder $D \times \mathbb{R}$, with the identification $(z, \phi) \sim (z', \phi')$ if $\phi' - \phi = n\pi$, $n \in \mathbb{Z}$, and $z' = (-1)^n z$.



The group S^1 acts on X, by $e^{i\vartheta} \cdot (z, \varphi) = (z, \vartheta + \varphi)$. (This is an example of a "Seifert circle bundle".) The group $\{\pm 1\}$ acts on D by $(-1) \cdot z = -z$, and $\{\pm 1\}$ is a subgroup of S^1 by $-1 \mapsto e^{i\pi}$. The embedding $D \to X$, $z \mapsto (z, 0)$, is equivariant with respect to $\{\pm 1\} \to S^1$, giving a morphism of groupoids

$$\{\pm 1\} \ltimes D \ \to\ S^1 \ltimes X.$$

EXERCISE 1.2. (a) Show that this morphism satisfies properties (i) and (ii), where the base category $\mathcal{S}$ is (Top) or (Diff). (b) Compute the isotropy groups of these actions at all points.

3. Triangles

Mike Artin has suggested that a quick way to get a feeling for stacks is to work out what the moduli space of ordinary triangles should be. As in all moduli problems, it is important to consider families of objects, in this case plane triangles up to isometry. If S is a topological space, a family of triangles over S will be a continuous and proper map $X \to S$, making X a fiber bundle over S with a continuously varying metric on fibers[3], such that each fiber is (isometric to) a triangle.

The classical moduli space of triangles would simply be the set T of plane triangles, up to isometry, suitably topologized. As a set, T consists of triples (a, b, c) of side lengths, satisfying (strictly) the triangle inequalities, up to reordering. As a space, T is a quotient of a subset of Euclidean space. Indeed, consider the open cone

$$\widetilde{T} = \{\, (a, b, c) \in \mathbb{R}^3_+ \mid a + b > c,\ \ b + c > a,\ \ c + a > b \,\}.$$

Then we have a map $\widetilde{T} \to T$, and T inherits a topology from $\widetilde{T}$, the quotient topology.

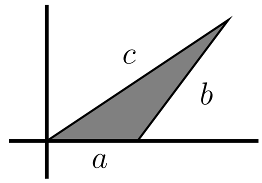
To phrase this moduli problem in the categorical language, we take $\mathcal{S}$ to be the category of topological spaces, and define a category $\mathfrak{T}$ whose objects are families of triangles $X \to S$. A morphism in $\mathfrak{T}$ from one family $X' \to S'$ to another family $X \to S$ is given by a pair of (continuous) maps $X' \to X$ and $S' \to S$ such that the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

commutes, and so that the induced maps on the fibers are isometries. The functor from $\mathfrak{T}$ to $\mathcal{S}$ is the evident one, as in the examples of Section 1.1.

The moduli problem becomes easier if we consider, instead, *ordered* triangles, ordering the sides (or, equivalently, their opposite vertices). Here the objects of the corresponding category $\widetilde{\mathfrak{T}}$ would be fibrations $X \to S$ as before, together with a triple (α, β, γ) of sections that pick out the three vertices of each fiber; the morphisms are required to be compatible with these sections. The moduli space is then the open cone $\widetilde{T} \subset \mathbb{R}^3$. There is a universal family $\widetilde{Y} \subset \widetilde{T} \times \mathbb{R}^2$, with its projection $\widetilde{Y} \to \widetilde{T}$, and with the fiber over (a, b, c) in $\widetilde{T}$ being the triangle

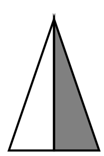
[3]That is, a continuous distance function $d\colon X \times_S X \to \mathbb{R}_{\geq 0}$ whose restriction to $X_s \times X_s$ is a metric on the fiber X_s, for every $s \in S$.



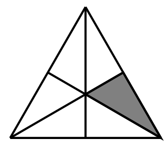
The essential point is that any triangle with labeled edges of lengths a, b, and c is *canonically* isometric to this one: given any family $X \to S$ with three vertex sections, there is a unique map from S to $\widetilde{T}$, and a unique isomorphism of X with the pullback of this universal family. Having this universal family means, in classical language, that $\widetilde{T}$ is a *fine moduli space*. In the language of stacks, there is an isomorphism of stacks (equivalence of categories) between $\widetilde{\mathfrak{T}}$ and $\underline{\widetilde{T}}$.

If we want a moduli space for unordered triangles, however, the situation is more complicated. The symmetric group $\mathfrak{S}_3$ acts (on the right) on $\widetilde{T}$ by permuting the coordinates, and the quotient space $T = \widetilde{T}/\mathfrak{S}_3$ is the obvious candidate for a moduli space of triangles. Its points, at least, do correspond to triangles up to isometry. The group $\mathfrak{S}_3$ also acts on $\widetilde{Y}$, compatibly with its projection to $\widetilde{T}$. We can therefore construct $Y = \widetilde{Y}/\mathfrak{S}_3$, with an induced map $Y \to T$. If one were trying to construct a universal family of plane triangles, this would be a first guess.

Any family of triangles $X \to S$ will determine a map from S to T, but the family may not be isomorphic (uniquely, or even at all) to the pullback of $Y \to T$. In classical language, then, this moduli space T is a *coarse*, but not a fine, moduli space, for $\mathfrak{T}$. For example, when S is the circle S^1 and $X \to S$ is a family of equilateral triangles that rotates the triangle by 120° in one revolution around the circle, then this is not a constant family even though the corresponding map from S to T is constant. For an isosceles triangle (taking S to be a point), say with sides of lengths 1, 2, and 2, corresponding to a point t in T, there are three points $(1,2,2)$, $(2,1,2)$, and $(2,2,1)$ in $\widetilde{T}$ lying over t; the action of the group includes flips over the altitude, and the fiber of Y over t is the quotient of the triangle by this flip:



For an equilateral triangle, there is only one point of $\widetilde{T}$ over the point t in T, and the fiber of Y over t is the quotient of the triangle by the action of $\mathfrak{S}_3$:



In fact, $Y \to T$ fails to satisfy the definition of family of triangles (e.g., Y is not a fiber bundle over T). The problems with $Y \to T$ arise from triangles with nontrivial automorphisms.

EXERCISE 1.3. Let $Y^\circ \to T^\circ$ be the restriction of $Y \to T$ to the locus T° of triangles with sides of distinct lengths. Show that $Y^\circ \to T^\circ$ is a fiber bundle and gives a universal family: T° is a fine moduli space for such triangles.

Given any family $X \to S$ of (unordered) triangles, let $\widetilde{S}$ be the space of pairs

$$(s, \text{ordering of the edges of } X_s).$$

Then $\widetilde{S} \to S$ is a 6-sheeted covering space, in fact, a principal bundle (torsor) under the symmetric group $\mathfrak{S}_3$. If $\widetilde{X} \to \widetilde{S}$ is the pullback of the given family $X \to S$ by the covering map $\widetilde{S} \to S$, we have a commutative diagram

$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & \widetilde{T} \arrow[d] \\ S \arrow[r] & T \end{tikzcd}$$

where the map $\widetilde{S} \to \widetilde{T}$ commutes with the action of $\mathfrak{S}_3$. This is exactly the data for an object of the stack $[\widetilde{T}/\mathfrak{S}_3]$ described in Section 1.1: the stack $\mathfrak{T}$ is isomorphic to the quotient stack $[\widetilde{T}/\mathfrak{S}_3]$. (The reader may verify that the functor from $\mathfrak{T}$ to $[\widetilde{T}/\mathfrak{S}_3]$ is an equivalence of categories.) The transformation groupoid $\widetilde{T} \times \mathfrak{S}_3 \rightrightarrows \widetilde{T}$ will be an atlas for this stack.

As in this example, it frequently happens that a coarse moduli space can be constructed as a quotient U/G of a space U by the action of a group G. This crude quotient space cannot capture the geometry of the moduli problem near points u of U where the stabilizer $G_u = \{g \in G |\ g \cdot u = u\}$ is not trivial. The stack is designed to remember some part of the group action. The group action is not part of the information carried by the stack, however. Indeed, if it were, we would just be studying equivariant spaces.

Here is quite a different atlas for the same stack. By a *plane triangle* we mean a triangle embedded in $\mathbb{R}^2$. Let G be the Lie group of isometries of $\mathbb{R}^2$, which is the 3-dimensional group generated by rotations, reflections, and translations. Let V be the space of (unordered) plane triangles, which is a 6-dimensional manifold.[4] We have a universal family $Z \subset \mathbb{R}^2 \times V$ of plane triangles over V. Note that G acts on the left on V, and on $\mathbb{R}^2 \times V$, preserving Z.

[4]This can be constructed as a quotient of the set $\widetilde{V}$ of noncollinear triples in $(\mathbb{R}^2)^3$ by the action of $\mathfrak{S}_3$. That V is a manifold follows from the general fact that this action is free.

We claim that the stack $\mathfrak{T}$ is isomorphic to the quotient stack $[G\backslash V]$. Indeed, if $X \to S$ is an object of $\mathfrak{T}$, there is a principal (left) G-bundle $E \to S$, whose fiber over s is the space of all isometric embeddings of the fiber X_s into $\mathbb{R}^2$. (Note that this G-torsor is trivial over any open set of S on which the $\mathfrak{S}_3$-covering $\widetilde{S} \to S$ is trivial.) We have a G-equivariant map from E to V, since any point of E determines a plane triangle. This gives a functor from $\mathfrak{T}$ to $[G\backslash V]$, which is an equivalence of categories. Summarizing, we have isomorphisms of stacks:

$$[G\backslash V] \cong \mathfrak{T} \cong [\widetilde{T}/\mathfrak{S}_3].$$

Note that the two corresponding atlases even have different dimensions. However, $\dim V - \dim G = 6 - 3$ and $\dim \widetilde{T} - \dim \mathfrak{S}_3 = 3 - 0$ are equal; this stack $\mathfrak{T}$ will be 3-dimensional.

We can also see a direct relation between the groupoid $G \times V \rightrightarrows V$ and the category $\mathfrak{T}$. Any family of triangles is locally planar: if $X \to S$ is a family of triangles, we can choose an open covering $\{U_\alpha\}$ of S, with maps $\phi_\alpha \colon U_\alpha \to V$ and an isomorphism of $X|_{U_\alpha}$ with the pullback of $Z \to V$. Compatible with these, there are, on $U_\alpha \cap U_\beta$, unique maps $\Phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$ such that $\phi_\beta(p) = \Phi_{\beta\alpha}(p) \cdot \phi_\alpha(p)$. This gives a map $\Phi \colon \coprod U_\alpha \cap U_\beta \to G \times V$, taking p in $U_\alpha \cap U_\beta$ to $(\Phi_{\beta\alpha}(p), \phi_\alpha(p))$.

Exercise 1.4. Show that $\{\phi_\alpha\}$ and $\{\Phi_{\alpha\beta}\}$ determine a morphism from the groupoid $\coprod U_\alpha \cap U_\beta \rightrightarrows \coprod U_\alpha$ to the groupoid $G \times V \rightrightarrows V$. The first is an atlas for S as in Example 1.2A, the second an atlas for $[G\backslash V]$.

The following exercise shows how the two atlases for $\mathfrak{T}$ are related.

Exercise 1.5. The set $\widetilde{V}$ of noncollinear triples in $(\mathbb{R}^2)^3$ has a right action of $\mathfrak{S}_3$ compatible with the left action of G. Construct morphisms of groupoids from the groupoid $G \times \widetilde{V} \times \mathfrak{S}_3 \rightrightarrows \widetilde{V}$ to the groupoid $G \times V \rightrightarrows V$ and to the groupoid $\widetilde{T} \times \mathfrak{S}_3 \rightrightarrows \widetilde{T}$, and show that they satisfy Conditions 1.3(i)–(ii).

Exercise 1.6. How do the results of this section change if one replaces isometry (congruence) of triangles by similarity?

4. Conics

We want to classify *conics*; for us a conic will be a curve which is isomorphic to the curve defined by a homogeneous polynomial of degree two in $\mathbb{P}^2$. Here we take $\mathcal{S}$ to be schemes over $\mathbb{C}$.

There are just three isomorphism classes of plane conics. Let x, y, z be the homogeneous coordinates on $\mathbb{P}^2$, and identify a plane conic with the homogeneous polynomial that defines it (identifying two polynomials if one is a nonzero multiple of the other). The isomorphism classes are

(1) N: nonsingular conics, e.g. $x^2 + y^2 + z^2$,
(2) L: pairs of two different lines, e.g. xy,
(3) D: double lines, e.g. x^2.

Therefore, in some sense the moduli space of plane conics is just a set $\{N, L, D\}$ of three points.

If M were a fine moduli space for conics, then the morphisms from a scheme S to M would be in one-to-one correspondence with the families of conics over S. If M were even a coarse moduli space, any family of conics over S would determine a morphism from S to M.

We can first see that if $\{N, L, D\}$ is such a moduli space, then it cannot carry the discrete topology. In the one parameter family defined by $xy + tz^2$, for $t \in \mathbb{C}$, the conic C_t is smooth for $t \neq 0$ and a pair of two different lines for $t = 0$. The corresponding map from $\mathbb{C}$ to $\{N, L, D\}$ sends $\mathbb{C} \smallsetminus 0$ to N and 0 to L; this shows that L must be in the closure of N. Similarly the family $x^2 + ty^2$, $t \in \mathbb{C}$, shows that D is in the closure of L. So we'd want the closed subsets of $\{N, L, D\}$ to be $\emptyset$, $\{D\}$, $\{D, L\}$, and $\{D, L, N\}$. This cannot be a fine moduli space, (nontrivial automorphisms of the conics prevent this), nor does it provide a course solution in algebraic geometry to the moduli problem. This illustrates the principle that the geometric points of a stack may tell us very little about it.

There a concrete description, familiar in algebraic geometry. One identifies the space of plane conics with the projective space $\mathbb{P}^5$ of homogeneous polynomials $ax^2 + bxy + cy^2 + dxz + eyz + fz^2$ of degree two in x, y, z modulo multiplication by nonzero scalars; so a plane conic defined by this polynomial is identified with the point $[a : b : c : d : e : f]$ in $\mathbb{P}^5$. The equation $ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$ defines a universal family $Y \subset \mathbb{P}^2 \times \mathbb{P}^5$ over $\mathbb{P}^5$. The group $G = PGL_3$ of projective linear transformations of $\mathbb{P}^2$ acts on the space of conics: an element of G defines an isomorphism $g\colon \mathbb{P}^2 \to \mathbb{P}^2$. For a conic Z in $\mathbb{P}^2$ the image $g(Z)$ is another conic, and the restriction $g|_Z\colon Z \to g(Z)$ is an isomorphism. Furthermore it is easy to see that two plane conics Z and W are isomorphic if and only if $W = g(Z)$ for a suitable $g \in G$, and that in this case the isomorphisms from Z to W are precisely the restrictions $h|_Z$ of those $h \in G$ such that $h(Z) = W$.[5] From this point of view, the moduli space of conics should be a quotient of $\mathbb{P}^5$ by the group G, and we may expect the moduli stack to be the quotient stack $[\mathbb{P}^5/G]$.

A categorical description of the stack of planar conics is a bit more complicated. A family of conics is a projective morphism $\pi\colon C \to S$, flat and of finite presentation, such that each geometric fiber is isomorphic to one of the three types of plane conics. Such a family comes with a $\mathbb{P}^2$-bundle $P \to S$, with C embedded into P as a closed subscheme[6]; locally, over an affine covering $\{U_\alpha\}$ of S, there are isomorphisms $P|_{U_\alpha} \cong \mathbb{P}^2 \times U_\alpha$ of $\mathbb{P}^2$-bundles, taking $C|_{U_\alpha}$ to the zeros of a degree 2 homogeneous polynomial which does not vanish identically at any point of U_α. If $E \to S$ is the bundle of local isomorphisms of P with $\mathbb{P}^2$, then E is a principal G-bundle over S, and we have a G-equivariant

[5] The group $G = PGL_{n+1}$ acts on the left on $\mathbb{P}^n$, so it acts on the right on the polynomials $\Gamma(\mathbb{P}^n, \mathcal{O}(m))$ of degree m by the formula $(F \cdot g)(x) = F(g \cdot x)$.

[6] In fact, P may be taken to be the projective bundle $\mathbb{P}(\mathcal{E})$ of lines in the rank 3 vector bundle $\mathcal{E} := \pi_*(\omega_{C/S}^\vee)$, where $\omega_{C/S}$ is the relative dualizing sheaf. A reader to whom this is unfamiliar can take this added structure of an embedding in a $\mathbb{P}^2$-bundle as part of the definition. Note that for a general base scheme S our notion of projective is that of [**EGA** II.5.5].

morphism from E to $\mathbb{P}^5$ that takes a point s to the image of $C_s \subset P_s$, which is a conic in $\mathbb{P}^2$, i.e., a point in $\mathbb{P}^5$. This pair $(E \to S, E \to \mathbb{P}^5)$ is an object of the category $[\mathbb{P}^5/G]$, and indicates why the stack of planar conics should be isomorphic to the quotient stack $[\mathbb{P}^5/G]$.

An important part of a moduli problem is to describe the automorphism groups of its objects. When the solution is a quotient by a group action, this is the same as describing the stabilizers of representative points. For conics, we have the three cases:

(1) N: $x^2+y^2+z^2$; the stabilizer consists of the complex orthogonal 3×3 matrices (i.e, those A such that ${}^tA \cdot A = I$, up to scalars . This group has dimension 3.

(2) L: $xy = 0$; the stabilizer consists of all invertible 3×3 matrices A modulo scalars, where A is of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \text{ or } \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ 0 & 0 & * \end{pmatrix}.$$

It has dimension 4.

(3) D: $x^2 = 0$; the stabilizer is the set of all matrices of the form

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Its dimension is 6.

In general, if a smooth algebraic group G of dimension k acts freely on a smooth variety of dimension d, then the quotient (when it exists) will be smooth of dimension equal to $d-k$. This is not true in the naive world if the action is not free, but it will remain true in the stack world. A smooth orbit for the action will correspond to a "point" in the quotient and the dimension of this point will be equal to the dimension of the orbit minus k.

In the case of conics, $G = PGL_3$ has dimension 8, and the action of G on $\mathbb{P}^5$ has three orbits corresponding to the isomorphism types N, L, D of conics. The corresponding orbits are smooth of dimensions 5, 4 and 2 respectively. Therefore the quotient consists of three points: an open one, N, of dimension $5-8=-3$; in its closure another point L of dimension $4-8=-4$; and in its closure the point D of dimension $2-8=-6$. Note that these dimensions are precisely the negatives of the dimensions of the automorphism groups of conics in N, L and D respectively. (For an algebraic group H, the dimension of the stack $BH = [\bullet/H]$ will be $-\dim H$.)

As with triangles, the atlas we have given is only one of many. For example, one could take U to be the conics passing through the point $[0:0:1]$. This is a hyperplane in $\mathbb{P}^5$, defined by the vanishing of the coefficient of z^2. Take R to be the subset of $U \times G$ consisting of those pairs (u, g) such that $u \cdot g$ is also in U. There is a natural groupoid structure s, t, e, m, i on U and R so that the inclusion of U in $\mathbb{P}^5$ and the inclusion of R in $\mathbb{P}^5 \times G$ determines a morphism of groupoids from $R \rightrightarrows U$ to $\mathbb{P}^5 \times G \rightrightarrows \mathbb{P}^5$; this morphism satisfies Condition 1.3(i)–(ii). Note that this groupoid is not of the form $U \times H \rightrightarrows U$, for any action of a group H on U.

5. Elliptic curves

Elliptic curves have been a fruitful area for the development of moduli problems, as well as stacks (e.g., [**71**], [**21**]). We will devote Chapter 12 to elliptic curves, including the cases of arbitrary characteristic and over $\mathbb{Z}$. Here we sketch a few of the ideas, working in the category $\mathcal{S}$ of schemes over $\mathbb{C}$.

It is known classically that an elliptic curve E over $\mathbb{C}$ is classified up to isomorphism by a value $j \in \mathbb{C}$ known as the *j-invariant*, and all complex numbers occur. The (coarse) moduli space for isomorphism classes of elliptic curves should therefore be $\mathbb{C}$ (the complex plane, or, to an algebraic geometer, the affine line $\mathbb{A}^1$). However, the j-line is not a fine moduli space, as we will soon see; and, in fact, no fine moduli space exists.

A family of elliptic curves is a smooth and proper morphism $C \to S$, whose geometric fibers are connected curves of genus 1, together with a section $\sigma\colon S \to C$. We often abbreviate this data to $C \to S$, or sometimes even C. A morphism from $C' \to S'$ to $C \to S$ is pair of morphisms $C' \to C$ and $S' \to S$ such that the diagrams

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array} \qquad\qquad \begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & & \downarrow \\ C' & \longrightarrow & C \end{array}$$

commute, and the first (and therefore the second) is cartesian. This determines the category $\mathcal{M}_{1,1}$, and the functor to $\mathcal{S}$ is the obvious one.

The section σ determines an origin in each fiber, which then gets the structure of an abelian variety; the section is called the zero section, or the identity section.[7] Note that every elliptic curve comes with an involution, written $p \mapsto -p$, that takes a point to its inverse with respect to this group structure. For instance, if $f(x)$ is a cubic polynomial over the complex numbers with 3 distinct roots, then $y^2 = f(x)$ is the equation of (an affine model of) an elliptic curve E. When $S = \operatorname{Spec}\mathbb{C}$ and C is the elliptic curve E, the identity section is the point at infinity, and the involution sends (x, y) to $(x, -y)$.

One reason that $\mathbb{A}^1$ is not a fine moduli space is that there are non-trivial families whose fibers (at closed points) are all isomorphic – so-called *isotrivial* families. The corresponding map from S to a moduli space would be constant, and, if the moduli space were fine, the family would have to be trivial.

Exercise 1.7. Fix a cubic polynomial $f(x)$ with 3 distinct roots, and let E be the elliptic curve defined by $y^2 = f(x)$. We take $S = \mathbb{A}^1 \smallsetminus \{0\}$, with coordinate t. Let $C \to S$ be the family of elliptic curves defined by the equation

$$ty^2 = f(x).$$

(1) Every fiber of this family is isomorphic to E.
(2) This family has only finitely many sections, hence is non-trivial.

Another reason that $\mathbb{A}^1$ cannot be a fine moduli space is that there are natural line bundles that one can obtain on a scheme S given any family of elliptic curves over

[7] In fact, the family gets the structure of a group scheme over S (see [**48**], §2).

S. One such line bundle assigns to a family $C \to S$ the normal bundle to the section $S \to C$. This assignment is natural in that sense that a morphism from $C' \to S'$ to $C \to S$ determines an isomorphism of the line bundle on S' with the pullback of the corresponding line bundle on S. (Such data, with an appropriate compatibility condition, is what is meant by a line bundle on the stack $\mathcal{M}_{1,1}$.) Such line bundles are not always trivial, as we will see. But there are no nontrivial (algebraic or analytic) line bundles on $\mathbb{A}^1$, so these line bundles cannot be pulled back via a morphism to $\mathbb{A}^1$.

One can study such line bundles without the formal language of stacks, and this is what Mumford did in [**71**]. He showed that there are exactly 12 such bundles (up to isomorphism), all tensor powers of the one just discussed. In stack language, this will say that $\mathrm{Pic}(\mathcal{M}_{1,1})$ is $\mathbb{Z}/12\mathbb{Z}$.

In this seminal paper, Mumford introduced the notion of a "modular family". This is a collection $\{\pi_\alpha \colon C_\alpha \to S_\alpha\}$ of families of elliptic curves, with the following property: each S_α must be a smooth curve, and, moreover, any first-order deformation of the fiber of π_α at $s \in S_\alpha$ must be captured by some tangent to S_α at s. The idea is that these $\{S_\alpha\}$ should be étale over the ideal moduli space (which cannot exist). This can be expressed by the assertion that, for any diagram

$$\begin{tikzcd}
\mathrm{Spec}(\Gamma/I) \arrow[r] \arrow[d, hook] & S_\alpha \arrow[d] \\
\mathrm{Spec}(\Gamma) \arrow[r] \arrow[ur, dashed] & \mathcal{M}_{1,1}
\end{tikzcd}$$

with Γ an Artin $\mathbb{C}$-algebra, I an ideal in Γ such that $I^2 = 0$, the dotted arrow can be filled in uniquely to make the diagram commute. Precisely, this says that for any family of elliptic curves $\pi \colon C \to \mathrm{Spec}(\Gamma)$, any morphism $\bar{f} \colon \mathrm{Spec}(\Gamma/I) \to S_\alpha$, and any isomorphism of families $\bar{\vartheta} \colon C \otimes_\Gamma \Gamma/I \to \mathrm{Spec}(\Gamma/I) \;{}_{\bar{f}}\!\times_{\pi_\alpha} C_\alpha$, there is a unique morphism $f \colon \mathrm{Spec}(\Gamma) \to S_\alpha$ lifting $\bar{f}$ and a unique isomorphism of families $\vartheta \colon C \to \mathrm{Spec}(\Gamma) \;{}_f\!\times_{\pi_\alpha} C_\alpha$ lifting $\bar{\vartheta}$.[8]

A modular family $\{\pi_\alpha \colon C_\alpha \to S_\alpha\}$ can be called a *covering* if every elliptic curve is isomorphic to some fiber of some π_α. (This makes $\{S_\alpha\}$ an étale covering of the nonexistent moduli space.)

We will see how a covering modular family determines an algebraic groupoid, which in fact is an atlas for the moduli stack $\mathcal{M}_{1,1}$.

To construct modular families, we need a few facts from the theory of elliptic curves, as found say in [**85**] when the base is a point, supplemented by [**19**] or [**48**] for families. Any elliptic curve can be embedded in the projective plane, with its chosen origin taken to the point $[0 : 1 : 0]$, and with an equation $y^2 z = x^3 + Axz^2 + Bz^3$, with A and B complex numbers such that the form on the right vanishes at three distinct points in

[8]This condition makes S_α what is called a universal deformation space at each of its points. Note that the first-order deformations of an elliptic curve E are parametrized by $H^1(E, \mathcal{T}_E) = H^1(E, \mathcal{O}_E)$, which is 1-dimensional. That each S_α must be smooth and 1-dimensional therefore follows from the lifting property to be a modular family, as it identifies complete local rings on S_α at $\mathbb{C}$-points with universal deformation rings for the fibers of π_α.

the projective line. We write this in affine coordinates:

$$y^2 = x^3 + Ax + B, \qquad (4A^3 + 27B^2 \neq 0).$$

This is a family over $E \subset \mathbb{P}^2 \times W$ over $W = \{(A, B) \in \mathbb{C}^2 \mid 4A^3 + 27B^2 \neq 0\}$, called a *Weierstrass family*.[9] The j-invariant is given by[10]

$$j = 1728 \cdot \frac{4A^3}{4A^3 + 27B^2}.$$

All j-invariants (elliptic curves) occur in this family, but it is 2-dimensional, so it cannot be a modular family. We will look at some 1-dimensional families by restricting this to various lines, first a diagonal line, and then a horizontal and a vertical line.

The family

$$C_0 : \qquad y^2 = x^3 + \frac{27}{4} \cdot \frac{j}{1728 - j}(x + 1)$$

over $S_0 = \mathbb{A}^1 \smallsetminus \{0, 1728\}$ has the virtue that the j-invariant of the fiber over j is j. However, this family cannot be extended smoothly across the two deleted points. In fact, any modular family that contains a curve with j-invariant 0 or 1728 must have curves with nearby j-invariants appearing multiple times. This is a hint of the "stackiness" of this moduli problem: 0 and 1728 are precisely the j-invariants of the elliptic curves which possess additional automorphisms besides the identity and the involution $p \mapsto -p$.

Consider the family $C_1 \to S_1$ with

$$C_1 : \qquad y^2 = x^3 + Ax + 1$$

and $A \in S_1 = \{A \in \mathbb{A}^1 \mid 4A^3 + 27 \neq 0\}$. This attains every j-invariant except $j = 1728$. The family $C_2 \to S_2$ with

$$C_2 : \qquad y^2 = x^3 + x + B$$

with $B \in S_2 = \{B \in \mathbb{A}^1 \mid 4 + 27B^2 \neq 0\}$ attains every j-invariant but $j = 0$.

Exercise 1.8. These two families satisfy the conditions to be modular families.

Together, these two families form a covering modular family.

Exercise 1.9. Show that the morphism from $S_1 \amalg S_2$ to the affine line given by the j-invariant is unramified except over 0 and 1728, and show that the ramification index is 3 over $j = 0$ and 2 over $j = 1728$.

To make an atlas, we want to glue S_1 and S_2, and we must keep track of where an elliptic curve appears in both families. In the stack world, we don't just take an equivalence relation on $S_1 \amalg S_2$; rather, we keep track of automorphisms. That is, for α and β in $\{1, 2\}$, we consider the scheme $R_{\alpha,\beta}$ that parametrizes isomorphisms between C_α and C_β. Loosely speaking,

$$R_{\alpha,\beta} = \{(u, v, \phi) \mid u \in S_\alpha, v \in S_\beta, \text{ and } \phi\colon (C_\alpha)_u \xrightarrow{\simeq} (C_\beta)_v\}.$$

[9]In fact, any family $C \to S$ of elliptic curves is, locally in the Zariski topology, isomorphic to the pullback of $E \to W$ by a morphism from S to W.

[10]In [**71**] Mumford replaces j by $1728 - j$.

There are projections $s\colon R_{\alpha,\beta} \to S_\alpha$, taking (u, v, ϕ) to u, and $t\colon R_{\alpha,\beta} \to S_\alpha$, taking (u, v, ϕ) to v. Define

$$U = S_1 \amalg S_2$$

and take R to be the disjoint union of these four $R_{\alpha,\beta}$:

$$R = R_{1,1} \amalg R_{1,2} \amalg R_{2,1} \amalg R_{2,2}.$$

Then we have maps s and t from R to U. The multiplication m comes by composing the isomorphisms, taking $(u, v, \phi) \times (v, w, \psi)$ to $(u, w, \psi \circ \phi)$. The identity e takes $u \in S_\alpha$ to (u, u, id), where id is the identity map on $(C_\alpha)_u$; and the inverse i takes (u, v, ϕ) to (v, u, ϕ^{-1}). It is a straightforward exercise to verify that this forms a groupoid $R \rightrightarrows U$. This will be an atlas for the stack $\mathcal{M}_{1,1}$.

If two elliptic curves are given in Weierstrass form, $y^2 = x^3 + Ax + B$ and $y^2 = x^3 + A'x + B'$, it is a general fact that any isomorphism between them must be of the form $(x, y) \mapsto (\lambda x, \mu y)$ for some λ, $\mu \in \mathbb{C}^*$ (see [**85**], §III.3, and see [**19**], §1 for the version in families). So, for instance, we can express $R_{1,1}$ as the scheme consisting of all $\{(A, A', \lambda, \mu)\}$ such that

$$\mu^2 x^3 + \mu^2 Ax + \mu^2 = \lambda^3 x^3 + \lambda A' x + 1.$$

In particular, $\mu^2 = 1$ and $\lambda^3 = 1$; setting $\gamma = \mu/\lambda$ we have $A' = \gamma^4 A$ and γ can be any sixth root of unity. Let ϕ_γ denote the map $(x, y) \mapsto (\gamma^2 x, \gamma^3 y)$. Then

$$R_{1,1} \cong S_1 \times \mu_6$$

by associating $(A, \gamma^4 A, \phi_\gamma)$ in $R_{1,1}$ to (A, γ) in $S_1 \times \mu_6$.

Exercise 1.10. Deduce, in a similar fashion, that $R_{2,2}$ is isomorphic to $S_2 \times \mu_4$ and that $R_{1,2}$ and $R_{2,1}$ can each be identified with the complement of 13 points in the affine line. (In fact the isomorphisms of curves can all be expressed conveniently in terms of the ϕ_γ.)

We want to see how this groupoid $R \rightrightarrows U$ can tell us about moduli of elliptic curves, i.e., about the category $\mathcal{M}_{1,1}$. We have a family $C \to U$, with $C = C_1 \amalg C_2$, and this contains every elliptic curve at least once. For any S and any map $\phi\colon S \to U$, we can pull back this family $C \to U$ to get a family on S, namely $C \times_U S \to S$. However, this fails two basic criteria to be a universal family:

(1) Two different maps $\phi_1\colon S \to U$ and $\phi_2\colon S \to U$ may determine isomorphic families on S.

(2) Some families over S may not be pullbacks from any morphisms from S to U.

As far as (1) is concerned, an isomorphism from the first pullback to the second determines (and is determined by) a morphism $\psi\colon S \to R$ that takes a point s to the given isomorphism from $C_{\phi_1(s)}$ to $C_{\phi_2(s)}$. In short, we have

$$\psi\colon S \to R \quad \text{with} \quad s \circ \psi = \phi_1 \quad \text{and} \quad t \circ \psi = \phi_2.$$

An extreme example of this occurs with $S = R$, $\phi_1 = s$, $\phi_2 = t$, in which case ψ is the identity.

Taking $S = S_0 = \mathbb{A}^1 \smallsetminus \{0, 1728\}$, the family $C_0 \to S$ is an example of the failure of (2): it is not the pullback from any map from S to U. However, it is *locally* a pullback: near any j in S, there is a disk Δ containing j and a morphism $\Delta \to U$ so that the restriction of the family to Δ is isomorphic to the pullback of the family $C \to U$. This works in the analytic category, but not in the algebraic category, if one uses the Zariski topology. Indeed, the only nonempty Zariski open sets in S are the complements of finite sets. But one can find a variety S', with a surjective morphism $\rho\colon S' \to S$, which is locally an analytic isomorphism — this makes it *étale* — together with a morphism $\phi\colon S' \to U$, with an isomorphism ϑ from the pullback of $C_0 \to S$ to S' via ρ with the pullback of $C \to U$ via ϕ.

Exercise 1.11. Show that $S' = \{\, a \in \mathbb{A}^1_{\mathbb{C}} \mid a \neq 0,\ 4a^6 + 27 \neq 0 \,\}$ is such a variety (with family $y^2 = x^3 + a^2x + 1$), and with the map $\rho\colon S' \to S$ given by $a \mapsto 1728 \cdot 4a^6/(4a^6 + 27)$, and $\phi\colon S' \to S_1 \subset U$ given by $a \mapsto A(a) = a^2$.

For such a "covering" $\rho\colon S' \to S$ (or S' a disjoint union of disks in the analytic case), we have a groupoid $S' \times_S S' \rightrightarrows S'$. For any point (s', s'') in $S' \times_S S'$, with s their common image in S, we have isomorphisms $C_{s'} \cong (C_0)_s \cong C_{s''}$.

Exercise 1.12. Show that these fiberwise isomorphisms are given by a (unique) global isomorphism of $(\phi \circ p_1)^*(C)$ with $(\phi \circ p_2)^*(C)$ on $S' \times_S S'$. This defines a morphism Φ from $S' \times_S S'$ to R with $s \circ \Phi = \phi \circ p_1$ and $t \circ \Phi = \phi \circ p_2$. Show that (ϕ, Φ) determines a morphism from the groupoid $S' \times_S S' \rightrightarrows S'$ to the groupoid $R \rightrightarrows U$.

Note that $S' \times_S S' \rightrightarrows S'$ is an atlas for S, and $R \rightrightarrows U$ is supposed to be an atlas for $\mathcal{M}_{1,1}$, so the groupoid morphism of the exercise can be regarded as a geometric realization of the morphism from (the stack corresponding to) S to the stack $\mathcal{M}_{1,1}$.

This picture can be reversed. Given an étale surjective map $\rho\colon S' \to S$, and a morphism (ϕ, Φ) from $S' \times_S S' \rightrightarrows S'$ to the groupoid $R \rightrightarrows U$, one gets a family $\phi^*(C)$ of elliptic curves on S' and an isomorphism $p_1^*(\phi^*C) \xrightarrow{\simeq} p_2^*(\phi^*C)$ on $S' \times_S S'$, satisfying a compatibility identity on $S' \times_S S' \times_S S'$. It is the theory of *descent* that implies that such a family is the pullback of a family on S (that is moreover unique up to unique isomorphism).

This example contains another fundamental insight of Grothendieck: Zariski open coverings $\{V_\alpha\}$ of a variety or scheme S should be replaced not just by $\coprod V_\alpha \to S$, but by arbitrary collections of étale morphisms $V_\alpha \to S$ whose (Zariski open) images cover S. When the base category is a category of schemes, the topology we will usually use — the *étale topology* — has these étale maps as its basic open sets.

Also by the theory of descent, the determination of $\mathrm{Pic}(\mathcal{M}_{1,1})$ can be reduced to the concrete computation of the the group of line bundles L on U equipped with isomorphisms $\varphi\colon s^*L \xrightarrow{\simeq} t^*L$ on R such that φ satisfies a natural compatibility condition on $R\ {}_t\!\times_s R$, up to isomorphism of such pairs (L, φ). The latter group is described by a finite amount of data: U has only trivial line bundles (since its components are Zariski open subsets of the affine line), and an isomorphism $s^*L \xrightarrow{\simeq} t^*L$ then given by an invertible function on R. So, $\mathrm{Pic}(\mathcal{M}_{1,1})$ is the quotient of the group of elements of $\mathcal{O}^*(R)$ satisfying the compatibility condition on $R\ {}_t\!\times_s R$ by the subgroup of elements of the form $t^*\chi/s^*\chi$, with $\chi \in \mathcal{O}^*(U)$. A tedious calculation yields an isomorphism $\operatorname{Pic}\mathcal{M}_{1,1} \cong \mathbb{Z}/12\mathbb{Z}$. This calculation will be carried out (using slightly different atlases) in Chapter 12.

In the analytic category, one has a modular family $E \to \mathbb{H}$ of elliptic curves over the upper half plane $\mathbb{H}$ whose fiber over τ in $\mathbb{H}$ is the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$, with Λ_τ the lattice $\mathbb{Z} + \mathbb{Z}\cdot\tau$. An isomorphism from E_τ to $E_{\tau'}$ is given by multiplication by a unique complex number ϑ such that $\vartheta \cdot \Lambda_\tau = \Lambda_{\tau'}$. A corresponding atlas is the groupoid $R \rightrightarrows \mathbb{H}$, where $R = \{(\tau, \tau', \vartheta) \in \mathbb{H} \times \mathbb{H} \times \mathbb{C} \mid \vartheta \cdot \Lambda_\tau = \Lambda_{\tau'}\}$. In fact, Mumford uses this analytic modular family in [**71**] to give a calculation of $\operatorname{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$ in the analytic category.

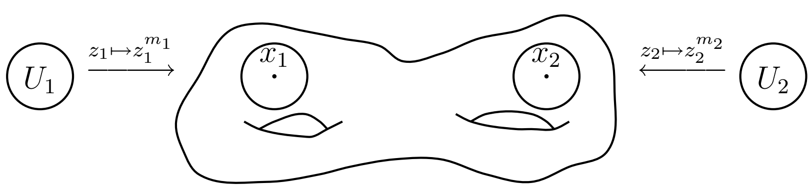
Exercise 1.13. Show that this analytic groupoid $R \rightrightarrows \mathbb{H}$ is isomorphic to the transformation groupoid $SL_2(\mathbb{Z}) \ltimes \mathbb{H}$ coming from the standard action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

6. Orbifolds

Orbifolds, sometimes called V-manifolds, provide another good introduction to some of the notions involved with stacks. In fact, the moduli stack of triangles, or any situation where a finite group acts on a manifold, gives rise to an orbifold. An orbifold is often described as a space that is locally a quotient of a manifold by a finite group, but this description is too crude: to give an orbifold, one must describe these local group actions, at least up to some equivalence. We will see that this extra data amounts to the difference between an ordinary space and a stack. (In fact, the underlying space corresponds to the coarse moduli space of the stack.)

As a simple example, let X be a Riemann surface, and let $x_1, \ldots, x_n$ be a finite set of points of X, and let $m_1, \ldots, m_n$ be positive integers, each greater than or equal to 2. Take a neighborhood V_i of x_i biholomorphic to a disk, and choose an isomorphism $V_i \cong U_i/G_i$, where U_i is a disk, and G_i is the cyclic group of m_i^{th} roots of unity, acting by rotation; take all the neighborhoods V_i to be disjoint.



At any other point x of X, choose any neighborhood of x biholomorphic to a disk and not containing any of the points x_i. These data determine an orbifold structure on the Riemann surface. Although the underlying (coarse) space is the original surface X, the orbifold structure is different, at any point x_i with $m_i > 1$. (See [**69**] for more on these Riemann surface orbifolds.)

For an explicit example, let $X = S^2 = \mathbb{C} \cup \{\infty\}$, with one point $p_1 = \infty$, and with $m_1 = m$. This is sometimes called the *m-teardrop*.

We turn next to a precise definition of an orbifold, following Haefliger [**43**], §4. (Compare Kawasaki's variation [**49**] of Satake's original [**80**].) We will define a complex analytic orbifold, although similar constructions work in other categories, cf. [**70**].

One starts with a topological space X. The data to give an orbifold structure to X consists of an open covering $\{V_\alpha\}$ of X, together with homeomorphisms $V_\alpha \cong G_\alpha \backslash U_\alpha$, where U_α is a connected complex manifold (usually taken to be an open set in $\mathbb{C}^n$), G_α is a finite group of analytic automorphisms of U_α, and $G_\alpha \backslash U_\alpha$ denotes the set of orbits, with the quotient topology inherited from U_α. (The action of G_α on U_α is assumed to be effective, i.e., $G_\alpha \subset \mathrm{Aut}(U_\alpha)$.) This data must satisfy the following compatibility condition: if $u \in U_\alpha$, and $u' \in U_\beta$ map to the same point in X, there must be neighborhoods W of u in U_α, and W' of u' in U_β, and a complex analytic isomorphism $\varphi\colon W \to W'$ taking u to u' and commuting with the projections to X:

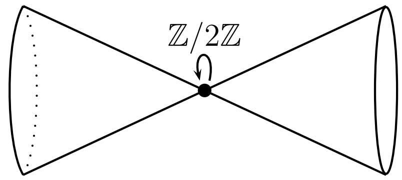
$$\begin{array}{ccccc} U_\alpha \supset W & \xrightarrow{\varphi} & W' \subset U_\beta \\ & \searrow \quad \swarrow & \\ & V_\alpha \subset X \supset V_\beta & \end{array}$$

Note that these germs are not part of the data for the orbifold, and they need not be unique; rather, their existence is a condition on the data. The same idea defines when two such data are compatible: the orbifold structure defined by the open covering $\{V'_\alpha\}$ with $V'_\alpha \cong G'_\alpha \backslash U'_\alpha$ is compatible if, whenever $u \in U_\alpha$ and $u' \in U'_{\alpha'}$ map to the same point in X, there exist neighborhoods of each point and a complex analytic isomorphism commuting with the projections to X. An **orbifold structure** on X is an equivalence class of orbifold data, where compatible data are called equivalent.

Each of the quotients $G_\alpha \backslash U_\alpha$ has the structure of complex analytic space in which the analytic functions on $V \subset V_\alpha$ are precisely the G_α-invariant analytic functions on the pre-image of V in U_α (cf. [**17**], §4). The structure sheaves of the V_α can be patched canonically so that X inherits a complex analytic structure; it will be the "coarse" space for the corresponding stack. If X is connected, the manifolds U_α all have the same dimension, called the *dimension* of the orbifold.

Given orbifold data on X, one can construct an analytic groupoid, as follows. Set $U = \coprod U_\alpha$, and set R to be the set of triples (u, u', φ), where u and u' are points in U with the same image in X, and φ is a germ of an isomorphism from a neighborhood of u to a neighborhood of u' over X. This R has a unique topology so that the two projections s and t from R to U (taking (u, u', φ) to u and u' respectively) are local homeomorphisms; this gives R the structure of a complex manifold. The other maps are easily defined: $e\colon U \to R$ takes u to (u, u, id), $i\colon R \to R$ takes (u, u', φ) to (u', u, φ^{-1}), and $m\colon R\, {}_t{\times}_s\, R \to R$ takes $(u, u', \varphi) \times (u', u'', \psi)$ to $(u, u'', \psi \circ \varphi)$. We will be able to regard an orbifold as a stack by means of this atlas.

The simplest example of orbifold is a finite group quotient. Here U is a manifold, G is a finite group with an effective action on U, and V is the quotient space $G \backslash U$. In this case R can be identified with $G \times U$ and we recover the transformation groupoid $G \ltimes U$. For instance, if $U = \mathbb{C}^2$ and $G = \mathbb{Z}/2\mathbb{Z}$, with the action of its generator given by $(x, y) \mapsto (-x, -y)$, then the quotient (analytic) space V is a quadric cone, isomorphic to the locus in $\mathbb{C}^3$ defined by the equation $uv = w^2$. In this case the orbifold quotient can be pictured as follows.



At the vertex of the cone there is a nontrivial orbifold structure (indicated by the arrow); the complement is a manifold.

EXERCISE 1.14. Construct a groupoid for the m-teardrop. Take $U = U_1 \coprod U_2$, with $U_1 = \mathbb{C}$ and $U_2 = D$ an open disk mapping to a neighborhood of ∞ by $z \mapsto 1/z^m$. Compute R, and s, t, m, e, and i.

Note that the canonical map from R to $U \times U$ is never injective, unless all the maps $U_\alpha \to X$ are local homeomorphisms, in which case X is a manifold with its trivial orbifold structure.

EXERCISE 1.15. For any u in U, the automorphism group $\mathrm{Aut}(u) = s^{-1}(u) \cap t^{-1}(u)$ is canonically isomorphic to the isotropy group $(G_\alpha)_u = \{g \in G_\alpha \mid g \cdot u = u\}$, if u is in U_α. The canonical morphism $R \to U \times U$ is injective if and only if all the isotropy groups are trivial.

For a point u in U_α, write G_u for the isotropy group $(G_\alpha)_u = \{g \in G_\alpha \mid g \cdot u = u\}$. Given one germ φ from u to u', over a point x in X, with $u \in U_\alpha$ and $u' \in U_\beta$, the other possible germs have the form $\varphi \circ g$, where g is in the isotropy group G_u; they also have the form $g' \circ \varphi$ for g' in $G_{u'}$. Fixing one such φ determines an isomorphism from G_u to $G_{u'}$, sending g to g' when $\varphi \circ g = g' \circ \varphi$. This means that one can assign an *isotropy group* G_x for each point x in X, defined to be G_u for any point u that maps to x. This group is determined only up to (inner) isomorphism, since changing φ gives another isomorphism of G_u with $G_{u'}$ differing by an inner automorphism. In fact, the map $g \mapsto g_*$, where g_* is the induced endomorphism of the tangent space $T_u U_\alpha \cong \mathbb{C}^n$, gives an embedding $G_u \hookrightarrow GL_n(\mathbb{C})$ (see [**17**], §4), so we have an embedding of G_x in $GL_n(\mathbb{C})$, unique up to conjugacy.

It is a general fact (cf. [**80**], p. 475), that any connected orbifold can be written globally as a quotient of a manifold M by a Lie group G, in fact, with $G = GL_n(\mathbb{C})$ in this complex case, with n the dimension of the orbifold. Let us work this out in the language of groupoids. Let $P_\alpha \to U_\alpha$ be the bundle of frames, with fiber over $u \in U_\alpha$ being the set of bases of the tangent space $T_u U_\alpha$. This is a principal right G-bundle, with action of $g = (g_{ij})$ on a frame $v = (v_1, \ldots, v_n)$ by $(v \cdot g)_i = \sum_j v_j g_{ji}$. The group G_α acts on the left on P_α, by $(\tau \cdot v)_i = \tau_*(v_i)$. This action is free, and commutes with the action of G. Therefore the quotient $M_\alpha = G_\alpha \backslash P_\alpha$ is a manifold, and G acts in the right on M_α. Let $\rho_\alpha \colon M_\alpha \to V_\alpha$ be the canonical projections. The orbifold data determine gluing maps from $\rho_\alpha^{-1}(V_\alpha \cap V_\beta)$ to $\rho_\beta^{-1}(V_\alpha \cap V_\beta)$, taking the class of a frame v to the class of the frame $\varphi_*(v)$, for any choice of local germ φ. These gluing data commute with the action of G, so we obtain a manifold M with a right action of $G = GL_n(\mathbb{C})$, and a projection from M to X that is constant on orbits.

To say that the orbifold is the same as the quotient $[M/G]$, we should compare the groupoid $M \times G \rightrightarrows M$ with the groupoid $R \rightrightarrows U$ defining the orbifold structure.

EXERCISE 1.16. Let $P = \coprod_\alpha P_\alpha$, with canonical projection $\pi \colon P \to U$. Let $Q = \{(v, v', \varphi) \mid v, v' \in P, \varphi$ a germ from $\pi(v)$ to $\pi(v')\}$. (a) Construct a groupoid $Q \rightrightarrows P$, with s and t taking (v, v', φ) to v and v' respectively, and $m((v, v', \varphi), (v', v'', \psi)) = (v, v'', \psi \circ \varphi)$. (b) Construct a morphism from $Q \rightrightarrows P$ to $R \rightrightarrows U$, taking (v, v', φ) to $(\pi(v), \pi(v'), \varphi)$, and verify that it satisfies Conditions 1.3(i)–(ii). (c) Construct a morphism from $Q \rightrightarrows P$ to $M \times G \rightrightarrows M$, taking (v, v', φ) to (v, g), where g is determined by the equation $v'_i = \sum_j \varphi_*(v_j) g_{ji}$, and show that this morphism satisfies the same two conditions.

The local charts on an orbifold are used to do analysis (see [**7**] and [**34**]). For example, a differential form is given by a compatible collection of G_α-invariant differential forms ω_α on U_α. In terms of the groupoid, this is a differential form ω on U such that $s^*(\omega) = t^*(\omega)$ on R. In fact, groupoids provide a useful setting for much of the study of orbifolds (see [**33**]).

It should perhaps be pointed out that some authors also use a more restricted notion of orbifold, where the groups G_α are not allowed to include any complex reflections (i.e. isomorphisms conjugate to those of the form $(z_1, \ldots, z_n) \mapsto (\zeta z_1, \ldots, z_n)$, where ζ is a root of unity, cf. [**78**]); in this case the coarse space X actually determines the orbifold. This rules out orbifold structures like the one we gave on a Riemann surface at the beginning of this section, however. We have seen a similar phenomenon for elliptic curves, where the j-line is a coarse moduli space, but the stack "remembers" the automorphisms of the elliptic curves.

The definition we have given here works also for differentiable or topological orbifolds, by replacing the word "complex analytic" by "differentiable" or "continuous", cf. [**70**]. One can give a corresponding definition in algebraic geometry, although here one must use étale neighborhoods to describe a notion of germ of an isomorphism. There are more general notions of orbifolds, cf. [**84**], where it is not required that the action of each G_α on U_α be effective. Both of these notions can be described more easily in the language of stacks.

7. Schemes, Functors, and Stacks

Before stacks, an approach to the study of families of algebraic objects was to consider a contravariant functor h from the base category of schemes $\mathcal{S}$ to the category of sets, with $h(S)$ being the set of isomorphism classes of families over S. For example, for $\mathcal{M}_g$, $h(S)$ was the set of families of curves $C \to S$, modulo isomorphism. The functor h is *representable* if there is a scheme X such that the functor h is naturally isomorphic to the functor of points (see Example 1.1A) h_X. One of the best known and most important examples of this is the functor that assigns to a scheme S the set of closed subschemes of $\mathbb{P}^n \times S$, flat over S; this is represented by a Hilbert scheme [**37**]. Most such functors, such as the one for moduli of curves, are not representable.

This approach is consistent with Grothendieck's idea of identifying a scheme with its functor of points (see Example 1.1A). Schemes, which generalize algebraic varieties, sit inside a larger category of (certain) functors, the top line in the following diagram of algebraic objects:

$$\begin{aligned}&\text{Algebraic varieties} \subset \text{Schemes} \subset \text{Algebraic spaces}\\ &\qquad\qquad\qquad\qquad \subset \text{Deligne–Mumford stacks} \subset \text{Artin stacks}\end{aligned}$$

Though they won't play a such a major role in this book, *algebraic spaces* are these functors [**4**] [**56**]; they form a class of algebraic objects which generalize schemes.[11]

The examples in this chapter have emphasized the point that geometric problems can lead to categories. These make up the bottom line of the diagram, the *algebraic stacks*. There are two different sets of axioms which make categories suitable for doing geometry. The focus of Part I of this book will be on *Deligne–Mumford stacks*, introduced by Deligne and Mumford in their stack-based proof of the irreducibility of moduli spaces of curves of genus g over arbitrary base fields [**20**]. In Part II of this book we will meet the more general *Artin stacks* [**5**].

Stabilizer groups were an important feature in the discussion surrounding the examples. In moduli problems, the stabilizer groups are the automorphism groups of the objects being parametrized. The main novelty of stacks, as opposed to varieties, schemes, or spaces, is the presence of nontrivial stabilizer groups (in fact, it will be shown that an algebraic stacks with no nontrivial stabilizers must be isomorphic to a scheme or an algebraic space). The distinction between Deligne–Mumford stacks and Artin stacks lies in the kind of stabilizer groups that are permitted. The stabilizer group at a geometric point of a Deligne–Mumford stack is always a finite group, whereas an Artin stack may have an arbitrary algebraic group (finite-type group scheme) as a geometric stabilizer. Stabilizer groups thus provide the answer to the question, "What makes something a stack and not a scheme or an algebraic space?"

Many of the examples presented in this chapter will end up being algebraic stacks. The stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}_g}$ are famous examples of Deligne–Mumford stacks. Stacks $[X/G]$, described in Example 1.1B in a topological setting, will be algebraic stacks when X is a scheme and G is an algebraic group; they are Artin stacks in general, and whether they are Deligne–Mumford stacks depends on what sorts of stabilizer groups they have. Conics form an Artin stack (positive-dimensional stabilizer groups), while $\mathcal{M}_{1,1}$ is an important example of a Deligne–Mumford stack (finite stabilizer groups). If a nonsingular complex variety is endowed with an orbifold structure, then it gives rise to a Deligne–Mumford stack. We will come back to these examples repeatedly throughout this book. Especially in the early chapters, this core collection of examples will serve as a counterbalance to the abstractions required in order to reach an answer to the question, "What is an algebraic stack?"

[11]There is actually a condition for a scheme to be an algebraic space: it must be *quasi-separated*, i.e., have quasi-compact diagonal (see the Glossary). This is really only a technical condition, since the schemes that one meets in practice are always quasi-separated.

Answers to Exercises

1.1. One axiom states that the composition of two arrows has the source of the first arrow as source and the target of the second arrow as target: $s \circ m = s \circ \mathrm{pr}_1$ and $t \circ m = t \circ \mathrm{pr}_2$ as maps $R \,{}_t\!\times_s R \to U$.

1.2. (a) $D \times S^1 \to X$, $(z, e^{i\vartheta}) \mapsto (z, \vartheta)$ is a 2-sheeted covering map, so satisfies the surjectivity requirement. (b) On X the group $\mathrm{Aut}(x)$ is trivial for x not on the central line, and it is $\{\pm 1\}$ for $x = (0, \vartheta)$.

1.3. The key fact is that $\widetilde{Y} \to Y$ and $\widetilde{T} \to T$, restricted to the pre-images of T°, are local homeomorphisms. With this, one sees that $Y^\circ \to T^\circ$ is a family of triangles. Any family of triangles with sides of distinct lengths is fiberwise uniquely identified with the pullback of $Y^\circ \to T^\circ$. That this gives a homeomorphism of families can be checked locally; locally we can make a choice of ordering of the vertices and argue as in the case of the universality property for $\widetilde{Y} \to \widetilde{T}$.

1.4. The commutative diagram

$$\begin{tikzcd}
 & (U_\alpha \cap U_\beta \cap U_\gamma) \,{}_{\phi_\alpha}\!\times_V Z \arrow[dr, "{(p,\Phi_{\gamma\alpha}(p)\cdot z)}"] \arrow[dd, "{(p,\Phi_{\beta\alpha}(p)\cdot z)}"] & \\
 & & (U_\alpha \cap U_\beta \cap U_\gamma) \,{}_{\phi_\gamma}\!\times_V Z \\
 & (U_\alpha \cap U_\beta \cap U_\gamma) \,{}_{\phi_\beta}\!\times_V Z \arrow[ur, "{(p,\Phi_{\gamma\beta}(p)\cdot z)}"] & \\
X|_{U_\alpha \cap U_\beta \cap U_\gamma} \arrow[uuur, "\sim"] \arrow[ur, "\sim"] \arrow[uurr, bend right, "\sim"'] & &
\end{tikzcd}$$

yields the crucial identity $\Phi_{\gamma\alpha}(p) = \Phi_{\gamma\beta}(p) \cdot \Phi_{\beta\alpha}(p)$.

1.5. The map of groupoids from $G \times \widetilde{V} \times \mathfrak{S}_3 \rightrightarrows \widetilde{V}$ to $G \times V \rightrightarrows V$ is given by $\phi(\tilde{v}) = v$ and $\Phi(g, \tilde{v}, \pi) = (g, v)$, where $g \in G$ and $\tilde{v} \in \widetilde{V}$, with v the triangle having vertices $\tilde{v}$. Given $(g, v) \in G \times V$, a point in $\widetilde{V} \times \widetilde{V}$ lying over $(v, g \cdot v) \in V \times V$ is determined by choosing an ordering $\tilde{v}$ of the vertices of v, and a re-ordering $\pi \in \mathfrak{S}_3$ of $g \cdot \tilde{v}$, hence Condition 1.3(i) is fulfilled. The map $\widetilde{V} \to V$ is a topological covering map; hence Condition 1.3(ii) is satisfied by choosing the identity element (e, v) of $G \times V$.

1.6. For the category $\mathfrak{T}$, take the same objects, but for morphisms allow the induced maps on fibers to be isometries followed by homotheties (multiplications by a positive scalar). Replace $\widetilde{T}$ by its intersection with the plane $a + b + c = 1$, and enlarge G by allowing homotheties. The resulting stack has dimension 2.

1.7. Under the substitution $s^2 = t$, sections correspond to maps $g\colon \mathbb{A}^1 \smallsetminus \{0\} \to E$ satisfying $g(-s) = -g(s)$, but g must be a constant map, so sections are in bijective correspondence with 2-torsion points of E.

1.8. That $C_1 \to S_1$ is modular can be shown using the criterion on p. 17: for any c in Γ, denote by $\bar{c}$ the image in Γ/I. Any elliptic curve over Γ has the form $y^2 = x^3 + ax + b$ for some a and b in Γ (see [**19**], §2). An isomorphism of $y^2 = x^3 + \overline{A}x + 1$ with $y^2 = x^3 + \bar{a}x + \bar{b}$ over Γ/I has the form $(x, y) \mapsto (\bar{\lambda}x, \bar{\mu}y)$ for units $\bar{\lambda}$ and $\bar{\mu}$ in Γ/I; these satisfy the equations $\bar{\lambda}^3 = \bar{\mu}^2 = \bar{b}$ and $\bar{a}\bar{\lambda} = \bar{\mu}^2\overline{A}$. Using the fact that 2 and 3 are invertible in Γ, one verifies that there are unique liftings λ, μ and A of $\bar{\lambda}$, $\bar{\mu}$ and $\overline{A}$ with $\lambda^3 = \mu^2 = b$ and $a\lambda = \mu^2 A$. Then (λ, μ) determine the required isomorphism of $y^2 = x^3 + Ax + 1$ with $y^2 = x^3 + ax + b$ over Γ, as required. A similar argument applies to the second family, solving equations $\lambda^3 = \mu^2 = a\lambda$ and $b = \mu^2 B$ for λ, μ and B.

1.9. For S_1, $j = 1728 \cdot 4A^3/(4A^3 + 27)$, and this has ramification index 3 over $j = 0$.

1.10.

$$\begin{aligned}
R_{1,1} &= \{(A, \lambda^2 A, \lambda, \mu) \mid A \in S_1, \lambda^3 = 1, \mu^2 = 1\};\\
R_{2,2} &= \{(B, \mu^2 B, \mu^2, \mu) \mid B \in S_2, \mu^4 = 1\};\\
R_{1,2} &= \{\rho^{-4}, \rho^6, \rho^2, \rho^3) \mid \rho \neq 0, \rho^{12} \neq -27/4\}.
\end{aligned}$$

1.11. Note that $\rho^*(C_0)$ is the family $y^2 = x^3 + a^6 x + a^6$, and an isomorphism ϑ from $\rho^*(C_0)$ to $\phi^*(C)$ is given by $(\lambda, \mu) = (a^{-2}, a^{-3})$.

1.12. The isomorphism on $S' \times_S S'$ can be constructed as the composite

$$\begin{tikzcd}
(\phi \circ p_1)^* C \arrow[d, equal] & (\rho \circ p_1)^* C_0 \arrow[r, equal] \arrow[d, equal] & (\rho \circ p_2)^* C_0 \arrow[d, equal] & (\phi \circ p_2)^* C \arrow[d, equal] \\
p_1^* \phi^* C & p_1^* \rho^* C_0 \arrow[l, "\sim"'] & p_2^* \rho^* C_0 \arrow[r, "\sim"] & p_2^* \phi^* C
\end{tikzcd}$$

1.13. An isomorphism is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \tau \mapsto (\tau, \tau', \vartheta)$, where $\tau' = \frac{a\tau + b}{c\tau + d}$ and $\vartheta = \frac{1}{c\tau + d}$. Note that although $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ have the same action on $\mathbb{H}$, the sign is determined by ϑ.

1.14. $R_{1,1} \cong U$, $R_{1,2} \cong R_{2,1} \cong D \setminus \{0\}$, $R_{2,2} \cong \mathbb{Z}/m\mathbb{Z} \times D$, with (k, z) corresponding to $(z, e^{2\pi ik/m}z, e^{2\pi ik/m})$ in $R_{2,2}$. The product on $R_{2,2}$ takes $(k, z) \times (l, e^{2\pi ik/m}z)$ to $(k + l, z)$.

1.15. For any $(u, u, \varphi) \in \mathrm{Aut}(u)$, with $u \in U_\alpha$, the given φ extends to an automorphism of U_α given by the action of unique $g \in G_\alpha$, such that $g(u) = u$.

1.16. The morphism of groupoids from $Q \rightrightarrows P$ to $R \rightrightarrows U$ is given by $\pi \colon P \to U$ and $(v, v', \varphi) \mapsto (\pi(v), \pi(v'), \varphi)$, and Condition 1.3(i) is immediate. The morphism of groupoids from $Q \rightrightarrows P$ to $M \times G \rightrightarrows M$ is given by $v \mapsto [v]$ and $(v, v', \varphi) \mapsto ([v], g)$ with g such that $\varphi_*(v) \cdot g = v'$. To verify Condition 1.3(i): say $v \in P_\alpha$ and $v' \in P_\beta$ are frames over points with the same image $x \in X$. Fix a germ φ from $u := \pi(v)$ to $\pi(v')$; for any other germ ψ the germ $\varphi^{-1} \circ \psi$ extends uniquely to the action of some $h \in (G_\alpha)_u$, and the condition now follows from the fact that $(G_\alpha)_u$ acts freely on $\pi^{-1}(u)$ with quotient $\rho_\alpha^{-1}(x)$. Since $P \to U$ and $P \to M$ have local sections (one is a bundle projection, the other is a surjective local homeomorphism), Condition 1.3(ii) is satisfied in both cases.

CHAPTER 2

Categories fibered in groupoids

This chapter sets up the first structures which will play a role in the theory of stacks. There is a base category, which for us will usually be a category of schemes. Over the base category we will consider categories where generally an object consists of an object of the base category plus some extra structure. Usually we are motivated by a moduli problem, so we could be considering a scheme S together with a geometric object such as a family of curves, on S.

We will start off by providing a host of examples of such categories to provide insight into the abstract definitions and constructions that follow. Most of these examples will end up being algebraic stacks. One feature that we will be able to observe immediately, however, is that we are always looking at objects or structures that can be pulled back along an arbitrary morphism of schemes. This statement is formalized by the notion of *fibered category*. Actually it is more important for the theory of stacks to consider a somewhat stronger notion, that of *categories fibered in groupoids*. For this, two basic axioms detailed in §2.3 assert that pullbacks of objects exist, up to a canonical isomorphism, and that these objects themselves are allowed to have additional automorphisms. It is this latter feature that makes *CFGs* (categories fibered in groupoids) well suited for the study of moduli problems. In the chapters that follow we will be developing the extra conditions to be satisfied for a CFG to be a stack, and eventually for a stack to be an algebraic stack.

1. The base category $\mathcal{S}$

We have seen that stacks are defined over a base category $\mathcal{S}$. Usually this will be a category of schemes, either all schemes (Sch) or schemes (Sch/Λ) over a fixed base scheme Λ. Often we take $\Lambda = \mathrm{Spec}(k)$, for k a field, or more generally a (commutative) base ring. This may be restricted to a smaller category, say schemes of finite type over k. For example, $\mathcal{S}$ may be taken to be the category of quasi-projective schemes over the complex numbers. For technical reasons, it is sometimes convenient to allow schemes that are merely locally of finite type over k. It is important, however, that $\mathcal{S}$ be closed under formation of arbitrary fiber products $X \times_Z Y$. In particular, one cannot limit oneself to reduced, irreducible varieties; nilpotent elements in the structure sheaves must be allowed. We write $X \times Y$ for $X \times_\Lambda Y$. All schemes will be understood to be in $\mathcal{S}$ unless otherwise stated. (See the Glossary for some basic notions about schemes and morphisms.)

We will sometimes abuse language by regarding Λ as a point, even if Λ is not Spec of a field. For example, we will say that G is an algebraic group instead of saying that G is a group scheme over Λ.

It can be useful, especially in a first reading, to take $\mathcal{S}$ to be the category of analytic spaces, where constructions are often easier, or even the category of topological spaces. Another variation is important, and is that taken in [**61**]: $\mathcal{S}$ can be the category of affine schemes. This is consistent with the point of view that any scheme or scheme-like object should be constructible from affine schemes, or that a scheme is determined by knowing all morphisms of affine schemes into it. We will not take this track, however, at least at this stage. (With this variation, one has to distinguish between schemes in $\mathcal{S}$ and general schemes.)

The base category $\mathcal{S}$ will come with a Grothendieck topology. For us this will mean that we have a notion of a **covering** of a scheme S, which is a collection of morphisms $\{U_\alpha \to S\}$, such that each point of S is in the image of at least one of these maps. (See the Glossary for precise definition.) The topologies that we may consider are: (1) the Zariski topology, where the $U_\alpha \to S$ are open embeddings; (2) the étale topology, where each of the maps is étale; (3) the smooth topology, where each map is smooth; (4) the flat topology, where each map is flat and locally of finite presentation. The relevant topology will generally be the étale topology; we will eventually see results that say we can just as well use the smooth or flat topology. (The Zariski topology is used only in examples.) A single map $U \to S$ is called a **covering map** if $\{U \to S\}$ is a covering. Any covering $\{U_\alpha \to S\}$ determines a covering map $U = \coprod U_\alpha \to S$, which can often be used in place of the covering. When we have a notion of triviality, we will say that something is trivial in the étale topology when its pullback to each U_α in such a covering is trivial; this will be equivalent to the single pullback to $U = \coprod U_\alpha$ being trivial.

In Chapters 2 and 3, in fact, the topology on $\mathcal{S}$ will be used only in some examples, for which one needs a notion of "locally trivial" in some topology. The general discussion here makes sense when $\mathcal{S}$ is any category with fiber products. The topology will come into play in a serious way in Chapter 4 when we state the definition of a stack. Only when the final axioms for a Deligne–Mumford stack are introduced in Chapter 5 will the fact that $\mathcal{S}$ is a category of schemes be used.

2. Examples

A stack is not any kind of space with some structure; rather, it is a *category*. A stack (over $\mathcal{S}$) is a category $\mathfrak{X}$ together with a functor $p\colon \mathfrak{X} \to \mathcal{S}$, satisfying some properties. A category together with a functor to another category, with an appropriate notion of pullbacks, is known as a **fibered category**. Our fibered categories will all be fibered over $\mathcal{S}$. First we look at some examples of such categories — many of which will turn out to be stacks, at least with appropriate added hypotheses (such as a condition to be locally trivial, or some stability condition). We will describe the objects and morphisms in the category $\mathfrak{X}$. Usually the compositions of morphisms will be obvious. The easy verifications that $\mathfrak{X}$ is a category, and p a functor, are left to the reader.

Example 2.1. Let X be a scheme (understood to be in $\mathcal{S}$). Then X determines a stack, which we will denote for now by $\underline{X}$. An object in the category $\underline{X}$ is a scheme S together with a morphism $f\colon S \to X$. A morphism from the object $f'\colon S' \to X$ to the object $f\colon S \to X$ is a morphism of schemes $g\colon S' \to S$ such that $f \circ g = f'$. Composites are defined in the obvious way. The functor $p\colon \underline{X} \to \mathcal{S}$ takes the object $f\colon S \to X$ to the scheme S; a morphism from $f'\colon S' \to X$ to $f\colon S \to X$ is taken to the corresponding morphism from S' to S.

One case deserves special mention. Let us consider $\mathcal{S} = (\mathrm{Sch}/\Lambda)$. When $X = \Lambda$, then $\underline{X}$ is the category $\mathcal{S}$ itself, and p is the identity functor. For then an object is a scheme S over Λ, i.e., equipped with a structure map $S \to \Lambda$, together with a morphism $f\colon S \to \Lambda$. This has to be a morphism over Λ, and that means that f must be equal to the structure map of S. In other words, an object of $\underline{\Lambda}$ is a scheme S with its structure map to Λ, i.e., an object of $\mathcal{S}$.

Example 2.2. For a nonnegative integer g, there is a category $\mathcal{M}_g$, the moduli stack of curves of genus g. The objects of $\mathcal{M}_g$ are smooth projective morphisms $\pi\colon C \to S$, whose geometric fibers are connected curves of genus g. A morphism from $\pi'\colon C' \to S'$ to $\pi\colon C \to S$ is a morphism from C' to C and a morphism from S' to S making a cartesian diagram with π' and π. If a fiber product $C_{S'} = C \times_S S'$ is fixed, this is the same as giving an isomorphism of $C' \to S'$ with $C_{S'} \to S'$. The map from $\mathcal{M}_g$ to $\mathcal{S}$ takes the family $\pi\colon C \to S$ to S, and a morphism to the constituent map $S' \to S$. Composites are defined in the evident way:

$$\begin{tikzcd} C'' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & C \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

noting that the outer diagram is cartesian if each of the inner diagrams is cartesian.

Example 2.3. Let G be an algebraic group (i.e., a group scheme over Λ). This defines a category BG, whose objects are principal G-bundles. A **principal G-bundle**, or **torsor**, is a pair of schemes S and E with a morphism from E to S and a right action $E \times G \to E$ of G on E. The trivial G-torsor over S is that with $E = S \times G$, with the right action of G on the second factor G only, and $E \to S$ the first projection. If $f\colon T \to S$ is any morphism, we have a pullback f^*E over T. This is defined by $f^*E = T \times_S E$, with induced map to T and induced action of G. We require that a G-torsor be locally trivial in the given topology on S. This means that there exists a covering map $f\colon T \to S$ such that the pullback f^*E is trivial (isomorphic to the trivial G-torsor on T). We will usually work with the étale topology, meaning f should be étale and surjective.

The category BG has these G-torsors as its objects. A morphism from a G-torsor $E' \to S'$ to a G-torsor $E \to S$ is given by a morphism $S' \to S$ and a G-equivariant morphism $E' \to E$ such that the diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is cartesian. As in Example 2.2, if a pullback of $E \to S$ by $S' \to S$ is fixed, this is the same as specifying an isomorphism of $E' \to S'$ with this pullback. Compositions and the mapping to $\mathcal{S}$ are defined as in Example 2.2.

EXAMPLE 2.4. Let $h \colon \mathcal{S} \to (\mathrm{Set})$ be any contravariant functor from our category of schemes to the category of sets. This determines a category which we denote for now by $\underline{h}$. The objects of $\underline{h}$ are pairs (S, α), with S a scheme in $\mathcal{S}$ and α an element of the set $h(S)$. A morphism from (S', α') to (S, α) is a map $\varphi \colon S' \to S$ in $\mathcal{S}$ such that $h(\varphi) \colon h(S) \to h(S')$ maps α to α'. The projection from $\underline{h}$ to $\mathcal{S}$ takes (S, α) to S.

Example 2.1 is a special case of Example 2.4. In fact, a scheme X determines a contravariant functor h_X from $\mathcal{S}$ to (Set), the functor of points $h_X(S) = \mathrm{Hom}_{\mathcal{S}}(S, X)$. Then $\underline{X}$ is the category $\underline{h}_X$.

Many of the stacks that are met in practice are variations of these four examples. Here are a few of these:

EXAMPLE 2.5. There is a category $\mathcal{M}_{g,n}$ of n-pointed curves of genus g. Its objects are smooth projective morphisms $\pi \colon C \to S$, whose geometric fibers are connected curves of genus g, together with disjoint sections $\sigma_1, \ldots, \sigma_n$. (These sections are morphisms $\sigma_i \colon S \to C$ such that $\pi \circ \sigma_i = \mathrm{id}_S$, which give n distinct points in each geometric fiber.) Morphisms are defined as in Example 2.2, with the added requirement that the sections of the first family are mapped to the sections of the second. The projection to $\mathcal{S}$ is defined as in Example 2.2.

Recall, an elliptic curve is a curve of genus 1 together with a chosen point (the identity element for the group structure). Then $\mathcal{M}_{1,1}$ is the category of elliptic curves.

EXAMPLE 2.6. Suppose an algebraic group G acts (on the right) on a scheme X. There is a category denoted $[X/G]$, whose objects are G-torsors $E \to S$ (with action $E \times G \to E$), together with an equivariant morphism from E to X. Morphisms are defined as in Example 2.3, with the additional condition that the map from E' to E must form a commutative triangle with the maps to X. The functor $p \colon [X/G] \to \mathcal{S}$ again maps $(E \to S,\ E \to X)$ to S. Note that when $X = \Lambda$, we recover the category BG, i.e., $[\Lambda/G] = BG$.

EXAMPLE 2.7. For a positive integer n, let $\mathcal{V}_n$ be the category of vector bundles of rank n. The objects are vector bundles $E \to S$, and the morphisms from $(E' \to S')$ to $(E \to S)$ are given by a cartesian diagram as in Example 2.3, that identifies $E' \to S'$ via a bundle isomorphism with a pullback bundle $E \times_S S' \to S'$. The functor to $\mathcal{S}$ takes $E \to S$ to S.

Example 2.8. For a positive integer n, let $\mathcal{C}_n$ be the category of covering spaces of degree n. An object is a finite étale morphism $X \to S$ of degree n, and a morphism from $X' \to S'$ to $X \to S$ is again a cartesian diagram.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be categories over $\mathcal{S}$. A **morphism** from $\mathfrak{X}$ to $\mathfrak{Y}$ is a functor $f\colon \mathfrak{X} \to \mathfrak{Y}$ commuting with the given functors to $\mathcal{S}$. Functors between two categories $\mathfrak{X}$ and $\mathfrak{Y}$ do not necessarily form a set; rather, they form a category. The objects are functors from $\mathfrak{X}$ to $\mathfrak{Y}$, and the morphisms are natural transformations between functors; recall that a natural transformation from f_1 to f_2 assigns to each object x in F a morphism from $f_1(x)$ to $f_2(x)$ in G, which is compatible with morphisms (see the Glossary). Given categories $\mathfrak{X}$ and $\mathfrak{Y}$, and projection functors $p\colon \mathfrak{X} \to \mathcal{S}$ and $q\colon \mathfrak{Y} \to \mathcal{S}$, we denote by $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$ the following category. The objects are functors $f\colon \mathfrak{X} \to \mathfrak{Y}$ satisfying $q \circ f = p$. The morphisms from f_1 to f_2 are natural isomorphisms from f_1 to f_2 such that, for all objects x in F, the isomorphism from $f_1(x)$ to $f_2(x)$ maps (via q) to the identity map in $\mathcal{S}$ from $p(x) = q(f_1(x))$ to $p(x) = q(f_2(x))$. A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of categories over $\mathcal{S}$ will be called an **isomorphism** if it is an equivalence of categories.

Example 2.9. Here are some examples of morphisms:

(1) A morphism $f\colon X \to Y$ of schemes determines a functor $\underline{f}\colon \underline{X} \to \underline{Y}$, that takes a scheme $S \to X$ over X to the composite $S \to X \to Y$. Conversely, if $\varphi\colon \underline{X} \to \underline{Y}$ is a functor over $\mathcal{S}$, applying φ to the identity map $X \to X$ (an object in $\underline{X}$), gives a map $f\colon X \to Y$ (the image object in $\underline{Y}$), and one verifies easily that $\varphi = \underline{f}$. In other words,

$$\mathrm{HOM}(\underline{X}, \underline{Y}) = \mathrm{Hom}_{\mathcal{S}}(X, Y).$$

(This means that the category on the left is just a set, meaning it has no maps besides identity maps.)

(2) A homomorphism $\varphi\colon G \to G'$ of algebraic groups determines a functor $BG \to BG'$ that takes a G-torsor $\pi\colon E \to S$ to the G'-torsor

$$E_{G'} = E \times^G G' = E \times G'/\{(x, \varphi(g)g') \sim (x \cdot g, g')\},$$ [1]

assuming that these quotient schemes exist (see Remark 2.17, below). This is given a right action of G' by $(x, g') \cdot h' = (x, g'h')$, and projection $E_{G'} \to S$ by $(x, g') \mapsto \pi(x)$. Note that if the pullback of $E \to S$ by a map $T \to S$ is isomorphic to the trivial bundle $T \times G \to T$, then the pullback of $E_{G'} \to S$ by the same map is isomorphic to the trivial bundle $T \times G' \to T$. In the familiar setting where $\Lambda = \mathrm{Spec}\, k$ for k a field, G an algebraic group over k, and $E \to S$ described by means of a covering $S' \to S$ and cocycle data $S' \times_S S' \to G$, then this cocycle data composed with φ serves as cocycle data for the G'-torsor $E_{G'}$.

(3) There is a canonical morphism from $\mathcal{V}_n$ to BGL_n that sends a vector bundle to its associated principal bundle of frames. A vector bundle $E \to S$ comes with a (right) GL_n-action. This induces an action on the n-fold product $E \times_S E \cdots \times_S$

[1] Here, as frequently throughout these notes, we use set-theoretic notation to describe various morphisms or compatibilities, trusting that the reader can construct the correct scheme-theoretic morphisms or commutative diagrams.

E (the diagonal action). The associated principal bundle is the open subscheme of $E \times_S E \cdots \times_S E$ of n-tuples of vectors that are linearly independent.

(4) If an algebraic group G acts on a scheme X, there is a canonical morphism from $\underline{X}$ to $[X/G]$. This takes an object $f\colon S \to X$ of X to the object with trivial torsor $S \times G \to S$, with map $S \times G \to X$ given by $(s, g) \mapsto f(s) \cdot g$.

(5) There is a canonical morphism from $\mathcal{M}_{g,n+1}$ to $\mathcal{M}_{g,n}$, that simply forgets the last section, and a morphism from $\mathcal{M}_{g,n}$ to $\mathcal{M}_g$ that forgets all the sections. The morphism from $\mathcal{M}_{g,1}$ to $\mathcal{M}_g$ can be regarded as the universal curve.

(6) There is a morphism from BG to $\mathcal{C}_n$, where $G = \mathfrak{S}_n$ is the symmetric group. This takes a G-torsor $E \to S$ to the covering $\{1, ..., n\} \times^G E \to S$.

EXERCISE 2.1. Verify that the map $BGL_n \to \mathcal{V}_n$ of (3) is an isomorphism, i.e., an equivalence of categories. Do the same for the map $B\mathfrak{S}_n \to \mathcal{C}_n$ of (6).

The above examples will be the most important for our discussions. However, we indicate next some of the many variations, a few of which will be discussed later. Some of these are related to an important goal in many moduli problems, that of constructing appropriate compactifications. Others are used to “rigidify” a given moduli problem.

EXAMPLE 2.10. A compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ ($g \geq 2$) by stable curves [**20**]. The objects are projective flat morphisms $\pi\colon C \to S$. Each geometric fiber of π must be a connected, reduced curve of arithmetic genus g, with at most nodes (ordinary double points) as singularities. There is a further stability condition, that any irreducible component of a fiber which is a nonsingular curve of genus 0 must meet other components in at least 3 points. This is a category over $\mathcal{S}$. (Eventually we will see that it is a Deligne–Mumford stack, proper over the base scheme.)

EXAMPLE 2.11. A compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. The objects are projective flat morphisms $\pi\colon C \to S$, together with n disjoint sections σ_i. Each geometric fiber of π must be a connected curve of arithmetic genus g, with at most nodes as singularities, and the n points picked out in the fiber by the sections must be nonsingular points. There is, again, a stability condition: we must have $2g + n - 2 > 0$, and any irreducible component of a geometric fiber of π that is nonsingular of genus 0 must have at least 3 markings, i.e., points at which it meets other components or points given by the sections.

EXAMPLE 2.12. Consider $\Lambda = \mathrm{Spec}(\mathbb{C})$. Let X be a smooth projective variety, and β a class in the homology group $H_2(X, \mathbb{Z})$. We have a category $\mathcal{M}_{g,n}(X, \beta)$, whose objects consist of smooth projective families of curves $\pi\colon C \to S$, together with n distinct sections σ_i as in Example 2.6, together with a morphism $\mu\colon C \to X$ with the property that, for each closed point s in S, the induced map $\mu_s\colon C_s = \pi^{-1}(s) \to X$ maps the fundamental class of the curve C_s to β, i.e., $\mu_*[C_s] = \beta$.

EXAMPLE 2.13. An important tool in quantum cohomology is Kontsevich's compactification $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of $\mathcal{M}_{g,n}, (X, \beta)$. The objects are $\pi\colon C \to S$ with σ_i as in Example 2.11, and $\mu\colon C \to X$ as in Example 2.12. In this case the stability condition is that any component that is nonsingular of genus 0 and is mapped to a point by μ

must have three markings among the points where it meets other components or the points given by the n sections.

EXAMPLE 2.14. Consider the category $\underline{\mathrm{Hilb}}_{g,r}$, whose objects consist of projective families $C \to S$ of curves as in Example 2.2 together with $N = (2r-1)(g-1)$ generating sections of the line bundle $\omega_{C/S}^{\otimes r}$, such that the induced map from C to $\mathbb{P}_S^{N-1}$ is a closed embedding; these are defined up to multiplication by scalars as in the preceding example. Here $\omega_{C/S}$ is the relative canonical line bundle, or dualizing sheaf; it is the sheaf of relative differentials $\Omega^1_{C/S}$ if C is smooth over S. We usually assume $g \geq 2$ and $r \geq 2$.

EXAMPLE 2.15. Consider smooth families of curves $\pi\colon C \to S$ over with a **Jacobi level n structure**. This is an isomorphism of each $H^1(C_s;\mathbb{Z}/n\mathbb{Z})$ with $(\mathbb{Z}/n\mathbb{Z})^{2g}$, that is symplectic, i.e., takes the cup product pairing (with values in $H^2(C_s;\mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z})$ to the canonical symplectic pairing on $(\mathbb{Z}/n\mathbb{Z})^{2g}$; these isomorphisms must vary nicely in the family, which means that they are given by a symplectic isomorphism of $R^1\pi_*(\mathbb{Z}/n\mathbb{Z})$ with the trivial sheaf $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

EXAMPLE 2.16. There is a category Qcoh, where an object of Qcoh over S in $\mathcal{S}$ is a quasicoherent sheaf $\mathcal{E}$ on S. A morphism from $\mathcal{E}'$ on S' to $\mathcal{E}$ on S over $f\colon S' \to S$ is a morphism of sheaves $\mathcal{E} \to f_*\mathcal{E}'$ on S such that the corresponding morphism $f^*\mathcal{E} \to \mathcal{E}'$ of sheaves on S' determined by adjunction is an isomorphism.

REMARK 2.17. We explain why the definition of morphism in Example 2.16 is phrased in terms of a morphism of sheaves $\mathcal{E} \to f_*\mathcal{E}'$, and not directly by means of an isomorphism isomorphism of sheaves $f^*\mathcal{E} \to \mathcal{E}'$ of sheaves on S'. The reason is that $f_*\mathcal{E}'$ is well-defined as a sheaf, while the pullback $f^*\mathcal{E}$ is only defined up to (canonical) isomorphism. It is important to be precise about what consistutes a morphism between two objects in any category; in this instance, the most convenient formulation is by means of the push-forward sheaf.

It happens quite frequently that an object of a category is defined only up to canonical isomorphism. This is the case, for instance, with fiber products in the category of schemes. It is also the case with some of the objects of HOM categories in Example 2.9. We now summarize the "fine print" concerning these examples.

In (2), the existence of the quotient scheme $E_{G'}$ is an honest mathematical requirement. It is satisfied when G and G' are affine group schemes (over the base Λ). Then the construction of $E_{G'}$, which can be achieved using descent (see Appendix A), will use the fact that $E \to S$ admits a local trivialization, making $E_{G'}$ locally a product with G'. The quotient $E_{G'}$, when it exists, is defined up to canonical isomorphism; hence (2) describes an object of $\mathrm{HOM}(BG, BG')$ up to canonical isomorphism. In (3) one could, with care, make sense of $E\times_S\cdots\times_S E$ as a well-defined scheme (Spec of a tensor product of sheaves of $\mathcal{O}_S$-algebras) and thereby obtain a particular object of $\mathrm{HOM}(\mathcal{V}_n, BGL_n)$. That this is possible is not important, and it is more natural to view the example as giving an object of $\mathrm{HOM}(\mathcal{V}_n, BGL_n)$ defined up to canonical isomorphism. Examples (4) and (6) describe objects of HOM categories up to canonical isomorphism, because of the use of fiber products and group quotients, respectively.

Example 2.18. Let X be a scheme (over a base Λ), and fix a functor $X \times -$ from (Sch/Λ) to (Sch/Λ). Then one has categories $\mathcal{V}_{X,n}$ and Coh_X, generalizing the example of vector bundles (Example 2.7). An object of $\mathcal{V}_{X,n}$ is a vector bundle of rank n on $X \times S$, with morphisms given by cartesian diagrams as usual. An object of Coh_X is a quasicoherent sheaf $\mathcal{E}$ on $X \times S$ that is finitely presented and flat over S. (This will be a coherent sheaf when X is a finite-type scheme and S is Noetherian, the case that is usually of interest.) A morphism from $\mathcal{E}'$ to $\mathcal{E}$ over $f\colon S' \to S$ is a morphism $\mathcal{E} \to (1_X \times f)_*\mathcal{E}'$ which by adjunction gives an isomorphism $(1_X \times f)^*\mathcal{E} \to \mathcal{E}'$. The functor, sending a vector bundle to its sheaf of sections, is a morphism $\mathcal{V}_{X,n} \to \mathrm{Coh}_X$. There are variants in which one imposes a stability condition.

One can also make stacks out of families of varieties of higher dimension. Important examples are principally polarized abelian varieties, $K3$ surfaces, etc. We will consider a few of these examples later.

3. CFGs over $\mathcal{S}$

The first requirement for a category $\mathfrak{X}$ with $p\colon \mathfrak{X} \to \mathcal{S}$ to be a stack is that it is a **category fibered in groupoids over** $\mathcal{S}$, which we will abbreviate to CFG, or CFG/$\mathcal{S}$. This means that the following two axioms must be satisfied:

Definition 2.19. A *category fibered in groupoids* over a base category $\mathcal{S}$ is a category $\mathfrak{X}$ with functor $p\colon \mathfrak{X} \to \mathcal{S}$ satisfying the following two axioms:

(1) For every morphism $f\colon T \to S$ in $\mathcal{S}$, and object s in $\mathfrak{X}$ with $p(s) = S$, there is an object t in $\mathfrak{X}$, with $p(t) = T$, and a morphism $\varphi\colon t \to s$ in $\mathfrak{X}$ such that $p(\varphi) = f$.

(2) Given a commutative diagram in $\mathcal{S}$

$$\begin{tikzcd} U \arrow[d, "g"'] \arrow[dr, "h"] & \\ T \arrow[r, "f"'] & S \end{tikzcd}$$

with $\varphi\colon t \to s$ in $\mathfrak{X}$ mapping to $f\colon T \to S$, and $\eta\colon u \to s$ in $\mathfrak{X}$ mapping to $h\colon U \to S$, there is a unique morphism $\gamma\colon u \to t$ in $\mathfrak{X}$ mapping to $g\colon U \to T$ such that $\eta = \varphi \circ \gamma$:

$$\begin{tikzcd} u \arrow[d, dashed, "\gamma"'] \arrow[dr, "\eta"] & \\ t \arrow[r, "\varphi"'] & s \end{tikzcd}$$

Axiom (2), applied with $U = T$, $h = f$, and $g = 1_T$, implies that the object t with $\varphi\colon t \to s$ guaranteed by the first axiom is determined up to canonical isomorphism. So Axiom (1) can be regarded as saying that pullbacks of objects exist, and Axiom (2) then tells us that these pullbacks are unique up to canonical isomorphism.

In the example $\mathcal{M}_g$, the pullback of a family $C \to S$ by $T \to S$ is the fibered product $C_T = T \times_S C \to T$ (which is unique up to canonical isomorphism). The verification of Axiom (2) comes down to the universality property of the fibered product. Given a diagram

$$\begin{tikzcd} C_U \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & C_T \arrow[r] \arrow[d] & C \arrow[d] \\ U \arrow[r] & T \arrow[r] & S \end{tikzcd}$$

with cartesian right-hand square and cartesian outer square, there is a unique dashed arrow making the left-hand square commute and making the curved arrow the composite of the top two horizontal arrows. Notice that the left-hand square must then be cartesian.

We leave it to the reader to verify, using similar reasoning, that these Axioms (1) and (2) are satisfied in each of the other examples that we have seen so far.

Exercise 2.2. Show that Axioms (1) and (2) are equivalent to (1) and

(2′) For every morphism $f\colon T \to S$ in $\mathcal{S}$, and morphisms $\varphi\colon t \to s$ and $\varphi'\colon t' \to s$ in F with $p(\varphi) = f = p(\varphi')$, there is a unique morphism $\vartheta\colon t \to t'$ in F over 1_T such that $\varphi = \varphi' \circ \vartheta$.

For an object S in $\mathcal{S}$, we denote by $\mathfrak{X}_S$ the subcategory of $\mathfrak{X}$ whose objects map to S, and whose morphisms map to the identity map 1_S. It follows from Axiom (2) that every morphism in $\mathfrak{X}_S$ is an isomorphism. (Given a morphism $\varphi\colon t \to s$ in F_S, take $u = s$, and $\eta = 1_s$ to get an inverse γ for φ.) Recall that a groupoid is a category in which every morphism is an isomorphism. This explains the terminology *category fibered in groupoids*: it follows from the axioms that the category $\mathfrak{X}_S$ is a groupoid. When two more axioms are satisfied, to be given in Chapter 4, a CFG qualifies as a full-fledged stack — which gives it the right to discard the awkward name "category fibered in groupoids." Still more will be required for a stack to be an algebraic stack, of either Deligne–Mumford or Artin type.

The next result provides the link between two notions of "S-valued points" of a stack. First, we have the fiber $\mathfrak{X}_S$ just introduced. In a moduli problem, where $\mathfrak{X}$ is a category of families of geometric objects, then $\mathfrak{X}_S$ will be the category of objects over S. But just as for schemes with its functor of points, we can consider the fibered category $\underline{S}$ and look at the category $\mathrm{HOM}(\underline{S}, \mathfrak{X})$. When $\mathfrak{X}$ is a category fibered in groupoids, these two notions are equivalent.

Proposition 2.20. *Let $\mathfrak{X}$ be a category fibered in groupoids over a base category $\mathcal{S}$. Let X be an object of $\mathcal{S}$. Then the functor from $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ to $\mathfrak{X}_X$, given by evaluation at the object $(X, 1_X)$ of $\underline{X}$, is surjective (on objects) and fully faithful. In particular it is an equivalence of categories.*

Proof. We show that the functor is fully faithul and essentially surjective. In fact, the functor is surjective. Given an object x of $\mathfrak{X}_X$, we apply Axiom (1) to every object $(S, f\colon S \to X)$ of $\underline{X}$ to obtain an associated object $x_{(S,f)}$ of $\mathfrak{X}_S$ and morphism $x_{(S,f)} \to x$ over $S \to X$. For X and the identity morphism 1_X we choose $x_{(X,1_X)} = x$. Whenever

$(T, g\colon T \to X)$ is another object of $\underline{X}$, with morphism $T \to S$ in $\underline{X}$, we obtain from Axiom (2) a unique morphism $x_{(T,g)} \to x_{(S,f)}$ making a commutative triangle with the morphisms $x_{(T,g)} \to x$ and $x_{(S,f)} \to x$. The association of the object $x_{(S,f)}$ to (S, f) and the morphism $x_{(T,g)} \to x_{(S,f)}$ to $T \to S$, is a functor from $\underline{X}$ to $\mathfrak{X}$; the verification of this uses the uniqueness assertion of Axiom (2). The functor is an object of $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ which, when evaluated at $(X, 1_X)$, produces x.

To see that the functor is fully faithful, consider a pair of objects (functors) h and h' in $\mathrm{HOM}(\underline{X}, \mathfrak{X})$. To give a morphism in $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ from h to h' is to give a morphism $h(S, f) \to h'(S, f)$ in $\mathfrak{X}_S$ for every object $(S, f\colon S \to X)$ of $\underline{X}$, such that for any object $(T, g\colon T \to X)$ in $\underline{X}$ and morphism $T \to S \to X$ in $\underline{X}$ the square

$$\begin{tikzcd}
h(T,g) \arrow[r] \arrow[d] & h(S,f) \arrow[d] \\
h'(T,g) \arrow[r] & h'(S,f)
\end{tikzcd}$$

commutes.

Set $x = h(X, 1_X)$ and $x' = h'(X, 1_X)$. Suppose that two morphisms α, $\beta\colon h \Rightarrow h'$ in the category $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ yield the same morphism $\varphi\colon x \to x'$ when evaluated at $(X, 1_X)$. Then the commutative square for $S \to X \to X$, where the second map is 1_X, yields

$$\begin{tikzcd}
h(S,f) \arrow[r] \arrow[d] & x \arrow[d, "\varphi"] \\
h'(S,f) \arrow[r] & x'
\end{tikzcd}$$

where the left-hand map is $\alpha(S, f)$ or $\beta(S, f)$. By the uniqueness assertion of Axiom (2), we have $\alpha(S, f) = \beta(S, f)$. Now let $\varphi\colon x \to x'$ be an arbitrary morphism in $\mathfrak{X}_X$; we need to exhibit a morphism $\alpha\colon h \Rightarrow h'$ in the category $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ which, when evaluted at $(X, 1_X)$, produces φ. Given an object (S, f) of $\underline{X}$, we define $\alpha(S, f)\colon h(S, f) \to h'(S, f)$, using Axiom (2), to be the unique morphism over 1_S whose composite with $h'(S, f) \to x'$ is equal to the composite $h(S, f) \to x \to x'$. Now given $T \to S$ in $\underline{X}$, we have a diagram

$$\begin{tikzcd}
h(T,g) \arrow[r] \arrow[d, "\alpha(T,g)"'] & h(S,f) \arrow[r] \arrow[d, "\alpha(S,f)"'] & x \arrow[d, "\varphi"] \\
h'(T,g) \arrow[r] & h'(S,f) \arrow[r] & x'
\end{tikzcd}$$

where the right-hand square and outer square commute. Again by Axiom (2) it follows that the left-hand square commutes. $\square$

If x is an object of $\mathfrak{X}$ over X, then we will frequently use the same symbol x to denote a morphism $\underline{X} \to \mathfrak{X}$ which yields x when evaluated at $(X, 1_X)$. So, for instance, if $\mathfrak{X} = BG$ and $E \to X$ is a G-torsor, then we will have $E\colon \underline{X} \to BG$. This morphism (functor between categories) is determined up to a canonical natural isomorphism of functors).

The proof of Proposition 2.20 makes heavy use of the existence of choices of pullbacks of a given object of $\mathfrak{X}$. It is convenient to formalize the existence of pullbacks in the form of a pullback (or change of base) functor. Let $f\colon T \to S$ be a morphism in $\mathcal{S}$. For every object s in $\mathfrak{X}_S$, fix an object t in $\mathfrak{X}_T$ with $t \to s$ as in Axiom (1); we then use the common notation $f^*(s)$, or sometimes s_T or $s|_T$, for this object t, and call it the pullback. If we have a morphism $\varphi\colon s' \to s$ in $\mathfrak{X}_S$, it follows from Axiom (2) that there is a unique morphism ψ from $f^*(s')$ to $f^*(s)$ in $\mathfrak{X}_T$ such that the diagram

$$\begin{array}{ccc} f^*(s') & \longrightarrow & s' \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ f^*(s) & \longrightarrow & s \end{array}$$

commutes; this map ψ is denoted $f^*(\varphi)$. These choices determine a functor f^* from $\mathfrak{X}_S$ to $\mathfrak{X}_T$, called the **change of base functor**. If $f\colon S \to S$ is the identify morphism, we choose f^* to be the identity. If $f\colon T \to S$ and $g\colon U \to T$, there is a canonical natural isomorphism of functors $g^* \circ f^* \cong (f \circ g)^*$ from $\mathfrak{X}_S$ to $\mathfrak{X}_U$. In addition, these isomorphisms satisfy the expected "cocycle" compatibility condition with a third map $h\colon V \to U$, specifically that the diagram

$$\begin{array}{ccccc} h^* \circ (g^* \circ f^*) & \cong & h^* \circ (f \circ g)^* & \cong & ((f \circ g) \circ h)^* \\ \| & & & & \| \\ (h^* \circ g^*) \circ f^* & \cong & (g \circ h)^* \circ f^* & \cong & (f \circ (g \circ h))^* \end{array}$$

commutes.

It is often simplest to think of $\mathfrak{X}$ as the collection of groupoids $\mathfrak{X}_S$, together with the pullbacks $f^*\colon \mathfrak{X}_S \to \mathfrak{X}_T$ for morphisms $f\colon T \to S$. This has all the essential information. By using the original definition as one category with a functor to $\mathcal{S}$, however, one avoids having to verify these cocycle conditions. Note that in Example 2.4 (and therefore also in Example 2.1), the only morphisms in the categories $\mathfrak{X}_S$ are the identity morphisms; it is the presence of nontrivial automorphisms in the other examples that make general stacks richer than ordinary schemes or functors to sets.

Remark 2.21. In the previous section we introduced the first of a series of axioms for a category to be a stack, namely that it should be a category fibered in groupoids over the base category. A more general notion, that of being a *fibered category*, appears in the literature; the difference is that the fibers are allowed to be arbitrary categories, rather than categories whose morphisms are all isomorphisms (groupoids). We outline the differences between the two notions here.

In a fibered category, Axiom (2) of Definition 2.19 is not required to hold for all diagrams and given morphisms. Rather, a morphism $\varphi\colon t \to s$ over $f\colon T \to S$ is defined to be *cartesian* if, for every U and u, the conclusion of Axiom (2) holds. Then, a fibered category is defined as a category $\mathfrak{X}$ with a functor to a base category $\mathcal{S}$, such that for every morphism $f\colon T \to S$ in $\mathcal{S}$ and object s in $\mathfrak{X}$ over S, there exists an object t in $\mathfrak{X}$ over T and a cartesian morphism $\varphi\colon t \to s$ in $\mathfrak{X}$ over f.

A CFG is then a fibered category in which all the morphisms are cartesian. Some of our examples of CFG sit inside larger fibered categories. In the stack of elliptic curves $\mathcal{M}_{1,1}$, for instance, we may allow arbitrary commutative diagrams $C' \to C$, $S' \to S$ as morphisms, rather than only cartesian diagrams. This produces a bigger category, which is a fibered category (this bigger category won't be an algebraic stack). Similarly, the categories $\mathcal{V}_n$, Qcoh, $\mathcal{V}_{X,n}$, and Coh_X sits inside larger categories, where we allow arbitrary morphisms, not only morphisms that identify a sheaf over S' with the pull-back of a sheaf over S.

Algebraic stacks are always categories fibered in groupoids, so we will not have much use for more general fibered categories. However we point out that the theory of descent (Appendix A) could be stated in the language of fibered categories. For instance, Theorem A.2 could be abbreviated to the statement that the fibered category of quasi-coherent sheaves over schemes is a stack (for the fpqc or fppf topology on schemes). In this book, we choose instead to present the results in Appendix A in an explicit manner, and we make the convention that stacks will be categories fibered in groupoids satisfying additional hypotheses.

An important remark is that a morphism of CFGs $f\colon \mathfrak{X} \to \mathfrak{Y}$ is an isomorphism (equivalence of categories) if and only if there exists a morphism in the other direction $g\colon \mathfrak{Y} \to \mathfrak{X}$, together with 2-isomorphisms $g \circ f \Rightarrow 1_{\mathfrak{X}}$ and $f \circ g \Rightarrow 1_{\mathfrak{Y}}$. Indeed, there is the familiar statement, stated as Proposition B.1, that a functor between categories is an equivalence of categories if and only if it is fully faithful and essentially surjective. Following the proof of Proposition B.1, we need to assign to each object t of $\mathcal{Y}$ (over some T in $\mathcal{S}$) an object $g(t)$ of $\mathcal{X}$. By essential surjectivity there exists an object $\tilde{t}$ of $\mathcal{X}$ and an isomorphism $f(\tilde{t}) \to t$; the isomorphism will be over some isomorphism $\varphi\colon \widetilde{T} \to T$, possibly not the identity. But then we define $g(t)$ to be $(\varphi^{-1})^*\tilde{t}$ and have an isomorphism $f(g(t)) \to t$ over 1_T. The rest of the verification can be copied from the proof of Proposition B.1.

4. 2-commutative diagrams

Given CFGs $\mathfrak{X}$ and $\mathfrak{Y}$ over a base category $\mathcal{S}$, we have seen that morphisms of CFGs from $\mathfrak{X}$ to $\mathfrak{Y}$ (which are functors) form a category $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$, with natural isomorphisms of functors as morphisms in $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$. It will come as no surprise, then, that the natural way to compare two morphisms in $\mathfrak{X}$ to $\mathfrak{Y}$ is to say that they are isomorphic. Often the morphisms will be canonically isomorphic. But it happen much more rarely that the morphisms will actually be equal.

This is particularly the case when the morphisms that we are comparing are gotten by composing other morphisms. Most "commutative" diagrams won't actually commute! Rather, they will commute "up to" a natural isomorphism. We give some examples of this.

Example 2.22. Here are some diagrams of CFGs.

(1) Let G, G', and G'' be affine algebraic groups, and let

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & G' \arrow[d] \\ & G'' \end{tikzcd}$$

be a commutative diagram of algebraic group homomorphisms. Then there is a diagram of CFGs

$$\begin{tikzcd} BG \arrow[r] \arrow[dr] & BG' \arrow[d] \\ & BG'' \end{tikzcd}$$

which commutes up to a canonical natural isomorphism. For instance, if $G'' = G$ and the homomorphsm $G \to G'' = G$ is the identity, then the composite $BG \to BG' \to BG$ will not be equal, but only naturally isomorphic, to the identity 1_{BG}.

(2) Consider the pair of morphisms from $\mathcal{M}_{g,2}$ to $\mathcal{M}_{g,1}$ which forget the first, resp. the second section. These fit into a diagram

$$\begin{tikzcd} \mathcal{M}_{g,2} \arrow[r] \arrow[d] & \mathcal{M}_{g,1} \arrow[d] \\ \mathcal{M}_{g,1} \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

an honest example of a diagram that actually commutes! However, there is a similar operation on n-pointed *stable* curves of genus g (Example 2.11), which forgets one of the sections σ_i of $\pi\colon C \to S$ and collapses any components of the fibers of π which are thereby made unstable. The corresponding diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,2} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,1} \arrow[d] \\ \overline{\mathcal{M}}_{g,1} \arrow[r] & \overline{\mathcal{M}}_g \end{tikzcd}$$

commutes up to a canonical natural isomorphism. That is, the results of forgetting and stabilizing the two markings in either order are canonically isomorphic.

Definition 2.23. A diagram of CFGs is said to be **2-commutative** if it commutes up to a given isomorphism in the relevant HOM category; it is **strictly commutative** if it commutes exactly. An isomorphism between two objects in $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$ is called a **2-morphism**, or **2-isomorphism**. (We recall, $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$ is a groupoid, i.e., every morphism in $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$ is an isomorphism.)

If $f,\ g\colon \mathfrak{X} \to \mathfrak{Y}$ are morphisms, a 2-morphism can be denoted by $f \Rightarrow g$. That a diagram is 2-commutative can be indicated by marking it with $\Rightarrow$. So, for instance,

$$\begin{tikzcd} BG \arrow[r] \arrow[dr] & BG' \arrow[d] \arrow[l, phantom, shift right=3, "\Rightarrow"] \\ & BG'' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \overline{\mathcal{M}}_{g,2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \overline{\mathcal{M}}_{g,1} \arrow[d] \\ \overline{\mathcal{M}}_{g,1} \arrow[r] & \overline{\mathcal{M}}_{g} \end{tikzcd}$$

In fact, CFGs over $\mathcal{S}$ form a 2-category, a richer structure than just a category. In a 2-category there are objects (in this case, CFGs), morphisms (which, for CFGs, are functors), and 2-morphisms, which for CFGs we have just introduced in Definition 2.23. The formalism of 2-categories is not necessary in these early chapters; the reader who wants to look ahead can turn to Appendix B. For now, the main point is that CFGs are part of a structure that is different from an ordinary category. So, for instance in the next section when we discuss fiber products of CFGs, we cannot just use the standard notion of fiber products in a category; a dedicated discussion of the topic will be required.

Exercise 2.3. Consider $\Lambda = \operatorname{Spec}(k)$ where k is an algebraically closed field. Let G be a finite group.

(i) Consider the morphism $f\colon \underline{\Lambda} \to BG$ which assigns to a scheme S the trivial G-torsor $S \times G$. Show that the automorphism group of f in $\mathrm{HOM}(\underline{\Lambda}, BG)$ can be identified with G.
(ii) Show that the automorphism group of 1_{BG} can be identified with the center $Z(G)$ of G.

Representative 2-commutative diagrams will be

$$\begin{tikzcd} \underline{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \underline{\Lambda} \arrow[d] \\ \underline{\Lambda} \arrow[r] & BG \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} BG \arrow[r, "1_{BG}"] \arrow[dr, "1_{BG}"'] & BG \arrow[d, "1_{BG}"] \arrow[l, phantom, shift right=3, "\Rightarrow"] \\ & BG \end{tikzcd}$$

Diagrams such as the first one will appear in the next section. The second diagram actually links up with a more advanced topic, group actions on a stack. A finite group H can act on BG, with every $h \in H$ acting by the identity map 1_{BG}, so that the "quotient" is the classifying stack of a group which is a nontrivial extension of H by G. (This will be a sort of quotient that generalizes how H can act on a point with stack quotient $[\bullet/H] = BH$.) In fact we get precisely the extensions which classically are classified by group cohomology $H^2(H, Z(G))$. The point is that the usual condition for a group action $x \cdot (hh') = (x \cdot h) \cdot h'$ is replaced by 2-commutative diagrams with a further requirement on the 2-morphisms $\Rightarrow$, and these will amount to the cocycle condition of group cohomology.

5. Fiber products of CFGs

We come to an important construction of CFGs, the fiber product. We will have *2-cartesian diagrams* which, just as for schemes, are diagrams which express the fact that one CFG is isomorphic to the fiber product of a pair of CFGs over a third CFG. Also, as in the usual setting, there will be a universal property characterizing such diagrams. However, this property relies heavily upon the notion of 2-morphism that has just been introduced. So for this reason we prefer to give a direct construction of the fiber product of CFGs, which will satisfy a "strict" universal property, and to define 2-cartesian diagrams as those involving a CFG that is isomorphic to "the" fiber product of the given CFGs. Afterwards, we will give the universal property as an optional remark.

Definition 2.24. Given $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$, all CFGs over $\mathcal{S}$, and morphisms $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$, the **fiber product** $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is the category whose objects are triples (x, y, α), where x is an object in $\mathfrak{X}$, y is an object in $\mathfrak{Y}$ (over the same S in $\mathcal{S}$), and α is an isomorphism from $f(x)$ to $g(y)$ in $\mathfrak{Z}$ (over the identity on S). A morphism from (x', y', α') to (x, y, α) is given by morphisms $x' \to x$ in $\mathfrak{X}$ and $y' \to y$ in $\mathfrak{Y}$ (over the same morphism $S' \to S$ in $\mathcal{S}$), such that the diagram

$$\begin{tikzcd} f(x') \arrow[r] \arrow[d, "\alpha'"'] & f(x) \arrow[d, "\alpha"] \\ g(y') \arrow[r] & g(y) \end{tikzcd}$$

commutes. Compositions of morphisms are defined in the obvious way, and there is an obvious projection from $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ to $\mathcal{S}$.

We have two canonical projections p and q from $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ to $\mathfrak{X}$ and $\mathfrak{Y}$. There is a 2-commutative diagram

$$\begin{tikzcd} & \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathfrak{X} \arrow[dr, "f"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathfrak{Y} \arrow[dl, "g"] \\ & \mathfrak{Z} & \end{tikzcd}$$

where $\Rightarrow$ indicates a 2-morphism $f \circ p \Rightarrow g \circ q$. This 2-morphism is given by α: for an object $\xi = (x, y, \alpha)$ in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$, we have $f \circ p(\xi) = f(x)$ and $g \circ q(\xi) = g(y)$, so α is an isomorphism of $f \circ p(\xi)$ with $g \circ q(\xi)$.

Example 2.25. Here are some examples of fiber products of CFGs.

(1) If X, Y, and Z are objects and $X \to Z$ and $Y \to Z$ are morphisms in $\mathcal{S}$, then

$$\underline{X} \times_{\underline{Z}} \underline{Y} \cong \underline{X \times_Z Y}.$$

Indeed, the fibers over any object S of both sides are sets, and the bijection between them is a result of the usual universal property of the fiber product.

(2) Recall that $\underline{\Lambda}$ is simply the base category $\mathcal{S} = (\mathrm{Sch}/\Lambda)$. The *product* of $\mathfrak{X}$ and $\mathfrak{Y}$ will be $\mathfrak{X} \times_{\underline{\Lambda}} \mathfrak{Y}$. An object is an object of $\mathfrak{X}$ and an object of $\mathfrak{Y}$ (over the same object S in $\mathcal{S}$). A morphism is a morphisms in $\mathfrak{X}$ and a morphism in $\mathfrak{Y}$ (over the same morphism in $\mathcal{S}$). This product will also be denoted $\mathfrak{X} \times \mathfrak{Y}$. In case $\mathfrak{Y} = \mathfrak{X}$ we have a *diagonal morphism* $\Delta_{\mathfrak{X}} \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$, sending an object s to (s, s) and a morphism φ to (φ, φ).

(3) Let G be an algebraic group. Consider a morphism $E \colon \underline{S} \to BG$ corresponding (Proposition 2.20) to a G-torsor $E \to S$. There is also the morphism $\mathrm{triv} \colon \underline{\Lambda} \to BG$ which assigns to every scheme T the trivial G-torsor $T \times G \to T$. The fiber product $\underline{S} \times_{BG} \underline{\Lambda}$, we claim, is isomorphic to $\underline{E}$. This fact will be expressed by saying there is a 2-cartesian diagram

$$\begin{tikzcd} \underline{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \underline{\Lambda} \arrow[d, "\mathrm{triv}"] \\ \underline{S} \arrow[r, "E"'] & BG \end{tikzcd}$$

Indeed, an object of the fiber product, over a scheme T, is a morphism $T \to S$ together with a G-equivariant isomorphism of E_T (the given pullback of E to T) with $T \times G$. The identity section $T \to T \times G$ corresponds, via this isomorphism, to a section of $E_T \to T$, giving rise to $T \to E_T \to E$, an object of $\underline{E}$. Conversely, given $T \to E$, we obtain a section of $E_T \to T$ from the fact that E_T is a fiber product of T with E over S. Finally, G-equivariance uniquely determines uniquely the isomorphism $E_T \cong T \times G$.

(4) We had noted in Example 2.9(5) that the forgetful morphism from $\mathcal{M}_{g,1}$ to $\mathcal{M}_g$ can be regarded as the universal curve. This example points out why. Let $C \to S$ be a family of curves of genus g. Then we claim $\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \cong \underline{C}$, i.e., there is a 2-cartesian diagram

$$\begin{tikzcd} \underline{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{M}_{g,1} \arrow[d, "\mathrm{forgetful}"] \\ \underline{S} \arrow[r, "C"'] & \mathcal{M}_g \end{tikzcd}$$

An object over T of the fiber product consists of a morphism $T \to S$, a family of curves $C' \to T$ with section σ, and an isomorphism $\vartheta \colon C_T \cong C'$ over T. The section σ, composed with ϑ and projection to C is a morphism $T \to C$, so we have a map

$$\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \to \underline{C}.$$

A map the other way assigns to $f \colon T \to C$ the family of curves $C_T \to T$ with section induced by f. The composition of these maps in one order is equal to $1_{\underline{C}}$, and in the other order is naturally isomorphic, by ϑ, to $1_{\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}}$.

Exercise 2.4. If each of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ is a CFG/$\mathcal{S}$, then $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is also a CFG/$\mathcal{S}$. [Hint: show that it satisfies the CFG axioms (1) and (2′) (Exercise 2.2).]

Remark 2.26. In any fiber product of the form $\underline{S} \times_{\mathfrak{X}} \underline{T}$, we get a category that has no nontrivial morphisms. This is because $\underline{S}$ and $\underline{T}$ are categories with no nontrivial morphisms. (By a CFG with "no nontrivial morphisms" we mean one whose fibers are sets, i.e., categories with only identity morphisms. Then, between any two objects t and s there will be precisely one morphism over $f\colon T \to S$ when $t = f^*(s)$ and otherwise no morphism over f.) So, in the discussion in (3) there is no mention of morphisms. Whereas, in (4) there are nontrivial morphisms in the fiber product: as a category, $\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$ is equivalent but not isomorphic to the category $\underline{C}$.

In fact, for any CFG $\mathfrak{X}$ the category $\mathrm{HOM}(\mathfrak{X}, \underline{S})$ is a set. Later we will see instances where, to a CFG $\mathfrak{X}$, we can associate a scheme M, with morphism $\mathfrak{X} \to \underline{M}$, inducing set bijections $\mathrm{HOM}(\mathfrak{X}, \underline{X}) = \mathrm{Hom}_{\mathcal{S}}(M, X)$ for all schemes X. For $\mathfrak{X} = \mathcal{M}_g$ we will be able to take $M = M_g$, the classical moduli space of curves of genus g.

The fiber product satisfies the following **strict universal property**: given maps $u\colon \mathfrak{W} \to \mathfrak{X}$, $v\colon \mathfrak{W} \to \mathfrak{Y}$, together with a natural isomorphism $f \circ u \Rightarrow g \circ v$, there is a unique map $(u, v)\colon \mathfrak{W} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ with $p \circ (u, v) = u$ and $q \circ (u, v) = v$, so that the natural isomorphism from $f \circ u$ to $g \circ v$ is the one determined by $f \circ p \Rightarrow g \circ q$ (by the identities $f \circ u = f \circ p \circ (u, v) \Rightarrow g \circ q \circ (u, v) = g \circ v$).

Notice that strict universal property involves maps to $\mathfrak{X}$ and $\mathfrak{Y}$ and a 2-morphism of the composite maps to $\mathfrak{Z}$. These are precisely the data to determine a 2-commutative diagram

(1)
$$\begin{tikzcd} & \mathfrak{W} \arrow[dl, "u"'] \arrow[dr, "v"] & \\ \mathfrak{X} \arrow[dr, "f"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathfrak{Y} \arrow[dl, "g"] \\ & \mathfrak{Z} & \end{tikzcd}$$

Now we say that the diagram (1) is **2-cartesian** (or is a fiber diagram, or a pullback diagram) if the morphism $(u, v)\colon \mathfrak{W} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ determined by the strict universal property, is an isomorphism of CFGs. We have met some 2-cartesian diagrams in Example 2.25. Here is one more.

Exercise 2.5. Given a CFG $\mathfrak{X}$ over $\mathcal{S}$, define the *inertia CFG* to be the following category, which will be denoted $I_{\mathfrak{X}}$. An object of $I_{\mathfrak{X}}$ is a pair (s, σ) where s is an object of $\mathfrak{X}$ (over some S in $\mathcal{S}$), and σ is an isomorphism $s \to s$ over 1_S. A morphism $(s', \sigma') \to (s, \sigma)$ is a morphism $s' \to s$ (over some $f\colon S' \to S$) such that $f^*(\sigma) = \sigma'$. There is a functor $i\colon I_{\mathfrak{X}} \to \mathfrak{X}$ which forgets σ.

(i) $I_{\mathfrak{X}}$ is a CFG.

(ii) There is a 2-cartesian diagram

$$\begin{tikzcd} I_{\mathfrak{X}} \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{X} \arrow[d, "\Delta_{\mathfrak{X}}"] \\ \mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}}"'] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

(iii) Let $\Lambda = \mathrm{Spec}(k)$ where k is an algebraically closed field, and let G be a finite group. For $\mathfrak{X} = BG$, we have $I_{\mathfrak{X}} \cong [G/G]$ where G acts on itself by conjugation.

The next two exercises gather some basic facts about fiber products which are familiar from the cas of schemes (or objects of a general category).

Exercise 2.6. Given morphisms $\mathfrak{X} \to \mathfrak{Y}$, $\mathfrak{Y} \to \mathfrak{Z}$, and $\mathfrak{W} \to \mathfrak{Z}$, construct an isomorphism of CFGs (equivalence of categories) between $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W})$ and $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W}$.

Exercise 2.7. Given $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$, we have morphisms $f \times g\colon \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z} \times \mathfrak{Z}$ and a diagonal morphism $\mathfrak{Z} \to \mathfrak{Z} \times \mathfrak{Z}$. Construct an isomorphism of CFGs

$$\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \cong (\mathfrak{X} \times \mathfrak{Y}) \times_{\mathfrak{Z} \times \mathfrak{Z}} \mathfrak{Z}.$$

The corresponding 2-cartesian diagrams are

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{W} \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Y} \arrow[r] & \mathfrak{Z} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Z} \arrow[d] \\ \mathfrak{X} \times \mathfrak{Y} \arrow[r] & \mathfrak{Z} \times \mathfrak{Z} \end{tikzcd}$$

Example 2.27. Suppose we are given morphisms $\mathfrak{X} \to \mathfrak{U} \leftarrow \mathfrak{Y} \to \mathfrak{V} \leftarrow \mathfrak{Z}$ of CFGs over $\mathcal{S}$. Define a category $\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z}$ whose objects are (x, y, z, α, β), with x, y, z objects in $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, respectively, over some S in $\mathcal{S}$; α is a map from the image of x to the image of y in $\mathfrak{U}$, and β is a map from the image of y to the image of z in $\mathfrak{V}$, all over 1_S. Morphisms are defined as in the case of fiber products. Then we have isomorphisms (these are, in fact, isomorphisms of categories)

$$\mathfrak{X} \times_{\mathfrak{U}} (\mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z}) \cong \mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z} \cong (\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y}) \times_{\mathfrak{V}} \mathfrak{Z}.$$

Projections to $\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y}$ and to $\mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z}$ give rise to a diagram

$$\begin{tikzcd}
\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z} \arrow[r] \arrow[d] & \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Z} \arrow[d] \\
\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Y} \arrow[r] \arrow[d] & \mathfrak{V} \\
\mathfrak{X} \arrow[r] & \mathfrak{U} &
\end{tikzcd}$$

The upper-left square (which, in fact, strictly commutes) is 2-cartesian; this can be seen by an application of Exercise 2.6.

REMARK 2.28. The strict universal property characterizes $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ up to an isomorphism of categories. This is too strict: the natural notion of isomorphism of CFGs is an equivalence of categories. There is a more natural universal property, which we should emphasize is not required for the material in this book. A diagram (1) is 2-cartesian if and only if the following universality condition is satisfied. Given a CFG $\mathfrak{U}$, define a category

$$\mathrm{HOM}\left(\mathfrak{U}, \quad \begin{matrix} & \mathfrak{Y} \\ & \downarrow \\ \mathfrak{X} \to & \mathfrak{Z} \end{matrix}\right)$$

whose objects are triples (m, n, δ) where $m\colon \mathfrak{U} \to \mathfrak{X}$ and $n\colon \mathfrak{U} \to \mathfrak{Y}$ are morphisms, and δ is a 2-morphism from $f \circ m$ to $g \circ n$. A morphism from (m, n, δ) to (m', n', δ') will consist of a pair of 2-morphisms $m \Rightarrow m'$ and $n \Rightarrow n'$ such that the composite 2-morphism $f \circ m \Rightarrow g \circ n \Rightarrow g \circ n'$ is equal to the composite $f \circ m \Rightarrow f \circ m' \Rightarrow g \circ n'$. There is a functor

$$\mathrm{HOM}(\mathfrak{U}, \mathfrak{W}) \longrightarrow \mathrm{HOM}\left(\mathfrak{U}, \quad \begin{matrix} & \mathfrak{Y} \\ & \downarrow \\ \mathfrak{X} \to & \mathfrak{Z} \end{matrix}\right)$$

which sends $h\colon \mathfrak{U} \to \mathfrak{W}$ to $(u \circ h, v \circ h, f \circ u \circ h \Rightarrow g \circ v \circ h)$ and $h \Rightarrow h'$ to the pair consisting of $u \circ h \Rightarrow u \circ h'$ and $v \circ h \Rightarrow v \circ h'$. The universality condition is that this functor should be an equivalence of categories for any CFG $\mathfrak{U}$.

Answers to Exercises

2.1. If one constructs a principal GL_n-bundle by means of transition functions, the vector bundle is constructed from the same transition functions. The same idea works in (6). In both cases, note that G is the automorphism group of the fiber.

2.2. To prove (2), chose by (1) some $\gamma_0\colon u \to t$ over g. Using (2′) for h, one obtains $\theta\colon u \to u$ over 1_U with $\eta = \varphi \circ \gamma_0 \circ \theta$. Then $\gamma = \gamma_0 \circ \theta$ is a solution. If γ and γ' were two solutions, applying (2′) to the morphism g, one finds $\tau\colon u \to u$ over 1_U with $\gamma' = \gamma \circ \tau$. Since $\varphi \circ \gamma \circ \tau = \varphi \circ \gamma$, the uniqueness for maps over h implies that $\tau = 1_u$.

2.3. For (i), by Proposition 2.20 this is the automorphism group of the trivial G-torsor G over Λ. This is G. Directly, $g_0 \in G$ corresponds to the automorphisms $(s,g) \mapsto (s,g_0g)$ of $S\times G$, for arbitrary S. For (ii), consider a 2-morphism $\alpha\colon 1_{BG} \to 1_{BG}$, that is, a specification of automorphisms of G-torsors $E \to S$ compatible with the morphisms in BG. Restricted to trivial G-torsors $S\times G$, these must be of the form $(s,g)\mapsto(s,g_0g)$ for some $g_0 \in G$. But every G-torsor is locally trivial, so α is completely determined by g_0, and it remains to see that g_0 is constrained to lie in the center $Z(G)$. For the trivial G-torsor G over Λ, the automorphism corresponding to g_0 is $g \mapsto g_0g$. For any $h \in G$ we have an isomorphism in BG sending G to G by $g \mapsto hg$. Compatibility forces $hg_0g = g_0hg$ for any $h \in G$, i.e., $g_0 \in Z(G)$. By descent, any $g_0 \in Z(G)$ determines an automorphism of an arbitrary G-torsor $E \to S$.

2.4. Given $h\colon T \to S$ and an object $s = (x,y,\alpha)$ in $\mathfrak{X}\times_{\mathfrak{Z}}\mathfrak{Y}$ over S, to find a morphism $t \to s$ over h, choose $x' \to x$ and $y' \to y$ over h, and use Axiom (2) for $\mathfrak{Z}$ to find a morphism $\alpha'\colon f(x') \to g(y')$ over 1_T so that the diagram

$$\begin{tikzcd} f(x') \arrow[r] \arrow[d, "\alpha'"'] & f(x) \arrow[d, "\alpha"] \\ g(y') \arrow[r] & g(y) \end{tikzcd}$$

commutes. Then we have $t = (x',y',\alpha') \to s$ over h. To prove Axiom ($2'$), suppose we have $(x,y,\alpha) \to (x_0,y_0,\alpha_0)$ and $(x',y',\alpha') \to (x_0,y_0,\alpha_0)$ over h. This means we have $x \to x_0$ and $x' \to x_0$ in $\mathfrak{X}$, $y \to y_0$ and $y' \to y_0$ in $\mathfrak{Y}$, all over h, and a commutative diagram

$$\begin{tikzcd} f(x) \arrow[r] \arrow[d, "\alpha_1"'] & f(x_0) \arrow[d, "\alpha"'] & f(x') \arrow[l] \arrow[d, "\alpha_2"'] \\ g(x) \arrow[r] & g(x_0) & g(x') \arrow[l] \end{tikzcd}$$

in $\mathfrak{Z}$, with the horizontal maps over h. From ($2'$) for $\mathfrak{X}$ and $\mathfrak{Y}$ we get morphisms $x \to x'$ and $y \to y'$. We need to know that the left square in the diagram

$$\begin{tikzcd} f(x) \arrow[r] \arrow[d, "\alpha"'] & f(x') \arrow[r] \arrow[d, "\alpha'"'] & f(x_0) \arrow[d, "\alpha"'] \\ g(x) \arrow[r] & g(x') \arrow[r] & g(x_0) \end{tikzcd}$$

commutes. This follows from the fact that the large rectangle commutes, and the uniqueness in $\mathfrak{Z}$ of maps from $f(x)$ to $f(x')$ over 1_T with given maps to $g(x_0)$ over h.

2.5. For (i), if $\varphi\colon t \to s$ is a morphism in $\mathfrak{X}$ over $f\colon T \to S$ then we have $(t, f^*(\sigma)) \to (s,\sigma)$ in $I_{\mathfrak{X}}$ over f. Let, now, $g\colon U \to T$ be a morphism, $h = f\circ f$, and morphisms $\varphi\colon (t,\tau) \to (s,\sigma)$ and $\eta\colon (u,\upsilon) \to (s,\sigma)$ in $I_{\mathfrak{X}}$ over f and h, respectively. Axiom (2) dictates a unique morphism $\gamma\colon u \to t$ in $\mathfrak{X}$. Since $\gamma^*(\tau) = \gamma^*(\varphi^*(\sigma)) = \eta^*(\sigma) = \upsilon$, we have $\gamma\colon (u,\upsilon) \to (t,\tau)$ in $I_{\mathfrak{X}}$. For (ii), we have a 2-morphism $\Delta_{\mathfrak{X}}\circ i \Rightarrow \Delta_{\mathfrak{X}}\circ i$, $(s,\sigma) \mapsto \sigma\times 1_s\colon (s,s) \to (s,s)$, hence a morphism $I_{\mathfrak{X}} \to \mathfrak{X}\times_{\mathfrak{X}\times\mathfrak{X}}\mathfrak{X}$. A map the other way is given by $(s, s', \sigma \times \sigma') \mapsto (s, \sigma'^{-1} \circ \sigma)$. One composition is the identify $1_{I_{\mathfrak{X}}}$. The other composition is naturally isomorphic to $1_{\mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}}$ by $(s, s', \sigma \times \sigma') \mapsto [1_s \times \sigma' \colon (s, s, \sigma'^{-1} \circ \sigma, 1_s) \to (s, s', \sigma \times \sigma')]$. For (iii), define $I_{BG} \to [G/G]$ by sending $(E \to S, \sigma \colon E \to E)$ to the torsor $E \to S$ together with map $E \to G$ which sends $e \in E$ to the unique $g \in G$ such that $\sigma(e) = e \cdot g$. This is an isomorphism of categories.

2.6. There is a morphism $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W}$ sending $(x, (y, z, \beta), \alpha)$ to $(x, z, \beta \circ g(\alpha))$, where g denotes the morphism $\mathfrak{Y} \to \mathfrak{Z}$, and a morphism $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W} \to \mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W})$ sending (x, z, γ) to $(x, (f(x), z, \gamma), 1_x)$, with f the morphism $\mathfrak{X} \to \mathfrak{Y}$. One composition is $1_{\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W}}$, while the other composition is naturally isomorphic to $1_{\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W})}$ by the pair consisting of the identity of the morphism $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{X}$ and the natural isomorphism from $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W} \to \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}$ to $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}$ given by $(x, (y, z, \beta), \alpha) \mapsto [(\alpha, 1_z) \colon (f(x), z, \beta \circ g(\alpha)) \cong (y, z, \beta)]$.

2.7. To an object (x, y, α) in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$, assign the object $((x, y), g(y), \alpha \times 1_{g(y)})$ in $(\mathfrak{X} \times \mathfrak{Y}) \times_{\mathfrak{Z} \times \mathfrak{Z}} \mathfrak{Z}$. To an object $((x, y), z, \alpha \times \beta)$ in $(\mathfrak{X} \times \mathfrak{Y}) \times_{\mathfrak{Z} \times \mathfrak{Z}} \mathfrak{Z}$, assign the object $(x, y, \beta^{-1} \circ \alpha)$ in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$. As in the previous exercise, the composition of these morphisms in one order is identity, and in the other order is naturally isomorphic to identity.

CHAPTER 3

Groupoids and Atlases

For an atlas for a stack, one has schemes R and U, with a morphism from R to $U \times U$ (not usually an embedding). Conditions are put on these data that, in case R is contained in $U \times U$, make R an equivalence relation. This is the notion of a *groupoid scheme*, which is the subject of this chapter (cf. [**30**]). These will be the *atlases* (or *groupoid presentations*) for stacks.

1. Groupoid schemes

A **groupoid scheme**, or **algebraic groupoid**,[1] consists of two schemes and five morphisms, satisfying several properties. One has a scheme U, a scheme R, two morphisms s and t from R to U, a morphism e from U to R, a morphism $m\colon R\ {}_t{\times}_s\, R \to R$ (where $R\ {}_t{\times}_s\, R$ denotes the fiber product $R \times_U R$ constructed from the two maps t and s), and a morphism $i\colon R \to R$, satisfying the five properties listed below.

All of this makes sense over an arbitrary base category $\mathcal{S}$, and then one defines a **groupoid object** in $\mathcal{S}$. A key example to keep in mind is a groupoid set. This notion coincides precisely with the notion of a small category in which all morphisms are isomorphisms. From this example one can in fact figure out what the axioms must be. Here are the axioms:

(1) The composites $U \xrightarrow{e} R \xrightarrow{s} U$ and $U \xrightarrow{e} R \xrightarrow{t} U$ are the identity maps on U:

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "1_U"'] & R \arrow[d, "s"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "1_U"'] & R \arrow[d, "t"] \\ & U \end{tikzcd}$$

(2) If pr_1 and pr_2 are the two projections from $R\ {}_t{\times}_s\, R$ to R, then $s \circ m = s \circ \mathrm{pr}_1$ and $t \circ m = t \circ \mathrm{pr}_2$:

$$\begin{tikzcd} R\ {}_t{\times}_s\, R \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"'] & R \arrow[d, "s"] \\ R \arrow[r, "s"'] & U \end{tikzcd} \qquad \begin{tikzcd} R\ {}_t{\times}_s\, R \arrow[r, "\mathrm{pr}_2"] \arrow[d, "m"'] & R \arrow[d, "t"] \\ R \arrow[r, "t"'] & U \end{tikzcd}$$

[1]The second choice of "algebraic groupoid" compares nicely with "algebraic group", and is common in the literature. It does conflict with the use of groupoid for a kind of category. In category language, cf. [**65**], what we call a groupoid scheme is an "internal groupoid in the category of schemes".

(3) (Associativity) The two maps $m \circ (1_R \times m)$ and $m \circ (m \times 1_R)$ from $R\ {}_t{\times}_s\ R\ {}_t{\times}_s\ R$ to R are equal:

$$\begin{tikzcd} R\ {}_t{\times}_s\ R\ {}_t{\times}_s\ R \arrow[r, "1_R \times m"] \arrow[d, "m \times 1_R"'] & R\ {}_t{\times}_s\ R \arrow[d, "m"] \\ R\ {}_t{\times}_s\ R \arrow[r, "m"'] & R \end{tikzcd}$$

(4) (Unit) The two maps $m \circ (e \circ s, 1_R)$ and $m \circ (1_R, e \circ t)$ from R to R are equal to the identity on R:

$$\begin{tikzcd} R \arrow[r, "(e \circ s, 1_R)"] \arrow[dr, "1_R"'] & R\ {}_t{\times}_s\ R \arrow[d, "m"] \\ & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "(1_R, e \circ t)"] \arrow[dr, "1_R"'] & R\ {}_t{\times}_s\ R \arrow[d, "m"] \\ & R \end{tikzcd}$$

(5) (Inverses) $i \circ i = 1_R$, $s \circ i = t$ and (therefore) $t \circ i = s$, and $m \circ (1_R, i) = e \circ s$ and $m \circ (i, 1_R) = e \circ t$:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "1_R"'] & R \arrow[d, "i"] \\ & R \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "s"'] & R \arrow[d, "t"] \\ & R \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "(1_R, i)"] \arrow[d, "s"'] & R\ {}_t{\times}_s\ R \arrow[d, "m"] \\ U \arrow[r, "e"'] & R \end{tikzcd} \quad \begin{tikzcd} R \arrow[r, "(i, 1_R)"] \arrow[d, "t"'] & R\ {}_t{\times}_s\ R \arrow[d, "m"] \\ U \arrow[r, "e"'] & R \end{tikzcd}$$

Note by (1) that s and t are always surjective. These axioms have some redundancy, but others could be added. For example:

Exercise 3.1. (a) Show that $i \circ e = e$. (b) Show that $m \circ (e, e) = e$. (c) Show that $m \circ (i \circ \mathrm{pr}_2, i \circ \mathrm{pr}_1) = i \circ m$. (d) Show that the diagrams of axiom (2) are cartesian.

Example 3.1. Any morphism $U \to X$ of schemes determines a groupoid scheme. For this, take $R = U \times_X U$, with s and t the two projections, e the diagonal map, and i the map switching the two factors. Identifying $R\ {}_t{\times}_s\ R$ with $U \times_X U \times_X U$, the map m is the projection onto the first and third factors.

Example 3.2. If $U = \Lambda$ (the base scheme) then the axioms for a groupoid scheme reduce to axioms for R to be a group scheme over Λ (with multiplication $m\colon R \times R \to R$, identity section $e\colon \Lambda \to R$, and inverse map $i\colon R \to R$).

Example 3.3. An important example of a groupoid scheme arises whenever an algebraic group G acts on the right on a scheme U. Set $R = U \times G$, and let $s\colon U \times G \to U$ be projection and $t\colon U \times G \to U$ the action. The map $e\colon U \to U \times G$ takes u to (u, e_G), where e_G is the identity element of G. The map i takes (u, g) to $(u \cdot g, g^{-1})$, and

$$m\big((u, g), (u \cdot g, h)\big) = (u, gh).$$

We may identify $R\ {}_t{\times}_s\ R$ with $U \times G \times G$ by the map $\big((u, g), (u \cdot g, h)\big) \mapsto (u, g, h)$. Under this identification, m becomes the map $(u, g, h) \mapsto (u, gh)$.

EXAMPLE 3.3′. For a left action, we have a groupoid scheme with $R = G \times U$, s the projection, t the action, $e(u) = (e_G, u)$, $i(g,u) = (g^{-1}, g \cdot u)$, and $m\big((g,u),(h, g \cdot u)\big) = (hg, u)$.

A groupoid scheme can be denoted (U, R, s, t, m, e, i), or more simply $R \rightrightarrows U$. We call the groupoid scheme an **étale groupoid scheme** if the two morphisms s and t are étale. Similar terminology is used for other adjectives such as smooth or flat. The groupoid scheme for $\mathcal{M}_{1,1}$ in Example 3.10 is an étale groupoid scheme. The groupoids $X \times G \rightrightarrows X$ and $G \times X \rightrightarrows X$ (Examples 3.9 and 3.9′) are étale groupoids when G is étale over the base field (e.g., a finite group), and are smooth groupoids when G is smooth (e.g., an algebraic group over a field).

A **morphism** of groupoid schemes from $(U', R', s', t', m', e', i')$ to (U, R, s, t, m, e, i) is a pair (ϕ, Φ), where $\phi\colon U' \to U$ and $\Phi\colon R' \to R$ are morphisms of schemes. These are required to be compatible with the structure morphisms defining each groupoid scheme, in the obvious sense: $s \circ \Phi = \phi \circ s'$, $t \circ \Phi = \phi \circ t'$, $e \circ \phi = \Phi \circ e'$, $m \circ (\Phi \times \Phi) = \Phi \circ m'$, and $i \circ \Phi = \Phi \circ i'$.

EXAMPLE 3.4. If G acts on U, and H acts on V, and $\theta\colon G \to H$ is a homomorphism of algebraic groups, and $\phi\colon U \to V$ is an equivariant map (so $\phi(u,g) = \phi(u) \cdot \theta(g)$ for $u \in U$ and $g \in G$), this determines a morphism (ϕ, Φ) from the groupoid scheme $U \times G \rightrightarrows U$ to $V \times H \rightrightarrows V$, with $\Phi(u,g) = (\phi(u), \theta(g))$.

2. Groupoids and CFGs

It will be possible to go back and forth between algebraic groupoids and CFGs. An algebraic groupoid will describe a CFG, much the way that a scheme can be described by patching. But this goes via a process that requires several steps; these will be presented over the course of this chapter and the next chapter. We start with the easier direction, that of producing an algebraic groupoid from a CFG.

PROPOSITION 3.5. *Let $\mathfrak{X}$ be a CFG, U a scheme, and $u\colon \underline{U} \to \mathfrak{X}$ a morphism. Assume given an isomorphism between $\underline{U} \times_{\mathfrak{X}} \underline{U}$ and $\underline{R}$ for some scheme R. Then R and U form a groupoid scheme, with the pair of projection maps*

$$s, t\colon \underline{R} \cong \underline{U} \times_{\mathfrak{X}} \underline{U} \rightrightarrows \underline{U}.$$

and the following additional maps:

(i) *$e\colon U \to R$ is the composite $\underline{U} \to \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}$, where the first map sends h to $(h, h, 1_{u(h)})$,*

(ii) *m is the map from $\underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}\ {}_t\times_{\underline{U},s} \underline{R}$ to $\underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}$ given by*

$$(h, h', h'', \varphi, \varphi') \mapsto (h, h'', \varphi' \circ \varphi),$$

(iii) *$i\colon \underline{R} \cong \underline{U} \times_{\mathfrak{X}} \underline{U} \to \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}$, where $(h, h', \varphi) \mapsto (h', h, \varphi^{-1})$.*

In the definition of m, notice that we make use of the isomorphism $\underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U} \cong \underline{R}\ {}_t\times_{\underline{U},s} \underline{R}$ of Example 2.27. We also repeatedly use the correspondence between morphisms of schemes and morphisms of the associated CFGs (Example 2.9(1)).

Proof. We have to verify the axioms. We verify the Associativity axiom (3) and leave the verification of the other axioms to the reader. Let

$$(h, h', h'', h''', \varphi, \varphi', \varphi'')$$

be an object of $\underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U} \times_{\mathfrak{X}} \underline{U}$, which is identified with $\underline{R}\ {}_t{\times}_s \underline{R}\ {}_t{\times}_s \underline{R}$. If we apply $m \times 1_R$, we get $(h, h'', h''', \varphi' \circ \varphi, \varphi'')$. Applying m produces $(h, h''', \varphi'' \circ (\varphi' \circ \varphi))$. If, intead, we apply $1_R \times m$ and then m, we get $(h, h''', (\varphi'' \circ \varphi') \circ \varphi)$. Since composition of morphisms in the category $\mathfrak{X}$ is associative, we have $\varphi'' \circ (\varphi' \circ \varphi) = (\varphi'' \circ \varphi') \circ \varphi$. So, the two maps $m \circ (1_R \times m)$ and $m \circ (m \times 1_R)$ are equal. $\square$

According to Proposition 2.20, to specify the morphism $u \colon \underline{U} \to \mathfrak{X}$ is equivalent to specifying an object of $\mathfrak{X}$ over U. We are adopting the notational convention to use the same symbol for both the object and the morphism; this means that for a given morphism $f \colon S \to U$ of schemes, $u(S, f)$ will be the (chosen) pullback $f^*(u)$. Now Proposition 3.5 can be interpreted as saying that an object u of $\mathcal{X}$ over a scheme U determines an algebraic groupoid, provided that the corresponding fiber product of $\underline{U}$ with itself over $\mathfrak{X}$ is isomorphic to a scheme.

Fiber products of schemes over a target CFG will occur so frequently, that they deserve a special notation. We introduce this now.

Definition 3.6. Let $\mathfrak{X}$ be a CFG over a base category $\mathcal{S}$ of schemes. Let U and V be schemes, and let u and v be objects of $\mathfrak{X}$ over U and over V, respectively. These determine morphisms (which are unique up to canonical 2-isomorphisms) $u \colon \underline{U} \to \mathfrak{X}$ and $v \colon \underline{V} \to \mathfrak{X}$. The **symmetry CFG** of u and v is the fiber product $\underline{U}\ {}_u{\times}_{\mathfrak{X},v} \underline{V}$. It will be denoted $\mathfrak{Sym}_{\mathfrak{X}}(u, v)$.

By Remark 2.26, the CFG $\mathfrak{Sym}_{\mathfrak{X}}(u, v)$ has sets as fibers. Concretely, the fiber over S is the set of triples (f, g, φ) where $f \colon S \to U$ and $g \colon S \to V$ are morphisms and $\varphi \colon f^*(u) \to g^*(v)$ is an isomorphism in $\mathfrak{X}_S$. To say that $\mathfrak{Sym}_{\mathfrak{X}}(u, v)$ is isomorphic to $\underline{T}$ for some scheme T is to say that the set of triples (f, g, φ) is naturally isomorphic to the set of morphisms $S \to T$. This condition only depends on the isomorphism class of u and the isomorphism class of v.

According to Proposition 3.5, now, if $\mathfrak{Sym}_{\mathfrak{X}}(u, u)$ is isomorphic to a scheme, then it determines an algebraic groupoid.

Definition 3.7. Let $\mathfrak{X}$ be a CFG, U a scheme, and u an object of $\mathfrak{X}_U$. If there is a scheme R and an isomorphism $\mathfrak{Sym}_{\mathfrak{X}}(u, u) \cong \underline{R}$, then the groupoid $R \rightrightarrows U$ of Proposition 3.5 will be called the **symmetry groupoid** of u.

The notation $\mathfrak{Sym}_{\mathfrak{X}}(u, u) \rightrightarrows U$ could be used to denote the symmetry groupoid, although we will tend to avoid doing this, because it hides the fact that there is a nontrivial hypothesis in Proposition 2.20, that the fiber product of $\underline{U}$ with itself over $\mathfrak{X}$ should be isomorphic to a scheme.

The notions of symmetry CFG and symmetry groupoid also make sense over an arbitrary base category $\mathcal{S}$. The symmetry groupoid of u, when it exists, will be a groupoid object in $\mathcal{S}$.

Here are some examples of symmetry groupoids.

Example 3.8. Suppose $\mathfrak{X} = \underline{X}$. Let $f\colon U \to X$ be a morphism, determining (by composition with f) a morphism $\underline{f}\colon \underline{U} \to \underline{X}$. The fiber product $\underline{U} \times_{\underline{X}} \underline{U}$ is isomorphic to $\underline{U \times_X U}$ by Example 2.25(1). So, the corresponding symmetry groupoid is the algebraic groupoid of Example 3.1.

Example 3.9. For $\mathfrak{X} = [X/G]$, we take $U = X$ with the trivial G-torsor $X \times G$ and action map $X \times G \to X$. The corresponding $f\colon \underline{X} \to [X/G]$ maps $h\colon S \to X$ to the torsor $S \times G \to S$ with equivariant map $S \times G \to X$, $(s, g) \mapsto h(s) \cdot g$ (cf. Example 2.9(4)). So an object of $\underline{X} \times_{[X/G]} \underline{X}$ consists of h, $h'\colon S \to X$, and G-equivariant $S \times G \to S \times G$, $(s, g) \mapsto (s, \varphi(s)g)$ (for some $\varphi\colon S \to G$) such that the diagram

$$\begin{tikzcd}
S \times G \arrow[rr, "h(s)\cdot g"] \arrow[dr, "{(s,\varphi(s)g)}"'] & & X \\
& S \times G \arrow[ur, "h'(s)\cdot g"'] &
\end{tikzcd}$$

commutes, i.e., $h'(s) \cdot \varphi(s) = h(s)$. In other words, $h'(s) = h(s) \cdot \varphi(s)^{-1}$. So, we identify $\underline{X} \times_{[X/G]} \underline{X}$ with $\underline{X} \times \underline{G}$ so that h, h', and $S \times G \xrightarrow{\sim} S \times G$ as above are sent to (h, φ^{-1}). Then, s is the first projection and t is the group action. We have $e = (1_X, e_G)$. To compute m, say $((h, \varphi^{-1}), (h', \varphi'^{-1}))$ is an object of $\underline{X} \times_{[X/G]} \underline{X} \times_{[X/G]} \underline{X}$. The composite isomorphism $S \times G \to S \times G$ is $(s, g) \mapsto (s, \varphi'(s)\varphi(s)g)$. The inverse of $\varphi'(s)\varphi(s)$ is $\varphi(s)^{-1}\varphi'(s)^{-1}$, so m sends $((h, \varphi^{-1}), (h', \varphi'^{-1}))$ to $(h, \varphi^{-1}\varphi'^{-1})$. Similarly, i sends (h, φ^{-1}) to $(h \cdot \varphi^{-1}, \varphi)$. We have reproduced the algebraic groupoid of Example 3.3.

In particular, for $\mathfrak{X} = BG$ we can take $U = \Lambda$ and obtain the groupoid scheme $G \rightrightarrows \Lambda$ with $m\colon G \times G \to G$ the multiplication of G and $i\colon G \to G$ the inverse.

Example 3.9′. There is a similar story for $\mathfrak{X} = [G\backslash X]$, again with $U = X$. We take f to be the morphism $\underline{X} \to [G\backslash X]$ which sends $h\colon S \to X$ to $G \times S \to S$ and $G \times S \to X$, $(g, s) \mapsto g \cdot h(s)$. An object of $\underline{X} \times_{[G\backslash X]} \underline{X}$ is given by h, h', and $G \times S \to G \times S$, $(g, s) \mapsto (g\varphi(s), s)$ such that $\varphi(s)h'(s) = h(s)$. Again it is convenient to rewrite this as $h'(s) = \varphi(s)^{-1}h(s)$, so that $\underline{X} \times_{[G\backslash X]} \underline{X} \cong \underline{G} \times \underline{X}$, with this object going to (φ^{-1}, h). So the algebraic groupoid is $G \times X \rightrightarrows X$ where s is projection, t is action, $e = (e_G, 1_X)$, $m((\varphi^{-1}, h), (\varphi'^{-1}, h')) = (\varphi'^{-1}\varphi^{-1}, h)$, and $i(\varphi^{-1}, h) = (\varphi, \varphi^{-1} \cdot h)$.

Example 3.10. Consider the CFG of elliptic curves $\mathfrak{X} = \mathcal{M}_{1,1}$ over $\mathcal{S} = (\mathrm{Sch}/\mathbb{C})$. Recall from §1.5 the modular families $C_\alpha \to S_\alpha$ of elliptic curves and the schemes $R_{\alpha,\beta}$ of pairs of points of S_α, S_β with isomorphism of the corresponding elliptic curves. The modular families are an object of $\mathcal{M}_{1,1}$ over $U := S_1 \amalg S_2$. We claim that $\underline{S_\alpha} \times_{\mathcal{M}_{1,1}} \underline{S_\beta} \cong \underline{R_{\alpha,\beta}}$. Then it follows that $\underline{U} \times_{\mathcal{M}_{1,1}} \underline{U} \cong \underline{R}$ where $R := \coprod_{1 \le i,j \le 2} R_{i,j}$, and we recover the algebraic groupoid $R \rightrightarrows U$ that we had claimed would be an atlas for $\mathcal{M}_{1,1}$.

The morphism $\underline{S}_\alpha \to \mathcal{M}_{1,1}$ associates to $T \to S_\alpha$ the family of elliptic curves $T \times_{S_\alpha} C_\alpha \to T$. This comes with a section $\sigma_\alpha\colon T \to T \times_{S_\alpha} C_\alpha$. We need to exhibit a bijection between isomorphisms $\varphi\colon T \times_{S_\alpha} C_\alpha \to T \times_{S_j} C_j$ over T and morphisms $T \to R_{\alpha,j}$. For this, it suffices to treat the case that T is affine, $T = \mathrm{Spec}(A)$. We have an isomorphism $\varphi^*\mathcal{O}(3\sigma_\beta) \cong \mathcal{O}(3\sigma_\alpha)$. This is determined up to automorphism of $\mathcal{O}(3\sigma_\alpha)$, i.e., an element of $\mathcal{O}^*(T \times_{S_\alpha} C_\alpha) = A^*$. Expressing each family of curves by an equation in Weierstrass form corresponds to a particular kind of choice of basis for the space of global sections of $\mathcal{O}(3\sigma_\beta)$, resp. $\mathcal{O}(3\sigma_\alpha)$. With respect to these bases, pullback of global sections by φ corresponds to an element of $PGL_3(A)$. The Weierstrass equations constrain this to be of diagonal form, say with diagonal entries λ, μ, 1, and we are reduced to the computations of §1.5.

The first step in turning a groupoid scheme into a stack is to associate, in a simple way, a CFG to a given groupoid scheme $R \rightrightarrows U$. This CFG will be denoted $[R \rightrightarrows U]^{\mathrm{pre}}$. It won't quite be a stack, but it will be a *prestack*, a term that will be defined in §4.1 and that explains our choice of notation. A further step called *stackification* will be required to turn this prestack into a stack. If the groupoid scheme satisfies certain hypotheses, then this stack will be an algebraic stack. These steps will make up the other half of the dictionary between algebraic stacks and groupoid schemes.

Definition 3.11. Let $R \rightrightarrows U$ be a groupoid scheme. The **associated prestack** will be the CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ defined as follows. An object over a scheme S is a morphism $g\colon S \to U$. A morphism over $f\colon S' \to S$ from $g'\colon S' \to U$ to $g\colon S \to U$ is a morphism $\gamma\colon S' \to R$ satisfying $s \circ \gamma = g'$ and $t \circ \gamma = g \circ f$. If $g\colon S \to U$ is an object of $[R \rightrightarrows U]^{\mathrm{pre}}$ then the identity morphism 1_g is given by $e \circ g\colon S \to R$. Composition of morphisms in $[R \rightrightarrows U]^{\mathrm{pre}}$ is defined using the multiplication map of the groupoid, as follows. If we have a pair of composable morphisms as pictured

$$\begin{tikzcd}
S'' \arrow[rrr, bend left, "\gamma'"] \arrow[rrrd, "g''"] \arrow[dr, "f'"'] & & & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
& S' \arrow[urr, "\gamma"] \arrow[rr, "g'"'] \arrow[dr, "f"'] & & U \\
& & S \arrow[ur, "g"'] &
\end{tikzcd}$$

then $t \circ \gamma' = g' \circ f' = s \circ \gamma \circ f'$, so there is an induced morphism $(\gamma', \gamma \circ f')\colon S' \to R\, {}_t\!\times_s R$. Now we define $\gamma \circ \gamma' = m(\gamma', \gamma \circ f')$.

Exercise 3.2. (i) With this definition, $[R \rightrightarrows U]^{\mathrm{pre}}$ is a category, i.e., composition of morphisms is associative. (ii) This category, with the obvious functor to $\mathcal{S}$, is a CFG, with the pullback of an object determined just by composing morphisms of schemes.

Example 3.12. Let G be an algebraic group, and consider the classifying stack BG. We have already seen that $\underline{\Lambda} \times_{BG} \underline{\Lambda} \cong \underline{G}$, and the associated groupoid scheme is $G \rightrightarrows \Lambda$. Now $[G \rightrightarrows \Lambda]^{\mathrm{pre}}$ is equivalent to the category of *trivial* G-torsors. Indeed, there is just one object over any S in $\mathcal{S}$, and isomorphisms from this object to itself correspond bijectively with morphisms $S \to G$.

We see concretely why an extra step is necessary to recover the stack BG. What goes wrong with $[G \rightrightarrows \Lambda]^{\mathrm{pre}}$ is that all the nontrivial G-torsors are missing! Indeed, the definition of G-torsor includes the requirement to be locally trivial for the given topology (in the first examples, this will be the étale topology). So far, the topology has not entered into any of our constructions. The topology will play an essential role in the definition of a stack, and in the procedure for recovering a stack from an atlas presentation.

3. Constructions with groupoid schemes

Groupoid schemes provide a means of carrying out explicit constructions which mirror what takes place in the world of stacks. Because of their explicit nature, carrying out these constructions provides a good way of getting a sense of how stacks behave. We focus on three concrete constructions.

EXAMPLE 3.13. There is an algebraic groupoid realization of fiber products. Let $R \rightrightarrows U$, $R' \rightrightarrows U'$, and $R'' \rightrightarrows U''$ be groupoid schemes, and let morphisms of groupoid schemes (ϕ', Φ') and (ϕ'', Φ'') to $R \rightrightarrows U$ from $R' \rightrightarrows U'$, resp. from $R'' \rightrightarrows U''$, be given. Then the category

$$[R' \rightrightarrows U']^{\mathrm{pre}} \times_{[R \rightrightarrows U]^{\mathrm{pre}}} [R'' \rightrightarrows U'']^{\mathrm{pre}} \tag{1}$$

is isomorphic to the category

$$[R' \times_U R \times_U R'' \rightrightarrows U' \times_U R \times_U U'']^{\mathrm{pre}}. \tag{2}$$

This is the groupoid where, in these fiber products the scheme to the left of U maps to U by a "target" map, and the scheme to the right maps by a "source" map (e.g., the first fiber product involves R' mapping to U by $t \circ \Phi'$) and where the source and target maps send (r', r, r'') to $(s'(r'), m(\Phi'(r'), r), s''(r''))$ and $(t'(r'), m(r, \Phi''(r'')), t''(r''))$, respectively. Indeed, an object over S of the fiber product (1) is a map $S \to U'$, a map $S \to U''$, and an isomorphism (map $S \to R$ whose composition with s is $S \to U' \to U$ and whose composition with t is $S \to U'' \to U$). That is precisely a morphism $S \to U' \times_U R \times_U U''$. A morphism over $\widetilde{S} \to S$ from $(\tilde{u}', \tilde{r}, \tilde{u}'')$ to (u', r, u'') will be a pair of morphisms, i.e., R'- and R''-valued points over T. Calling these r' and r'', the compatibility condition is the commutativity of the following square:

$$\begin{tikzcd}
\phi'(\tilde{u}') \arrow[r, "\Phi'(r')"] \arrow[d, "\tilde{r}"'] & \phi'(u') \arrow[d, "r"] \arrow[dl, dashed] \\
\phi''(\tilde{u}'') \arrow[r, "\Phi''(r'')"'] & \phi''(u'')
\end{tikzcd}$$

The middle factor R in $R' \times_U R \times_U R''$ corresponds to a choice of dotted arrow, so that $\tilde{r}$ and r are recovered as compositions of arrows — this accounts for the appearance of m in the formulas for the source and target maps of the groupoid scheme (2). We have an isomorphism on the level of objects and morphisms; from this the reader can work out the formulas for the identity, multiplication, and inverse in (2).

EXERCISE 3.3. Work this out explicitly in the case that the morphisms (ϕ', Φ') and (ϕ'', Φ'') are both the morphism $(e_G, 1_\Lambda)$ from $\Lambda \rightrightarrows \Lambda$ to $G \rightrightarrows \Lambda$. The result should be a groupoid presentation for $\underline{G}$ (in the sense of Example 3.8). Notice that the correct answer is *not* $\Lambda \times_G \Lambda \rightrightarrows \Lambda \times_\Lambda \Lambda$.

Example 3.14. Given $R' \rightrightarrows U'$ and $R' \rightrightarrows U$, a morphism $\gamma\colon U' \to R$ will associate to every object of $[R' \rightrightarrows U']^{\mathrm{pre}}$ a *morphism* in $[R \rightrightarrows U]^{\mathrm{pre}}$ (by composition with γ). We can give an explicit description of the category

$$\mathrm{HOM}([R' \rightrightarrows U']^{\mathrm{pre}}, [R \rightrightarrows U]^{\mathrm{pre}}). \tag{3}$$

The objects are morphisms (ϕ, Φ) of groupoid schemes. The morphisms from (ϕ, Φ) to $(\tilde{\phi}, \widetilde{\Phi})$ corespond bijectively with morphisms of schemes $\gamma\colon U' \to R$ satisfying $s \circ \gamma = \phi$, $t \circ \gamma = \tilde{\phi}$, and $m(\gamma \circ s, \widetilde{\Phi}) = m(\Phi, \gamma \circ t)$.

Every morphism of groupoid schemes will determine a morphism of stacks. The converse is not true, as we have seen. For instance, given a G-torsor on a scheme X, a corresponding morphism $\underline{X} \to BG$ will be (isomorphic to) one that comes from a morphism from $X \rightrightarrows X$ to $G \rightrightarrows \Lambda$ only when the given G-torsor is trivial. However, if we take $U \to X$ to be an étale cover which trivializes the given G-torsor, and $R = U \times_X U$ (Example 3.1), then $R \rightrightarrows U$ will be a different atlas for $\underline{X}$ (Example 3.8), and there will exist a morphism from $R \rightrightarrows U$ to $G \rightrightarrows \Lambda$ reproducing, up to isomorphism, the given morphism $\underline{X} \to BG$.

Example 3.14 will yield, as an application, a method for computing the set of 2-morphisms between a pair of morphisms of stacks. If f and g are morphisms of stacks $\mathfrak{X}' \to \mathfrak{X}$, represented concretely by morphisms of groupoids from $R' \rightrightarrows U'$ to $R \rightrightarrows U$, then any 2-morphism $f \Rightarrow g$ will come from a unique morphism in the HOM-category (3) of associated CFGs. Now we know that this HOM-category can be described entirely in terms of morphisms of schemes.

Example 3.15. Here is a general construction which produces a new groupoid scheme starting with a groupoid scheme $R \rightrightarrows U$ and a morphism $U' \to U$. This construction encapsulates refinements of atlases (e.g., when $U' \to U$ is an étale covering map), as well as the atlases of sub-CFGs (e.g., when U' is an open subscheme of U). First, we recall that a morphism of groupoid schemes $R' \rightrightarrows U'$ to $R \rightrightarrows U$ that satisfies Conditions 1.3(i)–(ii) corresponds, morally at least, to an isomorphism of stacks. Given $R \rightrightarrows U$ and an arbitrary morphism $U' \to U$, we choose the R' dictated by Condition 1.3(i). Then we obtain a groupoid scheme $R' \rightrightarrows U'$, where $R' = R \times_{U \times U} (U' \times U')$, the projection to $U' \times U'$ is (s', t'), and $m'((r, u_1', u_2'), (\tilde{r}, u_2', u_3')) = (m(r, \tilde{r}), u_1', u_3')$.

Exercise 3.4. In case $R \rightrightarrows U$ is the atlas for $\mathcal{M}_{1,1}$ (Example 3.10) and $U' = \mathrm{Spec}(\mathbb{C})$ is a point mapping to $u_0 \in U$, then R' will be $\mathrm{Aut}(E_0)$ where E_0 is the elliptic curve corresponding to the point u_0.

To continue the discussion of Example 3.15, suppose now that $U' \hookrightarrow U$ is the inclusion of a (locally closed) subscheme, and suppose further that

$$s^{-1}(U') = t^{-1}(U'). \tag{4}$$

Then R' will be equal to $s^{-1}(U')$. When $R \rightrightarrows U$ is an atlas for an algebraic stack $\mathfrak{X}$, there will be a dictionary between algebraic substacks of $\mathfrak{X}$ and subschemes U' of U satisfying (4). Substacks of an algebraic stack will play a role analogous to subschemes of a scheme. For instance, for $[X/G]$, these will be the $[Y/G]$ as Y ranges over the G-invariant subschemes of X.

Just as a complex algebraic variety is made up of a collection of points (satisfying the defining equations of the variety), a stack satisfying appropriate hypotheses (e.g., a reduced Deligne–Mumford stack, separated and of finite type over Spec($\mathbb{C}$)) will have "points", each of which is a copy of BG for some finite group G. The moduli stacks of curves and complex orbifolds described the Introduction can be thought of in this way, and one can thus get a rough picture of what a stack "looks like".

Condition (4) is significant; it fails, e.g., if U' is a point of U (for general $R \rightrightarrows U$). The next exercise indicates how to "saturate" U' to a bigger subscheme which will satisfy (4). So, for instance, in Exercise 3.4, a point of U will determine a point-like closed substack, isomorphic to $B(\mathrm{Aut}(E_0))$. The next exercise will tell us how to produce the corresponding closed subscheme of U that satisfies (4).

EXERCISE 3.5. Let U_0' be a subscheme of U. Define $U' = t(s^{-1}(U_0'))$. This satisfies (4), at least as subsets of R. [Hint: both sides are equal to $\mathrm{pr}_2((s \circ \mathrm{pr}_1)^{-1}(U_0'))$.]

In good cases, the U' produced in Exercise 3.5 will make sense as a subscheme of U, and (4) will hold as an equality of subschemes of R. For instance, usually s and t will be étale, or smooth, or flat and locally of finite presentation. Any of these is enough to guarantee that s and t are *open* morphisms. Then, whenever U_0' is an open subscheme of U, the scheme U' produced in Exercise 3.5 will be a larger open subscheme of U. It will satisfy the equality (4): both $s^{-1}(U')$ and $t^{-1}(U')$ will be equal to the same open subscheme $R' \subset R$. There will be a corresponding open substack, having a groupoid presentation $R' \rightrightarrows U'$. It will further emerge that the groupoid scheme $R_0' := R \times_{U \times U} (U_0' \times U_0') \rightrightarrows U_0'$ that arises by applying the construction of Example 3.15 to the morphism $U_0' \to U$, is *another* groupoid presentation for this substack.

In the next chapter, when we have the stackification $[R \rightrightarrows U]$ of $[R \rightrightarrows U]^{\mathrm{pre}}$ at our disposal, we will give statements concerning these stacks $[R \rightrightarrows U]$ which are analogous to the statements appearing in Examples 3.13 through 3.15.

Answers to Exercises

3.1. (a) $i \circ e = m \circ (e \circ s \circ (i \circ e), i \circ e) = m(e \circ t \circ e, i \circ e) = m(e, i \circ e) = e$. (b) $m \circ (e, e) = m \circ (e \circ s \circ e, e) = m \circ (e \circ s, 1_R) \circ e = e$. (c) Copy a proof that $(f \cdot g)^{-1} = g^{-1} \cdot f^{-1}$ in a group, or better, in a groupoid, with arrows $f \colon x \to y$, $g \colon y \to z$. Start with the analogue of $(g \cdot f)^{-1} \cdot (g \cdot f) = 1_x$, which is the identity $m \circ (m, i \circ m) = e \circ s \circ m$. Corresponding to $((g \cdot f)^{-1} \cdot g) \cdot f = 1_x$, we have the identity $m \circ (\mathrm{pr}_1, m \circ (\mathrm{pr}_2, i \circ m)) = e \circ s \circ m$ [which is true since $m \circ (\mathrm{pr}_1, m \circ (\mathrm{pr}_2, i \circ m)) = m \circ (m \circ (\mathrm{pr}_1, \mathrm{pr}_2), i \circ m)) = m \circ (m, i \circ m)$]. Next write $(g \cdot f)^{-1} \cdot g = f^{-1}$, and prove the identity $m \circ (\mathrm{pr}_2, i \circ m) = i \circ \mathrm{pr}_1$. Finally, multiply on the right by g^{-1}. This gives $m \circ (i \circ \mathrm{pr}_2, i \circ \mathrm{pr}_1) = m \circ (i \circ \mathrm{pr}_2, m \circ (\mathrm{pr}_2, i \circ m)) = m \circ (m \circ (i \circ \mathrm{pr}_2, \mathrm{pr}_2), i \circ m)) = m \circ (e \circ t \circ \mathrm{pr}_2, i \circ m)) = m \circ (e \circ t \circ \mathrm{pr}_2, i \circ m)) = m \circ (e \circ s \circ i \circ m, i \circ m)) = i \circ m$, as required. (d) The left-hand square comes from the following diagram with fiber squares

$$\begin{tikzcd}
R\,{}_t\!\times_s R \arrow[r, "(i\circ \mathrm{pr}_1, m)", "\sim"'] \arrow[d, "\mathrm{pr}_1"'] & R\,{}_t\!\times_s R \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & R \arrow[d, "s"] \\
R \arrow[r, "i", "\sim"'] & R \arrow[r, "t"] & R
\end{tikzcd}$$

while a similar diagram takes care of the right-hand square.

3.2. Given $S''' \to S'' \to S' \to S$, a triple of composable morphisms gives rise to $(\gamma'', \gamma' \circ f'', \gamma \circ f' \circ f'') \to R\,{}_t\!\times_s R\,{}_t\!\times_s R\,{}_t\!\times_s R$. Applying $m \circ (1_R \times m) = m \circ (m \times 1_R)$ gives associativity of composition of morphisms in $[R \rightrightarrows U]^{\mathrm{pre}}$.

3.3. This is $\Lambda \times_\Lambda G \times_\Lambda \Lambda \rightrightarrows \Lambda \times_\Lambda G \times_\Lambda \Lambda$, which is $G \rightrightarrows G$.

3.4. We have $R' = (s,t)^{-1}(u_0, u_0)$, which is the automorphism group of E_0.

3.5. We have $s^{-1}(t(s^{-1}(U_0'))) = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(s^{-1}(U_0'))) = \mathrm{pr}_2(m^{-1}(s^{-1}(U_0')))$. The last step, showing this equals $t^{-1}(t(s^{-1}(U_0')))$, uses the fact from Exercise 3.1(d) that $R\,{}_t\!\times_s R$ is, by (pr_2, m), isomorphic to the fiber product of $t\colon R \to U$ with itself.

CHAPTER 4

Stacks and Stackification

In this chapter we will endow our base category $\mathcal{S}$ with an additional structure, a *Grothendieck topology.* Using Grothendieck topologies it makes sense to speak of sheaves over a *category.* To a topological space, for instance, one associates the category of all open sets with inclusion maps. A sheaf over a topological space is given in terms of this category. Reflecting upon the definition of sheaves, one discovers that the essential notion needed to write down what a sheaf is, is that of a *covering family.* For the category associated with a topological space, these are just families of inclusions that cover the image.

Several of our examples drawn from algebraic geometry have illustrated the shortcomings of the usual Zariski topology. For many groups G (e.g., finite groups), G-torsors are almost never locally trivial for the Zariski topology. When we encountered modular families of elliptic curves and noted the property that every family can be analytically locally obtained from these by pullback, it was shown by example that this statement is no longer true in the algebraic setting if we use the Zariski topology. What works in both of these cases is to interpret "locally" to mean locally for the *étale topology*, which is a Grothendieck topology. For the formal definition of a Grothendieck topology see the Glossary.

A category endowed with a Grothendieck topology is called a *site.* Thus it makes sense to speak of sheaves on a site. We use the topology on $\mathcal{S}$ to define what it means for a CFG over $\mathcal{S}$ to be a *stack.* In fact, there will be two stack axioms. If a CFG satisfies only the first it is called a *prestack.* If the CFG is that associated with a presheaf as in Example 2.4, then the stack axioms reduce to the sheaf axioms. In this way, stacks appear as generalizations of sheaves.

In our "key case", where $\mathcal{S}$ is the category of schemes over the base scheme Λ, we put the étale topology on $\mathcal{S}$. In this topology a family of maps $\{U_i \to S\}$ is a covering family if all structure maps $U_i \to S$ are étale and $\coprod_i U_i \to S$ is surjective.

1. The stack axioms

Definition 4.1. A **site** is a category together with a Grotendieck topology. When the choice of Grothendieck topology on a category $\mathcal{S}$ is understood, we will often omit explicit mention of it and refer, e.g., to a CFG over a site $\mathcal{S}$.

Let $\mathcal{F} \to \mathcal{S}$ be a CFG over a site $\mathcal{S}$. The key case is that of $\mathcal{S}$ being the category of schemes over Λ with the étale topology, where Λ a fixed base scheme. The reader may consider only this case at a first reading. Another good example to keep in mind is the category of open subsets of a topological space, and inclusions of open subsets.

Let x and y be objects of $\mathcal{F}$ over the same object S of $\mathcal{S}$. Recall, if T is any scheme over S, with structural map $f\colon T \to S$, there are pullbacks $f^*(x)$ and $f^*(y)$, each defined up to canonical isomorphism. We define a presheaf $\underline{\mathrm{Isom}}_{\mathcal{F}}(x,y)$ by setting

$$\underline{\mathrm{Isom}}_{\mathcal{F}}(x,y)(T) = \{\text{isomorphisms from } f^*(x) \text{ to } f^*(y) \text{ in } \mathcal{F}_T\}.$$

Because this is defined for T equipped with a structural map to S, it is a presheaf on the category $\underline{S}$. (More common notation, in this context, would be $\mathcal{S}/S$, the *slice category* of schemes over S.) The definition appears to depend on choices of pullbacks $f^*(x)$ and $f^*(y)$, but if $\bar{x}$ and $\bar{y}$ denote other choices of pullbacks, then using the canonical isomorphisms $\bar{x} \cong f^*(x)$ and $\bar{y} \cong f^*(y)$ we can canonically identify the set of isomorphisms $\bar{x} \to \bar{y}$ with the set of isomorphisms $f^*(x) \to f^*(y)$. To be a presheaf means there are restriction maps: given $g\colon U \to S$ and a morphism $h\colon U \to T$ such that $g = f \circ h$, then there is a unique morphism $\psi\colon g^*(x) \to g^*(y)$ in $\mathcal{F}_U$ whose composite with $g^*(y) \to f^*(y)$ is equal to the composite $g^*(x) \to f^*(x) \xrightarrow{\varphi} f^*(y)$ for some given $\varphi \in \underline{\mathrm{Isom}}_{\mathcal{F}}(x,y)(T)$. Then $h^*\varphi$ is this morphism ψ.

The category $\underline{S}$ inherits a Grothendieck topology from $\mathcal{S}$, in which a set of morphisms in $\underline{S}$ is deemed a covering family if the collection of underlying morphisms of schemes in $\mathcal{S}$ is a covering family. The first of the stack axioms is that this presheaf is a sheaf for this inherited topology.

Axiom 1. If $\{T_\alpha \to T\}$ is a covering family in the category of schemes over S, then

$$\underline{\mathrm{Isom}}_{\mathcal{F}}(x,y)(T) \to \prod_{\alpha} \underline{\mathrm{Isom}}_{\mathcal{F}}(x,y)(T_\alpha) \rightrightarrows \prod_{\alpha,\beta} \underline{\mathrm{Isom}}_{\mathcal{F}}(x,y)(T_\alpha \times_T T_\beta)$$

is an exact sequence of sets.[1]

Definition 4.2. A CFG $\mathcal{F}$ which satisfies Axiom 1 (for every S and pair of objects x and y over S) is called a **prestack**.

Let us verify Axiom 1 in some of our examples. For $\mathcal{M}_g$, given two families $\pi\colon C \to S$ and $\pi'\colon C' \to S$, and given $T \to S$, then $\underline{\mathrm{Isom}}_{\mathcal{M}_g}(\pi,\pi')(T)$ is the set of isomorphisms $T \times_S C \to T \times_S C'$ over T. Let $\{T_\alpha \to T\}$ be a covering family. Let $\{T_\alpha \times_S C \to T_\alpha \times_S C'\}$ be isomorphisms. The condition to pull back by either projection to the same element of $\underline{\mathrm{Isom}}_{\mathcal{M}_g}(\pi,\pi')(T_\alpha \times_T T_\beta)$ for every α and β is equivalent to the equality of the two composite morphisms, involving the two projections:

$$\coprod_{\alpha,\beta} T_\alpha \times_T T_\beta \times_S C \rightrightarrows \coprod_{\alpha} T_\alpha \times_S C \to T \times_S C'.$$

By descent for morphisms to a scheme (Proposition A.13), this condition implies there is a unique morphism $T \times_S C \to T \times_S C'$ whose composite with $\coprod T_\alpha \times_S C \to T \times_S C$ is that indicated above. This morphism is an isomorphism because it becomes an isomorphism after faithfully flat base change (by [**EGA** IV.2.7.1]).

To verify that BG is a prestack, we use descent for morphisms as above to prove the existence of an isomorphism given one locally. It remains to see that it is G-equivariant,

[1] A sequence of sets $A \to B \rightrightarrows C$ is exact if A is mapped bijectively onto the set of elements in B which have the same image in C by the two maps from B to C.

but this amounts to checking equalities of morphisms, and again we use descent for morphisms (actually just the uniqueness part of descent for morphisms). The same argument applies for $\mathcal{M}_{g,n}$: we use descent for morphisms to produce the isomorphism, and then the uniqueness assertion of descent for morphisms to check compatibilities of the sections. Similar arguments apply for $[X/G]$, $\mathcal{V}_n$ (vector bundles), $\mathcal{C}_n$ (n-sheeted coverings) and the other variants of moduli stacks $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

The same kind of argument also applies to the CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ obtained from a groupoid scheme $R \rightrightarrows U$. In this CFG, we recall, a morphism is given by a map to R. So, if $R \rightrightarrows U$ is any groupoid scheme, the CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ is a prestack.

2. Stacks

A prestack is a *stack* if it satisfies a descent-type hypothesis, to the effect that an object can be constructed locally by gluing. We make use of projection maps $p_1\colon T_\alpha \times_T T_\beta \to T_\alpha$ and $p_2\colon T_\alpha \times_T T_\beta \to T_\beta$, or for $T' \to T$, projection maps $p_1, p_2\colon T'' \to T'$ where $T'' = T' \times_T T'$.

Axiom 2. If $\{T_\alpha \to T\}$ is a covering family, then given any collection of objects t_α over T_α and isomorphisms $\varphi_{\alpha\beta}\colon p_1^* t_\alpha \to p_2^* t_\beta$ over $T_\alpha \times_T T_\beta$ satisfying the cocycle condition, there is an object x over T and for each α, an isomorphism $\lambda_\alpha\colon x_\alpha \to t_\alpha$, where x_α denotes a pullback to T_α. These isomorphisms are required to satisfy the natural compatibility condition on $T_\alpha \times_T T_\beta$.

The cocycle condition states that, with projections $p_{12}\colon T_\alpha \times_T T_\beta \times_T T_\gamma \to T_\alpha \times_T T_\beta$, etc., the diagram

$$\begin{tikzcd}
p_{12}^*p_1^*t_\alpha \arrow[r, "p_{12}^*\varphi_{\alpha\beta}"] \arrow[d, equal] & p_{12}^*p_2^*t_\beta \arrow[r, equal] & p_{23}^*p_1^*t_\beta \arrow[d, "p_{23}^*\varphi_{\beta\gamma}"] \\
p_{13}^*p_1^*t_\alpha \arrow[r, "p_{13}^*\varphi_{\alpha\gamma}"'] & p_{13}^*p_2^*t_\gamma \arrow[r, equal] & p_{23}^*p_2^*t_\gamma
\end{tikzcd}$$

commutes, where the equal signs denote canonical isomorphisms of pullbacks. The natural compatibility condition on $T_\alpha \times_T T_\beta$ is the commutativity of the following diagram

$$\begin{tikzcd}
p_1^*x_\alpha \arrow[r, "p_1^*\lambda_\alpha"] \arrow[d, equal] & p_1^*t_\alpha \arrow[d, "\varphi_{\alpha\beta}"] \\
p_2^*x_\beta \arrow[r, "p_2^*\lambda_\beta"'] & p_2^*t_\beta
\end{tikzcd}$$

(again involving a canonical isomorphism of pullbacks denoted with an equal sign).

Definition 4.3. A CFG $\mathcal{F}$ is a **stack** if it satisfies both Axiom 1 and Axiom 2.

In Axiom 2, the tuple of objects x_α, together with isomorphisms $\varphi_{\alpha\beta}$ satisfying the cocycle condition, is called a **descent datum**. If the conclusion of Axiom 2 holds, we say the descent datum is **effective**. The condition in Axiom 2 can alternatively be expressed by writing T' for the disjoint union of all the T_α, and then speaking of a single object over T' and an isomorphism of its pullbacks to $T'' = T' \times_T T'$. In practice this gives an equivalent formulation of Axiom 2. But this involves two subtleties. First, the disjoint union of the T_α is not required to exist in the category $\mathcal{S}$. Second, if we restate Axiom 2 using only one-element covering families, then there will be no provision requiring a choice of objects t_i over T_i, for some collection of i, to determine an object over $\coprod T_i$. The category of schemes has arbitrary disjoint unions. The second issue is avoided when the following hypothesis is satisfied.

Hypothesis. We suppose $\mathcal{S}$ is the category of schemes over Λ with the étale topology, and $\mathcal{F}$ is a CFG which satisfies: for any collection of schemes S_α, if we set $S = \coprod_\alpha S_\alpha$, then some (or equivalently, any) choice of change of base functors $\mathcal{F}_S \to \mathcal{F}_{S_\alpha}$ determines an equivalence of categories

$$\mathcal{F}_S \to \prod_\alpha \mathcal{F}_{S_\alpha}.$$

(The product category, on the right, is the category whose objects are tuples of objects in $\mathcal{F}_{S_\alpha}$ for each α, and whose morphisms are tuples of morphisms.)

Assuming the Hypothesis, Axiom 2 is equivalent to:

Axiom 2′. If $f\colon T' \to T$ is a covering map (meaning that $\{f\}$ is a one-element covering family), and x' is any object over T', with isomorphism $\varphi\colon p_1^*x' \to p_2^*x'$ satisfying the cocycle condition $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$, then there exists an object x over T and an isomorphism $f^*x \to x'$ over T' such that $p_2^*\lambda = \varphi \circ p_1^*\lambda$.

The Hypothesis is satisfied by all of CFGs over schemes that we have seen as examples. An advantage of Axiom 2′ is that it nicely parallels many assertions from the theory of descent (Appendix A). For instance, when G is an affine group scheme over the base scheme Λ, then Axioms 1 and 2′ are implied by (a) and (b), respectively, of Corollary A.16. This gives us the first of several examples of stacks that we now list:

(1) BG is a stack, for any affine group scheme G over Λ.
(2) $\underline{X}$ is a stack for any scheme X: Axioms 1 and 2 follow from descent for morphisms to X.
(3) More generally, if $\mathcal{F} = \underline{h}$ is the CFG associated to a presheaf h on $\mathcal{S}$, then the stack axioms for $\mathcal{F}$ are equivalent to the sheaf axioms.
(4) Combining the first two examples, the CFG $[X/G]$ is a stack, for any affine group scheme G acting on a scheme X, respectively.
(5) The following are stacks: $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ for $g \geq 2$; $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ for $2g+n \geq 3$; $\overline{\mathcal{M}}_{g,n}(X,\beta)$. To show these are stacks, use Proposition A.18 to verify Axiom 2′, applied to the relative dualizing sheaf of a family of (stable or smooth) curves; the relative dualizing sheaf twisted by the sections; or twisted by the pullback of an ample line bundle on the projective variety X.
(6) The CFG $\mathcal{V}_n$ is a stack by Proposition A.11. The CFG $\mathcal{S}_n$ is a stack by Proposition A.12.

The hypothesis that G is an affine group scheme is a convenient one because it is satisfied for the most common linear algebraic groups (GL_n, PGL_n, etc.), as well as for finite groups. However the assertions can be generalized to the case G quasi-affine by appealing to the more general descent result Proposition A.17.

Exercise 4.1. Which of the following CFGs are prestacks? Which are stacks? (a) the CFG of families of (smooth) genus 0 curves. (b) The category of finite flat covers $E \to S$ of degree d (for some integer d). (c) The CFG associated with the presheaf whose sections on S are the isomorphism classes of families of elliptic curves over S. (d) The category of projectivized vector bundles $\mathbb{P}(V) \to S$.

If $\mathcal{F}$ and $\mathcal{G}$ are stacks, then a morphism of stacks from $\mathcal{F}$ to $\mathcal{G}$ will be just a morphism $f\colon \mathcal{F} \to \mathcal{G}$ of CFGs. We say f is an isomorphism when it is an isomorphism of CFGs (i.e., an equivalence of categories). If $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are stacks, and $\mathcal{F} \to \mathcal{H}$ and $\mathcal{G} \to \mathcal{H}$ are morphisms, then $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is a stack. Checking this involves just routine verificaion of axioms, e.g., a descent datum for $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ consists of a descent datum for $\mathcal{F}$, a descent datum for $\mathcal{G}$, and compatible isomorphisms in $\mathcal{H}$; by Axiom 2 for $\mathcal{F}$ and $\mathcal{G}$ we produce objects and by Axiom 1 for $\mathcal{H}$ we produce an isomorphism, which taken together show that the descent datum for $\mathcal{F} \times_{\mathcal{H}} \mathcal{G}$ is effective.

Remark 4.4. The stack axioms for a CFG on a general site can be stated in a way that avoids any reference to the presheaf $\underline{\mathrm{Isom}}_{\mathcal{F}}(x, y)$. It will be convenient to have these reformulations below, in §4.4. For Axiom 1, the CFG axioms let us identify isomorphisms $f^*(x) \to f^*(y)$ in $\mathcal{F}_T$ with morphisms $f^*(x) \to y$ over $f\colon T \to S$. Moreover the axiom applies in case x is an object over T rather than over S (by applying the axiom as stated to the objects x and $f^*(y)$ over T). So the axiom is equivalent to:

For any $f\colon T \to S$, objects y over S and t over T, covering family $\{T_\alpha \to T\}$ and morphisms $t_\alpha \to t$ over $T_\alpha \to T$, let $t_{\alpha\beta} \to t_\alpha$ and $t_{\alpha\beta} \to t_\beta$ be morphisms over the respective projections from $T_{\alpha\beta} := T_\alpha \times_T T_\beta$ such that the composite morphisms to t are equal. Then, composition with $t_\alpha \to t$ induces a bijection between morphisms $t \to y$ over f and tuples $(t_\alpha \to y)_\alpha$ over $T_\alpha \to S$ such that the diagram

$$\begin{tikzcd} t_{\alpha\beta} \arrow[r] \arrow[d] & t_\beta \arrow[d] \\ t_\alpha \arrow[r] & y \end{tikzcd}$$

commutes, for every α and β.

There is also a restatement of Axiom 2. Pullbacks in a CFG are only determined up to isomorphism, so there is no loss of generality in assuming every isomorphism $\varphi_{\alpha\beta}$ in the axiom to be the identity. This means objects $t_{\alpha\beta}$ are given, each with a morphism to t_α identifying $t_{\alpha\beta}$ with $p_1^* t_\alpha$ and a morphism to t_β identifying $t_{\alpha\beta}$ also with $p_2^* t_\beta$. We introduce $T_{\alpha\beta\gamma} := T_\alpha \times_T T_\beta \times_T T_\gamma$. If $t_{\alpha\beta\gamma} \to t_{\alpha\beta}$ is a morphism over $p_{12}\colon T_{\alpha\beta\gamma} \to T_{\alpha\beta}$, then the CFG axioms dictate a unique $t_{\alpha\beta\gamma} \to t_{\beta\gamma}$ over p_{23} making a commutative square with t_β. Axiom 2, restated, is that in the diagram

$$\begin{tikzcd} & t_{\alpha\beta\gamma} \arrow[dl] \arrow[d, shift right] \arrow[d, shift left] \arrow[dr] & \\ t_{\alpha\beta} \arrow[d] \arrow[dr] & t_{\alpha\gamma} \arrow[dl] \arrow[dr] & t_{\beta\gamma} \arrow[dl] \arrow[d] \\ t_\alpha & t_\beta & t_\gamma \end{tikzcd}$$

if the two curved arrows, defined to be the unique morphisms over p_{13} making the left-hand square resp. right-hand square commutative, are equal (this is the cocycle condition), then there exists an object t over T and morphisms $t_\alpha \to t$ such that the composites $t_{\alpha\beta} \to t_\alpha \to t$ and $t_{\alpha\beta} \to t_\beta \to t$ are equal, for every α and β.

3. Stacks from groupoid schemes

Given a groupoid scheme $R \rightrightarrows U$, we saw that the associated CFG $[R \rightrightarrows U]^{\mathrm{pre}}$ is a prestack. For instance, Example 3.10 shows that $[G \rightrightarrows \Lambda]^{\mathrm{pre}}$ is equivalent to the category of trivial G-torsors. A general G-torsor is only locally trivial, which means that Axiom 2 will fail.

In this case, it is clear how to proceed. We enlarge the category to contain all the locally trivial G-torsors, and we obtain the category BG which is (at least for an interesting class of group schemes G) a stack. In this section, we imitate this construction for a general groupoid scheme. There will be a notion of $(R \rightrightarrows U)$-torsor. This will be a stack when the groupoid satisfies a hypothesis that allows a result from the theory of descent to be applied, akin to requiring G to be affine or at least quasi-affine over the base scheme.

There are two main ideas that underlie the definition we are about to give. First, we are trying to imitate the case of BG, with its natural morphism $\underline{\Lambda} \to BG$ (corresponding to the trivial G-torsor $S \times G \to S$ for every S in $\mathcal{S}$). If T is a scheme and $\underline{T} \to BG$ is a morphism, corresponding to the G-torsor $E \to T$, then as we saw in Example 2.25(3) we have a 2-cartesian diagram

$$\begin{tikzcd} \underline{E} \arrow[r] \arrow[d] & \underline{\Lambda} \arrow[d] \\ \underline{T} \arrow[r] & BG \end{tikzcd}$$

So in the case of general $R \rightrightarrows U$ representing (eventually) a stack $\mathcal{F}$ we should be aiming to construct $\underline{T} \times_{\mathcal{F}} \underline{U}$. The second main idea is to focus on the diagonal $(s,t)\colon R \rightrightarrows U \times U$, which will captures some intrinsic properties of the stack. For instance, in the case of $\mathcal{F} = \underline{X}$, arbitrary U, and $R = U \times_X U$ (Example 3.1), the morphism $R \to U \times U$ is obtained by base change from the diagonal of X. Hence properties of X that are encoded in the diagonal show up as properties of $R \rightrightarrows U \times U$ (e.g., X separated implies $R \to U \times U$ is a closed embedding).

Consider the prestack $[R \rightrightarrows U]^{\mathrm{pre}}$. A typical descent datum, as in Axiom 2$'$, would consist of a morphism $\varphi\colon T' \to U$ (object over T') and a morphism $\Phi\colon T'' \to R$ satisfying $s \circ \Phi = \varphi \circ p_1$ and $t \circ \Phi = \varphi \circ p_2$ (morphism from the pullback by $p_1\colon T'' \to T'$ to the pullback by $p_2\colon T'' \to T'$), required to satisfy the cocycle condition; this, it can be checked, is precisely the condition for (φ, Φ) to be a morphism of groupoid schemes from $T'' \to T'$ to $R \rightrightarrows U$. One approach to get a stack would be add all such morphisms of groupoid schemes as extra objects. This would need to be done carefully, e.g., there need to be isomorphisms between objects over T defined relative to different covering schemes T'. We will comment on such an approach below, in Remark 4.15. The conceptually simpler approach that we take is to put restrictive hypotheses on $T' \to T$ to make it more canonical; these will eventually identify it with projection from $\underline{T} \times_{\mathcal{F}} \underline{U}$.

Locally, we can say what this needs to be. Over T', we have — at least on some formal level — identifications of $\underline{T'} \times_{\mathcal{F}} \underline{U}$ with $(\underline{T'} \times \underline{U}) \times_{\mathcal{F}\times\mathcal{F}} \mathcal{F}$, and then with $(T' \times U) \times_{U\times U} R$. That is, we want to replace $T' \to T$ with a scheme over T which, upon pullback by $T' \to T$, becomes isomorphic to $T' \times_{U,s} R$. This is a scheme over $T' \times U$ (by the first projection to T', and by the second projection composed with t to U) which is obtained by base change from the relative diagonal $R \to U \times U$. We will be able to apply descent and get a scheme over $T \times U$, provided that $R \to U \times U$ belongs to a class of morphisms which satisfies effective descent. We focus on the class of *quasi-affine* morphisms, for which the needed descent result is Proposition A.17.

Below we use notation for projections pr_1, pr_2, pr_{12}, etc., from fiber products, and we adopt the convention that in any $\times_U R$ the map is the source map s and in any $R \times_U$ the map is the target t.

DEFINITION 4.5. Let $R \rightrightarrows U$ be a groupoid scheme. Then an **étale locally trivial** $(R \rightrightarrows U)$**-torsor** over T consists of a scheme E over T with a morphism $\varphi\colon E \to U$ and a morphism $a\colon E \times_U R \to E$ over T which satisfy $\varphi \circ a = t \circ \mathrm{pr}_2$. Further, we require:

$$\begin{array}{ll} \text{(identity)} & a \circ (1_E, e \circ \varphi) = 1_E, \\ \text{(multiplication)} & a \circ (1_E \times m) = a \circ (a \times 1_R), \\ \text{(inverse)} & a \circ (a, i \circ \mathrm{pr}_2) = \mathrm{pr}_1 . \end{array}$$

There must exist étale surjective $T' \to T$ with a map $T' \to U$ and an isomorphism $\lambda\colon T' \times_T E \to T' \times_U R$ over T' such that $t \circ \mathrm{pr}_2 \circ \lambda = \varphi \circ \mathrm{pr}_2$ and $\lambda \circ (1_T \times a) = (1_T \times m) \circ (\lambda \times 1_R)$.

The conditions of Definition 4.5 can be expressed by the commutativity of the diagrams

$$\begin{tikzcd} E \arrow[r, "(1_E, e\circ\varphi)"] \arrow[dr, "1_E"'] & E \times_U R \arrow[d, "a"] \\ & E \end{tikzcd} \qquad \begin{tikzcd} E \times_U R \times_U R \arrow[r, "1_E \times m"] \arrow[d, "a \times 1_R"'] & E \times_U R \arrow[d, "a"] \\ E \times_U R \arrow[r, "a"] & E \end{tikzcd} \qquad \begin{tikzcd} E \times_U R \arrow[r, "(a, i\circ\mathrm{pr}_2)"] \arrow[dr, "\mathrm{pr}_1"'] & E \times_U R \arrow[d, "a"] \\ & E \end{tikzcd}$$

Together, they imply that

(1)
$$\begin{tikzcd} E \times_U R \arrow[r, "\mathrm{pr}_2"] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "a"] & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ E \arrow[r, "\varphi"] & U \end{tikzcd}$$

is a morphism of groupoid schemes. The groupoid scheme structure on the left is with identity morphism $(1_E, e \circ \varphi)$, inverse morphism $(a, i \circ \mathrm{pr}_2)$, and multiplication $m\colon (E \times_U R)\ {}_a\!\times_{\mathrm{pr}_1} (E \times_U R) \to E \times_U R$ defined by $((x, r), (a(x, r), r')) \mapsto (x, m(r, r'))$.

We note that the square (1) is cartesian with either pair of vertical morphisms (the left-hand morphisms, by construction, and the right-hand morphisms, by an application of the "inverse" condition of Definition 4.5).

The local triviality requirement dictates morphisms of groupoid schemes

$$\begin{tikzcd} T' \times_T E \times_U R \arrow[r, "\lambda\times 1_R"] \arrow[d, shift right, "\mathrm{pr}_{12}"'] \arrow[d, shift left, "1_{T'}\times a"] & T' \times_U R \times_U R \arrow[r, "\mathrm{pr}_3"] \arrow[d, shift right, "\mathrm{pr}_{12}"'] \arrow[d, shift left, "1_{T'}\times m"] & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ T' \times_T E \arrow[r, "\lambda"] & T' \times_U R \arrow[r, "t\circ \mathrm{pr}_2"] & U \end{tikzcd}$$

such that the composite morphism of groupoid schemes is the same as that which we obtain by making a base change of the morphisms $E \times_U R \rightrightarrows E$ by $T' \to T$ and composing with the morphism of groupoid schemes in (1). In particular, the large outer square is cartesian by either pair of vertical maps. The right half of the diagram has a natural groupoid scheme structure derived from the morphism $T' \to U$. The right-hand square is cartesian (by either pair of vertical maps). In the left-hand square the horizontal maps are isomorphisms. The maps pr_{12} and $1_T \times m$ identify $T' \times_U R \times_U R$ with the fiber product of $T' \times_U R$ with itself over T'. Consequently, the maps pr_{12} and $1_T \times a$ identify $T' \times_T E \times_U R$ with the fiber product of $T' \times_T E$ with itself over T'. Since this isomorphism is derived by base change by the étale surjective morphism $T' \to T$, we see that we have an isomorphism

$$(\mathrm{pr}_1, a)\colon E \times_U R \xrightarrow{\sim} E \times_T E$$

(it is an isomorphism because it becomes an isomorphism after after étale surjective base change). In other words, $E \times_U R \rightrightarrows E$ is an atlas for T, and (1) is a morphism of groupoid schemes to $R \rightrightarrows U$. Notice that this is a morphism of groupoid schemes which induces cartesian diagrams (by selecting the left-hand pair or right-hand pair of vertical maps in the diagram).

There is an obvious CFG structure on $(R \rightrightarrows U)$-torsors, in which a morphism over $S \to T$ is a diagram

$$\begin{tikzcd} D \times_U R \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & E \times_U R \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & R \arrow[d, shift right] \arrow[d, shift left] \\ D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & U \\ S \arrow[r] & T & \end{tikzcd}$$

such that the bottom square is cartesian (and hence so is the upper-left square, by either set of vertical morphisms).

Proposition 4.6. *Let $R \rightrightarrows U$ be a groupoid scheme over Λ. Assume that the relative diagonal $(s, t)\colon R \to U \times U$ is quasi-affine. Then the CFG of $(R \rightrightarrows U)$-torsors is a stack for the étale topology on schemes over the base scheme Λ.*

Proof. In any object (1) in the category of $(R \rightrightarrows U)$-torsors, the scheme E is quasi-affine over $T \times U$ (by its structure map to T and map φ to U). This is because after base change by $T' \times U \to T \times U$, it is identified with a morphism $T' \times_U R \to T' \times U$ which is obtained via base change from the quasi-affine relative diagonal (s,t):

$$\begin{tikzcd} T' \times_U R \arrow[r] \arrow[d] & R \arrow[d, "{(s,t)}"] \\ T' \times U \arrow[r] & U \times U \end{tikzcd}$$

Now we imitate the use of descent for schemes in Corollary A.16. Given a descent datum for an étale cover $T' \to T$, descent for schemes quasi-affine over their base schemes (Corollary A.17) provides a scheme E quasi-affine over $T \times U$. The action map $a\colon E \times_U R \to E$ is obtained using descent for morphisms. The compatibility of a with the map $E \to U$ and the additional conditions (identity, multiplication, inverse) all hold since they hold after base change by $T' \to T$. Moreover, the local triviality condition holds since it holds on the cover. $\square$

Definition 4.7. Let $R \rightrightarrows U$ be a groupoid scheme whose relative diagonal $R \to U \times U$ is quasi-affine. Then the **associated stack**, denoted $[R \rightrightarrows U]$, is defined to be the category of $(R \rightrightarrows U)$-torsors (which is indeed a stack, according to Proposition 4.6).

The "second half" of the dictionary between groupoid schemes and algebraic stacks, mentioned in the previous chapter, will be the statement that a given algebraic stack $\mathcal{F}$ is isomorphic to the stack $[R \rightrightarrows U]$ coming from a suitable cover $\underline{U} \to \mathcal{F}$. (As in Proposition 3.5, R is taken so $\underline{R} \cong \underline{U} \times_{\mathcal{F}} \underline{U}$.) So these are an important class of stacks to understand.

The hypothesis (s,t) quasi-affine is satisfied in many common situations. We list a few of these. First, whenever s and t are étale, and (s,t) is quasi-compact and separated, then it is quasi-affine; this is because any quasi-finite separated morphism is quasi-affine [**EGA** IV.18.12.12]. If s and t are themselves quasi-affine, then (s,t) will be quasi-affine provided R is quasi-separated, because it can be factored through $R \times R$. For instance, the groupoid scheme $X \times G \rightrightarrows X$ arising from a group action has this property whenever G is a group scheme, quasi-affine over Λ, and X is quasi-separated over Λ. (Any locally Noetherian scheme is quasi-separated over an arbitrary base scheme, so this applies to most group actions met in practice.)

Exercise 4.2. Show that BG is isomorphic (as a CFG) to the stack $[G \rightrightarrows \Lambda]$, where G is an affine group scheme over Λ. Do the same for $[X/G]$.

4. Stackification via torsors

Let $R \rightrightarrows U$ be a groupoid scheme. There is a natural morphism of CFGs from $[R \rightrightarrows U]^{\mathrm{pre}}$ to $[R \rightrightarrows U]$. To an object $T \to U$ in $[R \rightrightarrows U]^{\mathrm{pre}}$ we associate the natural groupoid structure that is obtained by pullback:

$$\begin{tikzcd} T\times_U R\times_U R \arrow[r, "\mathrm{pr}_3"] \arrow[d, shift right, "\mathrm{pr}_{12}"'] \arrow[d, shift left, "1_T\times m"] & R \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ T\times_U R \arrow[r, "t\circ\mathrm{pr}_2"] & U \end{tikzcd} \tag{2}$$

The scheme $T\times_U R$ with morphism $t\circ\mathrm{pr}_2$ to U and morphism $1_T\times m\colon T\times_R U\times_U R\to T\times_U R$ are an object of $[R\rightrightarrows U]$. Given a morphism $\gamma\colon T'\to R$ over $f\colon T'\to T$, between objects $g'\colon T'\to U$ and $g\colon T\to U$ (meaning, we recall, $s\circ\gamma=g'$ and $t\circ\gamma=g\circ f$), we define $\Gamma\colon T'\times_U R\to R$ by $\Gamma=m(i\circ\gamma\times 1_R)$. Then we send the morphism γ to the morphism

$$\begin{tikzcd} T'\times_U R\times_U R \arrow[r, "(f\circ\mathrm{pr}_1,\Gamma)\times 1_R"] \arrow[d, shift right, "\mathrm{pr}_{12}"'] \arrow[d, shift left, "1_{T'}\times m"] & T\times_U R\times_U R \arrow[d, shift right, "\mathrm{pr}_{12}"'] \arrow[d, shift left, "1_T\times m"] \\ T'\times_U R \arrow[r, "(f\circ\mathrm{pr}_1,\Gamma)"] & T\times_U R \end{tikzcd} \tag{3}$$

in $[R\rightrightarrows U]$.

In the case $[G\rightrightarrows\Lambda]$, this associates to an object T (i.e., $T\to\Lambda$) the trivial G-torsor $T\times G\to T$. A morphism $\gamma\colon T\to G$ over 1_T is sent to the G-equivariant isomorphism $T\times G\to T\times G$, $(x,g)\mapsto(x,\gamma(x)^{-1}g)$.

Exercise 4.3. Consider the descent datum of $[R\rightrightarrows U]^{\mathrm{pre}}$ over T given by a morphism of groupoid schemes $(\varphi,\Phi)\colon(T''\to T')\to(R\rightrightarrows U)$ (as discussed in §4.3). According to Proposition 4.6, this descent datum must be effective in $[R\rightrightarrows U]$, so it must lead to an object $E\to T$. Show that E can be obtained from the following recipe. The scheme $T'\times_U R$ is quasi-affine over $T'\times U$. The two pullbacks to $T''\times U$ can be identified with $T''\ {}_{s\circ\Phi}\!\times_s R$ and $T''\ {}_{t\circ\Phi}\!\times_s R$. Then the isomorphism

$$T''\ {}_{s\circ\Phi}\!\times_U R\xrightarrow{\psi}T''\ {}_{t\circ\Phi}\!\times_U R$$

given by $\psi=(\mathrm{pr}_1,m(i\circ\Phi\times 1_R))$ is a descent datum which by Proposition A.17 determines E, quasi-affine over $T\times U$, and an isomorphism $\lambda\colon T'\times_T E\to T'\times_U R$. The action $a\colon E\times_U R\to E$ is then determined by descent for morphisms of schemes by the requirement that $a\circ pr_{23}\colon T'\times_T E\times_U R\to E$ is equal to the composite

$$T'\times_T E\times_U R\xrightarrow{\lambda\times 1_R}T'\times_U R\times_U R\xrightarrow{1_{T'}\times m}T'\times_U R\xrightarrow{\lambda^{-1}}T'\times_T E\xrightarrow{\mathrm{pr}_2}E.$$

Definition 4.8. Let $\mathcal{F}_0$ be a CFG/$\mathcal{S}$. Then a **stackification** of $\mathcal{F}_0$ is a stack $\mathcal{F}$ together with a morphism of CFGs $b\colon\mathcal{F}_0\to\mathcal{F}$ such that for any stack $\mathcal{G}$ the functor

$$\mathrm{HOM}(\mathcal{F},\mathcal{G})\to\mathrm{HOM}(\mathcal{F}_0,\mathcal{G})$$

induced by composition with b is an equivalence of categories.

We wish to assert that the stack of $(R\rightrightarrows U)$-torsors is a stackification of $[R\rightrightarrows U]^{\mathrm{pre}}$. To do this, we first need a preliminary result.

Proposition 4.9. *Let $\mathcal{F}_0$ be a CFG, let $\mathcal{F}$ be a stack, and let $b\colon \mathcal{F}_0 \to \mathcal{F}$ be a morphism of CFGs which, as a functor, is full and faithful. If, for every object T in $\mathcal{S}$ and every object x of $\mathcal{F}$ over T there exists a covering family $\{T_\alpha \to T\}$ and for every α an object t_α in $\mathcal{F}_0$ and a morphism $b(t_\alpha) \to x$ over $T_\alpha \to T$, then $\mathcal{F}$ is a stackification of $\mathcal{F}_0$ (by the morphism b).*

Proof. To show that $\mathrm{HOM}(\mathcal{F},\mathcal{G}) \to \mathrm{HOM}(\mathcal{F}_0,\mathcal{G})$ is essentially surjective, we suppose that $f_0\colon \mathcal{F}_0 \to \mathcal{G}$ is given. Let x be as in the statement of the proposition. So there is a covering family $\{T_\alpha \to T\}$, with morphisms $b(t_\alpha) \to x$. Set $T_{\alpha\beta} = T_\alpha \times_T T_\beta$ and $T_{\alpha\beta\gamma} = T_\alpha \times_T T_\beta \times_T T_\gamma$. Choose $t_{\alpha\beta} \to t_\alpha$ over $T_{\alpha\beta} \to T_\alpha$. By the CFG axioms there is a unique morphism $b(t_{\alpha\beta}) \to b(t_\beta)$ over $T_{\alpha\beta} \to T_\beta$ whose composite with the morphism to x is equal to the composite $b(t_{\alpha\beta}) \to b(t_\alpha) \to x$, and since b is fully faithful this is the image under b of a morphism $t_{\alpha\beta} \to t_\beta$. Now let $t_{\alpha\beta\gamma} \to t_{\alpha\beta}$ be any morphism over $T_{\alpha\beta\gamma} \to T_{\alpha\beta}$ and let $t_{\alpha\beta\gamma} \to t_{\beta\gamma}$ be the unique morphism over $T_{\alpha\beta\gamma} \to T_{\beta\gamma}$ making a commutative diagram with t_β. The unique morphism $t_{\alpha\beta\gamma} \to t_{\alpha\gamma}$ over $T_{\alpha\beta\gamma} \to T_{\alpha\gamma}$ making a commutative diagram with t_α transforms via b to the unique morphism $b(t_{\alpha\beta\gamma}) \to b(t_{\alpha\gamma})$ whose composite with the morphism to x is equal to $b(t_{\alpha\beta\gamma}) \to b(t_{\alpha\beta}) \to x$. This in turn is equal to the composite $b(t_{\alpha\beta\gamma}) \to b(t_{\beta\gamma}) \to x$, equal to the image under b of the unique morphism $t_{\alpha\beta\gamma} \to t_{\alpha\gamma}$ making a commutative diagram with t_γ. So we have a commutative diagram

$$\begin{tikzcd}
& t_{\alpha\beta\gamma} \arrow[dl] \arrow[d] \arrow[dr] & \\
t_{\alpha\beta} \arrow[d] \arrow[dr] & t_{\alpha\gamma} \arrow[dl] \arrow[dr] & t_{\beta\gamma} \arrow[dl] \arrow[d] \\
t_\alpha & t_\beta & t_\gamma
\end{tikzcd}$$

If we apply f_0 we get a similar commutative diagram in $\mathcal{G}$. Since $\mathcal{G}$ is a stack, it follows from Axiom 2 (restated, as in Remark 4.4) that there is an object y, together with morphisms $f_0(t_\alpha) \to y$, making commutative diagrams with the $f_0(t_{\alpha\beta})$. We define $f(x) = y$.

It remains to specify how morphisms transform under f, and to supply a natural isomorphism $f \circ b \Rightarrow f_0$. Let $S \to T$ be a morphism, x an object of $\mathcal{F}$ over T, and u an object of $\mathcal{F}$ over S. Suppose $f(x) = y$, with $\{T_\alpha \to T\}$, t_α, $t_{\alpha\beta} \to t_\alpha$, $t_{\alpha\beta} \to t_\beta$, and $f_0(t_\alpha) \to y$ are as above. Suppose $f(u) = v$, with $\{S_\gamma \to S\}$, s_γ, $s_{\gamma\delta} \to s_\gamma$, $s_{\gamma\delta} \to s_\delta$, and $f_0(s_\gamma) \to v$, analogously. Now consider a morphism $u \to x$ over $S \to T$. We define $f(u \to x)$ to be the morphism $v \to y$ characterized as follows. For each α and γ let $S_{\alpha\gamma\varepsilon}$ be a collection of objects over $S_\gamma \times_T T_\alpha$ (with ε in an indexing set that depends on α and γ) such that for each fixed γ the collection $\{S_{\alpha\gamma\varepsilon} \to S_\gamma\}$ is a covering family. An example of such a family is $\{S_\gamma \times_T T_\alpha \to S_\gamma\}$. For each α, γ, and ε, let $s_{\alpha\gamma\varepsilon} \to s_\gamma$ be a morphism over $S_{\alpha\gamma\varepsilon} \to S_\gamma$ Now there is a unique morphism $b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha)$ over $S_{\alpha\gamma\varepsilon} \to T_\alpha$ whose composite with $b(t_\alpha) \to x$ is equal to the composite $b(s_{\alpha\gamma\varepsilon}) \to b(s_\gamma) \to u \to x$. As b is fully faithful this is b applied to a unique morphism $s_{\alpha\gamma\varepsilon} \to t_\alpha$. We may apply f_0 and compose with the morphism $f_0(t_\alpha) \to y$ to get a morphism

$$f_0(s_{\alpha\gamma\varepsilon}) \to y.$$

This, we claim, gives rise to a unique morphism $v \to y$ by Axiom 1, restated as in Remark 4.4; this we take to be $f(u \to x)$.

To verify this we consider $S_{\alpha\gamma\varepsilon\beta\delta\eta} := S_{\alpha\gamma\varepsilon} \times_S S_{\beta\delta\eta}$, with $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\alpha\gamma\varepsilon}$ lying over the first projection map and as usual a unique morphism $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\beta\delta\eta}$ whose image under b makes a commutative diagram with v. This last morphism can be obtained by starting with the unique morphism $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\gamma\delta}$ over $S_{\alpha\gamma\varepsilon\beta\delta\eta} \to S_{\gamma\delta}$ making a commutative diagram with s_γ, and then selecting the unique morphism over $S_{\alpha\gamma\varepsilon\beta\delta\eta} \to S_{\alpha\gamma\varepsilon}$ making the diagram below with s_δ commute

$$\begin{tikzcd}
s_{\gamma\delta} \arrow[d] & s_{\alpha\gamma\varepsilon\beta\delta\eta} \arrow[l, dashed] \arrow[d] \\
s_\gamma & s_{\alpha\gamma\varepsilon} \arrow[l]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
s_{\gamma\delta} \arrow[d] & s_{\alpha\gamma\varepsilon\beta\delta\eta} \arrow[l] \arrow[d, dashed] \\
s_\delta & s_{\beta\delta\eta} \arrow[l]
\end{tikzcd}$$

Then a diagram chase shows that after applying f_0 it gives a commutative diagram with v. To be able to apply Axiom 1 to get a morphism $v \to y$, we must verify that the two morphisms $f_0(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to y$ in the following diagram, going via $f_0(s_{\alpha\gamma\varepsilon})$ and via $f_0(s_{\beta\delta\eta})$, are equal:

$$\begin{tikzcd}
& f_0(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \arrow[dl] \arrow[d] \arrow[dr, dashed] & \\
f_0(s_{\alpha\gamma\varepsilon}) \arrow[dr] & f_0(s_{\beta\delta\eta}) \arrow[dr] & f_0(t_{\alpha\beta}) \arrow[dl] \arrow[d] \\
& f_0(t_\alpha) \arrow[d] & f_0(t_\beta) \arrow[dl] \\
& y &
\end{tikzcd}$$

There is a unique morphism $b(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to b(t_{\alpha\beta})$ whose composite with the morphism to x is equal to the composite $b(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to b(s_{\gamma\delta}) \to u \to x$. This morphism is b applied to some morphism $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to t_{\alpha\beta}$. If we can show that the composite with the map to t_α is equal to $s_{\alpha\gamma\varepsilon\beta\delta\eta} \to s_{\alpha\gamma\varepsilon} \to t_\alpha$, and analogously for the maps to t_β, then the dotted arrow above will make both of the upper squares in the diagram commute and the required equality of morphisms will follow. By faithfulness it suffices to verify these assertions after applying b. For the first assertion, both composites, further composed with $b(t_\alpha) \to x$, are equal to the composite $b(s_{\alpha\gamma\varepsilon\beta\delta\eta}) \to b(s_\gamma) \to u \to x$, and hence they are equal. Similarly, both composites to $b(t_\beta)$ are equal.

We claim that the morphism $v \to y$ thus produced is independent of the choice of covering families $\{S_{\alpha\gamma\varepsilon} \to S_\gamma\}$. For this, it suffices to check that the morphism produced is unchanged by refinement (i.e., replacing $S_{\alpha\gamma\varepsilon} \to S_\gamma$ by $S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon} \to S_\gamma$, where $\{S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon}\}$ is a covering family) and also unchanged by a change of maps to the T_α, i.e., re-indexing $S_{\alpha\gamma\varepsilon}$ as $S_{\alpha\beta\gamma\lambda}$ and replacing $S_{\alpha\beta\gamma\lambda} \to T_\alpha$ with some other map $S_{\alpha\beta\gamma\lambda} \to T_\beta$. We leave the case of refinements as an exercise: the first step is that, if we choose $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon}$ over $S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon}$, then the morphism $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(t_\alpha)$ that is stipulated above is the composite $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha)$. Let us treat changes of maps to the T_α. The maps $S_{\alpha\beta\gamma\lambda} \to T_\alpha$ and $S_{\alpha\beta\gamma\lambda} \to T_\beta$ determine a map $S_{\alpha\beta\gamma\lambda} \to T_{\alpha\beta}$. There is a unique morphism $b(s_{\alpha\beta\gamma\lambda}) \to b(t_{\alpha\beta})$ whose composite with the map to x is equal to the composite $b(s_{\alpha\beta\gamma\lambda}) \to b(s_\gamma) \to u \to x$. Therefore the composite morphism $b(s_{\alpha\beta\gamma\lambda}) \to b(t_{\alpha\beta}) \to b(t_\alpha)$ is the morphism stipulated in the construction. Analogously, the composite morphism to $b(t_\beta)$ is as stipulated in the construction applied to $S_{\alpha\beta\gamma\lambda} \to T_\beta$. The maps $f_0(s_{\alpha\beta\gamma\lambda}) \to f_0(t_\alpha) \to y$ and $f_0(s_{\alpha\beta\gamma\lambda}) \to f_0(t_\beta) \to y$ are then equal, since they both factor through $f_0(t_{\alpha\beta})$.

We have, indeed, produced a functor f. Indeed, if $w \to u$ is a morphism in $\mathcal{F}_0$ over $R \to S$, with image $z \to v$ in $\mathcal{G}$, then the composite $z \to v \to y$ is seen to satisfy the criteria characterizing the image under f of the composite $w \to u \to x$. For, if $\{R_\pi \to R\}$ is a covering family, with $R_{\gamma\pi\iota} \to R_\pi \times_S S_\gamma$, then we take $R_{\alpha\gamma\varepsilon\pi\iota}$ over $R_{\gamma\pi\iota} \times_{S_\gamma} S_{\gamma\alpha\varepsilon}$ such that for every fixed π, γ, and ι, $\{R_{\alpha\gamma\varepsilon\pi\iota} \to R_{\gamma\pi\iota}\}$ is a covering family. We use the characterization of $f(w \to x)$ coming from the $R_{\alpha\gamma\varepsilon\pi\iota}$. Let us suppose $f(w \to u)$ is constructed using $r_{\gamma\pi\iota} \to r_\pi$ and $r_{\gamma\pi\iota} \to s_\gamma$. Now choose $r_{\alpha\gamma\varepsilon\pi\iota} \to r_{\gamma\pi\iota}$ over $R_{\alpha\gamma\varepsilon\pi\iota} \to R_{\gamma\pi\iota}$, and let $r_{\alpha\gamma\varepsilon\pi\iota} \to s_{\alpha\gamma\varepsilon}$ be the unique morphism over $R_{\alpha\gamma\varepsilon\pi\iota} \to S_{\alpha\gamma\varepsilon}$ whose composite with $s_{\alpha\gamma\varepsilon} \to s_\gamma$ is equal to the composite $r_{\alpha\gamma\varepsilon\pi\iota} \to r_{\gamma\pi\iota} \to s_\gamma$. We have

$$\begin{tikzcd}
 & & r_{\alpha\gamma\varepsilon\pi\iota} \arrow[dl] \arrow[dr] & & \\
 & r_{\gamma\pi\iota} \arrow[dl] \arrow[dr] & & s_{\alpha\gamma\varepsilon} \arrow[dl] \arrow[dr] & \\
r_\pi & & s_\gamma & & t_\alpha
\end{tikzcd}$$

which, upon applying b, commutes with $w \to u \to x$, and upon applying f_0, commutes with $z \to v \to y$, so the latter is equal to $f(w \to x)$. It follows immediately from the construction that $f(1_x) = 1_y$. When $x = b(t)$ we have $f_0(b(t))$ in $\mathcal{G}$, with maps $f_0(t_\alpha) \to f_0(b(t))$ such that both composites $f_0(t_{\alpha\beta}) \to f_0(b(t))$ are equal. Hence there is a unique isomorphism $f_0(b(t)) \to y$ compatible with the morphisms from the $f_0(t_\alpha)$. If $u = b(s)$ and we have a morphism $u \to x$ equal to b applied to some $s \to t$, then the composite $v \xrightarrow{\sim} f_0(b(s)) \to f_0(b(t)) \xrightarrow{\sim} y$ satisfies the criterion which characterizes $f(u \to x)$. Hence we have a natural isomorphism $f_0 \circ b \Rightarrow f$.

We now show that the functor between HOM categories is fully faithful. Let f and g be morphisms $\mathcal{F} \to \mathcal{G}$, and let $f_0 = f \circ b$ and $g_0 = g \circ b$. Given a natural isomorphism $f_0 \Rightarrow g_0$ we need to show it is produced by a unique natural isomorphism $f \Rightarrow g$. Let $b(t_\alpha) \to x$ be as in the hypothesis. If we set $y = f(x)$ and $z = g(x)$ then the given natural isomorphism yields morphisms $f_0(t_\alpha) \to g_0(t_\alpha) \to z$. With $t_{\alpha\beta} \to t_\alpha$ and $t_{\alpha\beta} \to t_\beta$ such that the composite morphisms to x are equal, we have the composite $f_0(t_{\alpha\beta}) \to f_0(t_\alpha) \to z$ equal to $f_0(t_{\alpha\beta}) \to g_0(t_{\alpha\beta}) \to z$, which is equal to $f_0(t_{\alpha\beta}) \to f_0(t_\beta) \to z$. So, by Axiom 1, restated as in Remark 4.4, there is a uniquely determined isomorphism $y \to z$. Naturality is the condition that $v \to y \to z$ is equal to $v \to w \to z$, where $u \to x$ is a morphism over some $S \to T$, and $v = f(u)$ and $w = g(u)$. We introduce $S_{\alpha\gamma\varepsilon}$ and $s_{\alpha\gamma\varepsilon}$ as above; notice that the image $v \to y$ of $u \to x$ under f has the property that the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y$ is equal to the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(t_\alpha) \to y$, and that a similar assertion holds with g_0 in place of f_0 and $w \to z$ in place of $v \to y$. To verify naturality, it suffices by Axiom 1 to verify that the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y \to z$ is equal to the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to w \to z$, for every α, γ, and ε This is now a routine diagram chase, using the naturality of $f_0 \Rightarrow g_0$ and the fact that the morphism $y \to z$, resp.

$v \to w$, is characterized by its fitting in a commutative diagram with $f_0(t_\alpha) \to g_0(t_\alpha)$, resp. with $f_0(s_\gamma) \to g_0(s_\gamma)$. □

EXERCISE 4.4. Supply the details to the argument that the morphism $v \to y$ (image of $u \to x$ under f) is unchanged by refinement.

PROPOSITION 4.10. *Let $R \rightrightarrows U$ be a groupoid scheme with quasi-affine relative diagonal $(s,t)\colon R \to U \times U$. Then the morphism $b\colon [R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$ defined in* (2) *and* (3) *is a stackification.*

PROOF. We use Proposition 4.9. By the definition of $[R \rightrightarrows U]$, every object is locally isomorphic to an object in the image of b. So, it remains only to show b is fully faithful. Morphisms in $[R \rightrightarrows U]^{\mathrm{pre}}$ from $g'\colon T' \to U$ to $g\colon T \to U$ over $f\colon T' \to T$ are morphisms $\gamma\colon T' \to R$ satisfying $s \circ \gamma = g'$ and $t \circ \gamma = g \circ f$. In $[R \rightrightarrows U]$, the morphisms over f from $b(g')$ to $b(g)$ are diagrams (3) in which $\Gamma\colon T' \times_U R \to R$ satisfies $s \circ \Gamma = g \circ f \circ \mathrm{pr}_1$, $t \circ \Gamma = t \circ \mathrm{pr}_2$, and $m(\Gamma \times 1_R) = \Gamma \circ (1_{T'} \times m)$. The functor b sends γ to $m \circ (i \circ \gamma \times 1_R)$. We define a map the other way sending Γ to $i \circ \Gamma \circ (1_{T'}, e \circ g')$. Composing the two maps in one order we have γ mapping to $i \circ m(i \circ \gamma, e \circ g') = i \circ m(i \circ \gamma, e \circ t \circ i \circ \gamma) = \gamma$. In other order, Γ is sent to

$$\begin{aligned} m \circ (\Gamma \times 1_R) \circ [(1_{T'}, e \circ g') \times 1_R] &= \Gamma \circ (1_{T'} \times m) \circ [(1_{T'}, e \circ g') \times 1_R] \\ &= \Gamma \circ (\mathrm{pr}_1, m(e \circ g' \times 1_R)) \\ &= \Gamma. \end{aligned}$$

Hence b is fully faithful. □

REMARK 4.11. The universal property of Definition 4.8 characterizes the stackification — not an isomorphism of categories (this would be to strong) — but instead up to an isomorphism of CFGs which is canonical up to a canonical 2-isomorphism. This means: if $\mathcal{F}'$ with $b'\colon \mathcal{F}_0 \to \mathcal{F}'$ also satisfies the criterion of Definition 4.8 then there exists an isomorphism $i\colon \mathcal{F} \to \mathcal{F}'$ and a 2-isomorphism $i \circ b \Rightarrow b'$. And if $j\colon \mathcal{F} \to \mathcal{F}'$ is another isomorphism, with $j \circ b \Rightarrow b'$, then there exists a unique 2-isomorphism $i \Rightarrow j$ such that the diagram

$$\begin{tikzcd} i \circ b \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\ j \circ b \arrow[r, Rightarrow] & b' \end{tikzcd}$$

commutes.

Now we can give statements about the stacks associated with a CFG which are the analogues of the statements for the corresponding prestacks that have appeared in Examples 3.11 through 3.13.

EXAMPLE 4.12. Given groupoid schemes $R \rightrightarrows U$, $R' \rightrightarrows U'$, and $R'' \rightrightarrows U''$, and morphisms of groupoid schemes $(R' \rightrightarrows U') \to (R \rightrightarrows U)$ and $(R'' \rightrightarrows U'') \to (R \rightrightarrows U)$, then we have

$$[R' \rightrightarrows U'] \times_{[R \rightrightarrows U]} [R'' \rightrightarrows U''] \cong [R' \times_U R \times_U R'' \rightrightarrows U' \times_U R \times_U U''].$$

Indeed, we have a morphism from the fiber product of the associated prestacks to $[R' \rightrightarrows U'] \times_{[R\rightrightarrows U]} [R'' \rightrightarrows U'']$ (by the universal property of the fiber product). This is readily seen to be fully faithful. Moreover any object of $[R' \rightrightarrows U'] \times_{[R\rightrightarrows U]} [R'' \rightrightarrows U'']$ becomes, after étale pullback, isomorphic to an object in the image of this morphism. So the criterion of Proposition 4.9 is satisfied. Combining the isomorphism of Example 3.11 with Remark 4.11, we get the promised isomorphism of stacks.

Example 4.13. A morphism of groupoid schemes from $R' \rightrightarrows U'$ to $R \rightrightarrows U$ determines, as we have seen, a morphism of associated prestacks. This in turn determines a morphism of stacks. Indeed, composition with the stackification morphism $b\colon [R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$ is a fully faithful functor (since b is a fully faithful functor) from $\mathrm{HOM}([R' \rightrightarrows U']^{\mathrm{pre}}, [R \rightrightarrows U]^{\mathrm{pre}})$ to $\mathrm{HOM}([R' \rightrightarrows U']^{\mathrm{pre}}, [R \rightrightarrows U])$. This latter category is equivalent to the category $\mathrm{HOM}([R' \rightrightarrows U'], [R \rightrightarrows U])$ by the stackification property. Consequently, to every morphism (ϕ, Φ) of groupoid schemes from $R' \rightrightarrows U'$ to $R \rightrightarrows U$ there corresponds a morphism $f\colon [R' \rightrightarrows U'] \to [R \rightrightarrows U]$ (determined up to 2-isomorphism). Let $(\tilde{\phi}, \widetilde{\Phi})$ be another, with corresponding $\tilde{f}$. Then there is a canonically induced bijection between the morphisms in $\mathrm{HOM}([R' \rightrightarrows U']^{\mathrm{pre}}, [R \rightrightarrows U]^{\mathrm{pre}})$ from (ϕ, Φ) to $(\tilde{\phi}, \widetilde{\Phi})$ (which were described concretely in Example 3.12) and the 2-morphisms $f \Rightarrow \tilde{f}$.

Example 4.14. Let a groupoid scheme $R \rightrightarrows U$ be given. If $R' \rightrightarrows U'$ is a groupoid scheme with morphism of groupoid scheme to $R \rightrightarrows U$ satisfying Condition 1.3(i)–(ii) (this means R' is isomorphic to $R \times_{U\times U} (U' \times U')$ and $U' \times_U R \to U$ is a covering map) then we claim that the corresponding

$$[R' \rightrightarrows U'] \to [R \rightrightarrows U]$$

is an isomorphism. If we follow the steps of the previous example, we see this is associated, by the stackification property for $[R' \rightrightarrows U']$, with the composite morphism $[R' \rightrightarrows U']^{\mathrm{pre}} \to [R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$. This is a composite of fully faithful functors, hence is fully faithful. An object of $[R \rightrightarrows U]$ is locally isomorphic to one coming from $[R \rightrightarrows U]^{\mathrm{pre}}$. This in turn is locally isomorphic to one coming from $[R' \rightrightarrows U']^{\mathrm{pre}}$. For this last step, start with $T \to U$. By Condition 1.3(ii), we have a covering map $T' := U' \times_U R \times_U T \to T$. There are morphisms $\bar{s}, \bar{t}\colon T' \to U$ corresponding to the maps s, t respectively from the middle factor R. They are isomorphic objects of $[R \rightrightarrows U]^{\mathrm{pre}}$ over T'. The latter is the pullback of the given object over T, while the former is in the image of the morphism $[R' \rightrightarrows U']^{\mathrm{pre}} \to [R \rightrightarrows U]^{\mathrm{pre}}$.

In particular, for X a scheme we have $[X \rightrightarrows X] \cong \underline{X}$ (easy), so if $U \to X$ is a covering map and $R = U \times_X U$ then $[R \rightrightarrows U] \cong \underline{X}$.

Remark 4.15. There is another point of view on the stackification of $[R \rightrightarrows U]^{\mathrm{pre}}$. Recall, we have seen that a typical descent datum for this prestack has the form of an étale cover $T' \to T$ and a morphism of groupoid schemes $(T'' \rightrightarrows T') \to (R \rightrightarrows U)$ (where $T'' = T' \times_T T'$). Each $T' \to T$ determines a category

$$\mathrm{HOM}([T'' \rightrightarrows T']^{\mathrm{pre}}, [R \rightrightarrows U]^{\mathrm{pre}})$$

explicitly the category whose objects are morphisms of groupoid schemes and morphisms are maps $T' \to R$ as described in Example 3.12. As the cover T' is refined this category captures more of the full stackification. Indeed one finds the claim, e.g. in [**89**], that a fiber $[R \rightrightarrows U]_T$ of the stackification will be a sort of direct limit over covering maps $T' \to T$ of the categories of morphisms of groupoid schemes from $T'' \rightrightarrows T'$ to $R \rightrightarrows U$. Certainly, any such morphism of groupoid schemes determines a morphism from $[T'' \rightrightarrows T']$ to $[R \rightrightarrows U]$ (Example 4.13). We have just seen that $[T'' \rightrightarrows T'] \cong \underline{T}$. So we have $\underline{T} \to [R \rightrightarrows U]$, i.e. (Proposition 2.20) an object of $[R \rightrightarrows U]_T$.

It is beyond the scope of this text to discuss limits of categories. Just as CFGs form a 2-category rather than just a category and therefore familiar constructions such as the fiber product must be handled with special care, so do categories (the 2-category of categories is one of the principal examples), making limits of categories a tricky topic. The concrete stackification presented in this chapter is sufficient to develop the theory of algebraic stacks. A more abstract stackification — one which applies to an arbitrary CFG — will be given in Part II.

Answers to Exercises

4.1. The CFG in (a) is a stack; we can apply descent for projective morphisms taking as relatively ample invertible sheaf the relative anticanonical sheaf a smooth family of genus 0 curves. Any finite morphism is affine, so (b) is a stack. In (c) the prestack axiom would say that two families of elliptic curves which are locally isomorphic must be isomorphic; the existence of isotrivial families of elliptic curves means that this CFG is not even a prestack. The CFG in (d) is a prestack by the usual argument, descent for morphisms to a target scheme, but it is not a stack: a conic over a non-algebraically closed field (of characteristic $\neq 2$) with no rational points is not the projectivization of a vector bundle, but it becomes isomorphic to $\mathbb{P}^1$ after a quadratic field extension.

4.2. In the case of $G \rightrightarrows \Lambda$, Definition 4.5 exactly reduces to the definition of G-torsor. In the case of $X \times G \rightrightarrows X$, we have $\varphi\colon E \to X$ which is required to satisfy $\varphi(e \cdot g) = \varphi(e) \cdot g$ for $e \in E$, $g \in G$, i.e., is G-equivariant.

4.3. $T' \times_U R$ and the displayed isomorphism are what result by applying the morphism $[R \rightrightarrows U]^{\mathrm{pre}} \to [R \rightrightarrows U]$ to the given descent datum.

4.4. Let $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon}$ be a morphism over $S_{\alpha\gamma\varepsilon\lambda} \to S_{\alpha\gamma\varepsilon}$, so that the composite $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon} \to s_\gamma$ lies over $S_{\alpha\gamma\varepsilon\lambda} \to S_\gamma$. Let $s_{\alpha\gamma\varepsilon} \to t_\alpha$ be the morphism that is stipulated in the proof of the proposition, i.e., such that $b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha) \to x$ is equal to the composite $b(s_{\alpha\gamma\varepsilon}) \to b(s_\gamma) \to u \to x$. Now the composite $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(s_{\alpha\gamma\varepsilon}) \to b(t_\alpha) \to x$ is equal to the composite $b(s_{\alpha\gamma\varepsilon\lambda}) \to b(s_{\alpha\gamma\varepsilon}) \to b(s_\gamma) \to u \to x$. So, in the construction applied to the refined cover, we must take as morphism over $S_{\alpha\gamma\varepsilon\lambda} \to T_\alpha$ the composite $s_{\alpha\gamma\varepsilon\lambda} \to s_{\alpha\gamma\varepsilon} \to t_\alpha$. The proposition produces $v \to y$ such that the composite $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y$ is equal to $f_0(s_{\alpha\gamma\varepsilon}) \to f_0(t_\alpha) \to y$. So the composite $f_0(s_{\alpha\gamma\varepsilon\lambda}) \to f_0(s_{\alpha\gamma\varepsilon}) \to f_0(s_\gamma) \to v \to y$ is equal to $f_0(s_{\alpha\gamma\varepsilon\lambda}) \to f_0(s_{\alpha\gamma\varepsilon}) \to f_0(t_\alpha) \to y$, and thus the same morphism $v \to y$ is the unique morphism dictated by Axiom 1 for the refined cover.

APPENDIX A

Descent Theory

1. History and motivation

The theory of descent in modern algebraic geometry was introduced by Grothendieck in the Séminaire Bourbaki [**35**], with details and proofs offered in SGA 1 [**38**, Exposé VIII]. The origins of the subject go back at least to Weil, although his (less general) results predate the modern language of schemes. This Appendix gives a self-contained treatment of some of the more important results which are generally gathered under the heading, "theory of descent." Some of the easier steps are left as exercises, but all of these are solved in the Answers.

The idea behind descent is that, under appropriate hypotheses, objects and morphisms over a scheme can be described locally. An object is described (uniquely up to canonical isomorphism) by an object on some cover, plus a gluing map satisfying a cocycle condition. A morphism between two objects thus specified is determined by giving a morphism locally (i.e., on the cover), which is compatible with the gluing maps.

Let us spell this out in the particular case of vector bundles on schemes, and for simplicity, we take our covers to be Zariski covers, fine enough that they give local trivializations. So let T be a scheme, and $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_T$-modules of some finite rank r. Then there exists a Zariski open cover (U_i) of T and isomorphisms

$$\lambda_i\colon \mathcal{E}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r} \tag{1}$$

of $\mathcal{O}_{U_i}$-modules for each i. If we set $U_{ij} = U_i \cap U_j$, then for any pair i and j, the isomorphisms λ_i and λ_j determine isomorphisms $\varphi_{ij}\colon \mathcal{O}_{U_{ij}}^{\oplus r} \to \mathcal{O}_{U_{ij}}^{\oplus r}$ via the diagram

$$\begin{tikzcd} & \mathcal{O}_{U_{ij}}^{\oplus r} \arrow[dd, "\varphi_{ij}"] \\ \mathcal{E}|_{U_{ij}} \arrow[ur, "\lambda_i|_{U_{ij}}"] \arrow[dr, "\lambda_j|_{U_{ij}}"'] & \\ & \mathcal{O}_{U_{ij}}^{\oplus r} \end{tikzcd} \tag{2}$$

Note that specifying the transition mappings φ_{ij} is the same as giving GL_r-valued transition functions on each U_{ij}. The φ_{ij} satisfy the *cocycle condition*: φ_{ii} is the identity map for every i, and for every triple i, j, k, if we set $U_{ijk} = U_i \cap U_j \cap U_k$, then we have

$$(\varphi_{jk}|_{U_{ijk}}) \circ (\varphi_{ij}|_{U_{ijk}}) = \varphi_{ik}|_{U_{ijk}}. \tag{3}$$

(Note that the condition that each φ_{ii} be the identity follows from the latter condition, applied to the triple i, i, i.)

Now *descent* for locally free sheaves in the Zariski topology is a collection of assertions which imply that, given an open cover (U_i), and a collection of morphisms φ_{ij} satisfying the cocycle condition (3), then there exists a locally free sheaf $\mathcal{E}$ (unique up to canonical isomorphism) together with local trivializations (1) such that the diagram (2) commutes for all i and j. There is a similar assertion for morphisms, to the effect that if the locally free sheaf $\mathcal{F}$ also admits local trivializations $\mu_i \colon \mathcal{F}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus s}$ and transition maps ψ_{ij}, then there is a bijection between morphisms $h \colon \mathcal{E} \to \mathcal{F}$ of locally free sheaves and collections of morphisms $h_i \colon \mathcal{O}_{U_i}^{\oplus r} \to \mathcal{O}_{U_i}^{\oplus s}$ for all i such that the diagram

$$\begin{tikzcd} \mathcal{O}_{U_{ij}}^{\oplus r} \arrow[r, "h_i|_{U_{ij}}"] \arrow[d, "\varphi_{ij}"'] & \mathcal{O}_{U_{ij}}^{\oplus s} \arrow[d, "\psi_{ij}"] \\ \mathcal{O}_{U_{ij}}^{\oplus r} \arrow[r, "h_j|_{U_{ij}}"'] & \mathcal{O}_{U_{ij}}^{\oplus s} \end{tikzcd}$$

commutes, for all i and j.

The assertions just spelled out are artificially restrictive. Indeed it is not necessary for $\mathcal{E}$ to be trivialized on the cover (U_i). In fact we need not restrict to locally free sheaves; the same considerations work in the context of arbitrary quasi-coherent sheaves. Given a Zariski open covering $\{U_i\}$ of T, and a collection $\mathcal{E}_i$ of quasi-coherent sheaves on U_i, with isomorphisms $\varphi_{ij} \colon \mathcal{E}_i|_{U_{ij}} \to \mathcal{E}_j|_{U_{ij}}$ of sheaves of $\mathcal{O}_{U_{ij}}$-modules, satisfying the cocycle condition (3), then there is a quasi-coherent sheaf $\mathcal{E}$ on T, with isomorphisms $\mathcal{E}|_{U_i} \to \mathcal{E}_i$, giving rise to these transition homomorphisms. And there is a similar version of descent for morphisms between quasi-coherent sheaves: if $\mathcal{E}$ comes from $\mathcal{E}_i$ and φ_{ij}, and $\mathcal{F}$ comes from $\mathcal{F}_i$ and ψ_{ij}, then there is a canonical bijection between morphisms $h \colon \mathcal{E} \to \mathcal{F}$ and collections $h_i \colon \mathcal{E}_i \to \mathcal{F}_i$ of morphisms such that $\psi_{ij} \circ h_i|_{U_{ij}} = h_j|_{U_{ij}} \circ \varphi_{ij}$ for all i, j. We will use this fact, which is a basic construction in algebraic geometry; a reference is [**EGA** 0.3.3.1].

These assertions can be stated more succinctly, avoiding all the indices, by defining T' to be the disjoint union of the open sets U_i, which comes with a canonical mapping $T' \to T$. The sheaves $\mathcal{E}_i$ determine a sheaf $\mathcal{E}'$ on T'. The transition functions φ_{ij} amount to an isomorphism

$$\varphi \colon p_1^*(\mathcal{E}') \to p_2^*(\mathcal{E}')$$

on $T' \times_T T'$, where p_1 and p_2 are the projections from $T' \times_T T'$ to T'. The cocycle condition asserts that $p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi)$ on $T' \times_T T' \times_T T'$, where

$$p_{ij} \colon T' \times_T T' \times_T T' \to T' \times_T T'$$

are the projections to the corresponding factors.

A key feature of Grothendieck's descent theory is that it extends from Zariski coverings to the more general étale and smooth coverings that are required for the theory of stacks. In fact, the appropriate morphisms to use are quite general flat morphisms.

Notation A.1. *Given a morphism $f \colon T' \to T$ of schemes, set $T'' = T' \times_T T'$, with its two projections p_1 and p_2 from T'' to T'. Let $T''' = T' \times_T T' \times_T T'$, which comes* *with three projections p_{12}, p_{13}, and p_{23} from T''' to T''. We also have three projections q_1, q_2, and q_3 from T''' to T', with $q_i = p_1 \circ p_{ij}$ and $q_j = p_2 \circ p_{ij}$, $1 \le i < j \le 3$.*

Descent for objects says that an object specified on a cover, together with a patching isomorphism satisfying a cocycle condition, determines an object defined on the base, and this object is unique up to canonical isomorphism. In more traditional terminology, every **descent datum** (pair consisting of an object defined on the cover, with a patching isomorphism satisfying the cocycle condition) is **effective** (determines an object on the base); the object on the base that realizes this effectivity is called a **solution** to the descent problem posed by the given datum.

Descent for morphisms says that, if we are given two sets of descent data, together with respective objects on the base (solutions to the descent data), then to give a morphism between these objects is the same as to give a morphism between the objects on the cover, subject to a compatibility condition.

The next theorem spells this out in the case of quasi-coherent sheaves on schemes. Following the statement are detailed explanations of its assertions. The next two sections are devoted to the proof of the theorem, while the rest of this appendix will discuss applications to other descent situations, especially those involving schemes instead of quasi-coherent sheaves.

THEOREM A.2. *Let $f\colon T' \to T$ be a flat morphism of schemes. Assume, further, that f is surjective and either quasi-compact or locally of finite presentation.* (a) *Let $\mathcal{E}'$ be a quasi-coherent sheaf on T' and $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ an isomorphism on T'' such that*

$$p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$$

on T'''. Then there exists a quasi-coherent sheaf $\mathcal{E}$ on T and an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ on T' satisfying

$$p_2^*\lambda = \varphi \circ p_1^*\lambda$$

on T''. Moreover the pair consisting of the sheaf $\mathcal{E}$ and the isomorphism λ is unique up to canonical isomorphism.
(b) *With notation as in* (a)*, suppose $(\mathcal{F}', \psi)$ is another descent datum with solution given by $\mathcal{F}$ and μ. Then, for every morphism $h'\colon \mathcal{E}' \to \mathcal{F}'$ on T' satisfying*

$$p_2^*h' \circ \varphi = \psi \circ p_1^*h'$$

on T'', there is a unique morphism $h\colon \mathcal{E} \to \mathcal{F}$ on T such that $\mu \circ f^*h = h' \circ \lambda$ on T'.

The hypotheses on the morphism f (flat, surjective, etc.) will be discussed in Section 3. The hypothesis in (a), that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$, means that the diagram

$$\begin{tikzcd}
p_{12}^*p_1^*\mathcal{E}' \arrow[r, "p_{12}^*\varphi"] \arrow[d, equal] & p_{12}^*p_2^*\mathcal{E}' \arrow[r, equal] & p_{23}^*p_1^*\mathcal{E}' \arrow[d, "p_{23}^*\varphi"] \\
p_{13}^*p_1^*\mathcal{E}' \arrow[r, "p_{13}^*\varphi"'] & p_{13}^*p_2^*\mathcal{E}' \arrow[r, equal] & p_{23}^*p_2^*\mathcal{E}'
\end{tikzcd}$$

commutes. The three equal signs denote canonical isomorphisms coming from the equalities $p_1 \circ p_{jk} = q_j = p_2 \circ p_{ij}$.

The conclusion in (a), that $p_2^*\lambda = \varphi \circ p_1^*\sigma$, means that the diagram

$$\begin{tikzcd} p_1^* f^* \mathcal{E} \arrow[r, "p_1^*\lambda"] \arrow[d, equal] & p_1^* \mathcal{E}' \arrow[d, "\varphi"] \\ p_2^* f^* \mathcal{E} \arrow[r, "p_2^*\lambda"'] & p_2^* \mathcal{E}' \end{tikzcd}$$

commutes.

We clarify what it means in (a) for the solution to be unique up to canonical isomorphism. Precisely, it means that if $\mathcal{F}$ is another quasi-coherent sheaf on T, and $\mu\colon f^*\mathcal{F} \to \mathcal{E}'$ is an isomorphism on T' satisfying $p_2^*\mu = \varphi \circ p_1^*\mu$ on T'', i.e., the diagram

$$\begin{tikzcd} p_1^* f^* \mathcal{F} \arrow[r, "p_1^*\mu"] \arrow[d, equal] & p_1^* \mathcal{E}' \arrow[d, "\varphi"] \\ p_2^* f^* \mathcal{F} \arrow[r, "p_2^*\mu"'] & p_2^* \mathcal{E}' \end{tikzcd}$$

commutes, then there is a unique isomorphism $h\colon \mathcal{E} \to \mathcal{F}$ such that $\mu \circ f^*h = \lambda$ on T', i.e., the diagram

$$\begin{tikzcd} f^*\mathcal{E} \arrow[rr, "f^*h"] \arrow[dr, "\lambda"'] & & f^*\mathcal{F} \arrow[dl, "\mu"] \\ & \mathcal{E}' & \end{tikzcd}$$

commutes. This uniqueness claim is in fact a special case of (b), applied to the identity morphism on $\mathcal{E}'$.

The hypothesis in (b), that $p_2^*h' \circ \varphi = \psi \circ p_1^*h'$, means that the diagram

$$\begin{tikzcd} p_1^* \mathcal{E}' \arrow[r, "\varphi"] \arrow[d, "p_1^*h'"'] & p_2^* \mathcal{E}' \arrow[d, "p_2^*h'"] \\ p_1^* \mathcal{F}' \arrow[r, "\psi"'] & p_2^* \mathcal{F}' \end{tikzcd}$$

commutes.

Finally, the conclusion in (b), that $\mu \circ f^*h = h' \circ \lambda$, means that the diagram

$$\begin{tikzcd} f^*\mathcal{E} \arrow[r, "f^*h"] \arrow[d, "\lambda"'] & f^*\mathcal{F} \arrow[d, "\mu"] \\ \mathcal{E}' \arrow[r, "h'"'] & \mathcal{F}' \end{tikzcd}$$

commutes.

2. The affine case

The general case of Theorem A.2 will be reduced to the affine case, which amounts to some elementary commutative algebra. This algebra is worked out in this section. No Noetherian or finiteness conditions on either rings or modules are required.

We are concerned with an arbitrary homomorphism $A \to A'$ of commutative rings with unit, which corresponds to a morphism $f\colon T' \to T$, with $T = \mathrm{Spec}(A)$ and $T' = \mathrm{Spec}(A')$. Let $A'' = A' \otimes_A A'$, and $A''' = A' \otimes_A A' \otimes_A A'$, so we have identifications $T'' = \mathrm{Spec}(A'')$ and $T''' = \mathrm{Spec}(A''')$. The projections p_1 and p_2 from T'' to T' correspond to the homomorphisms $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$ from A' to $A' \otimes_A A'$. Similarly, the projections p_{12}, p_{13}, and p_{23} from T''' to T'' correspond to the mappings from $A' \otimes_A A'$ to $A''' = A' \otimes_A A' \otimes_A A'$ that take $x \otimes y$ to $x \otimes y \otimes 1$, $x \otimes 1 \otimes y$, and $1 \otimes x \otimes y$, respectively. Projections q_1, q_2, and q_3 from T''' to T' correspond to mappings $A' \to A'''$ given by $x \mapsto x \otimes 1 \otimes 1$, $x \mapsto 1 \otimes x \otimes 1$, and $1 \otimes 1 \otimes x$, respectively.

Definition A.3. A homomorphism $A \to A'$ of commutative rings with unit is **flat** if, for any exact sequence $M_1 \to M_2 \to M_3$ of A-modules, the induced sequence $A' \otimes_A M_1 \to A' \otimes_A M_2 \to A' \otimes_A M_3$ (of A'-modules) is exact. The homomorphism is called **faithfully flat** if it is flat and the corresponding map $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective.

Exercise A.1. (1) Show that a flat homomorphism $A \to A'$ is faithfully flat if and only if, for any nonzero A-module M, $A' \otimes_A M \neq 0$. (2) Show that a homomorphism $A \to A'$ is faithfully flat if and only if the following condition is satisfied: a sequence $M' \to M \to M''$ of A-modules is exact if and only if the sequence $A' \otimes_A M' \to A' \otimes_A M \to A' \otimes_A M''$ is exact.

Exercise A.2. Suppose $A \to A'$ is faithfully flat. (1) Show that a homomorphism $M \to N$ of A-modules is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if the homomorphism $A' \otimes_A M \to A' \otimes_A N$ is a monomorphism (resp. epimorphism, resp. isomorphism). (2) Show that an A-module M is finitely generated (resp. finitely presented, resp. flat, resp. locally free of finite rank n) if and only if the A'-module $A' \otimes_A M$ is finitely generated (resp. finitely presented, resp. flat, resp. locally free of finite rank n).

For any homomorphism $A \to A'$, and any A-module M, there is a canonical homomorphism $\gamma\colon M \to A' \otimes_A M$, taking u to $1 \otimes u$. There are two canonical homomorphisms $A' \otimes_A M \to A' \otimes_A A' \otimes_A M$, taking $x \otimes u$ to $x \otimes 1 \otimes u$ and $1 \otimes x \otimes u$, corresponding to the two projections p_1 and p_2.

Lemma A.4. *Let M be an A-module. If $A \to A'$ is faithfully flat, then*

$$M \xrightarrow{\gamma} A' \otimes_A M \rightrightarrows A' \otimes_A A' \otimes_A M$$

is exact, that is, the canonical homomorphism γ maps M isomorphically to the set of elements in $A' \otimes_A M$ that have the same image in $A' \otimes_A A' \otimes_A M$ by the two projection homomorphisms. Equivalently, if one defines a homomorphism δ from $A' \otimes_A M$ to

$A' \otimes_A A' \otimes_A M$ by the formula $\delta(x \otimes u) = 1 \otimes x \otimes u - x \otimes 1 \otimes u$, then the sequence

$$0 \to M \xrightarrow{\gamma} A' \otimes_A M \xrightarrow{\delta} A' \otimes_A A' \otimes_A M$$

of A-modules is exact.

Proof. By Exercise A.2 (2), it suffices to show that the sequence becomes exact after tensoring it (on the left) over A by A', i.e., that the sequence

$$0 \longrightarrow A' \otimes_A M \xrightarrow{A' \otimes \gamma} A' \otimes_A A' \otimes_A M \xrightarrow{A' \otimes \delta} A' \otimes_A A' \otimes_A A' \otimes_A M$$

is exact. Let $\mu \colon A' \otimes_A A' \to A'$ be the multiplication map, $\mu(x \otimes y) = xy$. The injectivity of the first map $A' \otimes \gamma$ is now immediate, since the mapping $\mu \otimes M \colon A' \otimes_A A' \otimes_A M \to A' \otimes_A M$ gives a left inverse to it. Suppose an element $\sum x_i \otimes y_i \otimes u_i$ is in the kernel of $A' \otimes \delta$, i.e.

$$\sum x_i \otimes 1 \otimes y_i \otimes u_i = \sum x_i \otimes y_i \otimes 1 \otimes u_i.$$

Applying μ to the first two factors yields

$$\sum x_i \otimes y_i \otimes u_i = \sum x_i y_i \otimes 1 \otimes u_i,$$

and $\sum x_i y_i \otimes 1 \otimes u_i$ is the image of $\sum x_i y_i \otimes u_i$ in $A' \otimes_A A' \otimes_A M$, as required. $\square$

The proof of this lemma is a common one in descent theory: one makes a faithfully flat base extension to achieve the situation where the covering map $T' \to T$ has a section, in which case the assertion proves itself.

Although we don't need it, a natural generalization of this lemma is true:

Exercise A.3. Define the *Amitsur complex* $T^\bullet = T^\bullet(A'/A)$ for a homomorphism $A \to A'$ by setting $T^0 = A$, and, for $n \geq 1$, T^n is the tensor product of n copies of A' over A. Define $\delta^n \colon T^n \to T^{n+1}$ by: δ^0 is the given map from A to A', and $\delta^n = \sum_{i=0}^{n} (-1)^i \epsilon_i$, where $\epsilon_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n$. This is a complex of A-modules. Show that, for any A-module M, if $A \to A'$ is faithfully flat, the complex $T^\bullet \otimes_A M$ is exact.

Descent for morphisms of modules amounts to the following easy consequence of the preceding lemma:

Lemma A.5. *If $A \to A'$ is faithfully flat, and M and N are A-modules, then the sequence*

$$\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_{A'}(A' \otimes_A M, A' \otimes_A N) \rightrightarrows \mathrm{Hom}_{A' \otimes_A A'}(A' \otimes_A A' \otimes_A M, A' \otimes_A A' \otimes_A N)$$

is exact.

Proof. The exactness of Lemma A.4, applied to N, together with the left exactness of Hom, gives the exactness of

$$\mathrm{Hom}_A(M,N) \to \mathrm{Hom}_A(M, A' \otimes_A N) \rightrightarrows \mathrm{Hom}_A(M, A' \otimes_A A' \otimes_A N).$$

Using the identifications $\mathrm{Hom}_A(M,P) = \mathrm{Hom}_B(B \otimes_A M, P)$ for any homomorphism $A \to B$ and any B-module P, first for $B = A'$ and then for $B = A' \otimes_A A'$, translates this exact sequence into the exact sequence of the lemma. $\square$

Now let M' be an A'-module. We have, as we recall, projection maps p_1 and p_2 from $\mathrm{Spec}(A'')$ to $\mathrm{Spec}(A')$, where $A'' = A' \otimes_A A'$. Hence we have pullbacks $p_1^*(M') = A'' \otimes_{p_1, A'} M'$ and $p_2^*(M') = A'' \otimes_{p_2, A'} M'$. The two pullbacks $p_1^*(M')$ and $p_2^*(M')$ can be identified with $M' \otimes_A A'$ and $A' \otimes_A M'$ respectively, where the actions of A'' on these modules are given by $(x \otimes y) \cdot (u \otimes z) = xu \otimes yz$ and $(x \otimes y) \cdot (z \otimes u) = xz \otimes yu$ respectively, with x, y, and z in A' and u in M'. Similarly, the three pullbacks of M' by q_1, q_2, and q_3 to A''' can be identified with $M' \otimes_A A' \otimes_A A'$, $A' \otimes_A M' \otimes_A A'$, and $A' \otimes_A A' \otimes_A M'$, respectively, again with the diagonal actions of $A''' = A' \otimes_A A' \otimes_A A'$.

Suppose $\varphi\colon M' \otimes_A A' = p_1^*(M') \to p_2^*(M') = A' \otimes_A M'$ is an isomorphism of A''-modules. This determines by the three pullbacks p_{ij}, isomorphisms

$$\varphi_{ij} = p_{ij}^*(\varphi) : \; q_i^*(M') = p_{ij}^*(p_1^*(M')) \;\to\; p_{ij}^*(p_2^*(M')) = q_j^*(M').$$

For example, φ_{12} is the map from $M' \otimes_A A' \otimes_A A'$ to $A' \otimes_A M' \otimes_A A'$ that takes $u \otimes x \otimes y$ to $\varphi(u \otimes x) \otimes y$; that is, if $\varphi(u \otimes x) = \sum x_i \otimes u_i$, then $\varphi_{12}(u \otimes x \otimes y) = \sum x_i \otimes u_i \otimes y$. Similarly, $\varphi_{13}(u \otimes y \otimes x) = \sum x_i \otimes y \otimes u_i$, and $\varphi_{23}(y \otimes u \otimes x) = \sum y \otimes x_i \otimes u_i$.

Descent for modules amounts to the following assertion:

LEMMA A.6. *Suppose $A \to A'$ is faithfully flat, M' is an A'-module, and $\varphi\colon M' \otimes_A A' \to A' \otimes_A M'$ is an isomorphism of A''-modules such that $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ from $q_1^*(M')$ to $q_3^*(M')$. Define the A-module M by*

$$M = \{\, u \in M' \mid \varphi(u \otimes 1) = 1 \otimes u \,\}.$$

Then the canonical homomorphism $\lambda\colon A' \otimes_A M \to M'$, $x \otimes u \mapsto x \cdot u$, is an isomorphism.

PROOF. Let $\tau\colon M' \to A' \otimes_A M'$ be defined by $\tau(u) = 1 \otimes u - \varphi(u \otimes 1)$. We have an exact sequence

$$0 \to M \to M' \xrightarrow{\tau} A' \otimes_A M'$$

Tensoring this on the right with A' over A gives the top row of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_A A' \arrow[r] \arrow[d, "\psi"] & M' \otimes_A A' \arrow[r] \arrow[d, "\varphi"] & A' \otimes_A M' \otimes_A A' \arrow[d, "A'\otimes\varphi"] \\
0 \arrow[r] & M' \arrow[r] & A' \otimes_A M' \arrow[r] & A' \otimes_A A' \otimes_A M'
\end{tikzcd}$$

The bottom row is the exact sequence from Lemma A.4, applied to the A-module M'. The map ψ is defined by $\psi(u \otimes x) = x \cdot u$, and we want to show ψ is an isomorphism. Since the rows are exact, and the right two vertical maps are isomorphisms, this conclusion will follow if we verify that the diagram is commutative.

The left square commutes since, for u in M and x in A', $\varphi(u \otimes x) = (1 \otimes x)\varphi(u \otimes 1) = (1 \otimes x)(1 \otimes u) = 1 \otimes xu$. To prove that the right diagram commutes, we must show that, for any u in M' and x in A', the element $u \otimes x$ in $M' \otimes_A A'$ has the same image by either route around the square. Let $\varphi(u \otimes 1) = \sum y_i \otimes v_i$, with $y_i \in A'$ and $v_i \in M'$. Then

$$\varphi(u \otimes x) = (1 \otimes x)\varphi(u \otimes 1) = \sum y_i \otimes x v_i,$$

so the image of $u \otimes x$ by the lower route is

$$\sum 1 \otimes y_i \otimes x v_i - \sum y_i \otimes 1 \otimes x v_i.$$

On the upper route, $u\otimes x$ maps to the right to $1\otimes u\otimes x - \varphi(u\otimes 1)\otimes x = 1\otimes u\otimes x - \sum y_i\otimes v_i\otimes x$, which maps down to

$$1\otimes\varphi(u\otimes x) - \sum y_i\otimes\varphi(v_i\otimes x) = \sum 1\otimes y_i\otimes xv_i - \sum y_i\otimes\varphi(v_i\otimes x).$$

We are therefore reduced to verifying that

$$\sum y_i\otimes\varphi(v_i\otimes x) = \sum y_i\otimes 1\otimes xv_i.$$

But this is exactly the assertion that $\varphi_{23}(\varphi_{12}(u\otimes 1\otimes x)) = \varphi_{13}(u\otimes 1\otimes x)$. $\square$

To complete the proof that the construction of this lemma solves the descent problem for modules, i.e., that it solves case (a) of Theorem A.2, we must verify that the identity $\varphi\circ p_1^*\lambda = p_2^*\lambda$ is satisfied. This amounts to verifying that the diagram

$$\begin{array}{ccc} A'\otimes_A M\otimes_A A' & \xrightarrow{p_1^*\lambda} & M'\otimes_A A' \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\varphi} \\ A'\otimes_A A'\otimes_A M & \xrightarrow{p_2^*\lambda} & A'\otimes_A M' \end{array}$$

commutes, where $\kappa(x\otimes u\otimes y) = x\otimes y\otimes u$. This amounts to the identity $x\otimes\lambda(y\otimes u) = \varphi(\lambda(x\otimes u)\otimes y)$, i.e., $x\otimes yu = \varphi(xu\otimes y)$, or $(x\otimes y)(1\otimes u) = (x\otimes y)\varphi(u\otimes 1)$, which follows from the fact that u is in M.

Similarly, we want Lemma A.5 to give a proof of (b) of Theorem A.2 in the affine case. This means that we have A'-modules M' and N', with isomorphisms

$$\varphi\colon M'\otimes_A A'\to A'\otimes_A M' \quad\text{ and }\quad \psi\colon N'\otimes_A A'\to A'\otimes_A N',$$

and we have A-modules M and N, with isomorphisms $\lambda\colon A'\otimes_A M\to M'$ and $\mu\colon A'\otimes_A N\to N'$, satisfying $\varphi\circ p_1^*\lambda = p_2^*\lambda$ and $\psi\circ p_1^*\mu = p_2^*\mu$. We are given a homomorphism $h'\colon M'\to N'$ of A'-modules, satisfying the identity $p_2^*(h')\circ\varphi = \psi\circ p_1^*(h')$. We must show that there is a unique homomorphism $h\colon M\to N$ of A-modules such that $\mu\circ(A'\otimes h) = h'\circ\lambda$. Set $g' = \mu^{-1}\circ h'\circ\lambda\colon A'\otimes_A M\to A'\otimes_A N$. If we show that $p_1^*(g') = p_2^*(g')$, then Lemma A.5 will produce a unique homomorphism $h\colon M\to N$ such that $g' = A'\otimes h$. This says that $h'\circ\lambda = \mu\circ(A'\otimes h)$, as required. To conclude the proof, we calculate:

$$\begin{aligned} p_1^*(g') = p_1^*(\mu^{-1}\circ h'\circ\lambda) &= p_1^*(\mu)^{-1}\circ p_1^*(h'\circ\lambda) = p_2^*(\mu)^{-1}\circ\psi\circ p_1^*(h')\circ p_1^*(\lambda) \\ &= p_2^*(\mu)^{-1}\circ p_2^*(h')\circ\varphi\circ p_1^*(\lambda) = p_2^*(\mu^{-1}\circ h')\circ p_2^*(\lambda) = p_2^*(g'), \end{aligned}$$

as required. The uniqueness assertion in (a) is a special case of (b), so the theorem is proved in the affine case.

The overall structure of the proofs in this section is worth noting, as it will be repeated below in the proof of Theorem A.2. First, we proved descent for morphisms in the case of objects pulled back from the base (Lemma A.5). Then we showed that every descent datum is effective (Lemma A.6). We saw as a formal consequence that descent for morphisms holds in the case of an arbitrary pair of descent data, each admitting a solution, and from this that the solution to any descent problem is unique up to canonical isomorphism.

3. The general case

In this section, we complete the proof of Theorem A.2. Recall that a morphism $f\colon T' \to T$ of schemes is **faithfully flat** if it is flat and surjective. It is not enough to assume f is faithfully flat for the conclusions of the theorem to hold, as we'll see below in Exercise A.6. To pass from the affine case (Lemmas A.5 and A.6) to the case of general schemes we'll need some additional hypothesis on the morphism f. In fact, there are two additional hypotheses that we may impose, and either one will suffice to establish descent for objects and morphisms, in the context of quasi-coherent sheaves:

(i) f is **fpqc**, that is, faithfully flat and *quasi-compact*. We recall this means that the pre-image, under f, of any affine open subset of the base is covered by finitely many affine open subsets.

(ii) f is **fppf**, that is, faithfully flat and *locally of finite presentation*. The important fact needed here is that every morphism that is flat and locally of finite presentation is open [**EGA** IV.2.4.6].

The notations fpqc and fppf come from the French terminology for the conditions on f (*fidèlement plat, quasi-compact* and *fidèlement plat, de présentation finie*).

As described at the end of the previous section, to prove Theorem A.2, it suffices to prove descent for morphisms of objects pulled back from the base and to show that every descent datum is effective. In other words, Theorem A.2 follows from the following pair of assertions.

Lemma A.7. *Assume $f\colon T' \to T$ is* (i) *fpqc or* (ii) *fppf. Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent sheaves on T. Then, for every morphism $h'\colon f^*\mathcal{E} \to f^*\mathcal{F}$ on T' such that $p_1^*h' = p_2^*h'$ on T'' there is a unique morphism $h\colon \mathcal{E} \to \mathcal{F}$ on T such that $f^*h = h'$.*

Lemma A.8. *Assume $f\colon T' \to T$ is* (i) *fpqc or* (ii) *fppf. Let $\mathcal{E}'$ be a quasi-coherent sheaf on T' and $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ an isomorphism on T'' such that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$ on T'''. Then there exists a quasi-coherent sheaf $\mathcal{E}$ on T and an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ on T' such that $p_2^*\lambda = \varphi \circ p_1^*\lambda$ on T''.*

Let us say that f satisfies **descent for morphisms** if the conclusion of Lemma A.7 is valid for f. Let us say that f satisfies **effective descent** if both the conclusion of Lemma A.7 and of Lemma A.8 are valid for f. We have proved in the previous section that every faithfully flat morphism of affine schemes satisfies effective descent. We saw in the first section that every Zariski open covering satisfies effective descent. These two facts will be combined to deduce what is claimed in Lemmas A.7 and A.8, namely that every morphism that is fpqc or fppf satisfies effective descent.

The argument rests on the following two claims. Let $f\colon S \to T$ and $g\colon R \to S$ be morphisms of schemes.

First claim: Suppose g satisfies descent for morphisms, and suppose for any morphism $g'\colon R' \to S'$ obtained from g by a base change with respect to an arbitrary morphism $S' \to S$ and any pair of quasi-coherent sheaves $\mathcal{E}'$ and $\mathcal{F}'$ on S', the map induced by pullback $g'^*\colon \operatorname{Hom}(\mathcal{E}', \mathcal{F}') \to \operatorname{Hom}(g'^*\mathcal{E}', g'^*\mathcal{F}')$ is injective. Then f satisfies descent for morphisms if and only if $f \circ g$ satisfies descent for morphisms.

To prove this, we consider the following diagram:

$$\tag{4}
\begin{tikzcd}
R \times_S R \arrow[r, "\ell"] \arrow[dr, shift left, "r_2"] \arrow[dr, shift right, "r_1"'] & R \times_T R \arrow[r, "k"] \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] & S \times_T S \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & \\
& R \arrow[r, "g"] & S \arrow[r, "f"] & T
\end{tikzcd}$$

Given quasi-coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on T, if $h''\colon g^*f^*\mathcal{E} \to g^*f^*\mathcal{F}$ satisfies $q_1^*h'' = q_2^*h''$, then $r_1^*h'' = \ell^*q_1^*h'' = \ell^*q_2^*h'' = r_2^*h''$, so by descent for morphisms for g there exists a unique $h'\colon f^*\mathcal{E} \to f^*\mathcal{F}$ such that $g^*h' = h''$. The morphism k factors as $R \times_T R \to R \times_T S \to S \times_T S$, a pair of morphisms each obtained from g by base change. Now since $k^*p_1^*h' = k^*p_2^*h'$ it follows that $p_1^*h' = p_2^*h'$. If descent for morphisms holds for f, it follows that there exists a unique morphism $h\colon \mathcal{E} \to \mathcal{F}$ such that $f^*h = h'$, and hence descent for morphisms holds for $f \circ g$. Conversely, suppose $f \circ g$ satisfies descent for morphisms. If we are given quasi-coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ on T, and if $h'\colon f^*\mathcal{E} \to f^*\mathcal{F}$ satisfies $p_1^*h' = p_2^*h'$, then $h'' := g^*h'$ satisfies $q_1^*h'' = q_2^*h''$, so there exists a morphism $h\colon \mathcal{E} \to \mathcal{F}$ satisfying $g^*f^*h = h''$, and hence $f^*h = h'$.

Exercise A.4. Use this first claim to show that every *affine* faithfully flat morphism of schemes satisfies descent for morphisms.

Second claim: Suppose g satisfies effective descent, and suppose any morphism obtained from g by base change satisfies descent for morphisms. Then f satisfies effective descent if and only if $f \circ g$ satisfies effective descent.

We refer to the diagram (4). For the "only if" portion of the claim, we suppose f satisfies effective descent. Now suppose we are given a quasi-coherent sheaf $\mathcal{E}''$ on R together with an isomorphism $\varphi'\colon q_1^*\mathcal{E}'' \to q_2^*\mathcal{E}''$ satisfying the cocycle condition

$$\pi_{13}^*\varphi' = \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi' \tag{5}$$

where $\pi_{ij}\colon R \times_T R \times_T R \to R \times_T R$ denote the various projections. By pulling back (5) by the morphism $R \times_S R \times_S R \to R \times_T R \times_T R$, we obtain the cocycle identity for the cover g. So, by effective descent for the morphism g, there exists a sheaf $\mathcal{E}'$ on S together with an isomorphism $\lambda'\colon g^*\mathcal{E}' \to \mathcal{E}''$ such that $r_2^*\lambda' = k^*\varphi' \circ r_1^*\lambda'$. Now we claim there exists a morphism $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ such that $q_2^*\lambda' \circ h^*\varphi = \varphi' \circ q_1^*\lambda'$. By the first claim, k (a composite of two pullbacks of g, as we saw in the proof of the first claim) satisfies descent for morphisms. Now consider the morphism $q_2^*\lambda'^{-1} \circ \varphi' \circ q_1^*\lambda'\colon k^*p_1^*\mathcal{E}' \to k^*p_2^*\mathcal{E}'$. For the existence of φ as promised we must check the agreement of the two pullback to $(R \times_T R) \times_{S \times_T S} (R \times_T R)$. This fiber product is identified, via the map which on points is given by $(w, x, y, z) \mapsto (w, y, z, x)$, with the fiber product $R \times_S R \times_T R \times_S R$, whereupon the agreement of the two pullbacks reduces to the identity

$$\pi_{14}^*\varphi' = \pi_{34}^*\varphi' \circ \pi_{23}^*\varphi' \circ \pi_{12}^*\varphi'. \tag{6}$$

In fact (6) is the pullback of a similar identity on $R \times_T R \times_T R \times_T R$, and the latter is deduced by combining the pullback of (5) by π_{123} with the pullback of (5) by π_{134} (here π_{ij} and π_{ijk} denote projections from quadruple fiber products). Now φ satisfies the cocycle condition for the covering map f, since the map $R \times_T R \times_T R \to S \times_T S \times_T S$ can be written as a composite of three morphisms, each obtained from g by base change, and via this map the cocycle condition we are claiming pulls back to (5). By effective descent for f there exists a quasi-coherent sheaf $\mathcal{E}$ on T with an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$. Hence effective descent holds for $f \circ g$.

Exercise A.5. Show, conversely, that under the hypotheses of the second claim, if $f \circ g$ satisfies effective descent, then effective descent holds for f.

We now complete the proof of Lemmas A.7 and A.8. We start by letting (T_i), $i \in I$, be an affine open cover of T. For each i, we let $(T'_{i,j})$, $j \in J_i$ be an affine open cover of $f^{-1}(T_i)$.

Suppose we are in case (i) of the lemmas, that is, f is faithfully flat and quasi-compact. Then the set J_i maybe taken to be finite, for every i. Now for each $i \in I$, the map $f_i\colon \coprod_{j\in J_i} T'_{i,j} \to T_i$ is a faithfully flat morphism of affine schemes. We consider the following commutative diagram

$$\begin{tikzcd} \coprod_{i,j} T'_{i,j} \arrow[r, "\coprod f_i"] \arrow[d] & \coprod_i T_i \arrow[d] \\ T' \arrow[r, "f"] & T \end{tikzcd} \tag{7}$$

The vertical maps are Zariski open coverings, and for such maps we know effective descent holds. By the affine case (Lemmas A.5 and A.6), effective descent holds for each morphism f_i, hence as well for the top map in (7). By Exercise A.4, any morphism obtained from the latter by base change satisfies descent for morphisms. So, by the second claim, the composite map $\coprod T'_{i,j} \to T$ in (7) satisfies effective descent. Again invoking the second claim, we conclude that effective descent holds for f.

We turn to case (ii) of the lemmas, where f is fppf and hence open. Fix $i \in I$. We let $U_{i,j} = f(T'_{i,j})$ for all $j \in J_i$, so $(U_{i,j})$ is a Zariski open covering of T_i. It follows[1] that each morphism $T'_{i,j} \to U_{i,j}$ is affine, so in particular is fpqc. By case (i) of the assertions, then, each map $T'_{i,j} \to U_{i,j}$ satisfies effective descent. Now, to conclude, we consider a square as in (7) but with $\coprod_{i,j} U_{i,j}$ in the upper right-hand corner, and we reason as above except we appeal to case (i) at the second step of the deduction.

Exercise A.6. Show that Theorem A.2 (a) fails for the covering map

$$\coprod_p \operatorname{Spec} \mathbb{Z}_p \to \operatorname{Spec} \mathbb{Z}.$$

Note that f is faithfully flat, but is neither fpqc nor fppf.

4. Categorical formulation

There is a category-theoretic approach to stating the above descent results. The proposition in this section outlines how the results appear in this language; one often sees them expressed this way in the literature.

[1]By [**EGA** II.1.6.2], any morphism from an affine scheme to a separated scheme is affine. Note that each U_{ij} is separated since it is an open subscheme of a separated, in fact affine, scheme T_i.

Fix schemes T and T' and a morphism $f\colon T' \to T$. Let $\mathcal{C}(T)$ be the category of quasi-coherent sheaves on T, with their usual morphisms as sheaves of $\mathcal{O}_T$-modules. Let $\mathcal{C}(T'/T)$ be the category whose objects are pairs $(\mathcal{E}', \varphi)$ of descent data, with a morphism from $(\mathcal{E}', \varphi)$ to $(\mathcal{F}', \psi)$ being a homomorphism $h'\colon \mathcal{E}' \to \mathcal{F}'$ such that $p_2^* h' \circ \varphi = \psi \circ p_1^* h'$. There is a canonical functor $\mathcal{C}(T) \to \mathcal{C}(T'/T)$, taking a quasi-coherent sheaf $\mathcal{E}$ on T to the pair consisting of the sheaf $f^*\mathcal{E}$ and the canonical isomorphism $p_1^* f^*\mathcal{E} \cong (f \circ p_1)^*(\mathcal{E}) = (f \circ p_2)^*(\mathcal{E}) \cong p_2^* f^* \mathcal{E}$; we will sometimes use *can* to denote this canonical isomorphism. (The cocycle condition $p_{13}^* can = p_{13}^* can \circ p_{23}^* can$ amounts to the compatibility of the canonical isomorphisms among pullbacks to T'''.)

PROPOSITION A.9. *If $f\colon T' \to T$ is an fpqc morphism or an fppf morphism of schemes, then the induced functor from the category $\mathcal{C}(T)$ of quasi-coherent sheaves of $\mathcal{O}_T$-modules to the category $\mathcal{C}(T'/T)$ of descent data is an equivalence of categories.*

PROOF. Let $(\mathcal{E}', \varphi)$ be an object of $\mathcal{C}(T'/T)$. To give an isomorphism $(f^*\mathcal{E}, can) \xrightarrow{\sim} (\mathcal{E}', \varphi)$ is, by definition, the same as to give an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$. So essential surjectivity of the functor is equivalent to the condition in Theorem A.2(a).

If we have isomorphisms $(f^*\mathcal{E}, can) \xrightarrow{\sim} (\mathcal{E}', \varphi)$ and $(f^*\mathcal{F}, can) \xrightarrow{\sim} (\mathcal{F}', \psi)$ then Theorem A.2(b) is the assertion that the map

$$\mathrm{Hom}_{\mathcal{C}(T)}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{C}(T'/T)}((\mathcal{E}', \varphi), (\mathcal{F}', \psi))$$

(obtained by applying the functor and composing with the isomorphisms) is bijective. In the case of identity morphisms $1_{(f^*\mathcal{E}, can)}$ and $1_{(f^*\mathcal{F}, can)}$, this is the condition for the functor to be fully faithful. □

REMARK A.10. There is a larger category $\mathcal{C}_0(T'/T)$ whose objects consist of pairs $(\mathcal{E}', \varphi)$ where $\mathcal{E}'$ is a quasi-coherent sheaf on T' and $\varphi\colon p_1^*\mathcal{E}' \to p_2\mathcal{E}'$ is an isomorphism. Morphisms in $\mathcal{C}_0(T'/T)$ are defined just as in $\mathcal{C}(T'/T)$, making $\mathcal{C}(T'/T)$ a full subcategory of $\mathcal{C}_0(T'/T)$. This subcategory has the property that, given an object $(\mathcal{E}', \varphi)$ of $\mathcal{C}(T'/T)$, if $(\mathcal{E}', \varphi) \to (\mathcal{F}', \psi)$ is an isomorphism in $\mathcal{C}_0(T'/T)$ then $(\mathcal{F}', \psi)$ is also in $\mathcal{C}(T'/T)$. The verification of this fact involves a diagram chase. This fact tells us that if the descent problem corresponding to an object $(\mathcal{F}', \psi)$ of $\mathcal{C}_0(T'/T)$ admits a solution, meaning that $\mathcal{F}'$ is isomorphic to a sheaf $f^*\mathcal{F}$ compatibly with ψ, then $(\mathcal{F}', \psi)$ lies in $\mathcal{C}(T'/T)$, i.e., ψ must satisfy the cocycle condition.

5. Faithfully flat descent

In this section we give some of the descent statements that are important for the theory of stacks. Most of these results are rather quick consequences of Theorem A.2. More challenging applications will be given in the last section. First we have a descent result for vector bundles.

PROPOSITION A.11. *Let $f\colon T' \to T$ be a morphism of schemes that is fpqc or fppf. Then:* (a) *Given a locally free sheaf of finite type $\mathcal{E}'$ on T' and an isomorphism $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ such that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$, there exists a locally free sheaf of finite type $\mathcal{E}$ on T and an isomorphism $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$, and these are*

unique up to canonical isomorphism.
(b) *With notation as in* (a)*, suppose* $(\mathcal{F}', \psi)$ *is another descent datum with solution given by* $\mathcal{F}$ *and* μ*. Then, for every morphism* $h' \colon \mathcal{E}' \to \mathcal{F}'$ *satisfying* $p_2^* h' \circ \varphi = \psi \circ p_1^* h'$ *there is a unique morphism* $h \colon \mathcal{E} \to \mathcal{F}$ *such that* $\mu \circ f^* h = h' \circ \lambda$*.*

Proof. This follows from Theorem A.2, coupled with Exercise A.2. □

Next we turn to descent for affine schemes.

Proposition A.12. *Let* $f \colon T' \to T$ *be a morphism of schemes that is fpqc or fppf.*
(a) *Given an affine morphism of schemes* $P' \to T'$ *and an isomorphism* $\varphi \colon P' \times_T T' \to T' \times_T P'$ *over* T'' *satisfying the cocycle condition, there exists an affine morphism* $P \to T$ *and isomorphism* $\lambda \colon T' \times_T P \to P'$ *over* T'*, unique up to canonical isomorphism, such that* $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$*.*
(b) *With notation as in* (a)*, suppose* (Q', ψ) *is another descent datum with solution given by* $Q \to T$ *and* μ*. Then, for every morphism* $h' \colon P' \to Q'$ *over* T' *satisfying* $(T' \times_T h') \circ \varphi = \psi \circ (h' \times_T T')$ *there is a unique morphism* $h \colon P \to Q$ *such that* $\mu \circ (T' \times_T h) = h' \circ \lambda$*.*

We will see that descent for morphisms reduces to the statement that the functor $\mathrm{Hom}(-, X)$ satisfies the sheaf axiom (for any fpqc or fppf cover), which holds for an arbitrary scheme X.

Proposition A.13. *Let* $f \colon T' \to T$ *be a morphism of schemes that is fpqc or fppf. Let* X *be a scheme. If* $g \colon T' \to X$ *is a morphism of schemes such that* $g \circ p_1 = g \circ p_2$*, then there is a unique morphism* $h \colon T \to X$ *such that* $h \circ f = g$*.*

The proof of this proposition requires some preparatory results. Below we denote by $f^\# \colon \mathcal{O}_T \to f_* \mathcal{O}_{T'}$ the morphism of structure sheaves induced by a morphism of schemes $f \colon T \to T'$. Let $p \colon T'' \to T$ be the composition $f \circ p_1 = f \circ p_2$.

Lemma A.14. *Suppose* $f \colon T' \to T$ *is fpqc or fppf. Then the sequence*

$$0 \longrightarrow \mathcal{O}_T \xrightarrow{f^\#} f_* \mathcal{O}_T \xrightarrow{f_* p_1^\# - f_* p_2^\#} p_* \mathcal{O}_{T''}$$

is exact.

Proof. By Theorem A.2(b) applied to $\mathcal{E} = \mathcal{F} = \mathcal{O}_T$ and adjointness of pushfoward and pullback, the sequence

$$0 \longrightarrow \Gamma(T, \mathcal{O}_T) \xrightarrow{f^\#} \Gamma(T, f_* \mathcal{O}_{T'}) \xrightarrow{f_* p_1^\# - f_* p_2^\#} \Gamma(T, p_* \mathcal{O}_{T''}) \tag{8}$$

is exact. The sequence (8) with T replaced by any open subscheme of T is still exact, so the sequence of sheaves is exact. □

Lemma A.15. *Suppose* $f \colon T' \to T$ *is fpqc. Let* Z *be a subset of* T *such that* $f^{-1}(Z)$ *is closed in* T'*. Then* Z *is a closed subset of* T*.*

Proof. Since f is surjective, it suffices to show that if $f^{-1}(Z)$ is closed, then

$$f^{-1}(\overline{Z}) = f^{-1}(Z) \tag{9}$$

where $\overline{Z}$ denotes the closure of Z. It suffices to verify (9) when T and T' are affine, so we may suppose $T = \operatorname{Spec} A$ and $T' = \operatorname{Spec} A'$. Introduce the ideals

$$I = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p} \qquad \text{and} \qquad I' = \bigcap_{\mathfrak{p}' \in f^{-1}(Z)} \mathfrak{p}',$$

corresponding to closed subsets $\overline{Z} \subset T$ and $f^{-1}(Z) \subset T'$, respectively. We have $I' \cap A = I$ (viewing A as a subring of A'). In other words, I fits into an exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A'/I'. \tag{10}$$

Tensoring (10) by A' identifies $I \otimes_A A'$ with the kernel of the composite of $A' \to A' \otimes_A A'$, $x \mapsto 1 \otimes x$, with the quotient map by the ideal $I' \otimes_A A'$. The ideal $A' \otimes_A I'$ has the same radical as the ideal $I' \otimes_A A'$, since $(f \circ p_1)^{-1}(Z)$ is the closed subset of T'' associated with both. Thus $\sqrt{I \otimes_A A'} = I'$, and (9) is established. □

Exercise A.7. If $f\colon T' \to T$ is any surjective morphism of schemes, then for any points x and y in T such that $f(x) = f(y)$, there exists $z \in T''$ such that $p_1(z) = x$ and $p_2(z) = y$.

Proof of Proposition A.13. By Exercise A.7, the map h that we are required to produce is completely determined on the set-theoretic level. By Lemma A.15 in the fpqc case, or by the fact that fppf maps are open, the map h is continuous. Finally, the required map of structure sheaves $h^{\#}\colon \mathcal{O}_X \to h_*\mathcal{O}_T$ is determined uniquely by looking at h_* applied to the exact sequence from Lemma A.14. □

Proof of Proposition A.12. To prove (a), we need to show is the existence of the solution to a descent problem. To give a scheme, affine over T, is the same as giving a quasi-coherent sheaf of $\mathcal{O}_T$-algebras. This sheaf (as a sheaf of modules) is constructed by descent for quasi-coherent sheaves, and is given the structure of $\mathcal{O}_T$-algebra (multiplication map) using descent for morphisms of quasi-coherent sheaves.

For (b), the exactness of

$$\operatorname{Hom}_T(P, Q) \to \operatorname{Hom}_{T'}(T' \times_T P, T' \times_T Q) \rightrightarrows \operatorname{Hom}_{T''}(T'' \times_T P, T'' \times_T Q) \tag{11}$$

is the same as the exactness of

$$\operatorname{Hom}_T(P, Q) \to \operatorname{Hom}_T(T' \times_T P, Q) \rightrightarrows \operatorname{Hom}_T(T'' \times_T P, Q),$$

which follows from Proposition A.13.

As in the previous sections, the existence of the solution to a descent problem plus exactness of (11) imply the full assertions of both statements of this proposition. □

Torsors for an affine group scheme provide an important example of affine morphisms of schemes. We recall that if G is a group scheme over T then a left G-torsor is a scheme E (the total space) with a map $E \to T$ (the structure map), together with a left G-action $a\colon G \times E \to E$ which upon pullback by some étale cover $T' \to T$ becomes isomorphic to the trivial G-torsor $G \times T'$ (with action of G on itself by left multiplication). Examples are any unramified two-sheeted cover (for $G = \mathbb{Z}/2$, i.e., the constant group scheme $T \times \mathbb{Z}/2 \to T$ over any T) and the complement of the zero section of a line bundle (for the multiplicative group $\mathbb{G}_m$). (The same applies to right instead of left actions.) A consequence of Proposition A.12 is that effective descent holds for G-torsors whenever G is an affine group scheme over the base scheme. Note that the action of G on E is given by a map of affine schemes, to which descent of morphisms applies.

COROLLARY A.16. *Let $f\colon T' \to T$ be a morphism of schemes that is fpqc or fppf. Let G be an affine group scheme over T. Then:* (a) *Given a G-torsor E' on T' and an isomorphism $\varphi\colon p_1^*E' \to p_2^*E'$ over T'' satisfying the cocycle condition over T''' there exists a G-torsor E on T and G-equivariant isomorphism $\lambda\colon f^*E \to E'$ over T', unique up to canonical isomorphism, such that $p_2^*\lambda = \varphi \circ p_1^*\lambda$.*
(b) *Let notation be as in* (a)*, and suppose (F', ψ) is another descent datum with solution given by F and μ. Then, for every G-equivariant isomorphism $h'\colon E' \to F'$ over T' satisfying $p_2^*h' \circ \varphi = \psi \circ p_1^*h'$ there is a unique G-equivariant isomorphism $h\colon E \to F$ over T such that $\mu \circ f^*h = h' \circ \lambda$.*

6. Non-effective descent: an example

In this section we show how descent can fail for proper morphisms. In the next section we will see how, with projective morphisms and suitable additional data, it is possible to overcome this problem.

Let T be a smooth projective threefold over the complex numbers which has a 2-to-1 étale cover $f\colon T' \to T$, such that there exists a nodal curve Z in T whose pre-image in T' consists of the union of two smooth curves E and F meeting transversely at two points that we denote P and Q.

Now form X' by modifying T' along $E \cup F$. Near P, we first blow up E, and then we blow up the proper transform of F. Near Q, we first blow up F, and then the proper transform of E. Away from $\{P, Q\}$, the order of blow-up is irrelevant, so we can glue these together to make a scheme X'.

Since T' is a 2-to-1 cover of T it has an involution that respects the map to T. Because of the order in which we performed the blow-ups, this involution actually extends to an involution of X'. Both the involution of T' and that of X' are without fixed points. We can express the problem of trying to form the quotient of X' by the involution as a descent problem. The pair consisting of the object $X' \to T'$ (in the category of schemes over T'), together with the isomorphism

$$X' \times_T T' \to T' \times_T X'$$

which is the identity map over the identity component of T'' and the involution over the other component, is a descent datum. This descent datum, we claim, is non-effective, i.e., there is no scheme quotient of X' by its involution.

Indeed, suppose X is a scheme over T with a map $\pi\colon X' \to X$ making

$$\begin{array}{ccc} X' & \longrightarrow & T' \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \longrightarrow & T \end{array}$$

a cartesian diagram. Consider, in X', the pre-images $A \cup B$ of P and $C \cup D$ of Q, where each of A, B, C, D is a rational curve. Now we make a calculation in the ring of cycles modulo algebraic equivalence [**27**, §10.3] on X. Denoting this ring by $B^*(X)$, we have, with a suitable labeling of the curves, equations $[B] = [C] + [D]$ and $[D] = [A] + [B]$ in $B^*(X')$, and hence

$$[A] + [C] = 0 \tag{12}$$

in $B^*(X)$. Since π is finite and flat of degree 2, we find from (12) that

$$2\pi_*[A] = \pi_*([A] + [C]) = 0 \tag{13}$$

in $B^*(X)$. This is impossible if X is a scheme. Indeed, if U denotes an affine neighborhood in X of the generic point of $\pi(A)$, and if we take Y to be a a generically chosen hypersurface of U, then the closure $\overline{Y}$ of Y in X meets $\pi(A)$ properly in at least one point. This means that $[\overline{Y}] \cdot \pi_*[A]$ is a zero-cycle class of positive degree, which is a contradiction to (13).

Of course, the quotient of X' by the involution exists as an analytic space. This analytic space quotient is, in effect, Hironaka's example of an algebraic space which is not a scheme (cf. [**47**, Exa. B.3.4.1]). So, effective descent fails for general schemes. The category of algebraic spaces, which contains quasi-separated schemes as a full subcategory, has the advantage over the category of schemes in that effective descent holds for general fppf morphisms. We remark that this descent property, stated in this text as Proposition **??**[2], relies on Artin's criterion for a stack to be algebraic (Theorem **??**[3]), whose proof is not easy!

EXERCISE A.8. Construct a non-effective descent datum with T a threefold over the real numbers and f the map induced by base change via $\mathbb{R} \to \mathbb{C}$. This demonstrates that there exist an algebraic space, separated and of finite type over a field, which is not a scheme, but which becomes a scheme after a finite extension of the base field.

7. Further descent results

Despite the failure of effective fppf (and fpqc) descent for general morphisms of schemes, there are restricted classes of morphisms of schemes for which effective descent is known to hold. We saw that affine morphisms form one such class of morphisms.

It is important for the theory of stacks that quasi-affine and (polarized) quasi-projective morphisms make up two other such classes. Let us recall that a morphism is quasi-affine if it can be factored as a quasi-compact open inclusion followed by an affine morphism. If $f\colon X \to Y$ is any separated quasi-compact morphism of schemes, then in the canonical factorization

$$X \xrightarrow{g} \operatorname{Spec} f_*\mathcal{O}_X \xrightarrow{h} Y, \tag{14}$$

g is an open inclusion if and only if $f = h \circ g$ is quasi-affine [**EGA** II.5.1.6]. Quasi-projective morphisms enjoy a similar characterization, factoring through $\operatorname{Proj}(\bigoplus f_*\mathcal{O}_X(n))$. The Proj construction relies on a choice of relative ample invertible sheaf $\mathcal{O}_X(1)$, which must be included as part of the descent datum.

[2]A reference to a statement in Part II of the book, which might not appear for a while.

[3]Another reference to Part II of the book.

Proposition A.17. *Let $f\colon T' \to T$ be a morphism of schemes that is fpqc or fppf. Then:* (a) *Given a quasi-affine morphism of schemes $P' \to T'$ and an isomorphism $\varphi\colon P' \times_T T' \to T' \times_T P'$ over T'' satisfying the cocycle condition, there exists a quasi-affine morphism $P \to T$ and isomorphism $\lambda\colon T' \times_T P \to P'$ over T', unique up to canonical isomorphism, such that $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$.*
(b) *With notation as in* (a)*, suppose (Q', ψ) is another descent datum with solution given by $Q \to T$ and μ. Then, for every morphism $h'\colon P' \to Q'$ over T' satisfying $(T' \times_T h') \circ \varphi = \psi \circ (h' \times_T T')$ there is a unique morphism $h\colon P \to Q$ such that $\mu \circ (T' \times_T h) = h' \circ \lambda$.*

Proof. Let t' denote the morphism $P' \to T'$. We have the canonical factorization

$$P' \to \operatorname{Spec}(t'_*\mathcal{O}_{P'}) \to T'. \tag{15}$$

Set $\mathcal{E}' = t'_*\mathcal{O}_{P'}$ and $\overline{P}' = \operatorname{Spec} \mathcal{E}'$. Since f is flat, we have a canonical isomorphism $p_1^*\mathcal{E}' \xrightarrow{\sim} (t' \times_T T')_*\mathcal{O}_{P' \times_T T'}$. Under this isomorphism, the morphisms we obtain by pulling back (15) by p_1,

$$P' \times_T T' \to \overline{P}' \times_T T' \to T'', \tag{16}$$

constitute the canonical factorization of $P' \times_T T' \to T''$. Similarly,

$$T' \times_T P' \to T' \times_T \overline{P}' \to T'', \tag{17}$$

gives the canonical factorization of $T' \times_T P' \to T''$, under the canonical isomorphism $p_2^*\mathcal{E}' \xrightarrow{\sim} (T' \times_T t')_*\mathcal{O}_{T' \times_T P'}$.

The isomorphism $\varphi\colon P' \times_T T' \to T' \times_T P'$ determines an isomorphism $\bar{\varphi}\colon \overline{P}' \times_T T' \to T' \times_T \overline{P}'$. Since φ satisfies the cocycle condition, so does $\bar{\varphi}$. Now by Proposition A.12, there is an affine morphism $\overline{P} \to T$ and an isomorphism $\lambda\colon T' \times_T \overline{P} \to \overline{P}'$ satisfying $T' \times_T \lambda = \bar{\varphi} \circ (\lambda \times_T T')$.

Since $\overline{P}'$ is isomorphic to $T' \times_T P'$, the morphism $\overline{P}' \to \overline{P}$ is fpqc if f is fpqc and is fppf if f is fppf. Moreover we can canonically identify $\overline{P}' \times_{\overline{P}} \overline{P}'$ with $\overline{P}' \times_T T'$. To do this, we start with the cube with cartesian faces and extend the top face with cartesian squares involving the isomorphism λ, as shown in the following diagram, where $\tilde{f}$ is used to denote the second projection from $T' \times_T \overline{P}$.

$$\begin{tikzcd}
 & & & & T' \times_T \overline{P}' \arrow[rr, "\overline{P}' \times_{T'} p_2"] \arrow[dl, "T' \times_T \lambda^{-1}"'] & & \overline{P}' \arrow[dl, "\lambda^{-1}"] \\
 & \overline{P}' \times_T T' \arrow[urrr, "\bar{\varphi}"] \arrow[rr, "\lambda^{-1} \times_T T'"] \arrow[dl, "\overline{P}' \times_{T'} p_1"'] & & T'' \times_T \overline{P} \arrow[rr] \arrow[dl] \arrow[dd] & & T' \times_T \overline{P} \arrow[dl, "\tilde{f}"'] \arrow[dd] & \\
\overline{P}' \arrow[rr, "\lambda^{-1}"] & & T' \times_T \overline{P} \arrow[rr, "\tilde{f}"] \arrow[dd] & & \overline{P} \arrow[dd] & & \\
 & & & T'' \arrow[rr, "p_2"] \arrow[dl, "p_1"'] & & T' \arrow[dl, "f"] & \\
 & & T' \arrow[rr, "f"'] & & T & &
\end{tikzcd}$$

By the condition on λ, the upper triangle commutes. Using Lemma A.15 in the case f is fpqc, or the fact that fppf morphisms are open, we see that there is a one-to-one correspondence between open subschemes $U \subset \overline{P}$ and open subschemes $U' \subset \overline{P}'$ satisfying

$$(\overline{P}' \times_{T'} p_1)^{-1}(U') = \bar{\varphi}^{-1}\big((\overline{P}' \times_{T'} p_2)^{-1}(U')\big). \tag{18}$$

In (15) we have P' realized as an open subscheme of $\overline{P}'$. The pre-image of P' by $\overline{P}' \times_{T'} p_1$, respectively by $\overline{P}' \times_{T'} p_2$, is the image of the open inclusion in (16), respectively (17). Now (18) holds since we have a commutative diagram

$$\begin{tikzcd}
P' \times_T T' \arrow[r] \arrow[d, "\varphi"'] & \overline{P}' \times_T T' \arrow[d, "\bar{\varphi}"] \\
T' \times_T P' \arrow[r] & T' \times_T \overline{P}'
\end{tikzcd}$$

So there is a unique open subscheme $P \subset \overline{P}$ satisfying $(\tilde{f} \circ \lambda^{-1})^{-1}(P) = P'$. Now the scheme P and the restriction of λ to $T' \times_T P$ constitute a solution to the descent problem posed by P' and φ. $\square$

The large diagram in the proof of this proposition illustrates a general principle. To give the descent datum (P', φ) is equivalent to giving a scheme P'' with morphism to T'' and an *equivalence relation*

$$(\tilde{p}_1, \tilde{p}_2) \colon P'' \to P' \times P'$$

compatible with $(p_1, p_2) \colon T'' \to T' \times T'$. The compatibility condition is that the diagram

$$\begin{tikzcd}
P'' \arrow[r, "\tilde{p}_i"] \arrow[d] & P' \arrow[d] \\
T'' \arrow[r, "p_i"] & T'
\end{tikzcd}$$

is cartesian for $i = 1$, 2. (To be an equivalence relation means that $(\tilde{p}_1, \tilde{p}_2)$ is a locally closed embedding[4] satisfying conditions that generalize the usual conditions when S is a set for a subset of $S \times S$ to be an equivalence relation.) We take P'' to be $P' \times_T T'$, with $\tilde{p}_1$ the projection map to P' and $\tilde{p}_2$ the composite of φ and the projection $T' \times_T P' \to P'$. In the language of equivalence relations, effectivity amounts to providing a scheme P over T and a map $P' \to P$ such that $P'' \cong P' \times_P P'$. Many descent problems can be stated in the language of equivalence relations (see [**38**]). In this appendix we stick to the language of descent data, though in the next result, descent for quasi-projective schemes, we employ the notation for the maps that we have just introduced:

$$\tilde{p}_1 \colon P' \times_T T' \to P', \tag{19}$$

$$\tilde{p}_2 \colon P' \times_T T' \xrightarrow{\sim} T' \times_T P' \to P'. \tag{20}$$

[4]The correct condition is really *monomorphism*, but the more restrictive condition suffices for the discussion of effectivity since the diagonal morphism of any scheme is a locally closed embedding.

PROPOSITION A.18. *Let $f\colon T' \to T$ be a morphism of schemes that is fpqc or fppf. Given a quasi-projective morphism of schemes $P' \to T'$, a relatively ample invertible sheaf $\mathcal{L}'$ on P', an isomorphism $\varphi\colon P' \times_T T' \to T' \times_T P'$ over T'' satisfying the cocycle condition, and an isomorphism $\omega\colon \tilde{p}_1^*\mathcal{L}' \to \tilde{p}_2^*\mathcal{L}'$ satisfying the cocycle condition on $P' \times_T T''$, where $\tilde{p}_1$ and $\tilde{p}_2$ are the maps of* (19)–(20)*, there exists a scheme P with quasi-projective morphism $P \to T$ and relatively ample invertible sheaf $\mathcal{L}$, an isomorphism $\lambda\colon T' \times_T P \to P'$ over T' and, with $\tilde{f}\colon P' \to P$ the composition of λ^{-1} and projection, an isomorphism $\chi\colon \tilde{f}^*\mathcal{L} \to \mathcal{L}'$; these satisfy $T' \times_T \lambda = \varphi \circ (\lambda \times_T T')$ and $\tilde{p}_2^*\chi = \omega \circ \tilde{p}_1^*\chi$. The solution to the descent problem is unique up to canonical isomorphism.*

As above, we set $P'' = P' \times_T T'$. If we further define $P''' = P' \times_T T''$ then the usual cocycle condition on φ is expressed by the commutativity of the triangle

$$\begin{tikzcd} P''' \arrow[rr] \arrow[dr] & & T' \times_T P'' \arrow[dl] \\ & T'' \times_T P' & \end{tikzcd}$$

where the maps are the ones obtained from φ by base change. There are projection maps $\tilde{p}_{12}$ and $\tilde{p}_{13}$ (obtained from p_{12} and p_{13} by base change) and $\tilde{p}_{23}$ (the composite $P''' \to T' \times_T P'' \to P''$). Now the cocycle condition on ω is that the diagram

$$\begin{tikzcd} \tilde{p}_{12}^*\tilde{p}_1^*\mathcal{L}' \arrow[r, "\tilde{p}_{12}^*\omega"] \arrow[d, equal] & \tilde{p}_{12}^*\tilde{p}_2^*\mathcal{L}' \arrow[r, equal] & \tilde{p}_{23}^*\tilde{p}_1^*\mathcal{L}' \arrow[d, "\tilde{p}_{23}^*\omega"] \\ \tilde{p}_{13}^*\tilde{p}_1^*\mathcal{L}' \arrow[r, "\tilde{p}_{13}^*\omega"] & \tilde{p}_{13}^*\tilde{p}_2^*\mathcal{L}' \arrow[r, equal] & \tilde{p}_{23}^*\tilde{p}_2^*\mathcal{L}' \end{tikzcd}$$

commutes.

The condition on χ is commutativity of the diagram

$$\begin{tikzcd} \tilde{p}_1^*\tilde{f}^*\mathcal{L} \arrow[r, "\tilde{p}_1^*\chi"] \arrow[d, equal] & \tilde{p}_1^*\mathcal{L}' \arrow[d, "\omega"] \\ \tilde{p}_2^*\tilde{f}^*\mathcal{L} \arrow[r, "\tilde{p}_2^*\chi"] & \tilde{p}_2^*\mathcal{L}' \end{tikzcd}$$

where the equality $\tilde{f} \circ \tilde{p}_1 = \tilde{f} \circ \tilde{p}_2$ is a consequence of the condition on λ (as detailed in the large commutative diagram in the proof of Proposition A.17).

Before we give the proof of this result, we recall that a quasi-projective morphism is a morphism of finite type which factors as an open embedding followed by a map of the form $\mathrm{Proj}(\mathcal{S}) \to X$ where $\mathcal{S}$ is a graded sheaf of quasi-coherent $\mathcal{O}_X$-algebras. Associated to a separated morphism of finite type of schemes $f\colon X \to Y$ and an invertible sheaf $\mathcal{L}$ on X is a graded sheaf of algebras $\mathcal{S} := \bigoplus_{n \geq 0} f^*(\mathcal{L}^{\otimes n})$, open subscheme $U \subset X$, and factorization of the restriction of f to U as

$$U \to \mathrm{Proj}(\mathcal{S}) \to Y.$$

Now [**EGA** II.4.6.3] states that the map f is quasi-projective if and only if $U = X$ and $X \to \mathrm{Proj}(\mathcal{S})$ is an open embedding. Further, for a morphism of schemes to be of finite type is a Zariski local condition, and this is a condition that holds for any morphism if it holds after fpqc base change ([**EGA** IV.2.7.1(v)]).

Proof of Proposition A.18. We introduce $P'' = P' \times_T T'$ as above, with morphism $t''\colon P'' \to P'$. Consider the composite isomorphism

$$p_1^* t'_* \mathcal{L}' \cong t''_* \tilde{p}_1^* \mathcal{L}' \xrightarrow{t''_* \omega} t''_* \tilde{p}_2^* \mathcal{L}' \cong p_2^* t'_* \mathcal{L}'$$

of two base-change isomorphisms and the pushforward of ω. We claim that $t'_*\mathcal{L}'$, together with this isomorphism, constitutes a descent datum, and hence determines by Theorem A.2 a quasi-coherent sheaf $\mathcal{S}_1$ on T. Verifying the cocycle condition amounts to writing down a large diagram whose commutativity results by (i) naturality of the base change isomorphism, (ii) the property that a composite of base change morphisms resulting from two commuting squares glued together equals the base change morphism coming from the large outer diagram (see the Glossary), and (iii) the cocycle condition on ω.

The same consideration applies as well to $\mathcal{L}'^{\otimes n}$ yielding a sheaf $\mathcal{S}_n$ on T, for all $n \geq 0$. So, we get a graded quasi-coherent sheaf $\mathcal{S} = \oplus_{n \geq 0} \mathcal{S}_n$ on T which is given an algebra structure by using descent for morphisms of quasi-coherent sheaves.

The remainder of the argument exactly parallels the proof of Proposition A.17. We have the canonical factorization of $P' \to T'$ through $\overline{P}' := \mathrm{Proj}(\bigoplus t'_* \mathcal{L}'^{\otimes n})$, with descent datum $\overline{\varphi}\colon \overline{P}' \times_T T' \to T' \times_T \overline{P}'$. A solution is given by $\overline{P} := \mathrm{Proj}(\mathcal{S})$. As before there is a uniquely determined open subscheme $P \subset \overline{P}$ whose pullback is the image of $P' \to \overline{P}'$. Now $P \to T$ with $T' \times_T P \to P'$ and the restriction of the invertible sheaf $\mathcal{O}_{\overline{P}}(1)$ to P constitute a solution to the descent problem. □

Proposition A.18 is used to show that various families of curves determine stacks. It is also are used to show that other moduli problems, such as abelian varieties with various kinds of polarization, give rise to stacks.

Remark A.19. The proof of Proposition A.17 is in fact the special case $\mathcal{L}' = \mathcal{O}_{P'}$ of the proof just given. In fact there is a common generalization of Propositions A.17 and A.18. This is the statement that effective descent holds for schemes equipped with relatively ample invertible sheaves. The proof is obtained by copying the proof of Proposition A.18 and changing "of finite type" to "quasi-compact" throughout.

A modern descent result – which is not needed in this book – stems from the study of principal bundles on curves. Consider a scheme T with a covering by two Zariski open subsets. Then, the cocycle condition on the transition mappings is vacuous, so any isomorphism of objects on the overlap determines an object on T. One might expect a similar result for the cover consisting of the formal neighborhood of a divisor on T and the complement of the divisor. So, for instance, a vector bundle on a curve C over a field k should be determined uniquely up to isomorphism by a vector bundle on the complement of a k-rational point x, a vector bundle on $\mathrm{Spec}\,\widehat{\mathcal{O}}_{x,C}$, and an isomorphism on the overlap. Here is the precise result:

Proposition A.20. *Let $T = \operatorname{Spec} A$ be an affine scheme, and let $f\colon T' \to T$ be the cover given by $T' = T_1' \amalg T_2'$, where $T_1' \subset T$ is the complement of the divisor corresponding to a non-zero-divisor $a \in A$ and T_2' is* Spec *of the completion of A with respect to the a-adic topology. Let $T'' = T_1' \times_T T_2'$ with projections p_i to T_i'. Given a quasi-coherent sheaves $\mathcal{E}_i'$ on T_i', for $i = 1$ and 2 such that $\mathcal{E}_2'$ is f-regular (i.e., such that multiplication by f induces an injective map $\mathcal{E}_2' \to \mathcal{E}_2'$) and an isomorphism $\varphi\colon p_1^*\mathcal{E}_1' \to p_2^*\mathcal{E}_2'$, there exists a locally free sheaf $\mathcal{E}$ on T, unique up to canonical isomorphism, with an isomorphism $\lambda'\colon f^*\mathcal{E} \to \mathcal{E}'$ satisfying $p_2^*\lambda = \varphi \circ p_1^*\lambda$.*

We remark that Proposition A.20 does not follow from faithfully flat descent. In fact, the map f is not even flat in general. What is true is that f is faithful, i.e., we have $f^*\mathcal{E} = 0$ if and only if $\mathcal{E} = 0$ for quasi-coherent $\mathcal{E}$. This "faithful descent" result is proved by Beauville and Laszlo in [**9**] and has important applications in conformal field theory (see [**8**] for a survey) and in the geometric Langlands program.

Answers to Exercises

A.1. (1) For $\Rightarrow$, a nonzero element of M determines an inclusion $A/I \to M$, hence an inclusion $A'/IA' \to A' \otimes_A M$. With $\mathfrak{m}$ any maximal ideal containing I, it suffices to show $A'/\mathfrak{m}A' \neq 0$, and this holds by surjectivity of $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$. For $\Leftarrow$, the crucial fact is that $\mathfrak{p} \in \operatorname{Spec} A$ implies $A/\mathfrak{p} \to A'/\mathfrak{p}A'$ is injective. Indeed, if the image in A' of some $a \in A \smallsetminus \mathfrak{p}$ lies in $\mathfrak{p}A'$ then $(\mathfrak{p} + aA)/\mathfrak{p}$ would be a nonzero A-module becoming zero under $A' \otimes_A -$. Now any maximal ideal of the localization $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \otimes_{A/\mathfrak{p}} A'/\mathfrak{p}A'$ gives an element of $\operatorname{Spec} A'$ that maps to $\mathfrak{p}$. The condition in (2) is readily shown to be equivalent to that given in (1); a reference is [**14**, Proposition I.3.1.1].

A.2. (1) If ρ is the homomorphism, look at the exact sequence

$$0 \to \operatorname{Ker}(\rho) \to M \to N \to \operatorname{Coker}(\rho) \to 0.$$

(2) If $A' \otimes_A M$ is finitely generated, one can find a finitely generated free A-module F and a morphism $F \to M$ such that $A' \otimes_A F \to A' \otimes_A M$ is surjective. Then (1) shows that $F \to M$ is surjective. The same argument on the kernel of $F \to M$ gives the corresponding assertion for finitely presented. The flat case follows directly from the definitions, and the last follows from the fact that locally free of finite rank is equivalent to flat and finitely presented. A reference for this last fact is [**14**, Corollary II.5.2.2].

A.3. It suffices to prove that $A' \otimes_A T^\bullet \otimes_A M$ is exact. One can prove this as in the lemma, or, more elegantly, by defining a chain homotopy $h^n\colon A' \otimes_A T^n \otimes_A M \to A' \otimes_A T^{n-1} \otimes_A M$ by the formula $h^n(x \otimes x_1 \otimes \cdots \otimes x_n \otimes m) = x \cdot x_1 \otimes x_2 \cdots \otimes x_n \otimes m$, and verifying that $h^{n+1} \circ \delta^n + \delta^{n-1} \circ h^n = 1_{A' \otimes T^n \otimes M}$.

A.4. Cover T by affines T_i, and let $S_i = f^{-1}(T_i)$. Descent for morphisms holds for each $S_i \to T_i$ by the affine case, hence as well for $\coprod S_i \to \coprod T_i$. Since Zariski coverings satisfy descent for morphisms, we may deduce descent for morphisms for $\coprod S_i \to T$, and then for $T' \to T$.

A.5. Given $\mathcal{E}'$ on S and $\varphi\colon p_1^*\mathcal{E}' \to p_2^*\mathcal{E}'$ satisfying the cocycle condition, pull back the cocycle condition via $R \times_T R \times_T R \to S \times_T S \times_T S$ and use effective descent for $f \circ g$ to conclude there exists $\mathcal{E}$ on T and $\lambda'\colon g^* f^* \mathcal{E} \to g^*\mathcal{E}'$ such that $q_2^*\lambda' = k^*\varphi \circ q_1^*\lambda'$. Since $k \circ \ell$ factors through the image of S in $S \times_T S$ (by the diagonal morphism), we have $r_2^*\lambda' = \ell^* q_2^*\lambda' = \ell^* q_1^*\lambda' = r_1^*\lambda'$, hence there exists $\lambda\colon f^*\mathcal{E} \to \mathcal{E}'$ such that $g^*\lambda = \lambda'$. Now $k^* p_2^*\lambda = q_2^* g^*\lambda = k^*\varphi \circ q_1^* g^*\lambda' = k^*(\varphi \circ p_1^*\lambda)$, hence $p_2^*\lambda = \varphi \circ p_1^*\lambda$.

A.6. A non-effective descent datum is given by multiplication by p/q on the trivial rank 1 free module on $\operatorname{Spec} \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q$, for every pair p and q of prime numbers.

A.7. What is true more generally is that if T_1 and T_2 are any schemes mapping to T, with $x \in T_1$ and $y \in T_2$ mapping to the same point $t \in T$, then the fiber product $T_1 \times_T T_2$ contains a point z with $p_1(z) = x$ and $p_2(z) = y$. Localizing, we may suppose we are in the affine case with x, y, and t all closed points. Passing to closed subschemes we are reduced to the assertion that the tensor product of two fields over a third field is a nonzero ring and hence contains a prime ideal.

A.8. Repeat the given construction using an irreducible curve defined over $\mathbb{R}$ which becomes the union of two irreducible components (meeting at nodes) after extending the base field to $\mathbb{C}$.

APPENDIX B

Categories and 2-categories

I do not believe in categories of any kind. *Duke Ellington*

We begin this appendix by reviewing some basic notions about categories. The second section defines and proves basic properties of 2-categories. These are applied in Section 3 to the study of adjoint functors. The fourth section has the main theorem, which spells out the appropriate notion of equivalence for 2-categories. Most of these notions and results are known in some form in general category theory. We have tried to present them in more concrete terms than usual, and hope that this, and a deficiency of references, will not offend category theorists. We expect geometers will find the going abstract enough; for a first reading, it should suffice to concentrate on the definitions, examples, and statements of the propositions.

In the last section we make a few remarks about set theoretic foundations, and the axiom of choice, which is used freely in the text. These are not designed to put us in any axiomatic set-theoretical framework, but rather to explain why we avoid doing this.

1. Categories

A **category** $\mathcal{C}$ has **objects** and **morphisms**, also called **maps** or **mappings** or **arrows**. To each morphism is associated two objects, its **source** and its **target**. We write $f\colon X \to Y$ to mean that f is a morphism with the object X as its source and the object Y as its target, and we say that f is a morphism **from** X **to** Y.[1] For any morphism f from X to Y, and any morphism g from Y to Z, there must be a morphism from X to Z, called the **composite** of f and g, and denoted $g \circ f$ or sometimes simply gf. The following properties must be satisfied:

(a) For any object X there is a morphism $1_X\colon X \to X$ such that $f \circ 1_X = f$ for all $f\colon X \to Y$ and $1_X \circ g = g$ for all $g\colon Y \to X$.
(b) For any $f\colon X \to Y$, $g\colon Y \to Z$, and $h\colon Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$.

Exercise B.1. Identity maps, if they exist, are unique.

A map $f\colon X \to Y$ is an **isomorphism** if there is a map $f^{-1}\colon Y \to X$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$.

[1] Although the notation $f\colon X \to Y$ is suggested by the functional notation of set theory, it does *not* mean that f assigns elements of Y to elements of X. In the category of schemes, for example, a morphism is much more than a function on underlying sets.

EXERCISE B.2. (1) An inverse, if it exists, is unique. (2) If $f\colon X \to Y$ is an isomorphism, and $g\colon Y \to Z$ is an isomorphism, then $g \circ f$ is an isomorphism, with inverse $f^{-1} \circ g^{-1}$.

A **subcategory** $\mathcal{C}'$ of a category $\mathcal{C}$ consists of some of the objects of $\mathcal{C}$ and some of the morphisms of $\mathcal{C}$, such that: (a) the source and target of any morphism in $\mathcal{C}'$ is in $\mathcal{C}'$; (b) if $f\colon X \to Y$ and $g\colon Y \to Z$ are in $\mathcal{C}'$, then $g \circ f$ is also in $\mathcal{C}'$; and (c) if an object X is in $\mathcal{C}'$, then 1_X is also in $\mathcal{C}'$. It follows that $\mathcal{C}'$ forms a category.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a (covariant) **functor** F from $\mathcal{C}$ to $\mathcal{D}$ assigns to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, and to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$, such that: (a) if $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$; (b) $F(1_X) = 1_{F(X)}$ for all objects X of $\mathcal{C}$. We write $F\colon \mathcal{C} \to \mathcal{D}$ to mean that F is a functor from $\mathcal{C}$ to $\mathcal{D}$.

EXERCISE B.3. (1) If $f\colon X \to Y$ is an isomorphism, then $F(f)$ is an isomorphism, with inverse $F(f^{-1})$. (2) In the definition of functor, the property that $F(1_X) = 1_{F(X)}$ could be replaced by the weaker property that $F(1_X)$ is an isomorphism, or that it has a left or a right inverse.

If $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are functors, their **composite**, denoted $G \circ F$ or GF, is the functor from $\mathcal{C}$ to $\mathcal{E}$ defined by $G \circ F(X) = G(F(X))$ and $G \circ F(f) = G(F(f))$. With this composition law, the categories form a category, denoted (Cat).

If F and G are functors from $\mathcal{C}$ to $\mathcal{D}$, a **natural transformation** θ from F to G assigns to each object X in $\mathcal{C}$ a morphism θ_X from $F(X)$ to $G(X)$ in $\mathcal{D}$, such that for any morphism $f\colon X \to Y$ in $\mathcal{C}$, $G(f) \circ \theta_X = \theta_Y \circ F(f)$, i.e., the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\theta_X"'] & F(Y) \arrow[d, "\theta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

commutes. The notation $\theta\colon F \Rightarrow G$ is used to indicate that θ is a natural transformation from F to G. It is a **natural isomorphism** if each θ_X is an isomorphism, in which case one writes $\theta\colon F \overset{\sim}{\Rightarrow} G$.

If F, G, and H are functors from $\mathcal{C}$ to $\mathcal{D}$, two natural transformations θ from F to G and η from G to H can be composed, giving a natural transformation $\eta \circ \theta$ from F to H. This is defined by setting $(\eta \circ \theta)_X = \eta_X \circ \theta_X$.

EXERCISE B.4. (1) For fixed categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathrm{HOM}(\mathcal{C}, \mathcal{D})$ (or $\mathrm{HOM}_{(\mathrm{Cat})}(\mathcal{C}, \mathcal{D})$) with objects the functors from $\mathcal{C}$ to $\mathcal{D}$, and with arrows from F to G the natural transformations. (2) If θ is a natural isomorphism from F to G, then θ^{-1}, defined by $(\theta^{-1})_X = (\theta_X)^{-1}$, is a natural isomorphism from G to F, with $\theta^{-1} \circ \theta = 1_F$ and $\theta \circ \theta^{-1} = 1_G$.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **strict isomorphism** if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ and $F \circ G$ are the identity functors $1_{\mathcal{C}}$ on $\mathcal{C}$ and $1_{\mathcal{D}}$ on $\mathcal{D}$.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an **equivalence** of categories if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms θ from $G \circ F$ to $1_{\mathcal{C}}$ and η from $F \circ G$ to $1_{\mathcal{D}}$. (Note that only the existence of G, θ, and η is required, and they need not be unique.)

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called **faithful** if for any morphisms $f\colon X \to Y$ and $g\colon X \to Y$ in $\mathcal{C}$, the equality of $F(f)$ and $F(g)$ implies the equality of f and g. A functor F is called **full** if, for any objects X and Y of $\mathcal{C}$, any morphism from $F(X)$ to $F(Y)$ in $\mathcal{D}$ has the form $F(f)$ for some $f\colon X \to Y$ in $\mathcal{C}$. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if, for every object X in $\mathcal{D}$, there is an object P in $\mathcal{C}$ and an isomorphism from $F(P)$ to X in $\mathcal{D}$.

The inclusion of a subcategory $\mathcal{C}'$ in a category $\mathcal{C}$ is always a faithful functor. If $\mathcal{C}'$ is obtained by choosing some of the objects of $\mathcal{C}$, and all morphisms between them, this inclusion is also full, and $\mathcal{C}'$ is called a **full subcategory**.

EXERCISE B.5. Suppose F and G are naturally isomorphic functors. Then F is faithful (resp. full, resp. essentially surjective) if and only if G is faithful (resp. full, resp. essentially surjective).

EXERCISE B.6. If $F\colon \mathcal{C} \to \mathcal{D}$ is full and faithful, and $f\colon X \to Y$ is a morphism in $\mathcal{C}$, show that f is an isomorphism if and only if $F(f)$ is an isomorphism.

PROPOSITION B.1. *A functor is an equivalence of categories if and only if it is full, faithful, and essentially surjective.*

PROOF. We sketch the proof of the implication $\Leftarrow$. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is the functor. For each object X of $\mathcal{D}$, choose (by an appropriate axiom of choice if necessary, cf. Section 5) an object $G(X)$ of $\mathcal{C}$ and an isomorphism $\eta_X\colon F(G(X)) \to X$ in $\mathcal{D}$. For a morphism $f\colon X \to Y$ in $\mathcal{D}$, there is a unique morphism $G(f)\colon G(X) \to G(Y)$ in $\mathcal{C}$ such that $F(G(f)) = {\eta_Y}^{-1} \circ f \circ \eta_X$. Verify that G is a functor. For an object P of $\mathcal{C}$, define $\theta_P\colon G(F(P)) \to P$ to be the morphism such that $F(\theta_P) = \eta_{F(P)}$, and verify that θ and η are natural isomorphisms. $\square$

EXERCISE B.7. Complete the proof of this proposition.

EXERCISE B.8. Show that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural isomorphisms θ from $G \circ F$ to $1_{\mathcal{C}}$ and η from $F \circ G$ to $1_{\mathcal{D}}$ such that $F(\theta_P) = \eta_{F(P)}$ for all objects P in $\mathcal{C}$ and $G(\eta_X) = \theta_{G(X)}$ for all objects X in $\mathcal{D}$. In this case the data $(F, G, \theta^{-1}, \eta)$ is what is called an *adjoint equivalence*, cf. [**65**, §IV.4] and Section 3.

EXERCISE B.9. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. (1) F and G faithful (resp. full, resp. essentially surjective) imply GF faithful (resp. full, resp. essentially surjective). (2) GF faithful implies F faithful; GF essentially surjective implies G essentially surjective; GF full and F essentially surjective implies G full; GF full and G full and faithful implies F full. (3) If GF is an equivalence of categories, and either F is essentially surjective or G is full and faithful, then F and G are both equivalences of categories.

EXAMPLE B.2. A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is a **skeleton** of $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic to exactly one object of $\mathcal{C}'$. The inclusion $\mathcal{C}' \to \mathcal{C}$ is then an equivalence of categories. For any category $\mathcal{C}$, the choice of one object from each isomorphism class of objects determines a skeleton $\mathcal{C}'$. For example, if $\mathcal{C}$ is the category of finite nonempty sets, the full subcategory whose objects are the sets $\{1, \ldots, n\}$ for $n \geq 1$ is a skeleton of $\mathcal{C}$.

Example B.3. The **product** $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}$ and $\mathcal{D}$ is the category whose objects are pairs (X, Y) of objects X of $\mathcal{C}$, Y of $\mathcal{D}$; a morphism $(f, g) \colon (X, Y) \to (X', Y')$ is a pair of morphisms $f \colon X \to X'$ in $\mathcal{C}$ and $g \colon Y \to Y'$ in $\mathcal{D}$, with composition induced by that in each category. One constructs similarly a product of any number of categories.

The **opposite** category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is obtained by reversing all the arrows of $\mathcal{C}$. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor F from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. This assigns to each object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, and to each morphism $f \colon X \to Y$ of $\mathcal{C}$ a morphism $F(f) \colon F(Y) \to F(X)$. These satisfy: if $f \colon X \to Y$ and $g \colon Y \to Z$, then $F(g \circ f) = F(f) \circ F(g)$, as well as $F(1_X) = 1_{F(X)}$ for all objects X.

Definition B.4. A commutative square

$$\begin{tikzcd} V \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

of objects and morphisms in a category $\mathcal{C}$ is called **cartesian** if it satisfies the following universal property. For any object U and morphisms $f \colon U \to X$ and $g \colon U \to Y$ such that $sf = tg$, there is a unique morphism $h \colon U \to V$ such that $ph = f$ and $qf = g$:

$$\begin{tikzcd} U \arrow[dr, dotted, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & V \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

It follows that V is unique up to canonical isomorphism: if V', $p' \colon V' \to X$, $q' \colon V' \to Y$ also satisfy the universal property, there is a unique isomorphism $\vartheta \colon V' \to V$ such that $p' = p\vartheta$ and $q' = q\vartheta$.

If the diagram is cartesian, one writes $V = X \times_Z Y$, and V is called the (or a) **fibered product** of X and Y over Z. If the morphisms s and t need to be specified, one writes $V = X \,{}_s{\times}_t\, Y$ or $V = X \,{}_s{\times}_{Z,t}\, Y$. The morphism $h \colon U \to X \times_Z Y$ determined by f and g is usually denoted (f, g). The projection $X \times_Z Y \to X$ is often called the **pullback** of the morphism $Y \to Z$ by $s \colon X \to Z$.

If the category $\mathcal{C}$ has a final object $\bullet$ (so each object of $\mathcal{C}$ has a unique morphism to $\bullet$) then the fibered product $X \times_\bullet Y$ is called the **product** of X and Y, and denoted $X \times Y$.

Exercise B.10. Construct fibered products for arbitrary morphisms $s\colon X \to Z$ and $t\colon Y \to Z$ in the category (Set) of sets and the category (Top) of topological spaces.

Exercise B.11. (1) Given morphisms $s\colon X \to Z$, $t\colon Y \to Z$, $s'\colon X' \to Z'$, $t'\colon Y' \to Z'$, and morphisms $f\colon X' \to X$, $g\colon Y' \to Y$, $h\colon Z' \to Z$, with $sf = hs'$ and $tg = ht'$, construct a canonical morphism $X' \times_{Z'} Y' \to X \times_Z Y$, whenever these fibered products exist. (2) For any morphism $f\colon X \to Y$, construct a canonical morphism $X \to X \times_Y X$, whenever this fibered product exists; it is called the **diagonal** morphism.

Exercise B.12. (1) For any morphism $s\colon X \to Y$, the fibered product $X \times_Y Y = X \;{}_s{\times}_{Y,1_Y} Y$ exists and is canonically isomorphic to X. (2) There is a canonical isomorphism of $X \;{}_s{\times}_{Z,t} Y$ with $Y \;{}_t{\times}_{Z,s} X$, with one existing if and only if the other exists. (3) Suppose $s\colon X \to Z$, $t\colon Y \to Z$, $u\colon Y \to W$, and $v\colon V \to W$ are given, and $X \times_Z Y$ and $Y \times_W V$ exist. If one of the fibered products $(X \times_Z Y) \times_Y (Y \times_W V)$, $(X \times_Z Y) \times_W V$ or $X \times_Z (Y \times_W V)$ exists, then all exist and are canonically isomorphic. This fibered product is also denoted $X \times_Z Y \times_W V$; it is characterized by a universal property for triples of morphisms $f\colon U \to X$, $g\colon U \to Y$, and $h\colon U \to V$ such that $sf = tg$ and $ug = vh$: there is a unique morphism $(f, g, h)\colon U \to X \times_Z Y \times_W V$ such that f, g, and h are recovered by composing (f, g, h) with the projections to the three factors. (4) Suppose $s\colon X \to Y$, $t\colon Y \to Z$, and $f\colon W \to Z$ are morphisms, and $Y \;{}_t{\times}_{Z,f} W$ exists. Then $X \;{}_{ts}{\times}_{Z,f} W$ exists if and only if $X \times_Y (Y \times_Z W)$ exists, and then they are canonically isomorphic. (5) Suppose morphisms $X \to Z$, $Y \to Z$, and $Z \to T$ are given, and $X \times_T Y$ and $Z \times_T Z$ exist. Then $X \times_Z Y$ exists if and only if $(X \times_T Y) \times_{Z \times_T Z} Z$ exists, and then they are canonically isomorphic; here $X \times_T Y \to Z \times_T Z$ is the canonical map, and $Z \to Z \times_T Z$ the diagonal map, of the preceding exercise. In particular, if $\mathcal{C}$ has a final object, there is a canonical isomorphism $X \times_Z Y \cong (X \times Y) \times_{Z \times Z} Z$, whenever these fibered products exist.

For an object X in a category $\mathcal{C}$, define a contravariant functor h_X from $\mathcal{C}$ to the category (Set) of sets, that takes an object S to the set $h_X(S) = \mathrm{Hom}(S, X)$ of morphisms from S to X, and takes a morphism $u\colon T \to S$ to the mapping $h_X(u)\colon h_X(T) \to h_X(S)$ which sends $g\colon S \to X$ to $g \circ u\colon T \to X$. The elements of $h_X(S)$ are called S-**valued points of** X.

Exercise B.13. For any functor $H\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, any object ζ in $H(X)$ determines a natural transformation from h_X to H; this assigns to an object S of $\mathcal{C}$ the map from $h_X(S)$ to $H(S)$ that takes $g\colon S \to X$ to $H(g)(\zeta)$. Show that every natural transformation from h_X to H arises in this way from a unique ζ in $H(X)$.

Any morphism $f\colon X \to Y$ in $\mathcal{C}$ determines a mapping from $h_X(S)$ to $h_Y(S)$ that takes $g\colon S \to X$ to $f \circ g\colon S \to Y$. This determines a covariant functor

$$\mathcal{C} \to \mathrm{HOM}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})).$$

Exercise B.14. Show that this functor is full and faithful.

A functor $H\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is **representable by an object** X of $\mathcal{C}$ if one has a natural isomorphism between h_X and H. This is given by an element ζ in $H(X)$ such that, for all S, the map $h_X(S) \to H$ that takes $g\colon S \to X$ to $H(g)(\zeta)$ is a bijection. (Note that one must specify both X and ζ to represent H.) By Exercise B.14, the object X that represents H is determined up to canonical isomorphism. This combination of ideas is known as *Yoneda's Lemma*.

If $F \to G$ and $H \to G$ are natural transformations between functors from $\mathcal{C}^{\mathrm{op}}$ to (Set), there is a **fibered product** $F \times_G H$, which takes an object S of $\mathcal{C}$ to the set

$$(F \times_G H)(S) \;=\; F(S) \times_{G(S)} H(S)$$

of pairs of elements in $F(S)$ and $H(S)$ with the same image in $G(S)$. A morphism $u\colon T \to S$ in $\mathcal{C}$ is sent to the map from $F(S) \times_{G(S)} H(S)$ to $F(T) \times_{G(T)} H(T)$ determined by $F(u)$ and $H(u)$. The fibered product $F \times_G H$ is a contravariant functor from $\mathcal{C}$ to (Set). It comes equipped with natural transformations (called projections) from $F \times_G H$ to F and to H; it is a fibered product in the category of contravariant functors from $\mathcal{C}$ to (Set).

Exercise B.15. A commutative diagram as in Definition B.4 is cartesian if and only if, for every object S in $\mathcal{C}$, the corresponding diagram of S-valued points is a cartesian diagram in the category of sets. That is, the map

$$h_V(S) \;\to\; h_X(S) \times_{h_Z(S)} h_Y(S)$$

is a bijection. Equivalently, the canonical natural transformation from h_V to $h_X \times_{h_Z} h_Y$ is a natural isomorphism.

A natural transformation $F \to G$ between contravariant functors from $\mathcal{C}$ to (Set) is called **representable** if, for every object X in $\mathcal{C}$ and natural transformation $h_X \to G$, the fibered product $F \times_G h_X$ is representable. If Y is an object representing $F \times_G h_X$, the projection from $F \times_G h_X$ to h_X determines a morphism from Y to X in $\mathcal{C}$.

Exercise B.16. If Y' is another object representing $F \times_G h_X$, the morphism from Y' to X determined by $F \to G$ factors uniquely into $Y' \to Y \to X$, where the first morphism is an isomorphism and the second is the morphism of the definition.

Exercise B.17. The composite of two representable natural transformations is representable. If $F \to G$ is representable, then $F \times_G H \to H$ is representable for any natural transformation $H \to G$. If $F \to G$ is representable, and H is representable, then $F \times_G H$ is representable for any $H \to G$.

If F and G are contravariant functors from $\mathcal{C}$ to (Set), F is a **subfunctor** of G if, for every object S of $\mathcal{C}$, $F(S)$ is a subset of $G(S)$, and, for every morphism $u\colon T \to S$, the map $F(u)$ from $F(S)$ to $F(T)$ is the restriction of the map $G(u)$ from $G(S)$ to $G(T)$.

Let $\pi\colon X \to Y$ be a morphism in a category $\mathcal{C}$, and assume that a fibered product $X \times_Y X$ exists, with projections p_1 and p_2 from $X \times_Y X$ to X. The morphism π makes Y a **quotient** of X if it satisfies the following universal mapping property: for any morphism $u\colon X \to Z$ such that the two morphisms $u \circ p_1$ and $u \circ p_2$ from $X \times_Y X$ to Z

are equal, there is a unique morphism $v\colon Y \to Z$ such that $u = v \circ \pi$:

$$\begin{tikzcd}
X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "\pi"'] \arrow[ddr, bend left, "u"] & \\
X \arrow[r, "\pi"] \arrow[drr, bend right, "u"'] & Y \arrow[dr, dotted, "v"] & \\
 & & Z
\end{tikzcd}$$

For π to make Y a quotient of X amounts to the fibered product square satisfying this dual *cocartesian* property as well as the cartesian property.

For more about categories, functors, and natural transformations, see [**65**]. For more on representable functors, see [**EGA** 0.8.1].

2. 2-categories

A **2-category** $\mathcal{C}$ has **objects** (denoted here X, Y, etc.), **morphisms**, sometimes called **1-morphisms** or **arrows** (denoted here f, g, etc.), and **2-morphisms** (denoted here α, β, etc.). Each morphism f has a source and target object, for which we write $f\colon X \to Y$ as before, and there are identity morphisms $1_X\colon X \to X$ for each object X, with compositions $g \circ f\colon X \to Z$ for each $f\colon X \to Y$ and $g\colon Y \to Z$. These objects and morphisms are required to satisfy the category axioms; this category is called the **underlying category** of the 2-category $\mathcal{C}$.

A 2-morphism α has a source morphism f and a target morphism g, with both f and g required to be morphisms with the same source and target. We write $\alpha\colon f \Rightarrow g$ to mean that α is a 2-morphism with source f and target g, and we say α is a 2-morphism **from** f **to** g. If f and g are morphisms from X to Y, this may be denoted

$$\begin{tikzcd}
X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & Y.
\end{tikzcd}$$

There are two operations on 2-morphisms. First, if $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$ are 2-morphisms, with f, g, and h all morphisms with the same source and target, there is a 2-morphism, denoted $\beta \circ \alpha$, from f to h:

$$\begin{tikzcd}
X \arrow[r, bend left=50, ""{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=50, ""{name=D}] \arrow[from=U, to=M, phantom, "\Downarrow\alpha"] \arrow[from=M, to=D, phantom, "\Downarrow\beta"] & Y \arrow[r, squiggly] & X \arrow[r, bend left, ""{name=A}] \arrow[r, bend right, ""{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\beta\circ\alpha"] & Y.
\end{tikzcd}$$

Second, if $\alpha\colon f \Rightarrow f'$, with f and f' morphisms from X to Y, and $\beta\colon g \Rightarrow g'$, with g and g' from Y to Z, then there is a 2-morphism $\beta * \alpha$ from $g \circ f$ to $g' \circ f'$:

$$\begin{tikzcd}
X \arrow[r, bend left, ""{name=A}] \arrow[r, bend right, ""{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, ""{name=C}] \arrow[r, bend right, ""{name=D}] \arrow[from=C, to=D, phantom, "\Downarrow\beta"] & Z \arrow[r, squiggly] & X \arrow[r, bend left, ""{name=E}] \arrow[r, bend right, ""{name=F}] \arrow[from=E, to=F, phantom, "\Downarrow\beta*\alpha"] & Z.
\end{tikzcd}$$

As the pictures indicate, these are sometimes called **vertical** and **horizontal** composition of 2-morphisms.[2] These operations are required to satisfy the following properties, each of which is an identity between 2-morphisms:

(a) If $\alpha\colon f \Rightarrow g$, $\beta\colon g \Rightarrow h$, and $\gamma\colon h \Rightarrow i$, then $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)\colon f \Rightarrow i$.
(b) For every morphism f, there is a 2-morphism $1_f\colon f \Rightarrow f$ such that $\alpha \circ 1_f = \alpha$ for all $\alpha\colon f \Rightarrow g$ and $1_f \circ \beta = \beta$ for all $\beta\colon g \Rightarrow f$. (This 1_f is unique.)
(c) For $\alpha\colon f \Rightarrow g$, with f and g from X to Y, $\alpha * 1_{1_X} = \alpha = 1_{1_Y} * \alpha$.
(d) For $f\colon X \to Y$ and $g\colon Y \to Z$, $1_g * 1_f = 1_{g \circ f}$.
(e) If $\alpha\colon f \Rightarrow f'$, with f and f' mapping X to Y, and $\beta\colon g \Rightarrow g'$, with g and g' mapping Y to Z, and $\gamma\colon h \Rightarrow h'$, with h and h' mapping Z to W, then $\gamma * (\beta * \alpha) = (\gamma * \beta) * \alpha$, as 2-morphisms from $h \circ g \circ f$ to $h' \circ g' \circ f'$.
(f) (Exchange) Given morphisms f, f', f'' from X to Y, morphisms g, g', g'' from Y to Z, and 2-morphisms $\alpha\colon f \Rightarrow f'$, $\alpha'\colon f' \Rightarrow f''$, $\beta\colon g \Rightarrow g'$, and $\beta'\colon g' \Rightarrow g''$, we have $(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$. In pictures:

$$\begin{tikzcd}
X \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\alpha'"] & Y \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta'"] \arrow[d, squiggly] & Z \arrow[r, squiggly] & X \arrow[rr, bend left=40] \arrow[rr] \arrow[rr, bend right=40] \arrow[rr, phantom, shift left=2.5ex, "\Downarrow\beta*\alpha"] \arrow[rr, phantom, shift right=2.5ex, "\Downarrow\beta'*\alpha'"] & {} \arrow[d, squiggly] & Z \\
X \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow\alpha'\circ\alpha"] & Y \arrow[r, bend left=40] \arrow[r, bend right=40] \arrow[r, phantom, "\Downarrow\beta'\circ\beta"] & Z \arrow[r, squiggly] & X \arrow[rr, bend left=30] \arrow[rr, bend right=30] \arrow[rr, phantom, "\Downarrow"] & {} & Z
\end{tikzcd}$$

It follows from (a) and (b) that, for any two objects X and Y, we have a category, denoted $\mathrm{HOM}(X, Y)$ (or $\mathrm{HOM}_{\mathcal{C}}(X, Y)$), whose objects are morphisms $f\colon X \to Y$, and whose arrows are 2-morphisms $\alpha\colon f \Rightarrow g$, composed by the vertical composition.

A 2-morphism $\alpha\colon f \Rightarrow g$ is a **2-isomorphism** if there is a 2-morphism $\alpha^{-1}\colon g \Rightarrow f$ with $\alpha^{-1} \circ \alpha = 1_f$ and $\alpha \circ \alpha^{-1} = 1_g$. Such α^{-1} is unique, if it exists. The notation $\alpha\colon f \overset{\sim}{\Rightarrow} g$ means that α is a 2-isomorphism. We say that morphisms f and g are **2-isomorphic** if there is a 2-isomorphism between them, and then we write $f \overset{\sim}{\Rightarrow} g$. Given any 2-category, one can throw away all 2-morphisms that are not 2-isomorphisms, with the result remaining a 2-category. (Almost all 2-morphisms appearing in this book are in fact 2-isomorphisms.)

Exercise B.18. (1) For $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$ as in the definition of $\beta * \alpha$, we have

$$\beta * \alpha = (\beta * 1_{f'}) \circ (1_g * \alpha) = (1_{g'} * \alpha) \circ (\beta * 1_f).$$

In particular, the $*$-product is determined by the $\circ$-product and the $*$-product for which one of the factors in an identity 2-morphism. (2) When $\beta * \alpha$ is defined, if α and β are 2-isomorphisms, then $\beta * \alpha$ is a 2-isomorphism, with inverse $\beta^{-1} * \alpha^{-1}$.

[2]The reader should be warned that the symbols $\circ$, $*$, $\bullet$, $\cdot$, as well as juxtaposition, and probably others, have been used for one or the other of these operations.

The 2-morphisms in a 2-category are sometimes called *2-cells*. In this case the morphisms are called *1-cells*, and the objects may be called *0-cells*.

A diagram

$$\begin{tikzcd} V \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

means that a 2-morphism $\alpha\colon f\,a \Rightarrow g\,b$ is specified. (If the arrow is pointed in the other direction, it indicates a 2-morphism from $g\,b$ to $f\,a$.) We say that the diagram **2-commutes** when a 2-isomorphism $\alpha\colon f\,a \overset{\sim}{\Rightarrow} g\,b$ is given. When $f\,a = g\,b$, the diagram is said to **strictly commute**, and α is taken to be $1_{f\,a} = 1_{g\,b}$; in this case the arrow $\Rightarrow$ in the diagram may be replaced by an equality sign $=$. The same terminology is used when the square is replaced by any polygon, with arrows starting at some vertex and moving in opposite directions toward another vertex.

The axioms, particularly the exchange property, allow one to compose 2-morphisms across diagrams, with the result being independent of choices. For example, given a diagram

$$\begin{tikzcd} S \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\overset{\alpha}{\Nearrow}"] & T \arrow[d, "c"'] \arrow[dr, bend left, "d"] \arrow[dr, phantom, "\overset{\beta}{\Nearrow}"] & \\ U \arrow[r, "e"] \arrow[dr, bend right, "g"'] \arrow[dr, phantom, "\overset{\gamma}{\Nearrow}"] & V \arrow[r, "f"] \arrow[d, "h"] \arrow[dr, phantom, "\overset{\delta}{\Nearrow}"] & W \arrow[d, "i"] \\ & X \arrow[r, "j"'] & Y \end{tikzcd}$$

one gets a 2-morphism from $j\,g\,b$ to $i\,d\,a$, by first doing γ, then α and δ (in either order), and finally doing β. Officially, this 2-morphism is

$$(1_i * \beta * 1_a) \circ (\delta * \alpha) \circ (1_j * \gamma * 1_b),$$

noting that $\delta * \alpha = (\delta * 1_{c\,a}) \circ (1_{j\,h} * \alpha) = (1_{i\,f} * \alpha) \circ (\delta * 1_{e\,b})$. Sometimes one can express an equality among 2-morphisms by saying that the results of such pastings of polygons around the sides of a solid polytope in 3-space are the same, but these diagrams (with their labels) are not easy to draw, nor are they easy to manipulate to prove identities. In fact, it is often useful to express an equality among 2-morphisms by an ordinary commutative diagram involving 2-morphisms in a HOM-category. For example, the above situation can be expressed by the diagram

$$\begin{tikzcd} & & j \circ h \circ c \circ a \arrow[dr, Rightarrow, "\delta"] & & \\ j \circ g \circ b \arrow[r, Rightarrow, "\gamma"] & j \circ h \circ e \circ b \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\delta"'] & & i \circ f \circ c \circ a \arrow[r, Rightarrow, "\beta"] & i \circ d \circ a \\ & & i \circ f \circ e \circ b \arrow[ur, Rightarrow, "\alpha"'] & & \end{tikzcd}$$

in the category HOM(S, Y), with the central square commuting. When no confusion is possible, we omit the identity 2-isomorphisms from the labels over double arrows; in this example, the γ over the first double arrow is short for $1_j * \gamma * 1_b$, and similarly for the others. Similarly, (1) of Exercise B.18 says that the diagrams

$$\begin{tikzcd} g \circ f \arrow[r, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta*\alpha"'] & g \circ f' \arrow[d, Rightarrow, "\beta"] \\ & g' \circ f' \end{tikzcd} \qquad \begin{tikzcd} g \circ f \arrow[d, Rightarrow, "\beta"'] \arrow[dr, Rightarrow, "\beta*\alpha"] & \\ g' \circ f \arrow[r, Rightarrow, "\alpha"'] & g' \circ f' \end{tikzcd}$$

commute.

Exercise B.19. Let $h\colon X \to X$ be a morphism in a 2-category, and let $\theta\colon 1_X \overset{\sim}{\Rightarrow} h$ be a 2-isomorphism. Show that $\theta * 1_h = 1_h * \theta$ from h to $h \circ h$, i.e., the diagram

$$\begin{tikzcd} h \arrow[r, equal] \arrow[d, equal] & 1_X \circ h \arrow[d, Rightarrow, "\theta"] \\ h \circ 1_X \arrow[r, Rightarrow, "\theta"'] & h \circ h \end{tikzcd}$$

commutes in the category HOM(X, X).

Exercise B.20. Properties (d) and (f) say that the assignment

$$\mathrm{HOM}(X, Y) \times \mathrm{HOM}(Y, Z) \to \mathrm{HOM}(X, Z)$$

that takes (f, g) to $g \circ f$, and (α, β) to $\beta * \alpha$, is a functor. Property (e) implies that, for X, Y, Z and W, the diagram of categories

$$\begin{tikzcd} \mathrm{HOM}(X, Y) \times \mathrm{HOM}(Y, Z) \times \mathrm{HOM}(Z, W) \arrow[r] \arrow[d] & \mathrm{HOM}(X, Z) \times \mathrm{HOM}(Z, W) \arrow[d] \\ \mathrm{HOM}(X, Y) \times \mathrm{HOM}(Y, W) \arrow[r] & \mathrm{HOM}(X, W) \end{tikzcd}$$

commutes. Property (c) implies that the composite functor

$$\mathrm{HOM}(X, Y) \to \mathrm{HOM}(X, X) \times \mathrm{HOM}(X, Y) \to \mathrm{HOM}(X, Y),$$

where the first takes f to $(1_X, f)$ and α to $(1_{1_X}, \alpha)$, is the identity functor.

We give several examples, starting with the prototype from geometry.

Example B.5. There is a 2-category (Top), whose objects are topological spaces, whose morphisms are continuous maps, and whose 2-morphisms come from homotopies — but here we must take appropriate equivalence classes. Given continuous maps f and g from X to Y, a **homotopy** from f to g is a continuous mapping

$$H\colon X \times [0, 1] \to Y$$

with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Call two homotopies H_0 and H_1 **equivalent** if there is a continuous mapping

$$K\colon X \times [0, 1] \times [0, 1] \to Y$$

with $K(x,t,0) = H_0(x,t)$, $K(x,t,1) = H_1(x,t)$, $K(x,0,u) = f(x)$, and $K(x,1,u) = g(x)$ for all $x \in X$, $t, u \in [0,1]$. (This is an equivalence relation.) A **2-morphism** from f to g is defined to be an equivalence class of homotopies from f to g.

If f, g, and h map X to Y, and H_1 is a homotopy from f to g, and H_2 a homotopy from g to h, define $H_2 \circ H_1$ by

$$H_2 \circ H_1(x,t) = \begin{cases} H_1(x,2t), & 0 \leq t \leq 1/2 \\ H_2(x,2t-1), & 1/2 \leq t \leq 1 \end{cases}.$$

This passes to equivalence of homotopies, so defines the vertical composition $\beta \circ \alpha$ of 2-morphisms. It is associative by the same calculation made to show the associativity of fundamental groups.

If f and f' map X to Y, and g and g' map Y to Z, and H_1 is a homotpy from f to f', and H_2 is a homotopy from g to g', define a homotopy $H_2 * H_1$ from $g \circ f$ to $g' \circ f'$ by

$$(H_2 * H_1)(x,t) = H_2(H_1(x,t),t) \qquad x \in X, \quad 0 \leq t \leq 1.$$

This passes to equivalence classes, defining the horizontal product $\beta * \alpha$ of 2-morphisms.

Exercise B.21. Verify that these operations make (Top) into a 2-category, in which all 2-morphisms are 2-isomorphisms.

Example B.6. There is a 2-category (CC) of chain complexes of abelian groups (and similarly, a 2-category (CC_R) of chain complexes of R-modules, for a commutative ring R). The objects are the usual chain complexes $C = C_\bullet$, with boundary homomorphisms $d_n \colon C_n \to C_{n-1}$ satisfying $d_{n-1} \circ d_n = 0$. A morphism $f = f_\bullet$ from C to D is a collection of homomorphisms $f_n \colon C_n \to D_n$, commuting with the boundary maps. A chain homotopy $\alpha = \alpha_\bullet$ from f to g is a collection of homomorphisms $\alpha_n \colon C_n \to D_{n+1}$ such that $d_{n+1} \circ \alpha_n + \alpha_{n-1} \circ d_n = g_n - f_n$ for all n. Call two chain homomotopies α and β from f to g **equivalent** if there is a collection $\theta = \theta_\bullet$ of homomorphisms $\theta_n \colon C_n \to D_{n+2}$ such that

$$d_{n+2} \circ \theta_n - \theta_{n-1} \circ d_n = \beta_n - \alpha_n$$

for all n. (This is an equivalence relation.) A **2-morphism** from f to g is an equivalence class of such chain homotopies.

If f, g, and h map C to D, α is a chain homotopy from f to g, and β is a chain homotopy from g to h, define the chain homotopy $\beta \circ \alpha$ from f to h by the formula $(\beta \circ \alpha)_n = \alpha_n + \beta_n$. This passes to equivalence classes, so defines a vertical composition of 2-morphisms. If f and f' map C to D, and g and g' map D to E, and α is a chain homotopy from f to f' and β is a chain homotopy from g to g', define the chain homotopy $\beta * \alpha$ from $g \circ f$ to $g' \circ f'$ by the formula

$$(\beta * \alpha)_n = g_{n+1} \circ \alpha_n + \beta_n \circ f'_n.$$

(This is equivalent to the alternative $\beta_n \circ f_n + g'_{n+1} \circ \alpha_n$.) This respects the equivalence, so defines a horizontal composition of 2-morphisms.

Exercise B.22. Verify that these objects, morphisms, and 2-morphisms satisfy the axioms to form a 2-category.

Example B.7. The 2-category (Grp) has groups as objects, group homomorphisms as morphisms, and, if f and g are homomorphisms from X to Y, a 2-morphism from f to g is an element y in Y such that

$$g(x) = y^{-1} \cdot f(x) \cdot y \quad \text{for all } x \in X.$$

If z gives a 2-morphism from g to h, the composition $z \circ y$ from f to h is given by $y \cdot z$. If $y\colon f \Rightarrow f'$, with f and f' from X to Y, and $z\colon g \Rightarrow g'$, with g and g' from Y to Z, then $z * y\colon g \circ f \Rightarrow g' \circ f'$ is given by the element $g(y) \cdot z = z \cdot g'(y)$ of Z.

Exercise B.23. Verify that these operations make (Grp) into a 2-category, in which all 2-morphisms are 2-isomorphisms.

The following example, with variations, is the key example for this text.

Example B.8. Categories form a 2-category (Cat). Its objects are categories $\mathcal{C}$, its morphisms are functors $F\colon \mathcal{C} \to \mathcal{D}$, and its 2-morphisms $\alpha\colon F \Rightarrow G$ are natural transformations from F to G. If $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ are natural transformations between functors from $\mathcal{C}$ to $\mathcal{D}$, then $\beta \circ \alpha\colon F \Rightarrow H$ is the natural transformation that takes an object X of $\mathcal{C}$ to the morphism $\beta_X \circ \alpha_X$ from $F(X)$ to $H(X)$. If F, F' are functors from $\mathcal{C}$ to $\mathcal{D}$, with $\alpha\colon F \Rightarrow F'$, and G, G' are functors from $\mathcal{D}$ to $\mathcal{E}$, with $\beta\colon G \Rightarrow G'$, define $\beta * \alpha\colon G \circ F \Rightarrow G' \circ F'$ to take the object X of $\mathcal{C}$ to the morphism

$$G'(\alpha_X) \circ \beta_{F(X)} = \beta_{F'(X)} \circ G(\alpha_X)$$

of $\mathcal{E}$.

Exercise B.24. Verify that these operations make (Cat) into a 2-category. (Not all 2-morphisms are 2-isomorphisms.)

Thinking of groupoids of sets as categories shows that groupoids of sets form a 2-category (Gpd). More generally:

Example B.9. Let $\mathcal{S}$ be a category. Let ($\mathcal{S}$-Gpd) be the category whose objects are $\mathcal{S}$-groupoids, whose morphisms are morphisms of $\mathcal{S}$-groupoids (see Chapter 3). If (φ, Φ) and (ψ, Ψ) are morphisms from $R' \rightrightarrows U'$ to $R \rightrightarrows U$, define a 2-morphism from (φ, Φ) to (ψ, Ψ) to be a morphism $\alpha\colon U' \to R$ in $\mathcal{S}$ such that $s \circ \alpha = \varphi$, $t \circ \alpha = \psi$, and the diagram

$$\begin{tikzcd}
R' \arrow[r, "(\alpha s', \Psi)"] \arrow[d, "(\Phi, \alpha t')"'] & R \,{}_t\!\times_s R \arrow[d, "m"] \\
R \,{}_t\!\times_s R \arrow[r, "m"] & R
\end{tikzcd}$$

commutes.

Exercise B.25. Make ($\mathcal{S}$-Gpd) into a 2-category, in which all 2-morphisms are 2-isomorphisms.

Exercise B.26. There is a category whose objects are sets, with an arrow from X to Y being a subset f of $X \times Y$. Define the composition of f with g from Y to Z to be the set of (x, z) in $X \times Z$ such that there is a y in Y with (x, y) in f and (y, z) in g.

This category can be enriched to a 2-category by defining a unique 2-cell from subsets f and g of $X \times Y$ if f is contained in g, with no 2-cell from f to g otherwise. Verify that this is a 2-category.

EXERCISE B.27. If $\mathcal{C}$ is a 2-category, construct a category $\overline{\mathcal{C}}$, whose objects are the same as the objects of $\mathcal{C}$, but whose morphisms from X to Y are equivalence classes of morphisms $f\colon X \to Y$ in $\mathcal{C}$, where f is equivalent to g if there is a 2-isomorphism from f to g. Show that this is an equivalence relation, and that $\overline{\mathcal{C}}$ is a category, with the canonical map from the underlying category of $\mathcal{C}$ to $\overline{\mathcal{C}}$ being a functor. Examples are: the category of topological spaces with homotopy classes of mappings; the category of groups with homomorphisms up to inner automorphism; the category of categories, with functors up to natural isomorphism. This category $\overline{\mathcal{C}}$ is sometimes called the *classifying category* of $\mathcal{C}$, see [**11**].

EXERCISE B.28. Any category $\mathcal{C}$ determines a 2-category, with the same objects and morphisms, and with the only 2-morphisms being identities 1_f, for morphisms f in $\mathcal{C}$.

We say that a 2-category **is a 1-category** if its only 2-morphisms are identities. In this spirit, one says that a category **is a 0-category** if its only morphisms are identity maps.

EXERCISE B.29. If $\mathcal{C}$ is a 2-category, a category $\mathcal{C}'$ can be constructed as follows. The objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$; the morphisms of $\mathcal{C}'$ from X to Y are the 2-morphisms $\alpha\colon f \Rightarrow g$, where f and g are maps from X to Y in $\mathcal{C}$. The composite of α followed by β is $\beta * \alpha$. Verify that $\mathcal{C}'$ is a category.

EXERCISE B.30. (1) If $\mathcal{C}$ is a 2-category, and $f\colon X \to Y$ a morphism in $\mathcal{C}$, we have, for every object S of $\mathcal{C}$, a functor

$$f^S\colon\ \mathrm{HOM}(S,X) \to \mathrm{HOM}(S,Y)$$

taking $h\colon S \to X$ to $f \circ h\colon S \to Y$, and $\alpha\colon h \Rightarrow h'$ to $1_f * \alpha\colon f \circ h \Rightarrow f \circ h'$. Similarly, there are functors

$$f_S\colon\ \mathrm{HOM}(Y,S) \to \mathrm{HOM}(X,S)$$

taking $h\colon Y \to S$ to $h \circ f\colon X \to S$ and $\alpha\colon h \Rightarrow h'$ to $\alpha * 1_f\colon h \circ f \Rightarrow h' \circ f$.

(2) If also $g\colon Y \to Z$, then $(g \circ f)^S = g^S \circ f^S$ and $(g \circ f)_S = f_S \circ g_S$. If $f = 1_X$, then $f^S = 1_{\mathrm{HOM}(S,X)}$ and $f_S = 1_{\mathrm{HOM}(X,S)}$. It follows that, if f is an isomorphism, then each functor f^S and f_S is an isomorphism of categories.

(3) If f and g are morphisms from X to Y, and $\sigma\colon f \Rightarrow g$ is a 2-morphism, then σ determines a natural transformation σ^S from f^S to g^S (taking $h\colon S \to X$ to $\sigma * 1_h$), and a natural transformation σ_S from f_S to g_S (taking $h\colon Y \to S$ to $1_h * \sigma$). If also $\tau\colon g \Rightarrow h$, then $(\tau \circ \sigma)^S = \tau^S \circ \sigma^S$. If $\sigma = 1_f$, then σ^S is the identity natural isomorphism on f^S. Hence, if σ is invertible, then σ^S is a natural isomorphism.

(4) For fixed objects X, Y, and S of $\mathcal{C}$, there is a functor $\mathrm{HOM}(X,Y) \to \mathrm{HOM}_{(\mathrm{Cat})}(\mathrm{HOM}(S,X), \mathrm{HOM}(S,Y))$ taking f to f^S and σ to σ^S.

Just as two topological spaces can be homotopy equivalent, there is a notion for two objects in any 2-category to be 2-isomorphic. In fact, there are several ways to say this:

Proposition B.10. *Let $f\colon X \to Y$ be a morphism in a 2-category $\mathcal{C}$. The following are equivalent:*

(1) *There is a morphism $g\colon Y \to X$ together with 2-isomorphisms $\phi\colon 1_X \overset{\sim}{\Rightarrow} g \circ f$ and $\psi\colon 1_Y \overset{\sim}{\Rightarrow} f \circ g$.*

(2) *There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\phi\colon 1_X \overset{\sim}{\Rightarrow} g \circ f$ and $\psi\colon 1_Y \overset{\sim}{\Rightarrow} f \circ g$ such that $1_f * \phi = \psi * 1_f$ (as 2-isomorphisms from f to fgf) and $\phi * 1_g = 1_g * \psi$ (as 2-isomorphisms from g to gfg). That is, the diagrams*

$$\begin{tikzcd} f \arrow[r, equal] \arrow[d, equal] & 1_Y \circ f \arrow[d, Rightarrow, "\psi"] \\ f \circ 1_X \arrow[r, Rightarrow, "\phi"'] & f \circ g \circ f \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, equal] \arrow[d, equal] & 1_X \circ g \arrow[d, Rightarrow, "\phi"] \\ g \circ 1_Y \arrow[r, Rightarrow, "\psi"'] & g \circ f \circ g \end{tikzcd}$$

commute, in the categories $\mathrm{HOM}(X, Y)$ *and* $\mathrm{HOM}(Y, X)$ *respectively.*

(3) *For every object S of $\mathcal{C}$, the functor $f^S\colon \mathrm{HOM}(S, X) \to \mathrm{HOM}(S, Y)$ is an equivalence of categories.*

(4) *The functors f^X and f^Y are equivalences of categories.*

Proof. We show first how (1) implies (3). By Exercise B.30, we have the functor $g^S\colon \mathrm{HOM}(S, Y) \to \mathrm{HOM}(S, X)$, and we have a natural isomorphism $\phi^S\colon 1_{\mathrm{HOM}(S,X)} \overset{\sim}{\Rightarrow} g^S \circ f^S$. Similarly, we have a natural isomorphism $\psi^S\colon 1_{\mathrm{HOM}(S,Y)} \overset{\sim}{\Rightarrow} f^S \circ g^S$.

Next we prove that (4) implies (2), which finishes the proof since (4) is a special case of (3) and (1) is a special case of (2). Since f^Y is essentially surjective, there is a morphism $g\colon Y \to X$ and a 2-isomorphism $\psi\colon 1_Y \overset{\sim}{\Rightarrow} f \circ g$. Since f^X is full and faithful, there is a unique 2-morphism $\phi\colon 1_X \Rightarrow g \circ f$ such that $f^X(\phi)$ is the 2-isomorphism $\psi * 1_f$ from $f = 1_Y \circ f$ to $f \circ g \circ f$; this ϕ is an isomorphism since $f^X(\phi)$ is an isomorphism (Exercise B.6). Since $f^X(\phi) = 1_f * \phi$, we have one of the required equations $1_f * \phi = \psi * 1_f$. To prove that the 2-morphisms $\phi * 1_g$ and $1_g * \psi$ from g to $g \circ f \circ g$ are equal in $\mathrm{HOM}(Y, X)$, it suffices to show that their images by the faithful functor f^Y are equal, i.e., to show that $1_f * (\phi * 1_g) = 1_f * (1_g * \psi)$. Now

$$\begin{aligned} 1_f * (\phi * 1_g) = (1_f * \phi) * 1_g = (\psi * 1_f) * 1_g = \psi * 1_{fg} \\ = 1_{fg} * \psi = (1_f * 1_g) * \psi = 1_f * (1_g * \psi), \end{aligned}$$

as required; the fourth equality used Exercise B.19. □

This proof shows that, given f, g, ϕ, and ψ, either of the equations $1_f * \phi = \psi * 1_f$ or $\phi * 1_g = 1_g * \psi$ implies the other.

Definition B.11. We call a morphism $f\colon X \to Y$ in a 2-category **2-invertible** or a **2-equivalence**, if it satisfies the conditions of the proposition. (We do not use the more natural term *2-isomorphism*, to avoid confusion with invertible 2-morphisms.)

On the other hand, if there exists a 2-invertible morphism $f\colon X \to Y$, then we call the objects X and Y **2-isomorphic**, as there is no danger of confusion in this context.

A triple (g, ϕ, η) satisfying the conditions of (2) may be called a **2-inverse** of f. A quadruple satisfying the conditions of (2) is sometimes called an **adjoint equivalence**. In practice, one uses (1) to check that a morphism is 2-invertible, but one uses the full data of (2) in making constructions.

Exercise B.31. Show that the conditions of the proposition are equivalent to each of the following:

(5) For every object S of $\mathcal{C}$, the functor $f_S\colon \mathrm{HOM}(Y,S) \to \mathrm{HOM}(X,S)$ is an equivalence of categories.
(6) The functors f_X and f_Y are equivalences of categories.
(7) The functors f_X and f^Y are essentially surjective.
(8) There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\phi\colon 1_X \overset{\sim}{\Rightarrow} g \circ f$ and $\eta\colon f \circ g \overset{\sim}{\Rightarrow} 1_Y$ such that the composition $f = f \circ 1_X \overset{\phi}{\Rightarrow} f \circ g \circ f \overset{\eta}{\Rightarrow} 1_Y \circ f = f$ is equal to 1_f, and the composition $g = 1_X \circ g \overset{\phi}{\Rightarrow} g \circ f \circ g \overset{\eta}{\Rightarrow} g \circ 1_Y = g$ is equal to 1_g.
(9) There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\psi\colon 1_Y \overset{\sim}{\Rightarrow} f \circ g$ and $\theta\colon g \circ f \overset{\sim}{\Rightarrow} 1_X$ such that the composition $f = 1_Y \circ f \overset{\psi}{\Rightarrow} f \circ g \circ f \overset{\theta}{\Rightarrow} f \circ 1_X = f$ is equal to 1_f, and the composition $g = g \circ 1_Y \overset{\psi}{\Rightarrow} g \circ f \circ g \overset{\theta}{\Rightarrow} 1_X \circ g = g$ is equal to 1_g.
(10) There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\theta\colon g \circ f \overset{\sim}{\Rightarrow} 1_X$ and $\eta\colon f \circ g \overset{\sim}{\Rightarrow} 1_Y$ such that the diagrams

$$\begin{tikzcd} f \circ g \circ f \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\eta"'] & f \circ 1_X \arrow[d, equal] \\ 1_Y \circ f \arrow[r, equal] & f \end{tikzcd} \qquad \begin{tikzcd} g \circ f \circ g \arrow[r, Rightarrow, "\theta"] \arrow[d, Rightarrow, "\eta"'] & 1_X \circ g \arrow[d, equal] \\ g \circ 1_Y \arrow[r, equal] & g \end{tikzcd}$$

commute.

It follows from Proposition B.20 in the next section, together with (9) of the preceding exercise, that if (g, ϕ, ψ) is a 2-inverse of f, then any other 2-inverse of f has the form (g', ϕ', ψ'), for a unique 2-isomorphism $\theta\colon g \overset{\sim}{\Rightarrow} g'$ with $\phi' = (\theta * 1_f) \circ \phi$ and $\psi' = (1_f * \theta) \circ \psi$.

Exercise B.32. In the 2-category (Top), two spaces are 2-isomorphic exactly when they have the same homotopy type. In the 2-category (Grp), two groups are 2-isomorphic if and only if they are isomorphic groups. In the 2-category (Cat), two categories are 2-isomorphic when they are equivalent.

Exercise B.33. Show that the condition of being 2-isomorphic is an equivalence relation on the objects of a 2-category.

When applied to the 2-category (Cat), Proposition B.10 and Exercise B.31 give a variety of criteria for a functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ to be an equivalence of categories. Note that the equivalence with (9) recovers the result of Exercise B.8. For the 2-category (Top), one recovers a criterion of Vogt [**90**]. The general statement, in the form that (1) implies (9), appears in [**63**], where it is attributed to a combination of folklore and R. Street.

Definition B.12. A **sub-2-category** $\mathcal{C}'$ of a 2-category $\mathcal{C}$ is obtained by selecting some of the objects, some of the morphisms, and some of the 2-morphisms, of $\mathcal{C}$, in such a way that all identities 1_X of selected objects and 1_f of selected morphisms are selected, and all composites $g \circ f$ and $\beta \circ \alpha$ of selected morphisms or 2-morphisms are selected, as is the product $\beta * \alpha$, whenever such composites or products are defined in $\mathcal{C}$. It is easy to verify that $\mathcal{C}'$ is a 2-category. A sub-2-category $\mathcal{C}'$ is called a **full sub-2-category** of $\mathcal{C}$ if any morphism in $\mathcal{C}$ between two objects of $\mathcal{C}'$ is in $\mathcal{C}'$, and any 2-morphism in $\mathcal{C}$ between two morphisms in $\mathcal{C}'$ is in $\mathcal{C}'$.

Example B.13. The 2-category (Grp) of groups forms a full sub-2-category of the 2-category (Gpd) of groupoids of sets, which in turn forms a full sub-2-category of the 2-category (Cat) of categories.

Most "mappings" from one 2-category to another will not preserve all the structure strictly; rather, the expected identities will be true only up to specified 2-isomorphisms. These "pseudofunctors" will be studied in Section 4. We include here a brief discussion of the stronger notion, called a 2-functor, as a warmup. A **2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ from one 2-category to another assigns to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$, and to each 2-morphism $\alpha\colon f \Rightarrow g$ in $\mathcal{C}$ a 2-morphism $F(\alpha)\colon F(f) \Rightarrow F(g)$ in $\mathcal{D}$, satisfying:

(a) $F(1_X) = 1_{F(X)}$ for all objects X of $\mathcal{C}$;
(b) $F(1_f) = 1_{F(f)}$ for all morphisms f of $\mathcal{C}$;
(c) $F(g \circ f) = F(g) \circ F(f)$ for $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathcal{C}$;
(d) $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$ for $\alpha\colon f \Rightarrow g$, $\beta\colon g \Rightarrow h$ in $\mathcal{C}$;
(e) $F(\beta * \alpha) = F(\beta) * F(\alpha)$ when $\beta * \alpha$ is defined in $\mathcal{C}$.

This gives a functor between the underlying categories (called the **underlying functor**). For objects X and Y of $\mathcal{C}$, it also gives a functor $\mathrm{HOM}(X,Y) \to \mathrm{HOM}(F(X),(F(Y))$ (by (b) and (d)). For example, the inclusion of a sub-2-category in a 2-category is a 2-functor.

Exercise B.34. Construct a 2-functor from the 2-category (Top) of topological spaces to the 2-category (Gpd) of groupoids, that takes a space X to its fundamental groupoid.

Example B.14. There is a 2-functor from the 2-category (Top) to the 2-category (CC) of chain complexes. This takes a topological space X to the chain complex $C_\bullet(X)$ of nondegenerate cubical chains.[3] A continuous mapping $f\colon X \to Y$ is sent to the chain mapping $f_\bullet\colon C_\bullet(X) \to C_\bullet(Y)$ that takes σ to $f \circ \sigma$. A homotopy $H\colon X \times [0,1] \to Y$

[3] $C_n(X)$ is the free module on the set of continuous maps $\sigma\colon [0,1]^n \to X$, modulo the submodule generated by those σ such that, for some $1 \le i \le n$, $\sigma(t_1, \ldots, t_n)$ is a constant function of t_i. The boundary $d_n\colon C_n(X) \to C_{n-1}(X)$ is defined by the formula $d_n = \sum_{i=1}^{n} (-1)^i (\partial_i^0 - \partial_i^1)$, where $\partial_i^\epsilon(\sigma)(t_1, \ldots, t_{n-1}) = \sigma(t_1, \ldots, t_{i-1}, \epsilon, t_i, \ldots, t_{n-1})$.

from f to g determines a chain homotopy α_H from $f_\bullet$ to $g_\bullet$, by the formula

$$\alpha_H(\sigma)(t_1, \ldots, t_{n+1}) = H(\sigma(t_2, \ldots, t_{n+1}), t_1).$$[4]

Exercise B.35. Verify that α_H is a chain homotopy. Show that equivalent homotopies from f to g determine equivalent chain homotopies from $f_\bullet$ to $g_\bullet$, so a 2-morphism in (Top) determines a 2-morphism in (CC). Show that taking X to $C_\bullet(X)$, f to $f_\bullet$, and an equivalence class of H's to the equivalence class of α_H's, determines a 2-functor from (Top) to (CC).

If F and G are 2-functors from a 2-category $\mathcal{C}$ to a 2-category $\mathcal{D}$, a **2-natural transformation** θ from F to G assigns to each object X of $\mathcal{C}$ a morphism $\theta_X \colon F(X) \to G(X)$ in $\mathcal{D}$, satisfying two properties. First, for all $f \colon X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\theta_X"'] & F(Y) \arrow[d, "\theta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

must commute. This says that θ is a natural transformation between the underlying functors on the underlying categories. The second property says that for any $f, g \colon X \to Y$ and 2-morphism $\alpha \colon f \Rightarrow g$ in $\mathcal{C}$, the two morphisms $1_{\theta_Y} * F(\alpha)$ and $G(\alpha) * 1_{\theta_X}$, pictured by

$$\begin{tikzcd} F(X) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\alpha)"] & F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "\theta_X"] & G(X) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(g)"'] \arrow[r, phantom, "\Downarrow G(\alpha)"] & G(Y) \end{tikzcd}$$

from $\theta_Y \circ F(f) = G(f) \circ \theta_X$ to $\theta_Y \circ F(g) = G(g) \circ \theta_X$ must be equal. A 2-natural transformation is a **2-natural isomorphism** if each θ_X is an isomorphism.

Exercise B.36. Define vertical and horizontal composition of 2-natural transformations, by the formulas: $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$; and $(\beta * \alpha)_X = G'(\alpha_X) \circ \beta_{F(X)} = \beta_{F'(X)} \circ G(\alpha_X)$, the latter when α (resp. β) is a 2-natural transformation from F to F' (resp. G to G'). Show that, with these operations, 2-categories, 2-functors, and 2-natural transformations form the objects, arrows, and 2-cells of a 2-category (2-Cat).

One can call a 2-functor $F \colon \mathcal{C} \to \mathcal{D}$ a **strict 2-isomorphism** if there is a 2-functor $G \colon \mathcal{D} \to \mathcal{C}$ with $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$. This notion is much too strong to be useful. Somewhat better is the following: A 2-functor $F \colon \mathcal{C} \to \mathcal{D}$ between 2-categories is a **2-equivalence** if there is a 2-functor $G \colon \mathcal{D} \to \mathcal{C}$ and there are 2-natural isomorphisms from $G \circ F$ to $1_{\mathcal{C}}$ and from $F \circ G$ to $1_{\mathcal{D}}$.

Exercise B.37. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a 2-functor. The following are equivalent: (1) F is a 2-equivalence. (2) F determines an equivalence between the underlying categories, and, for all objects X and Y of $\mathcal{C}$, the induced functor $\mathrm{HOM}(X, Y) \to \mathrm{HOM}(F(X), F(Y))$ is a *strict* isomorphism of categories.

[4]Readers who prefer simplices may use the method of acyclic models to obtain a similar 2-functor involving simplicial complexes.

These 2-functors are relatively rare in the world of 2-categories, and the notion of "isomorphism" that appears in the preceding exercise is too strong to be very useful; a more flexible notation is discussed in Section 4.

DEFINITION B.15. The **opposite** 2-category $\mathcal{C}^{\mathrm{op}}$ of a 2-category $\mathcal{C}$ is obtained by reversing the direction of the 1- morphisms, keeping the direction of the 2-morphisms the same. Thus if f and g are morphisms from X to Y in $\mathcal{C}$, and α is a 2-morphism from f to g, then in $\mathcal{C}^{\mathrm{op}}$ there are morphisms f and g from Y to X, with α a 2-morphism from f to g.

DEFINITION B.16. A 2-commutative diagram

$$\begin{tikzcd} V \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\theta\Nearrow"] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

(with θ a 2-isomorphism from $s\,p$ to $t\,q$) is said to be **2-cartesian** if it satisfies the following universal property: For any morphisms $f\colon U \to X$ and $g\colon U \to Y$ and 2-isomorphism $\phi\colon sf \xRightarrow{\sim} tg$, there is a morphism $h\colon U \to V$ and 2-isomorphisms $\alpha\colon f \xRightarrow{\sim} ph$ and $\beta\colon qh \xRightarrow{\sim} g$ such that $\phi = (1_t * \beta) \circ (\theta * 1_h) \circ (1_s * \alpha)$:

$$\begin{tikzcd} U \arrow[dr, dotted, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & V \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\theta\Nearrow"] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

In addition, we must have the following uniqueness: if $h'\colon U \to V$ and $\alpha'\colon f \xRightarrow{\sim} ph'$ and $\beta'\colon qh' \xRightarrow{\sim} g$ also have $\phi = (1_t * \beta') \circ (\theta * 1_{h'}) \circ (1_s * \alpha')$, then there is a *unique* 2-isomorphism $\rho\colon h \xRightarrow{\sim} h'$ such that $\alpha' = (1_p * \rho) \circ \alpha$ and $\beta = \beta' \circ (1_q * \rho)$. In this case we will call V a **fibered product** of X and Y over Z, and write $V = X \times_Z Y$, but note that the morphisms, and especially the 2-isomorphism, are understood to be part of the structure.

EXERCISE B.38. Given a diagram

$$\begin{tikzcd} X' \arrow[r, "s'"] \arrow[d] \arrow[dr, phantom, "\alpha\Nearrow"] & Y' \arrow[r, "t'"] \arrow[d] \arrow[dr, phantom, "\beta\Nearrow"] & Z' \arrow[d] \\ X \arrow[r, "s"'] & Y \arrow[r, "t"'] & Z \end{tikzcd}$$

if the two squares are 2-cartesian, show that the resulting diagram

$$\begin{tikzcd}
X' \arrow[r, "t's'"] \arrow[d] \arrow[dr, phantom, "\gamma\,\Nearrow"] & Z' \arrow[d] \\
Y \arrow[r, "ts"'] & Z
\end{tikzcd}$$

is also 2-cartesian, with $\gamma = (\beta * 1_{s'}) \circ (1_t * \alpha)$. State and prove analogues of the other parts of Exercises B.11 and B.12 for 2-cartesian diagrams.

Definition B.17. $^{(*)}$ The notion of a quotient in a 2-category is more complicated than that in an ordinary category. To define it, we need some notation for some fibered products. Let $\pi\colon X \to Y$ be a morphism in a 2-category $\mathcal{C}$, and assume there is a fibered product $X_1 = X \times_Y X$, with its projections p_1 and p_2 from X_1 to X and 2-isomorphism $\theta\colon \pi\circ p_1 \Rightarrow \pi\circ p_2$. In addition, assume that there is a fibered product $X_2 = X \times_Y X \times_Y X$, with its projections $q_1, q_2, q_3\colon X_2 \to X$, with associated 2-isomorphisms

$$\theta_{12}\colon \pi \circ q_1 \Rightarrow \pi \circ q_2, \quad \theta_{23}\colon \pi \circ q_2 \Rightarrow \pi \circ q_3.$$

Set $\theta_{13} = \theta_{23} \circ \theta_{12}\colon \pi \circ q_1 \Rightarrow \pi \circ q_3$. For $1 \le i < j \le 3$ we have projections $p_{ij}\colon X_2 \to X_1$, with 2-isomorphisms $\alpha_{ij}\colon q_i \Rightarrow p_1 \circ p_{ij}$ and $\alpha_{ji}\colon q_j \Rightarrow p_2 \circ p_{ij}$, such that the diagrams

$$\begin{tikzcd}
\pi \circ q_i \arrow[r, Rightarrow, "\alpha_{ij}"] \arrow[d, Rightarrow, "\theta_{ij}"'] & \pi \circ p_1 \circ p_{ij} \arrow[d, Rightarrow, "\theta"] \\
\pi \circ q_j \arrow[r, Rightarrow, "\alpha_{ji}"'] & \pi \circ p_2 \circ p_{ij}
\end{tikzcd}$$

commute. Define $\alpha_1 = \alpha_{13} \circ \alpha_{12}^{-1}\colon \pi \circ p_1 \circ p_{12} \Rightarrow \pi \circ p_1 \circ p_{13}$, $\alpha_2 = \alpha_{23} \circ \alpha_{21}^{-1}\colon \pi \circ p_2 \circ p_{12} \Rightarrow \pi \circ p_1 \circ p_{23}$, and $\alpha_3 = \alpha_{32} \circ \alpha_{31}^{-1}\colon \pi \circ p_2 \circ p_{13} \Rightarrow \pi \circ p_2 \circ p_{23}$.[5]

We say that $\pi\colon X \to Y$ makes Y a **2-quotient** of X if it satisfies the following universal mapping property. For any morphism $u\colon X \to Z$, and any 2-isomorphism $\tau\colon u \circ p_1 \Rightarrow u\colon p_2$, such that the diagram

$$\begin{tikzcd}
u \circ p_1 \circ p_{12} \arrow[r, Rightarrow, "\alpha_1"] \arrow[d, Rightarrow, "\tau"'] & u \circ p_1 \circ p_{13} \arrow[r, Rightarrow, "\tau"] & u \circ p_2 \circ p_{13} \arrow[d, Rightarrow, "\alpha_3"] \\
u \circ p_2 \circ p_{12} \arrow[r, Rightarrow, "\alpha_2"'] & u \circ p_1 \circ p_{23} \arrow[r, Rightarrow, "\tau"'] & u \circ p_2 \circ p_{23}
\end{tikzcd}$$

[5]This data may be assembled into a cube, with X_2 on one vertex, Y on the opposite vertex, three copies of X_1 on vertices adjacent to X_2, three copies of X adjacent to Y, with the various projections along the edges, and the 2-isomorphisms across the sides. To say that $X \to Y$ is a 2-quotient can be thought of as an appropriate "2-cocartesian" property of this cube, which amounts to a descent criterion.

commutes, there is a morphism $v\colon Y \to Z$ and a 2-isomorphism $\rho\colon u \Rightarrow v \circ \pi$ such that the diagram

$$\begin{tikzcd} u \circ p_1 \arrow[r, Rightarrow, "\rho"] \arrow[d, Rightarrow, "\tau"'] & v \circ \pi \circ p_1 \arrow[d, Rightarrow, "\theta"] \\ u \circ p_2 \arrow[r, Rightarrow, "\rho"'] & v \circ \pi \circ p_2 \end{tikzcd}$$

commutes. This must satisfy the following uniqueness property: if $v'\colon Y \to Z$ and $\rho'\colon u \Rightarrow v' \circ \pi$ are another morphism and 2-isomorphism satisfying the same properties, there is a unique 2-isomorphism $\zeta\colon v \Rightarrow v'$ such that the diagram

$$\begin{tikzcd} u \arrow[r, Rightarrow, "\rho"] \arrow[dr, Rightarrow, "\rho'"'] & v \circ \pi \arrow[d, Rightarrow, "\zeta"] \\ & v' \circ \pi \end{tikzcd}$$

commutes.

Definition B.18. The **opposite** 2-category $\mathcal{C}^{\mathrm{op}}$ of a 2-category $\mathcal{C}$ is obtained by reversing the direction of the 1- morphisms, keeping the direction of the 2-morphisms the same. Thus if f and g are morphisms from X to Y in $\mathcal{C}$, and α is a 2-morphism from f to g, then in $\mathcal{C}^{\mathrm{op}}$ there are morphisms f and g from Y to X, with α a 2-morphism from f to g.

3. Adjoints

Adjointness of functors is a familiar notion from category theory. Recall, we say that functors $F\colon \mathcal{X} \to \mathcal{Y}$ and $G\colon \mathcal{Y} \to \mathcal{X}$ are adjoint functors, if for every pair of objects X of $\mathcal{X}$ and Y of $\mathcal{Y}$, we have a bijection

$$\mathrm{Hom}_{\mathcal{X}}(GY, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Y}}(Y, FX), \tag{1}$$

natural in X and Y. More specifically, we say that F is right adjoint to G, and G is left adjoint to F. For instance, let $\pi\colon S \to T$ be a continuous map of topological spaces, so we have functors π^{-1} and π_* between the categories of sheaves on S and on T. Then π_* is right adjoint to π^{-1}. In algebraic geometry, when π is a morphism of schemes and our categories are of sheaves of $\mathcal{O}_S$ and $\mathcal{O}_T$ modules, then π_* is right adjoint to π^*; the latter defined by $\pi^*\mathcal{F} = (\pi^{-1}\mathcal{F}) \otimes_{\pi^{-1}\mathcal{O}_T} \mathcal{O}_S$. What is less familiar (and difficult to find in the literature) are results to the effect that adjointness repects base change. The goal of this section is to develop the machinery to arrive at such results in a natural way.

The reader familiar with category theory is probably aware of some equivalent formulations of the notion of adjointness. For instance, the bijection (1) is completely determined by the universal map $Y \to F(GY)$, that is, the image of 1_{GY} under (1) when $X = GY$. There is, similarly, a universal map $G(FX) \to X$. Conversely, a pair of natural transformations $1_{\mathcal{Y}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow 1_{\mathcal{X}}$ satisfying conditions analogous to (2), below, uniquely determines the adjointness relation between F and G. The connections among the various notions of adjointness will be spelled out in detail in section 3.2.

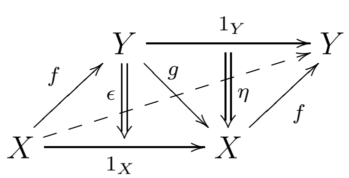
Here we use the language of 2-categories to develop the concept of adjointness and properties relating adjointness with base change. Specializing to the 2-category (Cat), we recover the usual notion of adjoint functors, and specializing further to the adjoint functors π_* and π^* (see Example B.22), we recover the properties concerning adjointness and base change alluded to above.

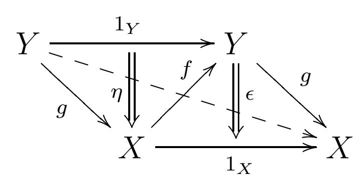
3.1. Adjunctions. We start with the notion of adjunction, in the form of a pair of functors and universal morphisms, abstracted to a general 2-category.

Definition B.19. Let X and Y be objects in a 2-category $\mathcal{C}$. An **adjunction** from X to Y is a quadruple (f, g, η, ϵ), consisting of two morphisms $f\colon X \to Y$ and $g\colon Y \to X$ and two 2-morphisms $\eta\colon 1_Y \Rightarrow f \circ g$ and $\epsilon\colon g \circ f \Rightarrow 1_X$, such that $(1_f * \epsilon) \circ (\eta * 1_f) = 1_f$ and $(\epsilon * 1_g) \circ (1_g * \eta) = 1_g$; that is, the following diagrams commute:

$$\text{(2)} \qquad \begin{tikzcd} 1_Y \circ f \arrow[r, Rightarrow, "\eta"] \arrow[d, equal] & f \circ g \circ f \arrow[d, Rightarrow, "\epsilon"] \\ f \arrow[r, equal] & f \circ 1_X \end{tikzcd} \qquad \begin{tikzcd} g \circ 1_Y \arrow[r, Rightarrow, "\eta"] \arrow[d, equal] & g \circ f \circ g \arrow[d, Rightarrow, "\epsilon"] \\ g \arrow[r, equal] & 1_X \circ g \end{tikzcd}$$

For those who prefer diagrams in 3 dimensions, we can rephrase (2) as the condition that in each diagram below, the "front" faces and "back" faces compose to the same 2-morphism. In the left-hand diagram, the dashed arrow is $f\colon X \to Y$, and in the right-hand diagram it is $g\colon Y \to X$, with the obvious 2-morphisms understood for the "back" faces that border the dashed arrow.





The 2-morphism η is called the **unit** of the adjunction, and ϵ the **counit**.

Exercise B.39. Suppose (f, g, η, ϵ) is an adjunction from X to Y in a 2-category $\mathcal{C}$, and V is any object in $\mathcal{C}$.

(1) If $a\colon V \to X$ and $b\colon V \to Y$ are morphisms in $\mathcal{C}$, there is a canonical bijection

$$\{\text{ 2-morphisms from } g \circ b \text{ to } a\,\} \longleftrightarrow \{\text{ 2-morphisms from } b \text{ to } f \circ a\}.$$

This takes a 2-morphism $\theta\colon g \circ b \Rightarrow a$ to the composite $b = 1_Y \circ b \overset{\eta}{\Rightarrow} f \circ g \circ b \overset{\theta}{\Rightarrow} f \circ a$; the inverse takes a 2-morphism $\pi\colon b \Rightarrow f \circ a$ to the composite $g \circ b \overset{\pi}{\Rightarrow} g \circ f \circ a \overset{\epsilon}{\Rightarrow} 1_X \circ a = a$. Verify that these are inverse bijections.

(2) There is an adjunction $(f^V, g^V, \eta^V, \epsilon^V)$ from $\mathrm{HOM}(V, X)$ to $\mathrm{HOM}(V, Y)$ in the 2-category (Cat). Here f^V, g^V , η^V, and ϵ^V are the functors and natural transformations defined in Exercise B.30, Similarly, there is an adjunction $(g_V, f_V, \eta_V, \epsilon_V)$ from $\mathrm{HOM}(Y, V)$ to $\mathrm{HOM}(X, V)$.

Proposition B.20. *If (f, g, η, ϵ) and $(f, g', \eta', \epsilon')$ are adjunctions from X to Y, there is a unique 2-isomorphism $\theta\colon g \overset{\sim}{\Rightarrow} g'$ such that $(1_f * \theta) \circ \eta = \eta'$ and $\epsilon' \circ (\theta * 1_f) = \epsilon$:*

$$\begin{tikzcd}
1_Y \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & f \circ g \arrow[d, Rightarrow, "\theta"] \\
& f \circ g'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
g \circ f \arrow[r, Rightarrow, "\epsilon"] \arrow[d, Rightarrow, "\theta"'] & 1_X \\
g' \circ f \arrow[ur, Rightarrow, "\epsilon'"'] &
\end{tikzcd}$$

Proof. Define θ to be the composition

$$g = g \circ 1_Y \overset{\eta'}{\Rightarrow} g \circ f \circ g' \overset{\epsilon}{\Rightarrow} 1_X \circ g' = g'.$$

To see that the first diagram commutes, consider the diagram

$$\begin{tikzcd}
1_Y \arrow[r, Rightarrow, "\eta'"] \arrow[d, Rightarrow, "\eta"'] & f \circ g' \arrow[r, equal] & 1_Y \circ f \circ g' \arrow[r, equal] \arrow[d, Rightarrow, "\eta"] & f \circ g' \arrow[dr, equal] & \\
f \circ g \arrow[r, equal] & f \circ g \circ 1_Y \arrow[r, Rightarrow, "\eta'"'] & f \circ g \circ f \circ g' \arrow[r, Rightarrow, "\epsilon"'] & f \circ 1_X \circ g' \arrow[r, equal] & f \circ g'
\end{tikzcd}$$

The left rectangle commutes by the exchange property, and the right trapezoid commutes by an adjunction property of η and ϵ. The bottom row is $1_f * \theta$. Similarly, the commutativity of the second diagram is seen from the diagram

$$\begin{tikzcd}
g \circ f \arrow[r, equal] \arrow[dr, equal] & g \circ 1_Y \circ f \arrow[r, Rightarrow, "\eta'"] & g \circ f \circ g' \circ f \arrow[r, Rightarrow, "\epsilon"] \arrow[d, Rightarrow, "\epsilon'"] & 1_X \circ g' \circ f \arrow[r, equal] & g' \circ f \arrow[d, Rightarrow, "\epsilon'"] \\
& g \circ f \arrow[r, equal] & g \circ f \circ 1_X \arrow[r, equal] & g \circ f \arrow[r, Rightarrow, "\epsilon"'] & 1_X
\end{tikzcd}$$

To see that θ is an isomorphism, define $\theta'\colon g' \Rightarrow g$ to be the composite

$$g' = g' \circ 1_Y \overset{\eta}{\Rightarrow} g' \circ f \circ g \overset{\epsilon'}{\Rightarrow} 1_X \circ g = g.$$

It suffices to show that $\theta' \circ \theta = 1_g$ and $\theta \circ \theta' = 1'_g$. By symmetry, θ' satisfies the identities $(1_f * \theta') \circ \eta' = \eta$ and $\epsilon \circ (\theta' * 1_f) = \epsilon'$. Hence the composite $\theta' \circ \theta$ satisfies the identities $(1_f * (\theta' \circ \theta)) \circ \eta = \eta$ and $\epsilon \circ ((\theta' \circ \theta) * 1_f) = \epsilon$, and similarly for $\theta \circ \theta'$. It therefore suffices to prove the following uniqueness assertion: if $\theta\colon g \Rightarrow g$ satisfies $(1_f * \theta) \circ \eta = \eta$ (and $\epsilon \circ (\theta * 1_f) = \epsilon$), then $\theta = 1_g$. For this, consider the diagram

$$\begin{tikzcd}
g \arrow[r, equal] \arrow[d, equal] & g \circ 1_Y \arrow[r, Rightarrow, "\eta"] & g \circ f \circ g \arrow[r, Rightarrow, "\epsilon"] \arrow[d, Rightarrow, "1_g * 1_f * \theta"] & 1_X \circ g \arrow[r, equal] \arrow[d, Rightarrow, "\theta"] & g \arrow[d, Rightarrow, "\theta"] \\
g \arrow[r, equal] & g \circ 1_Y \arrow[r, Rightarrow, "\eta"'] & g \circ f \circ g \arrow[r, Rightarrow, "\epsilon"'] & 1_X \circ g \arrow[r, equal] & g
\end{tikzcd}$$

The left rectangle commutes by assumption, the middle square commutes by the exchange property, and the right square commutes by property (c) of 2-categories. Reading around the diagram, one finds $1_g = \theta \circ 1_g$, so $\theta = 1_g$, as required. $\square$

Given a morphism $f\colon X \to Y$ in a 2-category, this proposition justifies the use of the notation $(f, f', \eta^f, \epsilon^f)$ for an adjunction from X to Y, and to call (f', η^f, ϵ^f) (or sometimes just f') a **left adjoint** of f. This notation is particularly useful when we want to compare adjoints for several morphisms. Whenever we have two composable morphisms, each with a left adjoint, the composite can be given the adjoint structure of the following exercise.

EXERCISE B.40. Suppose $(f, f', \eta^f, \epsilon^f)$ is an adjunction from X to Y, and $(g, g', \eta^g, \epsilon^g)$ is an adjunction from Y to Z. Define an adjunction $(g \circ f, f' \circ g', \eta^{gf}, \epsilon^{gf})$ from X to Z, where η^{gf} is the composite

$$1_Z \overset{\eta^g}{\Rightarrow} g \circ g' = g \circ 1_Y \circ g' \overset{\eta^f}{\Rightarrow} g \circ f \circ f' \circ g',$$

and ϵ^{gf} is the composite

$$f' \circ g' \circ g \circ f \overset{\epsilon^g}{\Rightarrow} f' \circ 1_Y \circ f = f' \circ f \overset{\epsilon^f}{\Rightarrow} 1_X.$$

Verify that $(g \circ f, f' \circ g', \eta^{gf}, \epsilon^{gf})$ is an adjunction from X to Z.

3.2. Adjoint functors. When applied to the 2-category (Cat) of categories, the notion of adjunction we have been discussing coincides with the usual notion of adjoint functors. In this context, an adjunction (F, G, η, ϵ) from a category $\mathcal{X}$ to a category $\mathcal{Y}$ consists of functors $F\colon \mathcal{X} \to \mathcal{Y}$, $G\colon \mathcal{Y} \to \mathcal{X}$, and natural transformations $\eta\colon 1_{\mathcal{Y}} \Rightarrow F \circ G$ and $\epsilon\colon G \circ F \Rightarrow 1_{\mathcal{X}}$, such that the composite $F = 1_{\mathcal{Y}} \circ F \overset{\eta}{\Rightarrow} F \circ G \circ F \overset{\epsilon}{\Rightarrow} F \circ 1_{\mathcal{X}} = F$ is the identity on F, and $G = G \circ 1_{\mathcal{Y}} \overset{\eta}{\Rightarrow} G \circ F \circ G \overset{\epsilon}{\Rightarrow} 1_{\mathcal{X}} \circ G = G$ is the identity on G. We say that G is a **left adjoint** of F, and F is a **right adjoint** of G, when this data is specified. If a given F has a left adjoint, it is unique up to a natural isomorphism, by Proposition B.20.

EXERCISE B.41. For every object X of $\mathcal{X}$, $F(\epsilon_X) \circ \eta_{F(X)} = 1_{F(X)}$. For every object Y of $\mathcal{Y}$, $\epsilon_{G(Y)} \circ G(\eta_Y) = 1_{G(Y)}$. For every morphism $a\colon X \to X'$ of $\mathcal{X}$, $a \circ \epsilon_X = \epsilon_{X'} \circ G(F(a))$. For every morphism $b\colon Y \to Y'$ of $\mathcal{Y}$, $F(G(b)) \circ \eta_Y = \eta_{Y'} \circ b$.

The usual definition an adjoint pair of functors prescribes, for every pair of objects $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, a bijection

$$\phi_{Y,X} :\ \mathrm{Hom}_{\mathcal{X}}(G(Y), X) \to \mathrm{Hom}_{\mathcal{Y}}(Y, F(X)),$$

between the morphisms from $G(Y)$ to X in $\mathcal{X}$ and the morphisms from Y to $F(X)$ in $\mathcal{Y}$, which is natural in X and Y; that is, for any morphisms $a\colon X \to X'$ in $\mathcal{X}$ and $b\colon Y' \to Y$ in $\mathcal{Y}$, the diagrams

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{X}}(G(Y), X) \arrow[r, "\phi_{Y,X}"] \arrow[d, "a_{G(Y)}"'] & \mathrm{Hom}_{\mathcal{Y}}(Y, F(X)) \arrow[d, "F(a)_Y"] \\
\mathrm{Hom}_{\mathcal{X}}(G(Y), X') \arrow[r, "\phi_{Y,X'}"'] & \mathrm{Hom}_{\mathcal{Y}}(Y, F(X'))
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{X}}(G(Y), X) \arrow[r, "\phi_{Y,X}"] \arrow[d, "G(b)^X"'] & \mathrm{Hom}_{\mathcal{Y}}(Y, F(X)) \arrow[d, "b^{F(X)}"] \\
\mathrm{Hom}_{\mathcal{X}}(G(Y'), X) \arrow[r, "\phi_{Y',X}"'] & \mathrm{Hom}_{\mathcal{Y}}(Y', F(X))
\end{tikzcd}$$

commute. Equivalently, for $c\colon G(Y) \to X$ in $\mathcal{X}$, and any $a\colon X \to X'$ and $b\colon Y' \to Y$,

$$\phi_{Y',X'}(a \circ c \circ G(b)) \ = \ F(a) \circ \phi_{Y,X}(c) \circ b.$$

These two notions of adjoints coincide, for fixed functors F and G. Given η and ϵ, define $\phi_{Y,X}\colon \operatorname{Hom}(G(Y),X) \to \operatorname{Hom}(Y,F(X))$ by the formula

$$\phi_{Y,X}(c) \;=\; F(c)\circ\eta_Y,$$

i.e., $\phi_{Y,X}(c)$ is the composite $Y \overset{\eta_Y}{\to} F(G(Y)) \overset{F(c)}{\to} F(X)$. The inverse map from $\operatorname{Hom}(Y,F(X))$ to $\operatorname{Hom}(G(Y),X)$ is defined by

$$\phi_{Y,X}{}^{-1}(d) \;=\; \epsilon_X\circ G(d),$$

i.e., $\phi_{Y,X}{}^{-1}(d)$ is the composite $G(Y) \overset{G(d)}{\to} G(F(X)) \overset{\epsilon_X}{\to} X$. Conversely, given natural bijections $\phi_{Y,X}$ for all X and Y, define η and ϵ by the formulas

$$\eta_Y \;=\; \phi_{Y,G(Y)}(1_{G(Y)}), \quad \epsilon_X \;=\; \phi_{F(X),X}{}^{-1}(1_{F(X)}).$$

Exercise B.42. Verify that the maps $\phi_{Y,X}$ and $\phi_{Y,X}{}^{-1}$ defined from η and ϵ are inverse bijections, natural in X and Y. Verify that the maps η_Y and ϵ_X defined from a collection $\{\phi_{Y,X}\}$ define natural transformations $\eta\colon 1_{\mathcal{Y}} \Rightarrow F\circ G$ and $\epsilon\colon G\circ F \Rightarrow 1_{\mathcal{X}}$, such that (F,G,η,ϵ) defines an adjoint from $\mathcal{X}$ to $\mathcal{Y}$. Verify that these correspondences $\{\phi_{Y,X}\} \leftrightarrow (\eta,\epsilon)$ are inverse bijections.

3.3. Base change.

Definition B.21. Suppose we have a 2-commutative diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\alpha\,\Nearrow"] & Y \arrow[d, "p"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in a 2-category, and that each of the morphisms p and q is part of an adjunction (p,p',η^p,ϵ^p) and (q,q',η^q,ϵ^q). Define a **base change** 2-morphism[6]

$$c_\alpha :\; p'\circ f \;\Rightarrow\; g\circ q'$$

to be the composite

$$p'\circ f = p'\circ f\circ 1_X \overset{\eta^q}{\Rightarrow} p'\circ f\circ q\circ q' \overset{\alpha}{\Rightarrow} p'\circ p\circ g\circ q' \overset{\epsilon^p}{\Rightarrow} 1_Y\circ g\circ q' = g\circ q'.$$

If f and g are also part of adjunctions (f,f',η^f,ϵ^f) and (g,g',η^g,ϵ^g), define a 2-morphism

$$\alpha' :\; g'\circ p' \;\Rightarrow\; q'\circ f'$$

to be the composite

$$g'\circ p' = g'\circ p'\circ 1_Z \overset{\eta^f}{\Rightarrow} g'\circ p'\circ f\circ f' \overset{c_\alpha}{\Rightarrow} g'\circ g\circ q'\circ f' \overset{\epsilon^g}{\Rightarrow} 1_W\circ q'\circ f = q'\circ f.$$

[6]In category theory, α and c_α are called *mates* of each other, see [**51**]. Category theorists often write adjunctions in the order (f',f,η,ϵ), with the left adjoint preceding the right adjoint.

Exercise B.43. (1) If p and q have left adjoints, show that the following diagram commutes:

$$\begin{tikzcd}
f \arrow[r, equal] \arrow[d, equal] & 1_Z \circ f \arrow[r, Rightarrow, "\eta^p"] & p \circ p' \circ f \arrow[d, Rightarrow, "c_\alpha"] \\
f \circ 1_X \arrow[r, Rightarrow, "\eta^q"'] & f \circ q \circ q' \arrow[r, Rightarrow, "\alpha"'] & p \circ g \circ q'
\end{tikzcd}$$

If f and g also have left adjoints, show that the following diagrams commute:

$$\begin{tikzcd}
p' \arrow[r, equal] \arrow[d, equal] & p' \circ 1_Z \arrow[r, Rightarrow, "\eta^f"] & p' \circ f \circ f' \arrow[d, Rightarrow, "c_\alpha"] \\
1_Y \circ p' \arrow[r, Rightarrow, "\eta^g"'] & g \circ g' \circ p' \arrow[r, Rightarrow, "\alpha'"'] & g \circ q' \circ f'
\end{tikzcd}
\qquad
\begin{tikzcd}
g' \circ p' \circ f \arrow[r, Rightarrow, "\alpha'"] \arrow[d, Rightarrow, "c_\alpha"'] & q' \circ f' \circ f \arrow[r, Rightarrow, "\epsilon^f"] & q' \circ 1_X \arrow[d, equal] \\
g' \circ g \circ q' \arrow[r, Rightarrow, "\epsilon^g"'] & 1_W \circ q' \arrow[r, equal] & q'
\end{tikzcd}$$

(2) Deduce that c_α is equal to the composite

$$p' \circ f = 1_Y \circ p' \circ f \overset{\eta^g}{\Rightarrow} g \circ g' \circ p' \circ f \overset{\alpha'}{\Rightarrow} g \circ q' \circ f' \circ f \overset{\epsilon^f}{\Rightarrow} g \circ q' \circ 1_X = g \circ q'.$$

(3) The correspondence of Exercise B.39, applied to the adjunction $(p, p', \eta^p, \epsilon^p)$ takes $c_\alpha \colon p' \circ f \Rightarrow g \circ q'$ to a 2-morphism from f to $p \circ g \circ q'$. Show that this morphism is the composite $f = f \circ 1_X \overset{\eta^q}{\Rightarrow} f \circ q \circ q' \overset{\alpha}{\Rightarrow} p \circ g \circ q'$. The inverse of the correspondence of Exercise B.39, applied to the adjunction $(g, g', \eta^g, \epsilon^g)$, produces a 2-morphism from $g' \circ p' \circ f$ to q'. Show that this is $g' \circ p' \circ f \overset{\alpha'}{\Rightarrow} q' \circ f' \circ f \overset{\epsilon^f}{\Rightarrow} q'$.

Exercise B.44. (1) Consider a diagram

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[d, "r"'] \arrow[dr, phantom, "\beta\Nearrow"] & W \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\alpha\Nearrow"] & Y \arrow[d, "p"] \\
V \arrow[r, "h"'] & X \arrow[r, "f"'] & Z
\end{tikzcd}$$

in a 2-category $\mathcal{C}$. Define $\gamma \colon (f \circ h) \circ r \Rightarrow p \circ (g \circ i)$ to be the composite

$$(f \circ h) \circ r = f \circ h \circ r \overset{\beta}{\Rightarrow} f \circ q \circ i \overset{\alpha}{\Rightarrow} p \circ g \circ i = p \circ (g \circ i).$$

If p, q, and r have left adjoints, show that the diagram

$$\begin{tikzcd}
p' \circ f \circ h \arrow[r, equal] \arrow[d, Rightarrow, "c_\alpha"'] & p' \circ (f \circ h) \arrow[dr, Rightarrow, "c_\gamma"] & \\
g \circ q' \circ h \arrow[r, Rightarrow, "c_\beta"'] & g \circ i \circ r' \arrow[r, equal] & (g \circ i) \circ r'
\end{tikzcd}$$

commutes.

(2) Dually, given a diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, "r"'] \arrow[dr, phantom, "\not\Leftarrow_\beta"] & W \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\not\Leftarrow_\alpha"] & Y \arrow[d, "p"] \\ V \arrow[r, "h"'] & X \arrow[r, "f"'] & Z \end{tikzcd}$$

define $\gamma\colon p\circ(g\circ i)\Rightarrow(f\circ h)\circ r$ to be the composite

$$p\circ(g\circ i)=p\circ g\circ i\overset{\alpha}{\Rightarrow} f\circ q\circ i\overset{\beta}{\Rightarrow} f\circ h\circ r=(f\circ h)\circ r.$$

If f, g, h, and i have adjoints, show that the following diagram commutes:

$$\begin{tikzcd} h'\circ f'\circ p \arrow[r, equal] \arrow[d, Rightarrow, "c_\alpha"'] & (f\circ h)'\circ p \arrow[drr, Rightarrow, "c_\gamma"] & & \\ h'\circ q\circ g' \arrow[r, Rightarrow, "c_\beta"'] & r\circ i'\circ g' \arrow[rr, equal] & & r\circ(g\circ i)' \end{tikzcd}$$

Example B.22. Let

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] & Y \arrow[d, "p"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

be a commutative diagram of schemes. Then there is a natural base change morphism

$$p^*f_*\mathcal{F}\to g_*q^*\mathcal{F}.$$

A detailed treatment of the construction and properties of base change morphisms for sheaves on schemes is given in the Glossary.

4. Pseudofunctors

In this section we consider 2-categories in their natural generality, where one rarely has equality of morphisms; in their place are identities among 2-isomorphisms. Although the definitions and assertions are natural enough, the verifications involve considerable diagram chasing, much of which is left to the interested (and determined) reader.

Definition B.23. If $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, a (covariant) **pseudofunctor** F from $\mathcal{C}$ to $\mathcal{D}$ assigns to each object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, to each morphism $f\colon X\to Y$ of $\mathcal{C}$ a morphism $F(f)\colon F(X)\to F(Y)$ in $\mathcal{D}$, and to each 2-morphism $\alpha\colon f\Rightarrow g$ a 2-morphism $F(\alpha)\colon F(f)\Rightarrow F(g)$. In addition, we must have:

(1) for morphisms $f\colon X\to Y$, $g\colon Y\to Z$ of $\mathcal{C}$, a 2-isomorphism

$$\gamma_{f,g}=\gamma^F_{f,g}:\ F(g\circ f)\overset{\sim}{\Rightarrow} F(g)\circ F(f);$$

(2) for each object X of $\mathcal{C}$, a 2-isomorphism

$$\delta_X=\delta^F_X:\ F(1_X)\overset{\sim}{\Rightarrow} 1_{F(X)}.$$

These must satisfy the following conditions:

(a) For morphisms $f\colon W \to X$, $g\colon X \to Y$, $h\colon Y \to Z$ in $\mathcal{C}$, we have the equality

$$(1_{F(h)} * \gamma_{f,g}) \circ \gamma_{gf,h} \;=\; (\gamma_{g,h} * 1_{F(f)}) \circ \gamma_{f,hg}$$

of 2-morphisms from $F(h \circ g \circ f)$ to $F(h) \circ F(g) \circ F(f)$:

$$\begin{tikzcd}
F(h \circ g \circ f) \arrow[r, Rightarrow, "\gamma_{f,hg}"] \arrow[d, Rightarrow, "\gamma_{gf,h}"'] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\gamma_{g,h}"] \\
F(h) \circ F(g \circ f) \arrow[r, Rightarrow, "\gamma_{f,g}"'] & F(h) \circ F(g) \circ F(f)
\end{tikzcd}$$

(This can also be described by saying that the two ways to move down in the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"'] \arrow[rr, bend left, "F(gf)"] \arrow[rrr, bend left=40, "F(hgf)"] & F(Y) \arrow[r, "F(g)"'] \arrow[rr, bend left, "F(hg)"] & F(Z) \arrow[r, "F(h)"'] & F(W)
\end{tikzcd}$$

agree.)

(b) For a morphism $f\colon X \to Y$ in $\mathcal{C}$, we have the equalities

$$(1_{F(f)} * \delta_X) \circ \gamma_{1_X,f} \;=\; 1_{F(f)} \;=\; (\delta_Y * 1_{F(f)}) \circ \gamma_{f,1_Y}$$

of 2-morphisms from $F(f)$ to $F(f) \circ 1_{F(X)} = F(f) = 1_{F(Y)} \circ F(f)$:

$$\begin{tikzcd}
F(f) \arrow[r, equal] \arrow[d, equal] & F(f) \circ 1_{F(X)} \\
F(f \circ 1_X) \arrow[r, Rightarrow, "\gamma_{1_X,f}"'] & F(f) \circ F(1_X) \arrow[u, Rightarrow, "\delta_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(f) \arrow[r, equal] \arrow[d, equal] & 1_{F(Y)} \circ F(f) \\
F(1_Y \circ f) \arrow[r, Rightarrow, "\gamma_{f,1_Y}"'] & F(1_Y) \circ F(f) \arrow[u, Rightarrow, "\delta_Y"']
\end{tikzcd}$$

(c) For any morphism f in $\mathcal{C}$, $F(1_f) = 1_{F(f)}$; and, if $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$ in $\mathcal{C}$, we have the equality

$$F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$$

of 2-morphisms from $F(f)$ to $F(h)$.

(d) If $f, f'\colon X \to Y$, $\alpha\colon f \Rightarrow f'$, $g, g'\colon Y \to Z$, and $\beta\colon g \Rightarrow g'$ in $\mathcal{C}$, then

$$(F(\beta) * F(\alpha)) \circ \gamma_{f,g} \;=\; \gamma_{f',g'} \circ F(\beta * \alpha),$$

an equality of 2-morphisms from $F(g \circ f)$ to $F(g') \circ F(f')$:

$$\begin{tikzcd}
F(g \circ f) \arrow[r, Rightarrow, "F(\beta*\alpha)"] \arrow[d, Rightarrow, "\gamma_{f,g}"'] & F(g' \circ f') \arrow[d, Rightarrow, "\gamma_{f',g'}"] \\
F(g) \circ F(f) \arrow[r, Rightarrow, "F(\beta)*F(\alpha)"'] & F(g') \circ F(f')
\end{tikzcd}$$

We write $F\colon \mathcal{C} \to \mathcal{D}$ to denote that F is a pseudofunctor from $\mathcal{C}$ to $\mathcal{D}$, with associated 2-isomorphisms δ^F_X and $\gamma^F_{f,g}$.

Note by (c) that a pseudofunctor determines (honest) functors

$$\mathrm{HOM}(X, Y) \longrightarrow \mathrm{HOM}(F(X), F(Y))$$

for any objects X and Y in $\mathcal{C}$. Note however that F does *not* induce a functor between the underlying categories of $\mathcal{C}$ and $\mathcal{D}$.

An important special case of this is when $\mathcal{C}$ is an ordinary category, regarded as a 2-category by specifying that its only 2-morphisms are identities. In this case there is no need to specify what F does to 2-morphisms in $\mathcal{C}$, and conditions (c) and (d) can be omitted.

In the text, the situation of a **contravariant pseudofunctor** from an ordinary category $\mathcal{C}$ to a 2-category $\mathcal{D}$ arises. This can be defined to be a pseudofunctor from the opposite category $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. Explicitly, the changes are: for a morphism $f\colon X \to Y$ one has $F(f)\colon F(Y) \to F(X)$, and for $f\colon X \to Y$, $g\colon Y \to Z$ one has $\gamma_{f,g}\colon F(g \circ f) \overset{\sim}{\Rightarrow} F(f) \circ F(g)$, and the two conditions become:

(a) $(\gamma_{f,g} * 1_{F(h)}) \circ \gamma_{gf,h} \;=\; (1_{F(f)} * \gamma_{g,h}) \circ \gamma_{f,hg}$;
(b) $(\delta_X * 1_{F(f)}) \circ \gamma_{f,1_X} \;=\; 1_{F(f)} \;=\; (1_{F(f)} * \delta_Y) \circ \gamma_{1_Y,f}$.

Exercise B.45. If $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are pseudofunctors, there is a composite pseudofunctor $G \circ F$. This takes an object X to $G(F(X))$, a morphism f to $G(F(f))$, and a 2-morphism ρ to $G(F(\rho))$; and one sets $\gamma^{G\circ F}_{f,g} = \gamma^{G}_{F(f),F(g)} \circ G(\gamma^{F}_{f,g})$ and $\delta^{GF}_X = \delta^{G}_{F(X)} \circ G(\delta^{F}_X)$. Verify that this defines a pseudofunctor. Show that 2-categories, with pseudofunctors as morphisms, form a category.

Exercise B.46. Construct a pseudofunctor B from the 2-category (Grp) of groups to the 2-category (Cat) that takes a group G to the category BG of G-torsors (where a group G is regarded as a topological group with the discrete topology).

Definition B.24. If F and G are pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$, a **pseudonatural transformation** α from F to G consists of

(1) For each object X in $\mathcal{C}$, a morphism $\alpha_X\colon F(X) \to G(X)$ in $\mathcal{D}$.
(2) For each morphism $f\colon X \to Y$ in $\mathcal{C}$, a 2-isomorphism

$$\tau_f = \tau^{\alpha}_f :\; G(f) \circ \alpha_X \overset{\sim}{\Rightarrow} \alpha_Y \circ F(f)$$

in $\mathcal{D}$. This is displayed in the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\tau_f \Nearrow"] & F(Y) \arrow[d, "\alpha_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

These must satisfy:

(a) For morphisms $f\colon X \to Y$, $g\colon Y \to Z$ of $\mathcal{C}$, we have an equality

$$(1_{\alpha_Z} * \gamma^{F}_{f,g}) \circ \tau_{gf} \;=\; (\tau_g * 1_{F(f)}) \circ (1_{G(g)} * \tau_f) \circ (\gamma^{G}_{f,g} * 1_{\alpha_X})$$

of 2-morphisms from $G(gf) \circ \alpha_X$ to $\alpha_Z \circ F(g) \circ F(f)$:

$$\begin{tikzcd}
G(g \circ f) \circ \alpha_X \arrow[r, Rightarrow, "\gamma^G_{f,g}"] \arrow[d, Rightarrow, "\tau_{gf}"'] & G(g) \circ G(f) \circ \alpha_X \arrow[r, Rightarrow, "\tau_f"] & G(g) \circ \alpha_Y \circ F(f) \arrow[d, Rightarrow, "\tau_g"] \\
\alpha_Z \circ F(g \circ f) \arrow[rr, Rightarrow, "\gamma^F_{f,g}"'] & & \alpha_Z \circ F(g) \circ F(f)
\end{tikzcd}$$

(b) For any object X of $\mathcal{C}$, we have the equality

$$(1_{\alpha_X} * \delta^F_X) \circ \tau_{1_X} \;=\; \delta^G_X * 1_{\alpha_X}$$

of 2-morphisms from $G(1_X) \circ \alpha_X$ to $\alpha_X \circ 1_{F(X)} = \alpha_X = 1_{G(X)} \circ \alpha_X$:

$$\begin{tikzcd}
G(1_X) \circ \alpha_X \arrow[r, Rightarrow, "\delta^G_X"] \arrow[d, Rightarrow, "\tau_{1_X}"'] & 1_{G(X)} \circ \alpha_X \arrow[d, equal] \\
\alpha_X \circ F(1_X) \arrow[r, Rightarrow, "\delta^F_X"'] & \alpha_X \circ 1_{F(X)}
\end{tikzcd}$$

(c) For $f, g \colon X \to Y$, and a 2-morphism $\rho \colon f \Rightarrow g$ in $\mathcal{C}$, we have the equality

$$\tau_g \circ (G(\rho) * 1_{\alpha_X}) \;=\; (1_{\alpha_Y} * F(\rho)) \circ \tau_f$$

of 2-morphisms from $G(f) \circ \alpha_X$ to $\alpha_Y \circ F(g)$:

$$\begin{tikzcd}
G(f) \circ \alpha_X \arrow[r, Rightarrow, "\tau_f"] \arrow[d, Rightarrow, "G(\rho)"'] & \alpha_Y \circ F(f) \arrow[d, Rightarrow, "F(\rho)"] \\
G(g) \circ \alpha_X \arrow[r, Rightarrow, "\tau_g"'] & \alpha_Y \circ F(g)
\end{tikzcd}$$

Exercise B.47. If F,G,H are pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$, one can compose pseudonatural transformations α from F to G and β from G to H to get a pseudonatural transformation $\beta \circ \alpha$ from F to H. This is defined by setting $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ and $\tau^{\beta\circ\alpha}_f = (1_{\beta_Y} * \tau^\alpha_f) \circ (\tau^\beta_f * 1_{\alpha_X})$. Show that this composition is associative, and has identities, so that the pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$ and the pseudonatural transformations between them form the objects and morphisms of a category.

Definition B.25. Suppose $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, F and G are pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$, and α and β are pseudonatural transformations from F to G. A **modification** Θ from α to β assigns to each object X of $\mathcal{C}$ a 2-morphism $\Theta_X \colon \alpha_X \Rightarrow \beta_X$:

$$\begin{tikzcd}
F(X) \arrow[r, bend left, "\alpha_X"] \arrow[r, bend right, "\beta_X"'] \arrow[r, phantom, "\Downarrow\Theta_X"] & G(X).
\end{tikzcd}$$

This must satisfy the property that for any morphisms $f, g \colon X \to Y$ and 2-morphism $\rho \colon f \Rightarrow g$ in $\mathcal{C}$, we have the equality

$$(\Theta_Y * F(\rho)) \circ \tau^\alpha_f \;=\; \tau^\beta_g \circ (G(\rho) * \Theta_X)$$

of 2-morphisms from $G(f) \circ \alpha_X$ to $\beta_Y \circ F(g)$:

$$\begin{tikzcd} G(f) \circ \alpha_X \arrow[r, Rightarrow, "G(\rho)*\Theta_X"] \arrow[d, Rightarrow, "\tau_f^\alpha"'] & G(g) \circ \beta_X \arrow[d, Rightarrow, "\tau_g^\beta"] \\ \alpha_Y \circ F(f) \arrow[r, Rightarrow, "\Theta_Y * F(\rho)"'] & \beta_Y \circ F(g) \end{tikzcd}$$

We write $\Theta\colon \alpha \Rrightarrow \beta$ to indicate that Θ is a modification from α to β. We call a modification an **isomodification** if each Θ_X is a 2-isomorphism. In this case we write $\Theta\colon \alpha \overset{\sim}{\Rrightarrow} \beta$.

Note that each of the conditions on pseudofunctors, pseudonatural transformations, and modifications is stated as an equality of 2-morphisms.

Exercise B.48. Show that the property of a modification in Definition B.25 follows from the property

$$(\Theta_Y * 1_{F(f)}) \circ \tau_f^\alpha \;=\; \tau_f^\beta \circ (1_{G(f)} * \Theta_X)$$

for any morphism $f\colon X \to Y$ in $\mathcal{C}$.

Exercise B.49. If $\Theta\colon \alpha \Rrightarrow \beta$ and $\Xi\colon \beta \Rrightarrow \gamma$, with α, β, and γ pseudonatural transformations from F to G, there is a modification $\Xi \circ \Theta\colon \alpha \Rrightarrow \gamma$, defined by $(\Xi \circ \Theta)_X = \Xi_X \circ \Theta_X$. If $\Theta\colon \alpha \Rrightarrow \alpha'$, with $\alpha, \alpha'\colon F \Rightarrow G$, and $\Xi\colon \beta \Rrightarrow \beta'$, with $\beta, \beta'\colon G \Rightarrow H$, there is a modification $\Xi * \Theta\colon \beta \circ \alpha \Rrightarrow \beta' \circ \alpha'$, defined by $(\Xi * \Theta)_X = \Xi_X * \Theta_X$. For fixed 2-categories $\mathcal{C}$ and $\mathcal{D}$, these operations, together with those of Exercise B.47, make the pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$ into the objects of a 2-category PSFUN$(\mathcal{C}, \mathcal{D})$, with arrows given by pseudonatural transformations, and 2-cells given by modifications.

When α and β are 2-natural transformations between 2-functors F and G, the condition on a modification simplifies to the equation $\Theta_Y * F(\rho) = G(\rho) * \Theta_X$.

Exercise B.50. $^{(*)}$ Given modifications $\Theta\colon \alpha \Rrightarrow \alpha'$ and $\Xi\colon \beta \Rrightarrow \beta'$ between 2-natural transformations $\alpha, \alpha'\colon F \Rightarrow F'$, $\beta, \beta'\colon G \Rightarrow G'$, with 2-functors $F, F'\colon \mathcal{C} \to \mathcal{D}$, $G, G'\colon \mathcal{D} \to \mathcal{E}$, define a modification $\Xi \diamond \Theta\colon \beta * \alpha \Rrightarrow \beta' * \alpha'$ by the formula

$$(\Xi \diamond \Theta)_X = G'(\Theta_X) * \Xi_{F(X)} = \Xi_{F'(X)} * G(\Theta_X).$$

Show that 2-categories, 2-functors, 2-natural transformations, and modifications form the objects, arrows, 2-cells, and 3-cells of a **3-category**: in addition to the underlying 2-category structure formed by the objects, arrows, and 2-cells, the three operations $\circ$, $*$, and $\diamond$ on 3-cells satisfy the following associativity, exchange, and unity identities:[7]

(a) $\Gamma \circ (\Xi \circ \Theta) = (\Gamma \circ \Xi) \circ \Theta$, $\Gamma * (\Xi * \Theta) = (\Gamma * \Xi) * \Theta$, and $\Gamma \diamond (\Xi \diamond \Theta) = (\Gamma \diamond \Xi) \diamond \Theta$;
(b) $(\Xi' \circ \Xi) * (\Theta' \circ \Theta) = (\Xi' * \Theta') \circ (\Xi * \Theta)$, $(\Xi' * \Xi) \diamond (\Theta' * \Theta) = (\Xi' \diamond \Theta') * (\Xi \diamond \Theta)$, $(\Xi' \circ \Xi) \diamond (\Theta' \circ \Theta) = (\Xi' \diamond \Theta') \circ (\Xi \diamond \Theta)$;

[7]In each identity it is assumed that one, and hence the other, side of the equation is defined.

(c) each 2-cell α has an identity 3-cell $1_\alpha\colon \alpha \Rrightarrow \alpha$, and the following identities are satisfied: $1_\beta * 1_\alpha = 1_{\beta\circ\alpha}$ when $\beta\circ\alpha$ is defined; $1_\beta \diamond 1_\alpha = 1_{\beta * \alpha}$ when $\beta * \alpha$ is defined; in addition, if $\Theta\colon \alpha \Rrightarrow \beta$, $\alpha, \beta\colon f \Rightarrow g$, $f, g\colon X \to Y$, then

$$\Theta \circ 1_\alpha = \Theta = 1_\beta \circ \Theta, \Theta * 1_{1_f} = \Theta = 1_{1_g} * \Theta, \Theta \diamond 1_{1_{1_X}} = \Theta = 1_{1_{1_Y}} \diamond \Theta.$$

A formal definition of 3-categories (equivalent to that in the preceding exercise) can be found in [**12**], §7.3, but note that 2-categories, pseudofunctors, and pseudonatural transformations, and modifications do *not* form a 3-category, only something weaker called a (Gray) tricategory [**32**]. In fact, in contrast with Exercise B.36, 2-categories, pseudofunctors, and pseudonatural transformations do not form a 2-category. In any case, we have will have no need for the formalism of 3-categories.

Note that a 2-functor F is just a pseudofunctor for which the associated 2-isomorphisms $\gamma^F_{f,g}$ and δ^F_X are identities. A 2-natural transformation $\alpha\colon F \Rightarrow G$ between 2-functors is a pseudonatural transformation such that τ^α_f is an identity for all morphisms f.

Exercise B.51. For fixed 2-categories $\mathcal{C}$ and $\mathcal{D}$, the 2-functors, 2-natural transformations, and modifications determine a sub-2-category 2-FUN$(\mathcal{C}, \mathcal{D})$ of the 2-category PSFUN$(\mathcal{C}, \mathcal{D})$.

Exercise B.52. For any 2-category $\mathcal{C}$, construct a 2-functor

$$\mathcal{C} \to \text{2-FUN}(\mathcal{C}^{\text{op}}, (\text{Cat})).$$

The following theorem gives the notion of “isomorphism” between 2-categories that one is likely to meet in practice.

Theorem B.26. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pseudofunctor between 2-categories. The following are equivalent:*

(1) (i) *Every object of $\mathcal{D}$ is 2-isomorphic to an object of the form $F(P)$ for some object P in $\mathcal{C}$;* (ii) *For any objects P and Q in $\mathcal{C}$, every morphism from $F(P)$ to $F(Q)$ is 2-isomorphic to a morphism of the form $F(a)$, for some morphism $a\colon P \to Q$ in $\mathcal{C}$;* (iii) *For morphisms a and b from P to Q in $\mathcal{C}$, any 2-morphism from $F(a)$ to $F(b)$ has the form $F(\rho)$ for a unique 2-morphism $\rho\colon a \Rightarrow b$ in $\mathcal{C}$.*

(2) *There is a pseudofunctor $G\colon \mathcal{D} \to \mathcal{C}$, together with four pseudonatural transformations:*

$$\alpha\colon G\circ F \Rightarrow 1_{\mathcal{C}}, \quad \alpha'\colon 1_{\mathcal{C}} \Rightarrow G\circ F, \quad \beta\colon F\circ G \Rightarrow 1_{\mathcal{D}}, \quad \beta'\colon 1_{\mathcal{D}} \Rightarrow F\circ G,$$

and four isomodifications:

$$\Theta\colon 1_{FG} \overset{\sim}{\Rrightarrow} \beta'\circ\beta, \quad \Theta'\colon 1_{1_{\mathcal{D}}} \overset{\sim}{\Rrightarrow} \beta\circ\beta', \quad \Xi\colon 1_{GF} \overset{\sim}{\Rrightarrow} \alpha'\circ\alpha, \quad \Xi'\colon 1_{1_{\mathcal{C}}} \overset{\sim}{\Rrightarrow} \alpha\circ\alpha'.$$

(3) *As in (2), but with the additional identities:*

$$1_\beta * \Theta = \Theta' * 1_\beta, \quad 1_{\beta'} * \Theta' = \Theta * 1_{\beta'}, \quad 1_\alpha * \Xi = \Xi' * 1_\alpha, \quad 1_{\alpha'} * \Xi' = \Xi * 1_{\alpha'}.$$

(These are modifications from β to $\beta\circ\beta'\circ\beta$, from β' to $\beta'\circ\beta\circ\beta'$, from α to $\alpha\circ\alpha'\circ\alpha$, and from α' to $\alpha'\circ\alpha\circ\alpha'$, respectively.)

Note that conditions (ii) and (iii) of (1) say that every induced functor $\mathrm{HOM}(P,Q) \to \mathrm{HOM}(F(P),F(Q))$ is a an equivalence of categories.

PROOF. This is at least a folk theorem in 2-category theory, cf. [**62**, §2.2], but since it is not easy to find a reference, we will sketch a proof. That (2) implies (1) is quite straightforward. First we show that (2) implies (i) of (1). Since Θ_Y is a 2-isomorphism $1_{FG(Y)} \Rightarrow \beta'_Y \circ \beta_Y$ and Θ'_Y is a 2-isomorphism $1_Y \Rightarrow \beta_Y \circ \beta'_Y$, it follows that β_Y and β'_Y are 2-equivalences, for any object Y of $\mathcal{D}$. In particular, Y is 2-isomorphic to $F(G(Y))$. Note, similarly, that α_X and α'_X are 2-equivalences, for any object X of $\mathcal{C}$. Now we show the remainder of (2) $\Rightarrow$ (1), namely, that every functor $\mathrm{HOM}(P,Q) \to \mathrm{HOM}(F(P),F(Q))$ determined by F is an equivalence of categories. Following this by the functor $\mathrm{HOM}(F(P),F(Q)) \to \mathrm{HOM}(GF(P),GF(Q))$ induced by G, and then by the functors

$$\mathrm{HOM}(GF(P),GF(Q)) \xrightarrow{(\alpha'_P)_{GF(Q)}} \mathrm{HOM}(P,GF(Q)) \xrightarrow{(\alpha_Q)^P} \mathrm{HOM}(P,Q),$$

each of which is an equivalence of categories by Proposition B.10 and Exercise B.31, one obtains a functor from $\mathrm{HOM}(P,Q)$ to itself. A natural isomorphism of this functor with the identity functor is given by sending $f\colon P \to Q$ to

$$\alpha_Q \circ GF(f) \circ \alpha'_P \overset{\tau_f^{\alpha'}}{\Rightarrow} \alpha_Q \circ \alpha'_Q \circ f \overset{{\Xi'_Q}^{-1}}{\Rightarrow} 1_Q \circ f = f;$$

the naturality is proved by a use of property (c) for the pseudonatural transformation α'. It follows that $\mathrm{HOM}(P,Q) \to \mathrm{HOM}(F(P),F(Q))$ is full and faithful, and, by the same applied to G, that $\mathrm{HOM}(F(P),F(Q)) \to \mathrm{HOM}(GF(P),GF(Q))$ is also full and faithful; it then follows from Exercise B.9 that each of them must also be an equivalence of categories.

We will show how to use (1) to construct all the data needed for (3). Then 25 identities among 2-isomorphisms must be verified to prove (3): 7 to prove that G is a pseudofunctor, 3 each to prove that α, α', β, and β' are pseudonatural transformations, 4 to verify that Θ, Θ', Ξ, and Ξ' are modifications, and 4 for the last conditions stated in (3). A few of these identities will be immediate from the construction, but most require – at least without a more sophisticated categorical language – tracing around rather large diagrams in various HOM categories. The key that makes it all work is a careful use of Proposition B.10.

For each object X of $\mathcal{D}$, use (i) and Proposition B.10 to choose an object $G(X)$ in $\mathcal{C}$ together with morphisms $\beta_X\colon F(G(X)) \to X$ and $\beta'_X\colon X \to F(G(X))$, and with 2-isomorphisms $\Theta_X\colon 1_{FG(X)} \overset{\sim}{\Rightarrow} \beta'_X \circ \beta_X$ and $\Theta'_X\colon 1_X \overset{\sim}{\Rightarrow} \beta_X \circ \beta'_X$, such that the two conditions

$$1_{\beta_X} * \Theta_X = \Theta'_X * 1_{\beta_X} \quad \text{and} \quad 1_{\beta'_X} * \Theta'_X = \Theta_X * 1_{\beta'_X}$$

are satisfied. For each morphism $f\colon X \to Y$ in $\mathcal{D}$, use (ii) to choose a morphism $G(f)\colon G(X) \to G(Y)$ in $\mathcal{C}$, together with a 2-isomorphism $\lambda_f\colon F(G(f)) \overset{\sim}{\Rightarrow} \beta'_Y \circ f \circ \beta_X$. Now for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{D}$, use (iii) to define $\gamma^G_{f,g}\colon G(g \circ f) \overset{\sim}{\Rightarrow} G(g) \circ G(f)$, by requiring $F(\gamma^G_{f,g})$ to make the following diagram commute:

$$\begin{tikzcd}
FG(g\circ f) \arrow[r, Rightarrow, "F(\gamma^G_{f,g})"] \arrow[d, Rightarrow, "\lambda_{gf}"'] & F(G(g)\circ G(f)) \arrow[r, Rightarrow, "\gamma^F_{G(f),G(g)}"] & FG(g)\circ FG(f) \arrow[d, Rightarrow, "\lambda_g * \lambda_f"] \\
\beta'_Z\circ g\circ f\circ \beta_X \arrow[r, equal] & \beta'_Z\circ g\circ 1_Y\circ f\circ\beta_X \arrow[r, Rightarrow, "\Theta'_Y"'] & \beta'_Z\circ g\circ \beta_Y\circ\beta'_Y\circ f\circ\beta_X
\end{tikzcd}$$

For each object X in $\mathcal{D}$, define $\delta^G_X\colon G(1_X) \overset{\sim}{\Rightarrow} 1_{G(X)}$ by requiring $F(\delta^G_X)$ to make the following diagram commute:

$$\begin{tikzcd}
FG(1_X) \arrow[r, Rightarrow, "F(\delta^G_X)"] \arrow[d, Rightarrow, "\lambda_{1_X}"'] & F(1_{G(X)}) \arrow[d, Rightarrow, "\delta^F_{G(X)}"] \\
\beta'_X\circ\beta_X & 1_{FG(X)} \arrow[l, Rightarrow, "\Theta_X"]
\end{tikzcd}$$

For $f\colon X\to Y$ in $\mathcal{D}$, define $\tau^\beta_f\colon f\circ\beta_X \overset{\sim}{\Rightarrow} \beta_Y\circ FG(f)$ and $\tau^{\beta'}_f\colon FG(f)\circ\beta'_X \overset{\sim}{\Rightarrow} \beta'_Y\circ f$ to make the following diagrams commute:

$$\begin{tikzcd}
f\circ\beta_X \arrow[r, Rightarrow, "\tau^\beta_f"] \arrow[d, equal] & \beta_Y\circ FG(f) \arrow[d, Rightarrow, "\lambda_f"] \\
1_Y\circ f\circ\beta_X \arrow[r, Rightarrow, "\Theta'_Y"'] & \beta_Y\circ\beta'_Y\circ f\circ\beta_X
\end{tikzcd}
\qquad
\begin{tikzcd}
FG(f)\circ\beta'_X \arrow[r, Rightarrow, "\tau^{\beta'}_f"] \arrow[d, Rightarrow, "\lambda_f"'] & \beta'_Y\circ f \arrow[d, equal] \\
\beta'_Y\circ f\circ\beta_X\circ\beta'_X & \beta'_Y\circ f\circ 1_X \arrow[l, Rightarrow, "\Theta'_X"]
\end{tikzcd}$$

For each object P in $\mathcal{C}$, use (ii) to choose a morphism $\alpha_P\colon GF(P)\to P$, together with a 2-isomorphism $\mu_P\colon F(\alpha_P)\overset{\sim}{\Rightarrow}\beta_{F(P)}$. For each morphism $a\colon P\to Q$ in $\mathcal{C}$, $\tau^\alpha_a\colon a\circ\alpha_P \overset{\sim}{\Rightarrow} \alpha_Q\circ GF(a)$ is determined so $F(\tau^\alpha_a)$ makes

$$\begin{tikzcd}
F(a\circ\alpha_P) \arrow[r, Rightarrow, "F(\tau^\alpha_a)"] \arrow[d, Rightarrow, "\gamma^F_{\alpha_P,a}"'] & F(\alpha_Q\circ GF(a)) \arrow[r, Rightarrow, "\gamma^F_{GF(a),\alpha_Q}"] & F(\alpha_Q)\circ FGF(a) \arrow[d, Rightarrow, "\mu_Q"] \\
F(a)\circ F(\alpha_P) \arrow[r, Rightarrow, "\mu_P"'] & F(a)\circ\beta_{F(P)} \arrow[r, Rightarrow, "\tau^\beta_{F(a)}"'] & \beta_{F(Q)}\circ FGF(a)
\end{tikzcd}$$

commute. Similarly choose $\alpha'_P\colon P\to GF(P)$ with $\mu'_P\colon F(\alpha'_P)\overset{\sim}{\Rightarrow}\beta'_{F(P)}$, and determine $\tau^{\alpha'}_a\colon GF(a)\circ\alpha'_P \overset{\sim}{\Rightarrow} \alpha'_Q\circ a$ so $F(\tau^{\alpha'}_a)$ makes

$$\begin{tikzcd}
F(GF(a)\circ\alpha'_P) \arrow[r, Rightarrow, "F(\tau^{\alpha'}_a)"] \arrow[d, Rightarrow, "\gamma^F_{\alpha'_P,GF(a)}"'] & F(\alpha'_Q\circ a) \arrow[r, Rightarrow, "\gamma^F_{a,\alpha'_Q}"] & F(\alpha'_Q)\circ F(a) \arrow[d, Rightarrow, "\mu'_Q"] \\
FGF(a)\circ F(\alpha'_P) \arrow[r, Rightarrow, "\mu'_P"'] & FGF(a)\circ\beta'_{F(P)} \arrow[r, Rightarrow, "\tau^{\beta'}_{F(a)}"'] & \beta'_{F(Q)}\circ F(a)
\end{tikzcd}$$

commute. For $f, g\colon X \to Y$, and $\rho\colon f \Rightarrow g$ in $\mathcal{D}$, define $G(\rho)\colon G(f) \Rightarrow G(g)$ by requiring that $F(G(\rho))$ makes the diagram

$$\begin{tikzcd}
FG(f) \arrow[r, Rightarrow, "F(G(\rho))"] \arrow[d, Rightarrow, "\lambda_f"'] & FG(g) \arrow[d, Rightarrow, "\lambda_f"] \\
\beta'_Y \circ f \circ \beta_X \arrow[r, Rightarrow, "\rho"'] & \beta'_Y \circ g \circ \beta_X
\end{tikzcd}$$

commute. Finally, define, for each object P in $\mathcal{C}$, 2-isomorphisms $\Xi_P\colon 1_{GF(P)} \overset{\sim}{\Rightarrow} \alpha'_P \circ \alpha_P$ and $\Xi'_P\colon 1_P \overset{\sim}{\Rightarrow} \alpha_P \circ \alpha'_P$, determined by the commutativity of the diagrams

$$\begin{tikzcd}
F(1_{GF(P)}) \arrow[r, Rightarrow, "F(\Xi_P)"] \arrow[d, Rightarrow, "\delta^F_{GF(P)}"'] & F(\alpha'_P \circ \alpha_P) \arrow[r, Rightarrow, "\gamma^F_{\alpha_P, \alpha'_P}"] & F(\alpha'_P) \circ F(\alpha_P) \arrow[d, Rightarrow, "\mu'_P * \mu_P"] \\
1_{FGF(P)} \arrow[rr, Rightarrow, "\Theta_{F(P)}"'] & & \beta'_{F(P)} \circ \beta_{F(P)}
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(1_P) \arrow[r, Rightarrow, "F(\Xi'_P)"] \arrow[d, Rightarrow, "\delta^F_P"'] & F(\alpha_P \circ \alpha'_P) \arrow[r, Rightarrow, "\gamma^F_{\alpha'_P, \alpha_P}"] & F(\alpha_P) \circ F(\alpha'_P) \arrow[d, Rightarrow, "\mu_P * \mu'_P"] \\
1_{F(P)} \arrow[rr, Rightarrow, "\Theta'_{F(P)}"'] & & \beta_{F(P)} \circ \beta'_{F(P)}
\end{tikzcd}$$

This completes the construction of the data. To prove each of the required identities, one writes it as a diagram, in which the maps are 2-isomorphisms between two morphisms, usually in $\mathcal{C}$, that should commute. By the faithfulness of F on the HOM categories, it suffices to prove this after applying F. One then uses the diagrams just constructed to see what this means, obtaining a large diagram that should commute. Finally, one finds a way to subdivide this large diagram into smaller diagrams that commute by properties of F and properties of 2-categories, especially the exchange property. □

Exercise B.53. Complete the proof of this proposition.

Definition B.27. A pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ is a **pseudoequivalence** if it satisfies the equivalent conditions of the proposition.

Exercise B.54. Show that the composite of two pseudoequivalences is a pseudoequivalence. Being pseudoequivalent is therefore an equivalence relation on 2-categories.

There are several generalizations of these notions and results. The conditions on pseudofunctors F and pseudonatural transformations α can be weakened, by allowing the associated $\gamma^F_{f,g}$, δ^F_X, and τ^α_f (or sometimes their inverses) to be only 2-morphisms, not 2-isomorphisms. Such are called *lax functors* and *lax natural transformations*. There is also a notion of a *bicategory*, in which the associativity equality $\gamma\circ(\beta\circ\alpha) = (\gamma\circ\beta)\circ\alpha$ of morphisms is replaced by a 2-isomorphism, and similarly for the equalities $f = f \circ 1_X =$

$1_Y \circ f$ for a morphism $f: X \to Y$. (In a bicategory, each $\mathrm{HOM}(X, Y)$ is a category, but removing the 2-cells from a bicategory does not leave a category.) Theorem B.26 generalizes to the setting of bicategories, cf. [**62**]. Bicategories, 3-categories, and lax notions are discussed in [**12**, §7].

5. Two remarks on set theory

The reader will have noticed that our discussion of categories is framed entirely in the language of "naive set theory" — indeed, sets were not mentioned at all, except in examples. In naive set theory, a set is prescribed by knowing the elements in the set. In this sense, the objects and morphisms of a category (and the 2-morphisms of a 2-category) form sets, since these objects, morphisms (and 2-morphisms) are assumed to be precisely defined. Thus a category is a permissible object in a category, since we know what a category is, and so we have the category (Cat) of categories, which is a 2-category (cf. B.8). And, of course, this leads to Bertrand Russell's famous paradox about the set of all sets.

This problem is not particular to stacks, and those who have made peace with this problem in other areas need not fret about it when studying stacks.[8] The fact that category theory plays a prominent role in the study of stacks, however, brings these problems closer to the surface. And, just as going from objects to categories causes the problem about the set of all sets, going from ordinary categories to 2-categories increases the difficulty. Our aim in this section is to discuss these problems briefly, and point out how to get around them. In this setting, one is not working with sets as collections of well-defined objects, but within an axiomatic set theory, usually taken today to be Zermelo-Fraenkel theory. We also discuss our use of the axiom of choice.

5.1. Categories. We have the need to consider three basic types of set-theoretic structures: sets, categories and 2-categories. If we would like to consider the category of all sets, for example, it is clear that categories cannot be sets. Similarly, if we need to consider the 2-category of all categories, then 2-categories cannot be categories. In spite of Russell's paradox, we would like to do all usual set-theoretic operations (products, disjoint unions, power sets etc.) not only with sets, but also with categories (less so with 2-categories). For this it is necessary to introduce some set-theoretic hierarchy.

We may require all sets to be sets in the usual sense, i.e., the sets given by the axioms of Zermelo-Fraenkel. Then we postulate the existence of *classes* and require categories to be classes. More precisely, the collection of all morphisms in a category is required to be a class — after all, a category can be thought of as its collection of morphisms with a partially defined binary operation, the objects are then defined in terms of the identity morphisms. We require classes to satisfy the same list of axioms as sets. Besides the existence of sets and classes, we moreover postulate the existence of *2-classes* and require 2-categories to be 2-classes; note that 2-categories may be identified with their collection of 2-arrows.

[8] We quote the footnote on the first page of the revised version [**EGA** I′]: "Nous considérons les catégories d'un point de vue 'naif', comme s'il s'agissait d'ensembles et renvoyons à SGA, 4, I pour les questions de logique liées à la théorie des catégories, et la justification du langage que nous utilisons."

Alternatively, (more precisely?) the required set-theoretic hierarchy may be introduced via the theory of *universes*, cf. [**65**], §I.6, and [SGA4] §I. One postulates the existence of three universes, $\mathfrak{U}_0$, $\mathfrak{U}_1$ and $\mathfrak{U}_2$ and requires that $\mathfrak{U}_0 \in \mathfrak{U}_1 \in \mathfrak{U}_2$. Recall that all universes are actually sets (and thus consist of sets). We then use all sets in $\mathfrak{U}_0$ as sets, all sets in $\mathfrak{U}_1$ as classes and all sets in $\mathfrak{U}_2$ as 2-classes. In other words, we require all sets (as the term is used in the book) to be in $\mathfrak{U}_0$, all categories to be in $\mathfrak{U}_1$ and all 2-categories to be in $\mathfrak{U}_2$. If a category happens to be in $\mathfrak{U}_0$, we call it **small**.

For many applications this notion of small is too restrictive. We find much more useful the notion of *essential* smallness. A category is called **essentially small** if its collection of isomorphism classes (i.e. its collection of objects up to isomorphism) is in $\mathfrak{U}_0$, i.e. is a set. When fibered categories are introduced, one can require the fibers to be essentially small. These and the other groupoids that appear can be taken to be essentially small.

We will not mention these set-theoretic issues in the main body of the book. The reader who thinks in this language is invited to check that none of our constructions with categories leave the universe $\mathfrak{U}_1$, and that every time we construct a set from a category we obtain a set in $\mathfrak{U}_0$.

5.2. The axiom of choice. The only essential way in which we use the axiom of choice is the following: we want a functor which is faithful, full, and essentially surjective to have an inverse (a functor in the other direction such that both compositions are naturally isomorphic to the identities). This is important because the natural notion of isomorphism between stacks is expressed by such an equivalence. Note that this actually uses the axiom of choice for *classes* of sets. This use of the axiom of choice can be avoided if one understands that whenever we call two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ *equivalent* , this means that there exists a chain of categories and equivalences

$$\mathcal{C}_1 \longleftarrow \mathcal{D}_1 \longrightarrow \mathcal{D}_2 \longleftarrow \ldots \longleftarrow \mathcal{D}_n \longrightarrow \mathcal{C}_2 \quad .$$

In effect, this amounts to localizing the 2-category of categories at the equivalences (For a discussion of similar localizations, see [**28**]). We prefer not to do this, but rather assume that all equivalences have inverses. In view of these remarks, this use of the axiom of choice may be viewed as merely a device of notational convenience.

We also remark that even though many general statements in the book formally require the axiom of choice for their validity, in any specific application one usually has more or less canonical choices at one's disposal. For example, the fibered product of schemes may be constructed in a canonical way, cf. [**EGA** I]; we do not need the axiom of choice to pick an object satisfying the universal mapping property of fibered product. Similarly, the construction of quotient sheaves can be carried out in a canonical way by sheafifying (using an explicit sheafification construction) a given quotient presheaf.

In short, the reader can choose whatever set-theoretic foundations he or she is comfortable with: we do not discuss them in the text. As with other areas of algebra or geometry, the notions and theorems about stacks change very little with changes in logical foundations.

Answers to Exercises

B.1. If $f \circ a = f$ for all f, and $b \circ g = g$ for all g, then $a = b \circ a = b$.

B.2. (1) If $g \circ f = 1_X$ and $f \circ h = 1_Y$, then $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$. (2) follows similarly from associativity.

B.7. For $\Rightarrow$, given G, θ, and η, a morphism $f\colon F(P) \to F(Q)$ has uniquely the form $f = F(a)$, where $a\colon P \to Q$ is given by $a = \theta_Q \circ G(f) \circ \theta_P^{-1}$. For any object X in $\mathcal{D}$, the isomorphism $\eta_X\colon FG(X) \to X$ shows that F is essentially surjective. Details can be found in [**65**, §IV.4].

B.8. This is exactly what the proof of the proposition produces.

B.10. Take $X \times_Z Y = \{(x, y) \in X \times Y \mid s(x) = t(y)\}$, with the induced topology in the topological case.

B.13. A natural transformation θ from h_X to H assigns to every object S of $\mathcal{C}$ a mapping θ_S from $h_X(S)$ to $H(S)$. Applied to $S = X$, one gets ζ as $\theta_X(1_X)$. For any S and $g\colon S \to X$, $\theta_S(g) = \theta_S(h_X(g)(1_X)) = H(g)(\theta_X(1_X)) = H(g)(\zeta)$, so θ is determined by ζ.

B.18. Both parts follow readily from the exchange property.

B.19. By (1) of the preceding exercise, $(\theta * 1_h) \circ (1_{1_X} * \theta) = (1_h * \theta) \circ (\theta * 1_{1_X})$. Since $1_{1_X} * \theta = \theta * 1_{1_X} = \theta$ is invertible, the required equation follows.

B.25. Given α from (φ, Φ) to (ψ, Ψ) and β from (ψ, Ψ) to (ω, Ω), define $\beta \circ \alpha\colon U' \to R$ to be the composite

$$U' \xrightarrow{(\alpha,\beta)} R\ {}_t\!\times_s R \xrightarrow{m} R.$$

Given (φ', Φ') and (ψ', Ψ') from $R'' \rightrightarrows U''$ to $R' \rightrightarrows U'$, and a 2-morphism β from (φ', Φ') to (ψ', Ψ'), define $\alpha * \beta\colon U'' \to R$ to be the composite

$$U'' \xrightarrow{(\Phi\beta,\alpha\psi')} R\ {}_t\!\times_s R \xrightarrow{m} R$$

(which is equal to $m \circ (\alpha\varphi', \Psi\beta)$).

B.30. In (3), the fact that σ^S is a natural transformation follows from the exchange property: given $\rho\colon h \Rightarrow h'$, $(1_g * \rho) \circ (\sigma * 1_h) = (\sigma * 1_{h'}) \circ (1_f * \rho)$.

B.31. That (1) implies (5) and (7) implies (2) are similar to the proofs that (1) implies (3) and (4) implies (2). To see that (7) implies (1), the essential surjectivity of f^Y, applied to 1_Y, provides a $g\colon Y \to X$ and a 2-isomorphism $f \circ g \overset{\sim}{\Rightarrow} 1_Y$; f_X essentially surjective, applied to 1_X, provides a $g'\colon Y \to X$ and a 2-isomorphism $f \circ g' \overset{\sim}{\Rightarrow} 1_X$. The equivalence of (2), (8), (9) and (10) is seen by taking η and ψ to be inverses of each other, and taking θ and ϕ to be inverses of each other.

B.34. If $H\colon X \times [0, 1] \to Y$ is a homotopy, define a 2-morphism by the mapping from $\pi(X)_0$ to $\pi(Y)_1$ that sends a point x in X to the path $t \mapsto H(x, t)$ in Y.

B.43. That the diagrams commute follows from several applications of the exchange property, together with the identity $(1_p * \epsilon^p) \circ (\eta^p * 1_p) = 1_p$ for the first diagram, $(\epsilon^q * 1_{q'}) \circ (1_{q'} * \eta^q) = 1_{q'}$ for the second, and $(1_f * \epsilon^f) \circ (\eta^f * 1_f) = 1_f$ for the third. For (2), consider the diagram

$$\begin{tikzcd}
p' \circ f \arrow[r, equal] \arrow[d, equal] & p' \circ 1_Z \circ f \arrow[r, Rightarrow, "\eta^f"] & p' \circ f \circ f' \circ f \arrow[r, Rightarrow, "\epsilon^f"] \arrow[d, Rightarrow, "c_\alpha"] & p' \circ f \circ 1_X \arrow[r, equal] \arrow[d, Rightarrow, "c_\alpha"] & p' \circ f \arrow[d, Rightarrow, "c_\alpha"] \\
1_Y \circ p' \circ f \arrow[r, Rightarrow, "\eta^g"'] & g \circ g' \circ p' \circ f \arrow[r, Rightarrow, "\alpha'"'] & g \circ q' \circ f' \circ f \arrow[r, Rightarrow, "\epsilon^f"'] & g \circ q' \circ 1_X \arrow[r, equal] & g \circ q'
\end{tikzcd}$$

The left rectangle commutes by the commutativity of the first diagram in (1), and the other squares commute by the exchange property. The map along the top is the identity on $p' \circ f$, by the defining property of $(f, f', \eta^f, \epsilon^f)$. The assertions of (3) follow from the other two commutative diagrams of (1).

B.35. If $K\colon X \times [0,1] \times [0,1] \to Y$ gives an equivalence from the homotopy H to H', then θ_K, defined by $\theta_K(\sigma)(t_1, \ldots, t_{n+2}) = K(\sigma(t_3, \ldots, t_{n+2}), t_2, t_1)$ gives an equivalence from α_H to $\alpha_{H'}$. If f, g, and h map X to Y, and H_1 is a homotopy from f to g, and H_2 is a homotopy from g to h, then θ_K defines an equivalence between $\alpha_{H_2} \circ \alpha_{H_1}$ and $\alpha_{H_2 \circ H_1}$, where K is defined by the formula $K(x, s, t) = H_1(x, s + 2t)$ if $s + 2t \le 1$, and $K(x, s, t) = H_2(x, (s + 2t - 1)/(s + 1))$ if $s + 2t \ge 1$.

B.44. These are proved with several applications of the exchange property, as well as the identity $(1_q * \epsilon^q) \circ (\eta^q * 1_q) = 1_q$ (for (1)).

B.46. This takes a homomorphism $f\colon G \to H$ to the functor $B(f)\colon BG \to BH$ that takes a (right) G torsor E to the H-torsor $E \times_f H = E \times H/\{(v \cdot x, y) \sim (v, f(x)y)\}$. If also $g\colon H \to K$, $\gamma^B_{f,g}$ takes E to the isomorphism from $E \times_{gf} K$ to $(E \times_f H) \times_g K$ that takes (v, z) to $((v, 1), z)$, and δ^B_G takes E to the isomorphism from $E \times_{\mathrm{id}} G$ to E that takes (v, x) to $v \cdot x$. If a in H gives a 2-morphism from f to g, for f and g homomorphisms from G to H, then $B(a)$ is the natural transformation from $B(f)$ to $B(g)$ that takes a G-torsor E to the map $E \times_f H$ to $E \times_g H$, $(v, y) \mapsto (v, a^{-1}y)$.

B.48. Use property (c) for β, together with the exchange property.

B.53. We will carry this out in one typical case, proving the first half of the pseudofunctor property (b) for G, with its associated 2-isomorphisms $\gamma^G_{f,g}$ and δ^G_X. That is, we show that the diagram

$$\begin{tikzcd}
G(f) \arrow[r, equal] \arrow[d, equal] & G(f) \circ 1_{G(X)} \\
G(f \circ 1_X) \arrow[r, Rightarrow, "\gamma^G_{1_X, f}"'] & G(f) \circ G(1_X) \arrow[u, Rightarrow, "\delta^G_X"']
\end{tikzcd}$$

commutes. Applying F, one needs the upper left square in the diagram

$$\begin{tikzcd}
FG(f) \arrow[r, equal] \arrow[d, equal] & F(G(f)\circ 1_{G(X)}) \arrow[r, Rightarrow, "\gamma^F_{1_{G(X)},G(f)}"] & FG(f)\circ F(1_{G(X)}) \arrow[r, Rightarrow, "\delta^F_{G(X)}"] & FG(f)\circ 1_{FG(X)} \arrow[d, Rightarrow, "\Theta_X"] \\
FG(f\circ 1_X) \arrow[r, Rightarrow, "F(\gamma^G_{1_X,f})"'] \arrow[d, Rightarrow, "\lambda_f"'] & F(G(f)\circ G(1_X)) \arrow[u, Rightarrow, "F(1_{G(f)}*\delta^G_X)"'] \arrow[r, Rightarrow, "\gamma^F_{G(1_X),G(f)}"'] & FG(f)\circ FG(1_X) \arrow[u, Rightarrow, "F(\delta^G_X)"'] \arrow[r, Rightarrow, "\lambda_{1_X}"'] \arrow[d, Rightarrow, "\lambda_f*\lambda_{1_X}"'] & FG(f)\circ\beta'_X\circ\beta_X \arrow[dl, Rightarrow, "\lambda_f"] \\
\beta'_Y\circ f\circ\beta_X \arrow[r, equal] & \beta'_Y\circ f\circ 1_X\circ\beta_X \arrow[r, Rightarrow, "\Theta'_X"'] & \beta'_Y\circ f\circ\beta_X\circ\beta'_X\circ\beta_X &
\end{tikzcd}$$

to commute. The upper center square commutes by property (d) for F; the upper right square commutes by the definition of γ^G_X; the lower right triangle commutes by the exchange property (e) for 2-categories. So we are reduced to showing that the outside diagram commutes. For this we fill in the diagram differently:

$$\begin{tikzcd}
FG(f) \arrow[r, equal] \arrow[d, equal] & F(G(f)\circ 1_{G(X)}) \arrow[r, Rightarrow, "\gamma^F_{1_{G(X)},G(f)}"] & FG(f)\circ F(1_{G(X)}) \arrow[d, Rightarrow, "\delta^F_{G(X)}"] & \\
FG(f) \arrow[rr, equal] \arrow[d, Rightarrow, "\lambda_f"'] & & FG(f)\circ 1_{FG(X)} \arrow[r, Rightarrow, "\Theta_X"] \arrow[d, Rightarrow, "\lambda_f"'] & FG(f)\circ\beta'_X\circ\beta_X \arrow[d, Rightarrow, "\lambda_f"] \\
\beta'_Y\circ f\circ\beta_X \arrow[rr, equal] \arrow[d, equal] & & \beta'_Y\circ f\circ\beta_X\circ 1_{FG(X)} \arrow[r, Rightarrow, "\Theta_X"'] & \beta'_Y\circ f\circ\beta_X\circ\beta'_X\circ\beta_X \arrow[d, equal] \\
\beta'_Y\circ f\circ 1_X\circ\beta_X \arrow[rrr, Rightarrow, "\Theta'_X"'] & & & \beta'_Y\circ f\circ\beta_X\circ\beta'_X\circ\beta_X
\end{tikzcd}$$

The top square commutes by property (b) for F; the next square down commutes by property (c) of a 2-category; the one to its right commutes by the exchange property (e). Finally, the lower rectangle commutes by the key property that

$$\begin{tikzcd}
\beta_X \arrow[r, equal] \arrow[d, equal] & \beta_X\circ 1_{FG(X)} \arrow[d, Rightarrow, "\Theta_X"] \\
1_X\circ\beta_X \arrow[r, Rightarrow, "\Theta'_X"'] & \beta_X\circ\beta'_X\circ\beta_X
\end{tikzcd}$$

commutes. Most of the other verifications are similar to this.

APPENDIX C

Groupoids

This appendix is written for two purposes. It can serve as a reference for facts about categories in which all morphisms are isomorphisms. More importantly, it can be regarded as a short text on groupoids and stacks of discrete spaces. In this way it can provide an introduction to many of the ideas and constructions that are made in the main text, without any algebro-geometric complications.

In this appendix, all categories are assumed to be small. This is not so much for set-theoretic reasons (cf. B §5, but rather to think and write about their objects and morphisms as discrete spaces of points.

If X is a category, we write X_0 for the set of its objects, X_1 for the set of its morphisms and $s, t\colon X_1 \to X_0$ for the source and target map. The notation $a\colon x \to y$ or $x \xrightarrow{a} y$ means that a is in X_1 and $s(a) = x$, $t(a) = y$. The set of morphisms from x to y is denoted $\mathrm{Hom}(x, y)$. The composition, or multiplication, is defined on the collection $X_2 = X_1 \, {}_t\!\times_{X_0 s} X_1$ of pairs (a, b) such that $t(a) = s(b)$. We write $b \circ a$ or $a{\cdot}b$ for the composition of a and b. We denote by $m\colon X_2 \to X_1$ the map that sends (a, b) to $a{\cdot}b$. There is also a map $e\colon X_0 \to X_1$ that takes every object x to the identity morphism id_x or 1_x on that object. In this appendix we generally denote the category by $X_\bullet$.

Exercise C.1. Show that the axioms for a category are equivalent to the following identities among s, t, m, and e: (i) $s \circ e = \mathrm{id}_{X_0} = t \circ e$; (ii) $s \circ m = s \circ p_1$ and $t \circ m = t \circ p_2$, where p_1 and p_2 are the projections from $X_1 \, {}_t\!\times_s X_1$ to X_1; (iii) $m \circ (m, 1) = m \circ (1, m)$ as maps from $X_1 \, {}_t\!\times_s X_1 \, {}_t\!\times_s X_1$ to X_1; (iv) $m \circ (s \circ e, 1) = \mathrm{id}_{X_1} = m \circ (1, t \circ e)$.

We pick a canonical one-element set and denote it pt.

1. Groupoids

Definition C.1. A category $X_\bullet$ is called a **groupoid** if every morphism $a \in X_1$ has an inverse. There exists therefore a map $i\colon X_1 \to X_1$ that takes a morphism to its inverse. The element $i(a)$ is often denoted a^{-1}.

Exercise C.2. A groupoid is a pair of sets X_0 and X_1, together with five maps s, t, m, e and i, satisfying the four identities of the preceding exercise, together with: (v) $s \circ i = t$ and $t \circ i = s$; (vi) $m \circ (1, i) = e \circ s$ and $m \circ (i, 1) = e \circ t$. Deduce from these identities the properties: (vii) $i \circ i = \mathrm{id}_{X_1}$; (viii) $i \circ e = e$; $m \circ (e, e) = e$; (ix) $i \circ m = m \circ (i \circ p_2, i \circ p_1)$. Show that e and i are uniquely determined by X_0, X_1, s, t, and m.

We will generally think of a groupoid $X_\bullet$ as a pair of sets (or discrete spaces) X_0 and X_1, with morphisms s, t, m, e, and i, satisfying these identities. Occasionally, however, we will use the categorical language, referring to elements of X_0 as *objects* and elements of X_1 as *arrows* or *morphisms.* The notation $X_1 \rightrightarrows X_0$ may be used in place of $X_\bullet$.

DEFINITION C.2. For any $x \in X_0$, the composition m defines a group structure on the set $\mathrm{Hom}(x,x) = \{a \in X_1 \mid s(a) = x, t(a) = x\}$. This group is denoted $\mathrm{Aut}(x)$, and it is called the **automorphism** or **isotropy** group of x.

A groupoid may be thought of as an approximation of a group, but where composition is not always defined.

Our first example is the prototype groupoid:

EXAMPLE C.3. Let X be a topological space. Define the **fundamental groupoid** $\pi(X)_\bullet$ by taking $\pi(X)_0 = X$ as the set of objects and

$$\pi(X)_1 = \{\gamma\colon [0,1] \to X \text{ continuous}\}/\sim$$

as the set of arrows. Here we write $\gamma \sim \gamma'$ for two paths in X if there exists a homotopy between γ and γ' fixing the endpoints. Then we define

$$s\colon \pi(X)_1 \longrightarrow \pi(X)_0, \qquad [\gamma] \longmapsto \gamma(0)$$

and

$$t\colon \pi(X)_1 \longrightarrow \pi(X)_0 \qquad [\gamma] \longmapsto \gamma(1).$$

Thus the paths γ and γ' are composable precisely if $\gamma(1) = \gamma'(0)$ and we have

$$\pi(X)_2 = \{([\gamma],[\gamma']) \in \pi(X)_1 \times \pi(X)_1 \mid \gamma(1) = \gamma'(0)\}.$$

The composition of $[\gamma]$ and $[\gamma']$ is defined to be the homotopy class of the path

$$(\gamma{\cdot}\gamma')(t) = \begin{cases} \gamma(2t) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1) & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases},$$

$$\bullet \xrightarrow{\gamma} \bullet \xrightarrow{\gamma'} \bullet$$
$$\gamma{\cdot}\gamma'$$

[There should be a nicely drawn picture of paths here.] Thus we have

$$m\colon \pi(X)_2 \longrightarrow \pi(X)_1, \qquad ([\gamma],[\gamma']) \longmapsto [\gamma{\cdot}\gamma'].$$

EXERCISE C.3. Prove that $\pi(X)_\bullet$ is a groupoid. In particular, determine the maps $e\colon \pi(X)_0 \to \pi(X)_1$ and $i\colon \pi(X)_1 \to \pi(X)_1$. More generally, for any subset A of X, construct a groupoid $\pi(X,A)_\bullet$, with $\pi(X,A)_0 = A$ and $\pi(X,A)_1$ the set of homotopy classes of paths with both endpoints in A.

It is useful to imagine any groupoid geometrically in terms of paths as suggested by this example. (It is in examples like this that the notation $a{\cdot}b$ is preferrable to the $b \circ a$ convention.)

The fundamental mathematical notions of *set* and *group* occur as extreme cases of groupoids:

EXAMPLE C.4. Every set X is a groupoid by taking the set of objects X_0 to be X and allowing only identity arrows, which amounts to taking $X_1 = X$, too. We consider every set as a groupoid in this way, if not mentioned otherwise.

Example C.5. Every group G is a groupoid by taking $X_0 = pt$ and declaring the automorphism group of the unique object of $X_\bullet$ to be G. Then $\mathrm{Aut}(x) = G = X_1$, if x denotes the unique element of pt. In this appendix we write $X_\bullet = BG_\bullet$ and call it the **classifying groupoid** of the group G.

The next example contains the previous two. It describes a much more typical groupoid:

Example C.6. If X is a right G-set, we define a groupoid $X \rtimes G$ by taking X as the set of objects of $X \rtimes G$ and declaring, for any two $x, y \in X$,

$$\mathrm{Hom}(x, y) = \{g \in G \mid xg = y\}.$$

Composition in $X \rtimes G$ is induced from multiplication in G.

More precisely, we have $(X \rtimes G)_0 = X$ and $(X \rtimes G)_1 = X \times G$. The source map $s\colon X \times G \to X$ is the first projection, the target map $t\colon X \times G \to X$ is the action map: $t(x, g) = xg$. The morphisms (x, g) and (y, h) are composable if and only if $y = xg$ and the multiplication is given by $(x, g)\cdot(y, h) = (x, gh)$:

$$\begin{tikzcd} x \arrow[r, "(x,g)"] \arrow[dr, "(x,gh)"'] & xg \arrow[d, "(xg,h)"] \\ & xgh \end{tikzcd}$$

Thus we may identify X_2 with $X \times G \times G$, with $(x, g) \times (xg, h)$ corresponding to (x, g, h), and write

$$m\colon X \times G \times G \longrightarrow X \times G, \quad (x, g, h) \longmapsto (x, gh).$$

The groupoid $X \rtimes G$ is called the **transformation groupoid** given by the G-set X.

Example C.7. If X is a left G-set, we get an associated groupoid by declaring

$$\mathrm{Hom}(x, y) = \{g \in G \mid gx = y\}.$$

Thus the pair (g, x) is considered as an arrow from x to gx. The source map is again the projection and the target map is the group action. We denote this groupoid by $G \ltimes X$. Note that the multiplication is given by $(g, x)\cdot(h, gx) = (hg, x)$, which reverses the order of the group elements.[1]

Exercise C.4. Suppose a set X has a left action of a group G and a right action of a group H, and these actions **commute**, i.e., $(gx)h = g(xh)$ for all $g \in G, x \in X, h \in H$; in this case we write gxh for this common element. Construct a **double transformation groupoid** $G \ltimes X \rtimes H$, of the form $G \times X \times H \rightrightarrows X$, with $s(g, x, h) = x$, $t(g, x, h) = gxh$, and $m((g, x, h), (g', gxh, h')) = (g'g, x, hh')$.

The next two examples go beyond group actions on sets:

[1]This notation is compatible with the composition notation $b \circ a$, which is useful in the common situation where an automorphism group of a mathematical structure is considered to act on the left, with the product given by composition.

Example C.8. If $R \subset X \times X$ is an equivalence relation on the set X, then we define an associated groupoid $R \rightrightarrows X$ by taking the two projections as source and target map: $s = p_1$, $t = p_2$. Composition is given by $(x,y)\cdot(y,z) = (x,z)$:

$$\begin{tikzcd} x \arrow[r, "{(x,y)}"] \arrow[dr, "{(x,z)}"'] & y \arrow[d, "{(y,z)}"] \\ & z \end{tikzcd}$$

For $x, y \in X$ there is at most one morphism from x to y and x and y are isomorphic in the groupoid $R \rightrightarrows X$ (meaning that there is an a in $X_1 = R$ with $s(a) = x$ and $t(a) = y$) if and only if $(x,y) \in R$, i.e., x and y are equivalent under the relation R.

Example C.9. Let $(G_i)_{i \in I}$ be a family of groups. Define an associated groupoid by taking as objects the set $X_0 = I$. We declare all objects to be pairwise non-isomorphic and define, for each $i \in I$, $\mathrm{Aut}(i) = G_i$. Then X_1 is the disjoint union $\coprod_{i \in I} G_i$ and $s = t$ maps $g \in G_i$ to i.

Example C.10. More generally, if $(X_\bullet(i))_{i \in I}$ is any family of groupoids, there is a **disjoint union** groupoid $X_\bullet = \coprod_i X_\bullet(i)$, with $X_0 = \coprod_i X_0(i)$ and $X_1 = \coprod_i X_1(i)$.

Example C.11. Let $X_0 \to Y$ be any map of sets. Define an associated groupoid $X_\bullet$ by defining X_1 to be the fibered product: $X_1 = X_0 \times_Y X_0$. The source is the first projection and the target is the second projection. Call this groupoid the **cross product groupoid** associated to $X_0 \to Y$. Note that this construction is a special case of an equivalence relation (Example C.8).

Example C.12. For any set X, there is a groupoid with $X_0 = X$, and $X_1 = X \times X$, with s and t the two projections, and $(x,y)\cdot(y,z) = (x,z)$. This is also an equivalence relation, with any two points being equivalent. This is sometimes called a **banal** groupoid. It is a special case of the preceding example, with $Y = pt$.

Definition C.13. Given a groupoid $X_\bullet$, a **subgroupoid** is given by subsets $Y_0 \subset X_0$ and $Y_1 \subset X_1$ such that: $s(Y_1) \subset Y_0$; $t(Y_1) \subset Y_0$; $e(Y_0) \subset Y_1$, $i(Y_1) \subset Y_1$, and $a, b \in Y_1$ with $t(a) = s(b)$ implies $a\cdot b \in Y_1$.

Exercise C.5. Let Z be any set. Construct a groupoid with X_0 the set of nonempty subsets of Z, and with $X_1 = \{(A, B, \phi) \mid A, B \in X_0 \text{ and } \phi\colon A \to B \text{ is a bijection}\}$, and multiplication given by $(A,B,\phi)\cdot(B,C,\psi) = (A, C, \psi \circ \phi)$.

Exercise C.6. Let Γ be a directed graph, which consists of a set V (of vertices) and a set E of edges, together with mappings $s, t\colon E \to V$. For any $a \in E$, define a symbol $\widetilde{a}$, called the *opposite edge* of a, and set $s(\widetilde{a}) = t(a)$ and $t(\widetilde{a}) = s(a)$. For each $v \in V$ define a symbol 1_v, with $s(1_v) = t(1_v) = v$. Construct a groupoid $F(\Gamma)_\bullet$, called the *free groupoid* on Γ, by setting $F(\Gamma)_0 = V$, and $F(\Gamma)_1$ is the (disjoint) union of $\{1_v \mid v \in V\}$ and the set of all sequences $(\alpha_1, \ldots, \alpha_n)$, with each α_i either an edge or an opposite edge, with $t(\alpha_i) = s(\alpha_{i+1})$, such that no successive pair (α_i, α_{i+1}) has the form $(a, \widetilde{a})$ or $(\widetilde{a}, a)$ for any edge a, $1 \le i < n$. Composition is defined by juxtaposition and canceling to eliminate successive pairs equal to an edge and its inverse. Verify that $F(\Gamma)_\bullet$ is a groupoid.

EXERCISE C.7. Let $X_\bullet$ be a groupoid in which the multiplication map $m\colon X_2 \to X_1$ is finite-to-one. For any commutative ring K with unity, let $A = K[X_\bullet]$ be the set of K-valued functions on X_1. Define a *convolution product* on A by the formula

$$(f * g)(c) = \sum_{a \cdot b = c} f(a) \cdot f(b),$$

the sum over all pairs $a, b \in X_1$ with $a{\bullet}b = c$. Show that, with the usual pointwise sum for addition, this makes A into an associative K-algebra with unity. If $X_\bullet = BG_\bullet$, this is the group algebra of G. (Extending this to infinite groupoids, with appropriate measures to replace the sums by integrals, is an active area (cf. [**18**]), as it leads to interesting $\mathbb{C}^*$-algebras.)

REMARK C.14. There is an obvious notion of isomorphism between groupoids $X_\bullet$ and $Y_\bullet$. It is given by a bijection between X_0 and Y_0 and a bijection between X_1 and Y_1, compatible with the structure maps s, t, m (and therefore e and i). This notion will be referred to as **strict isomorphism**, since it is too strong for most purposes. We will define a more supple notion of isomorphism in the next section.

EXERCISE C.8. Any left action of a groups G on a set X determines a right action of G on X by setting $x \cdot g = g^{-1}x$. Show that the map which is the identity on X, and maps $G \times X$ to $X \times G$ by $(g, x) \mapsto (x, g^{-1})$, determines a strict isomorphism of $G \ltimes X$ with $X \rtimes G$.

EXERCISE C.9. Let $X_\bullet$ be a groupoid. Define the groupoid $\widetilde{X}_\bullet$ by reversing the direction of arrows. In other words, $\widetilde{X}_0 = X_0$, $\widetilde{X}_1 = X_1$, $\tilde{s} = t$, $\tilde{t} = s$, $\widetilde{X}_2 = \{(x, y) \in X_1 \times X_1 \mid (y, x) \in X_2\}$ and $\widetilde{m}(x, y) = m(y, x)$. This is a groupoid (with $\tilde{e} = e$ and $\tilde{i} = i$). Show that $\widetilde{X}_\bullet$ is strictly isomorphic to $X_\bullet$ by sending an element of X_1 to its inverse, and the identity on X_0. This is called the **opposite groupoid** of $X_\bullet$, and is often denoted $X_\bullet^{\mathrm{opp}}$.

EXERCISE C.10. For a left action of a group G on a set X, define a groupoid with $X_0 = X$, $X_1 = G \times X$, with $s(g, x) = g \cdot x$, $t(g, x) = x$, and $m((g, h \cdot x), (h, x)) = (h \cdot g, x)$. Show that this is a groupoid, strictly isomorphic to the opposite groupoid of $G \ltimes X$. Similarly for a right action of G on X, there is a groupoid with $X_0 = X$, $X_1 = X \times G$, with $s(x, g) = x \cdot g$, $t(s, g) = x$, and $(x \cdot h, g){\bullet}(x, h) = (x, h \cdot g)$; this is strictly isomorphic to the opposite groupoid of $X \rtimes G$.

The preceding exercises show that, although there are several possible conventions for constructing transformation groupoids of actions of a group on a set, they all give strictly (and canonically) isomorphic groupoids.

EXERCISE C.11. By a **right action** of a group G on a groupoid $X_\bullet$ is meant a right action of G on X_1 and on X_0, so that s, t are equivariant[2], and satisfying $ag{\bullet}bg = (a{\bullet}b)g$

[2]A mapping $f\colon U \to V$ of right G-sets is **equivariant** if $f(ug) = f(u)g$ for all $u \in U$ and $g \in G$.

for $a, b \in X_1$ with $t(a) = s(b)$, and $g \in G$; that is, m is equivariant with repect to the diagonal action on X_2. It follows that e and i are also equivariant. Construct a groupoid $X_1 \times G \rightrightarrows X_0$, denoted $X_\bullet \rtimes G$, by defining $s(a, g) = s(a)$, $t(a, g) = t(ag) = t(a)g$, and $(a, g)\cdot(b, g') = (a\cdot bg^{-1}, gg')$. Verify that $X_\bullet \rtimes G$ is a groupoid. Construct a groupoid $G \ltimes X_\bullet$ for a left action.

Exercise C.12. Suppose a groupoid $X_\bullet$ has a left action of a group G, and a right action of a group H, and the actions commute, i.e., $(gx)h = g(xh)$ for $g \in G$, $h \in H$, and $x \in X_0$ or X_1. There is a natural right action of H on $G \ltimes X_\bullet$, and a left action of G on $X \rtimes H$. Construct a strict isomorphism between the groupoids $(G \ltimes X_\bullet) \rtimes H$ and $G \ltimes (X_\bullet \rtimes H)$.

Exercise C.13. $(^*)$[3] For every groupoid $X_\bullet$, construct a topological space X and a subset A so that $X_\bullet$ is strictly isomorphic to the fundamental groupoid $\pi(X, A)_\bullet$.

Let us consider two basic properties of groupoids:

Definition C.15. A groupoid $X_\bullet$ is called **rigid** if for all $x \in X_0$ we have $\mathrm{Aut}(x) = \{\mathrm{id}_x\}$.

A groupoid $X_\bullet$ is called **transitive** if for all $x, y \in X_0$ there is an $a \in X_1$ with $s(a) = x$ and $t(a) = y$.

Exercise C.14. For a topological space X, $\pi(X)_\bullet$ is rigid if and only if every closed path in X is homotopic to a trivial path, and $\pi(X)_\bullet$ is transitive if and only if X is path-connected.

Exercise C.15. For group actions, the transformation groupoid is rigid exactly when the action is free, and the groupoid is transitive when the action is transitive.

Exercise C.16. Show that every equivalence relation is rigid. Conversely, every rigid groupoid is strictly isomorphic to an equivalence relation.

Definition C.16. A groupoid is canonically and strictly isomorphic to a disjoint union of transitive groupoids, called its **components**. Call two points x and y of X_0 **equivalent** if there is some $a \in X_1$ with $s(a) = x$ and $t(z) = y$, and write $x \cong y$ if this is the case. This is an equivalence relation, defined by the image of X_1 in $X_0 \times X_0$ by the map (s, t). There is a component for each equivalence class; write $X_0/\cong$ for the set of equivalence classes.

Exercise C.17. If $s(a) = x$ and $t(a) = y$, the map $g \mapsto a^{-1}\cdot g\cdot a$ determines an isomorphism from $\mathrm{Aut}(x)$ to $\mathrm{Aut}(y)$. Replacing a by another a' with $s(a') = x$ and $t(a') = y$ gives another isomorphism from $\mathrm{Aut}(x)$ to $\mathrm{Aut}(y)$ that differs from the first by an inner automorphism. Hence there is a group, well-defined up to inner automorphism, associated to each equivalence class of a groupoid: the automorphism group $\mathrm{Aut}(x)$ of any of its points.

Exercise C.18. The free groupoid of a graph is rigid if and only if the graph has no loops. It is transitive when the graph is connected.

[3]The $(*)$ means that this is a more difficult exercise, which isn't central to understanding.

Next we show how to count in groupoids.

Definition C.17. A groupoid $X_\bullet$ is called *finite* if:

(1) the set of equivalence classes $X_0/\cong$ is finite;
(2) for every object $x \in X_0$ the automorphism group $\mathrm{Aut}(x)$ is finite.

If $X_\bullet$ is a finite groupoid, we define its **mass** to be

$$\#X_\bullet = \sum_{x \in X_0/\cong} \frac{1}{\#\,\mathrm{Aut}(x)},$$

where the sum is taken over a set of representatives of the objects modulo isomorphism. More generally, if each $\mathrm{Aut}(x)$ is finite, and the sums $\sum \frac{1}{\#\,\mathrm{Aut}(x)}$ have a least upper bound, as x varies over representatives of finite subsets of $X_0/\cong$, define the mass $\#X_\bullet$ to be this least upper bound, and call $X_\bullet$ **tame**.

Exercise C.19. Show that if G is a finite group and X a finite G-set, then $X \rtimes G$ is finite and

$$\#X \rtimes G = \frac{\#X}{\#G}.$$

Exercise C.20. $(*)$ Let F be a finite field with q elements. Consider the groupoid $X_\bullet$ of vector bundles over $\mathbb{P}^1_F$ which are of rank 2 and degree 0. The objects of this groupoid are all such vector bundles, the morphisms are all isomorphisms of these vector bundles. Show that this groupoid is tame but not finite, and find its mass.

Definition C.18. A **vector bundle** E on a groupoid $X_\bullet$ assigns to each $x \in X_0$ a vector space E_x, and to each $a \in X_1$ from x to y a linear isomorphism $a_*\colon E_x \to E_y$, satisfying the compatibility: for all $(a,b) \in X_2$, $(a{\bullet}b)_* = b_* \circ a_*$, i.e., with $z = t(b)$, the diagram

$$\begin{tikzcd} E_x \arrow[r, "a_*"] \arrow[dr, "(a\bullet b)_*"'] & E_y \arrow[d, "b_*"] \\ & E_z \end{tikzcd}$$

commutes. For example, a vector bundle on $BG_\bullet$ is the same as a representation of the group G.

Exercise C.21. If E is a vector bundle on $X_\bullet$, construct a groupoid $E_\bullet$ with $E_0 = \coprod_{x\in X_0} E_x$, and $E_1 = \{(a,v,w) \mid a \in X_1, v \in E_{sa}, w \in E_{ta}, a_*(v) = w\}$.

2. Morphisms of groupoids

Definition C.19. A **morphism** of groupoids $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is a pair of maps $\phi_0\colon X_0 \to Y_0$, $\phi_1\colon X_1 \to Y_1$, compatible with source, target and composition. In the language of categories, this is the same as a functor.

Example C.20. A continuous map of topological spaces $f\colon X \to Y$ gives rise to a morphism of fundamental groupoids

$$\pi(f)_\bullet\colon \pi(X)_\bullet \longrightarrow \pi(Y)_\bullet \quad .$$

EXAMPLE C.21. Let X and Y be sets. Then the set maps from X to Y are the same as the groupoid morphisms from X to Y.

EXAMPLE C.22. If G and H are groups, then the groupoid morphisms $BG_\bullet \to BH_\bullet$ are the group homomorphisms $G \to H$.

EXAMPLE C.23. Let X be a right G-set and Y a right H-set. Then a morphism $X \rtimes G \to Y \rtimes H$ is given by a pair (ϕ, ψ), where $\phi\colon X \to Y$ and $\psi\colon X \times G \to H$, such that:

(i) for all $x \in X$ and $g \in G$, $\phi(x)\psi(x,g) = \phi(xg)$;
(ii) for all $x \in X$ and g and g' in G, $\psi(x,g)\psi(xg,g') = \psi(x,gg')$.

The pair (ϕ, ψ) induces a groupoid morphism $X \rtimes G \to Y \rtimes H$ by $\phi\colon X \to Y$ on objects and

$$X \times G \longrightarrow Y \times H, \qquad (x,g) \longmapsto (\phi(x), \psi(x,g))$$

on arrows. Every groupoid morphism $X \rtimes G \to Y \rtimes H$ comes about in this way. In particular, if $\rho\colon G \to H$ is a group homomorphism, and $\phi\colon X \to Y$ is *equivariant* with respect to ρ (i.e., $\phi(xg) = \phi(x)\rho(g)$ for $x \in X$ and $g \in G$), then (ϕ, ψ) defines a morphism of groupoids, where $\psi(x,g) = \rho(g)$ for $x \in X$, $g \in G$.

For example, for any right G-set X, the map from X to a point determines a morphism from $X \rtimes G$ to $BG_\bullet$.

EXERCISE C.22. A morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ determines a mapping $X_0/\cong \to Y_0/\cong$ of equivalence classes. It also determines a group homomorphism $\mathrm{Aut}(x) \to \mathrm{Aut}(\phi_0(x))$ for every $x \in X_0$, taking a to $\phi_1(a)$.

EXERCISE C.23. If $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is a morphism, and E is a vector bundle on $Y_\bullet$, construct a pullback vector bundle $\phi_\bullet^*(E)$ on $X_\bullet$.

EXERCISE C.24. If $X_\bullet$ and $Y_\bullet$ are equivalence relations, any map $f\colon X_0 \to Y_0$ satisfying $x \sim y \Rightarrow f(x) \sim f(y)$ determines a morphism of groupoids $X_\bullet \to Y_\bullet$, and every morphism from $X_\bullet$ to $Y_\bullet$ arises from a unique such map.

EXAMPLE C.24. If a group G acts (on the right) on a set X, there is a canonical morphism $\pi\colon X \to X \rtimes G$ from the (groupoid of the set) X to the transformation groupoid.

EXERCISE C.25. Let $F(\Gamma)_\bullet$ be the free groupoid on a graph Γ, as in Exercise C.6. For any groupoid $X_\bullet$, show that any pair of maps $V \to X_0$ and $E \to X_1$ comuting with s and t determines a morphism of groupoids from $F(\Gamma)_\bullet$ to $X_\bullet$.

EXERCISE C.26. If $X_\bullet$ and $Y_\bullet$ are groupoids, their (direct) **product** $X_\bullet \times Y_\bullet$ has objects $X_0 \times Y_0$ and arrows $X_1 \times Y_1$, with s, t, and m defined component-wise. More generally, if $X(i)_\bullet$ is a family of groupoids, one has a product groupoid $\prod X(i)_\bullet$.

Of course, morphisms of groupoids may be composed, and we get in this way a category of groupoids (with isomorphisms being the strict isomorphisms considered above). But this point of view is too narrow. In the next section we shall enlarge this category of groupoids to a 2-category.

Exercise C.27. Given morphisms $X_\bullet \to Z_\bullet$ and $Y_\bullet \to Z_\bullet$ of groupoids, construct a groupoid $V_\bullet$ with $V_0 = X_0 \times_{Z_0} Y_0$ and $V_1 = X_1 \times_{Z_1} Y_1$. Show that this is a fibered product in the category of groupoids. (This will *not* be the fibered product in the 2-category of groupoids.)

Exercise C.28. If X is a set and $Y_\bullet$ is a groupoid, a morphism from X to $Y_\bullet$ is given by a mapping of sets from X to Y_0. A morphism from $Y_\bullet$ to X is given by a mapping of sets from $Y_0/\cong$ to X. In categorical language, the functor from (Set) to (Gpd) that takes a set to its groupoid has a right adjoint from (Gpd) to (Set) that takes $Y_\bullet$ to Y_0, and it has a left adjoint from (Gpd) to (Set) that takes $Y_\bullet$ to $Y_0/\cong$.

3. 2-Isomorphisms

Definition C.25. Let $\phi_\bullet$ and $\psi_\bullet$ be morphisms of groupoids from $X_\bullet$ to $Y_\bullet$. A **2-isomorphism** from $\phi_\bullet$ to $\psi_\bullet$ is a mapping $\theta\colon X_0 \to Y_1$ satisfying the following properties:

(1) for all $x \in X_0$: $s(\theta(x)) = \phi_0(x)$ and $t(\theta(x)) = \psi_0(x)$;
(2) for all $a \in X_1$: $\theta(s(a)){\cdot}\psi_1(a) = \phi_1(a){\cdot}\theta(t(a))$.

If $x \xrightarrow{a} y$, we therefore have a commutative diagram

$$\begin{tikzcd} \phi_0(x) \arrow[r, "\phi_1(a)"] \arrow[d, "\theta(x)"'] & \phi_0(y) \arrow[d, "\theta(y)"] \\ \psi_0(x) \arrow[r, "\psi_1(a)"] & \psi_0(y) \end{tikzcd}$$

In the language of categories, this says exactly that θ is a natural isomorphism from the functor $\phi_\bullet$ to the functor $\psi_\bullet$. We write $\theta\colon \phi_\bullet \Rightarrow \psi_\bullet$ to mean that θ is a 2-isomorphism from $\phi_\bullet$ to $\psi_\bullet$.

Example C.26. Consider two continuous maps $f, g\colon X \to Y$ of topological spaces and the groupoid morphisms $\pi(f)_\bullet, \pi(g)_\bullet\colon \pi(X)_\bullet \to \pi(Y)_\bullet$ they induce. Every homotopy $H\colon X \times [0,1] \to Y$ from f to g induces a 2-isomorphism $\pi(H)\colon \pi(f)_\bullet \Rightarrow \pi(g)_\bullet$, which assigns to x in X the homotopy class of the path $t \mapsto H(x,t)$ in Y.

Exercise C.29. Verify that this is a 2-isomorphism from $\pi(f)_\bullet$ to $\pi(g)_\bullet$.

Definition C.27. For a groupoid morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ define the 2-isomorphism $1_{\phi_\bullet}\colon \phi_\bullet \Rightarrow \phi_\bullet$ by $x \mapsto e(\phi_0(x))$ from X_0 to Y_1. For $\phi_\bullet, \psi_\bullet, \chi_\bullet$ from $X_\bullet$ to $Y_\bullet$, and $\alpha\colon \phi_\bullet \Rightarrow \psi_\bullet$ and $\beta\colon \psi_\bullet \Rightarrow \chi_\bullet$, define $\beta \circ \alpha\colon \phi_\bullet \Rightarrow \chi_\bullet$ by the formula $x \to \alpha(x){\cdot}\beta(x)$.

Exercise C.30. Show that these definitions define 2-morphisms. Prove that composition is associative, the identities defined behave as identities with respect to composition of 2-isomorphisms, and that every 2-isomorphism is invertible. Conclude that for given groupoids $X_\bullet$ and $Y_\bullet$ the morphisms from $X_\bullet$ to $Y_\bullet$ together with the 2-isomorphisms between them form a groupoid, denoted

$$\mathrm{HOM}(X_\bullet, Y_\bullet).$$

EXAMPLE C.28. The only 2-isomorphisms between set maps are identities. For sets X, Y, the groupoid $\mathrm{HOM}(X,Y)$ is the set $\mathrm{Hom}(X,Y)$ of maps from X to Y.

EXERCISE C.31. If Y is a set, then $\mathrm{HOM}(X_\bullet, Y)$ is strictly isomorphic to the set $\mathrm{Hom}(X_0/\cong, Y)$ of maps from $X_0/\cong$ to Y. If $Y_\bullet$ is rigid, then $\mathrm{HOM}(X_\bullet, Y_\bullet)$ is also rigid. If X is a set, then $\mathrm{HOM}(X, Y_\bullet)$ is strictly isomorphic to the groupoid $U_\bullet$ with U_0 the set of maps from X to Y_0 and U_1 the set of maps from X to Y_1.

In particular, for any groupoid $X_\bullet$ there is a canonical morphism

$$\pi\colon X_0 \to X_\bullet$$

from the set X_0 to the groupoid $X_\bullet$. Although this map can be regarded as an inclusion, we will see that it acts more like a projection. There is also a canonical morphism, called the *canonical map*,

$$\rho\colon X_\bullet \to X_0/\cong$$

from the groupoid $X_\bullet$ to the set $X_0/\cong$.

EXERCISE C.32. Let $X_\bullet$ and $Y_\bullet$ be equivalence relations and $f_\bullet, g_\bullet\colon X_\bullet \to Y_\bullet$ morphisms, given by $f_0, g_0\colon X_0 \to Y_0$. There exists a 2-isomorphism $\theta\colon f_\bullet \Rightarrow g_\bullet$ if and only if $f_0(x) \sim g_0(x)$ for all $x \in X_0$, and such a 2-isomorphism is unique if it exists. It follows that the groupoid $\mathrm{HOM}(X_\bullet, Y_\bullet)$ is an equivalence relation, whose set of equivalence classes has a canonical bijection with the set of maps from $X_0/\cong$ to $Y_0/\cong$.

EXAMPLE C.29. Let G and H be groups, $\phi, \psi\colon G \to H$ group homomorphisms. Denote by $\phi_\bullet$ and $\psi_\bullet$ the associated morphisms of groupoids $BG_\bullet \to BH_\bullet$. The 2-isomorphisms from $\phi_\bullet$ to $\psi_\bullet$ are the elements $h \in H$ satisfying $\psi(g) = h^{-1}\phi(g)h$, for all $g \in G$.

The groupoid $\mathrm{HOM}(BG_\bullet, BH_\bullet)$ is strictly isomorphic to the transformation groupoid $\mathrm{Hom}(G,H) \rtimes H$, where H acts on the group homomorphisms from G to H by conjugation $(\phi \cdot h)(g) = h^{-1}\phi(g)h$.

EXAMPLE C.30. Given a G-set X and an H-set Y, and two morphisms (ϕ, ψ) and (ϕ', ψ') from $X \rtimes G$ to $Y \rtimes H$, as in Exercise C.23, a 2-isomorphism from the former to the latter is a map $\theta\colon X \to H$ satisfying: (i) $\phi'(x) = \phi(x)\theta(x)$ for all $x \in X$; (ii) $\psi'(x,g) = \theta(x)^{-1}\psi(x,g)\theta(xg)$ for all $x \in X$ and $g \in G$. In the equivariant case, where $\psi(x,g) = \rho(g)$ and $\psi'(x,g) = \rho'(g)$ for group homomorphisms ρ and ρ' from G to H, the second condition becomes $\rho'(g) = \theta(x)^{-1}\rho(g)\theta(x)$ for all x and g. Show that (ϕ, ψ) is 2-isomorphic to an equivariant map exactly when there is a map $\theta\colon X \to H$ such that for all $g \in G$, the map $x \mapsto \theta(x)^{-1}\psi(x,g)\theta(xg)$ is independent of $x \in X$. [Are there cases where every morphism $X \rtimes G \to Y \rtimes H$ is 2-isomorphic to an equivariant map?]

EXERCISE C.33. We have seen that a morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ determines a homomorphism from $\mathrm{Aut}(x)$ to $\mathrm{Aut}(\phi_0(x)$ for every $x \in X_0$. A 2-isomorphism $\theta\colon \phi_\bullet \Rightarrow \psi_\bullet$ determines an isomorphism $\mathrm{Aut}(\phi_0(x)) \to \mathrm{Aut}(\psi_0(x))$, taking g to $\theta(x)^{-1}{\cdot}g{\cdot}\theta(x)$. This gives a commutative diagram

$$\begin{tikzcd} & \mathrm{Aut}(\phi_0(x)) \arrow[dd] \\ \mathrm{Aut}(x) \arrow[ur] \arrow[dr] & \\ & \mathrm{Aut}(\psi_0(x)) \end{tikzcd}$$

DEFINITION C.31. Given $\phi_\bullet, \phi'_\bullet \colon X_\bullet \to Y_\bullet$, $\alpha \colon \phi_\bullet \Rightarrow \phi'_\bullet$, and $\psi_\bullet, \psi'_\bullet \colon Y_\bullet \to Z_\bullet$, $\beta \colon \psi_\bullet \Rightarrow \psi'_\bullet$, there is a 2-isomorphism $\beta * \alpha$ from $\psi_\bullet \circ \phi_\bullet$ to $\psi'_\bullet \circ \phi'_\bullet$, that maps x in X_0 to

$$\psi_1(\alpha(x)) \boldsymbol{\cdot} \beta(\phi'_0(x)) = \beta(\phi_0(x)) \boldsymbol{\cdot} \psi'_1(\alpha(x))$$

in Z_1.

EXERCISE C.34. Verify that this defines a 2-isomorphism as claimed. Verify that groupoids, morphisms, and 2-isomorphisms form a 2-category, i.e., that the axioms of Appendix B, §2 are satisfied.

EXERCISE C.35. Let $I_\bullet$ be the banal groupoid $\{0,1\} \times \{0,1\} \rightrightarrows \{0,1\}$. For any groupoids $X_\bullet$ and $Y_\bullet$, construct a bijection between the morphisms

$$X_\bullet \times I_\bullet \longrightarrow Y_\bullet$$

and the triples $(\phi_\bullet, \psi_\bullet, \theta)$, where $\phi_\bullet$ and $\psi_\bullet$ are morphisms from $X_\bullet$ to $Y_\bullet$ and θ is a 2-isomorphism from $\phi_\bullet$ to $\psi_\bullet$.

4. Isomorphisms

DEFINITION C.32. A morphism of groupoids $\phi_\bullet \colon X_\bullet \to Y_\bullet$ is an **isomorphism of groupoids** if there exists a morphism $\psi_\bullet \colon Y_\bullet \to X_\bullet$, such that $\psi_\bullet \circ \phi_\bullet \cong \mathrm{id}_{X_\bullet}$ and $\phi_\bullet \circ \psi_\bullet \cong \mathrm{id}_{Y_\bullet}$, where '$\cong$' means the existence of a 2-isomorphism between the morphisms.

EXAMPLE C.33. Homotopy equivalent topological spaces have isomorphic fundamental groupoids: a homotopy equivalence $f \colon X \to Y$ determines an isomorphism $\pi(f)_\bullet \colon \pi(X)_\bullet \to \pi(Y)_\bullet$.

EXERCISE C.36. Let X be a path connected topological space and $x \in X$ a base point. Let $G = \pi_1(X, x)$ be the fundamental group of X. Then the fundamental groupoid $\pi(X)_\bullet$ is isomorphic to $BG_\bullet$.

EXERCISE C.37. Prove that every transitive groupoid is isomorphic to a groupoid of the form $BG_\bullet$, for a group G. Every groupoid is isomorphic to a disjoint union $\coprod BG(i)_\bullet$, for some groups $G(i)$.

EXERCISE C.38. Let $X_\bullet$ be an equivalence relation, and let $Y = X_0/\cong$ be the set of equivalence classes. (a) Show that the canonical map $X_\bullet \to Y$ is an isomorphism of groupoids. In particular, if a group G acts freely on a set X, the transformation groupoid $X \rtimes G$ is isomorphic to the set of orbits. (b) Show that if Z is any set, an isomorphism $X_\bullet \to Z$ determines a bijection between $Y = X_0/\cong$ and Z.

Exercise C.39. If $X_\bullet$ and $Y_\bullet$ are isomorphic groupoids, show that $X_\bullet$ is rigid (resp. transitive) if and only if $Y_\bullet$ is rigid (resp. transitive).

Exercise C.40. A groupoid is rigid if and only if it is isomorphic to a set.

Exercise C.41. If $\phi_\bullet\colon X_\bullet \to Y_\bullet$ and $\psi_\bullet\colon Y_\bullet \to Z_\bullet$ are isomorphisms, then the composition $\psi_\bullet \circ \phi_\bullet\colon X_\bullet \to Z_\bullet$ is an isomorphism.

Exercise C.42. Suppose a set X has a left action of a group G and a right action of a group H, and these actions commute. Show that, if both actions are free, then the groupoids $G \ltimes (X/H)$ and $(G\backslash X) \rtimes H$ are isomorphic. For example, if H is a subgroup of a group G, then the groupoid $BH_\bullet$ is isomorphic to $G \ltimes (G/H)$.

Proposition C.34. *A morphism of groupoids $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is an isomorphism if and only if it satisfies the following two conditions:*

(1) *For every $x, x' \in X_0$ and $b \in Y_1$ with $s(b) = \phi_0(x)$ and $t(b) = \phi_0(x')$, there is a unique $a \in X_1$ with $s(a) = x$, $t(a) = x'$, and $\phi_1(a) = b$. That is, the diagram*

$$\begin{CD} X_1 @>{(s,t)}>> X_0 \times X_0 \\ @V{\phi_1}VV @VV{\phi_0\times\phi_0}V \\ Y_1 @>{(s,t)}>> Y_0 \times Y_0 \end{CD}$$

is a cartesian diagram of sets;

(2) *For every $y \in Y_0$, there is an $x \in X_0$ and a $b \in Y_1$ with $\phi_0(x) = s(b)$ and $t(b) = y$. That is, the map*

$$X_0 \;{}_{\phi_0}\!\times_{Y_0,s} Y_1 \to Y_0,$$

taking (x, b) to $t(b)$ is surjective. Equivalently, the induced map $X_0/\!\cong\, \to Y_0/\!\cong$ is surjective.

In the language of categories, the first condition says exactly that the functor $\phi_\bullet$ is faithful and full, and the second condition says that it is essentially surjective. A morphism of groupoids satisfying the first condition is said to be **injective**, and one satisfying the second will be called **surjective**.

The proof is largely left as an exercise, as it is the same as the corresponding result in category theory (Apendix B, §1). We remark only that the essential step in constructing a morphism $\psi_\bullet\colon Y_\bullet \to X_\bullet$ back is to *choose*, for each $y \in Y_0$, an $x_y \in X_0$ *and* a $b_y \in Y_1$ with $s(b_y) = \phi_0(x_y)$ and $t(b_y) = y$. Then set $\psi_0(y) = x_y$, and, for c in Y_1, set $\psi_1(c)$ to be the arrow from $\psi_0(s(c))$ to $\psi_0(t(c))$ such that $\phi_1(\psi_1(c)) = b_{s(c)}{\cdot}c{\cdot}{b_{t(c)}}^{-1}$.

Note that the second condition is automatic whenever ϕ_0 is surjective. In this case one need only choose x_y in X_0 with $\phi_0(x_y) = y$, and then one can take $b_y = e(y)$.

Remark C.35. The choices in this proof are important, not so much to point out the necessary use of an axiom of choice, but because they show that the inverse of an isomorphism may be far from canonical. This has serious consequences when the groupoids have a geometric structure on them. Set theoretic surjections have sections (by the axiom of choice). But geometric surjections, even nice ones like projections of fiber bundles, do not generally have sections. In particular, the classification of geometric groupoids is not as simple as it is for set-theoretic groupoids:

COROLLARY C.36. *Every groupoid is isomorphic to a family of groups as in Example C.9.*

EXERCISE C.43. A morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is an isomorphism if and only if the induced map $X_0/\cong \to Y_0/\cong$ is bijective and the induced maps $\mathrm{Aut}(x) \to \mathrm{Aut}(\phi_0(x))$ are isomorphisms for all $x \in X_0$.

EXERCISE C.44. If $X_\bullet$ and $Y_\bullet$ are isomorphic groupoids, show that $X_\bullet$ is finite (resp. tame) if and only if $Y_\bullet$ is finite (resp. tame), in which case they have the same mass.

EXERCISE C.45. A groupoid is rigid if and only if it is isomorphic to a set.

EXERCISE C.46. A banal groupoid is isomorphic to a point $pt \rightrightarrows pt$.

EXERCISE C.47. Suppose a set X has a left action of a group G and a right action of a group H, and these actions commute. Show that the canonical morphisms from the double transformation groupoid $G \ltimes X \rtimes H$ to $G \ltimes (X/H)$ (resp. $(G\backslash X) \rtimes H$) is an isomorphism if and only if the action of H (resp. G) on X is free. Deduce the result of Exercise C.42.

EXERCISE C.48. Construct a groupoid $X_\bullet$ from a set Z as in Exercise C.5. Let G be the group of bijections of Z with itself. There is a canonical surjective morphism from $G \ltimes X_0$ to $X_\bullet$, taking (σ, A) to $(A, \sigma(A), \sigma|_A)$. For which Z is this an isomorphism?

EXERCISE C.49. Any linear map $L\colon V \to W$ of vector spaces (or abelian groups) determines an action of V on W by translation: $v \cdot w = L(v) + w$, and so we have the transformation groupoid $V \ltimes W$. If $L'\colon V' \to W'$ is another, a pair of linear maps $\phi_V\colon V \to V'$, $\phi_W\colon W \to W'$, such that $L' \circ \phi_V = \phi_W \circ L$ determines a homomorphism $\phi_\bullet\colon V \ltimes W \to V' \ltimes W'$. (a) Show that $\phi_\bullet$ is an isomorphism if and only if the induced maps $\mathrm{Ker}(L) \to \mathrm{Ker}(L')$ and $\mathrm{Coker}(L) \to \mathrm{Coker}(L')$ are isomorphisms. (b) Show that $V \ltimes W$ is isomorphic to the groupoid $\mathrm{Ker}(L) \ltimes \mathrm{Coker}(L)$ (with the trivial action), which is isomorphic to the product of $B(\mathrm{Ker}(L))_\bullet$ and the set $\mathrm{Coker}(L)$.

EXERCISE C.50. If a group G acts on the right on groupoids $X_\bullet$ and $Y_\bullet$, a morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is G-**equivariant** if ϕ_0 and ϕ_1 are G-equivariant. There is then an induced morphism $X_\bullet \rtimes G \to Y_\bullet \rtimes G$. Show that, if $\phi_\bullet$ is an isomorphism, then this induced morphism is also an isomorphism.

5. Fibered products

Let

$$\begin{tikzcd} & Y_\bullet \arrow[d, "\psi_\bullet"] \\ X_\bullet \arrow[r, "\phi_\bullet"] & Z_\bullet \end{tikzcd}$$

be a diagram of groupoids. We shall construct

(i) a groupoid $W_\bullet$;
(ii) two morphisms of groupoids $p_\bullet\colon W_\bullet \to X_\bullet$ and $q_\bullet\colon W_\bullet \to Y_\bullet$;
(iii) a 2-isomorphism θ between the compositions $W_\bullet \to X_\bullet \to Z_\bullet$ and $W_\bullet \to Y_\bullet \to Z_\bullet$.

The data $(W_\bullet, p_\bullet, q_\bullet, \theta)$ will be called the **fibered product** of $X_\bullet$ and $Y_\bullet$ over $Z_\bullet$, notation $W_\bullet = X_\bullet \times_{Z_\bullet} Y_\bullet$.

(1)
$$\begin{tikzcd} W_\bullet \arrow[r, "q_\bullet"] \arrow[d, "p_\bullet"'] \arrow[dr, phantom, "\theta\nearrow"] & Y_\bullet \arrow[d, "\psi_\bullet"] \\ X_\bullet \arrow[r, "\phi_\bullet"'] & Z_\bullet \end{tikzcd}$$

The objects of $W_\bullet$ are triples (x, y, c), where x and y are objects of $X_\bullet$ and $Z_\bullet$, respectively, and c is a morphism in $Z_\bullet$, between $\phi_0(x)$ and $\psi_0(y)$:

$$\phi_0(x) \xrightarrow{c} \psi_0(y)$$

Given two such objects (x, y, c) and (x', y', c') define a morphism from (x, y, c) to (x', y', c') to be a pair (a, b), $x \xrightarrow{a} x'$, $y \xrightarrow{b} y'$, such that

$$\begin{tikzcd} \phi_0(x) \arrow[r, "\phi_1(a)"] \arrow[d, "c"'] & \phi_0(x') \arrow[d, "c'"] \\ \psi_0(y) \arrow[r, "\psi_1(b)"] & \psi_0(y') \end{tikzcd}$$

commutes in $Z_\bullet$. Composition in $W_\bullet$ is induced from composition in $X_\bullet$ and $Y_\bullet$.

The two projections $p_\bullet$ and $q_\bullet$ are defined by projecting onto the first and second components, respectively (both on objects and morphisms).

To define $\theta\colon \phi_\bullet \circ p_\bullet \to \psi_\bullet \circ q_\bullet$, take $\theta\colon X_0 \to Y_1$ to be the map $(x, y, c) \mapsto c$.

This fibered product satisfies a **universal mapping property**: given a groupoid $V_\bullet$ and two morphisms $f_\bullet\colon V_\bullet \to X_\bullet$ and $g_\bullet\colon V_\bullet \to Y_\bullet$, together with a 2-isomorphism $\tau\colon \phi_c om \circ f_\bullet \Rightarrow \psi_c om \circ g_\bullet$, there is a unique morphism $h_\bullet = (f_\bullet, g_\bullet)\colon V_\bullet \to X_\bullet \times_{Z_\bullet} Y_\bullet$ such that $f_\bullet = p_\bullet \circ h_\bullet$, $g_\bullet = q_\bullet \circ h_\bullet$, and τ is determined from θ by $\tau = \theta * 1_{h_\bullet}$. In fact, $h_\bullet$ is defined by $h_0(v) = (f_0(v), g_0(v), \tau(v))$ for $v \in V_0$, and $h_1(d) = (f_1(d), g_1(d))$ for $d \in V_1$.

A **2-commutative diagram**

$$\begin{tikzcd} V_\bullet \arrow[r, "g_\bullet"] \arrow[d, "f_\bullet"'] \arrow[dr, phantom, "\tau\nearrow"] & Y_\bullet \arrow[d, "\psi_\bullet"] \\ X_\bullet \arrow[r, "\phi_\bullet"'] & Z_\bullet \end{tikzcd}$$

means that a 2-isomorphism τ from $\phi_\bullet \circ f_\bullet$ to $\psi_\bullet \circ g_\bullet$ is specified. It **strictly commutes** in case $\phi_\bullet \circ f_\bullet = \psi_\bullet \circ g_\bullet$. In this case the 2-isomorphism is taken to be $\epsilon\colon V_0 \to Z_1$ given by $\epsilon = e \circ \phi_o \circ f_0 = e \circ \psi_o \circ g_0$.

Exercise C.51. Show that a 2-commutative diagram strictly commutes exactly when the 2-isomorphism $\theta\colon V_0 \to Z_1$ factors through Z_0, i.e., $\theta = e \circ \theta_0$ for some map $\theta_0\colon V_0 \to Z_0$.

The diagram is called **2-cartesian** if it is 2-commutative and the induced mapping $(f_\bullet, g_\bullet)\colon V_\bullet \to X_\bullet \times_{Z_\bullet} Y_\bullet$ is an isomorphism. Such a $V_\bullet$ will not satisfy the same universal property as the fibered product we have constructed; but it does satisfy a universal property in an appropriate 2-categorical sense (see Exercise C.56). The universal property just described is easier to use in practise.

The diagram is called **strictly 2-cartesian** if the induced mapping $(f_\bullet, g_\bullet)\colon V_\bullet \to X_\bullet \times_{Z_\bullet} Y_\bullet$ is a strict isomorphism.

Example C.37. Let X be a right G-set and $X \rtimes G$ the associated transformation groupoid. Consider the diagram

$$\begin{tikzcd} X \times G \arrow[r, "\sigma"] \arrow[d, "p_1"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & X \rtimes G, \end{tikzcd}$$

where σ is the action map and π is the canonical map. This diagram does not strictly commute, so we consider the 2-isomorphism $\eta\colon \pi \circ p_1 \to \pi \circ \sigma$ given by the identity map on $X \times G$. This gives a 2-commutative diagram

(2)
$$\begin{tikzcd} X \times G \arrow[r, "\sigma"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\eta\Nearrow"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & X \rtimes G, \end{tikzcd}$$

and it is not difficult to see that the corresponding map from the set $X \times G$ to the fibered product $X \times_{X \rtimes G} X$ is a *strict* isomorphism. Thus $X \rtimes G$ can be considered to be a quotient of X by G, but a much better quotient than the set-theoretic quotient, because the set-theoretic quotient does not make the corresponding Diagram (2) a cartesian diagram of sets (or groupoids).

Remark C.38. Diagram (2) also has a 'dual' property, which expresses the fact that $X \rtimes G$ is a quotient of X by the action of G. This property is that for every set S and every morphism $f\colon X \to S$, such that $f \circ p_1 = f \circ \sigma$ there exists a unique morphism $\overline{f}\colon X \rtimes G \to S$ such that $\overline{f} \circ \pi = f$:

$$\begin{tikzcd} X \times G \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "\pi"] \arrow[dr, "f"'] & X \rtimes G \arrow[d, dotted, "\overline{f}"] \\ & & S \end{tikzcd}$$

We refer to this property as the *cocartesian* property of Diagram (2). There is also a more complicated version of this property for an arbitrary groupoid in place of the set S, for which we refer to Exercise C.53.

Exercise C.52. Generalize the previous example by replacing the transformation groupoid $X \rtimes G$ by an arbitrary groupoid $X_\bullet$. In other words, construct a 2-cartesian diagram

$$\begin{tikzcd} X_1 \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\eta}{\nearrow}"] & X_0 \arrow[d, "\pi"] \\ X_0 \arrow[r, "\pi"] & X_\bullet . \end{tikzcd}$$

Show in fact that X_1 is strictly isomorphic to the fibered product $X_0 \times_{X_\bullet} X_0$. This diagram also has a cocartesian property with respect to maps into sets S.

Exercise C.53. Show that for any groupoid $X_\bullet$, the morphism $\pi\colon X_0 \to X_\bullet$ makes $X_\bullet$ a 2-quotient of X_0 in the 2-category (Gpd) (in the sense of Definition B.17).

Exercise C.54. If $X_\bullet = X$ and $Y_\bullet = Y$ are sets, so $\phi_\bullet$ and $\psi_\bullet$ are given by maps $f\colon X \to Z_0$ and $g\colon Y \to Z_0$, then the fibered product $X \times_{Z_\bullet} Y$ is strictly isomorphic to the set

$$W = \{\,(x, y, c) \in X \times Y \times Z_1 \mid s(c) = f(x) \text{ and } t(c) = g(y)\,\}.$$

In the preceding exercise, if $Y = X$ and $g = f$, one gets a 2-cartesian diagram

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\nearrow"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Z_\bullet , \end{tikzcd}$$

with $W = \{(y_1, y_2, a) \in Y \times Y \times Z_1 \mid f(y_1) \xrightarrow{a} f(y_2)\}$, and $\theta\colon W \to Z_1$ is the third projection.

Example C.39. Let $W_\bullet$ be the fibered product $(X \rtimes G) \times_{BG_\bullet} pt$. From the construction of the fibered product we can identify W_0 with $X \times G$ and W_1 with $X \times G \times G$, with $s(x, g, h) = (x, gh)$, $t(x, g, h) = (xg, h)$, and $(x, g, h)\cdot(xg, g', h') = (x, gg', h')$.

Exercise C.55. Show that the canonical morphism $\alpha_\bullet\colon X \to W_\bullet$, defined by $\alpha_0(x) = (x, e)$ and $\alpha_1(x) = (x, e, e)$, satisfies the conditions of Proposition C.34, so $\alpha_\bullet$ is an isomorphism. Thus the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\pi"'] & pt \arrow[d] \\ X \rtimes G \arrow[r] & BG_\bullet \end{tikzcd}$$

is 2-cartesian. Construct a morphism $\beta_\bullet\colon W_\bullet \to X$ by the formulas $\beta_0(x, g) = xg$ and $\beta_1(x, g, h) = xgh$. Verify that $\beta_\bullet \circ \alpha_\bullet = 1_X$, and construct a 2-isomorphism from $\alpha_\bullet \circ \beta_\bullet$ to $1_{W_\bullet}$.

Note that $X \to X \rtimes G$ is the "general" quotient by G. Thus we see that every quotient by G is a pullback from the quotient of pt by G (which is $BG_\bullet$). This justifies calling $pt \to BG_\bullet$ the *universal* quotient by G.

EXERCISE C.56. $^{(*)}$ Show that a 2-commutative diagram is 2-cartesian as defined here if and only if it is 2-cartesian in the the 2-category of groupoids, i.e., it satisfies the universal property of Appendix B, Definition B.17.

Note how this universal mapping property characterizes the fibered product $W_\bullet$ up to an isomorphism which is unique up to a unique 2-isomorphism. This is the natural analogue in a 2-category of the usual 'unique up to unique isomorphism' in an ordinary category.

5.1. Square morphisms.

DEFINITION C.40. A morphism of groupoids $\phi_\bullet\colon X_\bullet \to Y_\bullet$ is called **square** if the diagrams

$$\begin{tikzcd} X_1 \arrow[r, "s"] \arrow[d, "\phi_1"'] & X_0 \arrow[d, "\phi_0"] \\ Y_1 \arrow[r, "s"] & Y_0 \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "t"] \arrow[d, "\phi_1"'] & X_0 \arrow[d, "\phi_0"] \\ Y_1 \arrow[r, "t"] & Y_0 \end{tikzcd}$$

are cartesian diagrams of sets. Since s and t are obtained from each other by the involution i, it suffices to verify that one of these diagrams is cartesian.

EXERCISE C.57. The morphism $X \rtimes G \to BG_\bullet$ of Example C.23 is square.

EXERCISE C.58. If $X_\bullet$ is a groupoid, then any square morphism $X_\bullet \to BG_\bullet$ makes $X_\bullet$ strictly isomorphic to a transformation groupoid associated to an action of G on X_0.

5.2. Restrictions and Pullbacks.

DEFINITION C.41. Let $X_\bullet$ be a groupoid, Y_0 a set and $\phi_0\colon Y_0 \to X_0$ a map. Define Y_1 to be the fibered product (of sets)

$$\begin{tikzcd} Y_1 \arrow[rr, "(s,t)"] \arrow[d, "\phi_1"'] & & Y_0 \times Y_0 \arrow[d, "\phi_0\times\phi_0"] \\ X_1 \arrow[rr, "(s,t)"] & & X_0 \times X_0. \end{tikzcd}$$

So an element of Y_1 is a triple $(y, y', a) \in Y_0 \times Y_0 \times X_1$ with $\phi_0(y) \xrightarrow{a} \phi_0(y')$. Define the structure of a groupoid on $Y_\bullet$ by the rule

$$(y, y', a){\bullet}(y', y'', b) = (y, y'', a{\bullet}b).$$

We get an induced morphism of groupoids $\phi_\bullet\colon Y_\bullet \to X_\bullet$, defined by $\phi_1(y, y', a) = a$.

The groupoid $Y_\bullet$ is called the **restriction** of $X_\bullet$ via $Y_0 \to X_0$; following [**50**], it may be denoted $X_\bullet|_{Y_0}$.[4]

[4][The word "pullback" and the notation $\phi_0^*(X_\bullet)$ might seem more appropriate, since "restriction" connotes some kind of subobject, but the word pullback is used for another concept.]

Note that by construction, $Y_\bullet \to X_\bullet$ is injective (full and faithful). It is an isomorphism exactly when the image of the map $Y_0 \to X_0$ intersects all isomorphism classes of $X_\bullet$, by Proposition C.34.

EXAMPLE C.42. Let X be a right G-set and $U \subset X$ a subset. The restriction of $X \rtimes G$ to U is not a transformation groupoid unless U is G-invariant. Thus we see that very natural constructions can lead out of the world of group actions.

EXAMPLE C.43. If $\pi(X)_\bullet$ is the fundamental group of a topological space X, and A is a subset of X, then the restriction of $\pi(X)_\bullet$ to A is the groupoid $\pi(X, A)_\bullet$.

EXERCISE C.59. Show that any morphism : $X_\bullet \to Y_\bullet$ factors canonically into $X_\bullet \to Y'_\bullet \to Y_\bullet$, with $X_0 \to Y'_0$ injective, and $Y'_\bullet \to Y_\bullet$ an isomorphism.

DEFINITION C.44. Let $X_\bullet$ be a groupoid, and $f \colon X_0 \to Z$ a map to a set Z such that $f \circ s = f \circ t$. For any map $Z' \to Z$, construct a **pullback** groupoid $X'_\bullet$ by setting $X'_0 = X_0 \times_Z Z'$, $X'_1 = X_1 \times_Z Z'$, with s' and t' induced by s and t, as is m' from m, by means of the isomorphism $X'_1 \;{}_{t'}{\times}_{X'_0,s'}\; X'_1 \cong (X_1 \;{}_{t}{\times}_{X_0,s}\; X_1) \times_Z Z'$.

EXERCISE C.60. Verify that $X'_\bullet$ is a groupoid. Show that the induced morphism $X'_\bullet \to X_\bullet$ is square.

5.3. Representable and gerbe-like morphisms.

DEFINITION C.45. A morphism $\phi_\bullet \colon X_\bullet \to Y_\bullet$ of groupoids is called **representable** if the induced mapping

$$(s, t, \phi_1) \colon X_1 \longrightarrow (X_0 \times X_0) \times_{Y_0 \times Y_0} Y_1$$

is injective; that is, $\phi_\bullet$ is faithful as a functor between categories. The morphism is said to be **gerbe-like** if this map (s, t, ϕ_1) is surjective, and the induced map $X_0/\cong \to Y_0/\cong$ is surjective; that is, $\phi_\bullet$ is a full and essentially surjective functor. So a representable and gerbe-like morphism is an isomorphism.

For any groupoid $X_\bullet$, the canonical morphism $X_0 \to X_\bullet$ is representable (but not usually injective). If $X'_\bullet$ is a pullback of $X_\bullet$, as defined in the last section, the map $X'_\bullet \to X_\bullet$ is representable.

The canonical morphism from $X_\bullet$ to $X_0/\cong$ is gerbe-like. Any surjective homomorphism $G \to H$ of groups determines a gerbe-like homomorphism $BG_\bullet \to BH_\bullet$.

EXERCISE C.61. Let $\phi_\bullet \colon X_\bullet \to Y_\bullet$ be a morphism of groupoids. The following are equivalent:

(i) $\phi_\bullet$ is representable;
(ii) For any set T and morphism $T \to Y_\bullet$, the fibered product $X_\bullet \times_{Y_\bullet} T$ is rigid;
(iii) For any rigid groupoid $T_\bullet$ and morphism $T_\bullet \to Y_\bullet$, the fibered product $X_\bullet \times_{Y_\bullet} T_\bullet$ is rigid;
(iv) For any 2-cartesian diagram

$$\begin{tikzcd} S_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha \nearrow"] & T_\bullet \arrow[d] \\ X_\bullet \arrow[r] & Y_\bullet \end{tikzcd}$$

with $T_\bullet$ rigid, $S_\bullet$ is also rigid.

(v) For any set T and morphism $T \to Y_\bullet$, there is a set S and a 2-cartesian diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & T \arrow[d] \\ X_\bullet \arrow[r] & Y_\bullet. \end{tikzcd}$$

Exercise C.62. Show that any morphism $X_\bullet \to Y_\bullet$ factors canonically into a gerbe-like morphism $X_\bullet \to Z_\bullet$ followed by a representable morphism $Z_\bullet \to Y_\bullet$.

Exercise C.63. For a morphism $\phi_\bullet \colon X_\bullet \to Y_\bullet$ of groupoids, show that the following are equivalent:

(i) $\phi_\bullet$ is gerbe-like;

(ii) For any morphism $pt \to Y_\bullet$ (given by $y \in Y_0$), the fibered product $X_\bullet \times_{Y_\bullet} pt$ is non-empty and transitive.

(iii) For any morphism $\psi_\bullet \colon\ pt \to Y_\bullet$, there is a group G and a 2-cartesian diagram

$$\begin{tikzcd} BG_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & pt \arrow[d, "\psi_\bullet"] \\ X_\bullet \arrow[r, "\phi_\bullet"] & Y_\bullet \end{tikzcd}$$

6. Simplicial constructions

We fix a groupoid $X_\bullet$ and explain several constructions of new groupoids out of $X_\bullet$. For any integer $n \geq 1$, denote by X_n the set of n composable morphisms in $X_\bullet$, i.e.,

$$X_n \;=\; \{\, (a_1, \ldots, a_n) \in (X_1)^n \mid t(a_i) = s(a_{i+1}) \text{ for } 1 \leq i < n \,\} :$$

$$* \xrightarrow{a_1} * \xrightarrow{a_2} \cdots \xrightarrow{a_n} *$$

6.1. Groupoid of diagrams. Let $X_\bullet$ be a groupoid. Define a new groupoid $X_\bullet\{n\}$, for $n \geq 1$ as follows. An object of $X_\bullet\{n\}$ is an n-tuple of composable arrows in $X_\bullet$, i.e., an element of X_n. A morphism in $X_\bullet\{n\}$ from $(a_1, \ldots, a_n) \in X_n$ to $(b_1, \ldots, b_n) \in X_n$ is a commutative diagram in $X_\bullet$

$$\begin{tikzcd} * \arrow[r, "a_1"] \arrow[d, "\phi_0"'] & * \arrow[r, "a_2"] \arrow[d, "\phi_1"'] & \cdots \arrow[r, "a_n"] & * \arrow[d, "\phi_n"] \\ * \arrow[r, "b_1"] & * \arrow[r, "b_2"] & \cdots \arrow[r, "b_n"] & * \end{tikzcd}$$

i.e., an $(n+1)$-tuple $(\phi_0, \ldots, \phi_n)$ of arrows in $X_\bullet$ such that $\phi_{i-1}{\cdot}b_i = a_i{\cdot}\phi_i$, for all $i = 1, \ldots, n$.

Composition in $X_\bullet\{n\}$ is defined by composing vertically:

$$(\phi_0, \ldots, \phi_n){\cdot}(\psi_0, \ldots, \psi_n) = (\phi_0{\cdot}\psi_0, \ldots, \phi_n{\cdot}\psi_n).$$

We call the groupoid $X_\bullet\{n\}$ the **groupoid of n-diagrams** of $X_\bullet$.

Exercise C.64. Construct a strict isomorphism between $X_\bullet\{1\}$ and the restriction of $X_\bullet$ by the map $s\colon X_1 \to X_0$. More generally, construct a strict isomorphism between $X_\bullet\{n\}$ and the restriction of $X_\bullet$ by the map from X_n to X_0 that takes $(a_1, \ldots, a_n)$ to $s(a_1)$. Conclude that all of the groupoids $X_\bullet\{n\}$ are isomorphic to $X_\bullet$.

Exercise C.65. Define a groupoid $V_\bullet^{(n)}$ with $V_0^{(n)} = X_n$, $V_1^{(n)} = X_{2n+1}$, $s(a_1, \ldots, a_n, c, b_1, \ldots, b_n) = (a_n^{-1}, \ldots, a_1^{-1})$, and $t(a_1, \ldots, a_n, c, b_1, \ldots, b_n) = (b_1, \ldots, b_n)$. Construct a strict isomorphism between $V_\bullet^{(n)}$ and $X_\bullet\{n\}$. Deduce that $X_\bullet\{n\}\{1\}$ is strictly isomorphic to $X_\bullet\{2n+1\}$. Prove more generally that $X_\bullet\{n\}\{m\}$ is strictly isomorphic to $X_\bullet\{(n+1)(m+1)-1\}$.

Definition C.46. Define the **shift of $X_\bullet$ by n** to be the subgroupoid $X_\bullet[n]$ of $X_\bullet\{n\}$ defined by

$$\begin{aligned}
&\big(X_\bullet[n]\big)_0 = \big(X_\bullet\{n\}\big)_0 = X_n \\
&\big(X_\bullet[n]\big)_1 = \{(\phi_0, \ldots, \phi_n) \in \big(X_\bullet\{n\}\big)_1 \mid \phi_1, \ldots, \phi_n \text{ are identity morphisms}\}.
\end{aligned}$$

Exercise C.66. (1) Define a groupoid $W_\bullet^{(n)}$ by $W_0^{(n)} = X_n$, $W_1^{(n)} = X_{n+1}$, with $s(a_1, \ldots, a_{n+1}) = (a_1{\cdot}a_2, a_3, \ldots, a_{n+1})$, $t(a_1, \ldots, a_{n+1}) = (a_2, a_3, \ldots, a_{n+1})$, and

$$(a_1, \ldots, a_{n+1}){\cdot}(b_1, \ldots, b_{n+1}) = (a_1{\cdot}b_1, b_2, \ldots, b_{n+1}).$$

(2) Construct a strict isomorphism between $W_\bullet^{(n)}$ and the cross product groupoid $X_n \times_{X_{n-1}} X_n \rightrightarrows X_n$, constructed from the morphism $X_n \to X_{n-1}$ that maps $(a_1, \ldots, a_n)$ to $(a_2, \ldots, a_n)$. (3) Show that $W_\bullet^{(n)}$ is strictly isomorphic to $X_\bullet\{n\}$.

Exercise C.67. Define a morphism $X_\bullet[n+1] \to X_\bullet[n]$ by leaving out the last component. Prove that this morphism is square.

Exercise C.68. (*) For $0 \le k \le n$, and $n \ge 2$, define $d_k\colon X_n \to X_{n-1}$ by the formulas $d_0(a_1, \ldots, a_n) = (a_2, \ldots, a_n)$, $d_k(a_1, \ldots, a_n) = (a_1, \ldots, a_k{\cdot}a_{k+1}, \ldots, a_n)$ for $0 < k < n$, and $d_n(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1})$. For any $1 \le k \le n$, construct a groupoid $U_\bullet = X_\bullet(n,k)$ with $U_0 = X_{n-1}$, $U_1 = X_n$, $s = d_k$, $t = d_{k-1}$, and

$$(a_1, \ldots, a_n){\cdot}(b_1, \ldots, b_n) = (a_1, \ldots, a_{k-1}, a_k{\cdot}b_k, b_{k+1}, \ldots, b_n).$$

(1) Show that $X_\bullet(n,k)$ is strictly isomorphic to $X_\bullet(n,l)$ for any $1 \le k, l \le n$. (2) The formulas $\phi_1(a_1, \ldots, a_n) = a_k$ and $\phi_0(a_1, \ldots, a_{n-1}) = s(a_k)$ determine a morphism $\phi_\bullet\colon U_\bullet \to X_\bullet$. Show that this morphism is faithful and essentially surjective, but not usually full.

6.2. Simplicial sets.

Definition C.47. A **simplicial set** X_* specifies a set X_n of **n-simplices** for each nonnegative integer n, together with **face** maps $d_i\colon X_n \to X_{n-1}$ for $0 \le i \le n$, and **degeneracy** maps $s_i\colon X_n \to X_{n+1}$ for $0 \le i \le n$, satisfying the following identities:

(a) $d_i d_j = d_{j-1} d_i$ for $i < j$;
(b) $s_i s_j = s_{j+1} s_i$ for $i \le j$;

$$\text{(c)}\ d_i s_j = \begin{cases} s_{j-1} d_i & \text{for} \quad i < j; \\ \text{id} & \text{for} \quad i = j, j+1; \\ s_j d_{i-1} & \text{for} \quad i > j+1. \end{cases}$$

A groupoid $X_\bullet$ determines a simplicial set X_*, called the **simplicial set of the groupoid**, whose set of n-simplices is the set X_n of composable morphisms $(a_1, \ldots, a_n)$ in $X_\bullet$, with X_0 the objects of $X_\bullet$. For $n = 1$, $d_0 = t$ and $d_1 = s$ are the two maps from X_1 to X_0, and $s_0 = e$ is the map from X_0 to X_1. The general maps are defined by:

$$d_i(a_1, \ldots, a_n) = \begin{cases} (a_2, \ldots, a_n) & \text{if} \quad i = 0; \\ (a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_n) & \text{if} \quad 0 < i < n; \\ (a_1, \ldots, a_{n-1}) & \text{if } i = n. \end{cases}$$

and

$$s_i(a_1, \ldots, a_n) = \begin{cases} (1_{s(a_1)}, a_1, \ldots, a_n) & \text{if} \quad i = 0; \\ (a_1, \ldots, a_i, 1_{t(a_i) = s(a_{i+1})}, a_{i+1}, \ldots, a_n) & \text{if} \quad 0 < i < n; \\ (a_1, \ldots, a_n, 1_{t(a_n)}) & \text{if} \quad i = n. \end{cases}$$

Exercise C.69. Verify (a), (b), and (c), so X_* is a simplicial set.

A **morphism** $\phi_*\colon X_* \to Y_*$ of simplicial sets is given by a mapping $\phi_n\colon X_n \to Y_n$ for each $n \geq 0$, commuting with the face and degeneracy operators. A morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ of groupoids determines a morphism $\phi_*\colon X_* \to Y_*$ of their simplicial sets, where ϕ_0 and ϕ_1 are the given maps, and $\phi_n(a_1, \ldots, a_n) = (\phi_1(a_1), \ldots, \phi_1(a_n))$ for $n \geq 1$. If ϕ_* and ψ_* are morphisms from X_* to Y_*, a **homotopy** h from ϕ_* to ψ_* is given by a collection of maps $h_i\colon X_n \to Y_{n+1}$ for all $0 \leq i \leq n$, satisfying:

(a) $d_0 h_0 = \phi_n$ and $d_{n+1} h_n = \psi_n$;

$$\text{(b)}\ d_i h_j = \begin{cases} h_{j-1} d_i & \text{if} \quad i < j; \\ d_j h_{j-1} & \text{if} \quad i = j > 0; \\ h_j d_{i-1} & \text{if} \quad i = n. \end{cases}$$

$$\text{(c)}\ s_i h_j = \begin{cases} h_{j+1} s_i & \text{if} \quad i \leq j; \\ h_j s_{i-1} & \text{if} \quad i > j. \end{cases}$$

Exercise C.70. If $\theta\colon X_0 \to Y_1$ gives a 2-isomorphism between morphisms $\phi_\bullet$ and $\psi_\bullet$ from a groupoid $X_\bullet$ to a groupoid $Y_\bullet$, show that the mappings $h_i\colon X_n \to Y_{n+1}$ defined by

$$h_i(a_1, \ldots, a_n) = (\phi_1(a_1), \ldots \phi_1(a_i), \theta(t(a_i)) = \theta(s(a_{i+1})), \psi_1(a_{i+1}), \ldots, \psi_1(a_n))$$

defines a homotopy from ψ_* to ϕ_*.

Definition C.48. A simplicial set X_* satisfies the **Kan condition** if, for every $0 \leq k \leq n$ and sequence $\sigma_0, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n$ of n $(n-1)$-simplices satisfying $d_i(\sigma_j) = d_{j-1}(\sigma_i)$ for all $i < j$ and $i \neq k \neq j$, there is a σ in X_n with $d_i(\sigma) = \sigma_i$ for all $i \neq k$. This condition is the simplicial analogue of the fact that the union of n faces of an n-simplex is a retract of the simplex. The Kan condition implies that the condition of being homotopic is an equivalence relation. It also implies that the homotopy groups of the geometric realization of the simplicial set can be computed combinatorially. For these and other facts about simplicial sets we refer to [**60**] and [**67**].

EXERCISE C.71. Show that the simplicial set of a groupoid satisfies the Kan condition. [For $k = 0$, and $\sigma_1 = (b_1, \ldots, b_{n-1})$ and $\sigma_2 = (c_1, \ldots, c_{n-1})$, the other σ_i are determined, and one may take $\sigma = (c_1, c_1^{-1}\bullet b_1, b_2, \ldots, b_{n-1})$. For $k = 1$, $\sigma_0 = (a_1, \ldots, a_{n-1})$ and $\sigma_2 = (c_1, \ldots, c_{n-1})$, take $\sigma = (c_1, a_1, a_2, \ldots, a_{n-1})$. For $k > 1$, $\sigma_0 = (a_1, \ldots, a_{n-1})$ and $\sigma_1 = (b_1, \ldots, b_{n-1})$, take $\sigma = (b_1\bullet a_1^{-1}, a_1, a_2, \ldots, a_{n-1})$.]

DEFINITION C.49. The **standard n-simplex** $\Delta(n)$ is defined by

$$\Delta(n) = \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1 \}.$$

regarded as a topological subspace of Euclidean space. For a simplicial set X_*, construct the topological space

$$X = \coprod_{n \geq 0} X_n \times \Delta(n).$$

Topologically, X is the disjoint union of copies of the standard n-simplex, with one for each n-simplex in X_*. Define the **geometric realization** $|X_*|$ of X_* to be the quotient space $X/\sim$ of X by the equivalence relation generated by all

$$(d_i(\sigma), (t_0, \ldots, t_{n-1})) \sim (\sigma, (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

for $\sigma \in X_n$, $(t_0, \ldots, t_{n-1}) \in \Delta(n-1)$, $0 \leq i \leq n$, and

$$(d_i(\sigma), (t_0, \ldots, t_{n+1})) \sim (\sigma, (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_{n+1})$$

for $\sigma \in X_n$, $(t_0, \ldots, t_{n+1}) \in \Delta(n+1)$, $0 \leq i \leq n$. An n-simplex σ in X_n is called **nondegenerate** if it does not have the form $s_i(\tau)$ for $\tau \in X_{n-1}$ and some $0 \leq i \leq n-1$. For each n-simplex σ there is a continuous mapping from $\Delta(n)$ to $|X_*|$ that takes $t \in \Delta(n)$ to the equivalence class of (σ, t). If σ is nondegenerate, this maps the interior of $\Delta(n)$ homeomorphically onto its image. The space $|X_*|$ is a CW-complex, with these images as its cells.

A morphism $\phi_*\colon X_* \to Y_*$ determines a continuous mapping $|\phi_*|\colon |X_*| \to |Y_*|$. Homotopic mappings of simplicial sets determine homotopic mappings between their geometric realizations.

EXERCISE C.72. Any topological space X determines a simplicial set $S_*(X)$, where $S_n(X)$ is the set of all continuous mappings σ from the standard n-simplex to X, with $(d_i\sigma)(t_0, \ldots, t_{n-1}) = \sigma(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n)$ and $(s_i\sigma)(t_0, \ldots, t_{n+1}) = \sigma(t_0, \ldots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \ldots, t_n)$, for $\sigma \in S_n(X)$ and $0 \leq i \leq n$. A continuous mapping $f\colon X \to Y$ determines a mapping $S_*(f)\colon S_*(X) \to S_*(Y)$ of simplicial sets, so we have a functor from (Top) to the category (Sss) of simplicial sets. This functor is a right adjoint to the geometric realization functor from (Sss) to (Top): if X_* is a simplicial set and Y is a topological space, there is a canonical bijection

$$\mathrm{Hom}(X_*, S_*(Y)) \longleftrightarrow \mathrm{Hom}(|X_*|, Y).$$

In fact, 2-isomorphisms of simplicial sets correspond to homotopies between spaces, so one has a strict isomorphism of categories $\mathrm{HOM}(X_*, S_*(Y)) \cong \mathrm{HOM}(|X_*|, Y)$. [See [**67**], §16.]

[What is the relation between a groupoid $X_\bullet$ and the (relative) fundamental groupoid $\pi(|X_*|, X_0)_\bullet$? Should we define product of simplicial sets? Should we point out that a simplicial set is the same thing as a contravariant functor from the category $\mathcal{V}$ to (Set), where $\mathcal{V}$ is the category with one object $\{0, \dots, n\}$ for each nonnegative integer, and with morphisms nondecreasing mappings between such sets. And/or say that both definitions make sense for (Set) replaced by any category? Define the simplicial set I_* and state that a homotopy is the same as $X_* \times I_* \to Y_*$ ([**67**], §6)?

There is a fancier 2-categorical notion in Barbara's chapter on group actions on stacks that could appear in this appendix? What else is needed in the text?]

Answers to Exercises

C.2. $e(x)$ is determined by the category properties (i)–(iv), as the identity of the monoid $\{a \in X_1 \mid s(a) = x, t(a) = x\}$. If $i(f){\cdot}f = et(f)$ and $f{\cdot}i(f) = es(f)$, then $i(f) = i(f){\cdot}(f{\cdot}i'(f)) = (i(f){\cdot}f){\cdot}i'(f) = i'(f)$. The proofs of identities (vii)–(ix) are similar to those in group theory.

C.6. The associativity is proved just as in the case of free groups.

C.7. The unity takes value 1 on $e(X_0)$ and 0 on the complement.

C.11. $G \ltimes X_\bullet$ is $G \times X_1 \rightrightarrows X_0$, with $s(g, a) = s(a)$, $t(g, a) = t(ga)$, and $(g, a){\cdot}(g', a') = (g'g, a{\cdot}g^{-1}a')$.

C.12. Each is (canonically) strictly isomorphic to a $G \ltimes X_\bullet \rtimes H$, which is the groupoid $G \times X_1 \times H \rightrightarrows X_0$, with $s(g, a, h) = s(a)$, $t(g, a, h) = t(gah)$, and $(g, a, h){\cdot}(g', a', h') = (g'g, a{\cdot}g^{-1}a'h^{-1}, hh')$.

C.13. The data $s, t \colon X_1 \to X_0$ determine a directed graph Γ. Form X by adjoining a disk for each identity map 1_x, $x \in X_0$, and a triangle for each $(a, b) \in X_2$: [pictures of disks bounding an arrow at x and a triangle with sides a, b, and a $\cdot$ b should be drawn here] Take A to be the set X_0 of vertices. See Section 6.2 for more general constructions.

C.17. If ϕ_a is defined by a and $\phi_{a'}$ is defined by a', then $\phi_{a'}(g) = z^{-1}\phi_a(g)z$, with $z = a^{-1}{\cdot}a'$.

C.20. The mass is $\frac{1}{(q+1)(q^3-1)}$.

C.21. This is the restriction of $X_\bullet$ from the canonical map from E_0 to X_0, cf. C.41.

C.29. To verify C.25, look at the map $(s, t) \mapsto H(a(s), t)$, which has $s \mapsto f(a(s))$ on the bottom, $s \mapsto g(a(s))$ on the top, $t \mapsto H(a(0), t)$ on the left side, and $t \mapsto H(a(1), t)$ on the right.

C.32. The only possible 2-isomorphism from $f_\bullet$ to $g_\bullet$ is given by $\theta(x) = (f_0(x), g_0(x)) \in Y_1 \subset Y_0 \times Y_0$.

C.35. A morphism from $X_\bullet \times I_\bullet$ to $Y_\bullet$ is given by a pair of maps $f_0, f_1 \colon X_0 \to Y_0$, and four maps $f_{00}, f_{01}, f_{10}, f_{11} \colon X_1 \to Y_1$, satisfying some identities. The bijection is given by

$$\phi_0 = f_0,\ \psi_0 = f_1,\ \phi_1 = f_{00},\ \psi_1 = f_{11},\ \theta = f_{01} \circ e,\ f_{01} = \phi_1 {\cdot} \theta t,\ f_{10} = \psi_1 {\cdot} i\theta t.$$

C.36. For each point y in X, choose a path a_y from x to y, and map a path γ in $\pi(X)_1$ from y to z to the homotopy class of $a_y {\cdot} \gamma {\cdot} a_z^{-1}$.

C.37. Choose $x_0 \in X_0$, and let $G = \mathrm{Aut}(x_0)$. Then $BG_\bullet$ is a subgroupoid of $X_\bullet$. Map $X_\bullet$ to $BG_\bullet$ by choosing $a_x \in X_1$ with $s(a_x) = x_0$, $t(a_x) = x$, with $a_{x_0} = e(x_0)$, and sending $b \in X_1$ to $a_x {\cdot} b {\cdot} {a_y}^{-1}$. The map $x \mapsto a_x$ is a 2-isomorphism from the composite $X_\bullet \to BG_\bullet \to X_\bullet$ to the identity on $X_\bullet$.

C.41. If α is a 2-isomorphism from $\phi'_\bullet \phi_\bullet$ to $1_{X_\bullet}$ and β is a 2-isomorphism from $\psi'_\bullet \psi_\bullet$ to $1_{Y_\bullet}$, then $\theta(x) = \phi'_1 \beta \phi_0(x) {\cdot} \alpha(x)$ defines a 2-isomorphism θ from $\phi'_\bullet \psi'_\bullet \psi_\bullet \phi_\bullet$ to $1_{X_\bullet}$. In the language of 2-categories, this is the composite of $1_{\phi'_\bullet} * \beta * 1_{\phi_\bullet}$ from $\phi'_\bullet \psi'_\bullet \psi_\bullet \phi_\bullet$ to $\phi'_\bullet 1_{Y_\bullet} \phi_\bullet = \phi'_\bullet \phi_\bullet$ and α from $\phi'_\bullet \phi_\bullet$ to $1_{X_\bullet}$.

C.42. Explicit isomorphisms between $G \ltimes (X/H)$ and $(G\backslash X) \rtimes H$, and 2-isomorphisms between their composites and the identities, can be constructed from choices of section of the maps $X \to X/H$ and $X \to G\backslash X$. See Exercise C.47.

C.48. When Z has at most two elements.

C.49. (a) Each is equivalent to the exactness of the sequence $0 \to V \to W \oplus V' \to W' \to 0$, the first taking v to $(L(v), \phi_V(v))$, the second taking (w, v') to $\phi_W(w) - L(v')$. (b) A splitting of $\mathrm{Ker}(L) \to V$ determines an isomorphism of $\mathrm{Ker}(L) \ltimes \mathrm{Coker}(L)$ to $V \ltimes W$, to which (a) applies; and similarly for a splitting $W \to \mathrm{Coker}(L)$. Without any splitting (for example for abelian groups), they are isomorphic because they both have components indexed by $\mathrm{Coker}(L)$, and all isotropy groups are $\mathrm{Ker}(L)$.

C.50. Apply the proposition.

C.53. Here $s = p_1$ and $t = p_2$ are the two projections from X_1 to X_0, with θ given by the identity map on X_1. And $X_2 = X_1 \ {}_t{\times}_{X_0,s}\ X_1$, with $q_1(a,b) = s(a)$, $q_2(a,b) = t(a) = s(b))$, $q_3(a,b) = t(b)$, $p_{12}(a,b) = a$, $p_{23}(a,b) = b$, $p_{13}(a,b) = a{\cdot}b$. Each θ_{ij} is given by a map from X_2 to X_1; in fact $\theta_{ij} = p_{ij}$. Each α_{ij}, α_{ji}, and α_i is an identity. A morphism $u_\bullet \colon X_0 \to Z_\bullet$ is given by map $u_0 \colon X_0 \to Z_0$, and $\tau : u_0 \circ s \overset{\tau}{\Rightarrow} u_0 \circ t$ is given by a map $\tau \colon X_1 \to Z_1$ with $s\tau = u_0 s$, $t\tau = u_0 t$, and $\tau(a{\cdot}b) = \tau(a){\cdot}\tau(b)$. The required $v_\bullet \colon X_\bullet \to Z_\bullet$ is defined by $v_0 = u_0$ and $v_1 = \tau$; and $\rho \colon u_\bullet \Rightarrow v_\bullet \circ \pi$ is given by the map $e \circ u_0 \colon X_0 \to Z_1$. For the uniqueness, if $v'_\bullet \colon X_\bullet \to Z_\bullet$ and $\rho' \colon u_\bullet \Rightarrow v'_\bullet \circ \pi$ are others, the 2-isomorphism $\zeta \colon v_\bullet \Rightarrow v'_\bullet$ is given by the map $\zeta = \rho' \colon X_0 \to Z_1$.

C.55. The 2-isomorphism is given by the mapping

$$\theta \colon X \times G \longrightarrow X \times G \times G, \qquad (x,g) \mapsto (xg, g^{-1}, g).$$

C.59. Given $\phi_\bullet\colon X_\bullet \to Y_\bullet$, take $Y'_0 = X_0 \times Y_0$, and define $Y'_\bullet$ to be the pullback of $Y_\bullet$ by means of the projection $X_0 \times Y_0 \to Y_0$, so $Y'_1 = Y_1 \times X_0 \times X_0$. Map X_0 to Y'_0 by the graph of ϕ_0 and X_1 to Y'_1 by $a \mapsto (\phi_1(a), s(a), t(a))$.

C.61. The equivalence of (ii) to (v) follows from Exercise C.40; that (i) implies (ii) follows from the construction of the fibered product $X_\bullet \times_{Y_\bullet} T$; that (ii) implies (i) is proved by taking $T = Y_0$ and ψ_0 the identity.

C.62. Factor the morphism into $X_\to\colon Y'_\bullet \to Y_\bullet$ as in Exercise C.59. Let $Z_0 = Y'_0 = X_0 \times Y_0$, and let Z_1 be the image of $X_1 \to Y'_1$. The canonical map from $X_\bullet$ to $Z_\bullet$ is gerbe-like, and the canonical map $Z_\bullet \to Y'_\bullet$ (and hence $Z_\bullet \to Y'_\bullet \to Y_\bullet$) is representable.

C.63. The equivalence of (i) and (ii) follows from the construction of fibered products, and the equivalence of (ii) and (iii) from Exercise C.37.

C.64. If $Y_\bullet$ is the restriction, with $Y_0 = X_n$, then Y_1 consists of triples (a, b, c) with $a, b \in X_n$, $c \in X_1$, $s(c) = s(a_1)$, and $t(c) = s(b_1)$. Let $Z_\bullet = X_\bullet\{n\}$. Map $Y_\bullet$ to $Z_\bullet$ by the identity $Y_0 = X_n = Z_0$ and map $Y_1 \to Z_1$ by $(a, b, c) \mapsto (\phi_0, \ldots, \phi_n)$, where $\phi_0 = c$ and $\phi_i = a_i^{-1}\boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_1^{-1}\boldsymbol{\cdot} c \boldsymbol{\cdot} b_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_i$ for $1 \le i \le n$.

C.65. The product in $V_\bullet^{(n)}$ is defined by

$$(a_1, \ldots, a_n, c, b_1, \ldots, b_n)\boldsymbol{\cdot}(b_n^{-1}, \ldots, b_1^{-1}, d, e_1, \ldots, e_n) = (a_1, \ldots, a_n, c\boldsymbol{\cdot} d, e_1, \ldots, e_n).$$

Set $Z_\bullet = X_\bullet\{n\}$. Map $V_\bullet = V_\bullet^{(n)}$ to $Z_\bullet$ by $V_0 = X_n = Z_0$ and V_1 to Z_1 by $(a_1, \ldots, a_n, c, b_1, \ldots, b_n) \mapsto (\phi_0, \ldots, \phi_n)$, where $\phi_0 = c$ and $\phi_i = a_{n+1-i}\boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_n \boldsymbol{\cdot} c \boldsymbol{\cdot} b_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} b_i$ for $1 \le i \le n$. There is a canonical isomorphism between $(V_\bullet^{(n)})^{(m)}$ and $V_\bullet^{((n+1)(m+1)-1)}$, both having objects identified with X_{mn+m+n} and arrows identified with $X_{2(mn+m+n)+1}$.

C.66. (1) The identity e takes $(a_1, \ldots, a_n)$ to $(1_{sa_1}, a_1, \ldots, a_n)$ and the inverse i takes $(a_1, \ldots, a_{n+1})$ to $(a_1^{-1}, a_1\boldsymbol{\cdot} a_2, a_3, \ldots, a_{n+1})$. (2) Let $Z_\bullet$ be the cross product groupoid, so $Z_0 = X_n = W_0^{(n)}$, and $Z_1 = \{((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \mid a_i = b_i \text{ for } i > 1\}$. Map Z_1 to $W_1^{(n)}$ by sending $((a_1, \ldots, a_n), (b_1, \ldots, b_n))$ to $(a_1\boldsymbol{\cdot} b_1^{-1}, b_1, \ldots, b_n)$. (3) We have $Z_0 = X_n = (X_\bullet[n])_0$, and $Z_1 \to (X_\bullet[n])_1$ by

$$((a_1, \ldots, a_n), (b_1, \ldots, a_b)) \mapsto (\phi_0, \ldots, \phi_n, a_1, \ldots, a_n, b_1, \ldots, b_n),$$

with $\phi_0 = a_1\boldsymbol{\cdot} b_1^{-1}$ and $\phi_i = 1_{sa_i}$ for $1 \le i \le n$.

C.67. Consider the morphism $W_\bullet^{(n+1)} \to W_\bullet^{(n)}$ that omits the last object on objects and arrows. This is easily checked to be square.

C.68. A strict isomorphism $\phi_\bullet$ from $X_\bullet(n, k)$ to $X_\bullet(n, k+1)$ is given by

$$\phi_1(a_1, \ldots, a_n) = (a_n^{-1}\boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_1^{-1}, a_1, \ldots, a_{n-1}),$$

with $\phi_0(a_1, \ldots, a_{n-1}) = (a_{n-1}^{-1}\boldsymbol{\cdot} \ldots \boldsymbol{\cdot} a_1^{-1}, a_1, \ldots, a_{n-2})$. [Should we omit this exercise?]

C.71. For $k = 0$, and $\sigma_1 = (b_1, \ldots, b_{n-1})$ and $\sigma_2 = (c_1, \ldots, c_{n-1})$, the other σ_i are determined, and one may take $\sigma = (c_1, c_1^{-1} \cdot b_1, b_2, \ldots, b_{n-1})$. For $k = 1$, $\sigma_0 = (a_1, \ldots, a_{n-1})$ and $\sigma_2 = (c_1, \ldots, c_{n-1})$, take $\sigma = (c_1, a_1, a_2, \ldots, a_{n-1})$. For $k > 1$, $\sigma_0 = (a_1, \ldots, a_{n-1})$ and $\sigma_1 = (b_1, \ldots, b_{n-1})$, take $\sigma = (b_1 \cdot a_1^{-1}, a_1, a_2, \ldots, a_{n-1})$.

Glossary

This glossary does not pretend to be a course in algebraic geometry. Its purpose is to put in one easily available place some of the notions and facts that are used in the text. It can also be used to test of your background: if you can read this glossary, even if all the assertions are not familiar, you should have enough background in algebraic geometry to read the text.

1. Schemes and fibered products

A **scheme** is a ringed space $(X, \mathcal{O}_X)$ that is locally of the form $\mathrm{Spec}(A)$, for A a commutative ring with unit. For most purposes in this book, one can restrict attention to the case where A has nice properties, such as Noetherian, or finitely generated over the ground ring or field.[5] Since fibered products play a prominent role, however, one cannot stay in a category of reduced or irreducible varieties.

An **open subscheme** of a scheme X is an open subset U of X with its structure sheaf $\mathcal{O}_U = \mathcal{O}_X \mid_U$. A **closed subscheme** Y of X is the support of a quasi-coherent sheaf $\mathcal{I}$ of ideals, with the structure sheaf $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$. A **subscheme** Y of X is given by a locally closed subspace Y of X, which is a closed subscheme of the open subscheme $U = X \smallsetminus (\overline{Y} \smallsetminus Y)$.

Given schemes X, Y, and Z, with morphisms $f : X \to Z$ and $g : Y \to Z$, there is a **fibered product**, which is a scheme $X \times_Z Y$, together with two projections $p : X \times_Z Y \to X$ and $q : X \times_Z Y \to Y$, with the $f \circ p = g \circ q$. The fibered product is determined by the following universal property: for any scheme S and morphisms $u : S \to X$ and $v : S \to Y$ such that $f \circ u = g \circ v$, there is a unique morphism $(u, v) : S \to X \times_Z Y$ such that $u = p \circ (u, v)$ and $v = q \circ (u, v)$. If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $Z = \mathrm{Spec}(C)$, then $X \times_Z Y = \mathrm{Spec}(A \otimes_C B)$. In general, $X \times_Z Y$ is constructed by patching (see below). For clarity, if other morphisms from X or Y to Z are in use, the notation $X\ {}_f{\times}_{Z,g}\, Y$ or $X\ {}_f{\times}_g\, Y$ may be used for this fibered product.

A diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

[5] We give the definitions in their natural generality, following [**EGA**]; much of the simpler situation with Noetherian hypotheses, which suffices for most applications, can be found in [**47**].

is called **cartesian** if it commutes, and the resulting morphism from X' to $Y' \times_Y X$ is an isomorphism. This agrees with the categorical notion of a cartesian diagram; in particular, X' is determined up to canonical isomorphism.

If $X \to Y$ is a family of some kind, then $X' = Y' \times_Y X \to Y'$ is called the **pullback** of the family for the morphism $Y' \to Y$. When $Y' = \mathrm{Spec}(\kappa(y))$, for y a point in Y, and $\kappa(y)$ the residue field of the local ring of Y at y, this pullback is the **fiber** of the family at y, and is denoted $f^{-1}(y)$. If Y is integral, with K the quotient field of its local rings, and $Y' = \mathrm{Spec}(K)$, the pullback is the **generic fiber**. When $Y' = \mathrm{Spec}(L)$, with L an algebraically closed field, the pullback is called a **geometric fiber** of the family.

For any morphism $f\colon X \to Y$, there is a canonical **diagonal** morphism $\Delta_f = (f, f)\colon X \to X \times_Y X$.

For any scheme X, there is a contravariant functor h_X from the category (Sch) of schemes to the category (Set) of sets, that takes a scheme S to the set $h_X(S) = \mathrm{Hom}(S, X)$ of morphisms from S to X. The elements of $h_X(S)$ are called S-valued points of X. For any scheme S, if $h_X(S)$ denotes the set of morphisms from S to X, there is a canonical bijection

$$h_{X\times_Z Y}(S) \leftrightarrow h_X(S) \times_{h_Z(S)} h_Y(S),$$

where the fibered product on the right is that of sets.

Schemes are often constructed by **recollement**, also called *gluing*, or *patching*. For this, one has a collection X_α of schemes, with an open subscheme $U_{\alpha\beta}$ of X_α for any pair α, β, so that $U_{\alpha\alpha} = X_\alpha$. In addition, one has isomorphisms $\vartheta_{\beta\alpha}$ of $U_{\alpha\beta}$ with $U_{\beta\alpha}$. These must satisfy the following compatibility condition: for any α, β, and γ, $\vartheta_{\beta\alpha}$ maps $U_{\alpha\beta} \cap U_{\alpha\gamma}$ isomorphically onto $U_{\beta\alpha} \cap U_{\beta\gamma}$, and the diagram

$$\begin{tikzcd}
U_{\alpha\beta} \cap U_{\alpha\gamma} \arrow[r, "\vartheta_{\beta\alpha}"] \arrow[d, equal] & U_{\beta\alpha} \cap U_{\beta\gamma} \arrow[r, equal] & U_{\beta\gamma} \cap U_{\beta\alpha} \arrow[d, "\vartheta_{\gamma\beta}"] \\
U_{\alpha\gamma} \cap U_{\alpha\beta} \arrow[r, "\vartheta_{\gamma\alpha}"'] & U_{\gamma\alpha} \cap U_{\gamma\beta} \arrow[r, equal] & U_{\gamma\beta} \cap U_{\gamma\alpha}
\end{tikzcd}$$

must commute. (It follows that $\vartheta_{\alpha\alpha}$ is the identity on X_α, and that $\vartheta_{\alpha\beta} \circ \vartheta_{\beta\alpha}$ is the identity on $U_{\alpha\beta}$.) Then there is a scheme X, with open embeddings $\varphi_\alpha\colon X_\alpha \to X$, such that: X is the union of the $\varphi(X_\alpha)$; for all α and β, $\varphi_\alpha(U_{\alpha\beta}) = \varphi_\alpha(X_\alpha) \cap \varphi_\beta(X_\beta)$; and $\varphi_\alpha = \varphi_\beta \circ \vartheta_{\beta\alpha}$ on $U_{\alpha\beta}$. The same construction works for any ringed spaces.

Let $\mathcal{C}$ be the category (Set) of schemes, or the category (Sch/Λ) of schemes over a base scheme Λ. A contravariant functor F from $\mathcal{C}$ to the category (Set) of sets is called a **sheaf** if, for every (Zariski) open covering $\{U_\alpha\}$ of a scheme X, the sequence

$$F(X) \to \prod F(U_\alpha) \rightrightarrows \prod F(U_\alpha \cap U_\beta)$$

is exact; that is, any element of $F(X)$ is determined by its restrictions to the open sets U_α, and a collection of elements in $F(U_\alpha)$ that agree on the overlaps $U_\alpha \cap U_\beta$ come from a unique element in $F(X)$.

A representable natural transformation $F \to G$ between contravariant functors from $\mathcal{C}$ to (Set) is called **open** if, for every scheme Z and natural transformation $h_Z \to G$, the functor $F \times_G h_Z$ is represented by a scheme Y, and the corresponding morphism from Y to Z is an open embedding.[6]

A collection of natural transformations $F_\alpha \to F$ is an **open covering** if each $F_\alpha \to F$ is representable and open, and, for any scheme Z and natural transformation $h_Z \to F$, the images of the schemes representing $F_\alpha \times_F h_Z$ in Z form an open covering of Z. With these definitions we have:

PROPOSITION (Grothendieck's representability theorem). *Let F be a contravariant functor from $\mathcal{C}$ to* (Set). *Suppose there is a family F_α of subfunctors of F, such that each F_α is representable, and the collection $F_\alpha \to F$ is an open covering. Then F is representable.*

To prove this, if X_α represents F_α, the fibered products $F_\alpha \times_F F_\beta$ determine open coverings $U_{\alpha\beta}$ of X_α, together with isomorphisms $\vartheta_{\beta\alpha}$ of $U_{\alpha\beta}$ with $U_{\beta\alpha}$. One verifies, using triple fibered products $F_\alpha \times_F F_\beta \times_F F_\gamma$, that these isomorphisms satisfy the compatibility conditions of recollement, so that X is constructed by gluing these X_α. For details of this verification, see [**EGA** I′.0.4.5.4]. The same argument works in the category of ringed spaces.

2. Morphisms

A morphism $f\colon X \to Y$ of schemes is an **embedding**[7] if f factors into an isomorphism $X \to X'$ followed by the inclusion $X' \to Y$ of a subscheme X' of Y. It is an **open embedding** if X' is an open subscheme of Y, and a **closed embedding** if X' is a closed subscheme of Y. For a general morphism $f\colon X \to Y$ of schemes, the diagonal $\Delta_f\colon X \to X \times_Y X$ is an embedding ([**EGA** I.5.3.9, Err_{III}.10]).

A morphism $f\colon X \to Y$ is **locally of finite type** if for every x in X, with $y = f(x)$, there are affine neighborhoods $U \cong \mathrm{Spec}(B)$ of x and $V \cong \mathrm{Spec}(A)$ of y, with $f(U) \subset V$, such that the induced map $A = \Gamma(V, \mathcal{O}_Y) \to B = \Gamma(U, \mathcal{O}_U)$ makes B a finitely generated A-algebra; that is, $B \cong A[X_1, \dots X_n]/I$ for some ideal I. The morphism is **locally of finite presentation** if one can find such neighborhoods with B of finite presentation over A; that is, $B \cong A[X_1, \dots X_n]/I$, with $I = (F_1, \dots, F_m)$ for some polynomials $F_i \in A[X_1, \dots X_n]/I$. When Y is locally Noetherian, these two notions coincide. A morphism f is **of finite type** if every point of Y has an affine open neighborhood $V \cong \mathrm{Spec}(A)$ such that $f^{-1}(V)$ is covered by a finite number of affine open sets $U \cong \mathrm{Spec}(B)$ with B a finitely generated A-algebra. This implies that the same property holds for every affine open subset of Y ([**EGA** I.6.3]).

Most ordinary morphisms, such as those between algebraic varieties over a field, will be of finite type. However, morphisms like $\mathrm{Spec}(K) \to X$, where K is the function field of an integral scheme X, or morphisms like $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{Q})$, although not of finite type, are often useful.

[6] Such a definition makes sense for any property of a morphism which is preserved by arbitrary pullbacks and by composing on either side by an isomorphism; all of the properties of morphisms defined in the next section have this property.

[7] We avoid the word "immersion" for this notion, since that word has such a different meaning in differential geometry.

A scheme X is **quasi-compact** if its underlying space has the property that every covering by open sets has a finite subcover; equivalently, X can be covered by a finite number of affine open subsets. (Note that $\mathrm{Spec}(A)$ is quasi-compact for any A, whether Noetherian or not.) A morphism $f\colon X \to Y$ of schemes is **quasi-compact** if $f^{-1}(U)$ is quasi-compact for every affine open subset U of Y. It suffices in fact that this property holds for every U in one affine open covering of Y ([**EGA** I.6.6]). A morphism is of finite type exactly when it is locally of finite type and quasi- compact.

A morphism $f\colon X \to Y$ is **separated** if the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is a closed embedding; equivalently, the image of Δ_f is a closed subset of $X \times_Y X$. E.g., every morphism of affine schemes is separated. The **valuative criterion for separatedness** asserts that f is separated if and only if (i) the diagonal $X \to X \times_Y X$ is quasi-compact, and (ii) for any valuation ring R, with quotient field L, and any morphisms $\mathrm{Spec}(R) \to Y$ and $\mathrm{Spec}(L) \to X$ such that the diagram

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] & Y \end{tikzcd}$$

commutes, there is *at most* one morphism from $\mathrm{Spec}(R)$ to X making the whole diagram commute; this criterion asserts that the canonical map

$$\mathrm{Hom}_Y(\mathrm{Spec}(R), X) \to \mathrm{Hom}_Y(\mathrm{Spec}(L), X)$$

is injective. When Y is locally Noetherian, one needs this test only when R is a discrete valuation ring. (See [**EGA** II.7.2.3] for these criteria.)

A morphism $f\colon X \to Y$ is **quasi-separated** if it satisfies the first condition (i) of the criterion for separatedness: the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is quasi-compact. Equivalently, for any affine open subsets U and V of X whose images are contained in an affine open subset of Y, the intersection $U \cap V$ is a finite union of affine open subsets.

A scheme X is called **separated** (resp. **quasi-separated**) if the morphism $X \to \mathrm{Spec}(\mathbb{Z})$ is separated (resp. quasi-separated). Every locally Noetherian scheme is quasi-separated.

A morphism $f\colon X \to Y$ is **proper** if it is separated, of finite type, and if, for any morphism $Y' \to Y$, the projection $X \times_Y Y' \to Y'$ is closed (i.e., the image of any closed subset is closed). The **valuative criterion for properness** asserts that f is proper if and only if (i) f is a separated morphism of finite type, and (ii) for any valuation ring R and morphisms as in the valuative criterion for separatedness, the canonical map $\mathrm{Hom}_Y(\mathrm{Spec}(R), X) \to \mathrm{Hom}_Y(\mathrm{Spec}(L), X)$ is surjective (and therefore bijective). When Y is locally Noetherian, it suffices to verify this criterion when R is a discrete valuation ring. (See [**EGA** II.7.3.8])

Recall that a homomorphism $A \to B$ of commutative rings is flat if the functor $M \to B \otimes_A M$ from A-modules to B-modules is (left) exact. A morphism $f\colon X \to Y$ of schemes is **flat** if for every point x in X, the local ring $\mathcal{O}_{x,X}$ is flat as a module over $\mathcal{O}_{y,Y}$. A morphism is **faithfully flat** if it is flat and surjective. For the morphism from

Spec(B) to Spec(A) coming from a homomorphism $A \to B$, this is equivalent to the flatness of B over A together with the assertion that the vanishing of $B \otimes_A M$ implies the vanishing of M, for any A-module M. A morphism is **fppf** if it is faithfully flat and locally of finite presentation. An important fact is that any fppf morphism is open, i.e., the image of any open set is open [**EGA** IV.2.4.6]. A morphism is **fpqc** if it is faithfully flat and quasi-compact.

A morphism $f\colon X \to Y$ is **unramified** if f is of locally of finite presentation and, for every x in X, with $y = f(x)$, one has $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$ and $\kappa(x)$ is a finite separable field extension of $\kappa(y)$. For f locally of finite presentation, this is equivalent to each of the following assertions: (i) the diagonal morphism $X \to X \times_Y X$ is an open embedding; (ii) the sheaf $\Omega^1_{X/Y}$ of relative differentials vanishes; (iii) for any nilpotent ideal I in a commutative ring Λ, and any morphism $\mathrm{Spec}(\Lambda/I) \to X$, there is *at most* one morphism from Spec(Λ) to X so that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Spec}(\Lambda/I) \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \mathrm{Spec}(\Lambda) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

That is, the canonical map

$$\mathrm{Hom}_Y(\mathrm{Spec}(\Lambda), X) \to \mathrm{Hom}_Y(\mathrm{Spec}(\Lambda/I), X)$$

is injective.

A morphism $f\colon X \to Y$ is **étale** if it is unramified and flat. Equivalently, f is locally of finite presentation and, with Λ and I as above, one can always fill in the diagram uniquely: the canonical map $\mathrm{Hom}_Y(\mathrm{Spec}(\Lambda), X) \to \mathrm{Hom}_Y(\mathrm{Spec}(\Lambda/I), X)$ is bijective.

A morphism $f\colon X \to Y$ is **smooth** if it is locally of finite presentation, flat, and, for any morphism $\mathrm{Spec}(L) \to Y$, with L a field, the fiber $X \times_Y \mathrm{Spec}(L)$ is regular, i.e., all its local rings are regular local rings. Equivalently, f is locally of finite presentation, and, with Λ and I as above, the canonical map $\mathrm{Hom}_Y(\mathrm{Spec}(\Lambda), X) \to \mathrm{Hom}_Y(\mathrm{Spec}(\Lambda/I), X)$ is surjective. Equivalently, any point on X has a neighborhood U, mapped to an open subset V of Y, such that there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & \mathrm{Spec}(A[X_1, \ldots, X_n]/(F_1, \ldots, F_m)) \arrow[d] \\ V \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd}$$

with the horizontal arrows open embeddings, and with $\mathrm{rank}(\partial F_i/\partial X_j) \equiv m$ on U. For f étale, one has the same local description but with $m = n$. A smooth morphism locally factors into a composition $U \to V \times \mathbb{A}^r \to V$, where the first map is étale and the second is the projection ([**EGA** IV.17.11.4]). Other characterizations and properties of unramified, étale, and smooth morphisms can be found in [**EGA** IV. §17].

A morphism is said to be **formally unramified**, resp. **formally étale**, resp. **formally smooth** if it satisfies the condition on liftings of morphisms $\mathrm{Spec}(\Lambda/I) \to X$ to $\mathrm{Spec}(\Lambda) \to X$ stated above for unramified, resp. étale, resp. smooth morphisms.

Smooth is equivalent to formally smooth and locally of finite presentation (and similarly for étale, unramified). We call particular attention to morphisms that are *formally unramified* and *locally of finite type*. This is a class of morphisms which naturally generalizes embeddings. By [**EGA** IV.17.2.1], a morphism is formally unramified if and only if it has trivial sheaf of relative differentials (see the section on differentials, below).

A morphism $f\colon X \to Y$ is **affine** if any point of Y has an affine open neighborhood V such that $f^{-1}(V)$ is an affine open subset of X. It follows that $f^{-1}(V)$ is affine for every affine open set V in Y. An affine morphism is separated and quasi-compact.

A morphism $f\colon X \to Y$ is **quasi-affine** if any point of Y has an affine open neighborhood V such that $f^{-1}(V)$ is isomorphic to a quasi-compact open subscheme of an affine scheme. Such a morphism is automatically separated and quasi-compact.

A morphism $f\colon X \to Y$ is **finite** if it is affine, and for any affine open $V \cong \operatorname{Spec}(A)$ of Y, $f^{-1}(V) \cong \operatorname{Spec}(B)$, with B finitely generated as an A-module. A morphism $f\colon X \to Y$ is **quasi-finite** if it is of finite type and each fiber $f^{-1}(y)$ is a finite set.

If Y is locally Noetherian, the following are equivalent: (i) f is finite; (ii) f is proper and affine; (iii) f is proper and quasi-finite [**EGA** III.4.4.2].

A morphism $f\colon X \to Y$ is **projective** if there is a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ of finite type, such that f factors into a closed embedding $X \to \operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$ followed by the canonical projection from $\operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$ to Y.

An **invertible sheaf** is a locally free sheaf of rank one. An invertible sheaf $\mathcal{L}$ on a quasi-compact scheme X is **ample** if, for any coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that, for all $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its sections. If $f\colon X \to Y$ is a quasi-compact morphism, an invertible sheaf $\mathcal{L}$ on X is **f-ample** if any point of Y has an affine open neighborhood U such that the restriction of $\mathcal{L}$ to $f^{-1}(U)$ is ample.

A morphism $f\colon X \to Y$ is **quasi-projective** if it is of finite type, and there is an f-ample invertible sheaf. If Y is quasi-compact (or its underlying space is Noetherian), this is equivalent to f factoring $X \to \operatorname{Proj}(\operatorname{Sym}(\mathcal{E})) \to Y$ as above, but with the first map only a locally closed embedding ([**EGA** II.5.3.2]). A projective morphism is proper and quasi-projective; the converse is true if the target scheme Y is quasi-compact or its underlying space is Noetherian ([**EGA** II.5.5.3]).

A morphism $f\colon X \to Y$ is an **epimorphism** if for any two morphisms g and h from Y to any scheme Z, $g \circ f = h \circ f$ implies $g = h$; that is, the canonical mapping from $\operatorname{Hom}(Y, Z)$ to $\operatorname{Hom}(X, Z)$ is always injective. It is an **effective epimorphism** if, whenever a morphism $\widetilde{g}\colon X \to Z$ is given such that $\widetilde{g} \circ p = \widetilde{g} \circ q$, where p and q are the two projections from $X \times_Y X$ to X, then there is a unique morphism $g\colon Y \to Z$ with $g \circ f = \widetilde{g}$; that is,

$$\operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z) \rightrightarrows \operatorname{Hom}(X \times_Y X, Z)$$

is exact. Any fppf or fpqc morphism is an effective epimorphism (see Appendix A).

A morphism $f\colon X \to Y$ is a **monomorphism** if for any morphisms g and h from any scheme S to X, $f \circ g = f \circ h$ implies $g = h$; that is, the map $\operatorname{Hom}(S, X) \to \operatorname{Hom}(S, Y)$ is always injective. Equivalently, the diagonal map $\Delta_f\colon X \to X \times_Y S$ is an isomorphism ([**EGA** I.5.2.8]). In particular, f is separated; if it is locally of finite presentation, it is unramified. A morphism is **radicial** if, whenever $S = \mathrm{Spec}(L)$, for L a field, the map $\mathrm{Hom}(S, X) \to \mathrm{Hom}(S, Y)$ is always injective; equivalently, the map is injective on the underlying sets of points, and, for every point x of X, the field extension $\kappa(f(x)) \subset \kappa(x)$ is purely inseparable ([**EGA** I.3.5.8]). If f is locally of finite type, it is a monomorphism if and only if it is unramified and radicial ([**EGA** IV.17.2.6]).

Each of the properties of morphisms $f\colon X \to Y$ listed here is preserved by arbitary base change (except epimorphism, where this is an extra condition, termed *universal* epimorphism). That is, if f has the property, and $Y' \to Y$ is an arbitrary morphism, then the pullback $X \times_Y Y' \to Y'$ also has the property. If $f\colon X \to Y$ and $f'\colon X' \to Y'$ each have one of these properties, and there are morphisms from Y and Y' to a scheme S, then the fibered product $X \times_S X' \to Y \times_S Y'$ also has the property. If $f\colon X \to Y$ and $g\colon Y \to Z$ each have one of these properties, the composition $g \circ f$ also has the property; however, in the case of projective or quasi-projective morphisms, one must assume that Z is quasi-compact (or its underlying space is Noetherian). Any isomorphism satisfies all the properties. There are also results that say if $g\colon Y \to Z$ satisfies an appropriate condition (often separated suffices), if $g \circ f$ has the property, then f has the property. And, when $Y' \to Y$ is surjective and satisfies an appropriate condition (faithfully flat and quasi-compact is common), if $X \times_Y Y' \to Y'$ satisfies a property, then f will satisfy the property.[8]

3. Differentials

We recall and collect some basic facts about sheaves of differentials. We start with affine schemes (algebras) and the algebraic properties of modules of differentials. Then we pass to schemes and their sheaves of relative differentials.

Attached to a surjective ring homomorphism $A \to B$, with kernel I, is the module I/I^2, which naturally has the structure of B-module. The associated sheaf is the conormal sheaf of $\mathrm{Spec}\, B$ in $\mathrm{Spec}\, A$. This construction applied to the relative diagonal gives the module of relative differentials.

Let B be an A-algebra. Then the **module of differentials** $\Omega_{B/A}$ is the sheaf I/I^2, where I is the kernel of the multiplication map $B \otimes_A B \to B$. It comes with a differential map $d\colon B \to \Omega_{B/A}$, defined by $df = 1 \otimes f - f \otimes 1$ for $f \in B$. This module satisfies the following universal property: for any B-module M and map $d'\colon B \to M$ which is additive, satisfies the Leibnitz rule $d'(fg) = f\,dg + g\,df$, and vanishes on A, there is a unique B-module homomorphism $\varphi\colon \Omega_{B/A} \to M$ such that $d' = \varphi \circ d$ (see [**47**, II.8]).

Considering, again, a surjective ring homomorphism $A \to B$ with kernel I, if $\varphi\colon A \to A'$ is an arbitrary ring homomorphism, we set $B' = B \otimes_A A'$, so $A' \to B'$ is also surjective. Let I' denote the kernel of $A' \to B'$. Then there is a morphism of B-modules $I/I^2 \to I'/I'^2$, induced by $f \mapsto \varphi(f)$. It is natural in the sense that given

[8]These and related results can be found in the following sections of [**EGA**], listed with the corresponding property: locally of finite type, IV.1.3.4; locally of finite presentation, IV.1.4.3; finite type, IV.1.5.4, quasi-compact, IV.1.1.2; separated, I.2.2; quasi- separated IV.1.2.2; proper II.5.4.2; flat IV.2.1; faithfully flat, IV.2.2.13; unramified, étale, and smooth, IV.17.3.3; affine, II.1.6.2; quasi-affine II.5.1.10; finite, II.6.1.5; quasi-finite, II.6.2.4; projective II.5.5.5; quasi-projective II.5.3.4; radicial I.3.5. General references for schemes and morphisms are: [**47**], [**24**], [**74**], [**68**], [**38**].

$\psi\colon A'' \to A'$ as well, with $B'' = B' \otimes_{A'} A'' \cong B \otimes_A A''$, then we get a commutative triangle

$$\begin{tikzcd} I/I^2 \arrow[rr] \arrow[dr] & & I'/I'^2 \arrow[dl] \\ & I''/I''^2 & \end{tikzcd}$$

This is because the right-hand map sends $\varphi(f)$ to $\psi(\varphi(f)) = \psi \circ \varphi(f)$.

We have a morphism of B-modules $I/I^2 \to I'/I'^2$, or just as well, a morphism of B'-modules $(I/I^2) \otimes_B B' \to I'/I'^2$. This is an isomorphism when A' is flat over A, and also when $A \to B$ is left inverse to some $B \to A$ and the base change is induced by some base change on B ([**EGA** IV.16.2.2–3]): when φ is flat we have

$$(I/I^2) \otimes_B B' \cong I'/I'^2,$$

and when $B \to A$ is a ring homomorphism, having the given $A \to B$ as left inverse, and B' is a B-algebra, then with $A' = A \otimes_B B'$ (with the induced $\varphi\colon A \to A'$),

$$(I/I^2) \otimes_B B' \cong I'/I'^2.$$

If A' and B are A-algebras, and we set $B' = B \otimes_A A'$, then we have

$$\Omega_{B/A} \otimes_B B' \cong \Omega_{B'/A'}.$$

In other words, formation of $\Omega_{B/A}$ commutes with arbitrary base change. A reference is [**EGA** 0.20.5.5]; a hint to following the (terse) proof is to apply the second conormal sheaf isomorphism above to the diagram

$$\begin{tikzcd} B' \arrow[rr] & & B' \otimes_{A'} B' \arrow[rr] & & B' & \\ & B \arrow[ul] \arrow[rr] & & B \otimes_A B \arrow[ul] \arrow[rr] & & B \arrow[ul] \\ A \arrow[uu] \arrow[rr] & & B' \arrow[uu] & & & \\ & A \arrow[ul] \arrow[uu] \arrow[rr] & & B \arrow[ul] \arrow[uu] & & \end{tikzcd}$$

There are two fundamental exact sequences on differentials. *First fundamental exact sequence:* ([**66**, Theorem 25.1], [**EGA** 0.20.5.7]) Given ring homomorphisms $A \to B$ and $B \to C$, this is the exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

This is exact on the left as well when C is a formally smooth B-algebra. *Second fundamental exact sequence:* ([**66**, Theorem 25.2], [**EGA** 0.20.5.14]) As above, with $B \to C$ surjective with kernel J, this is the exact sequence

$$J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0.$$

If C is formally smooth over A then this is also left exact. These sequences are functorial [**EGA** 0.20.5.7.3], [**EGA** 0.20.5.11.3]. Explicitly, this means that if A' is an A-algebra, and we set $B' = B \otimes_A A'$ and $C' = C \otimes_A A'$ then the first fundamental exact sequence of the primed rings fits into a commutative diagram with that above, and the same is true for the second fundamental exact sequence when $B \to C$ is surjective.

If $f\colon X \to Y$ is a locally closed embedding of schemes, then f factors through some open subscheme $Y' \subset Y$ with $X \to Y'$ a closed embedding. Now there is a quasi-coherent sheaf of ideals $\mathcal{I}$ on Y' which defines the image of X as a subscheme. We call $f^*(\mathcal{I}/\mathcal{I}^2)$ the **conormal sheaf** to the embedding of X in Y. It is denoted $\mathcal{N}^*_{X/Y}$, or $\mathcal{N}^*_f$. If the restriction of f to an affine open subset $\operatorname{Spec} B$ of X is closed embedding to some affine open $\operatorname{Spec} A$ in Y then the restriction of $\mathcal{N}^*_{X/Y}$ to $\operatorname{Spec} B$ is the quasi-coherent sheaf associated to I/I^2, where I is the kernel of $A \to B$.

For a general map of schemes $X \to Y$, the relative diagonal $X \to X \times_Y X$ is an embedding The conormal sheaf to the relative diagonal is the **sheaf of relative differentials** of X over Y. It is denoted $\Omega_{X/Y}$, or Ω_f if f denotes the map of schemes.

Consider a cartesian diagram of schemes

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then we have an isomorphism

$$h^*\Omega_{X/Y} \cong \Omega_{X'/Y'}.$$

When f is a locally closed embedding, there is an induced morphism

$$h^*\mathcal{N}^*_{X/Y} \to \mathcal{N}^*_{X'/Y'}.$$

These are natural in the sense that in each case a morphism $Y'' \to Y'$ gives rise to a commutative triangle of sheaves on $X'' = X \times_Y Y''$ (this is immediate from the algebraic preliminaries for the morphism of conormal sheaves, and is established for the isomorphism of sheaves of differentials by extending the large diagram in Exercise **??**). The morphism of conormal sheaves is an isomorphism when g is flat.

The fundamental exact sequences, for schemes, read as follows. If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, then there is an exact sequence of sheaves on X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

If f is formally smooth then this is left exact as well. When f is a locally closed embedding, we have an exact sequence of sheaves on X

$$\mathcal{N}^*_{X/Y} \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0.$$

This is also left exact when $g \circ f$ is formally smooth. These exact sequences are natural in the sense that if $Z' \to Z$ is a morphism and we set $X' = X \times_Z Z'$ and $Y' = Y \times_Z Z'$ then there are commutative diagrams relating the sequences for the primed and unprimed schemes.

The sheaf Ω_f, quasi-coherent in general, is of finite type when f is locally of finite type [**EGA** IV.16.3.9]. If f is formally smooth and locally of finite type then Ω_f is locally free of finite type [**EGA** IV.17.2.3(i)]. For a smooth morphism (i.e., one that is formally smooth and locally of finite presentation) we use the notation of relative dimension, a locally constant function on the source. This is the rank of Ω_f.

Let X and Y be schemes over a base scheme S. Consider the fiber product $X \times_S Y$, with projections p to X and q to Y. Then ([**EGA** IV.16.4.23])

$$p^*\Omega_{X/S} \otimes q^*\Omega_{Y/S} \cong \Omega_{X\times_S Y/S}.$$

The following is an important consequence of the second fundamental exact sequence, for schemes. Let $f\colon X \to Y$ be a smooth morphism. Let $s\colon Y \to X$ be a section of f (so $f \circ s = 1_Y$). Then

$$\mathcal{N}_s^* \cong s^*\Omega_{X/Y}.$$

In this situation we have, further, that s is a **regular embedding**, meaning that in a neighborhood of a point $y \in Y$ if we let r denote the relative dimension of f at $s(y)$ then $s(Y)$ is defined near $s(y)$ by r equations, forming a regular sequence in the local ring $\mathcal{O}_{s(y),X}$. This fact follows from [**EGA** IV.17.12.1].

4. Grothendieck topologies

A **Grothendieck topology** $\mathcal{T}$ on a category $\mathcal{S}$ consists of a set $\mathrm{Cov}(\mathcal{T})$ of families of maps $\{\varphi_\alpha\colon U_\alpha \to U\}_{\alpha\in\mathcal{A}}$, with each φ_α a morphism in $\mathcal{S}$; these families, called **coverings**, must satisfy the following conditions:

(1) If $\varphi\colon V \to U$ is an isomorphism in $\mathcal{S}$, then $\{\varphi\colon V \to U\}$ is a covering.

(2) If $\{U_\alpha \to U\}_{\alpha\in\mathcal{A}}$ is a covering, and $\{V_{\alpha\beta} \to U_\alpha\}_{\beta\in\mathcal{B}_\alpha}$ is a covering for each α, then the family $\{V_{\alpha\beta} \to U\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}_\alpha}$, obtained by composition, is a covering.

(3) If $\{U_\alpha \to U\}_{\alpha\in\mathcal{A}}$ is a covering, and $V \to U$ is any morphism in $\mathcal{S}$, each fiber product $U_\alpha \times_U V$ must exist in $\mathcal{S}$, and $\{U_\alpha \times_U V \to V\}_{\alpha\in\mathcal{A}}$ is a covering.

A category with a Grothendieck topology is called a **site**.

When $\mathcal{S}$ is the category (Top) of topological spaces, taking the coverings of a space U by a family of open subspaces U_α forms a Grothendieck topology. Similarly when $\mathcal{S}$ is any category of schemes which contains any open subscheme of any scheme in it, one has the **Zariski topology**, where a covering is a family of Zariski open subsets U_α of U, with each φ_α the inclusion of U_α in U, such that U is the union of these open sets. The examples of most importance in this text are the **étale topology** and the **smooth topology**; in these the morphisms φ_α are taken to be étale resp. smooth, with the condition that U is the union of the images of the U_α. Similarly one has the **flat topology**, also called the **fppf topology**, where one requires that the morphisms in a covering are faithfully flat and locally of finite presentation.

In each of these topologies, if $\{U_\alpha \to U\}$ is a covering, then the morphism $V = \coprod U_\alpha \to U$ is an fppf morphism, which means that descent (Appendix A) can be applied. Note also that if U is a disjoint union of open schemes U_α, the family $\{U_\alpha \to U\}$ is a covering in any of these topologies.

Although this text does not require any sophisticated knowledge of Grothendieck topologies, more can be found in [**3**] and [**68**].

5. Sheaves and base change

Our aim here is to describe the basic base change homomorphisms for sheaves, including compatibilities for successive base changes, for which we could not find complete references. We will see that these compatibilites follow formally from properties of adjoint functors that appear in Appendix B.

For a sheaf $\mathcal{F}$ on a space X, we denote its sections over an open subset U of X either by $\mathcal{F}(U)$ or $\Gamma(U, \mathcal{F})$. The **stalk** $\mathcal{F}_x$ of $\mathcal{F}$ at a point x in X is the direct limit $\varinjlim \mathcal{F}(U)$, as U varies over open neighborhoods of x. For any continuous map $f\colon X \to Y$ of topological spaces, and a sheaf $\mathcal{F}$ on X, there is a **pushforward sheaf** $f_*(\mathcal{F})$ on Y, whose sections over an open subset V of Y are defined by the formula

$$f_*(\mathcal{F})(V) \;=\; \mathcal{F}(f^{-1}(V)).$$

This pushforward is functorial: if also $g\colon Y \to Z$, then

$$(g \circ f)_*(\mathcal{F}) = g_*(f_*(\mathcal{F})).$$

If $\mathcal{G}$ is a sheaf on Y, there is a sheaf $f^{-1}(\mathcal{G})$, whose sections over an open U in X are defined to be those collections of elements $(s'_x)_{x\in U}$, with s'_x in the stalk $\mathcal{G}_{f(x)}$, such that for any x_0 in U there is a neighborhood V of $f(x_0)$ in Y, a section $s \in \mathcal{G}(V)$ and a neighborhood W of x_0 contained in $U \cap f^{-1}(V)$, such that s'_x is the germ defined by s at $f(x)$ for all x in W. This gives a functor from sheaves on Y to sheaves on X, which is a left adjoint to f_*. That is, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, there is a canonical bijection

$$\mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}) \leftrightarrow \mathrm{Hom}(\mathcal{G}, f_*(\mathcal{F})).$$

In fact, an element on each side of this display can be identified with a collection of maps from $\mathcal{G}(V)$ to $\mathcal{F}(U)$, for all open $U \subset X$ and $V \subset Y$ with $f(U) \subset V$, such that whenever $U' \subset U$ and $V' \subset V$, with $f(U') \subset V'$, the diagram

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{G}(V') \arrow[r] & \mathcal{F}(U') \end{tikzcd}$$

commutes. This bijection is natural in morphisms of sheaves on X and Y, and makes f^{-1} a left adjoint of f_*, and f_* a right adjoint of f^{-1}, see [**EGA** 0.3.5, 0.3.7].

The same formula for the pushforward works when f is a morphism of ringed spaces, and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, in which case $f_*(\mathcal{F})$ is a sheaf of $\mathcal{O}_Y$-modules. Here f_* defines a functor $f_*\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ from the category $\mathcal{S}(X)$ of sheaves of $\mathcal{O}_X$-modules to the category $\mathcal{S}(Y)$ of sheaves of $\mathcal{O}_Y$-modules. A left adjoint to this functor f_* is denoted f^*; this is constructed to be the functor that takes a sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules to the sheaf

$$f^*(\mathcal{G}) := \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{G}).$$

(Here, to be precise, one should make a choice of this tensor product.) This gives a functor $f^*\colon \mathcal{S}(Y) \to \mathcal{S}(X)$ from sheaves of $\mathcal{O}_Y$-modules to sheaves of $\mathcal{O}_X$-modules, and one again has a canonical bijection

$$\mathrm{Hom}(f^*(\mathcal{G}), \mathcal{F}) \leftrightarrow \mathrm{Hom}(\mathcal{G}, f_*(\mathcal{F})),$$

making f_* and f^* adjoint functors ([**EGA** 0.3.5, 0.4.4]), [**47**, §II.5]).

Because of the choice of tensor product in the definition, if $g\colon Y \to Z$ is another morphism, and $\mathcal{H}$ is a sheaf of $\mathcal{O}_Z$-modules, then $(g \circ f)^*(\mathcal{H})$ is not strictly equal to $f^*(g^*(\mathcal{H}))$, but there is a canonical isomorphism between them.

This adjoint pair comes equipped with canonical natural transformations $\epsilon = \epsilon^f\colon 1_{\mathcal{S}(Y)} \Rightarrow f_* \circ f^*$, and $\delta = \delta^f\colon f^* \circ f_* \Rightarrow 1_{\mathcal{S}(X)}$. For a sheaf $\mathcal{G}$ on Y we have a canonical morphism $\epsilon\colon \mathcal{G} \to f_*(f^*(\mathcal{G}))$, functorial in $\mathcal{G}$; and for a sheaf $\mathcal{F}$ on X, we have a canonical morphism $\delta\colon f^*(f_*(\mathcal{F})) \to \mathcal{F}$, functorial in $\mathcal{F}$. Explicitly, a section of $\mathcal{G}$ on an open V in Y determines a section of $f^*(\mathcal{G})$ on $f^{-1}(V)$, and hence a section of $f_*(f^*(\mathcal{G}))$ on V. A section of $f^*(f_*(\mathcal{F}))$ on an open U of X determines an element of the stalk $\mathcal{F}_x$ at all x in U, and these come from a section of $\mathcal{F}$ on U.

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is **quasi-coherent** if, for all x in X, there is a neighborhood U of x and a presentation $\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F} \to 0$, for some (not necessarily finite) index sets I and J. If $\mathcal{G}$ is quasi-coherent on Y, then $f^*(\mathcal{G})$ is always quasi-coherent on X. If f is a quasi-compact and quasi-separated morphism of schemes, and $\mathcal{F}$ is quasi-coherent on X, then $f_*(\mathcal{F})$ is quasi-coherent on Y (see [**EGA** I.9.2.1]). With these hypotheses, f_* and f^* are adjoint functors between quasi-coherent sheaves on X and quasi-coherent sheaves on Y.

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is of **finite type** if any point has an open neighborhood U on which there is a surjection $\mathcal{O}_U^n \to \mathcal{F}\,|_U \to 0$ for some integer n, The sheaf is **coherent** if, in addition, for all open subsets U of X, the kernel of any homomorphism $\mathcal{O}_U^m \to \mathcal{F}$ of $\mathcal{O}_U$-modules is of finite type. If a scheme X is **locally Noetherian**, i.e., it has a covering by open subschemes isomorphic to the Spec's of Noetherian rings, then $\mathcal{O}_X$ is coherent. For any morphism $f\colon X \to Y$, if $\mathcal{G}$ is coherent on Y, and if $\mathcal{O}_X$ is coherent on X, then $f^*(\mathcal{G})$ is coherent on X ([**EGA** 0.5.3]). If $f\colon X \to Y$ is a proper morphism, with Y locally Noetherian, and $\mathcal{F}$ is a coherent sheaf on X, then $f_*(\mathcal{F})$ is a coherent sheaf on Y; in fact, all the higher direct images $R^n f_*(\mathcal{F})$ are coherent ([**EGA** III.3.2.1]).

Now consider a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Z \end{array}$$

of ringed spaces. There is an associated **base change** map

$$p^*(f_*(\mathcal{F})) \;\to\; g_*(q^*(\mathcal{F}))$$

for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules.[9] This is a formal consequence of adunction (Section B.3). To construct this base change map, consider the diagram

$$\begin{tikzcd}
\mathcal{S}(W) \arrow[r, "g_*"] \arrow[d, "q_*"'] & \mathcal{S}(Y) \arrow[d, "p_*"] \\
\mathcal{S}(X) \arrow[r, "f_*"'] & \mathcal{S}(Z)
\end{tikzcd}$$

This diagram strictly commutes: $f_* \circ q_* = (f \circ q)_* = (p \circ g)_* = p_* \circ g_*$. Therefore we can take $\alpha\colon f_* \circ q_* \Rightarrow p_* \circ g_*$ to be the identity map. From Definition B.21, this gives a base change natural transformation (2-morphism) $c = c_\alpha$ from $p^* \circ f_*$ to $g_* \circ q^*$. The base change transformation $p^* \circ f_* \Rightarrow g_* \circ q^*$ is chararacterized either by the fact that its adjoint with respect to p is the transformation $f_* = f_* \circ 1_{\mathcal{S}(X)} \overset{\epsilon^q}{\Rightarrow} f_* \circ q_* \circ q^* = p_* \circ g_* \circ q^*$, or that its adjoint with respect to g is the composite $g^* \circ p^* \circ f_* \overset{\alpha'}{\Rightarrow} q^* \circ f^* \circ f_* \overset{\delta f}{\Rightarrow} q^*$ (by Exercise B.43(3)).[10]

In this setting, these formal adjoint constructions can be made explicit. If $U \subset Y$ and $U' \subset Y'$ are open subsets with $p(U') \subset U$, a section s in $(f_*\mathcal{F})(U)$ determines a section s' in $p^*(f_*\mathcal{F})(U')$, and also a section s'' in $(g_*(q^*(\mathcal{F})))(U') = (q^*(\mathcal{F}))(q^{-1}(U'))$. Show that the base change map c takes s' to s'', and show that c is determined by this property. The corresponding $\alpha'\colon g^* \circ p^* \overset{\sim}{\Rightarrow} q^* \circ f^*$ agrees with the composition of the canonical isomorphisms $g^* \circ p^* \cong (p \circ g)^* = (f \circ q)^* \cong q^* \circ f^*$ (cf. [**EGA** 0.3.5.5]).

Consider a commutative diagram of ringed spaces,

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[d, "r"'] \arrow[dr, phantom, "\beta"] & W \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\alpha"] & Y \arrow[d, "p"] \\
V \arrow[r, "h"'] & X \arrow[r, "f"'] & Z
\end{tikzcd}$$

with two commuting squares labeled with α and β, and label the outside square with γ:

$$\begin{tikzcd}
U \arrow[r, "g\circ i"] \arrow[d, "r"'] \arrow[dr, phantom, "\gamma"] & Y \arrow[d, "p"] \\
V \arrow[r, "f\circ h"'] & Z
\end{tikzcd}$$

It follows from Exercise B.44 that the following diagram commutes:

$$\begin{tikzcd}
p * \circ f_* \circ h_* \arrow[r, equal] \arrow[d, Rightarrow, "c_\alpha"'] & p^* \circ (f \circ h)_* \arrow[drr, Rightarrow, "c_\gamma"] & & \\
g_* \circ q^* \circ h_* \arrow[r, Rightarrow, "c_\beta"'] & g_* \circ i_* \circ r^* \arrow[rr, equal] & & (g \circ i)_* \circ r^*
\end{tikzcd}$$

[9] The same works for topological spaces, replacing pullbacks g^* by g^{-1}.

[10] See [**42**, XII.4, XVII.2.1] for a discussion about this point.

One has also the opposite base change maps $c'_\alpha\colon f^*\circ p_* \Rightarrow q_*\circ g^*$, $c'_\beta\colon h^*\circ q_* \Rightarrow r_*\circ i^*$, and $c'_\gamma\colon (f\circ h)^*\circ p_* \Rightarrow r_*\circ(g\circ i)^*$. The same Exercise B.44 gives a commutative diagram

$$\begin{tikzcd} h^*\circ f^*\circ p_* \arrow[r, Rightarrow, no head, "\sim"] \arrow[d, Rightarrow, "c'_\alpha"'] & (f\circ h)^*\circ p_* \arrow[dr, Rightarrow, "c'_\gamma"] & \\ h^*\circ q_*\circ g^* \arrow[r, Rightarrow, "c'_\beta"'] & r_*\circ i^*\circ g^* \arrow[r, Rightarrow, no head, "\sim"] & r_*\circ(g\circ i)^* \end{tikzcd}$$

(cf. [**42**, XII.4.4]).

From Exercise B.43(1), we get commutative diagrams

$$\begin{tikzcd} p^* \arrow[r, Rightarrow, "\epsilon^f"] \arrow[d, Rightarrow, "\epsilon^g"'] & p^*\,f_*\,f^* \arrow[d, Rightarrow, "c"] \\ g_*\,g^*\,p^* \arrow[r, Rightarrow, "\alpha'"'] & g_*\,q^*\,f^* \end{tikzcd} \qquad \begin{tikzcd} g^*\,p^*\,f_* \arrow[r, Rightarrow, "\alpha'"] \arrow[d, Rightarrow, "c"'] & q^*\,f^*\,f_* \arrow[d, Rightarrow, "\delta^f"] \\ g^*\,g_*\,q^* \arrow[r, Rightarrow, "\delta^g"'] & q^* \end{tikzcd} \qquad \begin{tikzcd} f_* \arrow[r, Rightarrow, "\epsilon^p"] \arrow[d, Rightarrow, "\epsilon^q"'] & p_*\,p^*\,f_* \arrow[d, Rightarrow, "c"] \\ f_*\,q_*\,q^* \arrow[r, Rightarrow, no head] & p_*\,g_*\,q^* \end{tikzcd}$$

The same adjoint formalism applies in the context of sheaves on arbitrary sites. It can also be applied with higher direct images. To see this, note that if $f\colon X \to Y$ and $g\colon Y \to Z$ are mappings, the Leray spectral sequence gives (edge homomorphism) mappings

$$R^n g_*(f_*\mathcal{F}) \to R^n(g\circ f)_*(\mathcal{F}) \to g_*(R^n f_*(\mathcal{F}))$$

(cf. [**EGA** III.12.2.5]). In particular, given a commutative diagram as above, and any $n \geq 0$, one has a natural transformation $\alpha\colon R^n f_*\circ q_* \Rightarrow p_*\circ R^n g_*$, given by

$$R^n f_*(q_*(\mathcal{F})) \to R^n(f\circ q)_*(\mathcal{F}) = R^n(p\circ g)_*(\mathcal{F}) \to p_*(R^n g_*(\mathcal{F})).$$

By the formal properties of adjoints, this determines a natural transformation c_α from $p^*\circ R^n f_*$ to $R^n g_*\circ q^*$. In particular we have homomorphisms

$$p^*(R^n f_*(\mathcal{F})) \to R^n g_*(q^*(\mathcal{F})),$$

which are natural in $\mathcal{F}$. One has the same compatibility as before, when two commutative diagrams are pasted together, again by formal properties of adjoint functors.[11]

The same formalism applies when one has adjoint functors Rf_* and Lf^* on derived categories (e.g. [**46**], Cor. 5.11), giving natural base change maps

$$(Lp^*)\circ(Rf_*)(\mathcal{F}^{\cdot}) \;\to\; (Rg_*)\circ(Lq^*)(\mathcal{F}^{\cdot}),$$

with the corresponding compatibilites when two commutative diagrams are combined.

[11] These base change maps agree with those constructed under additional hypotheses in [**EGA** III.1.4.15], [**EGA** IV.1.7.21], and [**47**, §III.9.3].

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