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GRAPH-DIFFERENT PERMUTATIONS*

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Abstract. For a finite graph G whose vertices are different natural numbers we call two infinite permutations of the natural numbers G-different if they have two adjacent vertices of G somewhere in the same position. The maximum number of pairwise G-different permutations of the naturals is always finite. We study this maximum as a graph invariant and relate it to a problem of the first two authors on colliding permutations. An improvement on the lower bound for the maximum number of pairwise colliding permutations is obtained.

Key words. extremal combinatorics, Shannon capacity of graphs, permutations

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1. Introduction.

In [1] the first two authors began to investigate the following mathematical puzzle. Call two permutations of $[n] := \{1, \ldots, n\}$ *colliding* if, represented by linear orderings of $[n]$, they put two consecutive elements of $[n]$ somewhere in the same position. For the maximum cardinality $\rho(n)$ of a set of pairwise colliding permutations of $[n]$ the following conjecture was formulated.

CONJECTURE 1 (see [1]). *For every $n \in \mathbb{N}$*

$$\rho(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

It was observed in [1] that the right-hand-side expression above is actually an upper bound for $\rho(n)$, while the best lower bound given there was a somewhat deceiving

$$35^{n/7-O(1)} \leq \rho(n). \tag{1}$$

The initial motivation for the present work was to improve on the above lower bound. However, our main purpose in this paper is to study the new graph invariant mentioned in the abstract. We will analyze its possible values for simple classes of graphs and then apply some of the results to obtain a new lower bound for $\rho(n)$.

For brevity's sake let us call a graph *natural* if its vertex set is a finite subset of $\mathbb{N}$, the set of all positive integers, and if the graph is simple (without loops and multiple edges). An infinite permutation of $\mathbb{N}$ is simply a linear ordering of all the elements of $\mathbb{N}$. (Instead of *infinite permutations of* $\mathbb{N}$ we will often say simply *infinite permutations* in what follows.) For an arbitrary natural graph $G = (V(G), E(G))$ we will call the infinite permutations $\pi = (\pi(1), \pi(2), \ldots, \pi(n), \ldots)$ and $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n), \ldots)$ *G-different* if there is at least one $i \in \mathbb{N}$ for which

$$\{\pi(i), \sigma(i)\} \in E(G).$$

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(We will use the same expression for a pair of finite sequences if at some coordinate they contain the two endpoints of an edge of G.) Let $\kappa(G)$ be the maximum cardinality of a set of infinite permutations any two elements of which are G-different. (It is easy to see that the finiteness of G implies that this number is finite as well; see Lemma 1 below.) Clearly, the value of κ is equal for isomorphic natural graphs. In this paper we will analyze this quantity for some elementary graphs and will apply some of the results to the problem of $\rho(n)$. We have been able to determine the value of $\kappa(G)$ only for some very small or simply structured graphs G. Thus, to further simplify matters, we ask questions about the extremal values of κ for graphs with a fixed number of edges (and, eventually, vertices). We define

$$K(\ell) = \max\{\kappa(G)\ ;\ \ |E(G)| = \ell\} \tag{2}$$

and

$$k(\ell) = \min\{\kappa(G)\ ;\ \ |E(G)| = \ell\} \tag{3}$$

as well as

$$K(v,\ell) = \max\{\kappa(G)\ ;\ |V(G)| = v,\ |E(G)| = \ell\}. \tag{4}$$

We also conjecture the following.

CONJECTURE 2. *For every $\ell \in \mathbb{N}$*

$$K(\ell) = 3^{\ell}.$$

In fact, we will show that $K(\ell)$ lies between 3^{ℓ} and 4^{ℓ} for every natural number ℓ. We will also see that $k(\ell)$ is linear in ℓ.

As we will explain, the values of $\kappa(P_r)$, where P_r is the r-vertex path, are relevant when investigating colliding permutations. Giving a lower bound on $\kappa(P_4)$, the lower bound of (1) will be improved to $10^{n/4-O(1)}$.

The concept of graph-different sequences from a fixed alphabet goes back to Shannon's classical paper on zero-error capacity [3]. As explained in the survey [2], a large body of problems in extremal combinatorics can be treated as zero-error problems in information theory. For the relationship of the present problems to zero-error information theory, we refer to [1].

2. Natural graphs and infinite permutations. Let G be a natural graph, and let again $\kappa(G)$ be the maximum cardinality of a set of infinite permutations any two elements of which are G-different, provided that this number is finite. It is easy to see that this is always the case. Let $\chi(G)$ denote the chromatic number of graph G.

LEMMA 1. *For every natural graph*

$$\kappa(G) \le (\chi(G))^{|V(G)|}$$

holds.

Proof. Let us consider a proper coloring $c : V(G) \to \{1,\dots,\chi(G)\}$ of G. Let us write $v = |V(G)|$ and denote by $W = [\chi(G)]^{\mathbb{N}}$ the set of infinite sequences over the alphabet $\{1,\dots,\chi(G)\}$. Let μ be the uniform probability measure on W. Let us consider a set C of pairwise G-different permutations. We assign to any $\pi \in C$ the set $W(\pi)$ of all those sequences of W that for all $u \in V(G)$ have the element $c(u)$ in the position where π contains u. By our hypothesis on C, the sets $W(\pi)$ are pairwise disjoint for the different elements $\pi \in C$, whence

$$1 = \mu(W) \geq \sum_{\pi \in C} \mu(W(\pi)) = \sum_{\pi \in C} \chi(G)^{-v} = |C|\chi(G)^{-v}. \qquad \square$$

In the rest of this section we first investigate $K(\ell)$ and $k(\ell)$. Subsequently our new lower bound on $\rho(n)$ will be proved.

Let us denote by $S(G)$ the set of nonisolated vertices of the graph G. We introduce a graph transformation that increases the value of κ.

PROPOSITION 1. *Let F and G be two graphs with G obtained from F upon deleting an arbitrary edge in $E(F)$ followed by the addition of two new vertices to $V(F)$ so that the latter form an additional edge in G. Then*

$$\kappa(F) \leq \kappa(G).$$

Proof. Let us consider the $m = \kappa(F)$ pairwise F-different infinite permutations of an arbitrary optimal configuration for F. Let t be large enough for the initial prefixes of length t of these infinite sequences to be pairwise F-different, and let q be the largest integer appearing in their coordinates. By the finiteness of $\kappa(F)$ such t and q exist. Without restricting generality we can suppose that the new edge of G is $\{c, d\}$ with both c and d being strictly larger than q. We also suppose that the edge that we will delete is $\{a, b\} \in E(F)$. Let us now suffix to each of our length-t sequences a new sequence of the same length t, where the suffix to a sequence $x_1x_2 \dots x_t$ is obtained from it by substituting every a with c and every b with d while the remaining coordinates are defined in an arbitrary manner but in such a way that the coordinates of the overall sequence of length $2t$ are all different. Clearly, the m new sequences of length $2t$ are G-different. To conclude, we can complete these new sequences to yield infinite permutations any way we like. $\square$

A straightforward consequence of the previous proposition is the following.

COROLLARY 1. $K(\ell) = \kappa(\ell K_2)$. $\square$

Thus we know that $K(\ell)$ is achieved by ℓ independent edges. It seems equally interesting to determine which graphs achieve $k(\ell)$. At first glance one might think that $S(F) \subseteq S(G)$ implies $\kappa(F) \leq \kappa(G)$, but this is false. In particular, complete graphs do not have minimum κ among graphs with the same number of edges. Yet, determining their κ value seems an interesting problem. As we will see, the right answer for what graphs achieve $k(\ell)$ turns out to be stars, at least for ℓ not too small. Below we will study the value of κ for complete graphs, stars, and paths. In particular, path graphs will take us back to the original puzzle about colliding permutations.

PROPOSITION 2. *For the complete graph K_n on n vertices*

$$\frac{(n+1)!}{2} \leq \kappa(K_n).$$

Proof. Consider the set of even permutations of $[n+1]$, and suppose $V(K_n) = [n]$. One can observe that these permutations are K_n-different. Indeed, if two arbitrary permutations of $[n+1]$ are not K_n-different, then they differ only in positions in which for some fixed $i \in [n]$ one has $n+1$ and the other has i. Thus any of these two permutations can be obtained from the other by exchanging the positions of $n+1$ and the corresponding i. But then the two permutations have different parity, and in particular they cannot both be even. In particular, the undesired relation does not occur between even permutations, and this gives us $\frac{(n+1)!}{2}$ permutations of $[n+1]$ that are K_n-different. Next extend each of these permutations to infinite ones by suffixing the remaining natural numbers in an arbitrary order. $\square$

PROPOSITION 3. *For the graph of ℓ independent edges we have*

$$3^\ell \le \kappa(\ell K_2) \le 4^\ell.$$

Proof. Notice that the graph ℓK_2 has chromatic number two, and its number of vertices is 2ℓ, whence our upper bound follows by Lemma 1.

To prove the lower bound, let us denote the edge set of our graph by $E(\ell K_2) = \{\{1,2\},\{3,4\},\dots,\{2\ell-1,2\ell\}\}$. Consider the set of cyclic permutations $C_1 = \{(12\star),(2\star1),(\star12)\}$ and for every $1 < i \le \ell$ the sets C_i obtained from C_1 by replacing 1 with $2i-1$ and 2 with $2i$. It is clear that for every $i \in [\ell]$ any two of the three ministrings in C_i "differ" in the edge $\{2i-1,2i\}$ of our graph ℓK_2, meaning that they have somewhere in the same position the two different endpoints of this edge. But this means that the 3^ℓ strings in their Cartesian product

$$C = \times_{i=1}^{\ell} C_i$$

are pairwise ℓK_2-different as requested. Replacing the symbol $\star$ in our strings in an arbitrary order with the different numbers from $[3\ell]-[2\ell]$, we obtain 3^ℓ permutations of $[3\ell]$ that continue to be pairwise ℓK_2-different. The extension to infinite permutations is as always. $\square$

The only infinite class of graphs for which we are able to determine κ are the stars, i.e., the complete bipartite graphs $K_{1,r}$. We have the following result.

PROPOSITION 4. *For every r*

$$\kappa(K_{1,r}) = 2r+1.$$

Proof. By Lemma 1 we know that $\kappa(K_{1,r}) < \infty$. Let us denote its value by m. Let us consider the vertices of $K_{1,r}$ to be the elements of $[r+1]$, and let 1 be the "central" vertex of degree r. It is obvious that in a set of m sequences (infinite permutations) achieving the maximum we are looking for, all the sequences must have the central vertex 1 in a different position. Let us consider our m sequences as vertices of a directed graph T in which $(a,b) \in E(T)$ if the sequence corresponding to a has a $j \in \{2,\dots,r+1\}$ in the same position where the 1 of the sequence corresponding to b is placed. Then, by definition, the directed graph T must contain a tournament, implying that

$$|E(T)| \ge \binom{m}{2}.$$

On the other hand, every $a \in V(T)$ has at most r outgoing edges. This means that

$$|E(T)| \le mr.$$

Comparing the last two inequalities, we get

$$m \le 2r+1.$$

To prove a matching lower bound, consider the following set of permutations of $[2r+1]$. For every $i \in [2r+1]$ let us define the coordinates of the ith sequence

$x_1(i)x_2(i)\dots x_{2r+1}(i)$ by $x_i(i) = 1$ and, in general, $x_{i+j}(i) = j + 1$ for any $0 \le j \le r$, where all the coordinate indices are considered modulo $2r + 1$. The remaining coordinates are defined in an arbitrary manner so that the resulting sequences define permutations of $[2r + 1]$. It is easily seen that this is a valid construction. In fact, observe that for any of our sequences the "useful" symbols, those of $[r + 1]$, corresponding to the vertices of the star graph, occupy $r + 1$ "cyclically" consecutive coordinates, forming cyclic intervals. Since $2(r + 1) > 2r + 1$, these intervals are pairwise intersecting, and thus for any two of them there must be a coordinate in the intersection for which the "left end" of one of the intervals is contained in the other. The resulting permutations can be considered as prefixes of infinite permutations in the usual obvious way. □

Now we are ready to return to the problem of determining $k(\ell)$, at least for large enough ℓ. The following easy lemma will be needed.

LEMMA 2. *If a finite graph F contains vertex disjoint subgraphs $F_1, \dots, F_s$, then*

$$\kappa(F) \ge \prod_{i=1}^{s} \kappa(F_i).$$

Proof. The proof is a straightforward generalization of the construction given in the proof of Proposition 3. Let $\hat{C}_i$ be a set of $\kappa(F_i)$ infinite sequences that are obtained from $\kappa(F_i)$ pairwise F_i-different permutations of $\mathbb{N}$ by substituting all natural numbers $i \notin V(F_i)$ by a $\star$. As the sequences in $\hat{C}_i$ contain only a finite number of elements different from $\star$, we can take some finite initial segment of all these sequences that already contains all elements of $V(F_i)$. Let C_i be the set of these finite sequences. Now consider the set

$$C := \times_{i=1}^{s} C_i$$

of finite sequences that each contain all vertices in $\bigcup_{i=1}^{s} V(F_i)$ exactly once and finitely many $\star$'s. These sequences are also pairwise F_i-different for some F_i, and thus they are pairwise F-different. By construction, their number is $\prod_{i=1}^{s} \kappa(F_i)$. Extending them to infinite permutations of $\mathbb{N}$, the statement is proved. □

It is straightforward from the previous lemma that if $F + G$ denotes the vertex disjoint union of graphs F and G, then $\kappa(F + G) \ge \kappa(F)\kappa(G)$. We do not know any example for strict inequality here. If equality was always true, that would immediately imply Conjecture 2.

Now we use Lemma 2 to prove our main result on $k(\ell)$.

PROPOSITION 5. *Let G be a natural graph with $n := |S(G)| > 20$ and $|E(G)| = \ell$. Then*

$$\kappa(G) \ge 2\ell + 1.$$

The value of $k(\ell)$ is achieved by the graph $K_{1,\ell}$ whenever $\ell > 150$.

Proof. Let G be a graph as in the statement, and let $\nu = \nu(G)$ denote the size of a largest matching in G.

First assume that $\nu \ge n/4$. Then by Proposition 3 and the obvious monotonicity of κ we have

$$\kappa(G) \ge \kappa(\nu K_2) \ge 3^{\nu}.$$

Since G is simple, we have $\ell \le \binom{n}{2}$, and thus $k(\ell) \le \kappa\left(K_{1,\binom{n}{2}}\right) = n(n-1) + 1$. So in this case (when $\nu \ge n/4$) it is enough to prove that

$$3^{\lceil n/4 \rceil} \ge n(n-1) + 1$$

holds. This is true if $n > 20$.

Next assume that $3 \le \nu < n/4$. Consider a largest matching of G consisting of edges $\{u_{2i-1}, u_{2i}\}$ with $i = 1, \dots, \nu$. The set $U := \{u_1, \dots, u_{2\nu}\}$ covers all edges of G, and thus $\ell \le \binom{2\nu}{2} + 2\nu(n - 2\nu)$. So we have

$$k(\ell) \le \kappa(K_{1,\ell}) \le 2\left[\binom{2\nu}{2} + 2\nu(n-2\nu)\right] + 1$$

in this case. On the other hand, for each vertex $a \in S(G) \setminus U$ there is an edge $\{a, u_i\}$ for some i. We also know that if a and b are two distinct vertices in $S(G) \setminus U$ and one of them is connected to u_{2j-1} (resp., u_{2j}) for some j, then the other one cannot be connected to u_{2j} (resp., u_{2j-1}), since otherwise replacing the matching edge $\{u_{2j-1}, u_{2j}\}$ with the other two edges of the path formed by the vertices a, u_{2j-1}, u_{2j}, b would result in a larger matching, a contradiction. Choosing an edge for each $a \in S(G) \setminus U$ that connects it to a vertex in U, we can form vertex disjoint star subgraphs $K_{1,\ell_1}, \dots, K_{1,\ell_\nu}$ of G, where $\ell_i \ge 1$ for all i and $\sum_{i=1}^{\nu} \ell_i = n - \nu$. Then by Lemma 2 and Proposition 4 we have $\kappa(G) \ge \prod_{i=1}^{\nu}(2\ell_i + 1)$. The latter product is minimal (with respect to the conditions on the ℓ_i's) if one ℓ_i, say ℓ_1, equals to $n - 2\nu + 1$ and $\ell_2 = \dots = \ell_\nu = 1$. Thus it is enough to prove that

$$3^{\nu-1}(2(n-2\nu+1)+1) > 2\left[\binom{2\nu}{2} + 2\nu(n-2\nu)\right] + 1,$$

as the left-hand side is a lower bound on $\kappa(G)$, while the right-hand side is an upper bound on $k(\ell)$. The latter inequality would be implied by

$$3^{\nu-1}(n-2\nu+1) > \binom{2\nu}{2} + 2\nu(n-2\nu) = \nu(2n-2\nu-1),$$

which, in turn, is equivalent to

$$\frac{3^{\nu-1}}{\nu} > \frac{2n-2\nu-1}{n-2\nu+1}.$$

The left-hand side of this last inequality is at least 3 if $\nu \ge 3$, while the right-hand side is strictly less than 3 for $\nu \le n/4$.

The only case not yet covered is that of $\nu < 3$. For $\nu = 1$ there is nothing to prove, since then G itself is a star. If $\nu = 2$, then let a largest matching be formed by the two edges $\{u_1, u_2\}$ and $\{u_3, u_4\}$, while once again let U denote the union of their vertices. Let $a_1, \dots, a_{n-4}$ be the rest of the nonisolated vertices of G, and note that $n - 4 > 16$. Assume that some a_i is connected to both u_1 and u_2, yielding a triangle. Then no a_j, $j \ne i$, can be connected to either of u_1 or u_2; otherwise we could form a larger matching. For similar reasons, if any a_j is connected to u_3, then no a_s, $s \ne j$, can be connected to u_4. (If some a_i forms a triangle with u_1, u_2 and some a_j with u_3 and u_4, then the remaining vertices a_s must be isolated, implying $n \le 6$, a contradiction.) Thus if a_i is connected to both u_1 and u_2, then the rest of the a_j's form a star centered at either u_3 or u_4. Thus in this case, using again Lemma 2, Propositions 2 and 4 imply $\kappa(G) \ge 12[2(n-4)+1] = 24n - 84$. The foregoing also implies $\ell \le n + 4$, thus $k(\ell) \le 2n + 9 < 24n - 84$, whenever $n \ge 5$. Clearly, the situation is similar if we exchange the role of the two matching edges.

Assuming that no triangle is formed, we can again attach each vertex in $S(G) \setminus U$ to one of the edges $\{u_1, u_2\}$ and $\{u_3, u_4\}$, whichever it is connected to. Two vertex disjoint stars can be formed this way, establishing the lower bound $\kappa(G) \ge$

$3(2(n-3)+1) = 6n-15$. For the number of edges we now get $\ell \leq 6+2(n-4) = 2n-2$ since the graph induces at most 6 edges on U. Thus we have $k(\ell) \leq 4n-3$, which is less then $6n-15$ if $n > 6$. This completes the proof of the first statement.

If a simple graph has at most 20 vertices, then its number of edges is at most 190, so the second statement immediately follows from the first one if $\ell > 190$. If the graph contains a K_6 subgraph, then by Proposition 2 we have $\kappa(G) \geq 7!/2 > 381 = \kappa(K_{1,190}) \geq k(\ell)$ if $\ell \leq 190$. Thus we may assume $K_6 \nsubseteq G$, and this implies by Turán's theorem that $\ell \leq 160$ if $n \leq 20$. But $\kappa(K_{1,160}) = 321 \leq 6!/2$, so if the conclusion is not true, we may also assume that G has no K_5 subgraph. Applying Turán's theorem again, this gives $\ell \leq 150$ for $n \leq 20$. Thus the statement is true whenever $\ell > 150$. ◻

Remark 1. We believe that the statement of Proposition 5 holds without any restriction on n or ℓ. Some improvement on our threshold on ℓ is easy to obtain. It seems to us, however, that proving the statement in full generality either leads to tedious case checkings or needs some new ideas.

The problem of determining κ seems interesting in itself; moreover, it helps to obtain better bounds for the original question on colliding permutations. To explain this, we introduce a notion connecting the two questions. Let $\kappa(G, n)$ be the maximum number of pairwise G-different permutations of $[n]$. Clearly,

$$\kappa(G) = \sup_n \kappa(G, n). \tag{5}$$

Notice that by the finiteness of $\kappa(G)$ the supremum above is always attained, so we could write maximum instead. Further, for the graph P_r, the path on r vertices, we have the following.

LEMMA 3. *For every $n > m > r$ the function ρ satisfies the recursion*

$$\rho(n) \geq \kappa(P_r, m)\rho(n-r).$$

Proof. We will call two arbitrary sequences of integers *colliding* if they have the same length and if somewhere in the same position they feature integers differing by 1. By the definition of $\kappa(P_r, m)$ we can construct this many sequences of length m such that in each of them every vertex of P_r appears exactly once, the other positions are occupied by the "dummy" symbol $\star$, and moreover these sequences are pairwise P_r-different. The latter implies that these sequences are pairwise colliding. Furthermore, we have, also by definition, $\rho(n-r)$ permutations of $[n-r]$ that are pairwise colliding. Let us "shift" these permutations by adding r to all of their coordinates. The new set of permutations of the set $r + [n-r] = [r+1, n]$ maintains the property that its elements are pairwise colliding. Next we execute our basic operation of "substituting" the permutations of the second set into those coordinates of any sequence $\boldsymbol{x}$ from the first set where the sequence $\boldsymbol{x}$ has a star. More precisely, consider any sequence $\boldsymbol{x} = x_1 x_2 \dots x_m$ from our first set, and let $S(\boldsymbol{x}) \in \binom{[m]}{m-r}$ be the set of those coordinates which are occupied by stars. Let further $\boldsymbol{y} = y_1 y_2 \dots y_{n-r}$ be an arbitrary sequence from our second set, i.e., a permutation of $[r+1, n]$. The sequence $\boldsymbol{z} = \boldsymbol{y} \to \boldsymbol{x}$ is a sequence of length n in which the first m coordinates are defined in the following manner. We have the equality $z_i = x_i$, if $i \leq m$ and $i \notin S(\boldsymbol{x})$. Suppose further that $S(\boldsymbol{x}) = \{j_1, j_2, \dots, j_{m-r}\}$. In the j_kth position we replace the symbol $\star$ by y_k. (For $i > m$ we set $z_i = y_{i-r}$.) Clearly, the resulting sequence is a permutation of $[n]$. Further, the so obtained $\kappa(P_r, m)\rho(n-r)$ permutations are pairwise colliding. ◻

Observe next the following equality.

LEMMA 4.

$$\kappa(P_4, 5) = 10.$$

Remark 2. The existence of 10 permutations of $\{1, \dots, 5\}$ with the requested properties is implicit in [1] since the construction of the 35 colliding permutations of $\{1, \dots, 7\}$ in that paper does contain such a set in some appropriate projection of its coordinates.

Proof. Let us consider the 10 permutations of $\{1, \dots, 5\}$ obtainable by considering the cyclic configurations of $(1, 2)$, $(4, 3)$ and the single element 5. We indeed have 10 different permutations by "cutting" in all the five possible ways both of the two cyclic configurations that three building blocks can define. (So these are 12435, 24351, 43512, 35124, 51243 and similarly the five cyclic shifts of the sequence 43125.) Let us further consider the graph P_4 (or, in fact, $P_4 + K_1$) with vertex set $\{1, \dots, 5\}$ and with edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. In other words, consecutive numbers are adjacent vertices, but 5 is isolated. It is easy to check that the 10 sequences above are P_4-different for the natural graph we defined.

To see that 10 is an upper bound it is enough to observe that the two even elements of $\{1, \dots, 5\}$ cannot be placed in the same two positions in two permutations belonging to a set of P_4-different permutations of $\{1, \dots, 5\}$. ∎

The above construction gives the following improved lower bound for the exponential asymptotics of $\rho(n)$.

PROPOSITION 6.

$$\lim_{n \to \infty} \rho^{\frac{1}{n}}(n) \geq 10^{\frac{1}{4}}.$$

Proof. A simple combination of our two preceding lemmas implies

$$\rho(n) \geq 10\rho(n - 4).$$

An iterated application of this inequality gives the desired result. ∎

To close this section, let us take another look at Lemma 4. We believe that, in fact, $\kappa(P_4) = \kappa(P_4, 5) = 10$ and more generally,

$$\kappa(P_v) = \kappa(P_v, v + 1) = \binom{v + 1}{\lfloor \frac{v+1}{2} \rfloor}$$

for even values of v. The original conjecture (see Conjecture 1) for $\rho(v)$ would be an immediate consequence of this conjecture. To see this, suppose first that v is even. Then $\rho(v+1) = \kappa(P_{v+1}, v+1) \geq \kappa(P_v, v+1)$, and this would imply Conjecture 1 for odd values of n right away. Now, since for even n

$$\rho(n) = \kappa(P_n, n) \geq 2\kappa(P_{n-2}, n - 1)$$

and likewise,

$$\binom{n}{\frac{n}{2}} = 2\binom{n - 1}{\lfloor \frac{n-1}{2} \rfloor},$$

the last two relations would lead us to settle the conjecture for n even. (The inequality above follows by putting an n to the end of each sequence in an optimal construction for $\kappa(P_{n-2}, n-1)$ and then doubling each sequence by considering also its variant, which one obtains by exchanging in it $n-1$ and n.)

We also believe that

$$K(v, v-1) = \kappa(P_v).$$

As a combination of the two conjectures above we would arrive at the next statement.

CONJECTURE 3. *For every even $v \in \mathbb{N}$*

$$K(v, v-1) = \binom{v+1}{\lfloor \frac{v+1}{2} \rfloor}.$$

3. Related problems.

3.1. A graph covering problem. We show that determining $K(\ell)$ is equivalent to a graph covering problem introduced below. The following standard definition is needed.

DEFINITION 1. *The (undirected) line graph $L(D)$ of the directed graph $D = (V, A)$ is defined by*

$$V(L(D)) = A,$$
$$E(L(D)) = \{\{(a, b), (c, d)\} : b = c \text{ or } a = d\}.$$

Let $\mathcal{L}$ denote the family of all finite simple graphs that are isomorphic to the line graph of some directed graph with possibly multiple edges.

Let the minimum number of graphs in $\mathcal{L}$, the edge sets of which together can cover the edges of the complete graph K_n, be denoted by $h(n)$.

PROPOSITION 7. *For any $M \in \mathbb{N}$, the minimum number ℓ for which $K(\ell) \geq M$ is equal to $h(M)$.*

Proof. Consider a construction attaining $K(\ell)$, that is, a graph G with ℓ edges and $K(\ell)$ infinite permutations that are G-different. Let this set of permutations be denoted by W, and let $\{a, b\}$ be one of the edges of G. Define a graph T_{a-b} on W as its vertex set where an edge is put between two permutations if and only if there is a position where one of them has a while the other has b. In other words, the two permutations are G-different by the edge $\{a, b\}$. Consider the graphs T_{a-b} for all edges of G. These all have the same vertex set, while the union of their edge sets clearly covers the complete graph K_M, where $M = K(\ell)$.

Next we show that all the graphs T_{a-b} belong to $\mathcal{L}$. To this end fix an edge $\{a, b\} \in V(G)$ and consider a graph D_{a-b} with its vertex set $V(D_{a-b})$ consisting of those positions where any of the permutations in W have a nonisolated vertex of G. Since G and W are finite, so is $V(D_{a-b})$. For each element of W we define an edge of D_{a-b}. For $\sigma \in W$, let i and j be the two positions where σ contains a and b, respectively. Then let σ be represented by the directed edge (i, j) in D_{a-b}. (If there is another permutation in W with a and b being in the same positions as in σ, then we have another arc (i, j) in D_{a-b} for this other permutation. Thus D_{a-b} is a directed multigraph.) Now it follows directly from the definitions that $T_{a-b} = L(D_{a-b})$, and thus T_{a-b} is indeed the line graph of a digraph. Together with the previous paragraph this proves $h(K(\ell)) \leq \ell$.

For the reverse inequality consider a covering of K_M with $h(M)$ graphs belonging to $\mathcal{L}$. Let the line graphs in this covering be $L_1, \ldots, L_{h(M)}$. We may assume that $V(L_i) = [M]$ for all i by extending the smaller vertex sets through the addition of isolated points. Let $D_1, \ldots, D_{h(M)}$ be directed graphs satisfying $L_i = L(D_i)$ for all i. By $E(D_i) = V(L_i) = [M]$ we can consider the edges of all D_i's labeled by $|E(D_i)|$ elements of $1, \ldots, M$. (If L_i had some isolated vertices, then the corresponding labels are not used.) Using these digraphs, we define M permutations $\sigma_1, \ldots, \sigma_M$ that are G-different for the graph $G = \ell K_2$ with $\ell = h(M)$. For all i define $t_i = |V(D_i)|$ and identify $V(D_i)$ with $[t_i]$. Consider D_1. If D_1 has an edge labeled r and this edge is (i, j), then put a 1 in position i of σ_r and put a 2 in position j of σ_r. Do similarly for all edges of D_1. Then consider D_2. If it has an edge labeled r which is (i', j'), then put a 3 in position $t_1 + i'$ of σ_r and put a 4 in position $t_1 + j'$ of σ_r. In general, if D_s has an edge labeled r which is (a, b), then put a $2s - 1$ in position $(\sum_{k=1}^{s-1} t_k) + a$ and a $2s$ in position $(\sum_{k=1}^{s-1} t_k) + b$ of σ_r. When this is done for all edges of all D_i's, then extend the obtained partial sequences to infinite permutations of $\mathbb{N}$ in an arbitrary manner. This way one obtains M permutations that are pairwise G-different. To see this consider two of these permutations, say, σ_q and σ_r. Look at the edge $\{q, r\}$ of our graph K_M that was covered by line graphs. Let L_i be the line graph that covered the edge $\{q, r\}$. Then D_i has an edge labeled q and another one labeled r in such a way that the head of the one is the tail of the other. This common point of these two edges defines a position of σ_q and σ_r where one of them has $2i - 1$ while the other has $2i$, making them G-different. $\square$

3.2. Fixed suborders. Here we consider a variant of the problem of the determination of $\kappa(G)$. We restrict attention to complete graphs.

Let $\kappa_{\mathrm{id}}(K_n)$ denote the maximum number of infinite permutations of $\mathbb{N}$ that are K_n-different and contain the first n positive integers (the vertices of K_n) in their natural order.

PROPOSITION 8. *For every $n \in \mathbb{N}$*

$$\kappa_{\mathrm{id}}(K_n) \geq C_n$$

holds, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number.

Proof. For $n = 1, 2$ we have equality: $\kappa_{\mathrm{id}}(K_1) = 1$, $\kappa_{\mathrm{id}}(K_2) = 2$. Set $a_0 = 1$ and $a_n := \kappa_{\mathrm{id}}(K_n)$. It is enough to prove that the numbers a_n satisfy the inequality

$$a_{n+1} \geq \sum_{i=0}^{n} a_i a_{n-i}$$

that has the well-known recursion of Catalan numbers on its right-hand side (cf. [4]). We will look at our infinite permutations as infinite sequences consisting of infinitely many $\star$'s and one of each of the symbols $1, 2, \ldots, n$, where the $\star$'s refer to all other symbols. Clearly, only the positions of the elements of $[n]$ are relevant with respect to the K_n-difference relation. Thus we will define the positions of the elements of $[n]$ and then let the $\star$'s be substituted by the other numbers in any way that will result in infinite permutations of $\mathbb{N}$.

Our construction is inductive. Assume that we already know that $a_k \geq C_k$ holds for $k \leq n$ and thus it suffices to prove it for $n + 1$. Fix a position of our permutations which is "far away," meaning that it is far enough for having enough earlier positions for the following construction. Call this position j. For each $i = 0, \ldots, n$ we construct $a_i a_{n-i}$ sequences having $i + 1$ at their position j. Any two of these sequences that have a different symbol at position j are K_n-different. For those sequences that have $i + 1$ at their position j do the following. Consider a construction of a_i pairwise K_i-different sequences consisting of symbols $1, \ldots, i, \star$, where the symbols in $[i]$ are all used somewhere in the first $j-1$ positions (this is possible if j is chosen large enough). Take the first $j-1$ coordinates of all these sequences, a_{n-i} times each, and continue each of them with an $i+1$ at the jth position. So we have $a_i a_{n-i}$ sequences of length j with $i+1$ at the jth position, each of these sequences are one of a_i possible types, and we have a_{n-i} copies from each type.

Now consider a_{n-i} sequences with the symbols $1, \dots, n-i, \star$ that are pairwise K_{n-i}-different, and shift each value in these sequences by $i+1$. (The latter means that we change each value k to $k+i+1$ in these sequences while $\star$'s remain $\star$'s.) For each type of the previous sequences take its a_{n-i} copies and suffix to each of them one of the current a_{n-i} different sequences. This way one gets $a_i a_{n-i}$ K_{n+1}-different sequences with symbol $i+1$ at position j. Doing this for all $i = 1, \dots, n$, one obtains $\sum_{i=0}^{n} a_i a_{n-i}$ K_{n+1}-different sequences, proving the desired inequality. ∎

Note added in proof. We learned from Graham Brightwell and Marianne Fairthorne that they have improved upon our lower bound in Proposition 6 using a similar approach and independently discovered some of the results (including Lemma 2 and Propositions 2 and 4) presented here.

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