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The Hopf algebra of finite topologies and T-partitions

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A noncommutative and noncocommutative Hopf algebra on finite topologies $\mathbf{H_T}$ is introduced and studied (freeness, cofreeness, self-duality...). Generalizing Stanley's definition of P-partitions associated to a special poset, we define the notion of T-partitions associated to a finite topology, and deduce a Hopf algebra morphism from $\mathbf{H_T}$ to the Hopf algebra of packed words **WQSym**. Generalizing Stanley's decomposition by linear extensions, we deduce a factorization of this morphism, which induces a combinatorial isomorphism from the shuffle product to the quasi-shuffle product of **WQSym**. It is strongly related to a partial order on packed words, here described and studied.

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Contents

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Introduction

In his thesis [16], Stanley introduced the notion of (P, ω, m)-partition associated to a (P, ω) poset. More precisely, a $(\mathcal{P}, \omega)$ poset, or equivalently a special poset, is a finite set $(\mathcal{P}, \leq_{\mathcal{P}}, \leq)$ with two orders, the second being total, see Section 1.2 for examples. A (P, ω, m)-partition, or, briefly, a P-partition, associated to a special poset $\mathcal{P}$ is a map $f : \mathcal{P} \longrightarrow \mathbb{N}$, such that:

(1) If $i \leq_{\mathcal{P}} j$ in $\mathcal{P}$, then $f(i) \leq f(j)$.
(2) If $i \leq_{\mathcal{P}} j$ and $i > j$ in $\mathcal{P}$, then $f(i) < f(j)$.

Stanley proved [16,8] that the set of P-partitions of $\mathcal{P}$ can be decomposed into a disjoint family of subsets indexed by the set of linear extensions of the partial order $\leq_{\mathcal{P}}$.

Special posets are organized as a Hopf algebra $\mathbf{H_{SP}}$, described in [11] as a subobject of the Hopf algebra of double posets, that is to say finite sets with two partial orders. Linear extensions are used to define a Hopf algebra morphism L from $\mathbf{H_{SP}}$ to the Malvenuto–Reutenauer Hopf algebra of permutations $\mathbf{FQSym}$ [9,10,1]. Considering P-partitions which are packed words (which allows to find all P-partitions), it is possible to define a Hopf algebra morphism Γ from $\mathbf{H_{SP}}$ to $\mathbf{WQSym}$, the Hopf algebra of packed words. Then Stanley's decomposition allows to define an injective Hopf algebra morphism $\varphi : \mathbf{FQSym} \longrightarrow \mathbf{WQSym}$, such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{H_{SP}} \arrow[r, "L"] \arrow[dr, "\Gamma"'] & \mathbf{FQSym} \arrow[d, "\varphi"] \\ & \mathbf{WQSym} \end{tikzcd}$$

Our aim in this the present text is a generalization of Stanley's theorem on P-partitions and its applications to combinatorial Hopf algebras. We here replace special posets by special preposets $(\mathcal{P}, \leq_{\mathcal{P}}, \leq)$, where $\leq_{\mathcal{P}}$ is a preorder, that is to say a reflexive and transitive relation, and $\leq$ is a total order. By Alexandroff's correspondence, these correspond to topologies on finite sets $[n] = \{1, \ldots, n\}$. A construction of a Hopf algebra on finite topologies (up to homeomorphism) is done in [4], where one also can find a brief historic of the subject. We apply the same construction here and obtain a Hopf algebra $\mathbf{H_T}$ on finite topologies, which is noncommutative and noncocommutative. It is algebraically studied in Section 2; we prove its freeness and cofreeness (Proposition 5 and Theorem 7), show that the Hopf algebra of special posets is both a subalgebra and a quotient of $\mathbf{H_T}$ via the construction of a family of Hopf algebra morphisms θ_q (Proposition 8). A (degenerate) Hopf pairing is also defined on $\mathbf{H_T}$, with the help of Zelevinsky's pictures, extending the pairing on special posets of [11]. The set of topologies on a given set is totally ordered by the refinement; using this ordering and a Möbius inversion, we define another basis of $\mathbf{H_T}$, called the *ribbon basis*. The product and the coproducts are described in this new basis (Theorem 12).

The notions of T-partitions and linear extensions of a topology are defined in Section 4. A T-partition of a topology $\mathcal{T}$ on the set $[n]$ is introduced in Definition 13. Namely, if $\leq_{\mathcal{T}}$ is the preorder associated to the topology $\mathcal{T}$, a generalized T-partition of $\mathcal{T}$ is a surjective map $f : [n] \longrightarrow [p]$ such that:

- if $i \leq_{\mathcal{T}} j$, then $f(i) \leq f(j)$.

The T-partition f is *strict* if:

- If $i \leq_{\mathcal{T}} j$, $i > j$ and not $j \leq_{\mathcal{T}} i$, then $f(i) < f(j)$.
- If $i < j < k$, $i \leq_{\mathcal{T}} k$, $k \leq_{\mathcal{T}} i$ and $f(i) = f(j) = f(k)$, then $i \leq_{\mathcal{T}} j$, $j \leq_{\mathcal{T}} i$, $j \leq_{\mathcal{T}} k$ and $k \leq_{\mathcal{T}} i$.

The last condition, which is empty for special posets, is necessary to obtain an equivalent of Stanley's decomposition, as it will be explained later. We now identify any T-partition f associated to the topology $\mathcal{T}$ on $[n]$ with the word $f(1) \ldots f(n)$. A family of Hopf algebra morphisms $\Gamma_{(q_1,q_2,q_3)}$ from $\mathbf{H_T}$ to $\mathbf{WQSym}$, parametrized by triples of scalars, is defined in Proposition 14. In particular, for any finite topology $\mathcal{T}$:

$$\Gamma_{(1,1,1)}(\mathcal{T}) = \sum_{f \text{ generalized } T\text{-partition of } \mathbf{T}} f, \quad \Gamma_{(1,0,0)}(\mathcal{T}) = \sum_{f \text{ strict } T\text{-partition of } \mathbf{T}} f.$$

Linear extensions are introduced in Definition 15. They are used to defined a Hopf algebra morphism $L : \mathbf{H_T} \longrightarrow \mathbf{WQSym}$, up to a change of the product of $\mathbf{WQSym}$: one has to replace its usual product by the shifted shuffling product $\sqcup\!\sqcup$, used in [5]. We then look for an equivalent of Stanley's decomposition theorem of P-partitions, reformulated in terms of Hopf algebras, that is to say we look for a Hopf algebra morphism $\varphi_{(q_1,q_2,q_3)}$ making the following diagram commute:

$$\begin{tikzcd} \mathbf{H_T} \arrow[r, "L"] \arrow[dr, "\Gamma_{(q_1,q_2,q_3)}"'] & (\mathbf{WQSym}, \sqcup\!\sqcup, \Delta) \arrow[d, "\varphi_{(q_1,q_2,q_3)}"] \\ & (\mathbf{WQSym}, ., \Delta) \end{tikzcd}$$

We prove in Proposition 21 that such a $\varphi_{(q_1,q_2,q_3)}$ exists if, and only if, $(q_1, q_2, q_3) = (1, 0, 0)$ or $(0, 1, 0)$, which justifies the introduction of strict T-partitions. The morphism $\varphi_{(1,0,0)}$ is defined in Proposition 19, with the help of a partial order on packed words introduced in Definition 17; the set decomposition of T-partitions is stated in Corollary 20. Finally, the partial order on packed words is studied in Section 4.5, with a combinatorial application in Corollary 26.

The text is organized as follows. The first section recalls the construction of the Hopf algebras **WQSym**, **FQSym** and $\mathbf{H_{SP}}$. The second section deals with the Hopf algebra of topologies and its algebraic study; the ribbon basis is the object of the third section. The equivalent of Stanley's decomposition, from a combinatorial and a Hopf algebraic point of view, is the object of the last section, together with the study of the partial order on packed words.

Notations.

- We work on a commutative base field $\mathbb{K}$, of any characteristic. Any vector space, coalgebra, algebra... of this text is taken over $\mathbb{K}$.
- For all $n \geq 0$, we put $[n] = \{1, \dots, n\}$. In particular, $[0] = \emptyset$. We denote by $\mathbb{N}_{>0}$ the set of strictly positive integers.

1. Reminders

1.1. **WQSym** *and* **FQSym**

Let us first recall the construction of **WQSym** [13].

- A packed word is a word f whose letters are in $\mathbb{N}_{>0}$, such that for all $1 \leq i \leq j$,

$$j \text{ appears in } f \Longrightarrow i \text{ appears in } f.$$

Here are the packed words of length ≤ 3:

$$1 = \emptyset; (1); (12), (21), (11); (123), (132), (213), (231), (312), (321),$$
$$(122), (212), (221), (112), (121), (211), (111).$$

- Let $f = f(1)\dots f(n)$ be a word whose letters are in $\mathbb{N}_{>0}$. There exists a unique increasing bijection ϕ from $\{f(1), \dots, f(n)\}$ into a set $[m]$. The packed word $Pack(f)$ is $\phi(f(1))\dots\phi(f(n))$.
- If f is a word whose letters are in $\mathbb{N}_{>0}$, and I is a subset of $\mathbb{N}_{>0}$, then $f_{|I}$ is the subword of f obtained by keeping only the letters of f which are in I.

As a vector space, a basis of **WQSym** is given by the set of packed words. Its product is defined as follows: if f and f' are packed words of respective lengths n and n':

$$f.f' = \sum_{\substack{f'' \text{ packed word of length } n+n',\\ Pack(f''(1)\dots f''(n))=f,\\ Pack(f''(n+1)\dots f''(n+n'))=f'}} f''.$$

For example:

$$\begin{aligned}(112).(12) &= (11212) + (11213) + (11214) + (11223) + (11224)\\ &\quad + (11234) + (11312) + (11323) + (11324) + (11423)\\ &\quad + (22312) + (22313) + (22314) + (22413) + (33412).\end{aligned}$$

The unit is the empty packed word $1 = \emptyset$.

If f is a packed word, its coproduct in **WQSym** is defined by:

$$\Delta(f) = \sum_{k=0}^{\max(f)} f_{|[k]} \otimes Pack(f_{|\mathbb{N}_{>0}\setminus[k]}).$$

For example:

$$\begin{aligned}\Delta((511423)) &= 1 \otimes (511423) + (1) \otimes (4312) + (112) \otimes (321)\\ &\quad + (1123) \otimes (21) + (11423) \otimes (1) + (511423) \otimes 1.\end{aligned}$$

Then $(\mathbf{WQSym}, ., \Delta)$ is a graded, connected Hopf algebra.

We denote by j the involution on packed words defined in the following way: if $f = f(1)\dots f(n)$ is a packed word of length n, there exists a unique decreasing bijection φ from $\{f(1), \dots, f(n)\}$ into a set $[l]$. We put $j(f) = \varphi(f(1))\dots\varphi(f(n))$. For example, $j((65133421)) = (12644356)$. The extension of j to **WQSym** is a Hopf algebra isomorphism from $(\mathbf{WQSym}, ., \Delta)$ to $(\mathbf{WQSym}, ., \Delta^{op})$.

In particular, permutations are packed words. Note that the subspace of **WQSym** generated by all the permutations is a coalgebra, but not a subalgebra: for example, $(1).(1) = (12) + (21) + (11)$. On the other side, the subspace of **WQSym** generated by packed words which are not permutations is a biideal, and the quotient of **WQSym** by this biideal is the Hopf algebra of permutations **FQSym** [9,1]. As a vector space, a basis of **FQSym** is given by the set of all permutations; if σ and σ' are two permutations of respective lengths n and n',

$$\sigma.\sigma' = \sum_{\substack{\sigma''\in\mathfrak{S}_{n+n'},\\ Pack(\sigma''(1)\ldots\sigma''(n))=\sigma,\\ Pack(\sigma''(n+1)\ldots\sigma''(n+n'))=\sigma'}} \sigma'' = \sum_{\epsilon\in Sh(n,n')} \epsilon\circ(\sigma\otimes\tau),$$

where $Sh(n,n')$ is the set of (n,n')-shuffles, that is to say permutations $\epsilon\in\mathfrak{S}_{n+n'}$ such that $\epsilon(1)<\ldots<\epsilon(n)$ and $\epsilon(n+1)<\ldots<\epsilon(n+n')$. For example:

$$\begin{aligned}(132).(21) &= (13254)+(14253)+(15243)+(14352)+(15342)\\ &\quad+(15432)+(24351)+(25341)+(25431)+(35421).\end{aligned}$$

If $\sigma\in\mathfrak{S}_n$, its coproduct is given by:

$$\Delta(\sigma)=\sum_{k=0}^{n}\sigma_{|[k]}\otimes Pack(\sigma_{|\mathbb{N}_{>0}\setminus[k]}).$$

For example:

$$\begin{aligned}\Delta((51423)) &= 1\otimes(51423)+(1)\otimes(4312)+(12)\otimes(321)\\ &\quad+(123)\otimes(21)+(1423)\otimes(1)+(51423)\otimes 1.\end{aligned}$$

The canonical epimorphism from **WQSym** to **FQSym** is denoted by ϖ.

We shall need the standardization map, which associates a permutation to any packed word. If $f=f(1)\ldots f(n)$ is a packed word, $Std(f)$ is the unique permutation $\sigma\in\mathfrak{S}_n$ such that for all $1\leq i,j\leq n$:

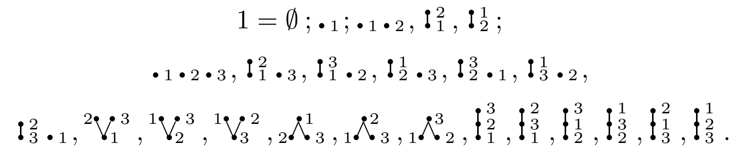
$$\begin{aligned}f(i)<f(j) &\Longrightarrow \sigma(i)<\sigma(j),\\ (f(i)=f(j) \text{ and } i<j) &\Longrightarrow \sigma(i)<\sigma(j).\end{aligned}$$

In particular, if f is a permutation, $Std(f)=f$. Here are examples of standardization of packed words which are not permutations:

$$\begin{array}{llll} Std(11)=(12), & Std(122)=(123), & Std(212)=(213), & Std(221)=(231),\\ Std(112)=(123), & Std(121)=(132), & Std(211)=(312), & Std(111)=(123).\end{array}$$

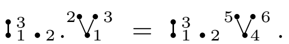
1.2. Special posets

Let us briefly recall the construction of the Hopf algebra on special posets [11,3]. A special (double) poset is a family $(P,\leq,\leq_{tot})$, where P is a finite set, $\leq$ is a partial order on P and $\leq_{tot}$ is a total order on P. For example, here are the special posets of cardinality ≤ 3: they are represented by the Hasse graph of $\leq$, the total order $\leq_{tot}$ is given by the indices of the vertices.



If $P = (P, \leq, \leq_{tot})$ and $Q = (Q, \leq, \leq_{tot})$ are two special posets, we define a special posets $P.Q$ in the following way:

- As a set, $P.Q = P \sqcup Q$.
- If $i, j \in P$, then $i \leq j$ in $P.Q$ if, and only if, $i \leq j$ in P, and $i \leq_{tot} j$ in $P.Q$ if, and only if, $i \leq_{tot} j$ in P.
- If $i, j \in Q$, then $i \leq j$ in $P.Q$ if, and only if, $i \leq j$ in Q, and $i \leq_{tot} j$ in $P.Q$ if, and only if, $i \leq_{tot} j$ in Q.
- If $i \in P$ and $j \in Q$, then i and j are not comparable for $\leq$, and $i \leq_{tot} j$.

For example, . The vector space generated by the set of (iso-classes) of special posets is denoted by $\mathbf{H_{SP}}$. This product is bilinearly extended to $\mathbf{H_{SP}}$, making it an associative algebra. The unit is the empty special poset $1 = \emptyset$.

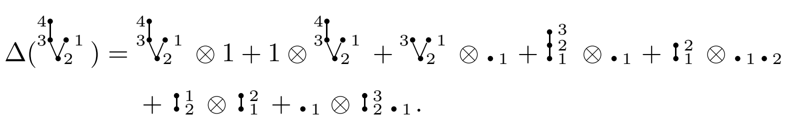
If P is a special poset and $I \subseteq P$, then by restriction I is a special poset. We shall say that I is an ideal of P if for all $i, j \in P$:

$$(i \in I \text{ and } i \leq j) \Longrightarrow j \in I.$$

We give $\mathbf{H_{SP}}$ the coproduct defined by:

$$\Delta(P) = \sum_{I \text{ ideal of } P} (P \setminus I) \otimes I.$$

For example:



Let $P = (P, \leq, \leq_{tot})$ be a special poset. A linear extension of P is a total order $\leq'$ extending the partial order $\leq$. Let $\leq'$ be a linear extension of P. Up to a unique isomorphism, we can assume that $P = [n]$ as a totally ordered set. For any $i \in [n]$, we denote by $\sigma(i)$ the index of i in the total order $\leq'$. Then $\sigma \in \mathfrak{S}_n$, and we now identify $\leq'$ and σ. The following map is a surjective Hopf algebra morphism:

$$L : \begin{cases} \mathbf{H_{SP}} \longrightarrow \mathbf{FQSym} \\ \quad P \longrightarrow \displaystyle\sum_{\sigma \text{ linear extension of } P} \sigma. \end{cases}$$

For example:

$$\begin{aligned}
&L({}_2\bigwedge^1{}_3) = (312) + (321), && L({}_1\bigwedge^2{}_3) = (132) + (231), && L({}_1\bigwedge^3{}_2) = (123) + (213),\\
&L({}^2\bigvee_1{}^3) = (123) + (132), && L({}^1\bigvee_2{}^3) = (213) + (312), && L({}^1\bigvee_3{}^2) = (231) + (321),\\
&L\left(\substack{3\\2\\1}\right) = (123), && L\left(\substack{3\\1\\2}\right) = (213), && L\left(\substack{2\\1\\3}\right) = (231),\\
&L\left(\substack{2\\3\\1}\right) = (132), && L\left(\substack{1\\3\\2}\right) = (312), && L\left(\substack{1\\2\\3}\right) = (321).
\end{aligned}$$

Let $\mathcal{P}$ be a special poset. With the help of the total order of $\mathcal{P}$, we identify $\mathcal{P}$ with the set $[n]$, where n is the cardinality of $\mathcal{P}$. A P-partition of $\mathcal{P}$ is a map $f : \mathcal{P} \longrightarrow [n]$ such that:

(1) If $i \leq_{\mathcal{P}} j$ in $\mathcal{P}$, then $f(i) \leq f(j)$.
(2) If $i \leq_{\mathcal{P}} j$ and $i > j$ in $\mathcal{P}$, then $f(i) < f(j)$.

We represent a P-partition of $\mathcal{P}$ by the word $f(1)\dots f(n)$. Obviously, if $w = w_1 \dots w_n$ is a word, it is a P-partition of the special poset $\mathcal{P}$ if, and only if, $Pack(w)$ is a P-partition of $\mathcal{P}$. We define:

$$\Gamma : \begin{cases} \mathbf{H_{SP}} \longrightarrow \mathbf{WQSym} \\ \quad \mathcal{P} \longrightarrow \displaystyle\sum_{w \text{ packed word, } P\text{-partition of } \mathcal{P}} w. \end{cases}$$

For example:

$$\begin{aligned}
\Gamma(\bullet_1) &= (1),\\
\Gamma(\bullet_1 \bullet_2) &= (12) + (21) + (11),\\
\Gamma\left(\substack{2\\1}\right) &= (12) + (11),\\
\Gamma\left(\substack{1\\2}\right) &= (21),\\
\Gamma({}^2\bigvee_1{}^3) &= (123) + (132) + (122) + (112) + (121) + (111),\\
\Gamma({}^1\bigvee_2{}^3) &= (213) + (312) + (212) + (211),\\
\Gamma({}^1\bigvee_3{}^2) &= (231) + (321) + (221).
\end{aligned}$$

We shall prove in Section 4 that Γ is a Hopf algebra morphism.

Remark. There is a natural surjective Hopf algebra morphism ϱ from **WQSym** to the Hopf algebra of quasisymmetric functions **QSym** [14]. For any special poset $\mathcal{P}$:

$$\varrho \circ \Gamma(\mathcal{P}) = \sum_{f\ P\text{-partition of } w} x_{f(1)} \ldots x_{f(n)} \in \mathbf{QSym} \subseteq \mathbb{Q}[[x_1, x_2, \ldots]].$$

So $\varrho \circ \Gamma(\mathcal{P})$ is the generating function of P in the sense of [16].

We shall also prove that Stanley's decomposition theorem can be reformulated in the following way: let us consider the map

$$\varphi : \begin{cases} \mathbf{FQSym} \longrightarrow \mathbf{WQSym} \\ \qquad \sigma \longrightarrow \displaystyle\sum_{w \text{ packed word}, Std(w)=\sigma} w. \end{cases}$$

Then φ is an injective Hopf algebra morphism, such that $\varphi \circ L = \Gamma$. Combinatorially speaking, for any special poset $\mathcal{P}$:

$$\{P\text{-partition of } \mathcal{P}\} = \bigsqcup_{\sigma \text{ linear extension of } \mathcal{P}} \{w \mid Pack(w) = \sigma\}.$$

1.3. Infinitesimal bialgebras

An infinitesimal bialgebra [7] is a triple (A, m, Δ) such that:

- (A, m) is a unitary, associative algebra.
- (A, Δ) is a counitary, coassociative algebra.
- For all $x, y \in A$, $\Delta(xy) = (x \otimes 1)\Delta(y) + \Delta(x)(1 \otimes y) - x \otimes y$.

The standard examples are the tensor algebras $T(V)$, with the concatenation product and the deconcatenation coproduct. By the rigidity theorem of [7], these are essentially the unique examples:

Theorem 1. *Let A be a graded, connected, infinitesimal bialgebra. Then A is isomorphic to $T(Prim(A))$ as an infinitesimal bialgebra.*

2. Topologies on a finite set

2.1. Notations and definitions

Let X be a set. Recall that a topology on X is a family $\mathcal{T}$ of subsets of X, called the open sets of $\mathcal{T}$, such that:

(1) $\emptyset$, $X \in \mathcal{T}$.
(2) The union of an arbitrary number of elements of $\mathcal{T}$ is in $\mathcal{T}$.
(3) The intersection of a finite number of elements of $\mathcal{T}$ is in $\mathcal{T}$.

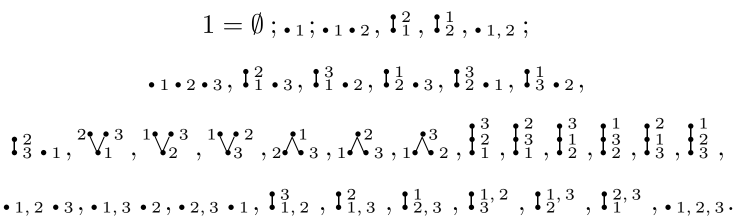
Let us recall from [2] the bijective correspondence between topologies on a finite set X and preorders on X:

(1) Let $\mathcal{T}$ be a topology on the finite set X. The relation $\leq_\mathcal{T}$ on X is defined by $i \leq_\mathcal{T} j$ if any open set of $\mathcal{T}$ which contains i also contains j. Then $\leq_\mathcal{T}$ is a preorder, that is to say a reflexive, transitive relation. Moreover, the open sets of $\mathcal{T}$ are the ideals of $\leq_\mathcal{T}$, that is to say the sets $I \subseteq X$ such that, for all $i, j \in X$:

$$(i \in I \text{ and } i \leq_\mathcal{T} j) \Longrightarrow j \in I.$$

(2) Conversely, if $\leq$ is a preorder on X, the ideals of $\leq$ form a topology on X denoted by $\mathcal{T}_\leq$. Moreover, $\leq_{\mathcal{T}_\leq} = \leq$, and $\mathcal{T}_{\leq_\mathcal{T}} = \mathcal{T}$. Hence, there is a bijection between the set of topologies on X and the set of preorders on X.
(3) Let $\mathcal{T}$ be a topology on X. The relation $\sim_\mathcal{T}$ on X, defined by $i \sim_\mathcal{T} j$ if $i \leq_\mathcal{T} j$ and $j \leq_\mathcal{T} i$, is an equivalence on X. Moreover, the set $X/\sim_\mathcal{T}$ is partially ordered by the relation defined by $\bar{i} \leq_\mathcal{T} \bar{j}$ if $i \leq j$. Consequently, we shall represent preorders on X (hence, topologies on X) by the Hasse diagram of $X/\sim_\mathcal{T}$, the vertices being the equivalence classes of $\sim_\mathcal{T}$.

For example, here are the topologies on $[n]$ for $n \leq 3$:



The number t_n of topologies on $[n]$ is given by the sequence A000798 in [15]:

n	1	2	3	4	5	6	7	8	9
t_n	1	4	29	355	6 942	209527	9 535 241	642 779 354	63 260 289 423

n	10	11	12
t_n	8 977 053 873 043	1 816 846 038 736 192	519 355 571 065 774 021

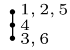
The set of topologies on $[n]$ will be denoted by $\mathbf{T}_n$, and we put $\mathbf{T} = \bigsqcup_{n \geq 0} \mathbf{T}_n$.

If $\mathcal{T}$ is a finite topology on a set X, then $\iota(\mathcal{T}) = \{X \setminus O \mid O \in \mathcal{T}\}$ is also a finite topology, on the same set X. Consequently, ι defines an involution of the set $\mathbf{T}$. The preorder associated to $\iota(\mathcal{T})$ is $\leq_{\iota(\mathcal{T})} = \geq_\mathcal{T}$.

Notations. Let f be a packed word of length n. We define a preorder $\leq_f$ on $[n]$ by:

$$i \leq_f j \text{ if } f(i) \leq f(j).$$

The associated topology is denoted by $\mathcal{T}_f$. The open sets of this topology are the subsets $f^{-1}(\{i, \dots, \max(f)\})$, $1 \leq i \leq \max(f)$, and $\emptyset$. For example:

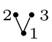
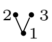
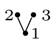
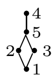
$\mathcal{T}_{(331231)} =$  .

2.2. Two products on finite topologies

Notations. Let $O \subseteq \mathbb{N}$ and let $n \in \mathbb{N}$. The set $O(+n)$ is the set $\{k + n \mid k \in O\}$.

Definition 2. Let $\mathcal{T} \in \mathbf{T}_n$ and $\mathcal{T}' \in \mathbf{T}_{n'}$.

(1) The topology $\mathcal{T}.\mathcal{T}'$ is the topology on $[n+n']$ which open sets are the sets $O \sqcup O'(+n)$, with $O \in \mathcal{T}$ and $O' \in \mathcal{T}'$.
(2) The topology $\mathcal{T} \downarrow \mathcal{T}'$ is the topology on $[n + n']$ which open sets are the sets $O \sqcup [n'](+n)$, with $O \in \mathcal{T}$, and $O'(+n)$, with $O' \in \mathcal{T}'$.

Example.  .  $=$   and  $\downarrow$  $=$  .

Proposition 3. *These two products are associative, with $\emptyset = 1$ as a common unit.*

Proof. Obviously, for any $\mathcal{T} \in \mathbf{T}$, $1.\mathcal{T} = \mathcal{T}.1 = 1 \downarrow \mathcal{T} = \mathcal{T} \downarrow 1 = \mathcal{T}$. Let $\mathcal{T} \in \mathbf{T}_n$, and $\mathcal{T}' \in \mathbf{T}_{n'}$. The preorder associated to $\mathcal{T}.\mathcal{T}'$ is:

$$\{(i,j) \mid i \leq_{\mathcal{T}} j\} \sqcup \{(i+n, j+n) \mid i \leq_{\mathcal{T}'} j\}.$$

The preorder associated to $\mathcal{T} \downarrow \mathcal{T}'$ is:

$$\{(i,j) \mid i \leq_{\mathcal{T}} j\} \sqcup \{(i+n, j+n) \mid i \leq_{\mathcal{T}'} j\} \sqcup \{(i,j) \mid 1 \leq i \leq n < j \leq n+n'\}.$$

Let $\mathcal{T} \in \mathbf{T}_n$, $\mathcal{T}' \in \mathbf{T}_{n'}$ and $\mathcal{T}'' \in \mathbf{T}_{n''}$. The preorders associated to $(\mathcal{T}.\mathcal{T}').\mathcal{T}''$ and to $\mathcal{T}.(\mathcal{T}'.\mathcal{T}'')$ are both equal to:

$$\{(i,j) \mid i \leq_{\mathcal{T}} j\} \sqcup \{(i+n, j+n) \mid i \leq_{\mathcal{T}'} j\} \sqcup \{(i+n+n', j+n+n') \mid i \leq_{\mathcal{T}''} j\}.$$

So $(\mathcal{T}.\mathcal{T}').\mathcal{T}'' = \mathcal{T}.(\mathcal{T}'.\mathcal{T}'')$. The preorders associated to $(\mathcal{T} \downarrow \mathcal{T}') \downarrow \mathcal{T}''$ and to $\mathcal{T} \downarrow (\mathcal{T}' \downarrow \mathcal{T}'')$ are both equal to:

$$\{(i,j) \mid i \leq_{\mathcal{T}} j\} \sqcup \{(i+n, j+n) \mid i \leq_{\mathcal{T}'} j\} \sqcup \{(i+n+n', j+n+n') \mid i \leq_{\mathcal{T}''} j\}$$
$$\sqcup \{(i,j) \mid 1 \leq i \leq n < j \leq n+n'+n''\} \sqcup \{(i,j) \mid n < i \leq n+n' < j \leq n+n'+n''\}.$$

So $(\mathcal{T} \downarrow \mathcal{T}') \downarrow \mathcal{T}'' = \mathcal{T} \downarrow (\mathcal{T}' \downarrow \mathcal{T}'')$. $\square$

Definition 4.

(1) We denote by $\mathbf{H_T}$ the vector space generated by $\mathbf{T}$. It is graded, the elements of $\mathbf{T}_n$ being homogeneous of degree n. We extend the two products defined earlier on $\mathbf{H_T}$.
(2) Let $\mathcal{T} \in \mathbf{T}$, different from 1.
  (a) We shall say that $\mathcal{T}$ is indecomposable if it cannot be written as $\mathcal{T} = \mathcal{T}'.\mathcal{T}''$, with $\mathcal{T}', \mathcal{T}'' \neq 1$.
  (b) We shall say that $\mathcal{T}$ is $\downarrow$-indecomposable if it cannot be written as $\mathcal{T} = \mathcal{T}' \downarrow \mathcal{T}''$, with $\mathcal{T}', \mathcal{T}'' \neq 1$.
  (c) We shall say that $\mathcal{T}$ is bi-indecomposable if it is both indecomposable and $\downarrow$-indecomposable.

Note that $(\mathbf{H}_{\mathcal{T}}, ., \downarrow)$ is a 2-associative algebra [7], that is to say an algebra with two associative products sharing the same unit.

Proposition 5.

(1) *The associative algebra $(\mathbf{H_T}, .)$ is freely generated by the set of indecomposable topologies.*
(2) *The associative algebra $(\mathbf{H_T}, \downarrow)$ is freely generated by the set of $\downarrow$-indecomposable topologies.*
(3) *The 2-associative algebra $(\mathbf{H_T}, ., \downarrow)$ is freely generated by the set of bi-indecomposable topologies.*

Proof. 1. An easy induction on the degree proves that any $\mathcal{T} \in \mathbf{T}$ can be written as $\mathcal{T} = \mathcal{T}_1. \ldots .\mathcal{T}_k$, with $\mathcal{T}_1, \ldots, \mathcal{T}_k$ indecomposable.

Let us assume that $\mathcal{T} = \mathcal{T}_1. \cdots .\mathcal{T}_k = \mathcal{T}'_1. \cdots .\mathcal{T}'_l$, with $\mathcal{T}_1, \ldots, \mathcal{T}_k, \mathcal{T}'_1, \ldots, \mathcal{T}'_l$ indecomposable topologies. Let m be the smallest integer ≥ 1 such that for all $1 \leq i \leq m < j \leq n$, i and j are not comparable for $\leq_{\mathcal{T}}$. By definition of the product ., for all $i \leq deg(\mathcal{T}_1)$, for all $j > deg(\mathcal{T}_1)$, i and j are not comparable for $\leq_{\mathcal{T}}$, so $m \leq deg(\mathcal{T}_1)$. Let $\mathcal{T}'$ be the restriction of the topology $\mathcal{T}_1$ to $\{1, \ldots, m\}$ and $\mathcal{T}''$ be the restriction of the topology $\mathcal{T}_1$ to $\{m+1, \ldots, deg(\mathcal{T}_1)\}$, reindexed to $\{1, \ldots, deg(\mathcal{T}_1) - m\}$. By definition of m, $\mathcal{T}_1 = \mathcal{T}'.\mathcal{T}''$. As $\mathcal{T}_1$ is indecomposable, $\mathcal{T}' = 1$ or $\mathcal{T}'' = 1$; as $m \geq 1$, $\mathcal{T}'' = 1$, so $\mathcal{T}' = \mathcal{T}_1$. Similarly, $\mathcal{T}' = \mathcal{T}'_1 = \mathcal{T}_1$. The restriction of $\mathcal{T}$ to $\{m+1, \ldots, deg(\mathcal{T})\}$, after a reindexation, gives $\mathcal{T}_2. \cdots .\mathcal{T}_k = \mathcal{T}'_2. \cdots .\mathcal{T}'_l$. We conclude by an induction on the degree of $\mathcal{T}$.

2. Similar proof. For the unicity of the decomposition, use the smallest integer $m \geq 1$ such that for all $i \leq m < j \leq n$, $i \leq_{\mathcal{T}} j$.

3. *First step.* Let $\mathcal{T} \in \mathbf{T}_n$, $n \geq 1$. Let us assume that $\mathcal{T}$ is not $\downarrow$-indecomposable. Then $\mathcal{T} = \mathcal{T}' \downarrow \mathcal{T}''$, with $\mathcal{T}', \mathcal{T}'' \neq 1$, so $1 \leq_{\mathcal{T}} n$: this implies that $\mathcal{T}$ is indecomposable. Hence, one, and only one, of the following assertions holds:

- $\mathcal{T}$ is indecomposable and not $\downarrow$-indecomposable.
- $\mathcal{T}$ is not indecomposable and $\downarrow$-indecomposable.
- $\mathcal{T}$ is bi-indecomposable.

Second step. Let $(A, .\ \downarrow)$ be a 2-associative algebra, and let $a_{\mathcal{T}} \in A$ for any bi-indecomposable $\mathcal{T} \in \mathbf{T}$. Let us prove that there exists a unique morphism of 2-associative algebras $\phi : \mathbf{H_T} \longrightarrow A$, such that $\phi(\mathcal{T}) = a_{\mathcal{T}}$ for all bi-indecomposable $\mathcal{T} \in \mathbf{T}$. The proof will follow, since $\mathbf{H_T}$ satisfies the universal property of the free 2-associative algebra generated by the bi-indecomposable elements. We define $\phi(\mathcal{T})$ for $\mathcal{T} \in \mathbf{T}$ by induction on $deg(\mathcal{T})$ in the following way:

(1) $\phi(1) = 1_A$.
(2) If $\mathcal{T}$ is bi-indecomposable, then $\phi(\mathcal{T}) = a_{\mathcal{T}}$.
(3) If $\mathcal{T}$ is indecomposable and not $\downarrow$-indecomposable, we write uniquely $\mathcal{T} = \mathcal{T}_1 \downarrow \ldots \downarrow \mathcal{T}_k$, with $k \geq 2$, $\mathcal{T}_1, \ldots, \mathcal{T}_k \in \mathbf{T}$, $\downarrow$-indecomposable. Then $\phi(\mathcal{T}) = \phi(\mathcal{T}_1) \downarrow \ldots \downarrow \phi(\mathcal{T}_k)$.
(4) If $\mathcal{T}$ is not indecomposable and $\downarrow$-indecomposable, we write uniquely $\mathcal{T} = \mathcal{T}_1 . \cdots . \mathcal{T}_k$, with $k \geq 2$, $\mathcal{T}_1, \ldots, \mathcal{T}_k \in \mathbf{T}$, indecomposable. Then $\phi(\mathcal{T}) = \phi(\mathcal{T}_1) . \cdots . \phi(\mathcal{T}_k)$.

By the first step, ϕ is well-defined. By the unicity of the decomposition into decomposables or $\downarrow$-indecomposables, ϕ is a morphism of 2-associative algebras. $\square$

We denote by $F(X)$ the generating formal series of all topologies on $[n]$, by $F_I(X)$ the formal series of indecomposable topologies on $[n]$, by $F_{\downarrow I}(X)$ the formal series of $\downarrow$-indecomposable topologies on $[n]$, and by $F_{BI}(X)$ the formal series on bi-indecomposable topologies on $[n]$. Then:

$$F_I(X) = F_{\downarrow I}(X) = \frac{F(X) - 1}{F(X)},\ F_{BI}(X) = \frac{-2 + 3F(X) - F(X)^2}{F(X)}.$$

This gives:

n	1	2	3	4	5	6	7	8	9
$I, \downarrow I$	1	3	22	292	6 120	193 594	9 070 536	622 336 756	61 915 861 962
BI	1	2	15	229	5 298	177 661	8 605 831	601 894 158	60 571 434 501

n	10	11	12
$I, \downarrow I$	8 846 814 822 932	1 798 543 906 246 948	515 674 104 905 890 202
BI	8 716 575 772 821	1 780 241 773 757 704	511 992 638 746 006 383

2.3. A coproduct on finite topologies

Notations.

(1) Let X be a finite, totally ordered set of cardinality n, and $\mathcal{T}$ a topology on X. There exists a unique increasing bijection ϕ from X to $[n]$. We denote by $Std(\mathcal{T})$ the topology on $[n]$ defined by:

$$Std(\mathcal{T}) = \{\phi(O) \mid O \in \mathcal{T}\}.$$

It is an element of $\mathbf{T}_n$.

(2) Let X be a finite set, and $\mathcal{T}$ be a topology on X. For any $Y \subseteq X$, we denote by $\mathcal{T}_{|Y}$ the topology induced by $\mathcal{T}$ on Y, that is to say:

$$\mathcal{T}_{|Y} = \{O \cap Y \mid O \in \mathcal{T}\}.$$

Note that if Y is an open set of $\mathcal{T}$, $\mathcal{T}_{|Y} = \{O \in \mathcal{T} \mid O \subseteq Y\}$.

Proposition 6. *Let $\mathcal{T} \in \mathbf{T}_n$, $n \geq 1$. We put:*

$$\Delta(\mathcal{T}) = \sum_{O \in \mathcal{T}} Std(\mathcal{T}_{|[n]\setminus O}) \otimes Std(\mathcal{T}_{|O}).$$

Then:

(1) $(\mathbf{H_T}, ., \Delta)$ *is a graded Hopf algebra.*
(2) $(\mathbf{H_T}, \downarrow, \Delta)$ *is a graded infinitesimal bialgebra.*
(3) *The involution ι defines a Hopf algebra isomorphism from* $(\mathbf{H_T}, ., \Delta)$ *to* $(\mathbf{H_T}, ., \Delta^{op})$.

Proof. Let $\mathcal{T} \in \mathbf{T}_n$, $n \geq 0$. Then:

$$\begin{aligned}
&(\Delta \otimes Id) \circ \Delta(\mathcal{T}) \\
&= \sum_{O \in \mathcal{T},\, O' \in \mathcal{T}_{|[n]\setminus O}} Std((\mathcal{T}_{|[n]\setminus O})_{|([n]\setminus O)\setminus O'}) \otimes Std((\mathcal{T}_{|[n]\setminus O})_{|O'}) \otimes Std(\mathcal{T}_{|O}) \\
&= \sum_{O \in \mathcal{T},\, O' \in \mathcal{T}_{|[n]\setminus O}} Std(\mathcal{T}_{|[n]\setminus(O \sqcup O')}) \otimes Std(\mathcal{T}_{|O'}) \otimes Std(\mathcal{T}_{|O}).
\end{aligned}$$

If $O \in \mathcal{T}$ and $O' \in \mathcal{T}_{|[n]\setminus O}$, then $O \sqcup O'$ is an open set of $\mathcal{T}$. Conversely, if $O_1 \subseteq O_2$ are open sets of $\mathcal{T}$, then $O_2 \setminus O_1 \in \mathcal{T}_{|[n]\setminus O_1}$. Putting $O_1 = O$ and $O_2 = O \sqcup O'$:

$$(\Delta \otimes Id) \circ \Delta(\mathcal{T}) = \sum_{O_1 \subseteq O_2 \in \mathcal{T}} Std(\mathcal{T}_{|[n]\setminus O_2}) \otimes Std(\mathcal{T}_{|O_2\setminus O_1}) \otimes Std(\mathcal{T}_{|O_1}).$$

Moreover:

$$(Id \otimes \Delta) \circ \Delta(\mathcal{T}) = \sum_{O \in \mathcal{T},\, O' \in \mathcal{T}_{|O}} Std(\mathcal{T}_{|[n]\setminus O}) \otimes Std(\mathcal{T}_{|O\setminus O'}) \otimes Std(\mathcal{T}_{|O'})$$

If O is an open set of $\mathcal{T}$ and O' is an open set of $\mathcal{T}_{|O}$, then O' is an open set of $\mathcal{T}$. Hence, putting $O_1 = O'$ and $O_2 = O$:

$$(Id \otimes \Delta) \circ \Delta(\mathcal{T}) = \sum_{O_1 \subseteq O_2 \in \mathcal{T}} Std(\mathcal{T}_{|[n]\setminus O_2}) \otimes Std(\mathcal{T}_{|O_2\setminus O_1}) \otimes Std(\mathcal{T}_{|O_1}).$$

This proves that Δ is coassociative. It is obviously homogeneous of degree 0. Moreover, $\Delta(1) = 1 \otimes 1$ and for any $\mathcal{T} \in \mathbf{T}_n$, $n \geq 1$:

$$\Delta(\mathcal{T}) = \mathcal{T} \otimes 1 + 1 \otimes \mathcal{T} + \sum_{\emptyset \subsetneq O \subsetneq [n]} Std(\mathcal{T}_{|[n]\setminus O}) \otimes Std(\mathcal{T}_{|O}).$$

So Δ has a counit.

Let $\mathcal{T} \in \mathbf{T}_n$, $\mathcal{T}' \in \mathbf{T}_{n'}$, $n, n' \geq 0$. By definition of $\mathcal{T}.\mathcal{T}'$:

$$\begin{aligned}
\Delta(\mathcal{T}.\mathcal{T}') &= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}'} Std((\mathcal{T}.\mathcal{T}')_{|[n+n']\setminus O.O'}) \otimes Std((\mathcal{T}.\mathcal{T}')_{|O.O'}) \\
&= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}'} Std(\mathcal{T}_{|[n]\setminus O}).Std(\mathcal{T}'_{[n']\setminus O'}) \otimes Std(\mathcal{T}_{|O}).Std(\mathcal{T}_{|O'}) \\
&= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}'} \left(Std(\mathcal{T}_{|[n]\setminus O}) \otimes Std(\mathcal{T}_{|O})\right) . \left(Std(\mathcal{T}'_{|[n']\setminus O'}) \otimes Std(\mathcal{T}_{|O'})\right) \\
&= \Delta(\mathcal{T}).\Delta(\mathcal{T}').
\end{aligned}$$

Hence, $(\mathbf{H}_{\mathcal{T}}, ., \Delta)$ is a Hopf algebra.

By definition of $\mathcal{T} \downarrow \mathcal{T}'$:

$$\begin{aligned}
\Delta(\mathcal{T} \downarrow \mathcal{T}') &= \sum_{O \in \mathcal{T}, O \neq \emptyset} Std\left((\mathcal{T} \downarrow \mathcal{T}')_{|[n+n']\setminus(O\downarrow[n'])}\right) \otimes Std\left((\mathcal{T} \downarrow \mathcal{T}')_{|O\downarrow[n']}\right) \\
&\quad + \sum_{O' \in \mathcal{T}', O' \neq [n']} Std\left((\mathcal{T} \downarrow \mathcal{T}')_{|[n+n']\setminus O'(+n)}\right) \otimes Std\left((\mathcal{T} \downarrow \mathcal{T}')_{|O'(+n)}\right) \\
&\quad + Std\left((\mathcal{T} \downarrow \mathcal{T}')_{|[n+n']\setminus[n'](+n)}\right) \otimes Std\left((\mathcal{T} \downarrow \mathcal{T}')_{[n'](+n)}\right) \\
&= \sum_{O \in \mathcal{T}, O \neq \emptyset} Std(\mathcal{T}_{|[n]\setminus O}) \otimes Std(\mathcal{T}_{|O}) \downarrow \mathcal{T}' \\
&\quad + \sum_{O' \in \mathcal{T}', O' \neq [n']} \mathcal{T} \downarrow Std(\mathcal{T}'_{|[n']\setminus O'}) \otimes Std(\mathcal{T}'_{|O'}) + \mathcal{T} \otimes \mathcal{T}' \\
&= \sum_{O \in \mathcal{T}, O \neq \emptyset} \left(Std(\mathcal{T}_{|[n]\setminus O}) \otimes Std(\mathcal{T}_{|O})\right) \downarrow (1 \otimes \mathcal{T}')
\end{aligned}$$

$$+\sum_{O'\in\mathcal{T}',O'\neq[n']}(\mathcal{T}\otimes 1)\downarrow\left(Std(\mathcal{T}'_{|[n']\setminus O'})\otimes Std(\mathcal{T}'_{|O'})\right)+\mathcal{T}\otimes\mathcal{T}'$$
$$=(\Delta(\mathcal{T})-\mathcal{T}\otimes 1)\downarrow(1\otimes\mathcal{T}')+(\mathcal{T}\otimes 1)\downarrow(\Delta(\mathcal{T})-1\otimes\mathcal{T}')+\mathcal{T}\otimes\mathcal{T}'$$
$$=\Delta(\mathcal{T})\downarrow(1\otimes\mathcal{T}')+(\mathcal{T}\otimes 1)\downarrow\Delta(\mathcal{T})-\mathcal{T}\otimes\mathcal{T}'.$$

Hence, $(\mathbf{H_T},\downarrow,\Delta)$ is an infinitesimal bialgebra.

For all $\mathcal{T},\mathcal{T}'\in\mathbf{T}$, $\iota(\mathcal{T}.\mathcal{T}')=\iota(\mathcal{T}).\iota(\mathcal{T}')$. Moreover:

$$\begin{aligned}\Delta(\iota(\mathcal{T}))&=\sum_{O\in\mathcal{T}}Std(\iota(\mathcal{T})_{|O})\otimes Std(\iota(\mathcal{T})_{|[n]\setminus O})\\&=\sum_{O\in\mathcal{T}}\iota(Std(\mathcal{T}_{|O}))\otimes\iota(Std(\mathcal{T}_{|[n]\setminus O}))\\&=(\iota\otimes\iota)\circ\Delta^{op}(\mathcal{T}).\end{aligned}$$

So ι is a Hopf algebra morphism from $\mathbf{H_T}$ to $\mathbf{H_T}^{cop}$. $\square$

As a consequence of Theorem 1:

Theorem 7. *The graded, connected coalgebra* $(\mathbf{H_T},\Delta)$ *is cofree, that is to say is isomorphic to the tensor algebra on the space of its primitive elements with the deconcatenation coproduct.*

Remark. Forgetting the total order on $[n]$, that is to say considering isoclasses of finite topologies, we obtain the Hopf algebra of finite spaces of [4] as a quotient of $\mathbf{H_T}$; the product $\downarrow$ induces the product $\succ$ on finite spaces.

2.4. Link with special posets

Let $\mathcal{T}\in\mathbf{T}$. We put:

$$c(\mathcal{T})=deg(\mathcal{T})-\sharp\{\text{equivalence classes of }\sim_{\mathcal{T}}\}.$$

Note that $c(\mathcal{T})\geq 0$. Moreover, $c(\mathcal{T})=0$ if, and only if, the relation $\sim_{\mathcal{T}}$ is the equality, or equivalently if the preorder $\leq_{\mathcal{T}}$ is an order, that is to say if $\mathcal{T}$ is T_0 [17].

If $\mathcal{T}\in\mathbf{T}_n$, $n\geq 0$, is T_0, then for any open set O of $\mathcal{T}$, $\mathcal{T}_{|O}$ and $\mathcal{T}_{|[n]\setminus O}$ are also T_0. Moreover, if $\mathcal{T}$ and $\mathcal{T}'$ are T_0, then $\mathcal{T}.\mathcal{T}'$ and $\mathcal{T}'\downarrow\mathcal{T}'$ also are. Hence, the subspace $\mathbf{H_{T_0}}$ of $\mathbf{H_T}$ generated by T_0 topologies is a Hopf subalgebra. Considering T_0 topologies as special posets, it is isomorphic to the Hopf algebra of special posets $\mathbf{H_{SP}}$: this defines an injective Hopf algebra morphism from $\mathbf{H_{SP}}$ to $\mathbf{H_T}$. We now identify $\mathbf{H_{SP}}$ with its image by this morphism, that is to say with $\mathbf{H_{T_0}}$. Note that $\mathbf{H_{SP}}$ is stable under $\downarrow$, so is a Hopf 2-associative subalgebra of $\mathbf{H_T}$.

Notation. Let $\mathcal{T} \in \mathbf{T}_n$, $n \geq 0$. We denote by $\overline{\mathcal{T}}$ the special poset $Std([n]/\sim_{\mathcal{T}})$, resulting on the set of equivalence classes of $\sim_{\mathcal{T}}$, where the elements of $[n]/\sim_{\mathcal{T}}$, that is to say the equivalence classes of $\sim_{\mathcal{T}}$, are totally ordered by the smallest element of each class. In this way, $\overline{\mathcal{T}}$ is a special poset.

Examples.

$$
\begin{array}{llll}
\overline{\bullet_{1,2}\,\bullet_3} = \bullet_1\bullet_2, & \overline{\mathbf{I}^{3}_{1,2}} = \mathbf{I}^{2}_{1}, & \overline{\mathbf{I}^{1,2}_{3}} = \mathbf{I}^{1}_{2}, & \overline{\bullet_{1,2}} = \bullet_1, \\
\overline{\bullet_{1,3}\,\bullet_2} = \bullet_1\bullet_2, & \overline{\mathbf{I}^{2}_{1,3}} = \mathbf{I}^{2}_{1}, & \overline{\mathbf{I}^{1,3}_{2}} = \mathbf{I}^{1}_{2}, & \overline{\bullet_{1,2,3}} = \bullet_1. \\
\overline{\bullet_{2,3}\,\bullet_1} = \bullet_1\bullet_2, & \overline{\mathbf{I}^{1}_{2,3}} = \mathbf{I}^{1}_{2}, & \overline{\mathbf{I}^{2,3}_{1}} = \mathbf{I}^{2}_{1}; &
\end{array}
$$

Proposition 8. *Let $q \in \mathbb{K}$. The following map is a surjective morphism of Hopf 2-associative algebras:*

$$
\theta_q : \left\{ \begin{array}{rcl} \mathbf{H_T} & \longrightarrow & \mathbf{H_{SP}} \\ \mathcal{T} & \longrightarrow & q^{c(\mathcal{T})}\overline{\mathcal{T}}. \end{array} \right.
$$

Proof. If $\mathcal{T}, \mathcal{T}' \in \mathbf{T}$, then $\overline{\mathcal{T}.\mathcal{T}'} = \overline{\mathcal{T}}.\overline{\mathcal{T}'}$ and $\overline{\mathcal{T} \downarrow \mathcal{T}'} = \overline{\mathcal{T}} \downarrow \overline{\mathcal{T}'}$. Moreover, $deg(\mathcal{T}.\mathcal{T}') = deg(\mathcal{T} \downarrow \mathcal{T}') = deg(\mathcal{T}) + deg(\mathcal{T}')$, and the number of equivalence classes of $\sim_{\mathcal{T}.\mathcal{T}'}$ and $\sim_{\mathcal{T}\downarrow\mathcal{T}'}$ are both equal to the sum of the number of equivalence classes of $\sim_{\mathcal{T}}$ and $\sim'_{\mathcal{T}}$. Hence, $c(\mathcal{T}.\mathcal{T}') = c(\mathcal{T} \downarrow \mathcal{T}') = c(\mathcal{T}) + c(\mathcal{T}')$, and:

$$
\begin{aligned}
\theta_q(\mathcal{T}.\mathcal{T}') &= q^{c(\mathcal{T}.\mathcal{T}')}\overline{\mathcal{T}.\mathcal{T}'} = q^{c(\mathcal{T})}q^{c(\mathcal{T}')}\overline{\mathcal{T}}.\overline{\mathcal{T}'} = \theta_q(\mathcal{T}).\theta_q(\mathcal{T}'), \\
\theta_q(\mathcal{T} \downarrow \mathcal{T}') &= q^{c(\mathcal{T}\downarrow\mathcal{T}')}\overline{\mathcal{T} \downarrow \mathcal{T}'} = q^{c(\mathcal{T})}q^{c(\mathcal{T}')}\overline{\mathcal{T}} \downarrow \overline{\mathcal{T}'} = \theta_q(\mathcal{T}) \downarrow \theta_q(\mathcal{T}'),
\end{aligned}
$$

so θ_q is a 2-associative algebra morphism. If $\mathcal{T} \in \mathbf{T}_n$, $n \geq 1$, then any open set of $\mathcal{T}$ is a union of equivalence classes of $\sim_{\mathcal{T}}$. So there is a bijection:

$$
\left\{ \begin{array}{rcl} \{\text{open sets of } \mathcal{T}\} & \longrightarrow & \{\text{ideals of } \overline{\mathcal{T}}\} \\ O & \longrightarrow & Std(O/\sim_{\mathcal{T}}) \end{array} \right.
$$

Moreover, $c(\mathcal{T}) = c(\mathcal{T}_{|[n]\setminus O}) + c(\mathcal{T}_{|O}) = c(Std(\mathcal{T}_{|[n]\setminus O})) + c(Std(\mathcal{T}_{|O}))$. If $\mathcal{T}$ has k equivalence classes, we obtain:

$$
\begin{aligned}
\Delta \circ \theta_q(\mathcal{T}) &= q^{c(\mathcal{T})}\Delta(\overline{\mathcal{T}}) \\
&= q^{c(\mathcal{T})} \sum_{O \in \mathcal{T}} (([n] \setminus O)/\sim_{\mathcal{T}}) \otimes (O/\sim_{\mathcal{T}}) \\
&= \sum_{O \in \mathcal{T}} q^{c(Std(\mathcal{T}_{|[n]\setminus O}))} q^{c(Std(\mathcal{T}_{|O}))} \overline{Std(\mathcal{T}_{|[n]\setminus O})} \otimes \overline{Std(\mathcal{T}_{|O})} \\
&= (\theta_q \otimes \theta_q) \circ \Delta(\mathcal{T}).
\end{aligned}
$$

If $\mathcal{T} \in \mathbf{T}$ is T_0, then $\overline{\mathcal{T}} = \mathcal{T}$ and $c(\mathcal{T}) = 0$, so $\theta_q(\mathcal{T}) = \mathcal{T}$: θ_q is surjective. $\square$

We obtain a commutative diagram of Hopf 2-associative algebras:

$$\begin{tikzcd} & \mathbf{H_T} \arrow[d, "\theta_q"] \\ \mathbf{H_{SP}} \arrow[ur, hook, "\iota"] \arrow[r, "Id"'] & \mathbf{H_{SP}} \end{tikzcd}$$

Examples.

$$\begin{array}{llll} \theta_q(\bullet_{1,2}\bullet_3) = q\bullet_1\bullet_2, & \theta_q(\mathbf{I}^{3}_{1,2}) = q\,\mathbf{I}^{2}_{1}, & \theta_q(\mathbf{I}^{1,2}_{3}) = q\,\mathbf{I}^{1}_{2}, & \theta_q(\bullet_{1,2}) = q\bullet_1, \\ \theta_q(\bullet_{1,3}\bullet_2) = q\bullet_1\bullet_2, & \theta_q(\mathbf{I}^{2}_{1,3}) = q\,\mathbf{I}^{2}_{1}, & \theta_q(\mathbf{I}^{1,3}_{2}) = q\,\mathbf{I}^{1}_{2}, & \theta_q(\bullet_{1,2,3}) = q^2\bullet_1, \\ \theta_q(\bullet_{2,3}\bullet_1) = q\bullet_1\bullet_2, & \theta_q(\mathbf{I}^{1}_{2,3}) = q\,\mathbf{I}^{1}_{2}, & \theta_q(\mathbf{I}^{2,3}_{1}) = q\,\mathbf{I}^{2}_{1}. & \end{array}$$

Remarks.

(1) In particular, for any $\mathcal{T} \in \mathbf{T}$:

$$\theta_0(\mathcal{T}) = \begin{cases} \mathcal{T} \text{ if } \mathcal{T} \text{ is } T_0, \\ 0 \text{ otherwise.} \end{cases}$$

(2) θ_q is homogeneous for the gradation of $\mathbf{H_T}$ by the cardinality if, and only if, $q = 0$. It is always homogeneous for the gradation by the number of equivalence classes (note that this gradation is not finite-dimensional).

2.5. Pictures and duality

The concept of pictures between tableaux was introduced by Zelevinsky in [18], and generalized to pictures between double posets by Malvenuto and Reutenauer in [11]. We now generalize this for finite topologies, to obtain a Hopf pairing on $\mathbf{H_T}$.

Notations. Let $\leq_{\mathcal{T}}$ be a preorder on $[n]$, and let $i, j \in [n]$. We shall write $i <_{\mathcal{T}} j$ if ($i \leq_{\mathcal{T}} j$ and not $j \leq_{\mathcal{T}} i$).

Definition 9. Let $\mathcal{T} \in \mathbf{T}_k$, $\mathcal{T} \in \mathbf{T}_l$, and let $f : [k] \longrightarrow [l]$. We shall say that f is a picture from $\mathcal{T}$ to $\mathcal{T}'$ if:

- f is bijective;
- For all $i, j \in [k]$, $i <_{\mathcal{T}} j \Longrightarrow f(i) < f(j)$;
- For all $i, j \in [k]$, $f(i) <_{\mathcal{T}'} f(j) \Longrightarrow i < j$;
- For all $i, j \in [k]$, $i \sim_{\mathcal{T}} j \Longleftrightarrow f(i) \sim_{\mathcal{T}'} f(j)$.

The set of pictures between $\mathcal{T}$ and $\mathcal{T}'$ is denoted by $Pic(\mathcal{T}, \mathcal{T}')$.

Proposition 10. *We define a pairing on* $\mathbf{H_T}$ *by* $\langle \mathcal{T}, \mathcal{T}' \rangle = \sharp Pic(\mathcal{T}, \mathcal{T}')$ *for all* $\mathcal{T}, \mathcal{T}' \in \mathbf{T}$*. Then this pairing, when extended by linearity, is a symmetric Hopf pairing. Moreover,* ι *is an isometry for this pairing.*

Proof. First, if $\mathcal{T}, \mathcal{T}' \in \mathbf{T}$, $Pic(\mathcal{T}', \mathcal{T}) = \{f^{-1} \mid f \in Pic(\mathcal{T}, \mathcal{T}')\}$, so $\langle \mathcal{T}', \mathcal{T} \rangle = \langle \mathcal{T}, \mathcal{T}' \rangle$: the pairing is symmetric.

We fix $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T} \in \mathbf{T}$, of respective degrees n_1, n_2 and n. Let $f \in Pic(\mathcal{T}_1.\mathcal{T}_2, \mathcal{T})$. Let $x \in [n_1 + n_2] \setminus [n_1]$ and $y \in [n_1 + n_2]$, such that $f(x) \leq_{\mathcal{T}} f(y)$. Two cases can occur:

- $f(x) <_{\mathcal{T}} f(y)$. Then $x < y$, so $y \in [n_1 + n_2] \setminus [n_1]$.
- $f(x) \sim_{\mathcal{T}} f(y)$. Then $x \sim_{\mathcal{T}_1.\mathcal{T}_2} y$. By definition of $\mathcal{T}_1.\mathcal{T}_2$, $[n_1 + n_2] \setminus [n_1]$ is an open set of $\mathcal{T}_1.\mathcal{T}_2$, so $y \in [n_1 + n_2] \setminus [n_1]$.

In both cases, $y \in [n_1+n_2]\setminus[n_1]$, so $f([n_1+n_2]\setminus[n_1])$ is an open set of $\mathcal{T}$, which we denote by O_f. Moreover, by restriction, $Std(f_{|[n_1]})$ is a picture between $\mathcal{T}_1$ and $Std(\mathcal{T}_{|[n]\setminus O_f})$ and $Std(f_{|[n_1+n_2]\setminus[n_1]})$ is a picture between $\mathcal{T}_2$ and $Std(\mathcal{T}_{|O})$. We can define a map:

$$\phi : \begin{cases} Pic(\mathcal{T}_1.\mathcal{T}_2, \mathcal{T}) \longrightarrow \displaystyle\bigsqcup_{O \in \mathcal{T}} Pic(\mathcal{T}_1, Std(\mathcal{T}_{|[n]\setminus O})) \times Pic(\mathcal{T}_2, Std(\mathcal{T}_{|O})) \\ \qquad\qquad\qquad f \longrightarrow (Std(f_{|[n_1]}), Std(f_{|[n_1+n_2]\setminus[n_1]})) \end{cases}$$

This map is clearly injective. Let $O \in \mathcal{T}$, $(f_1, f_2) \in Pic(\mathcal{T}_1, Std(\mathcal{T}_{|[n]\setminus O})) \times Pic(\mathcal{T}_2, Std(\mathcal{T}_{|O}))$. Let f be the unique bijection $[n_1+n_2] \longrightarrow [n]$ such that $f([n_1+n_2]\setminus[n_1]) = O$, $f_1 = Std(f_{|[n_1]})$ and $f_2 = Std(f_{|[n_1+n_2]\setminus[n_1]})$. Let us prove that f is a picture between $\mathcal{T}_1.\mathcal{T}_2$ and $\mathcal{T}$.

- If $i <_{\mathcal{T}_1.\mathcal{T}_2} j$ in $[n_1 + n_2]$, then $(i, j) \in [n_1]^2$ or $(i, j) \in ([n_1 + n_2] \setminus [n_1])^2$. In the first case, $f_1(i) < f_1(j)$, so $f(i) < f(j)$. In the second case, $f_2(i - n_1) < f_2(j - n_1)$, so $f(i) < f(j)$.
- If $f(i) <_{\mathcal{T}} f(j)$ in $\mathcal{T}$, as $f([n_1 + n_2] \setminus [n_1]) = O$ is an open set, $(i, j) \in [n_1]^2$ or $(i, j) \in ([n_1 + n_2] \setminus [n_1])^2$ or $(i, j) \in [n_1] \times ([n_1 + n_2] \setminus [n_1])$. In the first case, $f_1(i) <_{Std(\mathcal{T}_{|[n]\setminus O})} f_1(j)$, so $i < j$. In the second case, $f_2(i - n_1) <_{Std(\mathcal{T}_{|O})} f_2(j - n_1)$, so $i - n_1 < j - n_1$ and $i < j$. In the last case, $i < j$.
- If $i \sim_{\mathcal{T}_1.\mathcal{T}_2} j$, by definition of $\mathcal{T}_1.\mathcal{T}_2$, $(i, j) \in [n_1]^2$ or $(i, j) \in ([n_1 + n_2] \setminus [n_1])^2$. In the first case, $f_1(i) \sim_{Std(\mathcal{T}_{|[n]\setminus O})} f_1(j)$, so $f(i) \sim_{\mathcal{T}} f(j)$. In the second case, $f_2(i - n_1) \sim_{Std(\mathcal{T}_{|O})} f_2(j - n_1)$, so $f(i) \sim_{\mathcal{T}} f(j)$.
- If $f(i) \sim_{\mathcal{T}} f(j)$, as $O = f([n_1 + n_2] \setminus [n_1])$ is an open set of $\mathcal{T}$, both O and $f([n_1]) = [n] \setminus O$ are stable under $\sim_{\mathcal{T}}$, so $(i, j) \in [n_1]^2$ or $(i, j) \in ([n_1 + n_2] \setminus [n_1])^2$. In the first case, $f_1(i) \sim_{Std(\mathcal{T}_{|[n]\setminus O})} f_1(j)$, so $i \sim_{\mathcal{T}_1} j$ and $i \sim_{\mathcal{T}_1.\mathcal{T}_2} j$. In the second case, $f_2(i - n_1) \sim_{Std(\mathcal{T}_{|O})} f_2(j - n_1)$, so $i - n_1 \sim_{\mathcal{T}_2} j - n_1$ and $i \sim_{\mathcal{T}_1.\mathcal{T}_2} j$.

Finally, ϕ is bijective. We obtain:

$$\begin{aligned}\langle \mathcal{T}_1.\mathcal{T}_2, \mathcal{T}\rangle &= \sharp Pic(\mathcal{T}_1.\mathcal{T}_2, \mathcal{T})\\ &= \sum_{O\in\mathcal{T}} \sharp Pic(\mathcal{T}_1, Std(\mathcal{T}_{|[n]\setminus O})) \times \sharp Pic(\mathcal{T}_2, Std(\mathcal{T}_{|O}))\\ &= \sum_{O\in\mathcal{T}} \langle \mathcal{T}_1, Std(\mathcal{T}_{|[n]\setminus O})\rangle\langle Pic(\mathcal{T}_2, Std(\mathcal{T}_{|O})\rangle\\ &= \langle \mathcal{T}_1 \otimes \mathcal{T}_2, \Delta(\mathcal{T})\rangle.\end{aligned}$$

So $\langle -, -\rangle$ is a Hopf pairing.

Let $\mathcal{T}, \mathcal{T}' \in \mathbf{T}$. It is not difficult to show that $Pic(\iota(\mathcal{T}), \iota(\mathcal{T}')) = j(Pic(\mathcal{T}, \mathcal{T}'))$, so $\langle \iota(\mathcal{T}), \iota(\mathcal{T}')\rangle = \langle \mathcal{T}, \mathcal{T}'\rangle$. $\square$

Remark. Here is the matrix of the pairing in degree 2:

	$\bullet_1\bullet_2$	$\mathfrak{l}^{2}_{1}$	$\mathfrak{l}^{1}_{2}$	$\bullet_{1,2}$
$\bullet_1\bullet_2$	2	1	1	0
$\mathfrak{l}^{2}_{1}$	1	1	0	0
$\mathfrak{l}^{1}_{2}$	1	0	1	0
$\bullet_{1,2}$	0	0	0	2

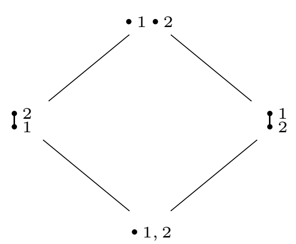
So this pairing is degenerated, as $\bullet_1\bullet_2 - \mathfrak{l}^{2}_{1} - \mathfrak{l}^{1}_{2}$ is in its kernel. The first values of the rank of the pairing in dimension n is given by the following array:

n	1	2	3	4
$Rk(\langle -, -\rangle_{\|(\mathbf{H}_{\mathcal{T}})_n})$	1	3	16	111
$dim(Ker(\langle -, -\rangle_{\|(\mathbf{H}_{\mathcal{T}})_n}))$	0	1	13	244

3. Ribbon basis

3.1. Definition

The set $\mathbf{T}_n$ of topologies on $[n]$ is partially ordered by the refinement of topologies: if $\mathcal{T}, \mathcal{T}' \in \mathbf{T}_n$, $\mathcal{T} \leq \mathcal{T}'$ if any open set of $\mathcal{T}$ is an open set of $\mathcal{T}'$. For example, here is the Hasse graph of $\mathbf{T}_2$:



Definition 11. We define a basis $(R_{\mathcal{T}})_{\mathcal{T}\in\mathbf{T}}$ of $\mathbf{H_T}$ in the following way: for all $\mathcal{T}\in\mathbf{T}_n$, $n\geq 0$,

$$\mathcal{T}=\sum_{\mathcal{T}'\leq\mathcal{T}} R_{\mathcal{T}'}.$$

Examples. If $\{i,j\}=\{1,2\}$:

$$R_{\bullet_{i,j}}=\bullet_{i,j},\quad R_{\mathtt{I}^{j}_{i}}=\mathtt{I}^{j}_{i}-\bullet_{i,j},\quad R_{\bullet_i\bullet_j}=\bullet_i\bullet_j-\mathtt{I}^{j}_{i}-\mathtt{I}^{i}_{j}+\bullet_{i,j}.$$

If $\{i,j,k\}=\{1,2,3\}$:

$$\begin{aligned}
R_{\bullet_{i,j,k}}&=\bullet_{i,j,k},\\
R_{\mathtt{I}^{k}_{i,j}}&=\mathtt{I}^{k}_{i,j}-\bullet_{i,j,k},\\
R_{\mathtt{I}^{j,k}_{i}}&=\mathtt{I}^{j,k}_{i}-\bullet_{i,j,k},\\
R_{\bullet_i\bullet_{j,k}}&=\bullet_i\bullet_{j,k}-\mathtt{I}^{j,k}_{i}-\mathtt{I}^{i}_{j,k}+\bullet_{i,j,k},\\
R_{\mathtt{I}\begin{smallmatrix}k\\j\\i\end{smallmatrix}}&=\mathtt{I}\begin{smallmatrix}k\\j\\i\end{smallmatrix}-\mathtt{I}^{j,k}_{i}-\mathtt{I}^{k}_{i,j}+\bullet_{i,j,k},\\
R_{{}^{j}\mathtt{V}^{k}_{i}}&={}^{j}\mathtt{V}^{k}_{i}-\mathtt{I}\begin{smallmatrix}k\\j\\i\end{smallmatrix}-\mathtt{I}\begin{smallmatrix}j\\k\\i\end{smallmatrix}+\mathtt{I}^{j,k}_{i},\\
R_{{}_{j}\Lambda^{i}{}_{k}}&={}_{j}\Lambda^{i}{}_{k}-\mathtt{I}\begin{smallmatrix}i\\j\\k\end{smallmatrix}-\mathtt{I}\begin{smallmatrix}i\\k\\j\end{smallmatrix}+\mathtt{I}^{i}_{j,k},\\
R_{\mathtt{I}^{j}_{i}\bullet_k}&=\mathtt{I}^{j}_{i}\bullet_k-{}^{j}\mathtt{V}^{k}_{i}-{}_{i}\Lambda^{j}{}_{k}+\mathtt{I}\begin{smallmatrix}j\\k\\i\end{smallmatrix}-\bullet_{i,j}\bullet_k+\mathtt{I}^{k}_{i,j}+\mathtt{I}^{i,j}_{k}-\bullet_{i,j,k},\\
R_{\bullet_i\bullet_j\bullet_k}&=\bullet_i\bullet_j\bullet_k-\mathtt{I}^{j}_{i}\bullet_k-\mathtt{I}^{k}_{i}\bullet_j-\mathtt{I}^{i}_{j}\bullet_k-\mathtt{I}^{k}_{j}\bullet_i-\mathtt{I}^{i}_{k}\bullet_j-\mathtt{I}^{j}_{k}\bullet_i\\
&\quad-\mathtt{I}^{j,k}_{i}-\mathtt{I}^{i,k}_{j}-\mathtt{I}^{i,j}_{k}-\mathtt{I}^{i}_{j,k}-\mathtt{I}^{j}_{i,k}-\mathtt{I}^{k}_{i,j}\\
&\quad+{}^{j}\mathtt{V}^{k}_{i}+{}^{i}\mathtt{V}^{k}_{j}+{}^{i}\mathtt{V}^{j}_{k}+{}_{j}\Lambda^{i}{}_{k}+{}_{i}\Lambda^{j}{}_{k}+{}_{i}\Lambda^{k}{}_{j}+2\bullet_{i,j,k}.
\end{aligned}$$

3.2. The products and the coproduct in the basis of ribbons

Notations.

(1) Let $\mathcal{T}\in\mathbf{T}_n$ and let $I,J\subseteq[n]$. We shall write $I\leq_{\mathcal{T}}J$ if for all $i\in I$, $j\in J$, $i\leq_{\mathcal{T}}j$.
(2) Let $\mathcal{T}\in\mathbf{T}_n$ and let $I,J\subseteq[n]$. We shall write $I<_{\mathcal{T}}J$ if for all $i\in I$, $j\in J$, $i<_{\mathcal{T}}j$.

Theorem 12.

(1) *Let $\mathcal{T}\in\mathbf{T}_k$, $\mathcal{T}'\in\mathbf{T}_l$, $k,l\geq 0$. Then:*

$$R_{\mathcal{T}}.R_{\mathcal{T}'}=\sum_{\substack{\mathcal{T}''\in\mathbf{T}_{k+l},\\ \mathcal{T}''_{|[k]}=\mathcal{T},\,Std(\mathcal{T}''_{|[k+l]\setminus[k]})=\mathcal{T}'}} R_{\mathcal{T}''}.$$

(2) *Let* $\mathcal{T} \in \mathbf{T}_k$*,* $\mathcal{T}' \in \mathbf{T}_l$*,* $k, l \geq 0$*. Then:*

$$R_{\mathcal{T}} \downarrow R_{\mathcal{T}'} = \sum_{\substack{\mathcal{T}'' \in \mathbf{T}_{k+l}, \\ \mathcal{T}''_{|[k]} = \mathcal{T},\, Std(\mathcal{T}''_{|[k+l]\setminus[k]}) = \mathcal{T}', \\ [k] \leq_{\mathcal{T}''} [k+l]\setminus[k]}} R_{\mathcal{T}''}.$$

(3) *For all* $\mathcal{T} \in \mathbf{T}_n$*,* $n \geq 0$*:*

$$\Delta(R_{\mathcal{T}}) = \sum_{\substack{O \in \mathcal{T}, \\ [n]\setminus O <_{\mathcal{T}} O}} R_{Std(\mathcal{T}_{|[n]\setminus O})} \otimes R_{Std(\mathcal{T}_{|O})}.$$

Proof. 1. *First step.* We first prove that for any $\mathcal{T}'' \in \mathbf{T}_{k+l}$, $\mathcal{T}'' \leq \mathcal{T}.\mathcal{T}'$ if, and only if, $\mathcal{T}''_{|[k]} \leq \mathcal{T}$ and $Std(\mathcal{T}''_{|[k+l]\setminus[k]}) \leq \mathcal{T}'$.

$\Longrightarrow$. As $\mathcal{T}'' \leq \mathcal{T}.\mathcal{T}'$, we obtain $\mathcal{T}''_{|[k]} \leq (\mathcal{T}.\mathcal{T}')_{|[k]} = \mathcal{T}$, and $Std(\mathcal{T}''_{|[k+l]\setminus[k]}) \leq Std((\mathcal{T}.\mathcal{T}')_{|[k+l]\setminus[k]}) = \mathcal{T}'$.

$\Longleftarrow$. Let I be an open set of $\mathcal{T}''$. Then $I_1 = I \cap [k]$ is an open set of $\mathcal{T}''_{|[k]}$, so I_1 is an open set of $\mathcal{T}$. Moreover, $I_2 = I \cap ([k+l] \setminus [k])(-k)$ is an open set of $Std(\mathcal{T}''_{|[k+l]\setminus[k]})$, so I_2 is an open set of $\mathcal{T}'$. By definition of $\mathcal{T}.\mathcal{T}'$, $I_1 \sqcup I_2[k] = I$ is an open set of $\mathcal{T}.\mathcal{T}'$, so $\mathcal{T}'' \leq \mathcal{T}.\mathcal{T}'$.

Second step. We define a product $\star$ on $\mathbf{H_T}$ by the formula:

$$R_{\mathcal{T}} \star R_{\mathcal{T}'} = \sum_{\substack{\mathcal{T}'' \in \mathbf{T}_{k+l}, \\ \mathcal{T}''_{|[k]} = \mathcal{T}, Std(\mathcal{T}''_{|[k+l]\setminus[k]}) = \mathcal{T}'}} R_{\mathcal{T}''},$$

for any $\mathcal{T} \in \mathbf{T}_k$, $\mathcal{T}' \in \mathbf{T}_l$, $k, l \geq 0$. Then:

$$\begin{aligned}
\mathcal{T} \star \mathcal{T}' &= \sum_{\mathcal{S} \leq \mathcal{T}, \mathcal{S}' \leq \mathcal{T}'} R_{\mathcal{S}} \star R_{\mathcal{S}'} \\
&= \sum_{\mathcal{S} \leq \mathcal{T}, \mathcal{S}' \leq \mathcal{T}'} \sum_{\substack{\mathcal{S}'' \in \mathbf{T}_{k+l}, \\ \mathcal{S}''_{|[k]} = \mathcal{S}, Std(\mathcal{S}''_{|[k+l]\setminus[k]}) = \mathcal{S}'}} R_{\mathcal{S}''} \\
&= \sum_{\substack{\mathcal{S}'' \in \mathbf{T}_{k+l}, \\ \mathcal{S}''_{|[k]} \leq \mathcal{T}, Std(\mathcal{S}''_{|[k+l]\setminus[k]}) \leq \mathcal{T}'}} R_{\mathcal{S}''} \\
&= \sum_{\substack{\mathcal{S}'' \in \mathbf{T}_{k+l}, \\ \mathcal{S}'' \leq \mathcal{T}.\mathcal{T}'}} R_{\mathcal{S}''} \\
&= \mathcal{T}.\mathcal{T}'.
\end{aligned}$$

We used the first step for the fourth equality. So $\star =$.

2. *First step.* We first prove that for any $\mathcal{T}'' \in \mathbf{T}_{k+l}$, $\mathcal{T}'' \leq \mathcal{T} \downarrow \mathcal{T}'$ if, and only if, $\mathcal{T}''_{|[k]} \leq \mathcal{T}$, $Std(\mathcal{T}''_{|[k+l]\setminus[k]}) \leq \mathcal{T}'$ and $[k] \leq_{\mathcal{T}''} [k+l] \setminus [k]$.

$\Longrightarrow$. By restriction, $\mathcal{T}''_{|[k]} \leq \mathcal{T}$, $Std(\mathcal{T}''_{|[k+l]\setminus[k]}) \leq \mathcal{T}'$. Let $x \in [k]$ and $y \in [k+l] \setminus [k]$, and let O be an open set of $\mathcal{T}''$ which contains x. It is also an open set of $\mathcal{T} \downarrow \mathcal{T}'$ which contains x, so it contains $[k+l] \setminus [k]$ by definition of $\mathcal{T} \downarrow \mathcal{T}'$. So $x \leq_{\mathcal{T}''} y$.

$\Longleftarrow$. As $[k] \leq_{\mathcal{T}''} [k+l] \setminus [k]$, any open sets of $\mathcal{T}''$ which contains an element of $[k]$ contains $[k+l] \setminus [k]$. Hence, there are two types of open sets in $\mathcal{T}''$:

- Open sets O contained in $[k+l] \setminus [k]$. Then $O(-k)$ is an open set of $Std(\mathcal{T}''_{|[k+l]\setminus[k]})$, so it is an open set of $\mathcal{T}'$, and finally O is an open set of $\mathcal{T} \downarrow \mathcal{T}'$.
- Open sets O which contain $[k+l] \setminus [k]$. Then $O \cap [k]$ is an open set of $\mathcal{T}''_{|[k]}$, so is an open set of $\mathcal{T}$. Hence, O is an open set of $\mathcal{T} \downarrow \mathcal{T}'$.

We obtain in this way that $\mathcal{T}'' \leq \mathcal{T} \downarrow \mathcal{T}'$.

Second step. Using the first step, we conclude as in the second step of the first point.

3. *First step.* Let $O \in \mathcal{T}$, $\mathcal{S} \leq \mathcal{T}_{|O}$ and $\mathcal{S}' \leq \mathcal{T}_{|[n]\setminus O}$. It is not difficult to see that there exists a unique topology $\mathcal{T}' \in \mathbf{T}_n$ such that $\mathcal{S} = \mathcal{T}'_{|O}$, $\mathcal{S}' = \mathcal{T}'_{|[n]\setminus O}$ and $[n] \setminus O <_{\mathcal{T}'} O$. It is:

$$\mathcal{T}' = \{\Omega \cup O \mid \Omega \in \mathcal{S}'\} \cup \mathcal{S}.$$

Second step. We define a coproduct on $\mathbf{H_T}$ by:

$$\delta(R_{\mathcal{T}}) = \sum_{\substack{O \in \mathcal{T},\\ [n]\setminus O <_{\mathcal{T}} O}} R_{Std(\mathcal{T}_{|[n]\setminus O})} \otimes R_{Std(\mathcal{T}_{|O})},$$

for any $\mathcal{T} \in \mathbf{T}$. Then:

$$\begin{aligned}
\delta(\mathcal{T}) &= \sum_{\mathcal{T}' \leq \mathcal{T}} \delta(R_{\mathcal{T}'})\\
&= \sum_{\mathcal{T}' \leq \mathcal{T}} \sum_{O \in \mathcal{T}',\, ([n]\setminus O) <_{\mathcal{T}'} O} R_{Std(\mathcal{T}'_{|[n]\setminus O})} \otimes R_{Std(\mathcal{T}'_{|O})}\\
&= \sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{T}' \leq \mathcal{T},\, O \in \mathcal{T}',\\ ([n]\setminus O) <_{\mathcal{T}'} O}} R_{Std(\mathcal{T}'_{|[n]\setminus O})} \otimes R_{Std(\mathcal{T}'_{|O})}\\
&= \sum_{O \in \mathcal{T}} \sum_{\substack{\mathcal{S} \leq Std(\mathcal{T}_{|[n]\setminus O}),\\ \mathcal{S}' \leq Std(\mathcal{T}_{|O})}} R_{\mathcal{S}} \otimes R_{\mathcal{S}'}\\
&= \sum_{O \in \mathcal{T}} Std(\mathcal{T}_{|[n]\setminus O}) \otimes Std(\mathcal{T}_{|O})\\
&= \Delta(\mathcal{T}).
\end{aligned}$$

We used the first step for the third equality. So $\delta = \Delta$. $\square$

4. Generalized T-partitions

4.1. Definition

Definition 13. Let $\mathcal{T} \in \mathbf{T}_n$.

(1) A generalized T-partition of $\mathcal{T}$ is a surjective map $f : [n] \longrightarrow [p]$ such that if $i \leq_{\mathcal{T}} j$ in $[n]$, then $f(i) \leq f(j)$ in $[p]$. If f is a generalized T-partition of $\mathcal{T}$, we shall represent it by the packed word $f(1) \ldots f(n)$.
(2) Let f a generalized T-partition of $\mathcal{T}$. We shall say that f is a (strict) T-partition if for all $i, j \in [n]$:
   - $i <_{\mathcal{T}} j$ and $i > j$ implies that $f(i) < f(j)$ in $[p]$.
   - If $i < j < k$, $i \sim_{\mathcal{T}} k$ and $f(i) = f(j) = f(k)$, then $i \sim_{\mathcal{T}} j$ and $j \sim_{\mathcal{T}} k$.
(3) The set of generalized T-partitions of $\mathcal{T}$ is denoted by $\mathcal{P}(\mathcal{T})$; the set of (strict) T-partitions of $\mathcal{T}$ is denoted by $\mathcal{P}_s(\mathcal{T})$.
(4) If $f \in \mathcal{P}(\mathcal{T})$, we put:

$$\begin{aligned}
\ell_1(f) &= \sharp\{(i,j) \in [n]^2 \mid i <_{\mathcal{T}} j,\ i < j, \text{ and } f(i) = f(j)\},\\
\ell_2(f) &= \sharp\{(i,j) \in [n]^2 \mid i <_{\mathcal{T}} j,\ i > j, \text{ and } f(i) = f(j)\},\\
\ell_3(f) &= \sharp\{(i,j,k) \in [n]^3 \mid i < j < k, i \sim_{\mathcal{T}} k, i \not\sim_{\mathcal{T}} j, j \not\sim_{\mathcal{T}} k \text{ and } f(i) = f(j) = f(k)\}.
\end{aligned}$$

Note that f is strict if, and only if, $\ell_2(f) = \ell_3(f) = 0$.

Example. Let $\mathcal{T} = {}^{1}\!\bullet\!\!\vee\!\!\bullet^{5}_{2,4}\ \bullet_{3}$. If f is a packed word of length 5, it is a generalized T-partition of $\mathcal{T}$ if, and only if, $f(2) = f(4) \leq f(1), f(5)$. Hence:

$$\mathcal{P}(\mathcal{T}) = \left\{ \begin{array}{c} (11111),(11112),(11211),(11212),(11213),(11312),(21111),\\ (21112),(21113),(21211),(21212),(21213),(21311),(21312),\\ (21313),(21314),(21413),(22122),(22123),(31112),(31211),\\ (31212),(31213),(31214),(31312),(31412),(32122),(32123),\\ (32124),(41213),(41312),(42123) \end{array} \right\}.$$

Moreover, f is a strict T-partition of $\mathcal{T}$ if, and only if, $f(2) = f(4) < f(1)$, $f(2) = f(4) \leq f(5)$, and $f(3) \neq f(2), f(4)$. Hence:

$$\mathcal{P}_s(\mathcal{T}) = \left\{ \begin{array}{c} (21211),(21212),(21213),(21311),(21312),(21313),(21314),\\ (21413),(31211),(31212),(31213),(31214),(31312),(31412),\\ (32122),(32123),(32124),(41213),(41312),(42123) \end{array} \right\}.$$

Remarks.

(1) Let $f \in \mathcal{P}(\mathcal{T})$. If $i \sim_{\mathcal{T}} j$ in $[n]$, then $i \leq_{\mathcal{T}} j$ and $j \leq_{\mathcal{T}} i$, so $f(i) \leq f(j)$ and $f(j) \leq f(i)$, and finally $f(i) = f(j)$.

(2) If $\mathcal{T} \in \mathbf{T}$ is T_0, then the set of strict T-partitions of $\mathcal{T}$ is the set of P-partitions of the poset $\mathcal{T}$, as defined in [6,16]. We shall now omit the term "strict" and simply write "T-partitions".

Proposition 14. *Let $q = (q_1, q_2, q_3) \in \mathbb{K}^3$. We define a linear map $\Gamma_q : \mathbf{H_T} \longrightarrow \mathbf{WQSym}$ in the following way: for all $\mathcal{T} \in \mathbf{T}_n$, $n \geq 0$,*

$$\Gamma_q(\mathcal{T}) = \sum_{f \in \mathcal{P}(\mathcal{T})} q_1^{\ell_1(f)} q_2^{\ell_2(f)} q_3^{\ell_3(f)} f(1) \dots f(n).$$

Then Γ_q is a homogeneous surjective Hopf algebra morphism. Moreover, $j \circ \Gamma_{(q_2,q_1,q_3)} = \Gamma_{(q_1,q_2,q_3)} \circ \iota$.

Proof. We shall use the following notations: if $\mathcal{T} \in \mathbf{T}$ and $f \in \mathcal{P}(\mathcal{T})$, we put $\ell(f) = (\ell_1(f), \ell_2(f), \ell_3(f))$ and $q^{\ell(f)} = q_1^{\ell_1(f)} q_2^{\ell_2(f)} q_3^{\ell_3(f)}$.

Let us first prove that Γ_q is an algebra morphism. Let $\mathcal{T}, \mathcal{T}' \in \mathbf{T}$, of respective degree n and n'. Let $f = f(1) \dots f(n+n')$ be a packed word. If $1 \leq i \leq n < j \leq n+n'$, then i and j are not comparable for $\leq_{\mathcal{T}.\mathcal{T}'}$. Hence, $f \in \mathcal{P}(\mathcal{T}.\mathcal{T}')$ if, and only if, $f' = Pack(f(1) \dots f(n)) \in \mathcal{P}(\mathcal{T})$ and $f'' = Pack(f(n+1) \dots f(n+n')) \in \mathcal{P}(\mathcal{T}')$. Moreover, $\ell(f) = \ell(f') + \ell(f'')$, so:

$$\begin{aligned}
\Gamma_q(\mathcal{T}.\mathcal{T}') &= \sum_{f' \in \mathcal{P}(\mathcal{T}), f'' \in \mathcal{P}(\mathcal{T}')} \sum_{\substack{Pack(f(1)\dots f(n)) = f' \\ Pack(f(n+1)\dots f(n+n')) = f''}} q^{\ell(f)} f(1) \dots f(n+n') \\
&= \sum_{f' \in \mathcal{P}(\mathcal{T}), f'' \in \mathcal{P}(\mathcal{T}')} q^{\ell(f')} q^{\ell(f'')} \sum_{\substack{Pack(f(1)\dots f(n)) = f' \\ Pack(f(n+1)\dots f(n+n')) = f''}} f(1) \dots f(n+n') \\
&= \sum_{f' \in \mathcal{P}(\mathcal{T}), f'' \in \mathcal{P}(\mathcal{T}')} q^{\ell(f')} q^{\ell(f'')} f'.f'' \\
&= \Gamma_q(\mathcal{T}).\Gamma_q(\mathcal{T}').
\end{aligned}$$

Let us now prove that Γ_q is a coalgebra morphism. Let $\mathcal{T} \in \mathbf{T}_n$. We consider the two following sets:

$$A = \bigsqcup_{O \in \mathcal{T}} \mathcal{P}(\mathcal{T}_{|[n] \setminus O}) \times \mathcal{P}(\mathcal{T}_{|O}), \quad B = \{(f, i) \mid f \in \mathcal{P}(\mathcal{T}), 0 \leq i \leq \max(f)\}.$$

Let $(f, g) \in \mathcal{P}(\mathcal{T}_{|[n] \setminus O}) \times \mathcal{P}(\mathcal{T}_{|O}) \subseteq A$. We define $h : [n] \longrightarrow \mathbb{N}$ by $h(i) = f(i)$ if $i \notin O$ and $h(i) = g(i) + \max(f)$ if $i \in O$. As f and g are packed words, $h(1) \dots h(n)$ also is. Let us assume that $i \leq_{\mathcal{T}} j$. Three cases are possible:

- $i, j \notin O$. As $f(i) \leq f(j)$, $h(i) \leq h(j)$.
- $i, j \in O$. As $g(i) \leq g(j)$, $h(i) \leq h(j)$.
- $i \notin O$ and $j \in O$. Then $h(i) = f(i) \leq \max(f) < \max(f) + g(j) = h(j)$.

Consequently, $h \in \mathcal{P}(\mathcal{T})$. Hence, we define a map $\theta : A \longrightarrow B$, sending (f,g) to $(h, \max(f))$.

θ is injective: if $\theta(f,g) = \theta(f',g') = (h,k)$, then $\max(f) = \max(f') = k$. Moreover:

$$O = h^{-1}(\{\max(f)+1, \ldots, \max(h)\}) = h^{-1}(\{k+1, \ldots, \max(h)\}) = O'.$$

Then $f = h_{|O} = h_{|O'} = f'$ and $g = Pack(h_{|[n]\setminus O}) = Pack(h_{|[n]\setminus O}) = g'$. Finally, $(f,g) = (f',g')$.

θ is surjective: let $(h,k) \in B$. We put $O = h^{-1}(\{k+1, \ldots \max(h)\})$. Let $i \in O$ and $j \in [n]$, such that $i \leq_{\mathcal{T}} j$. As $h \in \mathcal{P}(\mathcal{T})$, $h(j) \geq h(i) > k$, so $j \in O$ and O is an open set of $\mathcal{T}$. Let $f = h_{|[n]\setminus O}$ and $g = Pack(h_{|O})$. By restriction, $f \in \mathcal{P}(\mathcal{T}_{|[n]\setminus O})$ and $g \in \mathcal{P}(\mathcal{T}_{|O})$. Moreover, as h is a packed word, $\max(f) = k$, and for all $i \in O$, $g(i) = h(i) - k$. This implies that $\theta(f,g) = (h,k)$.

Let $(f,g) \in \mathcal{P}(\mathcal{T}_{|[n]\setminus O}) \times \mathcal{P}(\mathcal{T}_{|O}) \subseteq A$, and let $\theta(f,g) = (h,k)$. If $i \notin O$ and $j \in O$, $h(i) < h(j)$. Hence:

$$\begin{aligned}
\ell_1(h) &= \sharp\{(i,j) \in ([n] \setminus O)^2 \mid i <_{\mathcal{T}} j,\, i < j \text{ and } h(i) = h(j)\} \\
&\quad + \sharp\{(i,j) \in O^2 \mid i <_{\mathcal{T}} j,\, i < j \text{ and } h(i) = h(j)\} \\
&= \sharp\{(i,j) \in ([n] \setminus O)^2 \mid i <_{\mathcal{T}} j,\, i < j \text{ and } f(i) = f(j)\} \\
&\quad + \sharp\{(i,j) \in O^2 \mid i <_{\mathcal{T}} j,\, i < j \text{ and } g(i) + \max(f) = g(j) + \max(f)\} \\
&= \ell_1(f) + \ell_1(g).
\end{aligned}$$

Similarly, $\ell_2(h) = \ell_2(f) + \ell_2(g)$ and $\ell_3(h) = \ell_3(f) + \ell_3(g)$. We obtain:

$$\begin{aligned}
\Delta \circ \Gamma_q(\mathcal{T}) &= \sum_{h \in \mathcal{P}(\mathcal{T})} \sum_{0 \leq k \leq \max(h)} q^{\ell(h)} h_{|h^{-1}(\{1,\ldots,k\})} \otimes Pack\left(h_{|h^{-1}(\{k+1,\ldots,\max(h)\})}\right) \\
&= \sum_{(h,k) \in B} q^{\ell(h)} h_{|h^{-1}(\{1,\ldots,k\})} \otimes Pack\left(h_{|h^{-1}(\{k+1,\ldots,\max(h)\})}\right) \\
&= \sum_{(f,g) \in A} q^{\ell(f)} q^{\ell(g)} f \otimes g \\
&= \sum_{O \in \mathcal{T}} \left(\sum_{f \in \mathcal{P}(\mathcal{T}_{|[n]\setminus O})} q^{\ell(f)} f \right) \otimes \left(\sum_{g \in \mathcal{P}(\mathcal{T}_{|O})} q^{\ell(g)} g \right) \\
&= (\Gamma_q \otimes \Gamma_q) \circ \Delta(\mathcal{T}).
\end{aligned}$$

Consequently, Γ_q is a Hopf algebra morphism.

Let f be a packed word of length n, and $\mathcal{T}_f$ be the associated topology, as defined in Section 2. Note that $f \in \mathcal{P}(\mathcal{T}_f)$; moreover, if $g \in \mathcal{P}(\mathcal{T}_f)$ is different from f, then $\max(g) < \max(f)$. Hence:

$$\Gamma_q(\mathcal{T}_f) = f + (\text{linear span of packed words } g \text{ of maximum} < \max(f)) .$$

In particular, if $f = (1\dots1)$, $\mathcal{T}_w = \bullet_{1,\dots,n}$, and $\Gamma_q(T_f) = (1\dots1)$. By a triangularity argument, Γ_q is surjective.

Let $\mathcal{T} \in \mathbf{T}$. It is not difficult to prove that $\mathcal{P}(\mathcal{T}) = j\,(\mathcal{P}(\iota(\mathcal{T})))$. Moreover, if $f \in \mathcal{P}(\iota(\mathcal{T}))$ and $g = j(f)$, then $\ell_1(f) = \ell_2(g)$, $\ell_2(f) = \ell_1(g)$ and $\ell_3(f) = \ell_3(g)$. So:

$$j \circ \Gamma_{(q_2,q_2,q_3)} \circ \iota(\mathcal{T}) = \sum_{g \in \mathcal{P}(\mathcal{T})} q_2^{\ell_2(g)} q_1^{\ell_1(g)} q_2^{\ell_2(g)} g = \Gamma_{(q_1,q_2,q_3)}(\mathcal{T}),$$

so $j \circ \Gamma_{(q_2,q_2,q_3)} \circ \iota = \Gamma_{(q_1,q_2,q_3)}$. $\square$

Examples.

$$\begin{aligned}
\Gamma_q(\bullet_1) &= (1),\\
\Gamma_q(\bullet_1 \bullet_2) &= (12) + (21) + (11),\\
\Gamma_q(\mathbf{I}^{2}_{1}) &= (12) + q_1(11),\\
\Gamma_q(\mathbf{I}^{1}_{2}) &= (21) + q_2(11),\\
\Gamma_q(\bullet_{1,2}) &= (11),\\
\Gamma_q({}^{2}\mathbf{V}^{3}_{1}) &= (123) + (132) + (122) + q_1(112) + q_1(121) + q_1^2(111),\\
\Gamma_q({}^{1}\mathbf{V}^{3}_{2}) &= (213) + (312) + (212) + q_2(112) + q_1(211) + q_1q_2(111),\\
\Gamma_q({}^{1}\mathbf{V}^{2}_{3}) &= (231) + (321) + (221) + q_2(121) + q_2(211) + q_2^2(111),\\
\Gamma_q(\mathbf{I}^{3}_{1,2}) &= (112) + q_1^2(111),\\
\Gamma_q(\mathbf{I}^{2}_{1,3}) &= (121) + q_1q_2q_3(111),\\
\Gamma_q(\mathbf{I}^{1}_{2,3}) &= (211) + q_2^2(111),\\
\Gamma_q(\mathbf{I}^{2,3}_{1}) &= (122) + q_1^2(111),\\
\Gamma_q(\mathbf{I}^{1,3}_{2}) &= (212) + q_1q_2q_3(111),\\
\Gamma_q(\mathbf{I}^{1,2}_{3}) &= (221) + q_2^2(111).
\end{aligned}$$

Remarks.

(1) In particular:

$$\Gamma_{(1,1,1)}(\mathcal{T}) = \sum_{f \in \mathcal{P}(\mathcal{T})} f(1)\dots f(n), \quad \Gamma_{(1,0,0)}(\mathcal{T}) = \sum_{f \in \mathcal{P}_s(\mathcal{T})} f(1)\dots f(n).$$

(2) The restriction of $\Gamma_{(1,0,0)}$ to $\mathbf{H_{SP}}$ is the map Γ defined in Section 1.2.

4.2. Linear extensions

Definition 15. Let $\mathcal{T} \in \mathbf{T}_n$, $n \geq 0$. A linear extension of $\mathcal{T}$ is an ordered partition $A = (A_1, \ldots, A_k)$ of $[n]$ such that:

- the equivalence classes of $\sim_{\mathcal{T}}$ are $A_1, \ldots, A_k$;
- if, in the poset $\overline{\mathcal{T}}$, $A_i \leq_{\mathcal{T}} A_j$, then $i \leq j$.

The set of linear extensions of $\mathcal{T}$ is denoted by $\mathcal{L}(\mathcal{T})$.

Notations. If $A = (A_1, \ldots, A_k)$ is a linear extension of $\mathcal{T} \in \mathbf{T}_n$, we bijectively associate to A the packed word $f = f(1) \ldots f(n)$ of length n such that for all $i \in [n]$, for all $j \in [k]$, $f(i) = j$ if, and only if $i \in A_j$. We shall now use the packed word f instead of A.

Example. Let $\mathcal{T} = {}^{3\bullet}\!\bigvee^{\bullet\,1,5,6}_{2,4}$. The linear extensions of $\mathcal{T}$ are $(\{2,4\}, \{3\}, \{1,5,6\})$ and $(\{2,4\}, \{1,5,6\}, \{3\})$, or, expressed in packed words, (312133) and (213122).

Remark. Let $\mathcal{T} \in \mathbf{T}_n$, $n \geq 0$, and $f = f(1) \ldots f(n)$ be a packed word of length n. Then $f \in \mathcal{L}(\mathcal{T})$ if, and only if:

(1) for all $i, j \in [n]$, $f(i) = f(j)$ if, and only if, $i \sim_{\mathcal{T}} j$;
(2) for all $i, j \in [n]$, $i <_{\mathcal{T}} j$ implies that $f(i) < f(j)$.

Hence, $\mathcal{L}(\mathcal{T}) \subseteq P(\mathcal{T})$ and more precisely:

$$\mathcal{L}(\mathcal{T}) = \{f \in \mathcal{P}(\mathcal{T}) \mid \max(f) = \sharp\overline{\mathcal{T}}\} = \{f \in \mathcal{P}_s(\mathcal{T}) \mid \max(f) = \sharp\overline{\mathcal{T}}\}.$$

Proposition 16.

(1) *Let f', f'' be two packed words, of respective length n and n'. We put:*

$$f' \sqcup f'' = \sum_{\substack{Pack(f(1)\ldots f(n)) = f' \\ Pack(f(n+1)\ldots f(n+n')) = f'' \\ \{f(1),\ldots f(n)\} \cap \{f(n+1),\ldots,f(n+n')\} = \emptyset}} f.$$

This defines a product on $\mathbf{WQSym}$*, such that* $(\mathbf{WQSym}, \sqcup, \Delta)$ *is a Hopf algebra.*

(2) *Let L be the following map:*

$$L : \begin{cases} \mathbf{H_T} \longrightarrow \mathbf{WQSym} \\ \mathcal{T} \longrightarrow \displaystyle\sum_{f \in \mathcal{L}(\mathcal{T})} f. \end{cases}$$

Then L is a surjective Hopf morphism from $\mathbf{H_T}$ to $(\mathbf{WQSym}, \sqcup, \Delta)$. Moreover, $j \circ L = L \circ \iota$.

Proof. Let $\mathcal{T},\mathcal{T}' \in \mathbf{T}$, of respective degree n and n'. Let $f = f(1)\ldots f(n+n')$ be a packed word. If $1 \leq i \leq n < j \leq n+n'$, then i and j are not comparable for $\leq_{\mathcal{T}.\mathcal{T}'}$. Hence, $f \in \mathcal{L}(\mathcal{T}.\mathcal{T}')$ if, and only if:

- $f' = Pack(f(1)\ldots f(n)) \in \mathcal{L}(\mathcal{T})$ and $f'' = Pack(f(n+1)\ldots f(n+n')) \in \mathcal{L}(\mathcal{T}')$.
- $\{f(1),\ldots f(n)\} \cap \{f(n+1),\ldots,f(n+n')\} = \emptyset$.

So:

$$\begin{aligned}
L(\mathcal{T}.\mathcal{T}') &= \sum_{f'\in\mathcal{L}(\mathcal{T}),f''\in\mathcal{L}(\mathcal{T}')} \sum_{\substack{Pack(f(1)\ldots f(n))=f' \\ Pack(f(n+1)\ldots f(n+n'))=f'' \\ \{f(1),\ldots f(n)\}\cap\{f(n+1),\ldots,f(n+n')\}=\emptyset}} f(1)\ldots f(n+n') \\
&= \sum_{f'\in\mathcal{L}(\mathcal{T}),f''\in\mathcal{L}(\mathcal{T}')} f \sqcup\!\!\sqcup f' \\
&= L(\mathcal{T}) \sqcup\!\!\sqcup L(\mathcal{T}').
\end{aligned}$$

Let $\mathcal{T} \in \mathbf{T}_n$. We consider the two following sets:

$$A = \bigsqcup_{O\in\mathcal{T}} \mathcal{L}(\mathcal{T}_{|[n]\setminus O}) \times \mathcal{L}(\mathcal{T}_{|O}), \quad B = \{(f,i) \mid f \in \mathcal{L}(\mathcal{T}), 0 \leq i \leq \max(f)\}.$$

Let $(f,g) \in \mathcal{L}(\mathcal{T}_{|[n]\setminus O}) \times \mathcal{L}(\mathcal{T}_{|O}) \subseteq A$. We define $h : [n] \longrightarrow \mathbb{N}$ by $h(i) = f(i)$ if $i \notin O$ and $h(i) = g(i) + \max(f)$ if $i \in O$. As f and g are packed words, $h(1)\ldots h(n)$ also is. Moreover, as O and $[n] \setminus O$ are union of equivalence classes of $\sim_{\mathcal{T}}$, for all $i,j \in [n]$, $h(i) = h(j)$ if, and only if, $i \sim_{\mathcal{T}} j$. Let us assume that $i \leq_{\mathcal{T}} j$. Three cases are possible:

- $i,j \notin O$. As $f(i) \leq f(j)$, $h(i) \leq h(j)$.
- $i,j \in O$. As $g(i) \leq g(j)$, $h(i) \leq h(j)$.
- $i \notin O$ and $j \in O$. Then $h(i) = f(i) \leq \max(f) < \max(f) + g(j) = h(j)$.

Consequently, $h \in \mathcal{L}(\mathcal{T})$. Hence, we define a map $\theta : A \longrightarrow B$, sending (f,g) to $(h,\max(f))$. The proof of the bijectivity of θ is similar to the proof for the case of generalized T-partitions. We obtain:

$$\begin{aligned}
\Delta \circ L(\mathcal{T}) &= \sum_{h\in\mathcal{L}(\mathcal{T})} \sum_{0\leq k\leq\max(h)} h_{|h^{-1}(\{1,\ldots,k\})} \otimes Pack\left(h_{|h^{-1}(\{k+1,\ldots,\max(h)\})}\right) \\
&= \sum_{(h,k)\in B} h_{|h^{-1}(\{1,\ldots,k\})} \otimes Pack\left(h_{|h^{-1}(\{k+1,\ldots,\max(h)\})}\right) \\
&= \sum_{(f,g)\in A} f \otimes g
\end{aligned}$$

$$= \sum_{O \in \mathcal{T}} \left(\sum_{f \in \mathcal{L}(\mathcal{T}_{|[n]\setminus O})} f \right) \otimes \left(\sum_{g \in \mathcal{L}(\mathcal{T}_{|O})} g \right)$$
$$= (L \otimes L) \circ \Delta(\mathcal{T}).$$

Let f be a packed word. Then $\mathcal{L}(\mathcal{T}_f) = \{f\}$, so $L(\mathcal{T}_f) = f$, that is L is surjective. As L is compatible with the products and the coproducts of both $\mathbf{H}_{\mathcal{T}}$ and $(\mathbf{WQSym}, \sqcup\!\sqcup, \Delta)$, its image, that is to say $(\mathbf{WQSym}, \sqcup\!\sqcup, \Delta)$, is a Hopf algebra, and L is a Hopf algebra morphism.

It is not difficult to prove that for all $\mathcal{T} \in \mathbf{T}$, $j(\mathcal{L}(\iota(\mathcal{T}))) = \mathcal{L}(\mathcal{T})$, which implies that $j \circ L \circ \iota = L$. $\square$

Remark. The product $\sqcup\!\sqcup$ is defined and used in [5].

Examples.

$$\begin{array}{rclrclrcl}
L(\bullet_1) &=& (1), & L(\bullet_1 \bullet_2) &=& (12)+(21), & L(\mathbf{I}^{2}_{1}) &=& (12),\\
L(\mathbf{I}^{1}_{2}) &=& (21), & L(\bullet_{1,2}) &=& (11), & L(\bullet_{1,2,3}) &=& (111),\\
L(\mathbf{I}^{3}_{1,2}) &=& (112), & L(\mathbf{I}^{2}_{1,3}) &=& (121), & L(\mathbf{I}^{1}_{2,3}) &=& (211),\\
L(\mathbf{I}^{1,2}_{3}) &=& (221), & L(\mathbf{I}^{1,3}_{2}) &=& (212), & L(\mathbf{I}^{2,3}_{1}) &=& (122),\\
L({}^{2}\mathbf{V}^{3}_{1}) &=& (123)+(132), & L({}^{1}\mathbf{V}^{3}_{2}) &=& (213)+(312), & L({}^{1}\mathbf{V}^{2}_{3}) &=& (231)+(321),\\
L(\mathbf{I}\begin{smallmatrix}3\\2\\1\end{smallmatrix}) &=& (123), & L(\mathbf{I}\begin{smallmatrix}3\\1\\2\end{smallmatrix}) &=& (213), & L(\mathbf{I}\begin{smallmatrix}2\\1\\3\end{smallmatrix}) &=& (231),\\
L(\mathbf{I}\begin{smallmatrix}2\\3\\1\end{smallmatrix}) &=& (132), & L(\mathbf{I}\begin{smallmatrix}1\\3\\2\end{smallmatrix}) &=& (312), & L(\mathbf{I}\begin{smallmatrix}1\\2\\3\end{smallmatrix}) &=& (321).
\end{array}$$

Here is an example of product $\sqcup\!\sqcup$:

$$(112) \sqcup\!\sqcup (12) = (11234) + (11324) + (11423) + (22314) + (22413) + (33412).$$

4.3. From linear extensions to T-partitions

Definition 17. Let f, g be two packed words of the same length n. We shall say that $g \leq f$ if:

- For all $i, j \in [n]$, $f(i) \leq f(j)$ implies that $g(i) \leq g(j)$.
- For all $i, j \in [n]$, $f(i) > f(j)$ and $i < j$ implies that $g(i) > g(j)$.
- For all $i, j \in [n]$, $f(i) = f(j)$ implies that $g(i) = g(j)$.

The relation $\leq$ is a partial order on the set of packed words of length n. Here are the Hasse graphs of these posets if $n = 2$ or $n = 3$:

$$\begin{tikzcd}[column sep=small]
(12) \arrow[d, no head] & (21)\ ; \\
(11) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
& (123) \arrow[dl, no head] \arrow[dr, no head] & & (132) \arrow[d, no head] & (213) \arrow[d, no head] & (231) \arrow[d, no head] & (312) \arrow[d, no head] & (321) \\
(122) \arrow[dr, no head] & & (112) \arrow[dl, no head] & (121) & (212) & (221) & (211) & \\
& (111) & & & & & &
\end{tikzcd}$$

Remark. This order is also introduced and used in [12].

Lemma 18. *Let f, g be packed words of length n. Then $g \leq f$ if, and only if, g is a T-partition of $\mathcal{T}_f$.*

Proof. $\Longrightarrow$. Let us assume that $g \leq f$. If $i \leq_{\mathcal{T}_f} j$, then $f(i) \leq f(j)$, so $g(i) \leq g(j)$. If $i <_{\mathcal{T}_f} j$ and $i > j$, then $f(i) < f(j)$, so $g(i) < g(j)$. We assume that $i < j < k$, $i \sim_{\mathcal{T}_f} k$ and $g(i) = g(j) = g(k)$. As $i \sim_{\mathcal{T}} k$, $f(i) = f(k)$. If $f(j) < f(i)$, as $g \leq f$, we obtain $g(j) < g(i)$: contradiction. So $f(i) \leq f(j)$. If $f(j) > f(k)$, as $g \leq f$, we obtain $g(j) > g(k)$: contradiction. So $f(j) \leq f(k)$, and finally, $f(i) = f(j) = f(k)$, so $i \sim_{\mathcal{T}_f} j$ and $j \sim_{\mathcal{T}_f} k$. Hence, $g \in \mathcal{P}_s(\mathcal{T}_f)$.

$\Longleftarrow$. Let us assume that $g \in \mathcal{P}_s(\mathcal{T}_f)$. If $f(i) \leq f(j)$, then $i \leq_{\mathcal{T}_f} j$, so $g(i) \leq g(j)$. If $f(i) = f(j)$, then $i \sim_{\mathcal{T}_f} j$, so $g(i) = g(j)$. If $f(i) < f(j)$ and $i > j$, then $i <_{\mathcal{T}_f} j$ and $i > j$, so $g(i) < g(j)$. This implies $g \leq f$. $\square$

Proposition 19. *We define:*

$$\varphi_{(1,0,0)} : \begin{cases} \mathbf{WQSym} \longrightarrow \mathbf{WQSym} \\ \qquad\quad f \longrightarrow \displaystyle\sum_{g \leq f} g. \end{cases}$$

Then $\varphi_{(1,0,0)}$ is a Hopf algebra isomorphism from $(\mathbf{WQSym}, \sqcup\!\sqcup, \Delta)$ to $(\mathbf{WQSym}, ., \Delta)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{H_T} \arrow[r, "L"] \arrow[dr, "\Gamma_{(1,0,0)}"'] & (\mathbf{WQSym}, \sqcup\!\sqcup, \Delta) \arrow[d, "\varphi_{(1,0,0)}"] \\
& (\mathbf{WQSym}, ., \Delta)
\end{tikzcd}$$

Proof. Let $\mathcal{T} \in \mathbf{T}_n$, $n \geq 0$.

First step. Let $f \in \mathcal{L}(\mathcal{T})$ and $g \leq f$; let us prove that $g \in \mathcal{P}_s(\mathcal{T})$. This is the generalization of Lemma 18 to any finite topology.

(1) If $i \leq_{\mathcal{T}} j$, then $f(i) \leq f(j)$, so $g(i) \leq g(j)$.
(2) If $i <_{\mathcal{T}} j$ and $j < i$, then $f(i) < f(j)$, so $g(i) < g(j)$.
(3) Let us assume that $i < j < k$, $i \sim_{\mathcal{T}} k$ and $g(i) = g(j) = g(k)$. If $i \not\sim_{\mathcal{T}} j$, then $f(i) \neq f(j)$. If $f(i) > f(j)$, as $g \leq f$, we obtain $g(i) > g(j)$: this is a contradiction. So $f(i) < f(j)$. As $\sim_{\mathcal{T}}$ is an equivalence, $j \not\sim_{\mathcal{T}} k$, and we obtain in the same way $f(j) < f(k)$. So $f(i) < f(k)$, and $i \not\sim_{\mathcal{T}} k$: this is a contradiction. As a consequence, $i \sim_{\mathcal{T}} j$ and $j \sim_{\mathcal{T}} k$.

This proves that $g \in \mathcal{P}_s(\mathcal{T})$.

Second step. Let us now consider an element $g \in \mathcal{P}_s(\mathcal{T})$. We want to prove that there exists a unique $f \in \mathcal{L}(\mathcal{T})$ such that $g \leq f$. For all $p \in [\max(g)]$, $g^{-1}(\{p\})$ is the union of equivalence classes of $\sim_{\mathcal{T}}$, and we put:

$$g^{-1}(\{p\}) = C_{p,1} \sqcup \ldots \sqcup C_{p,k_p}.$$

Moreover, as $g \in P_s(\mathcal{T})$, necessarily the $C_{p,r}$ are intervals. Hence, we assume that for all p:

$$C_{p,1} < \ldots < C_{p,k_p},$$

which means for all $r < s$, all the elements of $C_{p,r}$ are smaller than all the elements of $C_{p,s}$.

Unicity. Let us assume there exists $f \in \mathcal{L}(\mathcal{T})$, such that $g \leq f$. The linear extension f is constant on $C_{p,r}$: we put $f(C_{p,r}) = \{c_{p,r}\}$. As $f \in \mathcal{L}(\mathcal{T})$, the $c_{p,r}$ are all distinct.

(1) If $r < s$ and $c_{p,r} > c_{p,s}$, choosing $i \in C_{p,r}$ and $j \in C_{p,s}$, then $i < j$ and $f(i) > f(j)$, so, as $g \leq f$, $p = g(i) > g(j) = p$: contradiction. So $c_{p,r} < c_{p,s}$.
(2) If $p < q$ and $c_{q,s} < c_{p,r}$, choosing $i \in C_{p,r}$ and $j \in C_{q,s}$, then $f(i) \geq f(j)$, which implies that $p = g(i) \geq g(j) = q$: contradiction. So $c_{p,r} < c_{q,s}$.

We finally obtain:

$$c_{1,1} < \ldots < c_{1,k_1} < \ldots < c_{\max(g),1} < \ldots < c_{\max(g),k_{\max(g)}},$$

which entirely determines f: f is unique.

Existence. Let us consider the packed word f determined by:

$$f(i) = k_1 + \ldots + k_{p-1} + r \text{ if } x \in C_{p,r}.$$

The values of f on two different subsets $C_{p,r}$ are different, so $i \sim_{\mathcal{T}} j$ if, and only if, $f(i) = f(j)$. If $i <_{\mathcal{T}} j$, assuming $i \in C_{p,r}$ and $j \in C_{q,s}$, then $p = g(x) \leq g(y) = q$. When $p < q$, then $f(i) < f(j)$. When $p = q$, if $j < i$ we should have, as $g \in \mathcal{P}_s(\mathcal{T})$, $g(i) < g(j)$: contradiction. So $i < j$, and $r < s$, so $f(x) < f(y)$. Finally, $f \in \mathcal{L}(\mathcal{T})$.

If $f(i) = f(j)$, then i and j are in the same $C_{p,r}$, so $g(i) = g(j)$. If $f(i) \leq f(j)$, assuming that $i \in C_{p,r}$ and $j \in C_{q,s}$, then $p \leq q$, so $p = g(x) \leq g(y) = q$. If $f(i) < f(j)$ and $i > j$ then $p \leq q$; if $p = q$, then $r > s$, so $f(i) > f(j)$: contradiction. So $p < q$, and $p = g(i) < g(j) = q$. We obtain that $g \leq f$.

As a conclusion:

$$\varphi_{(1,0,0)} \circ L(\mathcal{T}) = \sum_{f \in \mathcal{L}(\mathcal{T})} \sum_{g \leq f} g(1) \ldots g(n) = \sum_{g \in \mathcal{P}_s(\mathcal{T})} g(1) \ldots g(n) = \Gamma_{(1,0,0)}(\mathcal{T}).$$

So $\varphi_{(1,0,0)} \circ L = \Gamma_{(1,0,0)}$.

Third step. Let $x, y \in \mathbf{WQSym}$. As L is surjective, there exists $x', y' \in \mathbf{H}_{\mathcal{T}}$ such that $L(x') = x$ and $L(y') = y$. Then:

$$\begin{aligned}
\varphi_{(1,0,0)}(x \sqcup\!\sqcup y) &= \varphi_{(1,0,0)}(L(x') \sqcup\!\sqcup L(y')) \\
&= \varphi_{(1,0,0)} \circ L(x'.y') \\
&= \Gamma_{(1,0,0)}(x'.y') \\
&= \Gamma_{(1,0,0)}(x').\Gamma_{(1,0,0)}(y') \\
&= \varphi_{(1,0,0)} \circ L(x').\varphi_{(1,0,0)} \circ L(y') \\
&= \varphi_{(1,0,0)}(x).\varphi_{(1,0,0)}(y).
\end{aligned}$$

Moreover:

$$\begin{aligned}
\Delta \circ \varphi_{(1,0,0)}(x) &= \Delta \circ \varphi_{(1,0,0)} \circ L(x') \\
&= \Delta \circ \Gamma_{(1,0,0)}(x') \\
&= (\Gamma_{(1,0,0)} \otimes \Gamma_{(1,0,0)}) \circ \Delta(x') \\
&= (\varphi_{(1,0,0)} \otimes \varphi_{(1,0,0)}) \circ (L \otimes L) \circ \Delta(x') \\
&= (\varphi_{(1,0,0)} \otimes \varphi_{(1,0,0)}) \circ \Delta(L(x')) \\
&= (\varphi_{(1,0,0)} \otimes \varphi_{(1,0,0)}) \circ \Delta(x).
\end{aligned}$$

So $\varphi_{(1,0,0)}$ is a Hopf algebra morphism.

Let $x \in \mathbf{WQSym}$. As $\Gamma_{(1,0,0)}$ is surjective, there exists $x' \in \mathbf{H_T}$, such that $\Gamma_{(1,0,0)}(x') = x$. Then $\varphi_{(1,0,0)}(L(x')) = x$: $\varphi_{(1,0,0)}$ is surjective. Since it is homogeneous and the homogeneous components of $\mathbf{WQSym}$ are finite dimensional, it is an isomorphism. $\square$

Examples.

$$\varphi_{(1,0,0)}((123)) = (123) + (122) + (112) + (111),$$

$$\begin{aligned}
\varphi_{(1,0,0)}((132)) &= (132) + (121), & \varphi_{(1,0,0)}((213)) &= (213) + (212),\\
\varphi_{(1,0,0)}((231)) &= (231) + (221), & \varphi_{(1,0,0)}((312)) &= (312) + (211),\\
\varphi_{(1,0,0)}((321)) &= (321), & \varphi_{(1,0,0)}((112)) &= (112) + (111),\\
\varphi_{(1,0,0)}((121)) &= (121), & \varphi_{(1,0,0)}((211)) &= (211),\\
\varphi_{(1,0,0)}((122)) &= (122) + (111), & \varphi_{(1,0,0)}((212)) &= (212),\\
\varphi_{(1,0,0)}((221)) &= (221), & \varphi_{(1,0,0)}((111)) &= (111).
\end{aligned}$$

Corollary 20. *For any finite topology $\mathcal{T}$ of degree n:*

$$\mathcal{P}_s(\mathcal{T}) = \bigsqcup_{f\in\mathcal{L}(\mathcal{T})} \{g \mid g \leq f\} = \bigsqcup_{f\in\mathcal{L}(\mathcal{T})} \mathcal{P}_s(\mathcal{T}_f).$$

Proof. This is the first step of the proof of Proposition 19. $\square$

So T-partitions only depend on linear extensions. In the case of special posets, that is to say of T_0 topologies, the previous corollary is proved in [16,6].

Remark. Let $\varphi_{(0,1,0)} = j \circ \varphi_{(1,0,0)} \circ j$. Then:

$$\varphi_{(0,1,0)} \circ L = j \circ \varphi_{(1,0,0)} \circ j \circ L = j \circ \varphi_{(1,0,0)} \circ L \circ \iota = j \circ \Gamma_{(1,0,0)} \circ \iota = \Gamma_{(0,1,0)}.$$

Moreover, for all packed word f:

$$\varphi_{(0,1,0)}(f) = \sum_{g\leq' f} g,$$

where the order relation $\leq'$ on packed words is defined by $g \leq' f$ if:

- For all $i, j \in [n]$, $f(i) \leq f(j)$ implies that $g(i) \leq g(j)$.
- For all $i, j \in [n]$, $f(i) < f(j)$ and $i < j$ implies that $g(i) < g(j)$.
- For all $i, j \in [n]$, $f(i) = f(j)$ implies that $g(i) = g(j)$.

Proposition 21. *Let $q \in \mathbb{K}^3$. There exists a linear endomorphism φ_q of* **WQSym** *such that $\varphi_q \circ L = \Gamma_q$ if, and only if, $q = (1, 0, 0)$ or $q = (0, 1, 0)$.*

Proof. $\Longrightarrow$. If φ_q exists, then:

$$\varphi_q((123)) = \varphi_q \circ L\left(\begin{smallmatrix}\bullet\,3\\ \bullet\,2\\ \bullet\,1\end{smallmatrix}\right) = \Gamma_q\left(\begin{smallmatrix}\bullet\,3\\ \bullet\,2\\ \bullet\,1\end{smallmatrix}\right) = (123) + q_1(112) + q_1(122) + q_1^3(111),$$

$$\varphi_q((213)) = \varphi_q \circ L\left(\begin{smallmatrix}\bullet\,3\\ \bullet\,1\\ \bullet\,2\end{smallmatrix}\right) = \Gamma_q\left(\begin{smallmatrix}\bullet\,3\\ \bullet\,1\\ \bullet\,2\end{smallmatrix}\right) = (213) + q_2(112) + q_1(212) + q_1^2 q_2(111),$$

$$\varphi_q((312)) = \varphi_q \circ L\left(\begin{smallmatrix}\bullet\,1\\ \bullet\,3\\ \bullet\,2\end{smallmatrix}\right) = \Gamma_q\left(\begin{smallmatrix}\bullet\,1\\ \bullet\,3\\ \bullet\,2\end{smallmatrix}\right) = (312) + q_1(211) + q_2(212) + q_1 q_2^2(111).$$

Moreover:

$$\begin{aligned}
\varphi_q \circ L(\bullet_1 \,\mathfrak{l}^{3}_{2}) &= \varphi_q((123) + (213) + (312)) \\
&= (123) + (213) + (312) + (q_1 + q_2)(112) + q_1(211) + q_1(122) \\
&\quad + (q_1 + q_2)(212) + (q_1^3 + q_1^2 q_2 + q_1 q_2^2)(111) \\
&= \Gamma_q(\bullet_1 \,\mathfrak{l}^{3}_{2}) \\
&= (123) + (213) + (312) + (112) + q_1(211) + q_1(122) + (212) + q_1(111).
\end{aligned}$$

Identifying these two expressions, the coefficient of (112) gives $q_1 + q_2 = 1$; the coefficient of (111) gives:

$$\begin{aligned}
q_1 = q_1^3 + q_1^2 q_2 + q_2 q_1^2 &\Longleftrightarrow q_1(q_1^2 + q_1 q_2 + q_2^2 - 1) = 0 \\
&\Longleftrightarrow q_1((q_1 + q_2)^2 - q_1 q_2 - 1) = 0 \\
&\Longleftrightarrow q_1^2 q_2 = 0.
\end{aligned}$$

So $(q_1, q_2) = (1, 0)$ or $(0, 1)$. Moreover:

$$\begin{aligned}
\varphi_q((121)) &= \varphi_q \circ L(\mathfrak{l}^{2}_{1,3}) = \Gamma_q(\mathfrak{l}^{2}_{1,3}) = (121), \\
\varphi_q((212)) &= \varphi_q \circ L(\mathfrak{l}^{1,3}_{2}) = \Gamma_q(\mathfrak{l}^{1,3}_{2}) = (212),
\end{aligned}$$

and:

$$\begin{aligned}
(121) + (212) = \varphi_q((121) + (212)) &= \varphi_q \circ L(\bullet_{1,3} \bullet_2) = \Gamma_q(\bullet_{1,3} \bullet_2) \\
&= (121) + (212) + q_3(111).
\end{aligned}$$

The coefficient of (111) gives $q_3 = 0$. So $q = (1, 0, 0)$ or $(0, 1, 0)$.

$\Longleftarrow$. This comes from Proposition 19. $\square$

4.4. Links with special posets

If $\mathcal{T} \in \mathbf{T}$ is T_0, then all its linear extensions are permutations. Seeing **FQSym** as a Hopf subalgebra of $(\mathbf{WQSym}, \sqcup\!\sqcup, \Delta)$, by restriction we obtain a Hopf algebra morphism $L : \mathbf{H_{SP}} \longrightarrow \mathbf{FQSym}$, which is the morphism L defined in Section 1.2.

If $\mathcal{T} \in \mathbf{T}$ is not T_0, then no generalized T-partition of $\mathcal{T}$ is a permutation, so $\varpi \circ \Gamma_q(\mathcal{T}) = 0$. Hence, we have a commutative diagram of Hopf algebras:

$$\begin{tikzcd}
\mathbf{H_T} \arrow[r, "\Gamma_q"] \arrow[d, "\theta_0"'] & \mathbf{WQSym} \arrow[d, "\varpi"] \\
\mathbf{H_{SP}} \arrow[r, "L"'] & \mathbf{FQSym}
\end{tikzcd}$$

If $\mathcal{T} \in \mathbf{T}$ is T_0, then a T-partition of $\mathcal{T}$ is a P-partition of the poset associated to $\mathcal{T}$, in Stanley's sense [16], and we obtain the commutative diagram:

$$\begin{tikzcd} & \mathbf{FQSym} \arrow[d, "\theta_{(1,0,0)}"] \\ \mathbf{H_{SP}} \arrow[ur, "L"] \arrow[r, "\Gamma"'] & \mathbf{WQSym} \end{tikzcd}$$

4.5. The order on packed words

Let us precise the properties of the order on packed words of Definition 17. We shall use the following notion, which, in some sense, is the generalization of the notion of ascents of permutations.

Definition 22. Let f be a packed word of length n, and let $i \in [n]$. We shall say that $i \in M(f)$ if:

- $f(i) < \max(f)$.
- For all $j \in [n]$, $f(j) = f(i) \Longrightarrow j \leq i$.
- For all $j \in [n]$, $f(j) = f(i) + 1 \Longrightarrow j > i$.

Example. If $f = (412133)$, then $M(f) = \{3\}$.

Remark. If f is a permutation, then $i \in M(f)$ if, and only if, $f^{-1}(f(i) + 1) > i$.

The aim of this section is to prove the following theorem:

Theorem 23.

(1) *Let f, g be two packed words. Then $f \leq g$ if, and only if, $Std(f) = Std(g)$ and $M(f) \subseteq M(g)$.*
(2) *For all $n \geq 1$, there is an isomorphism of posets:*

$$\Phi : \begin{cases} (\{\text{packed words of length } n\}, \leq) \longrightarrow \bigsqcup_{\sigma \in \mathfrak{S}_n} (\{\text{subsets of } M(\sigma)\}, \subseteq) \\ \qquad\qquad\qquad\qquad\qquad f \longrightarrow M(f) \subseteq M(Std(f)). \end{cases}$$

Hence, for all $n \geq 1$, the poset of packed words of length n is a disjoint union of posets, indexed by $\mathfrak{S}_n$, the part indexed by σ being isomorphic to the poset of subsets of $M(\sigma)$, partially ordered by the inclusion.

Lemma 24.

(1) *For any packed word g, $g \leq Std(g)$.*
(2) *Let f, g be packed words. If $f \leq g$, then $Std(f) = Std(g)$.*

Proof. We put $\sigma = Std(f)$, $\tau = Std(g)$ and, for all $p \in [\max(g)]$, $g^{-1}(\{p\}) = C_p$.

1. Let $i, j \in [n]$. We assume that $i \in C_p$ and $j \in C_q$. If $\tau(i) \leq \tau(j)$, by definition of the standardization, $p \leq q$, so $g(i) = p \leq q = g(j)$. If $\tau(i) = \tau(j)$, as τ is a permutation, $i = j$, and $g(i) = g(j)$. If $\tau(i) < \tau(j)$ and $j > i$, then $p \leq q$. As τ is increasing on C_p by definition of the standardization, $p = q$ is impossible. So $p < q$, and $g(i) = p < q = g(j)$. We obtain $g \leq \tau$.

2. As $f \leq g$, f is constant on C_p for all p. We put $f(C_p) = \{c_p\}$. If $i \in C_p$ and $j \in C_q$, with $p < q$, then $g(i) = p \leq q = g(j)$, so $f(i) \leq f(j)$: $c_p \leq c_q$. If $c_p = c_q$ and $p < q$, let $i \in C_p$ and $j \in C_q$. If $j < i$, as $g(i) = p < q = g(j)$ and $f \leq g$, $c_p = f(i) < f(j) = c_q$: contradiction. Hence, if $c_p = c_q$, for all $i \in C_p$, for all $j \in C_q$, $i < j$, which is shortly denoted by $C_p < C_q$.

As f is constant on C_p, σ is increasing on C_p. If $p < q$ and $c_p \neq c_q$, then $c_p < c_q$. By definition of the standardization, for all $i \in C_p$, $j \in C_q$, $\sigma(i) < \sigma(j)$. If $p < q$ and $c_p = c_q$, then for all $i \in C_p$, $j \in C_q$, $i < j$. As f is constant on $C_p \sqcup C_q$, $\sigma(i) < \sigma(j)$. Finally:

- σ is increasing on C_p for all p.
- If $p < q$, $i \in C_p$ and $j \in C_q$, $\sigma(i) < \sigma(j)$.

So $\sigma = Std(g)$. $\square$

Lemma 25. *Let $\sigma \in \mathfrak{S}_n$, $n \geq 1$. The following map is bijective:*

$$\phi_\sigma : \begin{cases} \{f \text{ packed word} \mid Std(f) = \sigma\} \longrightarrow \{I \mid I \subseteq M(\sigma)\} \\ \qquad\qquad\qquad\qquad\qquad\quad f \longrightarrow M(f). \end{cases}$$

Proof. Let f be a packed word such that $Std(f) = \sigma$. We put $f^{-1}(\{p\}) = C_p$ for all $p \in [\max(f)]$. Let $i \in M(f)$. Assume that $i \in C_p$. Then $p < \max(f)$, i is the greatest element of C_p, and, if j is the smallest element of C_{p+1}, $i < j$. By definition of the standardization, $\sigma(j) = \sigma(i) + 1$. As $j > i$, $i \in M(\sigma)$, so $M(f) \subseteq M(\sigma)$, and ϕ_σ is well-defined.

We define a map $\psi_\sigma : \{I \mid I \subseteq M(\sigma)\} \longrightarrow \{f \text{ packed word} \mid Std(f) = \sigma\}$ in the following way. If $I \subseteq M(\sigma)$, we define $f(\sigma^{-1}(i))$ by induction: $f(\sigma^{-1}(1)) = 1$, and, for all $i \in [n-1]$:

- If $\sigma^{-1}(i) \in I$ or if $\sigma^{-1}(i) \notin M(\sigma)$, then $f(\sigma^{-1}(i+1)) = f(\sigma^{-1}(i)) + 1$.
- If $\sigma^{-1}(i) \in M(\sigma) \setminus I$, $f(\sigma^{-1}(i+1)) = f(\sigma^{-1}(i))$.

Clearly, f is a packed word. Let us prove that $Std(f) = \sigma$. For all $p \in [\max(f)]$, we put $f^{-1}(\{p\}) = C_p$. By definition of f, for all p, there exist $i_p \leq j_p$ such that $C_p = \sigma^{-1}(\{i_p, \ldots, j_p\})$, and $\sigma^{-1}(i_p), \ldots, \sigma^{-1}(j_p - 1) \in M(\sigma)$, which implies:

$$\sigma^{-1}(i_p) < \sigma^{-1}(i_p + 1) < \ldots < \sigma^{-1}(j_p - 1) < \sigma^{-1}(j_p).$$

We obtain that σ^{-1} is increasing on $i_p, \ldots, j_p$, so σ is increasing on C_p. Moreover, if $p < q$, $i \in C_p$ and $j \in C_q$, by definition of f, putting $\sigma^{-1}(i) = k$ and $\sigma^{-i}(j) = l$, $k < l$. Consequently, $\sigma(i) = k < l = \sigma(j)$. We obtain that $Std(f) = \sigma$. We can put $\psi_\sigma(I) = f$, and then ψ_σ is a well-defined map.

Let $I \subseteq M(\sigma)$, and $f = \psi_\sigma(I)$. For all $p \in [\max(f)]$, we put $f^{-1}(\{p\}) = C_p$. If $i \in M(f)$, then i is the greatest element of C_p with $p = f(i) < \max(f)$, and if j is the smallest element of C_{p+1}, then $i < j$. As $\sigma = Std(f)$, $\sigma(j) = \sigma(i) + 1$, so $i \in M(\sigma)$, and $M(f) \subseteq M(\sigma)$. By definition of f, $i \in I$ or $i \notin M(\sigma)$, so $i \in I$: $M(f) \subseteq I$. Let $i \in I$. We put $k = \sigma(i)$. Then $f(\sigma^{-1}(k+1)) = f(\sigma^{-1}(k)) + 1$. By definition of f, i is the greatest element of C_p, with $p = f(i)$ and $j = \sigma^{-1}(k+1)$ is the smallest element of C_{p+1}. As $\sigma^{-1}(k) = i \in M(\sigma)$, $\sigma^{-1}(k+1) = j > \sigma^{-1}(k) = i$: $i \in M(f)$. We obtain $M(f) = I$, that is to say $\phi_\sigma \circ \psi_\sigma(I) = I$.

Let f be a packed word such that $Std(f) = \sigma$, $I = M(f)$ and $g = \psi_\sigma(I)$. For all $p \in [\max(f)]$, we put $f^{-1}(\{p\}) = C_p$. Let us prove that $f(\sigma^{-1}(i)) = g(\sigma^{-1}(i))$ for all i by induction. If $i = 1$, as $\sigma = Std(f)$, $f(\sigma^{-1}(1)) = 1 = g(\sigma^{-1}(1))$. Let us assume that $f(\sigma^{-1}(i)) = g(\sigma^{-1}(i))$. We obtain three different cases.

(1) If $\sigma^{-1}(i) \in I$, then $\sigma^{-1}(i)$ is the greatest element of C_p, with $p = f(\sigma^{-1}(i))$, and if j is the smallest element of C_{p+1}, then $i < j$. As $Std(f) = \sigma$, $j = \sigma^{-1}(\sigma(\sigma^{-1}(i))+1) = \sigma^{-1}(i+1)$, and $f(\sigma^{-1}(i+1)) = p+1 = f(\sigma^{-1}(i)) + 1 = g(\sigma^{-1}(i+1))$.
(2) If $\sigma^{-1}(i) \notin M(\sigma)$, then $\sigma^{-1}(i+1) < \sigma^{-1}(i)$. As $\sigma = Std(f)$, necessarily $\sigma^{-1}(i)$ is the greatest element of C_p and $\sigma^{-1}(i+1)$ is the smallest element of C_{p+1}. We obtain $f(\sigma^{-1}(i+1)) = p+1 = f(\sigma^{-1}(i)) + 1 = g(\sigma^{-1}(i+1))$.
(3) If $\sigma^{-1}(i) \in M(\sigma) \setminus I$, then $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. As $\sigma = Std(f)$ and $i \notin I$, $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ are in the same C_p, so $f(\sigma^{-1}(i+1)) = p = f(\sigma^{-1}(i)) = g(\sigma^{-1}(i+1))$.

As a conclusion, $g = f$, so $\psi_\sigma \circ \phi_\sigma(f) = f$. $\square$

Proof of Theorem 23. 1. $\Longrightarrow$. If $f \leq g$, by Lemma 24, $Std(f) = Std(g)$. We denote by σ this permutation. If $I = M(f)$ and $J = M(g)$, then $f = \psi_\sigma(I)$ and $g = \psi_\sigma(J)$. For all $p \in [\max(f)]$, we put $f^{-1}(\{p\}) = C_p$. For all $q \in [\max(g)]$, we put $g^{-1}(\{q\}) = C'_q$.

Let $k \in I$. We put $\sigma(k) = i$. By construction of $\psi_\sigma(I)$, $k = \sigma^{-1}(i)$ is the greatest letter of C_p for $p = f(k)$, and if $l = \sigma^{-1}(i+1)$ is the smallest letter of C_{p+1}, then $k < l$. Consequently, if $k' \in C_p$, $l' \in C_{p+1}$, then $k' \leq k < l \leq l'$. If $g(k') \geq g(l')$, as $f \leq g$, we should have $f(k') > f(l')$: this is a contradiction, as $f(k') = p$ and $f(l') = p+1$. So $g(k') < g(l')$. Moreover, f is constant on C'_q for all q, as $f \leq g$. If $k \in C'_q$, then $C'_q \subseteq C_p$

with $p = f(k)$. Moreover, $l \in C_{p+1}$, so $l \notin C'_q$. As $Std(g) = \sigma$, $l = \sigma^{-1}(i+1) \in C'_{q+1}$, which implies $C'_{q+1} \subseteq C_{p+1}$. So for all $k' \in C_q$, $l' \in C_{q+1}$, $k' < l'$: $k \in M(g) = J$, and $I \subseteq J$.

1. $\Longleftarrow$. We put $I = M(f)$, $J = M(g)$, such that $f = \psi_\sigma(I)$ and $g = \psi_\sigma(J)$, with $\sigma = Std(f) = Std(g)$.

- As $I \subseteq J$, the change of values of f in the definition of $\psi_\sigma(I)$ are also change of values of g in the definition of $\psi_\sigma(J)$; consequently, if $g(i) = g(j)$, then $f(i) = f(j)$.
- If $g(k) \leq g(l)$, we put $\sigma(k) = i$ and $\sigma(l) = j$. By construction of $\psi_\sigma(J)$, $i < j$. By construction of $\psi_\sigma(I)$, $f(k) = f(\sigma^{-1}(i)) =\leq f(\sigma^{-1}(j)) = f(l)$.
- If $g(k) < g(l)$ and $k > l$, we put $\sigma(k) = i$ and $\sigma(l) = j$. Then the interval $\{i, \ldots, j-1\}$ contains an element which does not belong to $M(\sigma)$ (otherwise, it would contain only elements of $M(\sigma)$, and then $k \leq l$). By definition of $\psi_\sigma(I)$, $f(k) < f(l)$.

Finally, $f \leq g$.

2. For all $\sigma \in \mathfrak{S}_n$, ϕ_σ is bijective: this implies that Φ is bijective. By the first point, Φ is an isomorphism of posets. $\Box$

Remark. In particular, if σ is a permutation, $\varphi_{(1,0,0)}(\sigma)$ is the sum of all packed words f such that $Pack(f) = \sigma$. This implies that the restriction of $\varphi_{(1,0,0)}$ to **FQSym** is the map φ defined in Section 1.2.

Corollary 26. *For all $n \geq 1$, for all $\sigma \in \mathfrak{S}_n$:*

$$\sharp\{w \text{ packed word of length } n \mid Std(w) = \sigma\} = 2^{\sharp M(\sigma)}.$$

For all $n \geq 1$:

$$\sharp\{\text{packed words of length } n\} = \sum_{\sigma \in \mathfrak{S}_n} 2^{\sharp M(\sigma)}.$$

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