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Evacuation of labelled graphs☆

Claudia Malvenuto*, Christophe Reutenauer

Département de Mathématiques et Informatique, Université du Québec à Montréal, Case Postale 8888, Succ. A, Montreal, Quebec H3C 3P8, Canada

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Abstract

In this note, we extend Schützenberger's evacuation of Young tableaux (Schützenberger, 1963), and naturally labelled posets (Schützenberger, 1972), to labelled graphs. It is shown that evacuation is an involution, and that in that in the dual evacuation, tracks and trajectories are interchanged.

Let $G=(V,E)$ be a simple undirected graph without loops; V is its set of vertices, and E its set of edges, with $E\subseteq\mathscr{P}_2(V)$. A *labelling* of G is a bijective map $\Phi: V\to\{1,\dots,n\}$, with $n=|V|$.

The (*canonical*) *track* $P(\Phi)$ of Φ is the subset $\{v_1,\dots,v_k\}$ of V with:

(i) $v_1=\Phi^{-1}(1)$;

(ii) for $i\geqslant 2$, $v_i=\Phi^{-1}(\min\{\Phi(v)\,|\,(v,v_{i-1})\in E \text{ and } \Phi(v)>\Phi(v_{i-1})\})$ if such a vector v_i exists;

(iii) if such v_i does not exist, then $k=i-1$.

The *promotion* of Φ is the mapping $\partial: \Phi\to\partial\Phi$ defined by:

$$\begin{aligned} &\partial\Phi(v)=\Phi(v)-1 && \text{if } v\notin P(\Phi),\\ &\partial\Phi(v_i)=\Phi(v_{i+1})-1 && \text{for } i=1,\dots,k-1,\\ &\partial\Phi(v_k)=n. \end{aligned} \tag{1}$$

Example. Consider (G,Φ) as in Fig. 1. The track is the set of circled vertices. The promotion of Φ is shown in Fig. 2.

With Φ as above and $s\in\{1,\dots,n\}$, the *s-promotion* of Φ is the mapping $\partial_s: \Phi\to\partial_s\Phi$ obtained by the previous construction restricted to the vertices labelled in $\{1,\dots,s\}$

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* Corresponding author.

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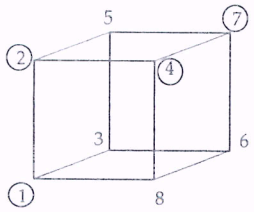

Fig. 1

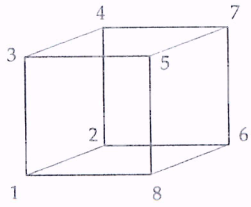

Fig. 2

and by keeping untouched the vertices labelled in $\{s+1, \ldots, n\}$. In other words, let $V' = \{v \in V \mid \Phi(v) \leqslant s\}$, $P' = P(\Phi) \cap V' = \{v_1, \ldots, v_l\}$. Then:

$$\partial_s \Phi(v) = \Phi(v) \quad \text{if } v \notin V',$$

and for v in V'

$$\partial_s \Phi(v) = \Phi(v) - 1 \qquad \text{if } v \notin P',$$

$$\partial_s \Phi(v_i) = \Phi(v_{i+1}) - 1 \qquad \text{for } i = 1, \ldots, l-1,$$

$$\partial_s \Phi(v_l) = s.$$

Observe that $\partial = \partial_n$ if $n = |V|$ and that $\partial_1 = \mathrm{id}|_\Phi$. Observe also that $\partial_n \Phi$ and $\partial_{n-1} \Phi$ differ at most on the vertices v_{k-1}, v_k and the vertex labelled n.

For later use, we call P' the *s-track* of Φ, denoted by $P_s(\Phi)$; it is simply the track obtained by considering only the vertices whose label is $\leqslant s$.

Example (*continued*). $(G, \partial_5 \Phi)$ is shown in Fig. 3.

Define $\Phi_1 = \Phi$, $\Phi_2 = \partial_n \Phi_1$, $\Phi_3 = \partial_{n-1} \partial_n \Phi_1, \ldots, \Phi_n = \partial_2 \ldots \partial_n \Phi_1$. We call *evacuation* the mapping $\Phi \to \Phi_n$.

Theorem 1. *Evacuation is an involution.*

This extends a result of Schützenberger: he introduced evacuation of Young tableaux in [4], and of naturally labelled posets in [5], and showed that this construction is an involution. Note that his definition of promotion is slightly different from ours, but the reader will easily convince himself of the equivalence of both constructions. For example, his definition of ∂ is obtained from ours by adding 1 to the right-hand sides of the two first equalities (1), and by omitting the third one ($\partial\Phi$ is thus defined on $V \setminus v_k$); hence, a promotion is a "slide along the track". Other variants were formulated by Knuth (see [3, pp. 57–59 and Ex. 30, p. 72]), Greene and Edelman [1, Section 5], Haiman [2, Section 4].

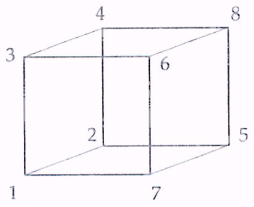

Fig. 3

The proof of the theorem becomes very simple once we use certain operators which were introduced by Haiman in the case of naturally labelled posets [2, proof of Lemma 2.7].

For $1 \leqslant i \leqslant n-1$ let r_i be the operator which exchanges the labels i and $i+1$ if and only if the corresponding vertices are not adjacent, that is:

$$r_i\Phi = \begin{cases} \Phi & \text{if } (\Phi^{-1}(i), \Phi^{-1}(i+1)) \in E, \\ (i, i+1) \circ \Phi & \text{otherwise,} \end{cases}$$

where $(i, i+1)$ is the usual transposition in the symmetric group.

Two easy but fundamental observations are that $r_i^2 = \mathrm{id}$ and that $|i-j| \geqslant 2$ implies $r_i r_j = r_j r_i$.

The promotion can now be expressed simply as the product of the r_i's, as shows the following lemma.

Lemma 1. $\partial_n \Phi = r_{n-1}\partial_{n-1}\Phi = r_{n-1}r_{n-2}\cdots r_1\Phi$.

Proof. We proceed by induction on n, the case $n=1$ being trivial. Let $n>1$, $P(\Phi)=(v_1, \ldots, v_k)$ and $\bar{v} = \Phi^{-1}(n)$. We observed that $\partial_n\Phi$ and $\partial_{n-1}\Phi$ can differ only on v_k, v_{k-1} and $\bar{v}$. Two cases have to be considered:

(i) $\bar{v} \in P(\Phi)$: in this case $\bar{v} = v_k$, since the sequence of integers $(\Phi(v_1), \ldots, \Phi(v_k))$ is increasing. We have:

$$\partial_n\Phi(v_{k-1}) = \Phi(v_k) - 1 = \Phi(\bar{v}) - 1 = n-1 = \partial_{n-1}\Phi(v_{k-1}),$$

$$\partial_n\Phi(v_k) = n = \partial_{n-1}\Phi(v_k).$$

Hence r_{n-1} is the identity on the labelling $\partial_{n-1}\Phi$, since $(v_{k-1}, v_k) \in E$ and v_{k-1}, v_k are labelled respectively, $n-1$, n by $\partial_{n-1}\Phi$. Thus $\partial_n\Phi = \partial_{n-1}\Phi = r_{n-1}\partial_{n-1}\Phi$.

(ii) $\bar{v} \notin P(\Phi)$: following the definition of ∂ we have:

$$\partial_n\Phi(\bar{v}) = n-1, \qquad \partial_{n-1}\Phi(\bar{v}) = n,$$

$$\partial_n\Phi(v_k) = n, \qquad \partial_{n-1}\Phi(v_k) = n-1,$$

$$\partial_n\Phi(v_{k-1}) = \Phi(v_k) - 1 = \partial_{n-1}\Phi(v_{k-1}).$$

As $\bar{v}$ is not in the canonical track, we must have $(\bar{v}, v_k) \notin E$; in this case r_{n-1} operates as the transposition $(n-1, n)$ on the labelling $\partial_{n-1}\Phi$ and again $r_{n-1}\partial_{n-1} = \partial_n$. Recalling that ∂_1 is in fact the identity, the second equality of the lemma follows immediately. □

This Lemma was inspired by the proof of Lemma 2.7 in [2].

Now let $c_1 = \mathrm{id}$ and for $j \geqslant 2$, let $c_j = r_{j-1} r_{j-2} \cdots r_1$. Thus $\partial_j \Phi = c_j \Phi$. We have $c_j = r_{j-1} c_{j-1}$ and the following commutations hold:

$$r_k c_j = c_j r_k \quad \text{if } k \geqslant j+1. \tag{2}$$

Proof of the Theorem 1. If $\Psi = \Phi_n$ denotes the labelling obtained after evacuation of Φ, then by Lemma 1, $\Psi = c_1 c_2 \cdots c_n \Phi$. We want to show that

$$\partial_1 \cdots \partial_n \Psi = \Phi,$$

that is

$$c_1 \cdots c_n = c_n^{-1} \cdots c_1^{-1}. \tag{3}$$

For $n = 1$ the equality is trivial.

For $n > 1$ we have:

$$\begin{aligned} c_1 c_2 \cdots c_n &= c_1 (r_1 c_1)(r_2 c_2) \cdots (r_{n-1} c_{n-1}) \\ &= (r_1 \cdots r_{n-1})(c_1 \cdots c_{n-1}) \quad \text{(by the commutation laws (2))} \\ &= c_n^{-1} c_{n-1}^{-1} \cdots c_1^{-1}. \quad \text{(by induction)} \end{aligned}$$ □

As for the promotion, we can express nicely the track of Φ through the operators $\{r_i\}$.

Lemma 2.

$$\begin{aligned} P(\Phi) &= \{\Phi^{-1}(1), (r_1\Phi)^{-1}(2), \ldots, (r_{n-1} \cdots r_1 \Phi)^{-1}(n)\} \\ &= \{(c_1\Phi)^{-1}(1), (c_2\Phi)^{-1}(2), \ldots, (c_n\Phi)^{-1}(n)\}. \end{aligned}$$

Proof. We prove the lemma by induction on n, with the additional fact that, if $P(\Phi) = \{v_1, \ldots, v_k\}$ as in the definition, then $v_k = (c_n\Phi)^{-1}(n)$. The case $n = 1$ is trivial.

For $n > 1$, let $\bar{v} = \Phi^{-1}(n)$, and let $G' = (V', E')$ be the graph obtained removing from G the vertex $\bar{v}$ and the edges containing it, with the labelling $\Phi' = \Phi|_{V'}$: the induction hypothesis will be applied below to Φ'. Observe that $\bar{v} = \Phi^{-1}(n) = (c_{n-1}\Phi)^{-1}(n)$, because $c_{n-1} = r_{n-2} \cdots r_1$ does not affect the vertex labelled n. We have two cases:

(i) $\bar{v} \notin P(\Phi)$, cf. Fig. 4. It is clear that $P(\Phi) = P(\Phi')$ and by induction

$$P(\Phi') = \{(c_1\Phi)^{-1}(1), \ldots, (c_{n-1}\Phi)^{-1}(n-1)\},$$

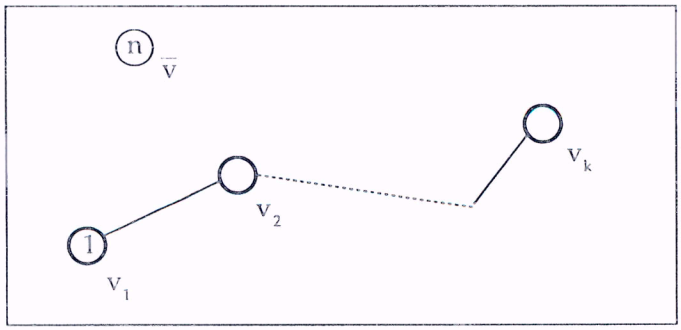

Fig. 4

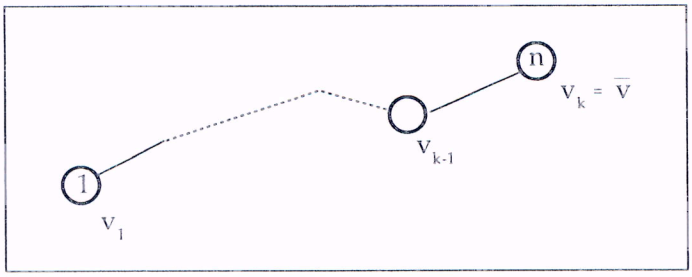

Fig. 5

where $v_k=(c_{n-1}\Phi)^{-1}(n-1)$. Note now that $(v_k, \bar{v})\notin E$, hence r_{n-1} exchanges the labels n and $n-1$ of $\bar{v}$ and v_k, so that $(c_n\Phi)^{-1}(n)=(r_{n-1}c_{n-1}\Phi)^{-1}(n)=v_k$ and

$$P(\Phi)=\{(c_1\Phi)^{-1}(1),\ldots,(c_n\Phi)^{-1}(n)\}.$$

(ii) $\bar{v}\in P(\Phi)$, cf. Fig. 5. We have $P(\Phi)=P(\Phi')\cup\{\bar{v}\}$. Again by induction $(c_{n-1}\Phi')^{-1}(n-1)=(c_{n-1}\Phi)^{-1}(n-1)=v_{k-1}$ and r_{n-1} is the identity on $c_{n-1}\Phi$, since $(v_{k-1},\bar{v})\in E$, so that $(r_{n-1}c_{n-1}\Phi)^{-1}(n)=(c_n\Phi)^{-1}(n)=\bar{v}$ as desired. □

Note that in this construction the track $P(\Phi)$ is a multiset consisting of n elements.

Recall the definition of $\Phi_1,\ldots,\Phi_q$ above. We call *qth track* in the evacuation of Φ the $(n-q+1)$-track of Φ_q, that is, $P_{n-q+1}(\Phi_q)$ with the previous notations. By Lemma 2, 9we have:

$$\begin{aligned}P_{n-q+1}(\Phi_q)&=\{(c_1\Phi_q)^{-1}(1),(c_2\Phi_q)^{-1}(2),\ldots(c_{n-q+1}\Phi_q)^{-1}(n-q+1)\}\\&=\{(c_j\Phi_q)^{-1}(j)\}_{1\le j\le n-q+1}.\end{aligned}$$

We now introduce the concept of *trajectory*. As in [5], the trajectory in Φ of an integer i is the set of vertices who have been labelled by that integer during the

evacuation. But in our version of the promotion, we have to take into account that at each new promotion j becomes $j-1$, for $j \geqslant 2$ (cf. the remark after Theorem 1). This justifies the following definition.

The *trajectory* of q in the evacuation Φ is the multiset:

$$T_q(\Phi) = \{\Phi_1^{-1}(q), \Phi_2^{-1}(q-1), \ldots, \Phi_q^{-1}(1)\} = \{\Phi_{q-j+1}^{-1}(j)\}_{1 \leqslant j \leqslant q}.$$

Let Ψ be the labelling obtained after the evacuation of Φ; hence by Theorem 1, Φ is the labelling obtained after the evacuation of Ψ.

Theorem 2. *For $q = 1, \ldots, n$, the qth track in the evacuation of Φ is equal to the trajectory of $n-q+1$ in the evacuation of Ψ, and vice versa.*

Proof. Let $\Psi_1 = \Psi$, $\Psi_2 = \partial_n \Psi_1$, $\Psi_3 = \partial_{n-1}\partial_n \Psi_1$, $\ldots$, $\Psi_n (= \Phi) = \partial_2 \cdots \partial_n \Psi_1$. We have to show that

$$P_{n-q+1}(\Phi_q) = T_{n-q+1}(\Psi)$$

i.e., for $j = 1, \ldots, n-q+1$, $(c_j \Phi_q)^{-1}(j) = (\Psi_{n-q-j+2})^{-1}(j)$.

We have:

$$\begin{aligned} \Psi_{n-q-j+2} &= c_{q+j} c_{q+j+1} \cdots c_n \Psi = (c_1 \cdots c_{q+j-1})^2 c_{q+j} \cdots c_n \Psi \quad \text{(by (3))} \\ &= c_1 \cdots c_{q+j-1} (c_1 \cdots c_n \Psi) \\ &= c_1 \cdots c_{q+j-1} \Phi \quad \text{(by Theorem 1).} \end{aligned}$$

Observe that for a labelling α and $k \neq j-1, j$, one has

$$(r_k \alpha)^{-1}(j) = \alpha^{-1}(j). \tag{4}$$

Since $c_1 \cdots c_{j-1}$ is a product of r_k with $k < j-1$, we deduce that

$$(c_1 \cdots c_{q+j-1} \Phi)^{-1}(j) = (c_j c_{j+1} \cdots c_{q+j-1} \Phi)^{-1}(j).$$

Furthermore, $c_j \Phi_q = c_j c_{n-q+2} \cdots c_{n-1} c_n \Phi$. Thus, all we have to show is that for $1 \leqslant j \leqslant n-q+1$:

$$(c_j c_{n-q+2} \cdots c_n \Phi)^{-1}(j) = (c_j c_{j+1} \cdots c_{q+j-1} \Phi)^{-1}(j). \tag{5}$$

By Lemma 3 below we have

$$c_j c_{n-q+2} \cdots c_n = d c_j c_{j+1} \cdots c_{q+j-1},$$

where d is a product of r_k's with $k \geqslant j+1$. Hence (4) implies (5). □

Lemma 3. *Let $n \geqslant 1$, $q \geqslant 1$, $1 \leqslant j \leqslant n-q+1$. Then*

$$c_j c_{n-q+2} c_{n-q+3} \cdots c_n = d c_j c_{j+1} \cdots c_{q+j-1},$$

where d is a product of r_k's with $k \geqslant j+1$.

Proof. Induction on $n+2-q-j=$ difference between the two first indices of the left-hand side of the equality. If this number is 1, there is nothing to prove, because in this case $n-q+2=j+1$ and $q+j-1=n$. Suppose now that $n+2-q-j\geqslant 2$. Then, since $c_i=r_{i-1}c_{i-1}$ for $i\geqslant 2$,

$$\begin{aligned} c_j c_{n-q+2} c_{n-q+3}\cdots c_n &= c_j(r_{n-q+1}c_{n-q+1})\cdots(r_{n-1}c_{n-1}) \\ &= c_j r_{n-q+1} r_{n-q+2}\cdots r_{n-1} c_{n-q+1} c_{n-q+2}\cdots c_{n-1} \\ &\qquad \text{(by the commutation laws (2))} \\ &= r_{n-q+1} r_{n-q+2}\cdots r_{n-1} c_j c_{n-q+1} c_{n-q+2}\cdots c_{n-1} \\ &\qquad \text{(by (2) again, because } n-q+1\geqslant j+1) \\ &= r_{n-q+1} r_{n-q+2}\cdots r_{n-1} d' c_j c_{j+1}\cdots c_{q+j-1}, \end{aligned}$$

where d' is a product of r_k's with $j+1\leqslant k$, by induction. Hence the Lemma follows, because $n-q+1\geqslant j+1$. □

Remark. It can be shown that a naturally labelled poset can be associated in a natural way to each labelled graph, so that the results of [5] apply. However, the point of the present note, is to give short proofs of these results, without using the previous paper.

Added in proof. The operators r_i on Young tableaux are particular cases of operators introduced by Bender–Knuth (Enumeration of plane partitions, J. Combin. Theory Ser. A 13 (1972) 40–54) and studied by Gansner (Discrete Math 30 (1980) 121–132).

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