Introduction to Domain Decomposition Methods

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Lecture 1

MOTIVATION

- DD can be used in the framework of any discretization method for PDEs (FEM, FV, FD, SEM) to make their algebraic solution more efficient on parallel computer platforms

- DDM allow the reformulation of a boundary-value problem on a partition of the computational domain into subdomains
  $\Rightarrow$ very convenient framework for the solution of heterogeneous or multiphysics problems, i.e. those that are governed by differential equations of different kinds in different subregions of the computational domain.

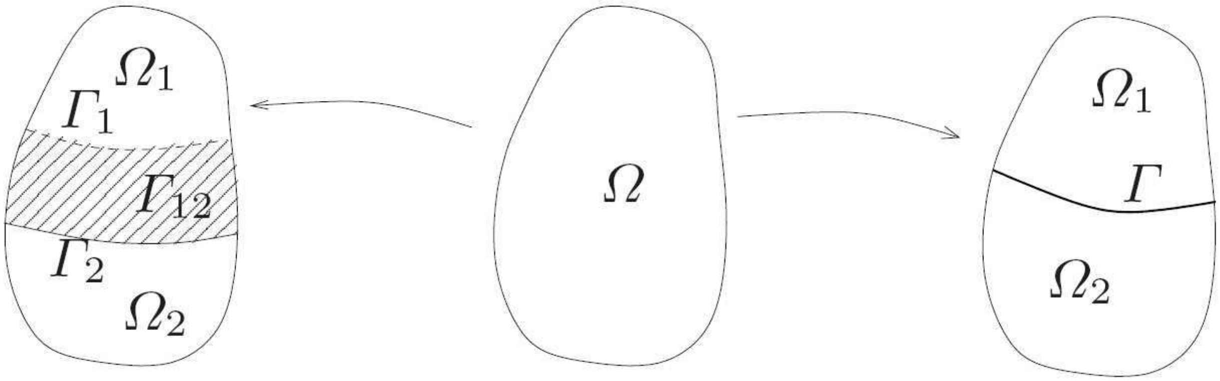
THE IDEA

By DDM methods the computational domain Ω where the bvp is set is subdivided into two or more subdomains on which discretized problems of smaller dimension are to be solved.

Parallel solution algorithms may be used.

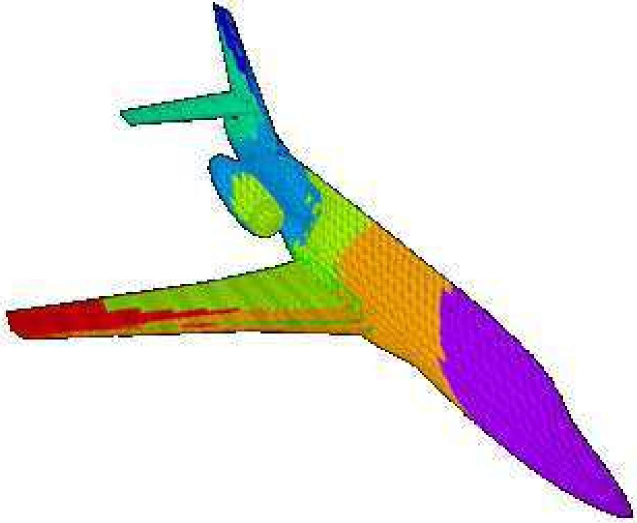
There are two ways of subdividing the computational domain:

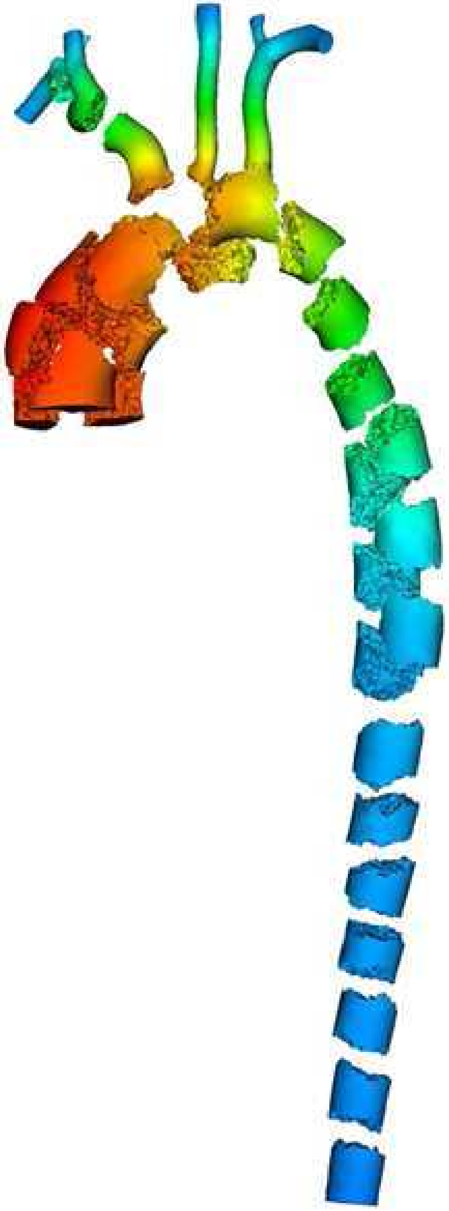
- with **disjoint** subdomains
- with **overlapping** subdomains



Correspondingly, different DD algorithms will be set up.

EXAMPLES OF SUBDIVISIONS IN APPLICATIONS





CLASSICAL ITERATIVE DD METHODS

MODEL PROBLEM

Consider the model problem:

find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

L is a generic second order elliptic operator.

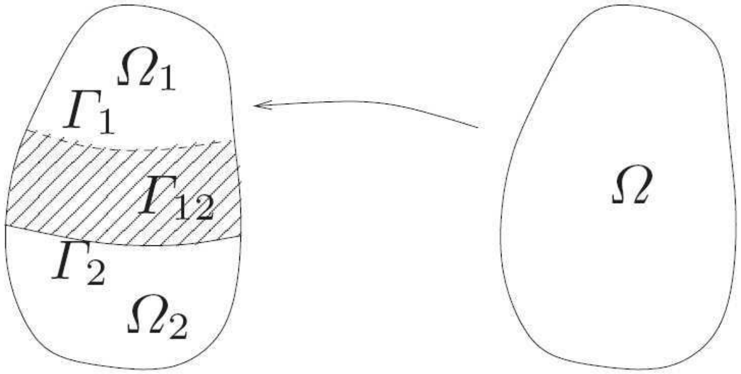
The weak formulation reads:

$$\text{find } u \in V = H_0^1(\Omega) : \quad a(u, v) = (f, v) \quad \forall v \in V,$$

where $a(\cdot, \cdot)$ is the bilinear form associated with L.

SCHWARZ METHODS

Consider a decomposition of Ω with overlap:



The iterative method: given $u_2^{(0)}$ on Γ_1, for $k \geq 1$:

$$\text{solve} \quad \begin{cases} Lu_1^{(k)} = f & \text{in } \Omega_1 \\ u_1^{(k)} = u_2^{(k-1)} & \text{on } \Gamma_1 \\ u_1^{(k)} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma_1 \end{cases}$$

$$\text{solve} \quad \begin{cases} Lu_2^{(k)} = f & \text{in } \Omega_2 \\ u_2^{(k)} = \begin{cases} u_1^{(k)} \\ u_1^{(k-1)} \end{cases} & \text{on } \Gamma_2 \\ u_2^{(k)} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma_2 \; . \end{cases}$$

Choice of the trace on Γ_2:

- if $u_1^{(k)} \Rightarrow$ *multiplicative Schwarz method*

- if $u_1^{(k-1)} \Rightarrow$ *additive Schwarz method*

We have two elliptic bvp with Dirichlet conditions in Ω_1 and Ω_2, and we wish the two sequences $\{u_1^{(k)}\}$ and $\{u_2^{(k)}\}$ to converge to the restrictions of the solution u of the original problem:

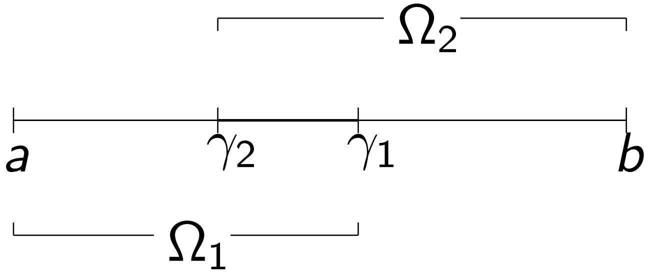
$$\lim_{k\to\infty} u_1^{(k)} = u_{|\Omega_1} \text{ and } \quad \lim_{k\to\infty} u_2^{(k)} = u_{|\Omega_2} \; .$$

The Schwarz method applied to the model problem always converges, with a rate that increases as the measure $|\Gamma_{12}|$ of the overlapping region Γ_{12} increases.

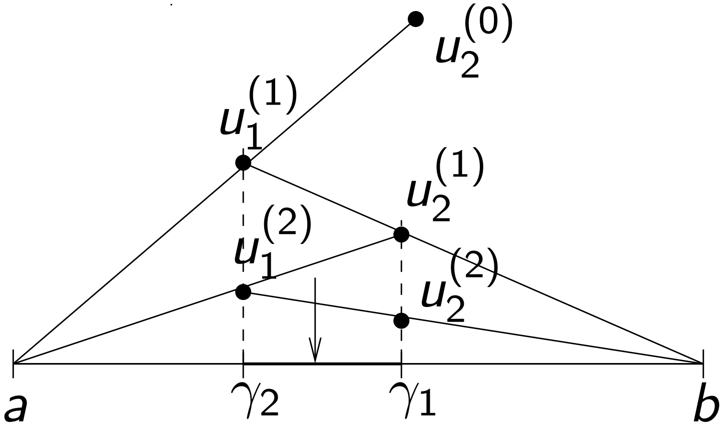
Example. Consider the model problem

$$\begin{cases} -u''(x) = 0 & a < x < b, \\ u(a) = u(b) = 0, & \end{cases}$$

where



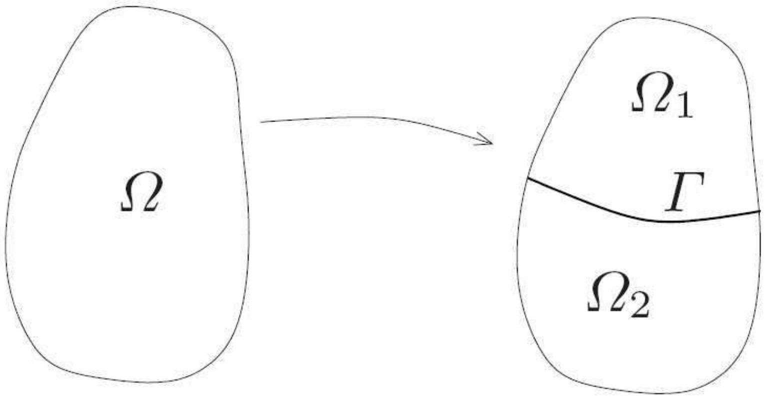
The solution is $u = 0$. We show a few iterations of the method:



Clearly, the method converges with a rate that reduces as the length of the interval (γ_2, γ_1) gets smaller.

NON-OVERLAPPING DECOMPOSITION

We partition now the domain Ω in two disjoint subdomains:



The following equivalence result holds.

Theorem

The solution u of the model problem is such that $u_{|\Omega_i} = u_i$ for $i = 1, 2$, where u_i is the solution to the problem

$$\begin{cases} Lu_i = f & \textit{in } \Omega_i \\ u_i = 0 & \textit{on } \partial\Omega_i \setminus \Gamma \end{cases}$$

with interface conditions

$$u_1 = u_2 \qquad \textit{and} \qquad \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \qquad \textit{on } \Gamma.$$

DIRICHLET-NEUMANN METHOD

Given $u_2^{(0)}$ on Γ, for $k \geq 1$ solve the problems:

$$\begin{cases} Lu_1^{(k)} = f & \text{in } \Omega_1 \\ u_1^{(k)} = u_2^{(k-1)} & \text{on } \Gamma \\ u_1^{(k)} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \ , \end{cases}$$

$$\begin{cases} Lu_2^{(k)} = f & \text{in } \Omega_2 \\ \dfrac{\partial u_2^{(k)}}{\partial n} = \dfrac{\partial u_1^{(k)}}{\partial n} & \text{on } \Gamma \\ u_2^{(k)} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \ . \end{cases}$$

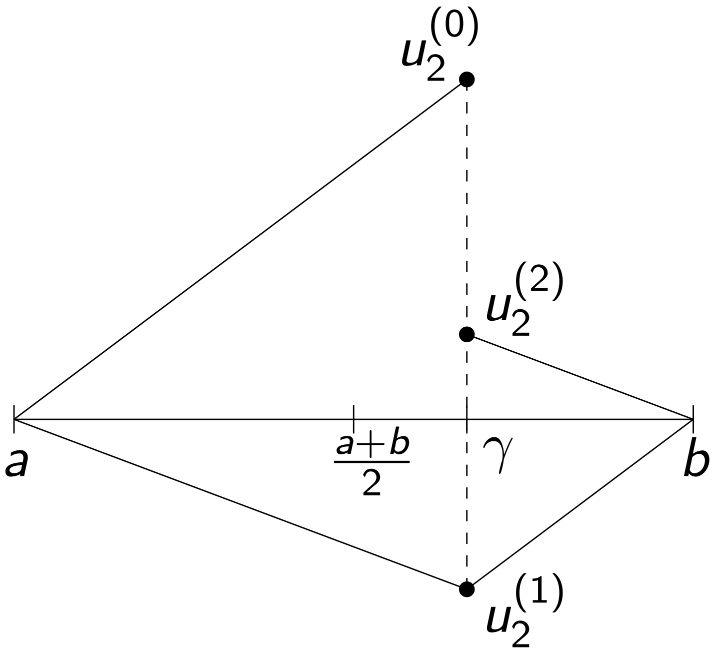
The equivalence theorem guarantees that when the two sequences $\{u_1^{(k)}\}$ and $\{u_2^{(k)}\}$ converge, then their limit will be necessarily the solution to the exact problem.
The DN algorithm is therefore consistent.
Its convergence however is not always guaranteed.

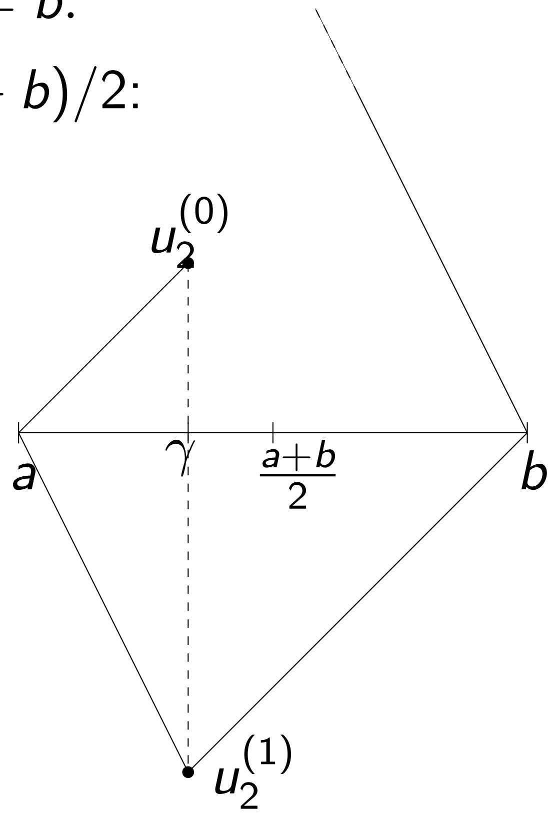
Example. Let $\Omega = (a, b)$, $\gamma \in (a, b)$, $L = -d^2/dx^2$ and $f = 0$. At every $k \geq 1$ the DN algorithm generates the two subproblems:

$$\begin{cases} -(u_1^{(k)})'' = 0 & a < x < \gamma \\ u_1^{(k)} = u_2^{(k-1)} & x = \gamma \\ u_1^{(k)} = 0 & x = a \end{cases}$$

$$\begin{cases} -(u_2^{(k)})'' = 0 & \gamma < x < b \\ (u_2^{(k)})' = (u_1^{(k)})' & x = \gamma \\ u_2^{(k)} = 0 & x = b. \end{cases}$$

The two sequences converge only if $\gamma > (a + b)/2$:





A variant of the DN algorithm can be set up by replacing the Dirichlet condition in the first subdomain by

$$u_1^{(k)} = \theta u_2^{(k-1)} + (1-\theta) u_1^{(k-1)} \quad \text{on } \Gamma \,,$$

that is by introducing a *relaxation* which depends on a positive parameter θ.
In such a way it is always possible to reduce the error between two subsequent iterates.

In the previous example, we can easily verify that, by choosing

$$\theta_{opt} = -\frac{u_1^{(k-1)}}{u_2^{(k-1)} - u_1^{(k-1)}} \,,$$

the algorithm converges to the exact solution in a single iteration.

More in general, there exists a suitable value $\theta_{max} < 1$ such that the DN algorithm converges for any possible choice of the relaxation parameter θ in the interval $(0, \theta_{max})$.

NEUMANN-NEUMANN ALGORITHM

Consider again a partition of Ω into two disjoint subdomains and denote by λ the (unknown) value of the solution u at their interface Γ.

Consider the following iterative algorithm: for any given $\lambda^{(0)}$ on Γ, for $k \geq 0$ and $i = 1, 2$, solve the following problems:

$$\begin{cases} -\triangle u_i^{(k+1)} = f & \text{in } \Omega_i \\ u_i^{(k+1)} = \lambda^{(k)} & \text{on } \Gamma \\ u_i^{(k+1)} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \ , \end{cases}$$

$$\begin{cases} -\triangle \psi_i^{(k+1)} = 0 & \text{in } \Omega_i \\ \dfrac{\partial \psi_i^{(k+1)}}{\partial n} = \dfrac{\partial u_1^{(k+1)}}{\partial n} - \dfrac{\partial u_2^{(k+1)}}{\partial n} & \text{on } \Gamma \\ \psi_i^{(k+1)} = 0 & \text{on } \partial\Omega_i \setminus \Gamma \ , \end{cases}$$

with

$$\lambda^{(k+1)} = \lambda^{(k)} - \theta \left(\sigma_1 \psi_{1|\Gamma}^{(k+1)} - \sigma_2 \psi_{2|\Gamma}^{(k+1)} \right) ,$$

where θ is a positive acceleration parameter, while σ_1 and σ_2 are two positive coefficients.

ROBIN-ROBIN ALGORITHM

For every $k \geq 0$ solve the following problems:

$$\begin{cases} -\triangle u_1^{(k+1)} = f & \text{in } \Omega_1 \\ u_1^{(k+1)} = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ \dfrac{\partial u_1^{(k+1)}}{\partial n} + \gamma_1 u_1^{(k+1)} = \dfrac{\partial u_2^{(k)}}{\partial n} + \gamma_1 u_2^{(k)} & \text{on } \Gamma, \end{cases}$$

then

$$\begin{cases} -\triangle u_2^{(k+1)} = f & \text{in } \Omega_2 \\ u_2^{(k+1)} = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\ \dfrac{\partial u_2^{(k+1)}}{\partial n} + \gamma_2 u_2^{(k+1)} = \dfrac{\partial u_1^{(k+1)}}{\partial n} + \gamma_2 u_1^{(k+1)} & \text{on } \Gamma, \end{cases}$$

where $u_2^{(0)}$ is assigned and γ_1, γ_2 are non-negative acceleration parameters that satisfy $\gamma_1 + \gamma_2 > 0$.

Aiming at parallelization, we could use $u_1^{(k)}$ instead of $u_1^{(k+1)}$.

THE STEKLOV-POINCARÉ INTERFACE EQUATION

MULTI-DOMAIN FORMULATION OF POISSON PROBLEM AND INTERFACE CONDITIONS

We consider now the model problem:

$$\begin{cases} -\triangle u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For a domain partitioned into two disjoint subdomains, we can write the equivalent multidomain formulation ($u_i = u_{|\Omega_i}$, $i = 1, 2$):

$$\begin{cases} -\triangle u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \\ -\triangle u_2 = f & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \\ u_1 = u_2 & \text{on } \Gamma \\ \dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n} & \text{on } \Gamma\,. \end{cases}$$

THE STEKLOV-POINCARÉ OPERATOR

Let λ be the unknown value of the solution u on the interface Γ: $\lambda = u_{|_\Gamma}$.

Should we know a priori the value λ on Γ, we could solve the following two independent boundary-value problems with Dirichlet condition on Γ $(i = 1, 2)$:

$$\begin{cases} -\triangle w_i = f & \text{in } \Omega_i, \\ w_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ w_i = \lambda & \text{on } \Gamma. \end{cases}$$

With the aim of obtaining the value λ on Γ, let us split w_i as follows

$$w_i = w_i^* + u_i^0,$$

where w_i^* and u_i^0 represent the solutions of the following problems $(i = 1, 2)$:

$$\begin{cases} -\triangle w_i^* = f & \text{in } \Omega_i, \\ w_i^* = 0 & \text{on } \partial\Omega_i \cap \partial\Omega, \\ w_i^* = 0 & \text{on } \Gamma, \end{cases}$$

and

$$\begin{cases} -\triangle u_i^0 = 0 & \text{in } \Omega_i, \\ u_i^0 = 0 & \text{on } \partial\Omega_i \cap \partial\Omega, \\ u_i^0 = \lambda & \text{on } \Gamma. \end{cases}$$

- The functions w_i^* depend solely on the source data $f \Rightarrow w_i^* = G_i f$ where G_i is a linear continuous operator

- u_i^0 depend solely on the value λ on $\Gamma \Rightarrow u_i^0 = H_i \lambda$, where H_i is the so-called harmonic extension operator of λ on the domain Ω_i.

We have that

$$u_i = w_i^* + u_i^0, \ (i = 1, 2,) \qquad \Leftrightarrow \qquad \frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n} \qquad \text{on } \Gamma.$$

Using the previously introduced notations:

$$\frac{\partial}{\partial n}(G_1 f + H_1 \lambda) = \frac{\partial}{\partial n}(G_2 f + H_2 \lambda) \ ,$$

and therefore

$$\left(\frac{\partial H_1}{\partial n} - \frac{\partial H_2}{\partial n}\right)\lambda = \left(\frac{\partial G_2}{\partial n} - \frac{\partial G_1}{\partial n}\right) f \qquad \text{on } \Gamma.$$

We have obtained the *Steklov-Poincaré* equation for the unknown λ on the interface Γ:

$$\boxed{S\lambda = \chi \qquad \text{on } \Gamma}$$

- S is the *Steklov-Poincaré* pseudo-differential operator:

$$S\mu = \frac{\partial}{\partial n} H_1\mu - \frac{\partial}{\partial n} H_2\mu = \sum_{i=1}^{2} \frac{\partial}{\partial n_i} H_i\mu$$

- χ is a linear functional which depends on f:

$$\chi = \frac{\partial}{\partial n} G_2 f - \frac{\partial}{\partial n} G_1 f = -\sum_{i=1}^{2} \frac{\partial}{\partial n_i} G_i f.$$

- The operator

$$S_i : \mu \to S_i\mu = \frac{\partial}{\partial n_i} (H_i\mu)\Big|_{\Gamma}, \qquad i = 1, 2,$$

  is called local Steklov-Poincaré operator which operates between the trace space

$$\Lambda = \{\mu \, : \, \exists v \in V \text{ s.t. } \mu = v_{|\Gamma}\} = H_{00}^{1/2}(\Gamma)$$

  and its dual Λ'.

EQUIVALENCE BETWEEN THE DD SCHEMES AND CLASSICAL ITERATIVE METHODS

- The DN method can be reinterpreted as a preconditioned Richardson method for the solution of the Steklov-Poincaré interface equation:

$$\boxed{P_{DN}(\lambda^{(k)} - \lambda^{(k-1)}) = \theta(\chi - S\lambda^{(k-1)})}$$

The preconditioning operator is $P_{DN} = S_2 = \partial(H_2\mu)/\partial n_2$.

- The NN method can also be interpreted as a preconditioned Richardson algorithm

$$\boxed{P_{NN}(\lambda^{(k)} - \lambda^{(k-1)}) = \theta(\chi - S\lambda^{(k-1)})}$$

with $P_{NN} = (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$.

- The Robin-Robin algorithm is equivalent to the following alternating direction (ADI) algorithm:

$$\begin{aligned}(\gamma_1 \mathcal{I}_\Lambda + S_1)\mu_1^{(k)} &= \chi + (\gamma_1 \mathcal{I}_\Lambda + S_2)\mu_2^{(k-1)},\\ (\gamma_2 \mathcal{I}_\Lambda + S_2)\mu_2^{(k)} &= \chi + (\gamma_2 \mathcal{I}_\Lambda + S_1)\mu_1^{(k-1)},\end{aligned}$$

  where $\mathcal{I}_\Lambda : \Lambda \rightarrow \Lambda'$ here denotes the Riesz isomorphism between the Hilbert space Λ and its dual Λ'.

  At convergence (for a convenient choice of γ_1 and γ_2), we have $\mu_1 = \mu_2 = \lambda$.

FINITE ELEMENT APPROXIMATION: MULTIDOMAIN FORMULATION

The Galerkin finite element approximation of the Poisson problem on a triangulation $\mathcal{T}_h$ of Ω reads

$$\text{find } u_h \in V_h : \quad a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h,$$

where

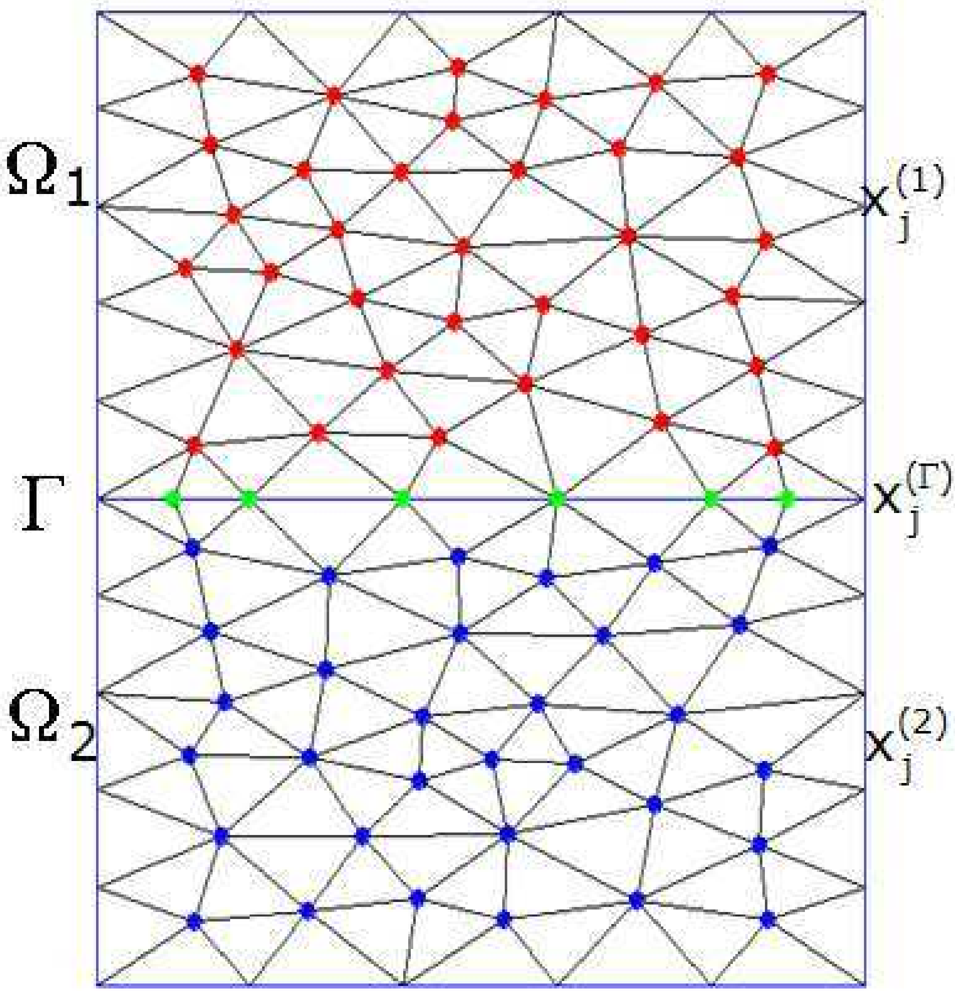
$$V_h = \{v_h \in C^0(\overline{\Omega}) \,:\, v_{h|_K} \in \mathbb{P}_r \quad r \geq 1,\ \forall K \in \mathcal{T}_h,\ v_h = 0 \text{ on } \partial\Omega\}$$

is the space of finite element functions of degree r with basis $\{\varphi_j\}_{j=1}^{N_h}$.

We partition the nodes of the triangulation as follows:

- $\{x_j^{(1)},\ 1 \leq j \leq N_1\}$ nodes in subdomain Ω_1
- $\{x_j^{(2)},\ 1 \leq j \leq N_2\}$ nodes in subdomain Ω_2
- $\{x_j^{(\Gamma)},\ 1 \leq j \leq N_\Gamma\}$ nodes on on the interface Γ

Example:



We split the basis functions accordingly:

- $\varphi_j^{(1)}$ functions associated to the nodes $x_j^{(1)}$
- $\varphi_j^{(2)}$ functions associated to the nodes $x_j^{(2)}$
- $\varphi_j^{(\Gamma)}$ functions associated to the nodes $x_j^{(\Gamma)}$ on the interface

Then, we can write:

$$\begin{aligned} u_h(x) &= \sum_{j=1}^{N_1} u_h(x_j^{(1)})\varphi_j^{(1)}(x) + \sum_{j=1}^{N_2} u_h(x_j^{(2)})\varphi_j^{(2)}(x) \\ &\quad + \sum_{j=1}^{N_\Gamma} u_h(x_j^{(\Gamma)})\varphi_j^{(\Gamma)}(x). \end{aligned}$$

Substituing in the Galerkin formulation, after some algebra, we end up with the linear system

$$A\mathbf{u} = \mathbf{f}, \ \text{ that is } \begin{bmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_\Gamma \end{bmatrix}$$

where $A_{\Gamma\Gamma} = \left(A^{(1)}_{\Gamma\Gamma} + A^{(2)}_{\Gamma\Gamma}\right)$ and $\mathbf{f}_\Gamma = \mathbf{f}_1^\Gamma + \mathbf{f}_2^\Gamma$.

Moreover,

$$\mathbf{u}_1 = \left(u_h(x_j^{(1)})\right), \ \mathbf{u}_2 = \left(u_h(x_j^{(2)})\right) \ \text{ and } \boldsymbol{\lambda} = \left(u_h(x_j^{(\Gamma)})\right).$$

THE SCHUR COMPLEMENT SYSTEM

Since λ represents the unknown value of u on Γ, its finite element correspondent is the vector $\boldsymbol{\lambda}$ of the values of u_h at the interface nodes.

By Gaussian elimination, we can obtain a new reduced system on the sole unknown $\boldsymbol{\lambda}$:

- Matrices A_{11} and A_{22} are invertible since they are associated with two homogeneous Dirichlet boundary-value problems for the Laplace operator:

$$\mathbf{u}_1 = A_{11}^{-1}\left(\mathbf{f}_1 - A_{1\Gamma}\boldsymbol{\lambda}\right) \quad \text{and} \quad \mathbf{u}_2 = A_{22}^{-1}\left(\mathbf{f}_2 - A_{2\Gamma}\boldsymbol{\lambda}\right).$$

- From the third equation we obtain:

$$\left[\left(A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma}\right) + \left(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}\right)\right]\boldsymbol{\lambda} = \\ \mathbf{f}_\Gamma - A_{\Gamma 1}A_{11}^{-1}\mathbf{f}_1 - A_{\Gamma 2}A_{22}^{-1}\mathbf{f}_2.$$

Setting

$$\Sigma = \Sigma_1 + \Sigma_2 \quad \text{with} \qquad \Sigma_i = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma i} A_{ii}^{-1} A_{i\Gamma} \qquad (i = 1, 2)$$

and

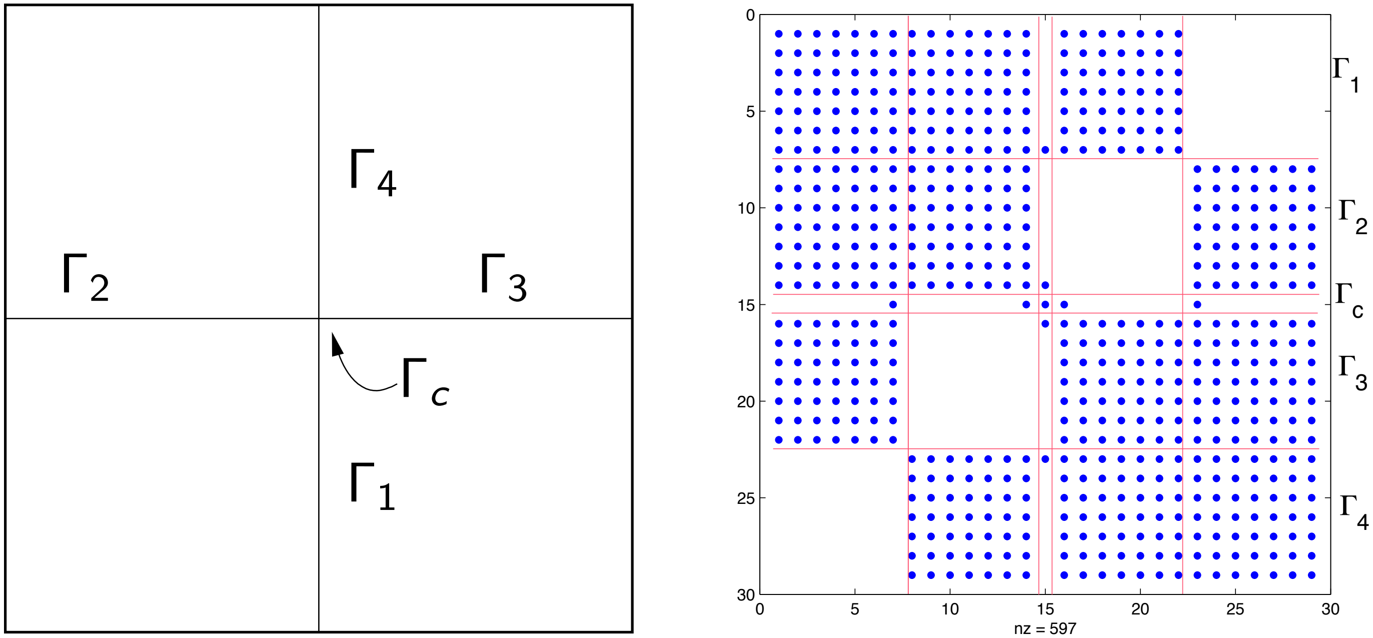
$$\boldsymbol{\chi}_\Gamma = \mathbf{f}_\Gamma - A_{\Gamma 1} A_{11}^{-1} \mathbf{f}_1 - A_{\Gamma 2} A_{22}^{-1} \mathbf{f}_2$$

we obtain the Schur complement system

$$\boxed{\Sigma \boldsymbol{\lambda} = \boldsymbol{\chi}_\Gamma}$$

- Σ and $\boldsymbol{\chi}_\Gamma$ approximate S and χ.
- Σ is the so-called *Schur complement* of A with respect to $\mathbf{u}_1$ and $\mathbf{u}_2$.
- Σ_i are the Schur complements related to the subdomains Ω_i $(i = 1, 2)$.

Example:



On the left, decomposition of $\Omega = (0,1) \times (0,1)$ into four square subdomains. On the right, sparsity pattern of the Schur complement matrix arising from the decomposition depicted on the left.

PROPERTIES OF THE SCHUR COMPLEMENT Σ

The Schur complement Σ inherits some of the properties of A:

- if A is singular, so is Σ;
- if A (respectively, A_{ii}) is symmetric, then Σ (respectively, Σ_i) is symmetric too;
- if A is positive definite, so is Σ.

Moreover, concerning the condition number, we have

- $\boxed{K_2(A) \simeq Ch^{-2}}$
- $\boxed{K_2(\Sigma) \simeq Ch^{-1}}$

EQUIVALENCE BETWEEN THE DN METHOD AND PRECONDITIONER RICHARDSON IN THE DISCRETE CASE

In the discrete case it is easy to prove that the DN method is equivalent to a preconditioned Richardson algorithm.

First, we write the algebraic DN method:

- Dirichlet problem:

$$A_{11}\mathbf{u}_1^{(k)} = \mathbf{f}_1 - A_{1\Gamma}\boldsymbol{\lambda}^{(k-1)},$$

- Neumann problem

$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_2^{(k)} \\ \boldsymbol{\lambda}^{(k-1/2)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_2 \\ \mathbf{f}_\Gamma - A_{\Gamma 1}\mathbf{u}_1^{(k)} - A_{\Gamma\Gamma}^{(1)}\boldsymbol{\lambda}^{(k-1)} \end{bmatrix}.$$

- Relaxation:

$$\boldsymbol{\lambda}^{(k)} = \theta\boldsymbol{\lambda}^{(k-1/2)} + (1-\theta)\boldsymbol{\lambda}^{(k-1)}.$$

By eliminating $\mathbf{u}_2^{(k)}$ we obtain:

$$\left(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right) \boldsymbol{\lambda}^{(k-1/2)} = \mathbf{f}_\Gamma - A_{\Gamma 1}\mathbf{u}_1^{(k)} - A_{\Gamma\Gamma}^{(1)} \boldsymbol{\lambda}^{(k-1)} - A_{\Gamma 2} A_{22}^{-1} \mathbf{f}_2 \,.$$

and using the definition of Σ_2 we have

$$\Sigma_2 \boldsymbol{\lambda}^{(k-1/2)} = \mathbf{f}_\Gamma - A_{\Gamma 1} A_{11}^{-1} \mathbf{f}_1 - A_{\Gamma 2} A_{22}^{-1} \mathbf{f}_2 - \left(A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}\right) \boldsymbol{\lambda}^{(k-1)} \,,$$

that is, owing to the definition of Σ_1 and χ_Γ:

$$\boldsymbol{\lambda}^{(k-1/2)} = \Sigma_2^{-1} \left(\chi_\Gamma - \Sigma_1 \boldsymbol{\lambda}^{(k-1)}\right).$$

Then,

$$\begin{aligned} \boldsymbol{\lambda}^{(k)} &= \theta \Sigma_2^{-1} \left(\chi_\Gamma - \Sigma_1 \boldsymbol{\lambda}^{(k-1)}\right) + (1-\theta)\boldsymbol{\lambda}^{(k-1)} \\ &= \theta \Sigma_2^{-1} \left(\chi_\Gamma - \Sigma \boldsymbol{\lambda}^{(k-1)} + \Sigma_2 \boldsymbol{\lambda}^{(k-1)}\right) + (1-\theta)\boldsymbol{\lambda}^{(k-1)} \end{aligned}$$

whence

$$\boxed{\Sigma_2(\boldsymbol{\lambda}^{(k)} - \boldsymbol{\lambda}^{(k-1)}) = \theta(\chi_\Gamma - \Sigma \boldsymbol{\lambda}^{(k-1)})}$$

Similarly, we can prove that the other iterative methods that we have illustrated are equivalent to preconditioned Richiardson methods for the Schur complement system with preconditioners:

- for the DN algorithm: $P_h = \Sigma_2$
- for the ND algorithm: $P_h = \Sigma_1$
- for the NN algorithm: $P_h = (\sigma_1 \Sigma_1^{-1} + \sigma_2 \Sigma_2^{-1})^{-1}$
- for the RR algorithm: $P_h = (\gamma_1 + \gamma_2)^{-1}(\gamma_1 I + \Sigma_1)(\gamma_2 I + \Sigma_2)$

All these preconditioners are optimal.

This is not true in the case of more subdomains, where the condition number of the Schur complement matrix Σ is

$$\boxed{\kappa_2(\Sigma) \leq C \frac{H}{hH_{min}^2}}$$

where H and H_{min} are the maximal and minimal diameters of the subdomains, respectively.

$\kappa(\Sigma)$	$H = 1/2$	$H = 1/4$	$H = 1/8$
h=1/8	9.77	14.83	25.27
h=1/16	21.49	35.25	58.60
h=1/32	44.09	75.10	137.73
h=1/64	91.98	155.19	290.43

Condition number of the Schur complement matrix.

NON-OVERLAPPING METHODS IN THE CASE OF MORE THAN TWO SUBDOMAINS

SCALABILITY

Definition *A preconditioner P_h of Σ is said to be scalable if the condition number of the preconditioned matrix $P_h^{-1}\Sigma$ is independent of the number of subdomains.*

Iterative methods using scalable preconditioners allow henceforth to achieve convergence rates independent of the subdomain number.

RESTRICTION AND EXTENSION OPERATORS

We introduce a *restriction operator* R_i that, to any vector $\mathbf{v}_h$ of nodal values on the global domain Ω, associates its restriction to the subdomain Ω_i:

$$R_i : \mathbf{v}_{h|_\Omega} \longrightarrow \mathbf{v}^i_{h|_{\Omega_i \cup \Gamma_i}} .$$

Moreover R_i^T

$$R_i^T : \mathbf{v}^i_{h|_{\Omega_i \cup \Gamma_i}} \longrightarrow \mathbf{v}_{h|_\Omega}$$

is the *prolongation* (or *extension-by-zero*) *operator*.

In algebraic form R_i can be represented by a matrix that coincides with the identity matrix in correspondence with the subdomain Ω_i:

$$R_i = \left[\begin{array}{ccc|c|ccc} 0 & \dots & 0 & \underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{\Omega_i} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & & 0 & \dots & 0 \end{array} \right] .$$

Similarly we can define the restriction and prolongation operators R_{Γ_i} and $R_{\Gamma_i}^T$ that act on the vector of the interface nodal values.

A possible preconditioner for Σ is

$$P_h = \sum_{i=1}^{M} R_{\Gamma_i}^T \Sigma_i R_{\Gamma_i}$$

More in general, we combine contributions of local subdmain preconditioners with that of a global contribution referring to a coarse grid whose elements are the subdomains themselves:

$$(P_h)^{-1} = \sum_{i=1}^{M} R_{\Gamma_i}^T P_{i,h}^{-1} R_{\Gamma_i} + R_{\Gamma}^T P_H^{-1} R_{\Gamma}$$

NEUMANN-NEUMANN PRECONDITIONER

The Neumann-Neumann preconditioner for more subdomains reads:

$$\boxed{(P_h^{NN})^{-1} = \sum_{i=1}^{M} R_{\Gamma_i}^T D_i \, \Sigma_i^* \, D_i R_{\Gamma_i}}$$

where Σ_i^* is either Σ_i^{-1} or an approximation of Σ_i^{-1}.

D_i is a diagonal matrix of positive weights

$$D_i = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

d_j is the number of subdomains that share the j-th node.

We have the following estimate:

$$\boxed{\kappa\left((P_h^{NN})^{-1}\Sigma\right) \leq CH^{-2}\left(1 + \log\frac{H}{h}\right)^2}$$

Example

$\kappa((P_h^{NN})^{-1}\Sigma)$	$H = 1/2$	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/16$	2.55	15.20	47.60	-
$h = 1/32$	3.45	20.67	76.46	194.65
$h = 1/64$	4.53	26.25	105.38	316.54
$h = 1/128$	5.79	31.95	134.02	438.02

Condition number of $(P_h^{NN})^{-1}\Sigma$.

BALANCED NEUMANN-NEUMANN PRECONDITIONER

- The Neumann-Neumann preconditioner of the Schur complement system is not scalable.

- A substantial improvement can be achieved by adding a coarse grid correction:

$$\boxed{(P_h^{BNN})^{-1} = \Sigma_H^{-1} + (I - \Sigma_H^{-1}\Sigma)(P_h^{NN})^{-1}(I - \Sigma\Sigma_H^{-1})}$$

  where $\Sigma_H^{-1} = R_\Gamma^T A_H^{-1} R_\Gamma$.

  This is called balanced Neumann-Neumann preconditioner.

- We can prove that

$$\boxed{\kappa_2\left((P_h^{BNN})^{-1}\Sigma\right) \leq C\left(1 + \log\frac{H}{h}\right)^2}$$

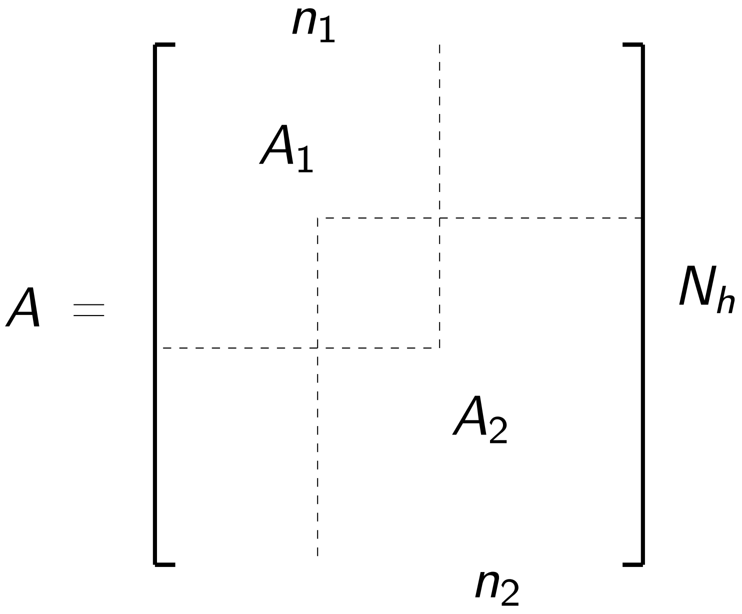
Example:

$\kappa((P_h^{BNN})^{-1}\Sigma)$	$H = 1/2$	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/16$	1.67	1.48	1.27	-
$h = 1/32$	2.17	2.03	1.47	1.29
$h = 1/64$	2.78	2.76	2.08	1.55
$h = 1/128$	3.51	3.67	2.81	2.07

ALGEBRAIC FORM OF SCHWARZ ITERATIVE METHODS

ALGEBRAIC FORM OF SCHWARZ METHODS FOR FE DISCRETIZATION

- Let $A\mathbf{u} = \mathbf{f}$ be the system associated to the finite element approximation of the Poisson problem.
- We still assume that Ω is decomposed in two overlapping subdomains Ω_1 and Ω_2.
- We denote by N_h the total number of interior nodes of Ω, and by n_1 and n_2 the interior nodes of Ω_1 and Ω_2: $N_h \leq n_1 + n_2$
- The stiffness matrix A contains two submatrices, say A_1 and A_2, which correspond to the local stiffness matrices associated to the Dirichlet problems in Ω_1 and Ω_2:



- We have:

$$A_i = R_i A R_i^T \in \mathbb{R}^{n_i \times n_i}$$

with

$$R_1^T = \left[\begin{array}{ccc} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ & & \\ & \mathbf{0} & \\ & & \end{array}\right] \in \mathbb{R}^{N_h \times n_1} \quad \text{and } R_2^T = \left[\begin{array}{ccc} & & \\ & \mathbf{0} & \\ & & \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{array}\right] \in \mathbb{R}^{N_h \times n_2} \; .$$

THE MULTIPLICATIVE SCHWARZ METHOD

Using these definitions, one iteration of the multiplicative Schwarz method for the system $A\mathbf{u} = \mathbf{f}$ can be represented as:

$$\mathbf{u}^{k+1/2} = \mathbf{u}^k + R_1^T A_1^{-1} R_1(\mathbf{f} - A\mathbf{u}^k)$$
$$\mathbf{u}^{k+1} = \mathbf{u}^{k+1/2} + R_2^T A_2^{-1} R_2(\mathbf{f} - A\mathbf{u}^{k+1/2}).$$

Setting

$$P_i = R_i^T A_i^{-1} R_i A$$

this corresponds to

$$\mathbf{u}^{k+1/2} = (I - P_1)\mathbf{u}^k + P_1\mathbf{u}$$

$$\mathbf{u}^{k+1} = (I - P_2)\mathbf{u}^{k+1/2} + P_2\mathbf{u} = (I - P_2)(I - P_1)\mathbf{u}^k + (P_1 + P_2 - P_2P_1)\mathbf{u}.$$

If we define, for $i = 1, 2$

$$Q_i := R_i^T A_i^{-1} R_i = P_i A^{-1}$$

then we obtain

$$\begin{aligned} \mathbf{u}^{k+1} &= \mathbf{u}^k + Q_1(\mathbf{f} - A\mathbf{u}^k) + Q_2[\mathbf{f} - A(\mathbf{u}^k + Q_1(\mathbf{f} - A\mathbf{u}^k))] \\ &= \mathbf{u}^k + (Q_1 + Q_2 - Q_2Q_1)(\mathbf{f} - A\mathbf{u}^k) \end{aligned}$$

THE ADDITIVE SCHWARZ METHOD

One iteration of the additive Schwarz method becomes:

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)(\mathbf{f} - A\mathbf{u}^{(k)})$$

and therefore

$$\mathbf{u}^{(k+1)} = (I - P_1 - P_2)\mathbf{u}^{(k)} + (P_1 + P_2)\mathbf{u}\,.$$

Using again

$$Q_i := R_i^T A_i^{-1} R_i = P_i A^{-1}$$

we obtain

$$\boxed{\mathbf{u}^{k+1} = \mathbf{u}^k + (Q_1 + Q_2)(\mathbf{f} - A\mathbf{u}^k).}$$

In the case of a decomposition of Ω into $M \geq 2$ overlapping subdomains $\{\Omega_i\}$ we have:

$$\boxed{\mathbf{u}^{k+1} = \mathbf{u}^k + \left(\sum_{i=1}^{M} Q_i\right)(\mathbf{f} - A\mathbf{u}^k)}$$

THE SCHWARZ METHOD AS A PRECONDITIONER

- By defining

  $$\boxed{P_{as} = \left(\sum_{i=1}^{M} Q_i\right)^{-1}}$$

  it follows that one iteration of the *additive Schwarz method* corresponds to a preconditioned Richardson iteration for system $A\mathbf{u} = \mathbf{f}$ with preconditioner P_{as} (*additive Schwarz preconditioner*).

- P_{as} is not optimal as the condition number blows up if the size of the subdomains reduces:

  $$\boxed{\kappa_2(P_{as}^{-1}A) \leq C\frac{1}{\delta H}}$$

  where C is a constant independent of h, H, δ.

Example

$\kappa(P_{as}^{-1}A)$	$H = 1/2$	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/16$	15.95	27.09	52.08	-
$h = 1/32$	31.69	54.52	104.85	207.67
$h = 1/64$	63.98	109.22	210.07	416.09
$h = 1/128$	127.99	218.48	420.04	832.57

This is due to the fact that the exchange of information occurs only between neighboring subdomains, since only local solves are involved by the application of $(P_{as})^{-1}$.

$\longrightarrow$ We have to introduce a "coarse" global problem over the whole domain to guarantee a mechanism of global communication among all subdomains.

TWO-LEVEL SCHWARZ PRECONDITIONERS

- As for the Neumann-Neumann method, we can introduce a coarse-grid mechanism that allows for a sudden information diffusion on the whole domain Ω:
    - consider the subdomains as macro-elements of a new coarse grid $\mathcal{T}_H$
    - build a corresponding stiffness matrix A_H
    - The matrix

      $$Q_0 = R_H^T A_H^{-1} R_H$$

      represents the coarse level correction for the two-level preconditioner, with R_H the restriction operator from the fine to the coarse grid.
- The two-level preconditioner P_{cas} is defined as:

$$\boxed{P_{cas}^{-1} = \sum_{i=0}^{M} Q_i}$$

We can prove that there exists a constant $C > 0$, independent of both h and H such that

$$\boxed{\kappa_2(P_{cas}^{-1}A) \leq C\frac{H}{\delta}}$$

If δ is a fraction of H, the preconditioner P_{cas} is scalable.

- Iterations on the original finite element system using P_{cas} converges with a rate independent of h and H (and therefore of the number of subdomains)
- Thanks to the additive structure of the preconditioner, the preconditioning step is fully parallel as it involves the solution of independent systems, one per each local matrix A_i.

Example

$\kappa(P_{cas}^{-1}A)$	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/32$	7.03	4.94	-
$h = 1/64$	12.73	7.59	4.98
$h = 1/128$	23.62	13.17	7.66
$h = 1/256$	45.33	24.34	13.28

If $H/\delta = constant$, this two-level preconditioner is either optimal and scalable.

Practical indications:

- for decompositions with a small number of subdomains, the single level Schwarz preconditioner P_{as} is very efficient;
- when the number M of subdomains gets large, using two-level preconditioners becomes crucial.

  In those cases in which the generation of a coarse grid is difficult, other algebraic techniques, like aggregation, can be adopted:

$$\boxed{\kappa_2(P_{aggre}^{-1}A) \leq C\left(1+\frac{H}{\delta}\right).}$$

$P_{aggre}^{-1}A$	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/16$	13.37	8.87	-
$h = 1/32$	26.93	17.71	9.82
$h = 1/64$	54.33	35.21	19.70
$h = 1/128$	109.39	70.22	39.07

REFERENCES

- B.F. Smith, P.E. Bjørstad, W.D. Gropp (1996): *Domain Decomposition*. Cambridge University Press, Cambridge.
- A. Quarteroni and A. Valli (1999): *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, Oxford.
- A. Toselli and O.B. Widlund (2005): *Domain Decomposition Methods – Algorithms and Theory*. Springer-Verlag, Berlin and Heidelberg.