A singular perturbation result with a fractional norm

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Abstract. Let I be an open bounded interval of $\mathbb{R}$ and W a non-negative continuous function vanishing only at $\alpha, \beta \in \mathbb{R}$. We investigate the asymptotic behaviour in terms of Γ-convergence of the following functional

$$G_\varepsilon(u) := \varepsilon^{p-2} \iint_{I \times I} \left| \frac{u(x) - u(y)}{x - y} \right|^p dxdy + \frac{1}{\varepsilon} \int_I W(u) dx \quad (p > 2),$$

as $\varepsilon \to 0$.

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1. Introduction

The classical variational model for phase transition is related to the so called Cahn-Hilliard functional

$$F_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 dx + \frac{1}{\varepsilon} \int_\Omega W(u)\, dx, \tag{1.1}$$

where W is a two well potential vanishing in two point, α and β.

The study of the Γ-limit of this functional, due to Modica and Mortola [15] (see also [14]), provided a connection between the singular perturbation of the two well potential and the (classical) surface tension model. They indeed proved that F_ε Γ-converges to the functional defined in $BV(\Omega, \{\alpha, \beta\})$ given by

$$c_W \operatorname{Per}(\{u = \alpha\});$$

i.e., its value is proportional to the measure of the surface which separates the two phases.

After this result it has been proved that many other kinds of singular perturbation give the same type of limit. For the case of local singular perturbation see for instance [6].

In the case of non local singular perturbation the first result is due to Alberti, Bouchitté and Seppecher. In [4] they consider the following 1-dimensional functional

$$\varepsilon \iint_{I\times I} \left| \frac{u(x)-u(y)}{x-y} \right|^2 dxdy + \lambda_\varepsilon \int_I W(u)\,dx, \tag{1.2}$$

where $\varepsilon \log \lambda_\varepsilon \to k \in (0,+\infty)$. Again the limit is defined in $BV(I,\{\alpha,\beta\})$ and is given by

$$F(u) = 2k(\beta-\alpha)^2 \mathcal{H}^0(Su), \quad u \in BV(I,\{\alpha,\beta\}),$$

where the jump set Su is the complement of the set of Lebesgue points of u and $\mathcal{H}^0$ denotes the measure that counts points.

Other kinds of similar non local phase transition problems in the case of both singular and regular kernels can be found in [2], [3], [5], [12] and [13].

The main difference between the non local energy with singular kernel (1.2) and the classical Modica-Mortola functional (1.1) is the optimal profile problem that approximately describes the shape of the optimal transitions. In fact, the asymptotical behaviour of (1.1) is characterized by the equipartition of the energy between the two terms in the functional and by a scaling property which provides an optimal profile problem that determines the constant c_W in the limit. Instead, the logarithmic natural scaling for functional (1.2) produces no equipartition of the energy, the limit comes only from the non local part of the energy, it does not depend on W, and any profile is optimal as far as the transition occurs on a layer of order ε.

In this paper, we study the following non local singularly perturbed energy

$$G_\varepsilon(u) := \varepsilon^{p-2} \iint_{I\times I} \left| \frac{u(x)-u(y)}{x-y} \right|^p dxdy + \frac{1}{\varepsilon} \int_I W(u)\,dx,$$

where W is the usual two well potential, with wells at α and β, and $p > 2$; ε^{p-2} being the natural scaling.

In contrast with what happens in the case of energy (1.2) (with $p = 2$) here the functional satisfies a useful scaling property and hence the limit is characterized by an optimal profile problem; i.e., G_ε Γ-converges to $\gamma_p \mathcal{H}^0(S_u)$, where γ_p is given by

$$\gamma_p := \inf \left\{ \iint_{\mathbb{R}\times\mathbb{R}} \left| \frac{v(x)-v(y)}{x-y} \right|^p dxdy + \int_{\mathbb{R}} W(v)\,dx : v \in W^{1-\frac{1}{p},p}_{\mathrm{loc}}(\mathbb{R}), \right.$$
$$\left. \lim_{x\to-\infty} v(x) = \alpha, \ \lim_{x\to+\infty} v(x) = \beta \right\}. \tag{1.3}$$

In this respect the case $p = 2$ represents the critical case in the context of this type of non local singular perturbations.

A similar dichotomy occurs in the case of Ginzburg-Landau problems (see for instance Alberti, Baldo and Orlandi [1] for the case $p = 2$ and Desenzani and Fragalà [10] for the case $p > 2$).

2. The Γ-Convergence Result

Let $p > 2$ be a real number and W a non-negative continuous function vanishing only at $\alpha, \beta \in \mathbb{R}$ $(0 < \alpha < \beta)$, with growth at least linear at infinity. By I we denote an open bounded interval or $\mathbb{R}$.

For every $\varepsilon > 0$ we consider the functional G_ε defined in the fractional Sobolev space $W^{1-\frac{1}{p},p}(I)$,

$$G_\varepsilon(u) := \varepsilon^{p-2} \iint_{I\times I} \left| \frac{u(x)-u(y)}{x-y} \right|^p dxdy + \frac{1}{\varepsilon} \int_I W(u)\, dx. \tag{2.1}$$

Notice that the first term of G_ε is the p-power of the semi-norm in $W^{1-\frac{1}{p},p}(I)$.

The asymptotic behaviour in term of Γ-convergence of G_ε is described by the functional

$$G(u) := \gamma_p \mathcal{H}^0(Su) \quad (\text{if } u \in BV(I, \{\alpha, \beta\}), \tag{2.2}$$

where γ_p is given by the optimal profile problem (1.3). (For details about Γ-convergence, introduced by De Giorgi and Franzoni in [9], see for instance [8] and [7]).

The Γ-convergence result is precisely stated in the following theorem.

Theorem 2.1. *Let $G_\varepsilon : W^{1-\frac{1}{p},p}(I) \to \mathbb{R}$ and $G : BV(I, \{\alpha, \beta\}) \to \mathbb{R}$ defined by (2.1) and (2.2).*

Then

(i) [Compactness] *Let $(u_\varepsilon) \subset W^{1-\frac{1}{p},p}(I)$ be a sequence such that $G_\varepsilon(u_\varepsilon)$ is bounded. Then (u_ε) is pre-compact in $L^1(I)$ and every cluster point belongs to $BV(I, \{\alpha, \beta\})$.*

(ii) [Lower Bound Inequality] *For every $u \in BV(I, \{\alpha, \beta\})$ and every sequence $(u_\varepsilon) \subset W^{1-\frac{1}{p},p}(I)$ such that $u_\varepsilon \to u$ in $L^1(I)$,*

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) \geq G(u).$$

(iii) [Upper Bound Inequality] *For every $u \in BV(I, \{\alpha, \beta\})$ there exists a sequence $(u_\varepsilon) \subset W^{1-\frac{1}{p},p}(I)$ such that $u_\varepsilon \to u$ in $L^1(I)$ and*

$$\limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) \leq G(u).$$

3. The Optimal Profile Problem

In this section we will study the main features of our functionals, namely the scaling property and the optimal profile problem.

It is useful to introduce the *localization* of the functional G_ε. For every open set $J \subseteq I$ and every function $u \in W^{1-\frac{1}{p},p}(J)$ we will denote

$$G_\varepsilon(u,J) := \varepsilon^{p-2} \iint_{J\times J} \left| \frac{u(x)-u(y)}{x-y} \right|^p dxdy + \frac{1}{\varepsilon} \int_J W(u)\, dx.$$

Clearly, $G_\varepsilon(u) = G_\varepsilon(u,I)$, for every $u \in W^{1-\frac{1}{p},p}(I)$.

Given $J \subseteq I$ and $u \in W^{1-\frac{1}{p},p}(J)$ we set $u^{(\varepsilon)}(x) := u(\varepsilon x)$ and $J/\varepsilon := \{x : \varepsilon x \in J\}$. By scaling it is immediately seen that

$$G_\varepsilon(u,J) = G_1(u^{(\varepsilon)}, J/\varepsilon). \tag{3.1}$$

In view of this scaling property it is now natural to consider the following *optimal profile* problem

$$\gamma_p := \inf \left\{ G_1(v,\mathbb{R}) : v \in W^{1-\frac{1}{p},p}_{\mathrm{loc}}(\mathbb{R}), \lim_{x\to-\infty} v(x) = \alpha, \lim_{x\to+\infty} v(x) = \beta \right\}. \tag{3.2}$$

The constant γ_p represents the minimal cost in the term of the non-scaled energy G_1 for a transition from α to β on the whole real line. By (3.1) γ_p will also give the cost of one jump from α to β.

Using a monotone rearrangement argument, we will prove that this minimum problem is not trivial and is achieved.

For every $u \in W^{1-\frac{1}{p},p}(J)$, with $J = (a,b)$, the non-decreasing rearrangement u^* of u in J, defined by

$$u^*(a+x) := \sup \left\{\lambda : |\{t \in (a,b) : u(t) < \lambda\}| \leq x \right\}, \quad \forall x \in (0, b-a), \tag{3.3}$$

satisfies

$$\iint_{J\times J} \left| \frac{u^*(x)-u^*(y)}{x-y} \right|^p dxdy \leq \iint_{J\times J} \left| \frac{u(x)-u(y)}{x-y} \right|^p dxdy \tag{3.4}$$

(see for instance [11], Theorem I.1).

Note that, since $\displaystyle\int_J W(u^*)dx = \int_J W(u)dx$, from (3.4) we get

$$G_\varepsilon(u^*,J) \leq G_\varepsilon(u,J).$$

This rearrangement result will be also used in the sequel to prove the compactness and the lower bound.

We are now in a position to prove the following proposition.

Proposition 3.1. *The constant γ_p is strictly positive.*

Proof. Fix $\delta > 0$ and fix $v \in W^{1-\frac{1}{p},p}_{\mathrm{loc}}(\mathbb{R})$ such that $\lim_{x\to-\infty} v(x) = \alpha$, $\lim_{x\to+\infty} v(x) = \beta$ and $G_1(v,\mathbb{R}) < +\infty$. Let us define

$$I_\alpha := \{x \in \mathbb{R} : v(x) \leq \alpha + \delta\} \quad \text{and} \quad I_\beta := \{x \in \mathbb{R} : v(x) \geq \beta - \delta\},$$

Denote by $J_\delta := \mathbb{R}\setminus(I_\alpha \cup I_\beta)$; we notice that, by the asymptotic behaviour of v, I_α, I_β and J_δ are not empty and J_δ is bounded, for every fixed $\delta \in (0,(\beta-\alpha)/2)$.

Now, let us consider the truncated function

$$v_\delta(x) := (v(x) \vee (\alpha+\delta)) \wedge (\beta-\delta) \quad \text{for every} \quad x \in \mathbb{R}.$$

It is easy to see that the non local energy decreases under truncation and then it follows that

$$\begin{aligned} G_1(v,\mathbb{R}) \;&\geq\; \iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{v_\delta(x)-v_\delta(y)}{x-y}\right|^p dxdy + \int_{\mathbb{R}} W(v)\,dx \\ &\geq\; \iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{v_\delta(x)-v_\delta(y)}{x-y}\right|^p dxdy + m_\delta |J_\delta|, \end{aligned} \tag{3.5}$$

where

$$m_\delta := \min_{s\in[\alpha+\delta,\beta-\delta]} W(s).$$

Let us define

$$x_\alpha := \min\{x : v(x) > \alpha+\delta\} \quad \text{and} \quad x_\beta := \max\{x : v(x) < \beta-\delta\};$$

since $v_\delta(x) = \alpha+\delta$ for every $x < x_\alpha$ and $v_\delta(x) = \beta-\delta$ for every $x > x_\beta$, for any interval $J \supset [x_\alpha, x_\beta]$ the non-decreasing rearrangement v_δ^* of v_δ in J defined by (3.1) does not depend on J and by (3.4) we have

$$\iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{v_\delta(x)-v_\delta(y)}{x-y}\right|^p dxdy \geq \iint_{J\times J} \left|\frac{v_\delta^*(x)-v_\delta^*(y)}{x-y}\right|^p dxdy$$

and hence

$$\begin{aligned} \iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{v_\delta(x)-v_\delta(y)}{x-y}\right|^p dxdy \;&\geq\; \iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{v_\delta^*(x)-v_\delta^*(y)}{x-y}\right|^p dxdy \\ &\geq\; \int_{-\infty}^{x_\alpha^*}\int_{x_\beta^*}^{+\infty} \left|\frac{v_\delta^*(x)-v_\delta^*(y)}{x-y}\right|^p dxdy, \end{aligned} \tag{3.6}$$

where $x_\alpha^* := \sup\{x : v_\delta^*(x) = \alpha+\delta\}$ and $x_\beta^* := \inf\{x : v_\delta^*(x) = \beta-\delta\}$.

By (3.5) and (3.6), it follows that

$$\begin{aligned} G_1(v,\mathbb{R}) \;&\geq\; (\beta-\alpha-2\delta)^p \int_{-\infty}^{x_\alpha^*}\int_{x_\beta^*}^{+\infty} \frac{dxdy}{|x-y|^p} + m_\delta|J_\delta| \\ &=\; \frac{(\beta-\alpha-2\delta)^p}{(p-1)(p-2)|J_\delta|^{p-2}} + m_\delta|J_\delta|. \end{aligned}$$

Finally, minimizing with respect to $|J_\delta|$, we obtain

$$G_1(v,\mathbb{R}) \geq \frac{(p-1)^{\frac{p-2}{p-1}}}{(p-2)} (\beta-\alpha-2\delta)^{\frac{p}{p-1}} m_\delta^{\frac{p-2}{p-1}} > 0$$

and, by to the arbitrariness of v, the proof is complete. □

In order to prove the upper bound it is convenient to introduce an auxiliary optimal profile problem. For every $T > 0$, we consider

$$\gamma_p^T := \inf\left\{G_1(v,\mathbb{R}) : v \in W_{\mathrm{loc}}^{1-\frac{1}{p},p}(\mathbb{R}), v(x) = \alpha \ \forall x \leq -T, v(x) = \beta \ \forall x \geq T\right\}. \tag{3.7}$$

By the compactness of the embedding of $W^{1-\frac{1}{p},p}((-2T,2T))$ in $L^p((-2T,2T))$, it is easy to prove that the minimum in (3.7) is achieved. By truncation and rearrangement it also follows that the minimum can be achieved by a function $\varphi^T \in W_{\mathrm{loc}}^{1-\frac{1}{p},p}(\mathbb{R})$ which is non-decreasing and satisfies $\alpha \leq \varphi^T \leq \beta$.

Proposition 3.2. *The sequence γ_p^T is non-increasing in T and $\lim_{T\to+\infty} \gamma_p^T = \gamma_p$.*

Proof. By the definition of γ_p^T, it immediately follows that γ_p^T is monotone and is greater than or equal to γ_p. Hence, the limit exists and satisfies

$$\lim_{T\to+\infty} \gamma_p^T \geq \gamma_p.$$

It remains to prove the reverse inequality. For every $\mu > 0$, let us fix $\psi \in W_{\mathrm{loc}}^{1-\frac{1}{p},p}(\mathbb{R})$ such that

$$\lim_{x\to-\infty} \psi(x) = \alpha, \quad \lim_{x\to+\infty} \psi(x) = \beta \quad \text{and} \quad G_1(\psi,\mathbb{R}) \leq \gamma_p + \mu.$$

Moreover, by truncation we may always assume that $\alpha \leq \psi \leq \beta$.

The idea is to modify ψ in order to construct a function φ which is a good competitor for γ_p^T. To this aim we consider

$$\Psi(x) := \int_{\mathbb{R}} \left|\frac{\psi(x)-\psi(y)}{x-y}\right|^p dy.$$

Since $\Psi \in L^1(\mathbb{R})$ we can choose a sequence $\{T_n\}_{n\in\mathbb{N}}$, with $T_n \to +\infty$, such that

$$\Psi(-T_n) \to 0 \quad \text{and} \quad \Psi(T_n) \to 0 \quad \text{as } n \to +\infty.$$

For every $\delta > 0$, due to the asymptotic behaviour of ψ, we can find $n_\delta \in \mathbb{N}$ such that

$$\psi(-T_n) \leq \alpha + \delta \quad \text{and} \quad \psi(T_n) \geq \beta - \delta, \quad \forall n \geq n_\delta. \tag{3.8}$$

For every $M > 0$, we define a function φ which coincides with ψ in $[-T_n, T_n]$, satisfies $\varphi(x) = \alpha$ if $x < -T_n - M$ and $\varphi(x) = \beta$ if $x > T_n + M$ and it is affine in

$(-T_n - M, -T_n)$ and $(T_n, T_n + M)$. Namely,

$$\varphi(x) := \begin{cases} \alpha & \text{if } x \in (-\infty, T_n - M], \\ \dfrac{\psi(-T_n) - \alpha}{M}(x + T_n) + \psi(-T_n) & \text{if } x \in (-T_n - M, -T_n), \\ \psi(x) & \text{if } x \in [-T_n, T_n], \\ \dfrac{\beta - \psi(T_n)}{M}(x - T_n) + \psi(T_n) & \text{if } x \in (T_n, T_n + M), \\ \beta & \text{if } x \in [T_n + M, +\infty). \end{cases}$$

Clearly, φ is a good competitor for $\gamma_p^{T_n+M}$. Let us compute its energy, denoting $J_n := (-T_n, T_n)$,

$$\begin{aligned} \gamma_p^{T_n+M} &\leq G_1(\varphi, \mathbb{R}) \\ &= G_1(\psi, J_n) + G_1(\varphi, \mathbb{R} \setminus J_n) + 2\iint_{(\mathbb{R}\setminus J_n)\times J_n} \left|\frac{\varphi(x) - \varphi(y)}{x - y}\right|^p dxdy \\ &\leq \gamma_p + \mu + \iint_{(\mathbb{R}\setminus J_n)\times(\mathbb{R}\setminus J_n)} \left|\frac{\varphi(x) - \varphi(y)}{x - y}\right|^p dxdy + \int_{\mathbb{R}\setminus J_n} W(\varphi)\, dx \\ &\quad +2\iint_{(\mathbb{R}\setminus J_n)\times J_n} \left|\frac{\varphi(x) - \varphi(y)}{x - y}\right|^p dxdy \\ &= \gamma_p + \mu + I_1 + I_2 + I_3. \end{aligned} \tag{3.9}$$

The first two integrals in the right hand side of (3.9) can be easily estimated as follows

$$\begin{aligned} I_1 := \iint_{(\mathbb{R}\setminus J_n)\times(\mathbb{R}\setminus J_n)} \left|\frac{\varphi(x) - \varphi(y)}{x - y}\right|^p dxdy &\leq (\beta - \alpha)^p \int_{-\infty}^{-T_n} \int_{T_n}^{+\infty} \frac{dxdy}{|x - y|^p} \\ &= \frac{(\beta - \alpha)^p}{(p - 1)(p - 2)(2T_n)^{p-2}} \end{aligned}$$

and

$$I_2 := \int_{\mathbb{R}/J_n} W(\varphi)\, dx \leq M\omega_\delta,$$

where

$$\omega_\delta := \max_{s\in[\alpha,\alpha+\delta]\cup[\beta-\delta,\beta]} W(s). \tag{3.10}$$

Instead, an upper bound for the last integral requires more attention in computation. Let us show it in details.

$$
\begin{aligned}
I_3 \;:=\; & 2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n}\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|^p dxdy+2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|^p dxdy\\
&+2\int_{T_n+M}^{+\infty}\int_{-T_n}^{T_n}\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|^p dxdy+2\int_{T_n}^{T_n+M}\int_{-T_n}^{T_n}\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|^p dxdy.
\end{aligned}
$$

We have

$$
\begin{aligned}
2\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n}\left|\frac{\psi(y)-\alpha}{x-y}\right|^p dxdy \;&\leq\; 2(\beta-\alpha)^p\int_{-\infty}^{-T_n-M}\int_{-T_n}^{T_n}\frac{dxdy}{|x-y|^p}\\
&=\; \frac{2(\beta-\alpha)^p}{(p-1)(p-2)M^{p-2}}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|^p dxdy\\
&=2\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\frac{\left|\psi(y)-\psi(-T_n)-\frac{\psi(-T_n)-\alpha}{M}(x+T_n)\right|^p}{|x-y|^p}\,dxdy\\
&\leq 2^p\int_{-T_n-M}^{-T_n}\Psi(-T_n)dx+2^p\frac{|\psi(-T_n)-\alpha|^p}{M^p}\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\left|\frac{x+T_n}{x-y}\right|^p dxdy\\
&\leq 2^pM\Psi(-T_n)+\frac{2^{p-1}\delta^p}{(p-1)M^{p-2}},\quad \forall n\geq n_\delta\,,
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\int_{-T_n-M}^{-T_n}\int_{-T_n}^{T_n}\frac{|x+T_n|^p}{|y-x|^p}dxdy \;&=\; \frac{1}{p-1}\int_{-T_n-M}^{-T_n}\left(|x+T_n|-\frac{|x+T_n|^p}{|T_n-x|^{p-2}}\right)dx\\
&\leq\; \frac{M^2}{2(p-1)}.
\end{aligned}
$$

Similarly, we can estimate the third and the fourth integrals of I_3 and we get

$$
I_3\leq 2^pM(\Psi(-T_n)+\Psi(T_n))+\frac{2^p\delta^p}{(p-1)M^{p-2}}+\frac{4(\beta-\alpha)^p}{(p-1)(p-2)M^{p-2}}.
$$

Finally, by (3.9), we obtain

$$
\gamma_p^{T_n+M}\leq\gamma_p+\mu+r_n+r_\delta+\frac{4(\beta-\alpha)^p}{(p-1)(p-2)M^{p-2}},\quad \forall n\geq n_\delta, \tag{3.11}
$$

where

$$r_n := \frac{2(\beta-\alpha)^p}{(p-1)(p-2)(2T_n)^{p-2}} + 2^p M\left(\Psi(-T_n)+\Psi(T_n)\right)$$

and

$$r_\delta := \frac{2^p}{(p-1)M^{p-2}}\delta^p + M\omega_\delta.$$

Taking the limit as $n \to +\infty$ and then $\delta \to 0$ and $M \to +\infty$, we get

$$\lim_{T\to+\infty} \gamma_p^T = \lim_{n\to+\infty} \gamma_p^{T_n+M} \leq \gamma_p + \mu,$$

which concludes the proof by the arbitrariness of μ. □

Let us conclude this section with the proof of the existence of an *optimal profile.*

Proposition 3.3. *The minimum for γ_p defined by* (3.2) *is achieved by a non-decreasing function φ satisfying $\alpha \leq \varphi \leq \beta$.*

Proof. Let $T > 0$ and let φ^T be a non-decreasing minimizer for γ_p^T. Since the functions φ^T are monotone and bounded, by Helly's theorem, there exist a subsequence φ^{T_k} of φ^T and a non-decreasing function φ, bounded by α and β, such that φ^{T_k} converges pointwise in $\mathbb{R}$ to φ. By Fatou's lemma and Proposition 3.2 we also have

$$\iint_{\mathbb{R}\times\mathbb{R}} \left|\frac{\varphi(x)-\varphi(y)}{x-y}\right|^p dxdy + \int_{\mathbb{R}} W(\varphi)\,dx \leq \lim_{k\to\infty} \gamma_p^{T_k} = \gamma_p\,.$$

This implies that φ is a minimizer for γ_p. □

4. Compactness

The proof of the compactness follows the lines of the proof of Alberti, Bouchitté and Seppecher in [4] and uses the following lemma which gives a (non-optimal) lower bound for G_ε.

Lemma 4.1. *Let $(u_\varepsilon) \subset W^{1-\frac{1}{p},p}(I)$ and let $J \subset I$ be an open interval. For every δ such that $0 < \delta < (\beta-\alpha)/2$, let us define*

$$A_\varepsilon := \{x \in I : u_\varepsilon(x) \leq \alpha+\delta\} \quad \text{and} \quad B_\varepsilon := \{x \in I : u_\varepsilon(x) \geq \beta-\delta\}.$$

Let us set

$$a_\varepsilon := \frac{|A_\varepsilon \cap J|}{|J|} \quad \text{and} \quad b_\varepsilon := \frac{|B_\varepsilon \cap J|}{|J|}. \tag{4.1}$$

Then

$$G_\varepsilon(u_\varepsilon, J) \geq \left(\frac{2(\beta-\alpha-2\delta)^p}{(p-1)(p-2)|J|^{p-2}}\left(1-\frac{1}{(1-a_\varepsilon)^{p-2}}-\frac{1}{(1-b_\varepsilon)^{p-2}}\right)\right)\varepsilon^{p-2}+c_\delta, \tag{4.2}$$

where c_δ does not depend on ε.

Proof. Let $x_0, y_0 \in \mathbb{R}$ be such that $J = (x_0, y_0)$; we obtain

$$\begin{aligned}
& G_\varepsilon(u_\varepsilon, J) \\
\geq\ & G_\varepsilon(u_\varepsilon^*, J) \\
\geq\ & 2\varepsilon^{p-2}(\beta-\alpha-2\delta)^p \iint_{[x_0, x_0+a_\varepsilon|J|]\times[y_0, y_0-b_\varepsilon|J|]} \frac{dxdy}{|y-x|^p} \\
& + \frac{1}{\varepsilon} m_\delta |J|(1-a_\varepsilon-b_\varepsilon) \\
=\ & \frac{2\varepsilon^{p-2}(\beta-\alpha-2\delta)^p}{(p-1)(p-2)|J|^{p-2}} \left(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} - \frac{1}{(1-b_\varepsilon)^{p-2}} + \frac{1}{(1-a_\varepsilon-b_\varepsilon)^{p-2}}\right) \\
& + \frac{1}{\varepsilon} m_\delta |J|(1-a_\varepsilon-b_\varepsilon),
\end{aligned}$$

where u_ε^* denote the non-decreasing rearrangement of u_ε in (x_0, y_0) defined by (3.3) and $m_\delta := \min\{W(s) : \alpha+\delta \leq s \leq \beta-\delta\}$.

Minimizing with respect to $|J|(1-a_\varepsilon-b_\varepsilon)$, we get

$$\begin{aligned}
G_\varepsilon(u_\varepsilon, J) \ \geq\ & \varepsilon^{p-2}\left(\frac{2(\beta-\alpha-2\delta)^p}{(p-1)(p-2)|J|^{p-2}}\left(1 - \frac{1}{(1-a_\varepsilon)^{p-2}} - \frac{1}{(1-b_\varepsilon)^{p-2}}\right)\right) \\
& + 2^{\frac{1}{p-1}} \frac{(p-1)^{\frac{p-2}{p-1}}}{p-2} (\beta-\alpha-2\delta)^{\frac{p}{p-1}} m_\delta^{\frac{p-2}{p-1}},
\end{aligned}$$

for every $0 < \delta < (\beta-\alpha)/2$, and hence (4.2) is proved. □

We are now in a position to prove the compactness result (i.e., Theorem 2.1, (i)).

Let $(u_\varepsilon) \subset W^{1-\frac{1}{p},p}(I)$ be a sequence with equi-bounded energy; i.e., a sequence satisfying $\sup_{\varepsilon>0} G_\varepsilon(u_\varepsilon, I) \leq C$. In particular

$$\int_I W(u_\varepsilon)\, dx \leq C\varepsilon$$

and this implies that

$$W(u_\varepsilon) \to 0 \text{ in } L^1(I). \tag{4.3}$$

Thanks to the growth assumption on W, (u_ε) is weakly relatively compact in $L^1(I)$; i.e., there exists $u \in L^1(I)$ such that (up to a subsequences) $u_\varepsilon \rightharpoonup u$ in $L^1(I)$. We have to prove that this convergence is strong in $L^1(I)$ and that $u \in BV(I, \{\alpha, \beta\})$. Let ν_x be the Young measure associated to (u_ε). Since $W \geq 0$, we have

$$\int_I \int_{\mathbb{R}} W(t)\, d\nu_x(t) \leq \liminf_{\varepsilon\to 0} \int_I W(u_\varepsilon)\, dx$$

(see for instance [16], Theorem 16). Hence, by (4.3), it follows that

$$\int_{\mathbb{R}} W(t)\, d\nu_x(t) = 0, \quad \text{a.e. } x \in I,$$

which implies the existence of a function θ on $[0,1]$ such that

$$\nu_x(dt) = \theta(x)\delta_\alpha(dt) + (1-\theta(x))\delta_\beta(dt)\,, \quad x \in I$$

and

$$u(x) = \theta(x)\alpha + (1-\theta(x))\beta\,, \quad x \in I.$$

It remains to prove that θ belongs to $BV(I,\{0,1\})$. Let us consider the set S of the points where the approximate limits of θ is neither 0 nor 1. For every $N \le \mathcal{H}^0(S)$ we can find N disjoint intervals $\{J_n\}_{n=1,\dots,N}$ such that $J_n \cap S \neq \emptyset$ and such that the quantities a^n_ε and b^n_ε, defined by (4.1) replacing J by J_n, satisfy

$$a^n_\varepsilon \to a^n \in (0,1) \quad \text{and} \quad b^n_\varepsilon \to b^n \in (0,1) \quad \text{as } \varepsilon \text{ goes to zero.}$$

We can now apply Lemma 4.1 in the interval J_n and, taking the limit as $\varepsilon \to 0$ in the inequality (4.2), we obtain

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, J_n) \ge c_\delta.$$

Finally, we use the sub-additivity of $G_\varepsilon(u,\cdot)$ and we get

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, I) \ge \sum_{n=1}^{N} \liminf G_\varepsilon(u_\varepsilon, J_n) \ge N c_\delta. \tag{4.4}$$

Since (u_ε) has equi-bounded energy, this implies that S is a finite set. Hence, $\theta \in BV(I,\{0,1\})$ and the proof of the compactness for G_ε is complete.

5. Lower Bound Inequality

In this section, we prove the Γ-liminf inequality. An optimal lower bound for $G_\varepsilon(u_\varepsilon)$ is a consequence of the following proposition.

Proposition 5.1. *Let J be an open interval of $\mathbb{R}$. Let (u_ε) be a non-decreasing sequence in $W^{1-\frac{1}{p},p}(J)$ and assume that there exist $\bar a, \bar b \in J$, $\bar a < \bar b$, such that for every $\delta > 0$ there exists ε_δ such that*

$$u_\varepsilon(\bar a) \le \alpha + \delta \quad \textit{and} \quad u_\varepsilon(\bar b) \ge \beta - \delta \quad \forall \varepsilon \le \varepsilon_\delta.$$

Then

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, J) \ge \gamma_p.$$

Proof. Let $J = (a,b)$. It is clearly enough to consider the case

$$\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon, (a,b)) < +\infty.$$

By a truncation argument, without loss of generality, we may also assume that

$$\alpha \le u_\varepsilon(x) \le \beta, \quad \forall x \in (a,b).$$

Let us define

$$U_\varepsilon(x) := \int_a^b \left|\frac{u_\varepsilon(x)-u_\varepsilon(y)}{x-y}\right|^p dy.$$

By the fact that

$$\liminf_{\varepsilon\to 0}\int_a^b U_\varepsilon(x)\,dx$$

is finite, we get that there exist $\tilde{x}\in(a,\bar{a})$ and $\tilde{y}\in(\bar{b},b)$ such that

$$\liminf_{\varepsilon\to 0} U_\varepsilon(\tilde{x})\le C \quad \text{and} \quad \liminf_{\varepsilon\to 0} U_\varepsilon(\tilde{y})\le C \quad \text{for some } C>0. \tag{5.1}$$

Fix $M>0$. We now extend u_ε in the whole $\mathbb{R}$ as follows

$$\tilde{u}_\epsilon(x) := \begin{cases} \alpha & \text{if } x\in(-\infty,\tilde{x}-M\varepsilon), \\ \dfrac{u_\varepsilon(\tilde{x})-\alpha}{M\varepsilon}(x-\tilde{x})+u_\varepsilon(\tilde{x}) & \text{if } x\in[\tilde{x}-M\varepsilon,\tilde{x}], \\ u_\varepsilon(x) & \text{if } x\in(\tilde{x},\tilde{y}), \\ \dfrac{\beta-u_\varepsilon(\tilde{y})}{M\varepsilon}(x-\tilde{y})+u_\varepsilon(\tilde{y}) & \text{if } x\in[\tilde{y},\tilde{y}+M\varepsilon], \\ \beta & \text{if } x\in(\tilde{y}+M\varepsilon,+\infty). \end{cases}$$

Denote $\tilde{J}:=(\tilde{x},\tilde{y})\subseteq(a,b)$. We have

$$\begin{aligned} G_\varepsilon(u_\varepsilon,\tilde{J}) &\ge \gamma_p - G_\varepsilon(\tilde{u}_\epsilon,\mathbb{R}\setminus\tilde{J}) - 2\varepsilon^{p-2}\iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}}\left|\frac{\tilde{u}_\epsilon(x)-\tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy \\ &= \gamma_p - \varepsilon^{p-2}\iint_{(\mathbb{R}\setminus\tilde{J})\times(\mathbb{R}\setminus\tilde{J})}\left|\frac{\tilde{u}_\epsilon(x)-\tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy - \frac{1}{\varepsilon}\int_{\mathbb{R}\setminus\tilde{J}} W(\tilde{u}_\epsilon)\,dx \\ &\quad -2\varepsilon^{p-2}\iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}}\left|\frac{\tilde{u}_\epsilon(x)-\tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy \\ &= \gamma_p - I_1 - I_2 - I_3. \end{aligned} \tag{5.2}$$

Using the definition of $\tilde{u}_\epsilon$, we easily get

$$\begin{aligned} I_1 &:= \varepsilon^{p-2}\iint_{(\mathbb{R}\setminus\tilde{J})\times(\mathbb{R}\setminus\tilde{J})}\left|\frac{\tilde{u}_\epsilon(x)-\tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy \\ &\le \varepsilon^{p-2}(\beta-\alpha)^p\iint_{(\mathbb{R}\setminus\tilde{J})\times(\mathbb{R}\setminus\tilde{J})}\frac{dxdy}{|x-y|^p} \\ &= \frac{(\beta-\alpha)^p}{(p-1)(p-2)|\tilde{J}|^{p-2}}\varepsilon^{p-2}. \end{aligned}$$

Moreover, since u_ε is non-decreasing,

$$u_\varepsilon(x) \le \alpha + \delta \quad \forall\, x \le \bar{a} \quad \text{and} \quad u_\varepsilon(x) \ge \beta - \delta \quad \forall\, x \ge \bar{b}$$

and, in particular,

$$I_2 := \varepsilon^{-1} \int_{\mathbb{R}\setminus \tilde{J}} W(\tilde{u}_\epsilon)\, dx \le M\omega_\delta,$$

where ω_δ is defined in (3.10).

Finally, using the fact that $u_\varepsilon(\tilde{x}) \le \alpha + \delta$ and $u_\varepsilon(\tilde{y}) \ge \beta - \delta$, we can estimate the third integral

$$\begin{aligned} I_3 \;:=\; & 2\varepsilon^{p-2} \iint_{(\mathbb{R}\setminus\tilde{J})\times\tilde{J}} \left|\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy = 2\varepsilon^{p-2} \int_{-\infty}^{\tilde{x}-M\varepsilon} \int_{\tilde{x}}^{\tilde{y}} \left|\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy \\ & + 2\varepsilon^{p-2} \int_{\tilde{x}-M\varepsilon}^{\tilde{x}} \int_{\tilde{x}}^{\tilde{y}} \left|\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy + 2\varepsilon^{p-2} \int_{\tilde{y}+M\varepsilon}^{+\infty} \int_{\tilde{x}}^{\tilde{y}} \left|\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy \\ & + 2\varepsilon^{p-2} \int_{\tilde{y}}^{\tilde{y}+M\varepsilon} \int_{\tilde{x}}^{\tilde{y}} \left|\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy. \end{aligned} \tag{5.3}$$

We have

$$\begin{aligned} 2\varepsilon^{p-2} \iint_{-\infty}^{\tilde{x}-M\varepsilon} \int_{\tilde{x}}^{\tilde{y}} \left|\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy \;&\le\; 2\varepsilon^{p-2}(\beta-\alpha)^p \int_{-\infty}^{\tilde{x}-M\varepsilon} \int_{\tilde{x}}^{\tilde{y}} \frac{dxdy}{|x-y|^p} \\ &\le\; \frac{2(\beta-\alpha)^p}{(p-1)(p-2)M^{p-2}}. \end{aligned}$$

Moreover,

$$\begin{aligned} & 2\varepsilon^{p-2} \int_{\tilde{x}-M\varepsilon}^{\tilde{x}} \int_{\tilde{x}}^{\tilde{y}} \left|\frac{\tilde{u}_\epsilon(x) - \tilde{u}_\epsilon(y)}{x-y}\right|^p dxdy \\ & \quad = 2\varepsilon^{p-2} \int_{\tilde{x}-M\varepsilon}^{\tilde{x}} \int_{\tilde{x}}^{\tilde{y}} \frac{|u_\varepsilon(y) - u_\varepsilon(\tilde{x}) - \frac{u_\varepsilon(\tilde{x})-\alpha}{M\varepsilon}(x-\tilde{x})|^p}{|x-y|^p} dxdy \\ & \quad \le 2^p \varepsilon^{p-2} \int_{\tilde{x}-M\varepsilon}^{\tilde{x}} U_\varepsilon(\tilde{x}) dx + 2^p \frac{|u_\varepsilon(\tilde{x}) - \alpha|^p}{M^p \varepsilon^2} \int_{\tilde{x}-M\varepsilon}^{\tilde{x}} \int_{\tilde{x}}^{\tilde{y}} \frac{|\tilde{x}-x|^p}{|x-y|^p} dxdy \\ & \quad \le 2^p \varepsilon^{p-1} M U_\varepsilon(\tilde{x}) + \frac{2^{p-1}\delta^p}{(p-1)M^{p-2}}, \end{aligned}$$

where we used that

$$\int_{\tilde{x}-M\varepsilon}^{\tilde{x}} \int_{\tilde{x}}^{\tilde{y}} \frac{|\tilde{x}-x|^p}{|x-y|^p} dxdy = \frac{1}{(p-1)} \int_{\tilde{x}-M\varepsilon}^{\tilde{x}} \left(|x-\tilde{x}| - \frac{|x-\tilde{x}|^p}{|\tilde{y}-x|^{p-1}} \right) dx \le \frac{(M\varepsilon)^2}{2(p-1)}.$$

Similarly, we can estimate the third and the fourth integrals of I_3 and we get

$$I_3 \leq 2^p M(U_\varepsilon(\tilde{x}) + U_\varepsilon(\tilde{y}))\varepsilon^{p-1} + \frac{2^p \delta^p}{(p-1)M^{p-2}} + \frac{4(\beta-\alpha)^p}{(p-1)(p-2)M^{p-2}}.$$

Hence, by (5.2), we obtain

$$\begin{aligned} G_\varepsilon(u_\varepsilon, \tilde{J}) \ \geq \ & \gamma_p - \left(\frac{(\beta-\alpha)^p}{(p-1)(p-2)|\tilde{J}|^{p-2}} + 2^p M(U_\varepsilon(\tilde{x}) + U_\varepsilon(\tilde{y}))\varepsilon \right) \varepsilon^{p-2} - r_\delta \\ & - \frac{4(\beta-\alpha)^p}{(p-1)(p-2)M^{p-2}}, \end{aligned}$$

with

$$r_\delta := \frac{2^p \delta^p}{(p-1)M^{p-2}} \delta^p + M\omega_\delta$$

vanishing as $\delta \to 0$.Thus, by (5.1) and taking the liminf as $\varepsilon \to 0$ and then as $\delta \to 0$, we get

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, \tilde{J}) \geq \gamma_p - \frac{4(\beta-\alpha)^p}{(p-1)(p-2)M^{p-2}},$$

which conclude the proof by the arbitrariness of M. □

Remark 5.2. Clearly an analogue proposition holds in the case of u_ε non-increasing satisfying the hypotheses with $\bar{a} > \bar{b}$.

In order to conclude, let us first observe that, thanks to the compactness result for G_ε, we may assume that the sequence (u_ε) converges in $L^1(I)$ to some $u \in BV(I, \{\alpha, \beta\})$. Hence, the jump set Su is finite and we can find $N := \mathcal{H}^0(Su)$ disjoint subintervals $\{I_i\}_{i=1,\dots,N}$ such that $Su \cap I_i \neq \emptyset$, for every $i = 1, ..., N$.

Now, let us consider the monotone rearrangement $u^*_{\varepsilon,i}$ of u_ε in I_i. The rearrangement $u^*_{\varepsilon,i}$ is non-decreasing if u is non-decreasing in I_i and non-increasing otherwise. With this choice clearly $u^*_{\varepsilon,i}$ converges to u in $L^1(I_i)$ and thus it satisfies the assumptions of Proposition 5.1 (se also Remark 5.2) with J replaced by I_i. Then, for every $i = 1, ..., N$, we may conclude that

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, I_i) \geq \liminf_{\varepsilon \to 0} G_\varepsilon(u^*_{\varepsilon,i}, I_i) \geq \gamma_p.$$

Finally, using the sub-additivity of $G_\varepsilon(u_\varepsilon, \cdot)$, we get

$$\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon, I) \geq \liminf_{\varepsilon \to 0} \sum_{i=1}^{N} G_\varepsilon(u_\varepsilon, I_i) \geq N\gamma_p = \gamma_p \mathcal{H}^0(Su)$$

and hence the lower bound states by Theorem 2.1, (ii), is proved. □

6. Upper Bound Inequality

In this section, we conclude the proof of the Theorem 2.1, proving the limsup inequality. Let us first construct an optimal sequence for u of the form

$$u(x) = \begin{cases} \alpha, & \text{if } x \le x_0, \\ \beta, & \text{if } x > x_0. \end{cases}$$

Let $T > 0$ be fixed and let $\varphi^T \in W^{1-\frac{1}{p},p}_{\mathrm{loc}}(\mathbb{R})$ be the minimizer for γ_p^T defined by (3.7); i.e.,

$$\varphi^T(x) = \alpha \quad \forall\, x \le -T, \quad \varphi^T(x) = \beta \quad \forall\, x \ge T \quad \text{and} \quad G_1(\varphi, \mathbb{R}) = \gamma_p^T.$$

Let us define, for every $\varepsilon > 0, u_\varepsilon(x) := \varphi^T\left(\dfrac{x - x_0}{\varepsilon}\right)$, for every $x \in I$. We have

$$u_\varepsilon \to u \text{ in } L^1(I)$$

and

$$\begin{aligned} G_\varepsilon(u_\varepsilon) &= \varepsilon^{p-2} \iint_{I\times I} \left| \frac{\varphi^T(\frac{x-x_0}{\varepsilon}) - \varphi^T(\frac{y-x_0}{\varepsilon})}{x-y} \right|^p dxdy + \frac{1}{\varepsilon}\int_I W\left(\varphi^T\left(\frac{x-x_0}{\varepsilon}\right)\right) dx \\ &= G_1(\varphi^T, (I - x_0)/\varepsilon) \le G_1(\varphi^T, \mathbb{R}) = \gamma_p^T. \end{aligned} \tag{6.1}$$

By Proposition 3.2 we get

$$\lim_{T\to+\infty} \limsup_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) \le \gamma_p.$$

Then by a diagonalization argument we can construct a sequence $\tilde{u}_\epsilon$ converging to u in $L^1(I)$, which satisfies

$$\limsup_{\varepsilon\to 0} G_\varepsilon(\tilde{u}_\epsilon) \le \gamma_p.$$

The optimal sequence for an arbitrary $u \in BV(I, \{\alpha, \beta\})$ can be easily obtained gluing the sequences constructed above for each single jump of u and taking into account that, thanks to the scaling ε^{p-2}, the long range interactions between two different recovery sequences decay as $\varepsilon \to 0$.

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