CAPACITY THEORY
FOR NON-SYMMETRIC ELLIPTIC OPERATORS

Gianni DAL MASO
Adriana GARRONI

Abstract

We study the properties of the capacity $\mathrm{cap}^L(A,\Omega)$ with respect to an arbitrary, possibly non-symmetric, elliptic operator L.

Ref. S.I.S.S.A. 179/M (Nov 93)

Introduction

The notion of capacity associated with a possibly non-symmetric elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_j u)\,,$$

with bounded measurable coefficients, was introduced by Stampacchia in [9] in order to study the behaviour of the Green's function of L. In the symmetric case the L-capacity $\mathrm{cap}^L(A,\Omega)$ of a set A in a bounded open set Ω can be defined as the infimum of

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}D_j u D_i u\Big)\,dx$$

over the set of all functions u in the Sobolev space $H_0^1(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of A. It follows easily from this definition that $\mathrm{cap}^L(A,\Omega)$ is increasing with respect to A and decreasing with respect to Ω. Moreover, using standard techniques, it is not difficult to prove that the set function $\mathrm{cap}^L(\cdot,\Omega)$ is countably subadditive and continuous along increasing sequences of subsets of Ω.

When the operator L is not symmetric, the definition of L-capacity is given in terms of the solution of a variational inequality. In this case very little seems to be known about the behaviour of $\mathrm{cap}^L(A,\Omega)$ as a function of A and Ω. At our knowledge, even the monotonicity properties have never been studied. Indeed Stampacchia proved only that, if $A \subseteq B \subset\subset \Omega$, then

$$\mathrm{cap}^L(A,\Omega) \leq K\,\mathrm{cap}^L(B,\Omega)\,,$$

where $K \geq 1$ is a constant depending on L, and $K > 1$ when L is not symmetric (see [9], Theorem 3.10). Since

$$\mathrm{cap}^{L_1}(A,\Omega) \leq \frac{\beta}{\alpha}\mathrm{cap}^{L_2}(A,\Omega)$$

for two elliptic operators L_1 and L_2 with the same ellipticity constants α and β (see [9], Theorem 3.11), the precise behaviour of the set function $\mathrm{cap}^L(\cdot,\Omega)$ is not important in those applications where only a rough estimate of $\mathrm{cap}^L(\cdot,\Omega)$ is needed, like the estimates for the Green's function and the Wiener condition for the regularity of boundary points (see [8] and [9]). Indeed in all these cases one can replace the non-symmetric operator

L by a simpler symmetric operator with the same ellipticity constants, and the previous estimate, together with the properties known in the symmetric case, are enough to obtain the desired results.

However, there are some problems where one can not replace $\mathrm{cap}^L(\cdot,\Omega)$ by an equivalent capacity. An example is given by the study of the asymptotic behaviour, as $h \to \infty$, of the solutions u_h of the Dirichlet problems

$$u_h \in H_0^1(\Omega_h)\,, \qquad Lu_h = f \quad \text{in } \Omega_h\,,$$

where Ω_h are open subsets of a given bounded domain Ω. When L is symmetric, it is possible to determine the behaviour of the sequence u_h by knowing the limit of $\mathrm{cap}^L(A\backslash\Omega_h,\Omega)$ for a sufficiently large class of subsets A of Ω (see [1] and [3]). Of course, in this problem $\mathrm{cap}^L(\cdot,\Omega)$ can not be replaced by an equivalent capacity, since the result actually depends on L. In order to extend this analysis to the case of non-symmetric operators, we need to know the properties of the set function $\mathrm{cap}^L(\cdot,\Omega)$ in the general case.

With this motivation in mind, in this paper we study the properties of $\mathrm{cap}^L(A,\Omega)$ for an arbitrary elliptic operator L. In particular we prove that $\mathrm{cap}^L(A,\Omega)$ is increasing with respect to A (Theorem 3.2) and decreasing with respect to Ω (Theorem 3.3). Moreover, we show that the set function $\mathrm{cap}^L(\cdot,\Omega)$ is strongly subadditive (Theorem 3.4) and continuous along increasing sequences of subsets of Ω (Theorem 3.5). These results together imply that $\mathrm{cap}^L(\cdot,\Omega)$ is countably subadditive (Theorem 3.6).

In view of the applications to the study of the asymptotic behaviour of the solutions of Dirichlet problems in varying domains (see [5]), we need a more symmetric treatment of the variables A and Ω in $\mathrm{cap}^L(A,\Omega)$. Therefore, for every pair of bounded sets A and B, with $A \subseteq B$, we define the L-capacity $\mathrm{cap}^L(A,B)$ of A in B by means of a variational inequality, which reduces to that used by Stampacchia when A is closed and B is open.

A crucial role in the proofs is played by the inner and outer L-capacitary distributions λ and ν. These are positive measures, supported by ∂A and ∂B, such that the L-capacitary potential u satisfies $Lu = \lambda - \nu$ (Theorem 2.6). Moreover we have $\mathrm{cap}^L(A,B) = \lambda(\partial A) = \nu(\partial B)$ (Proposition 2.9). With the aid of these properties we prove that $\mathrm{cap}^L(A,B) = \mathrm{cap}^{L^*}(A,B)$, where L^* is the adjoint operator (Theorem 3.1). This result is essential in our proof of the other properties of $\mathrm{cap}^L(\cdot,\Omega)$ mentioned above.

We conclude the paper with an Appendix which shows that, although $\lambda - \nu \in$

$H^{-1}(\mathbf{R}^n)$ when $\mathrm{cap}^L(A,B) < +\infty$, the single measures λ and ν may not belong to $H^{-1}(\mathbf{R}^n)$ when A is not relatively compact in the interior of B.

Acknowledgments

This work is part of the Project EURHomogenization, Contract SC1-CT91-0732 of the Program SCIENCE of the Commission of the European Communities, and of the Research Project "Irregular Variational Problems" of the Italian National Research Council.

1. Preliminaries

In the sequel U is always an *open* (possibly unbounded) subset of $\mathbf{R}^n$, $n \geq 2$, while Ω is always a *bounded open* subset of $\mathbf{R}^n$. We denote by $H_0^1(U)$ and $H^1(U)$ the usual Sobolev spaces, and by $H^{-1}(U)$ the dual of $H_0^1(U)$. By $H^1_{loc}(U)$ we denote the space of all functions that belong to $H^1(V)$ for every open set $V \subset\subset U$. For every set E we denote by E^c the complement of E in $\mathbf{R}^n$, i.e., $E^c = \mathbf{R}^n \setminus E$, and by 1_E the characteristic function of E, defined by $1_E(x) = 1$ for $x \in E$, and $1_E(x) = 0$ for $x \notin E$.

If $E \subseteq \Omega$, the (*harmonic*) *capacity* of E in Ω, denoted by $\mathrm{cap}(E,\Omega)$, is defined as the infimum of

$$\int_\Omega |Du|^2\, dx$$

over the set of all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E.

We say that a set $E \subseteq \mathbf{R}^n$ has *capacity zero* if $\mathrm{cap}(E\cap\Omega,\Omega) = 0$ for every bounded open set $\Omega \subseteq \mathbf{R}^n$. It is easy to prove that, if E is contained in a bounded open set Ω, then E has capacity zero if and only if $\mathrm{cap}(E,\Omega) = 0$. We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (abbreviated as q.e.) in a set E if it holds for all $x \in E$ except for a subset N of E of capacity zero. The expression *almost everywhere* (abbreviated as a.e.) refers, as usual, to the analogous property for the Lebesgue measure. A function $u\colon \Omega \to \mathbf{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists a set $A \subseteq \Omega$, with $\mathrm{cap}(A,\Omega) < \varepsilon$, such that the restriction of u to $\Omega\setminus A$ is continuous. A function $u\colon U \to \mathbf{R}$ is said to be *quasi continuous* on U if its restriction to every bounded open set $\Omega \subseteq U$ is quasi continuous on Ω.

It is well known that every $u \in H^1(U)$ has a quasi continuous representative, which is uniquely defined up to a set of capacity zero. In the sequel we shall always identify

u with its quasi continuous representative, so that the pointwise values of a function $u \in H^1(U)$ are defined quasi everywhere in U. With this convention we have

$$\operatorname{cap}(E,\Omega) = \min\Big\{\int_\Omega |Du|^2 dx \,:\, u \in H^1_0(\Omega)\,,\ u \geq 1 \text{ q.e. in } E\Big\}\,.$$

If u and v are two functions in $H^1(U)$ and $u \leq v$ a.e. in U, then $u \leq v$ q.e. in U. It can be proved that a function $u \in H^1(\mathbf{R}^n)$ belongs to $H^1_0(U)$ if and only if $u = 0$ q.e. in U^c. Finally we recall that, if a sequence (u_h) converges to u in $H^1_0(U)$, then a subsequence of (u_h) converges to u q.e. in U. For all these properties of quasi continuous representatives of Sobolev functions we refer to [10], Chapter 3.

A subset A of Ω is said to be *quasi open* in Ω if for every $\varepsilon > 0$ there exists an open subset A_ε of Ω, with $\operatorname{cap}(A_\varepsilon,\Omega) < \varepsilon$, such that $A \cup A_\varepsilon$ is open. It is easy to see that, if A is quasi open in Ω, then $A \cap \Omega'$ is quasi open in Ω' for every open set $\Omega' \subseteq \Omega$. When U is unbounded, a subset A of U is said to be quasi open in B if $A \cap \Omega$ is quasi open in Ω for every bounded open set $\Omega \subseteq U$. It is easy to see that if a function $u\colon U \to \mathbf{R}$ is quasi continuous, then the set $\{u > c\} = \{x \in U : u(x) > c\}$ is quasi open for every $c \in \mathbf{R}$.

Lemma 1.1. *Let (u_h) be a bounded sequence of $H^1_0(U)$ that converges pointwise q.e. to a function u. Then u is (the quasi continuous representative of) a function of $H^1_0(U)$ and (u_h) converge to u weakly in $H^1_0(U)$.*

Proof. Let $\varphi_h = \inf_{k\geq h} u_k$ and $\psi_h = \sup_{k \geq h} u_k$. It is easy to see that $\varphi_h \nearrow u$ q.e. in U and $\psi_h \searrow u$ q.e. in U. Moreover $\varphi_h \leq u_k \leq \psi_h$, for every $h \leq k$. Now, for every h, the set $K_h = \{v \in H^1_0(U) : \varphi_h \leq v \leq \psi_h \text{ q.e. in } U\}$ is convex and closed, thus K_h is weakly closed. Since (u_h) is bounded in $H^1_0(U)$, a subsequence of (u_h) converges weakly in $H^1_0(U)$ to a function v. Then $v \in K_h$, so that $\varphi_h \leq v \leq \psi_h$ q.e. for every h. This implies $u = v$ q.e. in U and concludes the proof of the lemma. ☐

Lemma 1.2. *For every quasi open subset A of U there exists an increasing sequence (v_h) of non-negative functions of $H^1_0(U)$, with $0 \leq v_h \leq 1_A$, converging to 1_A pointwise q.e. in U.*

Proof. See [2], Lemma 1.4, or [4], Lemma 2.1. ☐

We say that a Radon measure ν on U belongs to $H^{-1}(U)$ if there exists $f \in H^{-1}(U)$ such that

$$\langle f, \varphi \rangle_U = \int_U \varphi \, d\nu \qquad \forall \varphi \in C_0^\infty(U)\,, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle_U$ denotes the duality pairing between $H^{-1}(U)$ and $H_0^1(U)$. We shall always identify f and ν. Note that, by the Riesz theorem, for every positive functional $f \in H^{-1}(U)$ there exists a positive Radon measure ν such that (1.1) holds.

Let $L\colon H^1(\mathbf{R}^n) \to H^{-1}(\mathbf{R}^n)$ be an elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij} D_j u)\,,$$

where (a_{ij}) is an $n{\times}n$ matrix of functions of $L^\infty(\mathbf{R}^n)$ satisfying the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \geq |\xi|^2 \tag{1.2}$$

for a.e. $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$. Let $L^*\colon H^1(\mathbf{R}^n) \to H^{-1}(\mathbf{R}^n)$ be the adjoint operator, defined by

$$L^* u = -\sum_{i,j=1}^{n} D_i(a_{ji} D_j u)$$

for every $u \in H^1(\mathbf{R}^n)$. In the sequel we will denote by $a(\cdot,\cdot)$ the bilinear form on $H^1(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$ associated with the operator L defined by

$$a(u,v) \;=\; \int_{\mathbf{R}^n} \Big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i v\Big)\, dx$$

for every $u, v \in H^1(\mathbf{R}^n)$, and by $a^*(\cdot,\cdot)$ the bilinear form associated with the adjoint operator L^*. Clearly we have that $a(u,v) = a^*(v,u)$ for every $u, v \in H^1(\mathbf{R}^n)$.

For every open set $U \subseteq \mathbf{R}^n$, we shall identify each function $u \in H_0^1(U)$ with the function of $H^1(\mathbf{R}^n)$ obtained by extending u by zero on U^c. Moreover we will denote by $a_U(\cdot,\cdot)$ the restriction of $a(\cdot,\cdot)$ to $H_0^1(U) \times H_0^1(U)$.

2. The L-capacity and the L-capacitary distributions.

Let A and B be two bounded subsets of $\mathbf{R}^n$, with $A \subseteq B$, and let U be an open set containing B. Let us consider the convex sets $K_A^B(U)$ and $H_A^B(U)$ defined by

$$K_A^B(U) = \{v \in H_0^1(U) : v = 1 \text{ q.e. in } A \text{ and } v = 0 \text{ q.e. in } U \backslash B\},$$
$$H_A^B(U) = \{v \in H_0^1(U) : v \geq 1 \text{ q.e. in } A \text{ and } v \leq 0 \text{ q.e. in } U \backslash B\}.$$

Clearly $K_A^B(U) = K_A^B(\mathbf{R}^n)$ for every open set U containing B. We say that A is *compatible* with B if the set $K_A^B(\mathbf{R}^n)$ is non-empty. In this case we shall consider the solution of the following variational inequality

$$\text{(2.1)} \qquad \begin{cases} u \in K_A^B(\mathbf{R}^n), \\ a(u, v-u) \geq 0 \qquad \forall v \in K_A^B(\mathbf{R}^n), \end{cases}$$

and we shall prove that it coincides with the solution of the problem

$$\text{(2.2)} \qquad \begin{cases} u \in H_A^B(\mathbf{R}^n), \\ a(u, v-u) \geq 0 \qquad \forall v \in H_A^B(\mathbf{R}^n). \end{cases}$$

Theorem 2.1. *Let A and B be two bounded sets, A compatible with B. Then problem (2.1) has a unique solution u. Moreover u coincides with the unique solution of (2.2) and $0 \leq u \leq 1$ q.e. in $\mathbf{R}^n$.*

Proof. Let Ω be a bounded open set containing B. We have already seen that $K_A^B(\mathbf{R}^n) = K_A^B(\Omega)$. Then problem (2.1) is equivalent to the problem

$$\begin{cases} u \in K_A^B(\Omega), \\ a_\Omega(u, v-u) \geq 0 \qquad \forall v \in K_A^B(\Omega), \end{cases}$$

that has a unique solution by Stampacchia's theorem (see [7], Theorem 2.1). In order to prove the second assertion, for every (possibly unbounded) open set U containing B we consider the variational inequality

$$\text{(2.3)} \qquad \begin{cases} w \in H_A^B(U), \\ a_U(w, v-w) \geq 0 \qquad \forall v \in H_A^B(U). \end{cases}$$

Let us prove that, if w is a solution of (2.3), then w coincides with the solution u of problem (2.1). To this aim it is sufficient to prove that $0 \leq w \leq 1$ q.e. in U. Let us consider the function $z = w \wedge 1$. Since $z \in H_A^B(U)$, by (1.2) and (2.3) we obtain

$$0 \leq a_U(w, z-w) = -\int_{\{w>1\}} \sum_{i,j=1}^n a_{ij} D_j w D_i w \, dx \leq -\int_{\{w>1\}} |Dw|^2 dx.$$

Thus, either $|\{w > 1\}| = 0$, and hence $w \leq 1$ q.e. in U, or $Dw = 0$ a.e. in $\{w > 1\}$. This implies that $D(w \vee 1) = 0$ a.e. in U, so $w \vee 1$ is constant in each connected component of U. Since $w \in H_0^1(U)$ we have $w \vee 1 = 1$ q.e. in U and hence $w \leq 1$ q.e. in U. In particular $w = 1$ q.e. in A.

Similarly, using $z = w \vee 0$ as test function in (2.3), we can prove that $w \geq 0$ q.e. in U and in particular $w = 0$ q.e. in $U \setminus B$. Therefore $w \in K_A^B(U) = K_A^B(\mathbf{R}^n)$. As $K_A^B(\mathbf{R}^n) \subseteq H_A^B(U)$, w is a solution of problem (2.1), and thus, by uniqueness, $w = u$ q.e. in $\mathbf{R}^n$.

It remains to prove the existence of a solution of problem (2.2). Let us fix a bounded open set Ω such that $B \subset\subset \Omega$. By Stampacchia's theorem there exists a unique solution of the problem (2.3) corresponding to $U = \Omega$ and, by the previous step, this solution coincides with u. We are now in a position to prove that u is a solution of (2.2). Let φ be a function in $C_0^\infty(\Omega)$ such that $\varphi = 1$ in B and $\varphi \geq 0$ in Ω. Then for every $v \in H_A^B(\mathbf{R}^n)$ we have $v\varphi \in H_A^B(\Omega)$ and, since $u = 0$ q.e. in B^c, by (2.3) we obtain

$$a(u, v - u) \;=\; \int_B \sum_{i,j=1}^n a_{ij} D_j u D_i(v - u)\, dx \;=\; \int_\Omega \sum_{i,j=1}^n a_{ij} D_j u D_i(\varphi v - u)\, dx \;\geq\; 0\,.$$

Thus u is a solution of problem (2.2) and the proof is complete. □

Definition 2.2. If A is compatible with B, the solution u of problem (2.1) is called the *L-capacitary potential of A in B* and the *L-capacity of A in B* is defined by

$$\mathrm{cap}^L(A, B) \;=\; a(u, u)\,.$$

If A is not compatible with B, we put $\mathrm{cap}^L(A, B) = +\infty$.

When A is closed and B is open this definition coincides with the definition of capacity given by Stampacchia (see [9]). If L is symmetric and B is open, then the L-capacitary potential is the solution of the minimum problem

$$\min\{a_B(v, v) \;:\; v \in H_0^1(B)\,,\; u \geq 1 \text{ q.e. in } A\}\,.$$

In particular, when L is the Laplace operator $-\Delta$, the L-capacity coincides with the harmonic capacity introduced in Section 1.

Remark 2.3. It is clear that if u is the solution of problem (2.1), then it remains a solution if we replace the set A with the set $\{u=1\}$ and the set B with the set $\{u>0\}$. So that

$$\mathrm{cap}^L(\{u=1\},\{u>0\}) = \mathrm{cap}^L(A,B).$$

Since $\{u=1\}$ is quasi closed and $\{u>0\}$ is quasi open, in many applications it is not restrictive to assume that B is quasi open and A is quasi closed.

For the capacitary potentials the following comparison principle holds.

Lemma 2.4. *Let $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$ be four bounded subsets of $\mathbf{R}^n$ such that A_1 (resp. A_2) is compatible with B_1 (resp. B_2). Let u_1 (resp. u_2) be the L-capacitary potential of A_1 (resp. A_2) in B_1 (resp. B_2). Then $u_1 \leq u_2$ q.e. in $\mathbf{R}^n$.*

Proof. This result is a direct consequence of an elementary comparison principle for two-obstacle problems ([7], Theorem 6.4, for the case of one obstacle, and [6], Lemma 2.1, in the general case). □

Remark 2.5. If A is compatible with B, and u is the capacitary potential of A in B, then

$$a(u,\varphi) = 0 \qquad \text{for every } \varphi \in H^1(\mathbf{R}^n) \text{ with } \varphi = 0 \text{ q.e. in } A \cup B^c. \tag{2.4}$$

Indeed the set of all these functions φ is non-empty (for instance it contains the function $u(1-u)$) and if we choose φ in this set we have that $u+\varphi$ and $u-\varphi$ belong to $K_A^B(\mathbf{R}^n)$; so that using $u+\varphi$ and $u-\varphi$ as test functions in (2.1) we obtain (2.4).

Theorem 2.6. *Let A and B be two bounded subsets of $\mathbf{R}^n$, A compatible with B, and let u be the L-capacitary potential of A in B. Then there exist two positive bounded Radon measures ν and λ such that $\nu - \lambda \in H^{-1}(\mathbf{R}^n)$ and*

$$Lu = \nu - \lambda \tag{2.5}$$

in the sense of distributions. Moreover, $\mathrm{supp}\,\nu \subseteq \partial A$, $\mathrm{supp}\,\lambda \subseteq \partial B$, $\nu(E) = \lambda(E) = 0$ *for every Borel set E of capacity zero, and*

$$a(u,v) = \int_{\mathbf{R}^n} v\,d\nu - \int_{\mathbf{R}^n} v\,d\lambda \qquad \forall v \in H^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n). \tag{2.6}$$

Finally, $u=1$ ν-a.e. in $\mathbf{R}^n$ and $u=0$ λ-a.e. in $\mathbf{R}^n$.

Proof. By Theorem 2.1 the function u coincides with the solution of problem (2.2). Let $\varphi \in C_0^\infty(\mathbf{R}^n)$, with $\varphi \geq 0$. Clearly $u\varphi + u \in H_A^B(\mathbf{R}^n)$; so that, by (2.2), we have

$$a(u, u\varphi) \geq 0 \qquad \forall \varphi \in C_0^\infty(\mathbf{R}^n),\ \varphi \geq 0 .$$

Thus, by the Riesz representation theorem, there exists a positive Radon measure ν on $\mathbf{R}^n$ such that

$$a(u, u\varphi) = \int_{\mathbf{R}^n} \varphi \, d\nu \qquad \forall \varphi \in C_0^\infty(\mathbf{R}^n) . \tag{2.7}$$

Similarly, for every $\varphi \in C_0^\infty(\mathbf{R}^n)$, with $\varphi \geq 0$, we have that $a(u, (1-u)\varphi) \leq 0$. So that there exists a positive Radon measure λ on $\mathbf{R}^n$ such that

$$a(u, (1-u)\varphi) = -\int_{\mathbf{R}^n} \varphi \, d\lambda \qquad \forall \varphi \in C_0^\infty(\mathbf{R}^n) . \tag{2.8}$$

Then, by (2.7) and (2.8), we obtain

$$a(u, \varphi) = a(u, u\varphi) + a(u, (1-u)\varphi) = \int_{\mathbf{R}^n} \varphi \, d\nu - \int_{\mathbf{R}^n} \varphi \, d\lambda \tag{2.9}$$

for every $\varphi \in C_0^\infty(\mathbf{R}^n)$. This implies that $Lu = \nu - \lambda$ in the sense of distributions and that $\nu - \lambda \in H^{-1}(\mathbf{R}^n)$.

For every $\varphi \in C_0^\infty(\mathbf{R}^n)$ with $\varphi = 0$ in A, by (2.4) and (2.7), we have that $\int_{\mathbf{R}^n} \varphi \, d\nu = 0$, thus $\operatorname{supp} \nu \subseteq \overline{A}$. In the same way, taking $\varphi \in C_0^\infty(\mathbf{R}^n)$ with $\varphi = 0$ in B^c, by (2.4) and (2.8) we obtain that $\operatorname{supp} \lambda \subseteq \big(\operatorname{int}(B)\big)^c$, where $\operatorname{int}(B)$ denotes the interior of B. Moreover, since $\operatorname{supp} \nu \subseteq \overline{A} \subseteq \overline{B}$ and $u = 0$ q.e. in B^c, for every $\varphi \in C_0^\infty(\mathbf{R}^n)$, with $\varphi = 0$ in $\overline{B}$, by (2.9) we have

$$\int_{\mathbf{R}^n} \varphi \, d\lambda = -a(u, \varphi) = 0 ,$$

thus $\operatorname{supp} \lambda \subseteq \partial B$. Similarly, since $Du = 0$ in $\operatorname{int}(A)$ and $\operatorname{supp} \lambda \subseteq \big(\operatorname{int}(A)\big)^c$, for every $\varphi \in C_0^\infty(\mathbf{R}^n)$, with $\varphi = 0$ in $\big(\operatorname{int}(A)\big)^c$, by (2.9) we get

$$\int_{\mathbf{R}^n} \varphi \, d\nu = a(u, \varphi) = 0 ,$$

hence $\operatorname{supp} \nu \subseteq \partial A$. In particular λ and ν are finite measures. Let us prove now that the measures ν and λ vanish on all sets of capacity zero. To this aim it is sufficient to prove that $\nu(C) = 0$ and $\lambda(C) = 0$ for every compact set C of capacity zero. Let us fix such a set C and let us consider a bounded open set Ω containing C. It is possible to construct a sequence (φ_h) of functions in $C_0^\infty(\Omega)$ such that $0 \le \varphi_h \le 1$ in Ω, $\varphi_h = 1$ in C, and (φ_h) converges to zero strongly in $H_0^1(\Omega)$. Then by (2.7), for every $h \in \mathbf{N}$, we have

$$\nu(C) \le \int_{\mathbf{R}^n} \varphi_h \, d\nu = a(u, u\varphi_h) .$$

Taking the limit as $h \to \infty$ we obtain $\nu(C) = 0$. In the same way we can prove that $\lambda(C) = 0$. Since λ and ν are finite, we have that every $\varphi \in H^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ belongs to $L^1_\lambda(\mathbf{R}^n)$ and to $L^1_\nu(\mathbf{R}^n)$, and thus, by an easy approximation argument, we obtain (2.10)

$$a(u, u\varphi) = \int_{\mathbf{R}^n} \varphi \, d\nu , \qquad a(u, (1-u)\varphi) = -\int_{\mathbf{R}^n} \varphi \, d\lambda \qquad \forall \varphi \in H^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) ,$$

which implies (2.6).

Finally let us consider the quasi open set $\{u < 1\}$. By Lemma 1.2 there exists an increasing sequence (v_h) of functions of $H^1(\mathbf{R}^n)$, with $0 \le v_h \le 1_{\{u<1\}}$, converging to $1_{\{u<1\}}$ q.e. in $\mathbf{R}^n$. Since $uv_h = 0$ q.e. in $A \cup B^c$, from (2.4) and (2.10) we obtain

$$0 = a(u, uv_h) = \int_{\mathbf{R}^n} v_h \, d\nu .$$

Taking the limit as $h \to \infty$ we have $\nu(\{u < 1\}) = 0$ and thus $u \ge 1$ ν-a.e. in $\mathbf{R}^n$. Since $u \le 1$ q.e. in $\mathbf{R}^n$ (Theorem 2.1), we have also $u \le 1$ ν-a.e. in $\mathbf{R}^n$, and hence $u = 1$ ν-a.e. in $\mathbf{R}^n$

Similarly, let (z_h) be an increasing sequence of functions of $H^1(\mathbf{R}^n)$, with $0 \le z_h \le 1_{\{u>0\}}$, converging to $1_{\{u>0\}}$ q.e. in $\mathbf{R}^n$. Then $(1-u)z_h = 0$ in $A \cup B^c$ and from (2.4) and (2.10) we obtain

$$\int_{\mathbf{R}^n} z_h \, d\lambda = 0 .$$

Taking the limit as $h \to \infty$ we conclude that $u = 0$ λ-a.e. in $\mathbf{R}^n$. □

The measures ν and λ defined by (2.5) are called the *inner* and the *outer L-capacitary distribution of A in B*.

Remark 2.7. It is easy to see that if A is relatively compact in the interior of B, then $\nu \in H^{-1}(\mathbf{R}^n)$ and $\lambda \in H^{-1}(\mathbf{R}^n)$. We shall see in the Appendix, with an explicit counterexample, that, given a bounded open set B, it is possible to construct an open set A contained in B and compatible with B, such that the inner and the outer capacitary distributions of A in B are not in $H^{-1}(\mathbf{R}^n)$.

Remark 2.8. Let V an open set such that $\overline{V} \cap \overline{A} = \emptyset$. Suppose that $V \cap \partial B$ is a C^1 manifold and that $V \cap B$ lies, locally, on one side of $V \cap \partial B$. Then

$$\lambda(E) = -\int_{\partial B \cap E} \frac{\partial u}{\partial n_L} \, d\sigma \qquad \text{for every Borel set } E \subseteq V,$$

where n_L is the (outer) conormal vector on ∂B associated with the operator L, and σ is the $(n-1)$-dimensional measure on ∂B.

Proposition 2.9. *Let A and B be two bounded subsets of $\mathbf{R}^n$, A compatible with B, and let ν and λ be the inner and the outer capacitary distributions of A in B. Then*

$$\operatorname{cap}^L(A,B) = \nu(\partial A) = \nu(\mathbf{R}^n) = \lambda(\partial B) = \lambda(\mathbf{R}^n). \tag{2.11}$$

Proof. Let u be the capacitary potential of A in B. Since, by Theorem 2.6, $u = 1$ ν-a.e. in $\mathbf{R}^n$ and $u = 0$ λ-a.e. in $\mathbf{R}^n$, by (2.6) we obtain

$$\operatorname{cap}^L(A,B) = a(u,u) = \int_{\mathbf{R}^n} u \, d\nu = \nu(\mathbf{R}^n) = \nu(\partial A),$$

where in the last equality we used the fact that $\operatorname{supp} \nu \subseteq \partial A$. In order to prove the other equalities in (2.11) let us consider a function $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that $\varphi = 1$ in $\overline{B}$. Since $u = 0$ q.e. in B^c, by (2.8) we have

$$\operatorname{cap}^L(A,B) = a(u,u) = -a(u,(1-u)\varphi) = \int_{\mathbf{R}^n} \varphi \, d\lambda = \lambda(\partial B) = \lambda(\mathbf{R}^n),$$

where in the last two equalities we used the fact that $\operatorname{supp} \lambda \subseteq \partial B$. □

Proposition 2.10. *Let u_1 and u_2 be two functions in $H^1_{loc}(\mathbf{R}^n)$. If $u_1 \le u_2$ q.e. in A, then $u_1 \le u_2$ ν-a.e. in $\mathbf{R}^n$; if $u_1 = u_2$ q.e. in A, then $u_1 = u_2$ ν-a.e. in $\mathbf{R}^n$. Likewise, if $u_1 \le u_2$ q.e. in B^c, then $u_1 \le u_2$ λ-a.e. in $\mathbf{R}^n$; if $u_1 = u_2$ q.e. in B^c, then $u_1 = u_2$ λ-a.e. in $\mathbf{R}^n$.*

Proof. Since ν and λ have compact support, it is not restrictive to suppose $u_1, u_2 \in H^1(\mathbf{R}^n)$. It is clearly enough to prove only the statements concerning inequalities. Let us prove the first assertion, assuming that $u_1 \le u_2$ q.e. in A. Let $v = (u_2 - u_1 + 1)^+ \wedge 1$. Then $0 \le v \le 1$ q.e. in $\mathbf{R}^n$ and $v = 1$ q.e. in A. So that it is sufficient to prove that $v \ge 1$ ν-a.e. in $\mathbf{R}^n$. Suppose that $\nu(\{v < 1\}) > 0$. Let u be the capacitary potential of A in B. We can use the function uv as test function in problem (2.1), hence $a(u,u) \le a(u,uv)$. By Proposition 2.9 and by (2.10), we obtain

$$\nu(\mathbf{R}^n) \;=\; \mathrm{cap}^L(A,B) \;=\; a(u,u) \;\le\; a(u,uv) \;=\; \int_{\mathbf{R}^n} v\,d\nu < \nu(\mathbf{R}^n)\,.$$

This contradiction implies that $\nu(\{v<1\})=0$, hence $v \ge 1$ ν-a.e. in $\mathbf{R}^n$.

In order to prove the assertion concerning B^c, we assume that $u_1 \le u_2$ q.e. in B^c and we consider, as above, the function $v=(u_2-u_1+1)^+\wedge 1$. In this case we have $0\le v\le 1$ q.e. in $\mathbf{R}^n$ and $v=1$ q.e. in B^c. Then, taking $(u-1)v+1$ as test function in (2.1), by Proposition 2.9 and by (2.10) we have

$$\lambda(\mathbf{R}^n) \;=\; a(u,u) \;\le\; a(u,(u-1)v+1) \;=\; a(u,(u-1)v) \;=\; \int_{\mathbf{R}^n} v\,d\lambda\,.$$

This implies that $\lambda(\{v<1\})=0$ and concludes the proof. □

Remark 2.11. Proposition 2.10 and (2.6) imply that

$$a(u,\varphi) \;=\; \int_{\mathbf{R}^n} \varphi\,d\nu \qquad \forall \varphi \in H^1_{loc}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)\,,\ \varphi=0 \text{ q.e. in } B^c\,,$$

and

$$a(u,\varphi) \;=\; -\int_{\mathbf{R}^n} \varphi\,d\lambda \qquad \forall \varphi \in H^1_{loc}(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)\,,\ \varphi=0 \text{ q.e. in } A\,.$$

3. The main properties of the L-capacity

In this section we study the properties of $\mathrm{cap}^L(A,B)$, considered as a function of the sets A and B. For the sake of simplicity, in the second part of the section, we keep B fixed and we consider only the dependence on A. Dual statements could be proved by exchanging the roles of A and B.

Theorem 3.1. *Let $A\subseteq B$ be two subsets of $\mathbf{R}^n$. Then $\mathrm{cap}^L(A,B)=\mathrm{cap}^{L^*}(A,B)$.*

Proof. We may assume that A is compatible with B, otherwise the conclusion is trivial. Let u (resp. u^*) be the capacitary potential of A in B with respect to L (resp. L^*), and let ν (resp. ν^*) the inner capacitary distribution of A in B relative to L (resp. L^*). Since u and u^* are equal to 1 q.e. in A and equal to 0 q.e. in B^c, by Proposition 2.10 and Remark 2.11 we obtain

$$\nu(\mathbf{R}^n) \;=\; \int_{\mathbf{R}^n} u^*\,d\nu \;=\; a(u,u^*) \;=\; a^*(u^*,u) \;=\; \int_{\mathbf{R}^n} u\,d\nu^* \;=\; \nu^*(\mathbf{R}^n)\,.$$

The conclusion follows from Proposition 2.9. □

We are now in a position to prove the main properties of the L-capacity. We begin with the monotonicity with respect to A.

Theorem 3.2. *Let $A_1 \subseteq A_2 \subseteq B$ be three subsets of $\mathbf{R}^n$. Then*

$$\mathrm{cap}^L(A_1, B) \leq \mathrm{cap}^L(A_2, B).$$

Proof. We may assume that A_2 (hence A_1) is compatible with B. Let u_1 (resp. u_2^*) be the capacitary potential of A_1 (resp. A_2) in B with respect to L (resp. L^*) and let ν_1 (resp. ν_2^*) be the corresponding inner capacitary distribution. Since $u_2^* = 1$ q.e. in $A_2 \supseteq A_1$ and $u_1 \leq 1$ q.e. in $\mathbf{R}^n$, while $u_2^* = u_1 = 0$ q.e. in B^c, by Proposition 2.10 and Remark 2.11 we have

$$\nu_1(\mathbf{R}^n) = \int_{\mathbf{R}^n} u_2^* \, d\nu_1 = a(u_1, u_2^*) = a^*(u_2^*, u_1) = \int_{\mathbf{R}^n} u_1 \, d\nu_2^* \leq \nu_2^*(\mathbf{R}^n).$$

The conclusion follows from Proposition 2.9 and Theorem 3.1. □

Theorem 3.3. *Let $A \subseteq B_1 \subseteq B_2$ be three subsets of $\mathbf{R}^n$. Then*

$$\mathrm{cap}^L(A, B_2) \leq \mathrm{cap}^L(A, B_1).$$

Proof. Clearly it is not restrictive to suppose that A is compatible with B_1 (hence with B_2). Let u_1 (resp. u_2^*) be the capacitary potential of A in B_1 (resp. B_2) with respect to L (resp. L^*) and let λ_1 (resp. λ_2^*) be the corresponding outer capacitary distribution. Since, by Proposition 2.10, $u_1 = 0$ λ_2^*-q.e. in $\mathbf{R}^n$, by Remark 2.11 we obtain

$$\begin{aligned} \lambda_2^*(\mathbf{R}^n) &= \int_{\mathbf{R}^n} (1 - u_1) \, d\lambda_2^* = a^*(u_2^*, u_1 - 1) = \\ &= a(u_1, u_2^* - 1) = \int_{\mathbf{R}^n} (1 - u_2^*) \, d\lambda_1 \leq \lambda_1(\mathbf{R}^n), \end{aligned}$$

and we conclude by Proposition 2.9 and Theorem 3.1. □

We prove now that cap^L is strongly subadditive with respect to A.

Theorem 3.4. *Let A_1 and A_2 be two subsets of B. Then*

$$\mathrm{cap}^L(A_1 \cap A_2, B) + \mathrm{cap}^L(A_1 \cup A_2, B) \le \mathrm{cap}^L(A_1, B) + \mathrm{cap}^L(A_2, B) .$$

Proof. We may assume that A_1 and A_2 are compatible with B. In this case $A_1 \cap A_2$ and $A_1 \cup A_2$ are compatible with B too. Let u_1^* (resp. u_2^*) be the capacitary potential of A_1 (resp. A_2) in B with respect to L^*, and let ν_1^* (resp. ν_2^*) be the corresponding inner capacitary distribution. Moreover, let $u_{A_1\cup A_2}$ (resp. $u_{A_1\cap A_2}$) be the capacitary potential of $A_1 \cup A_2$ (resp. $A_1 \cap A_2$) in B with respect to L, and let $\nu_{A_1\cup A_2}$ (resp. $\nu_{A_1\cap A_2}$) be the corresponding inner capacitary distribution. Using the fact that $u_1^* \wedge u_2^* + u_1^* \vee u_2^* = u_1^* + u_2^*$, by Proposition 2.10 and Remark 2.11 we obtain

$$\begin{aligned} \nu_{A_1\cap A_2}(\mathbf{R}^n) + \nu_{A_1\cup A_2}(\mathbf{R}^n) &= \int_{\mathbf{R}^n} (u_1^* \wedge u_2^*)\, d\nu_{A_1\cap A_2} + \\ + \int_{\mathbf{R}^n} (u_1^* \vee u_2^*)\, d\nu_{A_1\cup A_2} &= a(u_{A_1\cap A_2}, u_1^* \wedge u_2^*) + a(u_{A_1\cup A_2}, u_1^* \vee u_2^*) = \\ &= a(u_{A_1\cap A_2} - u_{A_1\cup A_2}, u_1^* \wedge u_2^*) + a(u_{A_1\cup A_2}, u_1^*) + a(u_{A_1\cup A_2}, u_2^*) = \\ &= a^*(u_1^* \wedge u_2^*, u_{A_1\cap A_2} - u_{A_1\cup A_2}) + \int_{\mathbf{R}^n} u_{A_1\cup A_2}\, d\nu_1^* + \int_{\mathbf{R}^n} u_{A_1\cup A_2}\, d\nu_2^* = \\ &= a^*(u_1^* \wedge u_2^*, u_{A_1\cap A_2} - u_{A_1\cup A_2}) + \nu_1^*(\mathbf{R}^n) + \nu_2^*(\mathbf{R}^n) . \end{aligned} \tag{3.1}$$

In order to conclude, let us prove that $a^*(u_1^* \wedge u_2^*, u_{A_1\cap A_2} - u_{A_1\cup A_2}) \le 0$. Let us fix a bounded open set $\Omega \supset\supset B$ and let us consider the set $H^B_\emptyset(\Omega)$ of all functions $\varphi \in H^1_0(\Omega)$ with $\varphi = 0$ q.e. in B^c. Since u_1^* and u_2^* are solutions of variational inequalities of the type (2.3) with $U = \Omega$, it is easy to see that

$$a_\Omega(u_1^*, \varphi) \ge 0 \qquad \text{and} \qquad a_\Omega(u_2^*, \varphi) \ge 0$$

for every $\varphi \in H^B_\emptyset(\Omega)$ with $\varphi \ge 0$ q.e. in Ω. If B is open this means that $L^* u_1^* \ge 0$ and $L^* u_2^* \ge 0$ in B in the sense of distributions, and this implies $L^*(u_1^* \wedge u_2^*) \ge 0$ in B in the sense of distributions (see [7], Theorem 6.6). If B is not open, we can repeat the proof of Theorem 6.6 of [7], replacing $H^1_0(B)$ with $H^B_\emptyset(\Omega)$, and we still obtain that $a_\Omega(u_1^* \wedge u_2^*, \varphi) \ge 0$ for every $\varphi \in H^B_\emptyset(\Omega)$ with $\varphi \ge 0$ q.e. in Ω. Moreover, by the comparison principle (Lemma 2.4) we have $u_{A_1\cup A_2} \ge u_{A_1\cap A_2}$, and hence $a^*_\Omega(u_1^* \wedge u_2^*, u_{A_1\cap A_2} - u_{A_1\cup A_2}) \le 0$. The conclusion of the theorem follows now from (3.1), Proposition 2.9, and Theorem 3.1. □

The following theorem proves that cap^L is continuous along increasing sequences.

Theorem 3.5. *Let* (A_h) *be an increasing sequence of subsets of* B, *and let* A *be their union. Then*

$$\mathrm{cap}^L(A,B) = \sup_h \mathrm{cap}^L(A_h,B).$$

Proof. By monotonicity (Theorem 3.2) we have $\mathrm{cap}^L(A_h,B) \le \mathrm{cap}^L(A,B)$. Therefore it is enough to prove that

$$\mathrm{cap}^L(A,B) \le \sup_h \mathrm{cap}^L(A_h,B).$$

We may assume that the right hand side is finite, so that each set A_h is compatible with B. Let u_h be the L-capacitary potential of A_h in B. By the comparison principle (Lemma 2.4) the sequence (u_h) is increasing. Therefore it converges pointwise q.e. in $\mathbf{R}^n$ to some function u. Let Ω be any bounded open set such that $B \subset\subset \Omega$. Since $a_\Omega(u_h,u_h) = \mathrm{cap}^L(A_h,B)$, and $\sup_h \mathrm{cap}^L(A_h,B) < +\infty$, the sequence (u_h) is bounded in $H^1_0(\Omega)$. By Lemma 1.1 u is (the quasi continuous representative of) a function of $H^1_0(\Omega)$ and (u_h) converges to u weakly in $H^1_0(\Omega)$. It is easy to see that $u=1$ q.e. in A and $u=0$ q.e. in $\Omega\backslash B$. Thus $u \in K^B_A(\Omega) = K^B_A(\mathbf{R}^n)$ and A is compatible with B.

As u_h satisfies

$$a(u_h, v-u_h) \ge 0 \tag{3.2}$$

for every $v \in K^B_{A_h}(\mathbf{R}^n)$, we have, in particular, that (3.2) holds for every h if $v \in K^B_A(\mathbf{R}^n)$. Hence for any such v, taking the limit in (3.2) as $h \to +\infty$, and using the weak lower semicontinuity of $w \mapsto a(w,w)$ we obtain

$$a(u,u) \le \liminf_{h\to\infty} a(u_h,u_h) \le \liminf_{h\to\infty} a(u_h,v) = a(u,v). \tag{3.3}$$

Thus u is the capacitary potential of A in B and by (3.3) we have

$$\mathrm{cap}^L(A,B) = a(u,u) \le \liminf_{h\to\infty} a(u_h,u_h) = \sup_h \mathrm{cap}^L(A_h,B).$$

This concludes the proof of the theorem. □

Finally we establish the countable subadditivity of the capacity cap^L.

Theorem 3.6. *Let* (A_h) *be a sequence of subset of* B *and* $A \subseteq \bigcup_h A_h$. *Then*

$$\mathrm{cap}^L(A,B) \le \sum_h \mathrm{cap}^L(A_h,B).$$

Proof. The conclusion follows easily from Theorems 3.2, 3.4, 3.5. □

4. Appendix

In this section L is the Laplace operator $-\Delta$, so that the L-capacity coincides with the harmonic capacity considered in Section 1. We construct two bounded open sets A and B, with $A \subseteq B$, such that:

(i) A is compatible with B, i.e., $\mathrm{cap}(A,B) < +\infty$;

(ii) the inner and outer capacitary distributions ν and λ of A in B do not belong to $H^{-1}(\mathbf{R}^n)$.

The set B is just a ball with radius $R > 0$ and with center on the positive x_1-axis at a distance R from the origin, so that $0 \in \partial B$. The set A is the union of a sequence (A_i) of disjoint open balls contained in B. Each ball A_i has center on the positive x_1-axis, radius r_i, and distance from the origin d_i, so that its center has distance $d_i + r_i$ from the origin. We assume that (d_i) and (r_i) tend to zero and that

$$d_{i+1} + 2r_{i+1} < d_i \qquad \forall i \in \mathbf{N}, \tag{4.1}$$

so the ball A_{i+1} lies on the left of the ball A_i. We have to choose the parameters d_i and r_i in such way that (i) and (ii) are satisfied. For every k let U_k be the union of the balls $A_1, \ldots, A_k$. Let us denote by u_k and v_k the capacitary potentials of A_k and U_k in B. Finally, let us fix a non-negative function $w \in C^\infty(\mathbf{R}^n \setminus \{0\}) \cap H^1(\mathbf{R}^n)$ such that $w(x) = \omega(|x|)$, with ω decreasing and with $\lim_{\rho\to 0} \omega(\rho) = +\infty$. We will show that it is possible to choose the parameters d_i and r_i in such a way that:

$$v_k \le \sum_{i=1}^{k} u_i \le \big(1 + \sum_{i=1}^{k-1} \frac{1}{2^k}\big) v_k \qquad \forall k \in \mathbf{N}, \tag{4.2}$$

$$-\sum_{i=1}^{\infty} \int_{\partial B} \frac{\partial u_i}{\partial n}\, d\sigma < +\infty, \tag{4.3}$$

$$-\sum_{i=1}^{\infty} \int_{\partial B} \frac{\partial u_i}{\partial n} w\, d\sigma = +\infty, \tag{4.4}$$

where n is the exterior unit normal to ∂B and σ is the surface measure on ∂B. By this choice of d_i and r_i we will obtain our result. Let us prove this fact. Since $u_i = v_k = 0$ on ∂B, by (4.2) we have

$$-\frac{\partial v_k}{\partial n} \le -\sum_{i=1}^{k} \frac{\partial u_i}{\partial n} \le -\big(1 + \sum_{i=1}^{k-1} \frac{1}{2^k}\big) \frac{\partial v_k}{\partial n} \qquad \text{on } \partial B. \tag{4.5}$$

Then, using Remark 2.8, Proposition 2.9, and Theorem 3.4, from (4.3) we obtain

$$\operatorname{cap}(U_k, B) \le \sum_{i=1}^{k} \operatorname{cap}(A_i, B) \le -\sum_{i=1}^{\infty} \int_{\partial B} \frac{\partial u_i}{\partial n}\, d\sigma < +\infty$$

for every $k \in \mathbf{N}$, and thus A is compatible with B by Theorem 3.5.

Let us denote by v the capacitary potential of A in B. Since, by the maximum principle, $v_k \le v$ in B and $v_k = v = 0$ on ∂B, we have that $-\frac{\partial v_k}{\partial n} \le -\frac{\partial v}{\partial n}$ on ∂B. Moreover, if we denote by λ the outer capacitary distribution of A in B, by Remark 2.8, we get

$$\lambda(E) = -\int_{\partial B \cap E} \frac{\partial v}{\partial n}\, d\sigma$$

for every Borel set E, with $0 \notin \overline{E}$. As $\{0\}$ has capacity zero, the previous formula holds for every Borel set. Since by (4.4) and (4.5)

$$-\int_{\partial B} \frac{\partial v_k}{\partial n} w\, d\sigma \to +\infty$$

as $k \to \infty$, we have

$$\int_{\partial B} w\, d\lambda = +\infty\,.$$

As $w \in H^1(\mathbf{R}^n)$, this implies that $\lambda \notin H^{-1}(\mathbf{R}^n)$. Since $\nu - \lambda \in H^{-1}(\mathbf{R}^n)$ (Theorem 2.6), we have also $\nu \notin H^{-1}(\mathbf{R}^n)$.

It remains to construct (d_i) and (r_i) such that (4.1), (4.2), (4.3), (4.4) are satisfied. From now on we will denote by B_r the open ball of center zero and radius r. Let us fix a sequence of positive numbers (ρ_i) converging to zero. If we choose A_i such that

$$-\sum_{i=1}^{\infty} \int_{\partial B \setminus B_{\rho_i}} \frac{\partial u_i}{\partial n}\, d\sigma < +\infty\,, \tag{4.6}$$

then the conditions

$$-\sum_{i=1}^{\infty} \int_{\partial B \cap B_{\rho_i}} \frac{\partial u_i}{\partial n}\, d\sigma < +\infty \tag{4.7}$$

and

$$-\sum_{i=1}^{\infty} \int_{\partial B \cap B_{\rho_i}} \frac{\partial u_i}{\partial n} w\, d\sigma = +\infty \tag{4.8}$$

clearly imply (4.3) and (4.4). To get (4.6) we need the following lemma.

Lemma 4.1. *There exist three functions $\alpha(\varepsilon)$, $\delta(\varepsilon)$, $\eta(\varepsilon)$, defined for $0<\varepsilon<1$ and converging to 0 as $\varepsilon\to 0$, such that, if E_ε is any subset of $B\cap B_{\alpha(\varepsilon)}$ compatible with B, and z_ε is the capacitary potential of E_ε in B, then*

(a) $z_\varepsilon \le \delta(\varepsilon)$ *q.e. in* $B\backslash B_\varepsilon$

(b) $\displaystyle\int_{\partial B\backslash B_\varepsilon} \frac{\partial z_\varepsilon}{\partial n}\,d\sigma \;\le\; \eta(\varepsilon)$ *for every* $0<\varepsilon<1$.

Proof. Let $\alpha(\varepsilon)=\exp(-1/\varepsilon)$ and let $\zeta_\varepsilon(|x|)$ be the capacitary potential of $B_{\alpha(\varepsilon)}$ in B_{2R}. We set $\delta(\varepsilon)=\zeta_\varepsilon(\varepsilon)$ and $\eta(\varepsilon)=\mathrm{cap}(B_{\alpha(\varepsilon)},B_\varepsilon)$. By direct computation we verify that $\delta(\varepsilon)$ and $\eta(\varepsilon)$ tend to 0 as ε tends to 0. Let $C_\varepsilon=B\cup B_\varepsilon$ and let w_ε the capacitary potential of $B_{\alpha(\varepsilon)}$ in C_ε. By Theorem 3.3 we have that

$$\mathrm{cap}(B_{\alpha(\varepsilon)},C_\varepsilon)\;\le\;\eta(\varepsilon)\,. \tag{4.9}$$

Let E_ε be a subset of $B\cap B_{\alpha(\varepsilon)}$ compatible with B, and let z_ε be the capacitary potential of E_ε in B. By the maximum principle we have $-\frac{\partial w_\varepsilon}{\partial n}\ge 0$ on ∂C_ε and $z_\varepsilon(x)\le w_\varepsilon(x)\le\zeta_\varepsilon(|x|)$ for $x\in B\backslash B_{\alpha(\varepsilon)}$. As $\zeta_\varepsilon(|x|)$ is decreasing with respect to $|x|$, we obtain (a). Since $z_\varepsilon=w_\varepsilon=0$ on $\partial B\backslash B_\varepsilon$, we obtain that $0\le-\frac{\partial z_\varepsilon}{\partial n}\le-\frac{\partial w_\varepsilon}{\partial n}$ on $\partial B\backslash B_\varepsilon$. Finally (4.9) together with Remark 2.8 and Proposition 2.9 implies

$$-\int_{\partial B\backslash B_\varepsilon}\frac{\partial z_\varepsilon}{\partial n}\,d\sigma\;\le\;-\int_{\partial B\backslash B_\varepsilon}\frac{\partial w_\varepsilon}{\partial n}\,d\sigma\;\le\;-\int_{\partial C_\varepsilon}\frac{\partial w_\varepsilon}{\partial n}\,d\sigma\;=\;\mathrm{cap}(B_{\alpha(\varepsilon)},C_\varepsilon)\;\le\;\eta(\varepsilon)\,.$$

□

Let us fix a sequence (ε_i) such that $0<\varepsilon_i<\rho_i$ and $\eta(\varepsilon_i)\le 1/2^i$. If

$$A_i\subseteq B_{\alpha(\varepsilon_i)}\,, \tag{4.10}$$

then by Lemma 4.1 we have

$$-\int_{\partial B\backslash B_{\rho_i}}\frac{\partial u_i}{\partial n}\,d\sigma\;\le\;\frac{1}{2^i}\,. \tag{4.11}$$

This yields (4.6). It remains to find additional conditions on d_i and r_i which imply (4.7) and (4.8).

Let us fix the following notation: $\gamma_i=\omega(\rho_i)$, where $w(x)=\omega(|x|)$ is the function which appears in (4.4), and

$$\psi_i(d_i,r_i)\;=\;-\int_{\partial B\cap B_{\rho_i}}\frac{\partial u_i}{\partial n}\,d\sigma\,.$$

Since ω is decreasing, to obtain (4.8) it is enough to prove that

$$\sum_{i=1}^{\infty} \psi_i(d_i, r_i)\gamma_i \;=\; +\infty\,. \tag{4.12}$$

Since $\gamma_i \to +\infty$ as $i \to +\infty$, there exists a subsequence (γ_{i_k}) such that

$$\frac{1}{\gamma_{i_k}} \;\le\; \frac{1}{2^k} \qquad \forall k \in \mathbf{N}\,.$$

If we define the sequence β_i by

$$\beta_i \;=\; \begin{cases} \frac{1}{\gamma_{i_k}}\,, & \text{if } i = i_k, \\ \frac{1}{\gamma_i} \wedge \frac{1}{2^i}\,, & \text{otherwise,} \end{cases}$$

then $\sum_i \beta_i\gamma_i = +\infty$ and $\sum_i \beta_i < +\infty$. Therefore, if we choose d_i and r_i satisfying

$$\psi_i(d_i, r_i) \;=\; \beta_i \qquad \forall i \in \mathbf{N}\,, \tag{4.13}$$

we obtain (4.7) and (4.12) , and hence also (4.8).

We are now in a position to construct the sequences (d_i) and (r_i) by induction. Suppose that d_i and r_i have already been fixed for every $i = 1, \dots, k-1$, and that they satisfy (4.1), (4.2), (4.10), (4.13) . Let us construct d_k and r_k. Since $u_i(0) = 0$ and u_i is continuous at 0, there exists S_k, $0 < S_k < d_{k-1}$, such that

$$0 \;\le\; \sum_{i=1}^{k-1} u_i(x) \;\le\; \sum_{i=1}^{k-1} \frac{1}{2^i} \qquad \forall x \in B \cap B_{S_k}\,. \tag{4.14}$$

Moreover, by Lemma 4.1(a), if S_k is small enough, then

$$0 \;\le\; u_k(x) \;\le\; \frac{1}{2^{k-1}} \qquad \forall x \in U_{k-1} = \bigcup_{i=1}^{k-1} A_i \tag{4.15}$$

for every pair d_k, r_k such that $A_k \subseteq B_{S_k}$, i.e., $d_k + 2r_k \le S_k$. As $u_k = 1$ in A_k and, by induction,

$$1 \;\le\; \sum_{i=1}^{k-1} u_i(x) \;\le\; 1 + \sum_{i=1}^{k-2} \frac{1}{2^i} \qquad \forall x \in U_{k-1}\,,$$

by (4.14) and (4.15) , we get

$$1 \;\le\; \sum_{i=1}^{k} u_i(x) \;\le\; 1 + \sum_{i=1}^{k-1} \frac{1}{2^i} \qquad \forall x \in U_k\,.$$

Then, taking into account that $v_k = 1$ in U_k, by the maximum principle we obtain (4.2) whenever $A_k \subseteq B_{S_k}$, i.e., $d_k + 2r_k \leq S_k$. We can choose R_k and D_k small enough such that $D_k + 2R_k \leq S_k < d_{k-1}$ and $A_k \subseteq B_{\alpha(\varepsilon_k)}$ (see (4.10)), i.e., $D_k + 2R_k < \alpha(\varepsilon_k)$. Then, for every $r_k \leq R_k$ and $d_k \leq D_k$, (4.1), (4.2), (4.10) are satisfied. It remains to find $r_k \leq R_k$ and $d_k \leq D_k$ such that (4.13) holds. Since $\operatorname{cap}(A_i, B)$ tends to $+\infty$ as $d_i \to 0$, Remark 2.8 and Proposition 2.9, together with (4.11), imply that $\psi_k(\delta, R_k)$ tends to $+\infty$ as $\delta \to 0$. Therefore it is possible to fix $d_k \leq D_k$ such that

$$\psi_k(d_k, R_k) \geq \frac{1}{\gamma_k}.$$

By the definition of β_k we have that $0 < \beta_k \leq \psi_k(d_k, R_k)$. As $\psi_k(d_k, \rho)$ decreases continuously to zero as $\rho \to 0$, it is possible to find $r_k \leq R_k$ such that $\psi_k(d_k, r_k) = \beta_k$. With this choice of d_k and r_k conditions (4.1), (4.2), (4.10), (4.13) are satisfied, and this concludes our construction.

References

[1] BUTTAZZO G., DAL MASO G., MOSCO U.: A derivation theorem for capacities with respect to a Radon measure. *J. Funct. Anal.* **71** (1987), 263-278.

[2] DAL MASO G.: On the integral representation of certain local functionals. *Ricerche Mat.* **32** (1983), 85-113.

[3] DAL MASO G.: Γ-convergence and μ-capacities. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **14** (1987), 423-464.

[4] DAL MASO G., GARRONI A.: New results on the asymptotic behavior of Dirichlet problems in perforated domains. *Math. Mod. Meth. Appl. Sci.*, to appear.

[5] DAL MASO G., GARRONI A.: The capacity method for asymptotic Dirichlet problems. To appear.

[6] DAL MASO G., MOSCO U., VIVALDI M.A.: A pointwise regularity theory for the two-obstacle problem. *Acta Math.* **163** (1989), 57-107.

[7] KINDERLEHRER D., STAMPACCHIA G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York, 1980.

[8] LITTMAN W., STAMPACCHIA G., WEINBERGER H.: Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 43-77.

[9] STAMPACCHIA G.: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189-258.

[10] ZIEMER W. P.: Weakly Differentiable Functions. Springer-Verlag, Berlin, 1989.