Γ-convergence for concentration problems

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1 Introduction

In this lectures we will consider a large class of variational problems in which a concentration phenomenon occurs. This is a typical phenomenon in problems with scaling invariance as, for instance, semilinear problems involving the critical Sobolev exponent. More precisely we will study the asymptotic behaviour of problems of the form

$$S_\varepsilon^F(\Omega) := \sup\left\{\varepsilon^{2^*}\int_\Omega F(u)\,dx\ :\ \int_\Omega |\nabla u|^2 dx \le \varepsilon^2\,,\ u=0 \text{ on } \partial\Omega\right\}, \quad (1)$$

when $\varepsilon \to 0^+$, where Ω is a bounded domain of $\mathbf{R}^n$, $n \geq 3$, and F is a nonnegative upper-semicontinuous function bounded from above by $c|t|^{2^*}$, with $2^* = \frac{2n}{n-2}$ being the critical Sobolev exponent.

If F is a smooth function the maximizers of (1) satisfy the following semilinear equation

$$\begin{cases} -\Delta u = \lambda_\varepsilon f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f = F'$ and the Lagrange multiplier λ_ε tends to $+\infty$ when $\varepsilon \to 0^+$.

The, by now, classical approach for this type of semilinear problems is the *concentration-compactness alternative* due to P.L. Lions [21]. For the case of smooth F with critical growth (e.g. $F(t) = |t|^{2^*}$) he proved that the sequence of maximizers either concentrates at a single point (in a sense that will be clear in Section 2) or is compact in $H_0^1(\Omega)$. In particular in the case $\Omega \neq \mathbf{R}^n$ one can exclude compactness and deduce that only concentration is allowed.

As a particular case of problem (1) with, possibly, non smooth (or degenerate) F, we can also obtain interesting free boundary problems as the *plasma problem* or the *Bernoulli free boundary problem* (see for instance [16], [14] and [9]). The latter can be consider as the weak Euler-Lagrange equation of the following variational problem

$$S_\varepsilon^V = \sup\left\{\varepsilon^{-2^*}|A| \;:\; A\subseteq \Omega\,,\ \mathrm{cap}(A,\Omega)\le\varepsilon^2\right\}\,,$$

where $\mathrm{cap}(A,\Omega)$ denotes the harmonic capacity of the set A. The critical case, corresponding to the choice $F(t)=|t|^{2^*}$, and the Bernoulli problem will be described in details in Section 2 and can be considered as two "extreme" particular cases for problem (1).

In order to deal with the general problem (1), the concentration-compactness alternative has been generalized by Flucher and Müller in [13]. Again one deduce concentration for the sequence of maximizers (or almost maximizers).

In this notes we propose a different method to deduce concentration and, more in general, to study the asymptotic behaviour of problem (1).

We will use the notion of Γ-convergence which is the natural convergence for functionals in order to deduce convergence of extrema.

The idea of Γ-convergence is to substitute a sequence of functionals $\{\mathcal{F}_\varepsilon\}$ by an effective *Γ-limit* functional $\mathcal{F}$ which captures the relevant features of the sequence $\{\mathcal{F}_\varepsilon\}$. In particular, sequences of (almost) maximizers converge to maximum points of $\mathcal{F}$. Therefore, in terms of Γ-convergence, we study the asymptotic behaviour of the functional

$$\mathcal{F}_\varepsilon(u)=\varepsilon^{-2^*}\int_\Omega F(\varepsilon u)\,dx\,,$$

with the constraint $\int_\Omega|\nabla u|^2dx\le 1$ and $u\in H^1_0(\Omega)$.

Analyzing the Γ-limit $\mathcal{F}$ we describe the asymptotic behaviour of $\mathcal{F}_\varepsilon(u_\varepsilon)$ along all weakly converging sequences and we also deduce concentration.

The approach of Γ-convergence for this kind of concentration phenomena is recent and has been successfully used also in the study of Ginsburg-Landau problem by Alberti, Baldo e Orlandi [1]. A delicate point, in general for Γ-convergence and here in particular, is the choice of the topology which should be strong enough in order to assure convergence of maxima and weak enough in order to detect concentration.

A second natural question for this type of concentration results is whether it is possible to characterize the concentration point, in particular whether its position can be influenced by the shape of the domain.

A crucial rôle in the identification of the concentration point is played by the Green's function for the Dirichlet problem with the Laplacian. More precisely we will see that the maximizing sequences concentrates at the harmonic centers of Ω (the minima of the Robin function; i.e., the diagonal of the regular part of the Green's function).

2 Two classical examples

We introduce the problem starting with two examples: the Sobolev inequality and the harmonic capacity.

2.1 Sobolev inequality

Let us fix a domain $\Omega \subset \mathbf{R}^n$, with $n \geq 3$. We know that the Sobolev space $H_0^1(\Omega)$ is embedded continuously in $L^p(\Omega)$ for every $p \leq 2^* = \frac{2n}{n-2}$; i.e., there exists a constant $C_p(\Omega)$, depending on p and Ω, such that

$$\int_\Omega |u|^p dx \leq C_p(\Omega) \left(\int_\Omega |\nabla u|^2 dx \right)^{\frac{p}{2}} \tag{1}$$

for every $u \in H_0^1(\Omega)$. The embedding is also compact for $p < 2^*$. It is well known that for $p = 2^*$ (the so called *critical case*) the embedding is not compact.

There is a large number of interesting and difficult analytical and geometrical problems involving the critical growth and many interesting phenomena arise from this lack of compactness.

Let us consider first the Sobolev inequality in $\mathbf{R}^n$

$$\int_{\mathbf{R}^n} |u|^{2^*} dx \leq S^* \left(\int_{\mathbf{R}^n} |\nabla u|^2 dx \right)^{\frac{2^*}{2}}, \qquad \forall u \in C_0^\infty(\mathbf{R}^n)\,. \tag{2}$$

Inequality (2) holds true for all functions in the Deny space $D^{1,2}(\mathbf{R}^n)$ obtained as the closure of $C_0^\infty(\mathbf{R}^n)$ with respect to the topology induced by the L^2 norm of the gradient ($D^{1,2}(\mathbf{R}^n) = \overline{C_0^\infty(\mathbf{R}^n)}^{\|\nabla u\|_{L^2}}$) and S^* is the best Sobolev constant.

Question 1 Is S^* achieved? In other words, does there exist a function $u \in D^{1,2}(\mathbf{R}^n)$ such that (2) holds with equality?

In the case of $\mathbf{R}^n$ the answer is *yes*. This problem has been solved by Talenti in '76 ([23]), and all the possible solutions have been completely characterized. Namely, all the solutions of the following variational problem

$$S^* = \max \left\{ \frac{\displaystyle\int_{\mathbf{R}^n} |u|^{2^*} dx}{\left(\displaystyle\int_{\mathbf{R}^n} |\nabla u|^2 \right)^{\frac{2^*}{2}}} \ : \ u \in D^{1,2}(\mathbf{R}^n) \right\},$$

or equivalently

$$S^* = \max \left\{ \int_{\mathbf{R}^n} |u|^{2^*} dx \ : \ u \in D^{1,2}(\mathbf{R}^n) \text{ and } \int_{\mathbf{R}^n} |\nabla u|^2 \leq 1 \right\}, \tag{3}$$

have been characterized.

Indeed, one can consider the Euler-Lagrange equation of (3) and, by a rearrangement argument, prove that

$$u_1(x) = \frac{1}{(c^2 + |x|^2)^{\frac{n-2}{2}}}\,, \tag{4}$$

is a solution. Here the constant c is a renormalization which gives

$$\int_{\mathbf{R}^n} |\nabla u_1|^2 dx = 1\,.$$

All other solutions can be obtained by scaling and translating u_1. Indeed, both the Dirichlet integral on $\mathbf{R}^n$ and the L^{2^*} norm are invariant under the following dilations and translations:

$$u(x) = \sigma^{-\frac{n}{2^*}} u_1\left(\frac{x-y}{\sigma}\right) \qquad \text{for } \ \sigma > 0 \ \text{ and } \ y \in \mathbf{R}^n\,.$$

Namely $|\nabla u(x)|^2 = \sigma^{-n}|\nabla u_1\left(\frac{x-y}{\sigma}\right)|^2$ and $|u(x)|^{2^*} = \sigma^{-n}|u_1\left(\frac{x-y}{\sigma}\right)|^{2^*}$, and thus, by a change of variables,

$$\int_{\mathbf{R}^n} |\nabla u|^2 dx = \int_{\mathbf{R}^n} |\nabla u_1|^2 dx \qquad \text{and} \qquad \int_{\mathbf{R}^n} |u|^{2^*} dx = \int_{\mathbf{R}^n} |u_1|^{2^*} dx\,.$$

Let us consider now the case $\Omega \neq \mathbf{R}^n$, for instance consider the case Ω bounded. In this case $H_0^1(\Omega) = D^{1,2}(\Omega)$ and the Sobolev inequality reads

$$\int_\Omega |u|^{2^*} dx \leq S^*(\Omega)\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{2^*}{2}} \qquad \forall u \in H_0^1(\Omega)\,. \tag{5}$$

Remark 1 Again by a scaling argument one can see that the best Sobolev constant does not depend on the domain; i.e.,

$$S^*(\Omega) = S^* \qquad \forall\ \Omega \subseteq \mathbf{R}^n\,.$$

Question 2 Is S^* achieved in Ω?

The answer in this case is *no*. In fact, otherwise, we would find solutions for (3) with compact support and hence not of the form (4). Thus an other question arise naturally.

Question 3 What we can expect from an *optimal sequence*; i.e., a sequence $u_\varepsilon \in H_0^1(\Omega)$ such that

$$\frac{\displaystyle\int_\Omega |u_\varepsilon|^{2^*} dx}{\left(\displaystyle\int_\Omega |\nabla u_\varepsilon|^2 dx\right)^{\frac{2^*}{2}}} = S^* + o(1)\,? \tag{6}$$

We may have a rather precise idea of the situation through the following example: take the sequence v_ε such that $v_\varepsilon(|x|) = \varepsilon^{-\frac{n}{2^*}} u_1(\frac{x}{\varepsilon})$ and $\Omega = B_R$ (B_R denotes the ball centered in the origin of radius R). Clearly we have

$$\lim_{\varepsilon\to 0} \int_{B_R} |v_\varepsilon(|x|)|^{2^*} dx = \lim_{\varepsilon\to 0} \int_{B_{\varepsilon R}} |u_1|^{2^*} dx = S^* .$$

In particular we get that the sequence $|v_\varepsilon|^{2^*}$ converges to $S^*\delta_0$ (δ_0 denotes the Dirac mass at zero) weakly* in the sense of measure. It is then easy to see that $u_\varepsilon(x) := (v_\varepsilon(|x|) - v_\varepsilon(R))_+ \in H^1_0(\Omega)$ satisfies (6). Moreover also u_ε concentrates all the energy at zero; i.e., $|u_\varepsilon|^{2^*} \overset{*}{\rightharpoonup} S^*\delta_0$ and $|\nabla u_\varepsilon|^2 \overset{*}{\rightharpoonup} \delta_0$.

The scaling invariance of the problem is responsible for this phenomenon of concentration and in general for the lack of compactness in the embedding theorem. This lack of compactness can be very well described, and this has been done by P.L. Lions ([21]) by means of the concentration-compactness principle. Generally speaking, it consists in the analysis of the possible ways a bounded sequence of measures can loose compactness. In the special case of the Sobolev embedding theorem he proves in particular the following *Concentration-compactness alternative.*

We fix a sequence $u_\varepsilon \in D^{1,2}(\Omega)$, with $\|\nabla u_\varepsilon\|_{L^2} \le 1$. Up to a subsequence there exists a function $u_0 \in D^{1,2}(\Omega)$ such that

$$u_\varepsilon \rightharpoonup u_0 \quad \text{in} \quad L^{2^*}(\Omega) \qquad \text{and} \qquad \nabla u_\varepsilon \rightharpoonup \nabla u_0 \quad \text{in} \quad L^2(\Omega)\,;$$

i.e., $u_\varepsilon \rightharpoonup u_0$ in $D^{1,2}(\Omega)$. We may also assume that there exist two measures $\mu^*, \nu^* \in \mathcal{M}(\overline{\Omega}) := (C(\overline{\Omega}))'$, such that

$$|u_\varepsilon|^{2^*} dx \overset{*}{\rightharpoonup} \nu^* \quad \text{in} \quad \mathcal{M}(\overline{\Omega}) \qquad \text{and} \qquad |\nabla u_\varepsilon|^2 dx \overset{*}{\rightharpoonup} \mu^* \quad \text{in} \quad \mathcal{M}(\overline{\Omega})\,.$$ [1]

In order to study the possible lack of compactness for u_ε the idea of P.L. Lions is to characterize the measures ν^* in terms of μ^*.

Remark 2 Note that if $\nu^* = |u_0|^{2^*} dx$ then we have compactness in L^{2^*} for u_ε, while if we also know that $\mu^* = |\nabla u_0|^2 dx$ we conclude strong convergence of u_ε in $D^{1,2}(\Omega)$.

In general, by the lower semi-continuity of the norm, we get

$$\mu^* \ge |\nabla u_0|^2 \, dx\,.$$

Thus we can isolate the atoms of μ^*, $\{x_i\}_{i\in J}$, and rewrite μ^* as follows

[1] Recall: We say that a sequence of measures $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$ if and only if

$$\lim_{\varepsilon\to 0} \int_{\overline{\Omega}} \varphi \, d\mu_\varepsilon = \int_{\overline{\Omega}} \varphi \, d\mu \qquad \forall\, \varphi \in C(\overline{\Omega})\,.$$

$$\mu^* = |\nabla u_0|^2 dx + \sum_{i\in J} \mu_i \delta_{x_i} + \widetilde{\mu}\,, \tag{7}$$

where μ_i denotes the positive weight of the atom x_i and $\widetilde{\mu}$ is the non-atomic part of $\mu^* - |\nabla u_0|^2 dx$.

Remark 3 Note that $\widetilde{\mu}$ in general may also contain a part which is absolutely continuous with respect to the Lebesgue measure.

Lemma 4 *([21]) Let $u_\varepsilon, u_0 \in D^{1,2}(\Omega)$ be such that $\int_\Omega |\nabla u_\varepsilon|^2 \leq 1$, $u_\varepsilon \rightharpoonup u_0$ in $D^{1,2}(\Omega)$, $|u_\varepsilon|^{2^*} dx \overset{*}{\rightharpoonup} \nu^*$ and $|\nabla u_\varepsilon|^2 dx \overset{*}{\rightharpoonup} \mu^*$ in $\mathcal{M}(\overline{\Omega})$ for some measures μ^* and ν^*. Assume μ^* be decomposed as in (7); then*

1. *there exist non-negative constants ν_i^* such that*
   i) $\nu^* = |u_0|^{2^*} dx + \sum_{i\in J} \nu_i^* \delta_{x_i}$;
   ii) $|u_\varepsilon - u_0|^{2^*} dx \overset{*}{\rightharpoonup} \sum_{i\in J} \nu_i^* \delta_{x_i}$ *in* $\mathcal{M}(\overline{\Omega})$;
   iii) $0 \leq \nu_i^* \leq S^* (\mu_i)^{\frac{n}{n-2}}$ *for all* $i \in J$.
2. *(Alternative) If $\nu^*(\overline{\Omega}) = S^*$ and $\mu^*(\overline{\Omega}) = 1$; i.e., u_ε is an optimal sequence for the Sobolev embedding, then one of the two following situations is possible*
   a) Concentration: there exists $x_0 \in \overline{\Omega}$ such that $\mu^* = \delta_{x_0}$ and $\nu^* = S^* \delta_{x_0}$;
   b) Compactness: $\nu^* = |u_0|^{2^*} dx$.

Note that in the alternative b), by the optimality of u_ε we also get $\mu^* = |\nabla u_0|^2 dx$ and hence $u_\varepsilon \to u_0$ in $D^{1,2}(\Omega)$.

Remarks 5

1. In the case $\Omega \neq \mathbf{R}^n$, by the fact that the Sobolev constant is never achieved in Ω, we deduce that only alternative a) is possible; i.e., concentration occurs.
2. Part 1 of Lemma 4 is obtained by a fine use of the Sobolev inequality. In particular note that ii) states that the only way a bounded sequence in $D^{1,2}(\Omega)$ can loose compactness in $L^{2^*}(\Omega)$ is by concentration (so an oscillating bounded sequence for which the gradients weak converge but do not concentrate, is always strongly convergent in $L^{2^*}(\Omega)$).

2.2 Capacity and Isoperimetric inequality for the capacity

We now introduce the notion of *harmonic capacity* for which we will see that a similar concentration phenomenon arises.

Definition 6 *(Capacity) Given $\Omega \subseteq \mathbf{R}^n$ and an open set $A \subset \Omega$, the capacity of A with respect to Ω is the following set function*

$$\mathrm{cap}(A, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^2 dx \; : \; u \geq 1 \; a.e. \; in \; \; A \, , \; u \in H_0^1(\Omega) \right\} . \tag{8}$$

Note that the constraint $u \geq 1$ a.e. in A in the definition of the capacity is convex and strongly closed in $H_0^1(\Omega)$ and hence it is closed in the weak topology. So that there exists a unique minimum point for problem (8). It is called the *capacitary potential of A with respect to Ω*.

Remarks 7

1. By a truncation argument we can show that the capacitary potential u_A satisfies $u_A = 1$ a.e. in A.
2. The capacitary potential is also a weak solution of the following Euler-Lagrange equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{A} \\ u = 1 & \text{on } \partial A \\ u = 0 & \text{on } \partial \Omega \, . \end{cases} \tag{9}$$

The capacity theory is a classical tool in potential theory. In the variational approach it is the natural object when studying problems in the Sobolev space $H_0^1(\Omega)$. It is involved in regularity results for elliptic problems, fine behaviour of Sobolev functions, etc. (for a very nice review of this subject see Frehse [17]).

In general it is not very easy to compute explicitly the capacity of a set. Let us consider the easiest case: two concentric balls.

Example 8 Fix $0 < r < R$ and compute the capacity of B_r with respect to B_R. From now on we will denote by

$$K(\rho) = \frac{\gamma_n}{\rho^{n-2}} \qquad \text{with } \gamma_n = \frac{1}{(n-2)|S^{n-1}|}$$

the *fundamental singularity of the Laplacian Δ in $\mathbf{R}^n$*, $n \geq 3$, and then $K(|x-y|)$ will be the *fundamental solution with singularity at y*. In particular $K(|x|)$ is harmonic outside zero. Moreover $K(|x|) - K(R) = 0$ on ∂B_R and $\frac{K(|x|)-K(R)}{K(r)-K(R)} = 1$ on ∂B_r. Thus we have that the capacitary potential of B_r with respect to B_R is given by

$$u(x) = \min \left\{ \frac{K(|x|) - K(R)}{K(r) - K(R)}, 1 \right\} .$$

Then, using the fact that $-\Delta K(|x|) = \delta_0$ in the sense of distributions, we get

$$\operatorname{cap}(B_r, B_R) = \int_{B_R} |\nabla u|^2 dx = \frac{1}{K(r) - K(R)} \int_{B_R \setminus B_r} \nabla K \nabla u \, dx$$
$$= \frac{1}{K(r) - K(R)} .$$

Similarly, we obtain

$$\operatorname{cap}(B_r, \mathbf{R}^n) = \frac{1}{K(r)} .$$

The notion of capacity can be extended to any subset E of Ω as follows

$$\operatorname{cap}(E, \Omega) = \inf\{\operatorname{cap}(A, \Omega) \ : \ A \ \text{open}, \ \ A \supseteq E\} .$$

Proposition 9 *Let A and B be two given subsets of Ω. The following properties hold.*

1. *(Monotonicity) If $A \subset B$ then*
$$\operatorname{cap}(A, \Omega) \leq \operatorname{cap}(B, \Omega) ;$$
2. *(Subadditivity)*
$$\operatorname{cap}(A \cup B, \Omega) \leq \operatorname{cap}(A, \Omega) + \operatorname{cap}(B, \Omega) ;$$
3. *(Scaling property) For any $\rho > 0$ we have*
$$\operatorname{cap}(\rho A, \rho \Omega) = \rho^{n-2} \operatorname{cap}(A, \Omega) ;$$
4. *If B is open and $A \subseteq B \subseteq \Omega$, then*
$$\frac{1}{\operatorname{cap}(A, \Omega)} \geq \frac{1}{\operatorname{cap}(A, B)} + \frac{1}{\operatorname{cap}(B, \Omega)}$$
and equality holds if and only if B is a superlevel of the capacitary potential u_A of A in Ω.

The above properties can be easily deduced from the definition of the capacity and making use of the capacitary potentials.

Note that in particular the capacity is an external measure, but in general it is not additive (it can be proved that it is a measure only on the class of zero capacity sets).

The "equivalent" of the Sobolev inequality in the case of the capacity is the so called *isoperimetric inequality for the capacity*: there exists a constant $S^V(\Omega)$ such that

$$|A| \leq S^V(\Omega) \left(\operatorname{cap}(A, \Omega)\right)^{\frac{2^*}{2}} \qquad \forall \ \ A \subseteq \Omega . \tag{10}$$

Question 4 Is this constant $S^V(\Omega)$ achieved by a non-trivial set A? In other words: there exists a subset A of Ω with $\operatorname{cap}(A, \Omega) > 0$, such that equality holds in (10)?

If $\Omega = \mathbf{R}^n$ the answer is again *yes*. In fact we can also compute it explicitly. It is given by

$$S^V := S^V(\mathbf{R}^n) = \max\left\{|A| : \quad \mathrm{cap}(A, \mathbf{R}^n) \leq 1\right\} .$$

It easy to see, by a symmetrization argument, that the maximum is achieved on balls; i.e. $A = B_R$. Imposing the constraint $\mathrm{cap}(B_R, \mathbf{R}^n) = 1$ the radius R can be computed explicitly and hence the constant S^V turns out to be the following

$$S^V = \omega_n \gamma_n^{\frac{n}{n-2}} , \tag{11}$$

where $\gamma_n = K(1)$ and $\omega_n = |B_1|$.

Remark 10 Note that, thanks to the scaling property of the capacity (see Proposition 9, 4)), the equality in the isoperimetric inequality (10) is achieved for all balls. Indeed, if R is such that $\mathrm{cap}(B_R, \mathbf{R}^n) = 1$ and $|B_R| = S^V$, then

$$|B_{\rho R}| = \rho^n |B_R| = S^V \rho^n = S^V \left(\mathrm{cap}(\rho B_R, \mathbf{R}^n)\right)^{\frac{n}{n-2}} .$$

In the case $\Omega \neq \mathbf{R}^n$ the answer is *no*. The constant S^V is not achieved by a non trivial set A for a reason which is similar to the one we saw in the Sobolev embedding case. Indeed also in this case it can be seen, using the scaling property of the capacity, that

$$S^V(\Omega) = S^V \qquad \forall\ \Omega .$$

Moreover whenever $\mathrm{cap}(\mathbf{R}^n \setminus \Omega, \mathbf{R}^n) \neq 0$ and $\mathrm{cap}(A, \Omega) \neq 0$, it can be seen, by using the maximum principle, that

$$\mathrm{cap}(A, \Omega) > \mathrm{cap}(A, \mathbf{R}^n)$$

and then, for any such A, the equality in (10) is not possible.

Thus also in this case we may wonder which is the behaviour of an optimal sequence; i.e., a sequence of sets A_ε such that

$$\frac{|A_\varepsilon|}{\left(\mathrm{cap}(A_\varepsilon, \Omega)\right)^{\frac{n}{n-2}}} = S^V + o(1) . \tag{12}$$

So, as above, to have an idea let us construct an optimal sequence by choosing $A_\varepsilon = B_{r_\varepsilon}$, with $r_\varepsilon \to 0$. Then, for any $R > 0$,

$$\left(\mathrm{cap}(A_\varepsilon, \Omega)\right)^{\frac{n}{n-2}} = \left(\mathrm{cap}(B_{r_\varepsilon}, \Omega)\right)^{\frac{n}{n-2}} = \left(\frac{r_\varepsilon}{R}\right)^n \left(\mathrm{cap}(B_R, \frac{R}{r_\varepsilon}\Omega)\right)^{\frac{n}{n-2}} .$$

Now it can be proved that

$$\lim_{\varepsilon \to 0} \mathrm{cap}(B_R, \frac{R}{r_\varepsilon}\Omega) = \mathrm{cap}(B_R, \mathbf{R}^n) ,$$

thus, if we choose R such that $|B_R| = S^V$, we clearly have that the sequence satisfies (12).

Notice that the balls B_{r_ε} shrink to the origin. Again we have a concentration phenomenon for the capacitary potentials similar to the case of the Sobolev embedding.

More in general a way to construct an optimal sequence for (10) is to consider the following variational problem

$$S_\varepsilon^V(\Omega) = \varepsilon^{-2^*} \max\left\{|A| \ : \ \ \mathrm{cap}(A, \Omega) = \varepsilon^2\right\} , \tag{13}$$

where the factor ε^{-2^*} is the right scaling passing from the capacity to the volume.

3 The general problem

We are now in a position to introduce a very general class of problems which includes and unifies the two examples shown above as two extreme case of the same phenomenon.

3.1 Variational Formulation

We will consider the following family of variational problem depending on a small parameter $\varepsilon > 0$

$$S_\varepsilon^F(\Omega) = \varepsilon^{-2^*} \sup\left\{\int_\Omega F(u)\, dx \ : \ u \in D^{1,2}(\Omega)\, , \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2\right\}, \tag{14}$$

with the following assumptions:

i) $0 \leq F(t) \leq c|t|^{2^*}$ for every $t \in \mathbf{R}$;
ii) $F \not\equiv 0$ and upper semi-continuous.

To simplify the exposition we also assume

iii) there exist the following two limits

$$F_0(t) = \lim_{t\to 0} \frac{F(t)}{|t|^{2^*}} \quad \text{and} \quad F_\infty(t) = \lim_{t\to\infty} \frac{F(t)}{|t|^{2^*}} .$$

With this formulation we recover the two examples seen before.

Examples 11

1. If $F(t) = |t|^{2^*}$, then (14) gives the Sobolev embedding problem and $S^F_\varepsilon(\Omega) = S^*$ for every $\Omega \subseteq \mathbf{R}^n$.
2. We are allowed to choose F discontinuous, then we recover the case of the capacity taking F of the form

$$F(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1. \end{cases}$$

We claim that, with this choice of F, problem (14) coincides with (13); i.e.,

$$S^V_\varepsilon(\Omega) = \sup\left\{ |\{u \geq 1\}| \ : \ u \in D^{1,2}(\Omega)\,, \ \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2 \right\} \tag{15}$$

Clearly

$$S^V_\varepsilon(\Omega) \geq \sup\left\{ |\{u \geq 1\}| \ : \ u \in D^{1,2}(\Omega)\,, \ \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2 \right\} ;$$

indeed any $u \in D^{1,2}(\Omega)$ satisfying $\int_\Omega |\nabla u|^2 dx \leq \varepsilon^2$ is a good competitor for the computation of the capacity of the set $A = \{u \geq 1\}$ and gives $\mathrm{cap}(A, \Omega) \leq \int_\Omega |\nabla u|^2 dx \leq \varepsilon^2$. On the other hand, if A is an open set such that $\mathrm{cap}(A, \Omega) \leq \varepsilon^2$ and u_A is the capacitary potential of A, then $A \subseteq \{u_A \geq 1\}$ and $\int_\Omega |\nabla u_A|^2 dx = \mathrm{cap}(A, \Omega) \leq \varepsilon^2$; hence we get (15).
3. If $F \in C^1$ we can compute the Euler-Lagrange equation and we have

$$\begin{cases} -\Delta u = \lambda_\varepsilon f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f = F'$ and λ_ε is the Lagrange multiplier (one can show that $\lambda_\varepsilon \approx \varepsilon^{-\frac{4}{n-2}}$).
4. In general the non-differentiability of F allows for free boundary problems. The easiest example is again the case of the capacity, whose Euler-Lagrange equation is the so called *Bernoulli free boundary problem*: look for an open set A and a function u which is the weak solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus A \\ u = 1 & \text{on } \partial A \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = q_\varepsilon & \text{on } \partial A, \end{cases}$$

where q_ε goes to infinity as $\varepsilon \to 0$ and play the role of the Lagrange multiplier. Another classical example covered by the problem is the so called *plasma problem* (see e.g. [16]).

3.2 Generalized Sobolev inequality

Let us see first some general facts related to the scaling properties of our problem. Denote by $S^F := S_1^F(\mathbf{R}^n)$; i.e.,

$$S^F = \sup\left\{\int_{\mathbf{R}^n} F(u)\,dx\ :\ \ u \in D^{1,2}(\mathbf{R}^n)\,,\ \ \int_{\mathbf{R}^n} |\nabla u|^2 dx \le 1\right\}. \qquad (16)$$

Lemma 12 *1. $S_\varepsilon^F(\Omega) \le S^F$ for every $\varepsilon > 0$ and for every $\Omega \subseteq \mathbf{R}^n$;*
2. The following Generalized Sobolev Inequality holds

$$\int_\Omega F(u)\,dx \le S^F\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{2^*}{2}} \qquad \forall\ u \in D^{1,2}(\Omega)\,;$$

3. $S^F \ge F_0 S^*$.

Remark 13 The above lemma is based on the following scaling argument: if $u \in D^{1,2}(\Omega)$ and $s > 0$, define

$$u^s(x) := u\left(\frac{x}{s}\right) \in D^{1,2}(s\Omega)\,.$$

Then we have

$$\int_{s\Omega} F(u^s)\,dx = s^n \int_\Omega F(u)\,dx \quad \text{and} \quad \int_{s\Omega} |\nabla u^s|^2\,dx = s^{n-2}\int_\Omega |\nabla u|^2\,dx\,.$$

In particular if we choose $s = \|\nabla u\|_{L^2}^{\frac{-2}{n-2}}$ then we also obtain $\int_{s\Omega} |\nabla u^s|^2 dx = 1$. Taking u^s as a competitor for (16), one get the generalized Sobolev inequality.

By a cut-off argument the following result can be also proved.

Proposition 14 $\lim_{\varepsilon\to 0} S_\varepsilon^F = S^F$.

The proof of Lemma 12 and Proposition 14 can be found in [13].

3.3 Concentration

From now on we will consider only the case of Ω bounded and we will write $H_0^1(\Omega)$ instead of $D^{1,2}(\Omega)$. Nevertheless most of the result we will present in this lectures have been obtained for general domains, possibly unbounded.

We are interested in the asymptotic behaviour of maximizing sequences for problem (14); i.e., sequences u_ε such that

$$\varepsilon^{-2^*}\int_\Omega F(u_\varepsilon)\,dx = S_\varepsilon^F + o(1)\,, \qquad (17)$$

and, possibly, in saying something about their optimal profile.

The case of $F \in C^1(\mathbf{R})$ has been studied by P.L. Lions [21] as an application of the concentration-compactness principle. The general case has been considered by M. Flucher and S. Müller in '99 [13] (see also the book of Flucher [10] and the references therein), still in the spirit of to concentration-compactness.

Theorem 15 *([13], Theorem 3) Let u_ε be a maximizing sequence (i.e. satisfying (17)), then there exists $x_0 \in \overline{\Omega}$ such that :*

i) The sequence u_ε concentrates at x_0 in the following sense

$$\frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_{x_0} \quad \text{and} \quad \frac{|u_\varepsilon|^{2^*}}{\varepsilon^{2^*}} \overset{*}{\rightharpoonup} S^F \delta_{x_0} \qquad \text{in } \mathcal{M}(\overline{\Omega})\,; \tag{18}$$

ii) Suppose that $S^F > \max\{F_0, F_\infty\}$. Then we identify the optimal profile for the maximizing sequences. Namely there exists a sequence x_ε converging to x_0 such that the sequence $w_\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon^{\frac{2}{n-2}} x)$ is compact in $D^{1,2}(\mathbf{R}^n)$ and every cluster point w is a solution for S^F; i.e., it is a maximum for problem (16).

Remark 16 Let us spend a few words about the condition in part ii) of the theorem. This is a natural condition for those who are familiar with the concentration-compactness approach. It guarantees the existence of a ground state and no concentration of optimal sequences for problem (16). In particular it gives a precise rate of concentration for maximizing sequences of the scaled problem S^F_ε (we will see that this condition can be slightly relaxed).

It is actually easy to see that $S^F \geq \max\{F_0, F_\infty\}$ is always true. It is enough to take an optimal function for S^*, e.g. u_1, and define

$$u^s(x) = s^{-\frac{n}{2^*}} u_1\left(\frac{x}{s}\right) .$$

Then from the generalized Sobolev inequality we obtain $S^F \geq F_0 S^*$ and $S^F \geq F_\infty S^*$ taking the limit as $s \to \infty$ and $s \to 0$ respectively. The idea is that the strict inequality, in applying the concentration-compactness principle to the sequence w_ε, is sufficient to rule out vanishing and concentration.

We will see a proof of the concentration result (part i) of Theorem 15) in terms of the variational convergence introduced by De Giorgi in '75 ([8]), the Γ-convergence.

3.4 Γ^+-convergence

We are considering maximum problems, so the natural variational convergence is Γ^+-convergence. Since in the literature it is mainly Γ^--convergence, the suitable convergence for minimum problems, that is used, we recall quickly the definition and the main properties of Γ^+-convergence.

In what follows X will be a metric space and τ will denote its topology.

Definition 17 *Let $\mathcal{F}_\varepsilon : X \to \overline{\mathbf{R}}$, with $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$, be a family of functionals. We say that the sequence $\mathcal{F}_\varepsilon$ Γ^+-converges to the functional $\mathcal{F} : X \to \overline{\mathbf{R}}$ with respect to τ,*

$$\mathcal{F}_\varepsilon \overset{\Gamma^+(X)}{\longrightarrow} \mathcal{F}\,,$$

if the following two properties are satisfied *i) for every sequence* $x_\varepsilon \xrightarrow{\tau} x$ *we have that*

$$\limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \le \mathcal{F}(x)\,;$$

ii) for every $x \in X$*, there exists a sequence* x_ε*, such that* $x_\varepsilon \xrightarrow{\tau} x$ *and*

$$\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) \ge \mathcal{F}(x)\,;$$

We usually refer to condition i) as the Γ^+*-limsup inequality* and to condition ii) as the *existence of a recovery sequence*.

The idea of Γ^+-convergence is that the functional $\mathcal{F}$ describes the main features of the sequence $\mathcal{F}_\varepsilon$ in terms of maxima; i.e., it gives the best (maximal) behaviour of the sequences $\mathcal{F}_\varepsilon(x_\varepsilon)$ along maximizing sequences x_ε.

Remarks 18

1. The Γ^+-limit $\mathcal{F}$ is upper-semicontinuous with respect to τ.
2. If the topology τ is separable, then Γ^+-convergence is compact.
3. If there exists a compact set $K \subseteq X$ such that

$$\sup_{x\in X} \mathcal{F}_\varepsilon(x) = \sup_{x\in K} \mathcal{F}_\varepsilon(x)\,,$$

   then
   a) $\lim_{\varepsilon\to 0}\sup_{x\in X} \mathcal{F}_\varepsilon(x) = \sup_{x\in X}\mathcal{F}(x) = \max_{x\in K}\mathcal{F}(x)$;
   b) given a maximizing sequence; i.e., any sequence x_ε such that

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0}\sup_{x\in X} \mathcal{F}_\varepsilon(x)\,,$$

   any cluster point x of x_ε is a maximum point for $\mathcal{F}$.

A rigorous and complete treatment of the Γ^--convergence for the case of minimum problems, whose definition is perfectly symmetric to the one we gave for Γ^+, can be found for instance in [7] or [4].

3.5 The concentration result in terms of Γ^+-convergence

It is convenient to modify the functional, rescaling the functions by ε and define

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \varepsilon^{-2^*} \displaystyle\int_\Omega F(\varepsilon u)\,dx & \text{if } \displaystyle\int_\Omega |\nabla u|^2 dx \le 1 \\ 0 & \text{otherwise in } L^{2^*}(\Omega). \end{cases} \tag{19}$$

The idea is to consider the functionals $\mathcal{F}_\varepsilon$ and find a suitable topology and a limit functional in order to capture the behaviour stated in Theorem 15 by Flucher and Müller.

Main requirements:

1) Choose a space X, to which possibly extend the functional $\mathcal{F}_\varepsilon$, rich enough to give information on the maxima;
2) Find a topology which is compact on X in order to get convergence of maxima.

In order to clarify our future choice let us go back to the approach of P.L. Lions. Let u_ε be a sequence satisfying $\int_\Omega |\nabla u_\varepsilon|^2 dx \leq 1$. Define $\mu_\varepsilon = |\nabla u_\varepsilon|^2 dx$ and try to describe the limit of $\varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\, dx$ in terms of the limit of μ_ε. Since $\int_\Omega |\nabla u_\varepsilon|^2 dx \leq 1$, we can always assume that u_ε converges weakly in $H^1_0(\Omega)$ to some function u and that μ_ε converges weakly* in $\mathcal{M}(\overline{\Omega})$ to some measure μ.

As above we may decompose μ as follows

$$\mu = |\nabla u|^2 dx + \sum_{i \in J} \mu_i \delta_{x_i} + \widetilde{\mu}\,, \tag{20}$$

where μ_i denotes the positive weight of the atom x_i and $\widetilde{\mu}$ is the non-atomic part of $\mu - |\nabla u|^2 dx$, and define

$$\nu_\varepsilon = \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\, dx\,.$$

By the generalized Sobolev inequality we have

$$\nu_\varepsilon(\Omega) \leq S^F \mu_\varepsilon(\Omega)$$

and thus we have that up to a subsequence ν_ε converges weakly* in $\mathcal{M}(\overline{\Omega})$ to some measure ν. On the other hand by Lemma 4 we know that

$$\nu^*_\varepsilon = |u_\varepsilon|^{2^*} dx \overset{*}{\rightharpoonup} \nu^* = |u|^{2^*} dx + \sum_{i \in J} \nu^*_i \delta_{x_i}\,;$$

so that by assumption i) on F we get

$$\nu \leq c\, \nu^*\,. \tag{21}$$

This implies that there exists a function $g \in L^1(\Omega)$ and non-negative numbers ν_i such that

$$\nu = g(x)\, dx + \sum_{i \in J} \nu_i \delta_{x_i}\,. \tag{22}$$

Then it is clear that, in order to describe the behaviour of $\varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\, dx$ the weak limit in $H^1_0(\Omega)$ of u_ε is not enough, but we need also to take into account "how" it converges weakly, which is detected by the weak* limit of the measures μ_ε. This suggests the choice of the space $X(\Omega)$ as

$$X(\Omega) \subset H^1_0(\Omega) \times \mathcal{M}(\overline{\Omega})$$

and precisely, in view of the constraint $\int_\Omega |\nabla u|^2 dx \le 1$, we will choose

$$X(\Omega) = \big\{(u,\mu) \in H^1_0(\Omega) \times \mathcal{M}(\overline{\Omega}) \,:\, \mu \ge |\nabla u|^2 dx\,,\ \mu(\overline{\Omega}) \le 1\big\}\,. \tag{23}$$

Consequently the topology τ will be chosen such that

$$(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u,\mu) \qquad \Longleftrightarrow \qquad \begin{cases} u_\varepsilon \rightharpoonup u & \text{in } w - L^{2^*}(\Omega) \\ \mu_\varepsilon \overset{*}{\rightharpoonup} \mu & \text{in } \mathcal{M}(\overline{\Omega}). \end{cases} \tag{24}$$

Remarks 19

1. With this choice of the topology the space $X(\Omega)$ is metric and separable.
2. It is possible to show that all the pairs $(u,\mu) \in X(\Omega)$ can be obtained as a limit with respect to τ of pairs of the form $(u_\varepsilon, |\nabla u_\varepsilon|^2)$.
3. The topology τ is compact in $X(\Omega)$, then the Γ^+-convergence of functionals in this space implies the convergence of maxima.

Now that we have set the right space we have to extend our functional to $X(\Omega)$ and this is done, by a little abuse of notation, in the natural way as follows

$$\mathcal{F}_\varepsilon(u,\mu) = \begin{cases} \varepsilon^{-2^*} \displaystyle\int_\Omega F(\varepsilon u)\, dx & \text{if } \mu = \displaystyle\int_\Omega |\nabla u|^2 dx, \\ \\ 0 & \text{otherwise in } X(\Omega). \end{cases} \tag{25}$$

Then we look for a functional $\mathcal{F}$ which is the Γ^+-limit of $\mathcal{F}_\varepsilon$ in $X(\Omega)$. Clearly this Γ^+-limit has to take into account both, the absolutely continuous part of μ and its atomic part.

Theorem 20 *([2], Theorem 3.1) There exists a functional $\mathcal{F} : X(\Omega) \to \mathbf{R}$ which is the Γ^+-limit of $\mathcal{F}_\varepsilon$ in $X(\Omega)$ and it is given by*

$$\mathcal{F}(u,\mu) := F_0 \int_\Omega |u|^{2^*} dx + S^F \sum_{i\in J} (\mu_i)^{\frac{2^*}{2}}\,. \tag{26}$$

Remark 21 By the Γ^+-convergence result we immediately deduce the concentration result. Indeed, as already observed, we have that

$$\sup_{(u,\mu)\in X(\Omega)} \mathcal{F}_\varepsilon(u,\mu) = S^F_\varepsilon \to \max_{(u,\mu)\in X(\Omega)} \mathcal{F}(u,\mu)\,;$$

and hence, since by Proposition 14 we have that $S^F_\varepsilon \to S^F$, we get $S^F = \max_{(u,\mu)\in X(\Omega)} \mathcal{F}(u,\mu)$. On the other hand by the Sobolev inequality, Lemma 12 and the convexity of the function $|t|^{\frac{2^*}{2}}$ we get

$$\begin{aligned}
\mathcal{F}(u,\mu) &= F_0\int_\Omega |u|^{2^*}\,dx + S^F\sum_{i\in J}(\mu_i)^{\frac{2^*}{2}} \\
\text{(Sobolev inequality)} &\le F_0 S^*\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{2^*}{2}} + S^F\sum_{i\in J}(\mu_i)^{\frac{2^*}{2}} \qquad (27)\\
{\scriptstyle (S^*F_0\le S^F)} &\le S^F\left[\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{2^*}{2}} + \sum_{i\in J}(\mu_i)^{\frac{2^*}{2}}\right]\\
&\le S^F\mu(\overline{\Omega}) \le S^F\,.
\end{aligned}$$

Since the Sobolev constant is not attained in Ω the first inequality is strict unless $u=0$ and the third inequality is strict unless $\mu=\delta_{x_0}$ for some $x_0\in\overline{\Omega}$. In other words

$$\mathcal{F}(u,\mu) = S^F = \max_{X(\Omega)}\mathcal{F} \qquad \Longleftrightarrow \qquad (u,\mu)=(0,\delta_{x_0})\,,$$

which correspond to the concentration of the maximizing sequence at x_0.

Proof (Theorem 20). We give a detailed proof of the Γ^+-limsup inequality (condition i) of Definition 17); i.e. we will prove that for every sequence $(u_\varepsilon,\mu_\varepsilon)\xrightarrow{\tau}(u,\mu)$ then

$$\limsup_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon)\le\mathcal{F}(u,\mu)\,. \qquad (28)$$

We can assume $(u_\varepsilon,\mu_\varepsilon)=(u_\varepsilon,|\nabla u_\varepsilon|^2)$ otherwise the proof is trivial. We already know that in this case

$$\mathcal{F}_\varepsilon(u_\varepsilon,\mu_\varepsilon)=\nu_\varepsilon(\overline{\Omega})=\varepsilon^{-2^*}\!\int_\Omega F(\varepsilon u_\varepsilon)\,dx\to\int_\Omega g\,dx+\sum_{i\in J}\nu_i=\nu(\overline{\Omega}). \qquad (29)$$

We will give the prove in several steps.

Step 1. *(Localization of the generalized Sobolev inequality) For every $\delta>0$ there exists a constant $k(\delta)>0$ such that if $0<r<R$ with $\frac{r}{R}\le k(\delta)$, then for every $x_0\in\Omega$*

$$\int_{B_r(x_0)}F(u)\,dx\le S^F\left(\int_{B_R(x)}|\nabla u|^2dx+\delta\int_\Omega|\nabla u|^2dx\right)^{\frac{2^*}{2}} \qquad (30)$$

for every $u\in H^1_0(\Omega)$.

In order to see this step let define the function

$$\varphi_r^R(x)=\max\left\{\frac{\log|x_0-x|-\log R}{\log r-\log R},1\right\}\,.$$

This function (the n-capacitary potential of the ball B_r with respect to B_R) has the following property

$$\int_{B_R(x_0)} |\nabla \varphi_r^R|^n dx \to 0 \quad \text{as} \quad \frac{r}{R} \to 0 \,.$$

Moreover it is a cut-off function between $B_r(x_0)$ and $B_R(x_0)$; i.e., $\varphi_r^R(x) = 1$ in $B_r(x_0)$ and $\varphi_r^R(x) = 0$ in $\mathbf{R}^n \setminus B_R(x_0)$.

Now by Hölder's inequality and the Sobolev inequality, for any $\beta > 0$ and for any $u \in H_0^1(\Omega)$, we get

$$\begin{aligned} &\int_{B_R(x_0)} |\nabla(\varphi_r^R u)|^2 dx \\ &\leq \left(1 + \frac{1}{\beta}\right) \int_{B_R(x_0)} |\nabla \varphi_r^R|^2 |u|^2 dx + (1+\beta) \int_{B_R(x_0)} |\nabla u|^2 dx \\ &\leq \left(1 + \frac{1}{\beta}\right) \left(\int_{B_R(x_0)} |\nabla \varphi_r^R|^n dx\right)^{\frac{2}{n}} \left(\int_{B_R(x_0)} |u|^{2^*} dx\right)^{\frac{n-2}{2}} \\ &\quad + (1+\beta) \int_{B_R(x_0)} |\nabla u|^2 dx \\ &\leq \int_{B_R(x_0)} |\nabla u|^2 dx + \left(\beta + \left(1 + \frac{1}{\beta}\right) S^* \left[\int_{B_R(x_0)} |\nabla \varphi_r^R|^n dx\right]^{\frac{2}{n}}\right) \int_\Omega |\nabla u|^2 dx \,. \end{aligned}$$

Now if $\beta \leq \delta/2$ and the ratio between r and R is small enough we get

$$\int_{B_R(x_0)} |\nabla(\varphi_r^R u)|^2 dx \leq \int_{B_R(x_0)} |\nabla u|^2 dx + \delta \int_\Omega |\nabla u|^2 dx.$$

Then the conclusion follows applying the Generalized Sobolev inequality to the function $\varphi_r^R u$.

Step 2. *We now prove that*

$$\nu_i \leq S^F (\mu_i)^{\frac{2^*}{2}} . \tag{31}$$

For any $\delta > 0$, for any atom $x_i \in \Omega$, with $i \in J$, and $\frac{r}{R} \leq k(\delta)$ we may apply the localization of the generalized Sobolev inequality (30) to the functions $\varepsilon u_\varepsilon$ and recalling that $\int_\Omega |\nabla u_\varepsilon|^2 dx \leq 1$ we get

$$\varepsilon^{-2^*} \int_{B_r(x_i)} F(\varepsilon u_\varepsilon) \, dx \leq S^F \left(\int_{B_R(x_i)} |\nabla u_\varepsilon|^2 dx + \delta\right) .$$

Taking the limit as $\varepsilon \to 0$ we have

$$\nu(B_r(x_i)) \leq S^F (\mu(B_R(x_i)) + \delta)^{\frac{2^*}{2}} ,$$

thus the conclusion follows taking the limit as $r \to 0$, by the arbitrariness of R and δ.

Step 3. *We finally prove that*

$$g \le F_0|u|^{2^*}. \tag{32}$$

Here the idea in that large values of u_ε do not contribute to the absolutely continuous part of ν.

Fix an open subset U of Ω and $\delta > 0$. Let $t_\delta > 0$ be such that

$$F(t) \le (F_0 + \delta)|t|^{2^*} \qquad |t| < t_\delta\,. \tag{33}$$

Since U is open we have that $\nu(U) \le \liminf_{\varepsilon\to 0} \int_U \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx$ and that $\nu^*(\overline{U}) \ge \limsup_{\varepsilon\to 0} \int_U |u_\varepsilon|^{2^*}\,dx$; hence, we get

$$\begin{aligned}\int_U g\,dx \le \nu(U) &\le \liminf_{\varepsilon\to 0} \int_U \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx \\ &\le \limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon|<t_\delta\}} \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx \\ &\quad + \limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon|\ge t_\delta\}} \varepsilon^{-2^*} F(\varepsilon u_\varepsilon)\,dx \\ &\le (F_0+\delta)\nu^*(\overline{U}) + c\limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon|\ge t_\delta\}} |u_\varepsilon|^{2^*}\,dx\,, \end{aligned} \tag{34}$$

where for the last inequality we used (33) and the growth condition for F. Now, by Lemma 4 and the fact that $\varepsilon u_\varepsilon$ converges to zero in measure we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int_{U\cap\{|\varepsilon u_\varepsilon|\ge t_\delta\}} |u_\varepsilon|^{2^*}\,dx \\ &\quad\le 2^{2^*} \limsup_{\varepsilon\to 0} \left(\int_{U\cap\{|\varepsilon u|\ge t_\delta\}} |u|^{2^*}\,dx + \int_U |u_\varepsilon - u|^{2^*}\,dx \right) \\ &\quad= 2^{2^*} \limsup_{\varepsilon\to 0} \int_U |u_\varepsilon - u|^{2^*}\,dx = \sum_{x_i\in U} \nu_i^* \end{aligned}$$

This together with (34) implies

$$g dx \le F_0|u|^{2^*}dx + \sum_{i\in J}(F_0 + 2^{2^*}c)\nu_i^*\delta_{x_i}$$

which gives (32). Then the prove of the Γ^+-limsup inequality is completed.

As for the prove of the existence of a recovery sequence we just sketch it and we refer to [2] for the details. The main steps are the following:

Step I. In the case $(u,\mu) = (u, |\nabla u|^2 + \widetilde{\mu})$, any sequence such that $(u_\varepsilon, |\nabla u_\varepsilon|^2)$ converges to $(u, |\nabla u|^2 + \widetilde{\mu})$ in the τ topology will do the job. Indeed, by

Lemma 4, we know that u_ε converges strongly in L^{2^*} to u, and this, together with the upper semi-continuity of F and the definition of F_0, implies that

$$\liminf_{\varepsilon\to 0}\int_\Omega \varepsilon^{-2^*}F(\varepsilon u_\varepsilon)\,dx \geq \int_\Omega F_0|u|^{2^*}\,dx\,.$$

Step II. By a localization argument we prove that the Γ^+-limit exists in every pair $(0,\delta_x)$, with $x\in\Omega$, and coincides with $S^F=\mathcal{F}(0,\delta_x)$. In particular this implies the existence of the recovery sequence for such class of pairs.

Step III. All the pairs of the form $(0,\sum_{i\in J}\mu_i\delta_{x_i})$, with $x_i\in\Omega$, are obtained by scaling the recovery sequences for the single atoms in a small ball around the atoms.

Step IV. The general case is recovered by a density lemma which permits to glue the different contributions described above.

4 Identification of the Concentration point

The next natural question, that we will address in this section, is the following: is there a special point of Ω which is "preferred" for the concentration? For instance, if problem (14) has a maximum for any ε, does the shape of the domain Ω influence the concentration of the sequence of maxima?

If we look at the example of the capacity it is clear that in order to maximize the volume for fixed capacity the optimal set has to stay "far" from the boundary. This "far" has to be understood in the sense of potential theory. The object that will play a crucial role to make this concept precise is the Green's function for the Dirichlet problem in Ω with the Laplacian.

Example 22 Let us briefly explicitly show that in the case of the *Volume functional* (see problem (13)), with $\Omega=B_R(0)$, the concentration of optimal sets is at the origin. Indeed let us denote by ρ_ε the radius such that $\mathrm{cap}(B_{\rho_\varepsilon}(0),B_R(0))=\varepsilon^2$ and assume that A_ε is a set for which the maximum for $S^V_\varepsilon(B_R(0))$ is achieved. Let u_ε be its capacitary potential and u^*_ε be its radial symmetrization. In particular we have that u^*_ε is the potential of the set

$$A^*_\varepsilon=\{u^*_\varepsilon\geq 1\}\,.$$

Moreover by symmetrization we also know that

$$\int_{B_R(0)}|\nabla u^*_\varepsilon|^2dx\leq\int_{B_R(0)}|\nabla u_\varepsilon|^2dx=\mathrm{cap}(A_\varepsilon,B_R(0))=\varepsilon^2$$

and the inequality is strict unless u_ε is itself radial. Thus if $A_\varepsilon\neq B_{\rho_\varepsilon}(0)$ and denoting by ρ^*_ε the radius of A^*_ε, then we have

$$\mathrm{cap}(B_{\rho_\varepsilon^*}(0), B_R(0)) < \varepsilon^2 \,,$$

and hence $\rho_\varepsilon^* < \rho_\varepsilon$, which contradicts the maximality of A_ε. In conclusion, recalling that $\mathrm{cap}(B_{\rho_\varepsilon}(0), B_R(0)) = \frac{1}{K(\rho_\varepsilon)-K(R)}$, we proved that the optimal sets are given by

$$B_{\rho_\varepsilon}(0) = \left\{K(|x|) - K(R) > \varepsilon^2\right\} \,, \tag{35}$$

and hence they concentrate at the origin.

In the example above we wrote the optimal sets in the form (35) in order to underline the fact that they are given by super-level sets of the Green's function of $-\Delta$ in $B_R(0)$ and with singularity at the origin. In fact in this case the Green's function is given by

$$G^0_{B_R(0)}(x) = K(|x|) - K(R) \,.$$

We will soon see that this is a general fact; we can construct optimal sets for problem $S_\varepsilon^V(\Omega)$ using the super-level sets of the Green's function in Ω and this is related with the remarkable fact that the rescaled potentials of concentrating sets converges to the Green's function.

4.1 The Green's function and the Robin function

Let us recall the definition and the main properties of the Green's function. Assume Ω be a bounded set with regular boundary (e.g. satisfying the property of the external ball or, more precisely, regular in the sense of Wiener).

Definition 23 *The Green's function $G_\Omega^{x_0}(x)$ for the Dirichlet problem, with the Laplace operator in the domain Ω and singularity x_0, is the function satisfying*

$$G_\Omega^{x_0} \in H^1(\Omega \setminus B_\rho(x_0)) \cap W_0^{1,p}(\Omega) \qquad \forall\ \rho > 0 \ \ \text{and} \ \ \forall\ p < \frac{n}{n-1}$$

and it is a solution in the sense of distribution of the following problem

$$\begin{cases} -\Delta G_\Omega^{x_0} = \delta_{x_0} & \text{in } \Omega \\ G_\Omega^{x_0} = 0 & \text{on } \partial\Omega. \end{cases} \tag{36}$$

The Green's function depends on Ω through its *regular part*. Actually we can rewrite it as

$$G_\Omega^{x_0}(x) = K(|x - x_0|) - H_\Omega(x_0, x) \,, \tag{37}$$

where $K(|\cdot - x_0|)$ is the fundamental solution and $H_\Omega(x_0, \cdot)$, the regular part, is the solution in the sense of H^1 of the following Dirichlet problem

$$\begin{cases} -\Delta H_\Omega(x_0,\cdot) = 0 & \text{in } \Omega \\ G_\Omega^{x_0}(x) = K(|x-x_0|) & \text{if } x \in \partial\Omega. \end{cases} \tag{38}$$

In order to make the ansatz suggested by Example 22 more precise let us fix $x_0 \in \Omega$ and let us consider the set $A_\varepsilon = \{G_\Omega^{x_0} > \varepsilon^{-2}\}$. First we compute $\operatorname{cap}(A_\varepsilon, \Omega)$. Since $G_\Omega^{x_0}$ is harmonic in $\Omega \setminus \{x_0\}$ and it is zero on the boundary of Ω, the potential u_ε of A_ε is given by

$$u_\varepsilon = \varepsilon^2 \left(G_\Omega^{x_0} \wedge \varepsilon^{-2}\right) \in W_0^{1,\infty}(\Omega)\,.$$

Then

$$\begin{aligned} \operatorname{cap}(A_\varepsilon, \Omega) &= \int_\Omega |\nabla u_\varepsilon|^2 dx = \varepsilon^4 \int_{\Omega \setminus A_\varepsilon} |\nabla G_\Omega^{x_0}|^2 dx \\ &= \varepsilon^4 \int_\Omega \nabla G_\Omega^{x_0} \nabla (G_\Omega^{x_0} \wedge \varepsilon^{-2})\, dx = \varepsilon^2\,; \end{aligned}$$

i.e., A_ε is a good competitor for problem $S_\varepsilon^V(\Omega)$. Let us see now heuristically how the volume of the set A_ε depends on x_0. Note that $A_\varepsilon = \{y \in \Omega : K(|x_0 - y|) - H_\Omega(x_0, y) > \varepsilon^{-2}\}$ is contained in a small ball centered in x_0. Indeed $H_\Omega(x_0,\cdot)$ is harmonic, and hence bounded, and $K(|x_0 - \cdot|)$ is radial around x_0. Moreover, by the uniform continuity of $H_\Omega(x_0,\cdot)$ we get that

$$H_\Omega(x_0, y) = H_\Omega(x_0, x_0) + o(1) \quad \text{as } \ |x_0 - y| \to 0\,. \tag{39}$$

Then we can rewrite

$$\begin{aligned} A_\varepsilon &= \left\{\gamma_n |x_0 - y|^{2-n} > \varepsilon^{-2} + H(x_0, y)\right\} \\ &= \left\{ |x_0 - y| < \left[\frac{\gamma_n}{\varepsilon^{-2} + H(x_0,y)}\right]^{\frac{1}{n-2}} \right\} \\ &= \left\{ |x_0 - y| < \varepsilon^{\frac{2}{n-2}} \left[\frac{\gamma_n}{1 + \varepsilon^2 (H(x_0,x_0) + o(1))}\right]^{\frac{1}{n-2}} \right\}\,. \end{aligned}$$

This, recalling that $S^V = \omega_n \gamma_n^{\frac{n}{n-2}}$, implies that

$$\begin{aligned} |A_\varepsilon| &= S^V \varepsilon^{2^*} \left[\frac{1}{1 + \varepsilon^2 H(x_0,x_0) + o(\varepsilon^2)}\right]^{\frac{n}{n-2}} \\ &= S^V \varepsilon^{2^*} \left[1 - \frac{n}{n-2}\varepsilon^2 H(x_0,x_0) + o(\varepsilon^2)\right]\,. \end{aligned}$$

Remark 24 In the computation above, the quantity which determines the volume of A_ε, when x_0 varies in Ω is the regular part of the Green's function, more precisely $H_\Omega(x_0, x_0)$.

Definition 25 *The diagonal of the regular part of the Green's function is called the Robin function of Ω; i.e.,*

$$\tau_\Omega(x) := H_\Omega(x, x)\,.$$

We call the harmonic radius of Ω at x, the positive number $\rho_\Omega(x)$ such that

$$K(\rho_\Omega(x)) = \tau_\Omega(x)\,.$$

The points of Ω where $\tau(x)$ attains its minimum (the maxima for $\rho_\Omega(x)$) are called the harmonic centers of Ω.

Remark 26 The computation that we did before shows that among all the super-level sets of the Green's function, with given capacity, the biggest are those corresponding to the singularity in the harmonic centers of Ω; and hence which concentrates at this points.

In the case of a ball $B_R = B_R(0)$ the Robin function and the harmonic radius can be computed explicitly and they are given by

$$\tau_{B_R}(x) = \gamma_n \left| R - \frac{|x|^2}{R} \right|^{2-n} \quad \text{and} \quad \rho_{B_R}(x) = R - \frac{|x|^2}{R}\,.$$

Hence τ_{B_R} attains its minimum at the origin and tends to infinity when x approaches the boundary.

Note that in general the harmonic radius of a domain Ω at a point x is the radius of the ball whose corresponding Robin function evaluated at the origin agrees with $\tau_\Omega(x)$.

Remarks 27

1. In general, if the boundary of Ω is regular then the Robin function $\tau_\Omega(x)$ tends to infinity as x approaches the boundary. Indeed in this case the regular part of the Green's function, $H_\Omega(x, \cdot)$, attains the boundary condition continuously. We will see that we can extend all these notions to the case of general domains (possibly irregular) and that in some cases this property may be false.
2. In the case of the ball the harmonic center is unique and coincides with the center of the ball. It has been proved by Cardaliaguet and Tahraoni [6] that for any bounded convex domain there exists only one harmonic center.
3. The role of the Green's function for similar problems involving the critical Sobolev exponent was first conjectured by Brezis and Pelletier [5]. In particular the conjecture says that the solutions of the following problem

$$\begin{cases} -\Delta u = u^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega\,, \end{cases} \tag{40}$$

with $p < 2^*$, concentrate at critical points of the Robin function of Ω as $p \to 2^*$. Namely the points x_p where a solution u_p of problem (40) attains the maximum, converge to a critical point of $\tau_\Omega(x)$. This conjecture has been then proved by Rey [22] and Han [18] independently. Later, Flucher and Wei [15] proved that the variational solutions; i.e., the maximizers of

$$S_p(\Omega) = \sup \left\{ \frac{\displaystyle\int_\Omega |u|^p dx}{\left(\displaystyle\int_\Omega |\nabla u|^2 dx\right)^{\frac{p}{2}}} \ : \ u \in H_0^1(\Omega) \ \ u \neq 0 \right\},$$

concentrate at the harmonic center of Ω.

In view of the example of the capacity, we expect for our general variational problem $S_\varepsilon^F(\Omega)$ a result similar to that described in Remark 27 (3).

We know that for any maximizing sequence u_ε, namely $\mathcal{F}_\varepsilon(u_\varepsilon) = S_\varepsilon^F(\Omega) + o(1)$, there exist a subsequence (still denoted by u_ε) and a point $x_0 \in \overline{\Omega}$ such that u_ε concentrates at x_0. On the other hand for any $x_0 \in \overline{\Omega}$ we can construct a maximizing sequence which concentrates at x_0.

Now the question is the following: if we look at maximizing sequences which converge faster, is the concentration point determined by the shape of the domain Ω? This problem has been considered in [11] by means of an asymptotic expansion of the energy. The same result can be stated in terms of Γ^+-convergence (see [2]). In the latter approach a way to select among maximizing sequences is to consider a first order expansion in Γ^+-convergence (a sort of Taylor expansion in Γ^+-convergence).

4.2 Asymptotic expansion in Γ^+-convergence

Assume that a sequence of functionals $\mathcal{F}_\varepsilon$, defined in a metric space X, Γ^+-converges with respect to the topology τ to some functional $\mathcal{F} : X \to \overline{\mathbf{R}}$. Suppose now that, following an ansatz, we know that

$$\sup_X \mathcal{F}_\varepsilon = \max_X \mathcal{F} + O(\lambda_\varepsilon)$$

for some $\lambda_\varepsilon = o(1)$. We then consider the following functional

$$\mathcal{F}_\varepsilon^1(x) = \frac{\mathcal{F}_\varepsilon - \max_X \mathcal{F}}{\lambda_\varepsilon}.$$

If the Γ^+-limit $\mathcal{F}^1$ of $\mathcal{F}_\varepsilon^1$ exists, then it clearly will be finite at most on the maxima of $\mathcal{F}$. Furthermore under the usual compactness condition for the topology, we deduce that

$$\lim_{\varepsilon \to 0} \sup_X \mathcal{F}_\varepsilon^1 = \lim_{\varepsilon \to 0} \frac{\sup_X \mathcal{F}_\varepsilon - \max_X \mathcal{F}}{\lambda_\varepsilon} = \max_X \mathcal{F}^1, \tag{41}$$

which gives an asymptotic expansion for the suprema of $\mathcal{F}_\varepsilon$; i.e.,

$$\sup_X \mathcal{F}_\varepsilon = \max_X \mathcal{F} + \lambda_\varepsilon \max_X \mathcal{F}^1 + o(\lambda_\varepsilon)\,. \tag{42}$$

In addition we have that any maximizing sequence x_ε for $\mathcal{F}^1_\varepsilon$; i.e., such that

$$\mathcal{F}^1_\varepsilon(x_\varepsilon) = \sup_X \mathcal{F}^1_\varepsilon + o(1)\,,$$

converges, up to a subsequence, to a maximum of $\mathcal{F}^1$. In particular such a maximizing sequence will also satisfy

$$\mathcal{F}_\varepsilon(x_\varepsilon) = \sup_X \mathcal{F}_\varepsilon + o(\lambda_\varepsilon)\,. \tag{43}$$

In this sense the first order Γ^+-limit selects, among all maximizing sequences for $\mathcal{F}_\varepsilon$, those which converge faster.

4.3 The result

In our case the scaling suggested by the computation for the super-level set of the Green's function is $\lambda_\varepsilon = \varepsilon^2$. Thus our next goal is to compute the Γ^+-limit of the following functionals

$$\mathcal{F}^1_\varepsilon(u,\mu) = \frac{\mathcal{F}_\varepsilon(u,\mu) - S^F}{\varepsilon^2}\,. \tag{44}$$

In order to obtain a non trivial limit for $\mathcal{F}^1_\varepsilon$ we have to assume an additional condition. Indeed we cannot aspect in general a precise rate of convergence of maximizing sequences. For instance in the critical case $F(t) = |t|^{2^*}$, we may have optimal sequences converging to S^* with any rate. We then need an assumption which forbids this scaling invariance effect. As we already briefly mentioned a sufficient condition is given by

$$S^F > \max\{F_0, F_\infty\} S^*\,. \tag{45}$$

Under this condition we know that there exists a ground state w for S^F. The qualitative behaviour of such a solution has been stated by Flucher and Müller in [12]. Among other things they prove that there exists a point $x_0 \in \mathbf{R}^n$ and a ball $B_{R_0}(x_0)$ such that w is radial around x_0 outside this ball. More precisely they find a constant W_∞ such that

$$w(x) = W_\infty K(|x - x_0|)(1 + o(r^{-2})) \qquad \forall\, x \,:\, r = |x - x_0| \geq R_0; \tag{46}$$

in other words w is asymptotically proportional to the fundamental solution. The constant W_∞ is given by

$$W_\infty^2 = \frac{2(n-1)}{n\, S^F} \int_{\mathbf{R}^n} \frac{F(w(x))}{K(|x|)}\, dx.$$

Set now $w_\infty > 0$ such that

$$w_\infty^2 := \frac{2(n-1)}{n\, S^F} \inf \left\{ \liminf_{k\to\infty} \int_{\mathbf{R}^n} \frac{F(w_k(x))}{K(|x|)}\, dx \right\}, \tag{47}$$

where the infimum is taken among radial maximizing sequences for S^F. By using the concentration compactness principle it is shown in [11] that condition (45) implies

$$0 < w_\infty < +\infty\,. \tag{48}$$

We now are ready to state the theorem on the identification of the concentration point in terms of Γ^+-convergence.

Theorem 28 *([11] and [2]) Under condition (45) we have that there exists the Γ^+-limit $\mathcal{F}^1$ of $\mathcal{F}^1_\varepsilon$ defined by (44) in the space $X(\Omega) = H^1_0(\Omega) \times \mathcal{M}(\overline{\Omega})$ and it is given by*

$$\mathcal{F}^1(u,\mu) = \begin{cases} -\dfrac{n}{n-2} S^F\, w_\infty\, \tau_\Omega(x_0) & \text{if } (u,\mu) = (0,\delta_{x_0}) \\ \\ -\infty & \text{otherwise.} \end{cases}$$

As a consequence of this Γ^+-convergence result we can deduce the result of concentration for sequences of *almost maximizers*; i.e., satisfying

$$\varepsilon^{-2^*} \int_\Omega F(\varepsilon u_\varepsilon)\, dx = S^F_\varepsilon(\Omega) + o(\varepsilon^2)\,. \tag{49}$$

Theorem 29 *Under condition (45), all sequences of almost maximizers, in the sense of (49), up to a subsequence, concentrate at a harmonic center of Ω.*

Proof. Theorem 28 implies that the sequence u_ε, being a maximizing sequence for $\mathcal{F}^1_\varepsilon$, up to a subsequence, must concentrate at a point $x_0 \in \overline{\Omega}$ which satisfies

$$\mathcal{F}^1(0,\delta_{x_0}) = \max_{X(\Omega)} \mathcal{F}^1(u,\mu)\,,$$

and this implies that $\tau_\Omega(x_0) = \min_\Omega \tau_\Omega$; hence we deduce the concentration at a harmonic center of Ω.

In order to obtain the result above, condition (45) can be relaxed assuming directly condition (48). More details about the importance of condition (48) can be found in [11]. Here we will consider in detail only the proof of Theorem 28 in the case of the Volume problem in (13). In this case we already saw that the solution for S^V is given by

$$w(x) = K(|x|) \wedge 1\,,$$

and hence $w_\infty = 1$.

In the following we will prove Theorem 28 in the capacity case. Namely we will prove

(i) (Existence of the recovery sequence) For every $x_0 \in \Omega$ there exists a sequence of sets A_ε which concentrates at x_0 such that $\operatorname{cap}(A_\varepsilon, \Omega) = \varepsilon^2$ and

$$\liminf_{\varepsilon \to 0} \frac{\varepsilon^{-2^*}|A_\varepsilon| - S^V}{\varepsilon^2} \geq -S^V \frac{n}{n-2} \tau_\Omega(x_0)$$

(ii) (Γ^+-limsup inequality) For every sequence of sets A_ε which concentrates at some x_0 and satisfies $\operatorname{cap}(A_\varepsilon, \Omega) \leq \varepsilon^2$, we have

$$\limsup_{\varepsilon \to 0} \frac{\varepsilon^{-2^*}|A_\varepsilon| - S^V}{\varepsilon^2} \leq -S^V \frac{n}{n-2} \tau_\Omega(x_0)\,.$$

Here with concentration of the sets A_ε at x_0, we mean

$$\frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \stackrel{*}{\rightharpoonup} \delta_{x_0}\,,$$

where u_ε denotes the capacitary potential of A_ε, or equivalently the pair $(u_\varepsilon/\varepsilon, |\nabla u_\varepsilon|^2/\varepsilon^2)$ converges to $(0, \delta_{x_0})$ in the topology of $X(\Omega)$.

Proof (i). The existence of the recovery sequence essentially has been already proved when we computed the volume of the super-level sets of the Green's function. Thus the recovery sequence is given by $A_\varepsilon = \{G_\Omega^{x_0} > \varepsilon^{-2}\}$ and we have

$$|A_\varepsilon| \geq \varepsilon^{2^*} S^V \left(1 - \frac{n}{n-2}\tau_\Omega(x_0)\varepsilon^2 + o(\varepsilon^2)\right).$$

The proof of the Γ^+-limsup inequality is based on an asymptotic formula for the capacity of the small sets. The crucial lemma is the following.

Lemma 30 *Let $x_0 \in \Omega$ and let A_ε be a sequence of subsets of Ω, with $|A_\varepsilon| > 0$, such that*

$$\frac{|\nabla u_\varepsilon|^2}{\operatorname{cap}(A_\varepsilon, \Omega)} \stackrel{*}{\rightharpoonup} \delta_{x_0}\,,$$

where u_ε is the corresponding capacitary potential, then

$$\liminf_{\varepsilon \to 0} \frac{1}{\operatorname{cap}(A_\varepsilon^*, \mathbf{R}^n)} + \frac{1}{\operatorname{cap}(A_\varepsilon, \Omega)} \geq \tau_\Omega(x_0)\,. \tag{50}$$

Remark 31 The assumption of concentration for the sets A_ε given in the lemma above can be given in the following weaker form

$$\frac{\chi_{A_\varepsilon}}{|A_\varepsilon|} \stackrel{*}{\rightharpoonup} \delta_{x_0}$$

(which is actually what we need for the proof of the theorem in the general case).

A complete proof of Lemma 30 can be found in [11], Lemma 16. Let us see now how we obtain the proof of the Γ^+-limsup inequality using the lemma.

Proof. (ii) Without loss of generality we may assume that A_ε satisfies

$$\limsup_{\varepsilon\to 0} \frac{\operatorname{cap}(A_\varepsilon, \Omega)}{\varepsilon^2} = 1\,.$$

Otherwise the sequence A_ε will not be a maximizing sequence and the Γ^+-limsup would be trivially satisfied. Then we can apply Lemma 30. By symmetrization we have

$$|A_\varepsilon^*| = |B_{\rho_\varepsilon}| = |A_\varepsilon| \qquad \text{with} \quad \rho_\varepsilon = \left(\frac{|A_\varepsilon|}{\omega_n}\right)^{\frac{1}{n}}$$

and hence

$$\operatorname{cap}(A_\varepsilon^*, \mathbf{R}^n) = \frac{1}{K(\rho_\varepsilon)} = \frac{1}{\gamma_n}\left(\frac{|A_\varepsilon|}{\omega_n}\right)^{\frac{n-2}{n}} = \left(\frac{|A_\varepsilon|}{S^V}\right)^{\frac{n-2}{n}}\,.$$

Thus Lemma 30 implies

$$\left(\frac{S^V}{|A_\varepsilon|}\right)^{\frac{n-2}{n}} - \frac{1}{\varepsilon^2} \geq \tau_\Omega(x_0) + o(1)\,.$$

Then

$$\frac{|A_\varepsilon|}{\varepsilon^{2^*}} \leq \frac{S^V}{[1 + (\tau_\Omega(x_0) + o(1))\varepsilon^2]^{\frac{n}{n-2}}} = S^V\left(1 - \frac{n}{n-2}(\tau_\Omega(x_0) + o(1))\varepsilon^2\right)\,,$$

which gives the Γ^+-limisup inequality.

In the proof of the identification of concentration points for the case of the Volume Functional, Lemma 30 plays a crucial role. It measures the contribution of the boundary of Ω in the computation of the capacity with respect to Ω. We will not give the proof of Theorem 28 in the general case. The proof of the Γ^+-limsup inequality uses similar ideas, but it requires a fine analysis of the asymptotic behaviour of a maximizing sequence and the corresponding super-level sets. We just give a few hints for the construction of the recovery sequence for the general case. This indeed gives us the occasion to recall a rearrangement technique which is one of the main tools to get lower bounds for this kind of problems. This is the classical technique of *harmonic transplantation*. Introduced by Hersch in '69 ([20]) it provides information somehow complementary to the information obtained by radial rearrangement.

4.4 Harmonic Transplantation

Definition 32 *Denote by G_B^0 the Green's function of the ball $B = B_R(0)$ with singularity at 0. Given an arbitrary radial function*

$$U \,:\, B_R(0) \to \mathbf{R}$$

we can write it as

$$U = \varphi \circ G_B^0\,,$$

for a suitable real function φ. Now fix $x_0 \in \Omega$. The harmonic transplantation of U from $(B_R(0), 0)$ to (Ω, x_0) is the function

$$u = \varphi \circ G_\Omega^{x_0} \,:\, \Omega \to \mathbf{R}\,.$$

Theorem 33 *The harmonic transplantation u of U from $(B_R(0), 0)$ to (Ω, x_0) satisfies*

1. *The Dirichlet integral is preserved; i.e.,*
$$\int_\Omega |\nabla u|^2 dx = \int_{B_R(0)} |\nabla U|^2 dx\,;$$
2. *If $R = \rho(x_0)$ is the harmonic radius of x_0 in Ω, then*
$$\int_\Omega F(u)\,dx \geq \int_{B_R(0)} F(U)\,dx$$
for every non-negative function F;
3. *If U_ε is a sequence of radial functions, with $U_\varepsilon = \varphi_\varepsilon \circ G_B^0$, which concentrate at 0 in the sense that $|\nabla U_\varepsilon|^2 \overset{*}{\rightharpoonup} \delta_0$, then the corresponding harmonic transplantation $u_\varepsilon = \varphi \circ G_\Omega^{x_0}$ from $(B_R, 0)$ to (Ω, x_0) are concentrates at x_0.*

Proof. Part 1 is a consequence of the co-area formula. Indeed, by using $G_\Omega^{x_0} \wedge t$ as test function in problem (36) and integrating by parts, it can be easily seen that

$$\int_{\{G_\Omega^{x_0} = t\}} |\nabla G_\Omega^{x_0}|\, d\mathcal{H}^{n-1} = 1 \qquad \forall\ t > 0 \quad \forall\ \Omega\,.$$

Then, using the level sets of the Green's function and recalling that $u = \varphi \circ G_\Omega^{x_0}$ and $U = \varphi \circ G_B^0$, we can write

$$\begin{aligned}\int_\Omega |\nabla u|^2 dx &= \int_\Omega |\varphi'(G_\Omega^{x_0})|^2 |\nabla G_\Omega^{x_0}|^2 dx \\ &= \int_0^{+\infty} |\varphi'(t)|^2 \int_{\{G_\Omega^{x_0}=t\}} |\nabla G_\Omega^{x_0}|\, d\mathcal{H}^{n-1}\, dt = \int_0^{+\infty} |\varphi'(t)|^2 dt\,,\end{aligned}$$

independently of Ω.

Part 2 is a consequence of the following *Mean Value Inequality*: *Fix $x_0 \in \mathbf{R}^n$, $t > 0$ and $\rho > 0$; then among all Ω such that $x_0 \in \Omega$ and $\rho = \rho_\Omega(x_0)$ the following quantity*

$$\int_{\partial\{G_\Omega^{x_0} > t\}} \frac{1}{|\nabla G_\Omega^{x_0}|}\, dx$$

is minimal for $\Omega = B_\rho(x_0)$. This fact can be proved using the isoperimetric inequality on the level sets of the Green's' function and the properties of the harmonic radius (see [3] for a detailed proof). Using this result we then have

$$\int_\Omega F(u)\,dx = \int_0^{+\infty} F(\varphi(t)) \int_{\partial\{G_\Omega^{x_0}>t\}} \frac{1}{|\nabla G_\Omega^{x_0}|}\,dx\,dt \geq \int_{B_{\rho(x_0)}} F(u)\,dx.$$

As we said harmonic transplantation is the basic tool in order to construct a recovery sequence in the general case. To understand how to make use of this tool, let us briefly show the main lines in the construction. Suppose that we are able to construct a recovery sequence in the special case $\Omega = B_R(0)$ with concentration at the center; namely suppose that we have a sequence $U_\varepsilon : B_R(0) \to \mathbf{R}$ of radial functions which concentrates at the origin, with $\int_{B_R} |\nabla U_\varepsilon|^2 dx = 1$ and satisfying

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon^{-2^*}\displaystyle\int_{B_R(0)} F(\varepsilon U_\varepsilon)\,dx - S^F}{\varepsilon^2} \geq -\frac{n}{n-2} w_\infty \tau_{B_R}(0)\,. \qquad (51)$$

Now given $\Omega \subseteq \mathbf{R}^n$, $x_0 \in \Omega$ and fix $R = \rho_\Omega(x_0)$ the harmonic radius of Ω at x_0, by harmonic transplantation we get a sequence $u_\varepsilon : \Omega \to \mathbf{R}$, which concentrates at x_0, such that

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = 1,$$

and from the form of the Robin function for a ball, the definition of the harmonic radius and (51) satisfies

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \frac{\varepsilon^{-2^*}\displaystyle\int_\Omega F(\varepsilon u_\varepsilon)\,dx - S^F}{\varepsilon^2} &\geq \liminf_{\varepsilon\to 0} \frac{\varepsilon^{-2^*}\displaystyle\int_{B_R(0)} F(\varepsilon U_\varepsilon)\,dx - S^F}{\varepsilon^2} \\
&\geq -\frac{n}{n-2} w_\infty \tau_{B_R}(0) \\
&= -\frac{n}{n-2} w_\infty K(\rho(x_0)) \\
&= -\frac{n}{n-2} w_\infty \tau_\Omega(x_0)\,;
\end{aligned}$$

hence $(u_\varepsilon, |\nabla u_\varepsilon|^2)$ is a recovery sequence for $(0, \delta_{x_0})$.

Remark 34 The case $\Omega = B_R(0)$, with concentration in the center can be done by hand, taking into account that the solution w for S^F is radial and behaves as the fundamental solution at ∞. Then one can construct U_ε truncating w and scaling it (see [11] for details).

5 Irregular domains

In this last section we will briefly consider the case of domains Ω with possibly irregular boundary. In order to deal with this case we should be able to define the Robin function and the harmonic radius up to the boundary for possibly irregular domains. Indeed if the domain is irregular in general one can not aspect the harmonic center being attained at an interior point (see Example 35).

With the following example we exhibit a domain whose harmonic center is at the boundary.

Example 35 Let $\Omega_0 = B_1(0)$ and let τ_{Ω_0} be the corresponding Robin function. The harmonic center for Ω_0 is 0 and τ_{Ω_0} is strictly convex. The idea is to construct two symmetric sequences of small balls centered in the points $(\frac{1}{2^n}, 0, \dots, 0)$ and $(-\frac{1}{2^n}, 0, \dots, 0)$ respectively with radii which go to zero, in a way that the set obtained from Ω_0 by subtracting a finite number of symmetric pairs of balls has its unique harmonic center in the origin.

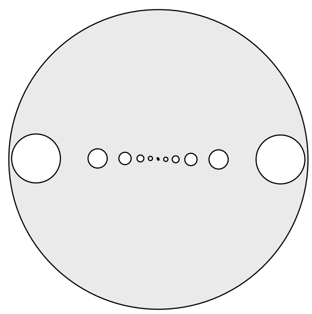

Fig. 1. The set $\Omega_n = \Omega_0 \setminus (\cup_{i=1}^n (\overline{B_{\rho_i}(x_i^+)} \cup \overline{B_{\rho_i}(x_i^-)}))$

Let us denote by $x_n^\pm = (\pm\frac{1}{2^n}, 0, \dots, 0)$, $n \in \mathbb{N}$ and let $\varepsilon_1 > 0$ be such that $0 < \varepsilon_1 < \min_{\Omega_0 \setminus B_{\frac{1}{4}}(0)} \tau_{\Omega_0} - \min_{B_{\frac{1}{4}}(0)} \tau_{\Omega_0}$. Fix $0 < \alpha < 1/2$, let $\rho_1 > 0$ and denote $\Omega_1 = \Omega_0 \setminus (\overline{B_{\rho_1}(x_1^+)} \cup \overline{B_{\rho_1}(x_1^-)})$. It is easy to check that τ_{Ω_1} converges uniformly to τ_{Ω_0}, as ρ_1 tends to zero, in $\Omega_0 \setminus (B_{\rho_1^\alpha}(x_1^+) \cup B_{\rho_1^\alpha}(x_1^-))$ and the same is true for the derivatives. Thus we can choose ρ_1 small enough such that τ_{Ω_1} is strictly convex on $\Omega_0 \setminus (B_{\rho_1^\alpha}(x_1^+) \cup B_{\rho_1^\alpha}(x_1^-))$, $B_{\rho_1^\alpha}(x_1^\pm) \cap B_{\frac{1}{4}}(x_1^\pm) = \emptyset$ and we have

$$\tau_{\Omega_0}(x) \le \tau_{\Omega_1}(x) \le \tau_{\Omega_0}(x) + \frac{\varepsilon_1}{2} \qquad \forall\, x \in \Omega_0 \setminus (B_{\rho_1}(x_1^+) \cup B_{\rho_1}(x_1^-))\,.$$

This implies that the harmonic center of Ω_1 is unique, and arguing by symmetry, we conclude that it is in the origin.

By induction we can construct a sequences $\{\rho_n\}$, such that the sets $\Omega_n = \Omega_0 \setminus (\cup_{i=1}^{n}(\overline{B_{\rho_i}(x_i^+)} \cup \overline{B_{\rho_i}(x_i^-)}))$ have a unique harmonic center at the origin. In particular $\mathrm{dist}(0, \partial\Omega_n) \to 0$ as $n \to \infty$.

Question: what happens in domains with irregular boundary? Can we still prove a concentration result like Theorem 4.3?

Note that all the techniques we used in the previous section make use of the fact that the concentration point is an interior point for Ω. Nevertheless the answer to the second question is *yes*, in order to state and prove the concentration result we need a good definition of the Robin function for irregular domains up to the boundary. This can be done in three steps.

Step 1. For any $x_0 \in \overline{\Omega}$ we can define the regular part of the Green's function $H_\Omega(x_0, \cdot)$ as the solution in the sense of Perron-Wiener-Brelot of the following Dirichlet problem

$$\begin{cases} \Delta_y H_\Omega(x_0, y) = 0 & \text{in } \Omega, \\ H_\Omega(x_0, y) = K(|x_0 - y|) & \text{on } \partial\Omega; \end{cases} \tag{52}$$

i.e., $H_\Omega(x_0, \cdot)$ is the infimum of all superharmonic functions u such that

$$\liminf_{\substack{z \to y \\ z \in \Omega}} u(z) \geq K(|x_0 - y|)$$

for every $y \in \partial\Omega$ (see [19]). Note that $K(|x_0 - \cdot|)$ is an admissible boundary condition in order to get a unique solution for problem (52).

Step 2. We may extend $H(x_0, \cdot)$ to the boundary of Ω as follows

$$\widetilde{H}_\Omega(x_0, y_0) = \liminf_{\substack{y \to y_0 \\ y \in \Omega}} H(x_0, y)\,.$$

Step 3. We now can define the Robin function up to the boundary as

$$\tau_\Omega(x_0) = \widetilde{H}_\Omega(x_0, x_0)\,.$$

With the definition above of τ_Ω we can state and prove Theorem 4.3 for any domain, possibly irregular.

Remark 36 One could be tempted to define the Robin function up to the boundary simply taking the lower semi-continuous extension of $\tau_\Omega(x) = H_\Omega(x, x)$ with $x \in \Omega$. In dimension $n \geq 5$ one can construct an example which shows that the two procedures do not give the same function; i.e., the Robin function defined by Step 3 can be strictly smaller at the boundary than its lower semi-continuous extension from Ω.

The definition of τ_Ω permits also to show that it satisfies the following properties.

Proposition 37 *a) τ_Ω is lower semi-continuous in $\overline{\Omega}$;*
b) For any $\rho > 0$ and any $x_0 \in \overline{\Omega}$ we denote by τ_ρ the Robin function corresponding to the domain $\Omega \cup B_\rho(x_0)$. Then we have that τ_ρ converges increasingly to τ_Ω as $\rho \to 0$.

Remark 38 Note that property b) in the proposition above is essential to extend the result to arbitrary domains. The idea is that it permits to consider a boundary point of Ω as an interior point for a slightly perturbed domain and hence use the same techniques used in the regular case. Indeed first it can be shown that with this definition of τ_Ω, harmonic transplantation works up to the boundary. Second, it permits to extend Lemma 30 for sets which concentrates at points x_0 of the boundary where $\tau_\Omega(x_0) < \infty$. In fact the lemma does not require regularity, but it require x_0 to be an interior point. Then it can be applied to the set $\Omega \cup B_\rho(x_0)$ and gives

$$\liminf_{\varepsilon\to 0} \frac{1}{\mathrm{cap}(A^*_\varepsilon, \mathbf{R}^n)} + \frac{1}{\mathrm{cap}(A_\varepsilon, \Omega \cup B_\rho(x_0))} \geq \tau_\rho(x_0)\,.$$

By the fact that $\mathrm{cap}(A_\varepsilon, \Omega \cup B_\rho(x_0)) < \mathrm{cap}(A_\varepsilon, \Omega)$ and the previous proposition we get

$$\liminf_{\varepsilon\to 0} \frac{1}{\mathrm{cap}(A^*_\varepsilon, \mathbf{R}^n)} + \frac{1}{\mathrm{cap}(A_\varepsilon, \Omega)} \geq \tau_\Omega(x_0)\,.$$

Remark 39 Proposition 37 shows that $\tau_{\Omega_n}(x)$ constructed in Example 35 converges to the Robin function $\tau_{\Omega_\infty}(x)$ for the set

$$\Omega_\infty = \Omega_0 \setminus \overline{(\cup_{i=1}^\infty (B_{\rho_i}(x_i^+) \cup B_{\rho_i}(x_i^-)))}\,,$$

for every $x \in \overline{\Omega}_\infty$. In particular, since $\{\tau_{\Omega_n}\}$ is an increasing sequence, 0 is the harmonic center of Ω_∞ and by construction belongs to the boundary of Ω_∞.

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