A NON HOMOGENEOUS EXTRA TERM FOR THE LIMIT OF DIRICHLET PROBLEMS IN PERFORATED DOMAINS

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Abstract: We study the asymptotic behaviour of Dirichlet problems on varying domains with a nonlinear elliptic differential operator. In the limit problem it appears a nonlinear extra term whose properties are strictly connected with the form of the differential operator. We investigate this connection when the domains have a periodic structure.

1. Introduction to the problem.

The contents of the present paper is related to the study of the asymptotic behaviour of the following problems

$$(1.1)\qquad \begin{cases} -\operatorname{div} a(x, Du_h) = f & \text{in } \Omega_h, \\ u_h \in W_0^{1,p}(\Omega_h)\,, \end{cases}$$

where Ω_h is a sequence of open subsets of a fixed bounded open set $\Omega \subset \mathbf{R}^n$, $a(x,\xi)$ is a Carathéodory function which satisfies standard conditions of monotonicity and of $p-1$ order growth $(1 < p < +\infty)$, and f belongs to $W^{-1,p'}(\Omega)$. For every $h \in \mathbf{N}$ we extend the function u_h to a function in the space $W_0^{1,p}(\Omega)$ by setting $u_h = 0$ on $\Omega \setminus \Omega_h$. By the monotonicity of a it is easy to prove that the sequence (u_h) is bounded in $W_0^{1,p}(\Omega)$ and then, up to a subsequence, converges weakly in $W_0^{1,p}(\Omega)$ to some function $u \in W_0^{1,p}(\Omega)$. The problem is to characterize u as the solution of some limit problem.

This problem has been studied by many authors in the literature under different assumptions on the differential operator and with different approaches (see for instance [10], [4], [11], [7], [5], [6], [2],...)

In the case of monotone operators G. Dal Maso and F. Murat in [8] (see also [5]) proved the following result: assume that the function $a(x,\xi)$ satisfies the homogeneity condition

$$(1.2)\qquad a(x, t\xi) = |t|^{p-2} t\, a(x,\xi) \qquad \forall \xi \in \mathbf{R}^n \ \text{ and } \ \forall t \in \mathbf{R}$$

(this assumption is clearly satisfied if $a(x,\xi) = |\xi|^{p-2}\xi$, i.e., when the operator $-\mathrm{div}\, a(x,Du)$ is the p-Laplacian operator $-\Delta_p$). Then, there exists a subsequence of indices, that we still denote by (h), and a non negative Borel measure μ such that for every $f \in W^{-1,p'}(\Omega)$ the solutions of (1.1) converge weakly in $W_0^{1,p}(\Omega)$ to the solution u of the problem

$$(1.3)\qquad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \\ \displaystyle\int_\Omega a(x,Du)Dv\,dx + \int_\Omega |u|^{p-2}uv\,d\mu = \langle f, v\rangle \qquad \forall v \in W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega)\,. \end{cases}$$

The measure μ which appears in problem (1.3) does not charge sets of p-capacity zero (for the definition and the properties of the p-capacity we refer to [12]).

Let us remark that the class of problems of the type (1.3) is quite large and includes, with a suitable choice of μ, the class of Dirichlet boundary valued problems on open subset of Ω. Namely if E is a closed subset of Ω and μ is defined as follows

$$\mu(B) = \begin{cases} 0\,, & \text{if } p\text{-Cap}(B\cap E) = 0, \\ +\infty\,, & \text{otherwise} \end{cases}$$

for every Borel set $B \subseteq \Omega$, then it is easy to see that problem (1.3) is equivalent to the problem

$$\begin{cases} -\mathrm{div}\, a(x,Du) = f & \text{in } \Omega\setminus E, \\ u \in W_0^{1,p}(\Omega\setminus E)\,, \end{cases}$$

If we assume that μ is a Radon measure, the problem (1.3) may be written as

$$(1.4)\qquad \begin{cases} -\mathrm{div}\ a(x,Du) + |u|^{p-2}u\mu = f & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega)\cap L^p_\mu(\Omega)\,, \end{cases}$$

where the equation above is understood in the sense of distribution. Comparing problems (1.1) and (1.3), we find that in the limit problem it appears an extra term, $|u|^{p-2}u\mu$, which is $(p-1)$-homogeneous.

In [3] we generalize the result of G. Dal Maso and F. Murat to the case in which the function $a(x,\xi)$ does not satisfy the assumption (1.2). More precisely we assume that the function $a(x,\xi)$ satisfies the following conditions if $2 \le p < +\infty$:

(i) there exists a constant $\alpha > 0$ such that

$$(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) \ge \alpha|\xi_1 - \xi_2|^p$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$ and for a.e. $x \in \Omega$;

(ii) there exist a constant $\beta > 0$ and a function $h \in L^p(\Omega)$ such that

$$|a(x,\xi_1) - a(x,\xi_2)| \le \beta(h(x) + |\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$ and for a.e. $x \in \Omega$;
(iii) $a(x, 0) = 0$ for a.e. $x \in \Omega$.

We make analogous assumptions in the case $1 < p < 2$.

Under these conditions on the operators $-\mathrm{div}\, a(x, \cdot)$ the following result holds.

Theorem 1.1. *Let (Ω_h) be a sequence of arbitrary open subsets of Ω. There exist a subsequence of (Ω_h), still denoted by (Ω_h), a non negative Borel measure μ and a function $F : \Omega \times \mathbf{R} \mapsto \mathbf{R}$, such that for every $f \in W^{-1,p'}(\Omega)$ the sequence (u_h) of solutions of the problems* (1.1) *converges weakly in $W_0^{1,p}(\Omega)$ to the solution u, of the following problem*

$$(1.5) \qquad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \\ \displaystyle\int_\Omega a(x, Du) Dv\, dx + \int_\Omega F(x, u) v\, d\mu = \langle f, v \rangle \qquad \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,. \end{cases}$$

Moreover the measure μ is zero on sets of p-capacity zero and the function $F(x, s)$ satisfies the following conditions if $2 \le p < +\infty$:
(I) *for every $s_1, s_2 \in \mathbf{R}$ and for every $x \in \Omega$ we have*

$$|F(x, s_1) - F(x, s_2)| \le L(|s_1| + |s_2|)^{\frac{p-2}{p-1}} |s_1 - s_2|^{\frac{1}{p-1}}\,;$$

(II) *for every $s_1, s_2 \in \mathbf{R}$ and for every $x \in \Omega$ we have*

$$(F(x, s_1) - F(x, s_2))(s_1 - s_2) \ge \alpha |s_1 - s_2|^p\,;$$

(III) $F(x, 0) = 0$ *for every $x \in \Omega$;*
Analogous conditions hold in the case $1 < p < 2$.

This result for general monotone operator without any homogeneity condition has been also proved in [11] and [2] under additional geometric assumptions on the sequence (Ω_h) which assure in particular that the measure μ which appears in the limit problem is a Radon measure.

Remark 1.2. The measure μ in problem (1.5) may be taken equal to the measure which defines the extra term when the operator A is the p-Laplacian operator.

Remark 1.3. The result above has a local character. In the sense that the function F and the measure μ which appear in the extra term in the limit problem do not depend on the domain Ω and on the fact that we consider solutions of Dirichlet problems with boundary value zero on $\partial\Omega$. Namely if (Ω_h) is the sequence given by Theorem 1.1 and (z_h) is a sequence of functions in $W^{1,p}(\Omega)$ satisfying

$$\int_\Omega a(x, Dz_h) Dv\, dx = \langle f, v \rangle$$

for every $v \in W_0^{1,p}(\Omega \cap \Omega_h)$ with compact support in Ω, which converges to some function z weakly in $W^{1,p}(\Omega)$, then z belongs to $L^p_\mu(\Omega')$ for every open set $\Omega' \subset\subset \Omega$ and

$$\int_\Omega a(x, Dz) Dv\, dx + \int_\Omega F(x, z) v\, d\mu = \langle f, v \rangle$$

for every $v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$ with compact support in Ω.

The goal of the present paper is to show in a particular case that the function $F(x,s)$ for a non homogeneous operator can be in general non homogeneous with respect to s (see Section 3). We preliminarily study (in Section 2) the homogenization problem (1.1) in the simple case in which $a(x,\xi)$ does not depend on x and the sequence (Ω_h) is given by $(\Omega \setminus E_h)$, where E_h is the union of closed balls of radius r_h whose centers are periodically distributed.

2. The periodic case.

In this section we consider the asymptotic behaviour of the problem (1.1) in the special case when the sequence (Ω_h) has a periodic structure.

For every $0 < \varepsilon < 1$ let us consider a partition of $\mathbf{R}^n$, with $n > p$, composed of semi-open cubes Q^i_ε, $i \in \mathbf{Z}^n$, of side length 2ε and center $x^i_\varepsilon = 2\varepsilon i$. Let $r = \varepsilon^{n/n-p}$ and for every $i \in \mathbf{Z}^n$ let B^i_r be the closed ball in Q^i_ε of radius r and center x^i_ε. By Q_ε and B_r we shall denote Q^0_ε and B^0_r, respectively.

Let $E_\varepsilon = \bigcup_i B^i_r$. It is well known (see [4] and [9]) that, under this special geometrical assumption, the limit problem which describes the asymptotic behaviour of the Dirichlet problems in $\Omega \setminus E_\varepsilon$ is determined by the measure $C_n dx$ in Ω, where C_n is a positive constant which depends only on n and p. Namely for every $f \in W^{-1,p'}(\Omega)$ if w_ε is the solution of

$$\begin{cases} -\Delta_p w_\varepsilon = f & \text{in } \Omega \setminus E_\varepsilon, \\ w_\varepsilon \in W_0^{1,p}(\Omega \setminus E_\varepsilon) \end{cases}$$

then w_ε converge weakly in $W_0^{1,p}(\Omega)$ to the solution w of the problem

$$\begin{cases} -\Delta_p w + |w|^{p-2} w = f & \text{in } \Omega, \\ w \in W_0^{1,p}(\Omega) \end{cases} \tag{2.1}$$

as $\varepsilon \to 0$.

Let us suppose now that the function $a(x,\xi)$ does not depend on x and let (r_h) be a sequence which converges to zero such that

$$\exists \lim_{h\to\infty} -(r_h)^{p-1} a(-r_h^{-1}\xi) \qquad \forall \xi \in \mathbf{R}^n .$$

Let us define the function $a_\infty(\xi)$ by

$$a_\infty(\xi) = \lim_{h\to\infty} -(r_h)^{p-1} a(-r_h^{-1}\xi) \qquad \forall \xi \in \mathbf{R}^n . \tag{2.2}$$

It is easy to see that a_∞ satisfies (i)–(iii). In the sequel (ε_h) will be the sequence of positive numbers converging to zero defined by

$$r_h = \varepsilon_h^{n/n-p} . \tag{2.3}$$

By Theorem 1.1 and by Remark 1.2 we can suppose that for every $f \in W^{-1,p'}(\Omega)$ the sequence (u_h) of the solutions of the problems

$$\tag{2.4} \begin{cases} -\operatorname{div} a(Du_h) = f & \text{in } \Omega \setminus E_{\varepsilon_h}, \\ u_h \in W_0^{1,p}(\Omega \setminus E_{\varepsilon_h}), \end{cases}$$

converges weakly in $W_0^{1,p}(\Omega)$ to the unique solution u of the problem

$$\tag{2.5} \begin{cases} -\operatorname{div} a(Du) + F(x,u) = f & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

where $F : \Omega \times \mathbf{R} \mapsto \mathbf{R}$ satisfied conditions (I)–(III).

We shall show that since the function a does not depend on x, the function F does not depend on x too (Lemma 2.1). Moreover we shall prove that, in this case, it is possible to construct F by means of the function a_∞ defined in (2.2) (Theorem 2.2).

Lemma 2.1. *The function $F(x,s)$ in problem (2.5) does not depend on x, i.e., $F(x,s) = F(s)$ for every $x \in \mathbf{R}^n$.*

Proof: Let us consider an open set $\Omega' \subset\subset \Omega$. Let $\varepsilon_0 = \operatorname{dist}(\partial\Omega, \partial\Omega')$, then for every $i \in \mathbf{Z}^n$, with $|i| = 1$, and for every $0 < \varepsilon \leq \varepsilon_0$ we have that $\tilde{\Omega} = \Omega' + \varepsilon i \subset\subset \Omega$.

Let $s \in \mathbf{R}$, let s_h^k be the solution of problem

$$\begin{cases} -\operatorname{div} a(Ds_h^k) = k(|s|^{p-2}s - |s_h^k|^{p-2}s_h^k) & \text{in } \Omega' \setminus E_{\varepsilon_h}, \\ s_h^k \in W_0^{1,p}(\Omega' \setminus E_{\varepsilon_h}), \end{cases}$$

and let $\tilde{s}_h^k$ be the solution of the analogous problem corresponding to $\tilde{\Omega}$. By the local character of Theorem 1.1 (see Remarks 1.3) we have that the sequences (s_h^k) and $(\tilde{s}_h^k)$, extended by zero in $(\Omega \setminus \Omega') \cup E_{\varepsilon_h}$ and in $(\Omega \setminus \tilde{\Omega}) \cup E_{\varepsilon_h}$, converge weakly in $W_0^{1,p}(\Omega)$ to the solutions s^k and $\tilde{s}^k$ of problems

$$\tag{2.6} \begin{cases} -\operatorname{div} a(Ds^k) + F(x,s^k) = k(|s|^{p-2}s - |s^k|^{p-2}s^k) & \text{in } \Omega', \\ s^k \in W_0^{1,p}(\Omega'), \end{cases}$$

and

$$\tag{2.7} \begin{cases} -\operatorname{div} a(D\tilde{s}^k) + F(x,\tilde{s}^k) = k(|s|^{p-2}s - |\tilde{s}^k|^{p-2}\tilde{s}^k) & \text{in } \tilde{\Omega}, \\ \tilde{s}^k \in W_0^{1,p}(\tilde{\Omega}), \end{cases}$$

Moreover a penalization result proved in [3] permits us to conclude that

$$\tag{2.8} \lim_{k\to\infty} \int_{\Omega'} \varphi |D(s^k - s)|^p dx + k \int_{\Omega'} \varphi |s^k - s|^p dx = 0$$

for every $\varphi \in C_0^\infty(\Omega')$, with $\varphi \geq 0$, and

$$\lim_{k\to\infty} \int_{\tilde{\Omega}} \varphi |D(\tilde{s}^k - s)|^p dx + k \int_{\tilde{\Omega}} \varphi |\tilde{s}^k - s|^p dx = 0 \tag{2.9}$$

for every $\varphi \in C_0^\infty(\tilde{\Omega})$, with $\varphi \geq 0$. By (2.7) we have

$$\int_{\tilde{\Omega}} a(D\tilde{s}^k) D\varphi \, dx + \int_{\tilde{\Omega}} F(x, \tilde{s}^k)\varphi \, dx = k \int_{\tilde{\Omega}} (|s|^{p-2}s - |\tilde{s}^k|^{p-2}\tilde{s}^k)\varphi \, dx$$

for every $\varphi \in C_0^\infty(\tilde{\Omega})$ and then by changing variables in (2.7) we obtain

$$\int_{\Omega'} a(D\tilde{s}^k(x+\varepsilon i)) D\varphi \, dx + \int_{\Omega'} F(x+\varepsilon i, \tilde{s}^k(x+\varepsilon i))\varphi \, dx =$$
$$= k \int_{\Omega'} (|s|^{p-2}s - |\tilde{s}^k(x+\varepsilon i)|^{p-2}\tilde{s}^k(x+\varepsilon i))\varphi \, dx$$

for every $v \in C_0^\infty(\Omega')$. Thus for every $\varphi \in C_0^\infty(\Omega')$ by (2.6) we have

$$\begin{aligned}
&\int_{\Omega'} |F(x+\varepsilon i, \tilde{s}^k(x+\varepsilon i)) - F(x, s^k(x))||\varphi| \, dx \leq \\
&\leq \int_{\Omega'} |a(D(\tilde{s}^k(x+\varepsilon i)) - a(Ds^k(x))||D\varphi| \, dx + \\
&+ k \int_{\Omega'} ||\tilde{s}^k(x+\varepsilon i)|^{p-2}\tilde{s}^k(x+\varepsilon i) - |s^k(x)|^{p-2}s^k(x)||\varphi| \, dx \,.
\end{aligned} \tag{2.10}$$

Since by (2.8) and (2.9) we easily obtain that the right hand side of (2.10) converges to zero as $k \to \infty$. By condition (I) the sequence $(F(x+\varepsilon i, \tilde{s}^k(x+\varepsilon i)))$ converges to $F(x+\varepsilon i, s)$ and $(F(x, s^k(x)))$ converges to $F(x,s)$ for a.e. x in a compact subset of Ω'; hence, from (2.10), we get

$$F(x+\varepsilon i, s) = F(x,s)$$

for every $x \in \Omega'$ and for every $\varepsilon \leq \varepsilon_0$; so that $F(x,s) = F(s)$ for every $x \in \Omega'$ and the conclusion follows from the arbitrariness of Ω' and s. □

From now on we shall denote by $H_p(\mathbf{R}^n)$ the space of all functions belonging to $L^{p^*}(\mathbf{R}^n)$, $1/p^* = 1/p - 1/n$, whose first order distribution derivatives belong to $L^p(\mathbf{R}^n)$.

We shall say that a function $u : \mathbf{R}^n \mapsto \mathbf{R}$ is Q_{ε_h}-periodic if $u(x + \varepsilon_h i) = u(x)$ for every $x \in \mathbf{R}^n$ and for every $i \in \mathbf{Z}^n$.

The following theorem gives an explicit representation of the function $F(s)$ in terms of the p-capacity of the closed unit ball B_1 in $\mathbf{R}^n$ relative to the operator $-\operatorname{div} a_\infty$. This result can be obtained as a particular case of the results proved by Skrypnik in a more general context (see [11]). For the sake of completeness we shall give here an alternative proof that holds in the particular case of a periodic structure.

We shall denote by C a positive constant which can change from line to line and which depends only on n, α, and β.

Theorem 2.2. *Let $s \in \mathbf{R}$ and let ζ be the solution of the problem*

$$\begin{cases} -\operatorname{div} a_\infty(D\zeta) = 0 & \text{in } \{|x| > 1\} \\ \zeta = s & \text{in } \{|x| \leq 1\} \\ \zeta \in H_p(\mathbf{R}^n)\,. \end{cases} \tag{2.11}$$

Then the function $F(\cdot)$ in (2.5) is given by the following formula

$$F(s) = \frac{1}{2^n} \int_{\mathbf{R}^n} a_\infty(D\zeta) Dv\, dx\,, \tag{2.12}$$

where v is an arbitrary function in $H_p(\mathbf{R}^n)$ such that $v = 1$ on $\{|x| \leq 1\}$.

Proof: Let $s_h \in W^{1,p}_{loc}(\mathbf{R}^n)$ be the solution of the problem

$$\begin{cases} -\operatorname{div} a(Ds_h) = F(s) & \text{in } Q_{\varepsilon_h} \setminus B_{r_h}, \\ s_h = 0 & \text{on } B_{r_h}, \\ s_h \ Q_{\varepsilon_h}\text{-periodic}\,, \end{cases} \tag{2.13}$$

where Q_{ε_h} is the cube of center 0 and side $2\varepsilon_h$ and B_{r_h} is the closed ball of center 0 and radius r_h. Let us prove that the sequence (s_h) converges to s weakly in $W^{1,p}(\Omega)$.

For every function $v \in W^{1,p}(\Omega)$ the following version of the Poincaré inequality holds

$$\int_\Omega |v|^p dx \leq \frac{K|\Omega|}{p\text{-Cap}(N(v),\Omega)} \int_\Omega |Dv|^p dx\,, \tag{2.14}$$

where K is a positive constant independent of v and Ω, $N(v) = \{x \in \Omega : v(x) = 0\}$, $|\Omega|$ denotes the Lebesgue measure of Ω, and p-Cap$(N(v),\Omega)$ is the p-capacity of the set $N(v)$ in Ω (see [12]). Since $\{x \in Q_{\varepsilon_h} : s_h(x) = 0\} \supseteq B_{r_h}$ by (2.14) we get

$$\int_{Q_{\varepsilon_h}} |s_h|^p dx \leq \frac{K(2\varepsilon_h)^n}{p\text{-Cap}(B_{r_h}, Q_{\varepsilon_h})} \int_{Q_{\varepsilon_h}} |Ds_h|^p dx \tag{2.15}$$

for every $h \in \mathbf{N}$. Moreover it is well known that p-Cap$(B_{r_h}, Q_{\varepsilon_h}) \geq p$-Cap$(B_{r_h}, B_{2\varepsilon_h}) = (r_h^{p-n} - (2\varepsilon_h)^{p-n})^{-1}$ and hence, by (2.3), p-Cap$(B_{r_h}, Q_{\varepsilon_h}) \geq \varepsilon_h^n$; so that by (2.15) we obtain

$$\int_{Q_{\varepsilon_h}} |s_h|^p dx \leq K \int_{Q_{\varepsilon_h}} |Ds_h|^p dx\,, \tag{2.16}$$

where K is positive constant independent of h. Taking s_h as a test function in (2.13), by Hölder inequality and (2.16), we have

$$\int_{Q_{\varepsilon_h}} |Ds_h|^p dx = F(s) \int_{Q_{\varepsilon_h}} s_h dx \leq$$

$$\leq F(s)(2\varepsilon_h)^{n/p'} \Big(\int_{Q_{\varepsilon_h}} |s_h|^p dx \Big)^{\frac{1}{p}} \leq K^{\frac{1}{p}} F(s)(2\varepsilon_h)^{n/p'} \Big(\int_{Q_{\varepsilon_h}} |Ds_h|^p dx \Big)^{\frac{1}{p}}$$

and hence

$$\frac{1}{(2\varepsilon_h)^n}\int_{Q_{\varepsilon_h}} |Ds_h|^p dx \,\leq\, CF(s)^{p'}\,. \tag{2.17}$$

Since (ε_h) tends to zero and (s_h) is Q_{ε_h}-periodic we have

$$\begin{cases} \displaystyle\int_\Omega |Ds_h|^p dx \,=\, \frac{|\Omega|+o_h}{(2\varepsilon_h)^n}\int_{Q_{\varepsilon_h}} |Ds_h|^p dx\,, \\ \displaystyle\int_\Omega |s_h|^p dx \,=\, \frac{|\Omega|+o_h}{(2\varepsilon_h)^n}\int_{Q_{\varepsilon_h}} |s_h|^p dx\,, \end{cases} \tag{2.18}$$

where $\lim_{h\to\infty} o_h = 0$. Thus by (2.16) we get

$$\int_\Omega |Ds_h|^p dx + \int_\Omega |s_h|^p dx \,\leq\, \frac{|\Omega|+o_h}{(2\varepsilon_h)^n}(1+K)\int_{Q_{\varepsilon_h}} |Ds_h|^p dx$$

and hence, by (2.17), we have that (s_h) is bounded in $W^{1,p}(\Omega)$. Then, up to a subsequence, (s_h) converges weakly in $W^{1,p}(\Omega)$ to some function v. Since s_h is Q_{ε_h}-periodic it is easy to check that v is constant, i.e., $v = c \in \mathbf{R}$. Moreover for every $\varphi \in C_0^\infty(\Omega)$ the function s_h satisfies

$$\int_\Omega a(Ds_h)D\varphi\, dx \,=\, F(s)\int_\Omega \varphi\, dx\,. \tag{2.19}$$

Indeed if $\varphi \in C_0^\infty(\Omega)$, then the function $\psi(x) = \sum_{i\in\mathbf{Z}^n}\varphi(x+\varepsilon_h i)$ is Q_{ε_h}-periodic and by (2.13) we have

$$\begin{aligned} \int_\Omega a(Ds_h)D\varphi\, dx \,&=\, \sum_{i\in\mathbf{Z}^n}\int_{Q^i_{\varepsilon_h}} a(Ds_h)D\varphi\, dx \,=\, \int_{Q_{\varepsilon_h}} a(Ds_h)D\psi\, dx \,= \\ &=\, \int_{Q_{\varepsilon_h}} F(s)\psi\, dx \,=\, \sum_{i\in\mathbf{Z}^n}\int_{Q^i_{\varepsilon_h}} F(s)\varphi\, dx \,=\, \int_\Omega F(s)\varphi\, dx\,. \end{aligned}$$

Thus by Remark 1.3 we have that

$$\int_\Omega F(c)\varphi\, dx \,=\, \int_\Omega F(s)\varphi\, dx$$

and hence by the monotonicity of F (condition (II)) we get $c = s$.

Let us consider now the function $z_h(x) = s_h(r_h x)$ and let us denote by Q_h the cube of center 0 and side $2\varepsilon_h/r_h$. By changing variables in (2.13) we obtain that z_h satisfies

$$\int_{Q_h} r_h^{-1}a(r_h^{-1}Dz_h)Dv\, dx \,=\, F(s)\int_{Q_h} v\, dx \tag{2.20}$$

for every v Q_h-periodic and $v=0$ on B_1. By (2.17) and (2.3) we have

$$\int_{Q_h} |Dz_h|^p dx = \frac{1}{r_h^n}\int_{Q_{\varepsilon_h}} r_h^p |Ds_h|^p dx = \frac{1}{\varepsilon_h^n}\int_{Q_{\varepsilon_h}} |Ds_h|^p dx \le 2^n C F(s)^{p'} . \tag{2.21}$$

Let us denote by $(z_h)_{Q_h} = \frac{1}{|Q_h|}\int_{Q_h} z_h dx$ the average of z_h on Q_h. Since by (2.18) we have

$$(z_h)_{Q_h} = \frac{r_h^n}{2^n\varepsilon_h^n}\int_{Q_h} z_h dx = \frac{1}{2^n\varepsilon_h^n}\int_{Q_{\varepsilon_h}} s_h dx = \frac{1}{(|\Omega|+o_h)}\int_\Omega s_h dx$$

and (s_h) converges to s strongly in $L^p(\Omega)$, we obtain that $(z_h)_{Q_h}$ converges to s. Moreover by the Sobolev inequality we have

$$\Big(\int_{Q_h} |z_h - (z_h)_{Q_h}|^{p^*} dx\Big)^{1/p^*} \le C\Big(\int_{Q_h} |Dz_h|^p dx\Big)^{1/p}, \tag{2.22}$$

where the constant C is independent of h. Then by (2.21) and (2.22), and by the fact that $(z_h)_{Q_h}$ converges to s we have that, up to a subsequence, the sequence (z_h) converges to some function $z \in W^{1,p}_{loc}(\Omega)$ weakly in $W^{1,p}(B)$ for every bounded open set $B \subseteq \mathbf{R}^n$. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be with compact support. Since for h large enough we have $\operatorname{supp}\varphi \subseteq Q_h$ we can take $(z_h - z)\varphi$ as test function in (2.20) and by (i) we get

$$\begin{aligned}
&\alpha\int_{\operatorname{supp}\varphi} |D(z_h - z)|^p \varphi\, dx \le \\
&\le \int_{\operatorname{supp}\varphi} (r_h)^{p-1}(a(r_h^{-1}Dz_h) - a(r_h^{-1}Dz))D(z_h - z)\varphi\, dx = \\
&= r_h^p\int_{\operatorname{supp}\varphi} F(s)(z_h - z)\varphi\, dx - \int_{\operatorname{supp}\varphi} (r_h)^{p-1} a(r_h^{-1}Dz_h)D\varphi(z_h - z)\, dx - \\
&- \int_{\operatorname{supp}\varphi} (r_h)^{p-1} a(r_h^{-1}Dz)D(z_h - z)\varphi\, dx .
\end{aligned} \tag{2.23}$$

Since, by (ii) and (2.2), the right hand side of (2.23) tends to zero as $h\to\infty$ we obtain that (z_h) converges to z strongly in $W^{1,p}_{loc}(\mathbf{R}^n)$. If we take as test function in (2.20) a function $v \in W^{1,p}(\mathbf{R}^n)$ with compact support and $v = 0$ in B_1, then we can take the limit as $h\to\infty$ and by (2.2) we have that

$$\int_{\mathbf{R}^n} a_\infty(-Dz)Dv = 0 \tag{2.24}$$

for every $v \in W^{1,p}(\mathbf{R}^n)$ with compact support and $v = 0$ in B_1.

Let now $\zeta = s - z$. By (2.21) and the fact that (z_h) converges to z strongly in $W^{1,p}_{loc}(\mathbf{R}^n)$ we have

$$\int_B |D\zeta|^p dx \le C$$

for every bounded open set $B \subseteq \mathbf{R}^n$ and hence $D\zeta \in L^p(\mathbf{R}^n, \mathbf{R}^n)$. Similarly by (2.22) and the fact that $(z_h)_{Q_h}$ converges to s we get that $\zeta \in L^{p^*}(\mathbf{R}^n)$ and hence $\zeta \in H_p(\mathbf{R}^n)$. Finally, since $\zeta = s$ on B_1, by (2.24) we have that ζ is the unique solution of problem (2.11).

Let us prove the representation formula (2.12). Let $v \in H_p(\mathbf{R}^n)$ be with compact support and $v = 1$ on B_1. Taking $1 - v$ as test function in problem (2.20) we have

$$\int_{Q_h} (r_h)^{p-1} a(r_h^{-1} D z_h) Dv \, dx \;=\; 2^n F(s) - r_h^p F(s) \int_{Q_h} v \, dx \,. \tag{2.25}$$

Taking the limit as $h \to \infty$ we obtain (2.12). If v has not compact support, it is enough to consider a function $\tilde{v} \in H_p(\mathbf{R}^n)$ with compact support and $\tilde{v} = 1$ on B_1. Using $v - \tilde{v}$ as test function in (2.11) we get

$$\int_{\mathbf{R}^n} a_\infty(D\zeta) Dv \;=\; \int_{\mathbf{R}^n} a_\infty(D\zeta) D\tilde{v} \;=\; 2^n F(s)$$

and this concludes the proof. □

Remark 2.3. The representation formula (2.12) assure that if the function a is asymptotically $(p-1)$-homogeneous, i.e.,

$$\exists \lim_{r \to 0} -r^{p-1} a(-r^{-1}\xi) \qquad \forall \xi \in \mathbf{R}^n \,, \tag{2.26}$$

then the limit problem (2.5) does not depend on the choice of the sequence (ε_h). More precisely if (2.26) is satisfied then the asymptotic behaviour of the solutions u_ε of the problems

$$\begin{cases} -\mathrm{div} a(Du_\varepsilon) \;=\; f \qquad \text{in } \Omega \setminus E_\varepsilon \,, \\ \\ u_\varepsilon = 0 \quad \text{on } \partial(\Omega \setminus E_\varepsilon) \ , \end{cases} \tag{2.27}$$

is completely described by the solution of problem (2.5). So that in this case we have the same homogenization phenomenon which is well known in the homogeneous case.

3. Example of non homogeneous extra term.

In this section, under special assumption for the function a_∞, we shall prove that the function F is $(p-1)$-homogeneous if and only if a_∞ is $(p-1)$-homogeneous. This result will permit us to exhibit simple examples where the function F is not homogeneous (Remark 3.3) and to show that in this case the homogenization phenomenon describe by Remark 2.3 does not occurs (Proposition 3.2).

Proposition 3.1. *Let us suppose that the function a_∞ defined by* (2.2) *is of the form*

$$a_\infty(\xi) \;=\; \gamma(|\xi|)\xi \tag{3.1}$$

where $\gamma : [0,+\infty] \mapsto [\alpha,\beta]$. Then the function $F(s)$ given by (2.12) is homogeneous of degree $(p-1)$ if and only if a_∞ is homogeneous of degree $(p-1)$.

Proof: If a_∞ is homogeneous of degree $(p-1)$ then the result is a direct consequence of formula (2.12) once we note that if ζ is the solution of problem (2.11) at the level $s \in \mathbf{R}$ then the function $t\zeta$, with $t \in \mathbf{R}$, is the solution of the same problem at the level ts.

Vice versa let $F(s)$ be homogeneous of degree $(p-1)$. Let ω_n be the $(n-1)$-dimensional measure of the unit sphere in $\mathbf{R}^n$ and let $\overline{H}_p = \{u : [0,+\infty) \mapsto \mathbf{R} : \int_1^{+\infty} r^{n-1}|u|^{p^*}dr + \int_1^{+\infty} r^{n-1}|u'|^p dr < +\infty\}$. By assumption (3.1) it is easy to see that the solution ζ of problem (2.11) is radially symmetric, i.e., $\zeta(x) = z(|x|)$ with $z \in \overline{H}_p$, and

$$F(s) = \frac{\omega_n}{2^n}\int_1^{+\infty} r^{n-1}\gamma(|z'|)z'v'dr \tag{3.2}$$

for every $v \in \overline{H}_p$ with $v(1) = 1$. Moreover for every $v \in \overline{H}_p$ with $v(1) = 0$ we have

$$\int_1^{+\infty} r^{n-1}\gamma(|z'|)z'v'dr = 0\,.$$

Then there exist a constant $t(s)$ depending on s such that $r^{n-1}\gamma(|z'|)z' = t(s)$. By (3.2) we have that $2^nF(s)/\omega_n = -t(s)$ and hence

$$\omega_n r^{n-1}\gamma(|z'|)z' = -2^nF(s)\,. \tag{3.3}$$

Let us denote by $P(r)$ the function defined by $P(r) = \gamma(|r|)r$ for every $r \in \mathbf{R}$. By (i) and (ii) we have that there exists the inverse function P^{-1} of P. Then by (3.3) we have

$$z'(r) = P^{-1}\Big(\frac{-2^nF(s)}{\omega_n r^{n-1}}\Big) \tag{3.4}$$

and, since $\displaystyle\int_1^{+\infty} z'dr = -s$ and $F(s) = |s|^{p-2}sF(1)$, we get

$$\int_1^{+\infty} G\Big(\frac{|s|^{p-2}s}{r^{n-1}}\Big)dr = -s\,, \tag{3.5}$$

where $G(t) = P^{-1}(-2^nF(1)t/\omega_n)$. By changing variables in (3.5), with $\rho = |s|^{p-2}s/r^{n-1}$, we obtain

$$\frac{1}{n-1}|s|^{\frac{p-n}{n-1}}s\int_0^{|s|^{p-2}s} G(\rho)|\rho|^{\frac{-n}{n-1}}d\rho = -s$$

and hence

$$\int_0^{|s|^{p-2}s} G(\rho)|\rho|^{\frac{-n}{n-1}}d\rho = -(n-1)|s|^{\frac{n-p}{n-1}}\,.$$

If we derive with respect to s we get $G(s)|s|^{\frac{-n(p-1)}{n-1}} = -(n-p)|s|^{\frac{1-p}{n-1}}s/|s|$ and hence

$$\gamma(|r|)r = \frac{2^n}{\omega_n(n-p)}F(1)|r|^{p-2}r\,,$$

which concludes the proof. □

Proposition 3.2. *Under the same assumption of Proposition* 3.1*, let us assume that the function a_∞ is not homogeneous. Then there exists a sequence (ε'_h) of positive number converging to zero such that the sequence $(u_{\varepsilon'_h})$ of the solutions of problems* (2.26) *corresponding to $\varepsilon = \varepsilon'_h$ do not converges weakly in $W_0^{1,p}(\Omega)$ to the solution of problem* (2.5)*.*

Proof: Let us prove the result by contradiction. Let u_ε be the solution of problem (2.26) and let us suppose that every subsequence of (u_ε) converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (2.5). Let (r_h) be a sequence such that $a_\infty(\xi) = \lim_{h\to\infty} -(r_h)^{p-1} a(-r_h^{-1}\xi)$ for every $\xi \in \mathbf{R}^n$, for every $t \in \mathbf{R}$, $t > 0$, let us define $r_h^t = t^{-1} r_h$, and let ε_h^t be defined by

$$r_h^t \;=\; (\varepsilon_h^t)^{n/(n-p)} \;=\; t^{-n/(n-p)} \varepsilon_h^{n/(n-p)}\,.$$

Since for every $t > 0$ the sequence $(u_{\varepsilon_h^t})$ converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (2.5), by Proposition 2.2 for every $s \in \mathbf{R}$ and for every $v \in H_p(\mathbf{R}^n)$, with $v = 1$ in $\{|x| \le 1\}$, we have

$$F(s) \;=\; \frac{1}{2^n} \int_{\mathbf{R}^n} a_\infty^t(D\zeta_t^s) Dv\, dx\,, \tag{3.6}$$

where for every $\xi \in \mathbf{R}^n$

$$a_\infty^t(\xi) \;=\; \lim_{h\to\infty} -(r_h^t)^{p-1} a(-(r_h^t)^{-1}\xi) \;=\; \frac{1}{t^{p-1}} a_\infty(t\xi) \tag{3.7}$$

and ζ_t^s is the solution of the problem

$$\begin{cases} -\mathrm{div}\, a_\infty^t(D\zeta_t^s) \;=\; 0 & \text{in } \{|x| > 1\} \\ \zeta_t^s = s & \text{in } \{|x| \le 1\} \\ \zeta_t^s \in H_p(\mathbf{R}^n)\,. & \end{cases} \tag{3.8}$$

Moreover, since $a_\infty(\xi) = \gamma(|\xi|)\xi$, for every $\xi \in \mathbf{R}^n$ we have that

$$a_\infty^{|t|}(\xi) \;=\; \frac{1}{|t|^{p-2}} \gamma(|t\xi|)\xi \;=\; \frac{1}{|t|^{p-2} t} a_\infty(t\xi) \quad \forall t \in \mathbf{R}\,,\; t \ne 0\,. \tag{3.9}$$

Now let $z_t^s = t^{-1}\zeta_1^s$ for every $s, t \in \mathbf{R}$, with $t \ne 0$. By (3.8) z_t^s is the solution of the problem

$$\begin{cases} -\mathrm{div}\, a_\infty^{|t|}(Dz_t^s) \;=\; 0 & \text{in } \{|x| > 1\} \\ z_t^s = s/t & \text{in } \{|x| \le 1\} \\ z_t^s \in H_p(\mathbf{R}^n)\,, & \end{cases}$$

then, by uniqueness, we deduce that $z_t^s = \zeta_{|t|}^{s/t}$ and hence

$$t\zeta_{|t|}^{s/t} \;=\; \zeta_1^s \qquad \forall s, t \in \mathbf{R}\,,\; t \ne 0\,.$$

Therefore by (3.6) and (3.9), for every $s, t \in \mathbf{R}$ with $t \neq 0$, we have

$$F(s) = \frac{1}{2^n} \int_{\mathbf{R}^n} a_\infty(D\zeta_1^s) Dv \, dx = \frac{|t|^{p-2}t}{2^n} \int_{\mathbf{R}^n} a_\infty^{|t|}(D\zeta_{|t|}^{s/t}) Dv \, dx = |t|^{p-2} t F(t^{-1}s)$$

for every $v \in H_p(\mathbf{R}^n)$, with $v = 1$ in $\{|x| \leq 1\}$. This implies that F is $(p-1)$-homogeneous and hence, by Proposition 3.1, a_∞ is $(p-1)$-homogeneous which is a contradiction. $\square$

Example 3.3. Let us consider the operator $-\mathrm{div}\,(\gamma(|Du|)Du)$ where $\gamma(|t|) = (2 + \sin(\log|t|))|t|^{p-2}$, that can be considered as a non linear perturbation of the p-Laplacian operator. It is easy to see that the function $a(\xi) = \gamma(|\xi|)\xi$ satisfies condition (i)–(iii). Moreover if we choose $r_h = \exp(-2\pi n)$, then $\gamma(r_h^{-1}|\xi|) = \gamma(|\xi|)$ for every $\xi \in \mathbf{R}^n$ and hence $a_\infty(\xi) = a(\xi)$ for every $\xi \in \mathbf{R}^n$. Since the function $a(\xi)$ is clearly non-homogeneous, by Proposition 3.1 we obtain that the function $F(s)$ which appears in the limit problem (2.5) is non-homogeneous. Moreover with this choice of $a(\xi)$, by Proposition 3.2, the limit problem for the sequence (u_{ε_h}) depend on the choice of the sequence (ε_h), i.e., the homogenization result which holds for the homogeneous case does not occur.

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