A CAPACITARY METHOD FOR THE ASYMPTOTIC ANALYSIS OF DIRICHLET PROBLEMS FOR MONOTONE OPERATORS

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Abstract

Given a non-linear elliptic equation of monotone type in a bounded open set $\Omega \subset \mathbf{R}^n$, we prove that the asymptotic behaviour, as $j \to \infty$, of the solutions of the Dirichlet problems corresponding to a sequence (Ω_j) of open sets contained in Ω is uniquely determined by the asymptotic behaviour, as $j \to \infty$, of suitable non-linear capacities of the sets $K \setminus \Omega_j$, where K runs in the family of all compact subsets of Ω.

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Ref. S.I.S.S.A. 36/96/M (March 96)

Introduction

Let Ω be a bounded open set in $\mathbf{R}^n$, with $n \geq 2$, and let $A: W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ be a quasi-linear monotone operator of the form

$$Au = -\mathrm{div}\big(a(x, Du)\big)\,,$$

where $2 \leq p \leq n$, $1/p + 1/q = 1$, and $a: \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the classical hypotheses: Carathéodory conditions, local Lipschitz continuity, and strong monotonicity (see (i), (ii), (iii) in Section 1). In this paper we examine the connection between the asymptotic behaviour of the solutions of Dirichlet problems for the operator A in perforated domains, which has been investigated in [6] in the most general situation, and a notion of non-linear capacity associated with A, whose properties have been studied in [18].

Given an arbitrary sequence (Ω_j) of open sets contained in Ω, we consider for every $f \in W^{-1,q}(\Omega)$ the sequence (u_j) of the solutions of the Dirichlet problems

$$\begin{cases} u_j \in W_0^{1,p}(\Omega_j)\,, \\ Au_j = f \ \text{ in } \ W^{-1,q}(\Omega_j)\,, \end{cases} \tag{0.1}$$

extended to Ω by setting $u_j = 0$ in $\Omega \backslash \Omega_j$. A compactness result proved in [6] guarantees that there exist a subsequence, still denoted by (Ω_j), a Borel function $b: \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying conditions (I), (II), (III) of Section 1, and a measure μ of the class $\tilde{\mathcal{M}}_0^p(\Omega)$ (Definition 1.2), such that for every $f \in W^{-1,q}(\Omega)$ the sequence (u_j) of the solutions of (0.1) converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \\ \displaystyle\int_\Omega \big(a(x, Du), Dv\big)\, dx + \int_\Omega b(x, u) v \, d\mu \ = \ \langle f, v \rangle \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \end{cases} \tag{0.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$. We refer to [13], [17], and [19] for a wide bibliography on this subject. The restriction $p \geq 2$ has been made here only to simplify the exposition. The case $1 < p < 2$ can be treated by similar arguments with minor changes in the hypotheses. The condition $p \leq n$ is due to the Sobolev Embedding Theorem. If $p > n$, then $W_0^{1,p}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and, consequently, the study of the asymptotic behaviour of the sequence (u_j) of the solutions of (0.1) is much easier and does not require the use of problems of the form (0.2).

In this paper we shall prove that the function b and the measure μ which appear in (0.2) can be obtained from the sequence (Ω_j) by using the notion of A-capacity studied in [18]. If K is a compact set contained in Ω and s is a real number, the *A-capacity* of K relative to the constant s is defined as

$$C_A(K,s) = \int_{\Omega\setminus K} \big(a(x,Du),Du\big)\,dx\,, \tag{0.3}$$

where u, the *A-potential* of K relative to the constant s, is the weak solution of the Dirichlet problem

$$\begin{cases} Au = 0 \ \text{ in } \ W^{-1,q}(\Omega\setminus K)\,, \\ u = s \ \text{ in } \ \partial K\,, \quad u = 0 \ \text{ in } \ \partial\Omega\,. \end{cases} \tag{0.4}$$

The last line in (0.4) means that $u-\varphi \in W_0^{1,p}(\Omega\setminus K)$, where φ is an arbitrary function in $C_0^\infty(\Omega)$ such that $\varphi = s$ in a neighbourhood of K. If $A_p u = -\mathrm{div}\big(|Du|^{p-2}Du\big)$ is the p-Laplacian, then we have

$$C_{A_p}(K,s) = |s|^p C_p(K)\,,$$

where $C_p(K)$ is the $(1,p)$-capacity of K in Ω (see Section 1).

The first connection between A-capacity and asymptotic behaviour of solutions of Dirichlet problems is given by the following theorem, which will be proved in Section 7.

Theorem 0.1. *Let (Ω_j) be a sequence of open sets contained in Ω. Suppose that for every $f \in W^{-1,q}(\Omega)$ the sequence (u_j) of the solutions of (0.1) converges weakly in $W_0^{1,p}(\Omega)$. Then*

$$\limsup_{j\to\infty} C_A(H\setminus\Omega_j,s) \,\le\, \liminf_{j\to\infty} C_A(K\setminus\Omega_j,s) \tag{0.5}$$

for every $s \in \mathbf{R}$ and for every pair H, K of compact subsets of Ω such that H is contained in the interior $\mathring{K}$ of K.

As the set functions $\alpha_j(K,s) = C_A(K\setminus\Omega_j,s)$ are increasing with respect to K (see [18]), condition (0.5) is equivalent, for every $s\in\mathbf{R}$, to the weak convergence of the sequence $\big(\alpha_j(\cdot,s)\big)$ according to the definition given in [21].

The main result of our paper is the converse of the previous theorem. Suppose that (Ω_j) satisfies (0.5). Then for every $s\in\mathbf{R}$ there exists an increasing set function $\alpha(\cdot,s)$ defined on the family of all compact subsets of Ω such that

$$\limsup_{j\to\infty} C_A(H\setminus\Omega_j,s) \,\le\, \alpha(K,s) \,\le\, \liminf_{j\to\infty} C_A(L\setminus\Omega_j,s) \tag{0.6}$$

whenever H, K, L are compact sets with $H \subset \mathring{K} \subset K \subset \mathring{L} \subset L \subset \Omega$. For instance, one can take

$$\alpha(K,s) = \liminf_{j\to\infty} C_A(K \setminus \Omega_j, s)$$

for every compact set $K \subset \Omega$. Let $\beta(\cdot,s)$ be the regularized version of $\alpha(\cdot,s)$ defined by

$$\begin{aligned} &\beta(U,s) = \sup\{\alpha(K,s):\ K \text{ compact},\ K \subset U\}, \qquad \text{if } U \text{ is an open set in } \Omega, \\ &\beta(B,s) = \inf\{\beta(U,s):\ U \text{ open},\ B \subset U \subset \Omega\}, \qquad \text{if } B \text{ is a Borel set in } \Omega. \end{aligned} \tag{0.7}$$

The set function $\beta(\cdot,s)$ can be interpreted as an *asymptotic capacity* relative to the sequence (Ω_j). By the properties of the A-capacity proved in [18] $\beta(\cdot,s)$ is increasing and countably subadditive. Therefore for every $s \in \mathbf{R}$ we can consider the least measure $\nu(\cdot,s)$ which is greater that or equal to $\beta(\cdot,s)$. According to [3] $\nu(\cdot,s)$ can be regarded as the *limiting capacity measure* relative to the sequence (Ω_j). It is easy to see that for every Borel set $B \subset \Omega$ we have

$$\nu(B,s) = \sup \sum_{i\in I} \beta(B_i, -s), \tag{0.8}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B.

We shall prove that the measure μ defined by

$$\mu(B) = \nu(B,1) \tag{0.9}$$

belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$ and that there exists a non-negative Borel function $b{:}\, \Omega\times\mathbf{R} \to \mathbf{R}$, which satisfies conditions (I), (II), (III) of Section 1, such that

$$\int_B b(x,s)\, d\mu = \frac{1}{s}\nu(B,s) \tag{0.10}$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$, $s \neq 0$.

The main result of this paper is the following theorem, which will be proved in Section 7.

Theorem 0.2. *Let (Ω_j) be a sequence of open subsets of Ω which satisfies (0.5). Then for every $f \in W^{-1,q}(\Omega)$ the sequence (u_j) of the solutions of (0.1) converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (0.2), where $b(x,s)$ and μ are defined by (0.6)–(0.10).*

This result can be simplified when the set function $\alpha(\cdot, s)$ which appears in (0.6) is bounded by a Radon measure. In addition to (0.5), assume that there exists a non-negative Radon measure λ on Ω such that

$$\limsup_{j\to\infty} C_A(K \setminus \Omega_j, 1) \leq \lambda(K) \tag{0.11}$$

for every compact set $K \subset \Omega$, and let $\beta(\cdot, s)$ be the set function defined by (0.6) and (0.7). Then for λ-a.e. $x \in \Omega$ and every $s \in \mathbf{R}$ the following limit exists:

$$\lim_{\rho\to 0} \frac{\beta(B_\rho(x), -s)}{\lambda(B_\rho(x))} = \psi(x, s)\,, \tag{0.12}$$

where $B_\rho(x)$ is the open ball with center x and radius ρ. Let μ be the Radon measure defined by

$$\mu(B) = \int_B \psi(x, 1)\, d\lambda \tag{0.13}$$

for every Borel set $B \subset \Omega$. Then μ belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$ and there exists a function $b\colon \Omega\times\mathbf{R} \to \mathbf{R}$, which satisfies conditions (I), (II), (III) of Section 1, such that

$$b(x, s) = \frac{1}{s}\frac{\psi(x, s)}{\psi(x, 1)} \tag{0.14}$$

for μ-a.e. $x \in \Omega$ and for every $s \in \mathbf{R}$.

The following theorem, which will be proved at the end of the paper, shows that the function b and the measure μ which appear in the limit problem (0.2) can be obtained by taking the derivative of the asymptotic capacity $\beta(\cdot, s)$ with respect to the measure λ.

Theorem 0.3. *Let (Ω_j) be a sequence of open subsets of Ω which satisfies (0.5) and (0.11) for a suitable non-negative Radon measure λ. Then for every $f \in W^{-1,q}(\Omega)$ the sequence (u_j) of the solutions of (0.1) converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (0.2), where $b(x, s)$ and μ are defined by (0.6), (0.7), and (0.12)–(0.14).*

When $a(x, \xi)$ is linear and symmetric with respect to ξ, these results are already known and can be found in [5] and [9]. The case $a(x, \xi) = \partial_\xi \psi(x, \xi)$, with $\psi(x, \xi)$ convex, even, and p-homogeneous with respect to ξ, is studied in [12]. When $a(x, \xi)$ is linear and non-symmetric the proof is more recent, and can be found in [14]. Our results are completely new when $a(x, \xi)$ is a general monotone operator and the sequence of sets

(Ω_j) satisfies only condition (0.5), which turns out to be necessary and sufficient for the convergence of the solutions of problems (0.1).

Under various additional hypotheses on the sequence of sets (Ω_j) or on the operator A, the notion of A-capacity has been used to determine the asymptotic behaviour of the solutions of problems (0.1) in [27]–[33], [3], [19], and [20].

To prove Theorem 0.2 we study carefully the set function $\beta(\cdot,s)$ defined by (0.6) and (0.7). We prove that, if the solutions of (0.1) converge weakly in $W_0^{1,p}(\Omega)$ to the solution of (0.2) for every $f \in W^{-1,q}(\Omega)$, then $\beta(\cdot,s)$ is uniquely determined by b and μ. To study the relationships between $\beta(\cdot,s)$ and the pair (b,μ) we introduce (Definition 2.1) the notion of $C_A^{b,\mu}$-capacity, which extends to the non-linear case the notion of μ-capacity introduced in [15] and [16]. For every $s \in \mathbf{R}$ and for every Borel set $B \subset\subset \Omega$ the $C_A^{b,\mu}$-*capacity* of B relative to the constant s is defined by

$$C_A^{b,\mu}(B,s) \;=\; \int_\Omega \big(a(x,Du),Du\big)\,dx + \int_B b(x,u-s)(u-s)\,d\mu\,,$$

where u is the solution of the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega)\,, \qquad u-s \in L^p_\mu(B)\,, \\ \displaystyle\int_\Omega \big(a(x,Du),Dv\big)\,dx + \int_B b(x,u-s)v\,d\mu \;=\; 0 \\ \forall v \in W_0^{1,p}(\Omega)\cap L^p_\mu(B)\,. \end{cases}$$

It is easy to see that, if the measure μ is infinite on every set of positive $(1,p)$-capacity, then $C_A^{b,\mu}(K,s) = C_A(K,s)$ for every compact set $K \subset \Omega$. In Sections 2–5 we study the main properties of $C_A^{b,\mu}$. In particular we prove that for every $s \in \mathbf{R}$ the set function $C_A^{b,\mu}(\cdot,s)$ is increasing (Theorem 3.4), continuous along increasing sequences of Borel sets (Theorem 4.1), continuous along decreasing sequences of compact sets (Theorem 4.3), and countably subadditive (Theorem 5.5). Moreover we prove that for every Borel set $B \subset\subset \Omega$

$$\begin{aligned} C_A^{b,\mu}(B,s) \;&=\; \sup\{C_A^{b,\mu}(K,s)\;:\;K \text{ compact}\,,\; K \subset B\} \;= \\ &=\; \inf\{C_A^{b,\mu}(U,s)\;:\;U \text{ open}\,,\; B \subset U \subset\subset \Omega\} \end{aligned} \tag{0.15}$$

(Theorems 5.1 and 5.6). These results, together with Theorem 7.3, show that, if the solutions of (0.1) converge weakly in $W_0^{1,p}(\Omega)$ to the solution of (0.2) for every $f \in W^{-1,q}(\Omega)$, then $\beta(B,s) = C_A^{b,\mu}(B,s)$ for every Borel set $B \subset\subset \Omega$ and for every $s \in \mathbf{R}$. In Section 6 we show that for every $s \in \mathbf{R}$ the measure $sb(x,s)\mu$ is the least measure which is greater than or equal to $C_A^{b,\mu}(\cdot,-s)$ (Theorem 6.1). This shows that, if the solutions of (0.1) converge to the solution of (0.2), then (0.10) is satisfied. The hypothesis that the solutions of (0.1) converge weakly in $W_0^{1,p}(\Omega)$ to the solution of (0.2) for every $f \in W^{-1,q}(\Omega)$, which has been crucial in our arguments, can be eventually omitted thanks to the compactness result proved in [6].

Finally, Theorem 0.3 can be obtained from Theorem 0.2 thanks to a general result on countably subadditive set functions proved in [1].

1. Preliminaries

Sobolev spaces and capacity. Let Ω be a bounded open subset of $\mathbf{R}^n$, with $n \geq 2$, and let $2 \leq p < +\infty$ and $1 < q \leq 2$, with $1/p + 1/q = 1$. The space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$, and the space $W^{-1,q}(\Omega)$ is the dual of $W_0^{1,p}(\Omega)$.

For every set $E \subset \Omega$ the $(1,p)$-*capacity* of E in Ω, denoted by $C_p(E)$, is defined as the infimum of $\int_\Omega |Du|^p\,dx$ over the set of all functions $u \in W_0^{1,p}(\Omega)$ such that $u \geq 1$ almost everywhere in a neighbourhood of E. If $E \subset\subset \Omega$, i.e., E is relatively compact in Ω, then $C_p(E) < +\infty$.

We say that a property $\mathcal{P}(x)$ holds C_p-*quasi everywhere* (abbreviated as C_p-*q.e.*) in a set E if it holds for all $x \in E$ except for a subset N of E with $C_p(N) = 0$. The expression *almost everywhere* (abbreviated as *a.e.*) refers, as usual, to the Lebesgue measure. A function $u\colon \Omega \to \mathbf{R}$ is said to be C_p-*quasi continuous* if for every $\varepsilon > 0$ there exists a set $E \subset \Omega$, with $C_p(E) < \varepsilon$, such that the restriction of u to $\Omega \setminus E$ is continuous.

It is well known that every $u \in W^{1,p}(\Omega)$ has a C_p-quasi continuous representative, which is uniquely defined up to a C_p-null set. In the sequel we shall always identify u with its C_p-quasi continuous representative, so that the pointwise values of a function $u \in W^{1,p}(\Omega)$ are defined C_p-quasi everywhere in Ω. We recall that, if a sequence (u_j) converges to u strongly in $W_0^{1,p}(\Omega)$, then a subsequence of (u_j) converges to u C_p-q.e. in Ω. For all these properties of C_p-quasi continuous representatives of Sobolev functions we refer to [22], Section 4.8, [26], Section 7.2.4, [24], Section 4, and [34], Chapter 3.

A subset U of Ω is said to be a C_p-*quasi open* (resp. C_p-*quasi closed*) if for every $\varepsilon > 0$ there exists an open (resp. closed) subset V of Ω such that $C_p(U \triangle V) < \varepsilon$, where $\triangle$ denotes the symmetric difference of sets. We shall frequently use the following lemma about the approximation of the characteristic function of a C_p-quasi open set. We recall that the characteristic function 1_E of a set $E \subset \Omega$ is defined by $1_E(x) = 1$, if $x \in E$, and by $1_E(x) = 0$, if $x \in \Omega \setminus E$.

Lemma 1.1. *For every C_p-quasi open set $U \subset \Omega$ there exists an increasing sequence (v_j) of non-negative functions of $W_0^{1,p}(\Omega)$ which converges to 1_U C_p-quasi everywhere in Ω.*

Proof. See [8], Lemma 1.5, or [13], Lemma 2.1. □

Measures. By a non-negative *Borel measure* on Ω we mean a countably additive set function defined in the Borel σ-field of Ω with values in $[0, +\infty]$. By a non-negative *Radon measure* on Ω we mean a non-negative Borel measure which is finite on every compact subset of Ω. We shall always identify a non-negative Borel measure with its completion. If μ is a non-negative Borel measure on Ω, we shall use $L^r_\mu(\Omega)$, $1 \le r \le +\infty$, to denote the usual Lebesgue space with respect to the measure μ. We adopt the standard notation $L^r(\Omega)$ when μ is the Lebesgue measure.

If E is a Borel subset of Ω, the measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(B) = \mu(B \cap E)$ for every Borel set $B \subset \Omega$. For every non-negative Borel function $f: \Omega \to [0, +\infty]$, the measure $f\mu$ is defined by $(f\mu)(B) = \int_B f\, d\mu$ for every Borel set $B \subset \Omega$.

Definition 1.2. By $\mathcal{M}_0^p(\Omega)$ we denote the set of all non-negative Borel measures μ on Ω such that $\mu(B) = 0$ for every Borel set $B \subset \Omega$ with $C_p(B) = 0$. By $\tilde{\mathcal{M}}_0^p(\Omega)$ we denote the class of measures $\mu \in \mathcal{M}_0^p(\Omega)$ such that

$$\mu(B) = \inf\{\mu(U) \ : \ U \ \ C_p\text{-quasi open}\,,\ B \subset U \subset \Omega\} \tag{1.1}$$

for every Borel set $B \subset \Omega$.

Property (1.1) is a weak regularity property of the measure μ. Since any C_p-quasi open set differs from a Borel set by a C_p-null set, every C_p-quasi open set is μ-measurable for every non-negative Borel measure μ which belongs to $\mathcal{M}_0^p(\Omega)$. Therefore $\mu(U)$ is well defined when U is C_p-quasi open, and condition (1.1) makes sense.

For every set $E \subset \Omega$ we consider the Borel measure ∞_E defined for every Borel set $B \subset \Omega$ by

$$\infty_E(B) = \begin{cases} 0\,, & \text{if } C_p(B \cap E) = 0, \\ +\infty\,, & \text{otherwise.} \end{cases} \tag{1.2}$$

It is easy to see that ∞_E belongs to $\mathcal{M}_0^p(\Omega)$, and that ∞_E belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$ if and only if E is C_p-quasi closed.

We introduce now an equivalence relation on $\mathcal{M}_0^p(\Omega)$.

Definition 1.3. We say that two measures λ and μ in $\mathcal{M}_0^p(\Omega)$ are *equivalent* if $\int_\Omega |u|^p d\lambda = \int_\Omega |u|^p d\mu$ for every $u \in W_0^{1,p}(\Omega)$.

Remark 1.4. For every measure $\mu \in \mathcal{M}_0^p(\Omega)$ let $\tilde{\mu}$ be the set function defined by

$$\tilde{\mu}(B) = \inf\{\mu(U) : U \ \ C_p\text{-quasi open}, \ B \subset U \subset \Omega\} \tag{1.3}$$

for every Borel set $B \subset \Omega$. Then $\tilde{\mu}$ is a Borel measure and belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$. It is the unique measure in $\tilde{\mathcal{M}}_0^p(\Omega)$ equivalent to μ and $\tilde{\mu} \geq \lambda$ for every $\lambda \in \mathcal{M}_0^p(\Omega)$ in the equivalence class of μ (see [9], Section 3). Moreover it is easy to see that μ_1, $\mu_2 \in \mathcal{M}_0^p(\Omega)$ are equivalent if and only if they agree on all C_p-quasi open sets $U \subset \Omega$ (see [2], Lemma 4.1, or [9], Theorem 2.6). Finally, if $\mu \in \mathcal{M}_0^p(\Omega)$ is a Radon measure, then $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$ and no other measure is equivalent to μ.

Although every measure $\mu \in \mathcal{M}_0^p(\Omega)$ is equivalent to the measure $\tilde{\mu} \in \tilde{\mathcal{M}}_0^p(\Omega)$, and the main results of the paper are valid only for measures of $\tilde{\mathcal{M}}_0^p(\Omega)$, we shall sometimes use also measures of $\mathcal{M}_0^p(\Omega)$ which do not belong to $\tilde{\mathcal{M}}_0^p(\Omega)$. An example is given by the measures of the form $\mu \llcorner E$, which play a crucial role in the proof of Theorem 6.1.

Remark 1.5. If $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$ and E is C_p-quasi closed, then $\mu \llcorner E \in \tilde{\mathcal{M}}_0^p(\Omega)$. This is not true, in general, if E is not C_p-quasi closed. It is easy to see that, if $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$ and f is a non-negative bounded Borel function such that $1/f$ is bounded, then the measure $f\mu$ belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$. This is not true, in general, if $1/f$ is not bounded.

Finally, we say that a (signed) Radon measure μ on Ω belongs to $W^{-1,q}(\Omega)$ if there exists $f \in W^{-1,q}(\Omega)$ such that

$$\langle f, \varphi \rangle = \int_\Omega \varphi \, d\mu \qquad \forall \varphi \in C_0^\infty(\Omega), \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$. We shall always identify f and μ. Note that, by the Riesz Theorem, for every non-negative functional $f \in W^{-1,q}(\Omega)$, there exists a non-negative Radon measure μ such that (1.4) holds. It is well known that every non-negative Radon measure which belongs to $W^{-1,q}(\Omega)$ belongs also to $\tilde{\mathcal{M}}_0^p(\Omega)$.

Relaxed Dirichlet problems. Given $\mu \in \mathcal{M}_0^p(\Omega)$, the space $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ is well defined, since all functions of $W_0^{1,p}(\Omega)$ are defined μ-almost everywhere in Ω. It is easy to see that $W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$, endowed with the sum of the two norms, is a Banach space (see, e.g., [4], Proposition 2.1). Therefore, by the classical theory of monotone operators (see, e.g., [25], Chapter 2, Theorem 2.1), for every $f \in W^{-1,q}(\Omega)$ there exists a unique solution u of the problem

$$(1.5) \qquad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\,, \\ \displaystyle\int_\Omega |Du|^{p-2} Du Dv\, dx \,+\, \int_\Omega |u|^{p-2} uv\, d\mu \;=\; \langle f, v\rangle \\ \forall v \in W_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\,. \end{cases}$$

Following the terminology introduced in [15] and [16], problems of this kind will be called *relaxed Dirichlet problems* for the p-Laplacian.

Using Theorem 4.5 of [24] it is easy to check that, if E is closed and μ is the measure ∞_E introduced in (1.2), then problem (1.5) reduces to the following boundary value problem for the p-Laplacian on $\Omega \setminus E$:

$$(1.6) \qquad \begin{cases} u \in W_0^{1,p}(\Omega \setminus E)\,, \\ -\mathrm{div}\big(|Du|^{p-2} Du\big) \;=\; f \;\text{ in }\; W^{-1,q}(\Omega \setminus E)\,, \end{cases}$$

in the sense that u is the solution of (1.5) if and only if the restriction of u to $\Omega \setminus E$ is the solution of (1.6) and $u = 0$ C_p-q.e. in E.

In the space $\mathcal{M}_0^p(\Omega)$ it is possible to introduce a notion of convergence related to the solutions of Dirichlet problems for the p-Laplacian (see [12]).

Definition 1.6. Let (μ_j) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $\mu \in \mathcal{M}_0^p(\Omega)$. We say that (μ_j) γ_p*-converges* to the measure μ if, for every $f \in W^{-1,q}(\Omega)$, the sequence (u_j) of the solutions of the problems

$$(1.7) \qquad \begin{cases} u_j \in W_0^{1,p}(\Omega) \cap L_{\mu_j}^p(\Omega)\,, \\ \displaystyle\int_\Omega |Du_j|^{p-2} Du_j Dv\, dx + \int_\Omega |u_j|^{p-2} u_j v\, d\mu_j \;=\; \langle f, v\rangle \\ \forall v \in W_0^{1,p}(\Omega) \cap L_{\mu_j}^p(\Omega) \end{cases}$$

converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of problem (1.5).

Remark 1.7. It is proved in [12] (see also Proposition 3.4 of [6] for a different proof) that a sequence (μ_j) in $\mathcal{M}_0^p(\Omega)$ γ_p-converges to a measure $\mu \in \mathcal{M}_0^p(\Omega)$ if and only if the following conditions are satisfied:

(a) for every $u \in W_0^{1,p}(\Omega)$ and for every sequence (u_j) converging to u weakly in $W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |Du|^p dx + \int_\Omega |u|^p d\mu \le \liminf_{j\to\infty}\Big(\int_\Omega |Du_j|^p dx + \int_\Omega |u_j|^p d\mu_j\Big);$$

(b) for every $u \in W_0^{1,p}(\Omega)$ there exists a sequence (u_j) converging to u weakly in $W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |Du|^p dx + \int_\Omega |u|^p d\mu = \lim_{j\to\infty}\Big(\int_\Omega |Du_j|^p dx + \int_\Omega |u_j|^p d\mu_j\Big).$$

Conditions (a) and (b) show the relationships between γ_p-convergence of measures and Γ-convergence of suitable functionals. For a general exposition of the properties of the Γ-convergence we refer to [10].

Remark 1.8. It is easy to see that, if the sequence (μ_j) γ_p-converges to μ, then (μ_j) γ_p-converges also to any other measure λ which is equivalent to μ in $\mathcal{M}_0^p(\Omega)$. In particular, by Remark 1.4, we can always suppose that the γ_p-limit of a sequence (μ_j) in $\mathcal{M}_0^p(\Omega)$ belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$.

Many properties of the measure $\mu \in \mathcal{M}_0^p(\Omega)$ can be studied by means of the solution w of the problem

$$\text{(1.8)}\qquad \begin{cases} w \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega), \\ \displaystyle\int_\Omega |Dw|^{p-2} Dw Dv\, dx + \int_\Omega |w|^{p-2} wv\, d\mu = \int_\Omega v\, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega). \end{cases}$$

Let $\tilde{\mu}$ be the measure introduced in (1.3).

Lemma 1.9. *If $\mu \in \mathcal{M}_0^p(\Omega)$ and w is the solution of (1.8), then the measure $\mu \llcorner \{w > 0\}$ is σ-finite on Ω and $\mu(U) = +\infty$ for every C_p-quasi open set $U \subset \Omega$ with $C_p(U \cap \{w = 0\}) > 0$. Consequently $\tilde{\mu}(B) = +\infty$ for every Borel set $B \subset \Omega$ with $C_p(B \cap \{w = 0\}) > 0$. If $u \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$, then $u = 0$ C_p-q.e. in $\{w = 0\}$.*

Proof. See [17], Lemma 5.3. □

Lemma 1.10. *Let $\mu \in \mathcal{M}_0^p(\Omega)$ and let w be the solution of problem (1.8). Then the set $\{w\psi : \psi \in C_0^\infty(\Omega)\}$ is dense in $W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$.*

Proof. See [17], Proposition 5.5. □

Remark 1.11. Let $\mu \in \mathcal{M}_0^p(\Omega)$ and let w be the solution of problem (1.8). Since the measure $\mu \llcorner \{w > \varepsilon\}$ is finite on Ω for every $\varepsilon > 0$, we have

$$\mu(B \cap \{w > \varepsilon\}) \;=\; \inf\{\mu(U \cap \{w > \varepsilon\}) : \; U \text{ open}\,, \; B \subset U \subset \Omega\}\,.$$

As $\{w > \varepsilon\}$ is C_p-quasi open, we have $\tilde{\mu}(B) = \mu(B)$ for every Borel set $B \subset \{w > \varepsilon\}$. Since $\varepsilon > 0$ is arbitrary, we obtain $\tilde{\mu}(B) = \mu(B)$ for every Borel set $B \subset \{w > 0\}$. By Lemma 1.9 this implies that

$$\tilde{\mu} \;=\; \mu \llcorner \{w > 0\} + \infty_{\{w=0\}}\,,$$

where ∞_E is the measure defined by (1.2).

The general non-linear problem. Throughout the paper $A\colon W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ denotes a fixed monotone operator of the form

$$Au = -\mathrm{div}\big(a(x, Du)\big)\,,$$

where $a\colon \Omega \times \mathbf{R}^n \mapsto \mathbf{R}^n$ is a Carathéodory function. We assume that there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

(i) $$\big(a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2\big) \geq c_1|\xi_1 - \xi_2|^p\,,$$

(ii) $$|a(x,\xi_1) - a(x,\xi_2)| \;\leq\; c_2(1 + |\xi_1| + |\xi_2|)^{p-2}|\xi_1 - \xi_2|\,,$$

(iii) $$a(x,0) = 0$$

for a.e. $x \in \Omega$ and for every ξ_1, $\xi_2 \in \mathbf{R}^n$. Our assumptions imply that

(iv) $$\big(a(x,\xi),\xi\big) \;\geq\; c_1|\xi|^p\,,$$

(v) $$|a(x,\xi)| \leq c_2(1 + |\xi|)^{p-1}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$. We can extend the function $a(x,\xi)$ to $\mathbf{R}^n \times \mathbf{R}^n$ preserving all properties listed above by setting, e.g., $a(x,\xi) = \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega a(y,\xi)\,dy$ for every $x \in \mathbf{R}^n \setminus \Omega$ and for every $\xi \in \mathbf{R}^n$. In the sequel we shall use the following lemma.

Lemma 1.12. *Let E be a closed set in $\mathbf{R}^n$, let U be an open set in $\mathbf{R}^n$, and let w_1 and w_2 be two functions in $W^{1,p}(\mathbf{R}^n)$ such that $w_1 = w_2$ C_p-q.e. in E and $w_1 \le w_2$ C_p-q.e. in U. Assume that $Aw_1 = Aw_2$ in $W^{-1,q}(U \setminus E)$. Then $Aw_1 \ge Aw_2$ in $W^{-1,q}(U)$.*

Proof. See [18], Lemma 2.5. □

Let $c_3 > 0$ and $c_4 > 0$ be two constants and let $0 < \sigma \le 1$. We define $\mathcal{F}(c_3, c_4, \sigma)$ as the set of all Borel functions $b: \Omega \times \mathbf{R} \to \mathbf{R}$ which satisfy the following properties:

$$\text{(I)} \qquad \big(b(x,t) - b(x,s)\big)(t-s) \ge c_3|t-s|^p\,,$$

$$\text{(II)} \qquad |b(x,t) - b(x,s)| \le c_4(|t| + |s|)^{p-1-\sigma}|t-s|^\sigma\,,$$

$$\text{(III)} \qquad b(x,0) = 0$$

for every $x \in \Omega$ and for every t, $s \in \mathbf{R}$. Our assumptions imply that

$$\text{(IV)} \qquad b(x,s)s \ge c_3|s|^p\,,$$

$$\text{(V)} \qquad |b(x,s)| \le c_4|s|^{p-1}$$

for every $x \in \Omega$ and for every $s \in \mathbf{R}$.

Let $\mu \in \mathcal{M}_0^p(\Omega)$, let $b \in \mathcal{F}(c_3, c_4, \sigma)$, and let $f \in W^{-1,q}(\Omega)$. We shall consider the following relaxed Dirichlet problem:

$$(1.9) \qquad \begin{cases} u \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \\ \displaystyle\int_\Omega \big(a(x,Du), Dv\big)\,dx + \int_\Omega b(x,u)v\,d\mu \;=\; \langle f, v\rangle \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,. \end{cases}$$

By the classical theory of monotone operators (see, e.g., [25], Chapter 2, Theorem 2.1) it is easy to prove that problem (1.9) has a unique solution u.

Remark 1.13. Using Theorem 4.5 of [24] it is easy to check that, if E is closed and μ is the measure ∞_E introduced in (1.2), then problem (1.9) reduces to the following boundary value problem on $\Omega \setminus E$:

$$(1.10) \qquad \begin{cases} u \in W_0^{1,p}(\Omega \setminus E)\,, \\ Au = f \;\text{ in }\; W^{-1,q}(\Omega \setminus E)\,, \end{cases}$$

in the sense that u is the solution of (1.9) if and only if the restriction of u to $\Omega \setminus E$ is the solution of (1.10) and $u = 0$ C_p-q.e. in E.

Definition 1.14. Let (μ_j) be a sequence in $\mathcal{M}_0^p(\Omega)$, let (b_j) be a sequence in $\mathcal{F}(c_3, c_4, \sigma)$, let $\mu \in \mathcal{M}_0^p(\Omega)$, and let $b \in \mathcal{F}(c_3, c_4, \sigma)$. Let Ω' an open subset of Ω, let $f \in W^{-1,p'}(\Omega')$, and let (u_j) be a sequence of solutions of the problems

$$\begin{cases} u_j \in W^{1,p}(\Omega') \cap L^p_{\mu_j}(\Omega'), \\ \displaystyle\int_{\Omega'} \big(a(x, Du_j), Dv\big)\, dx + \int_{\Omega'} b(x, u_j) v \, d\mu_j = \langle f, v \rangle \\ \forall v \in W_0^{1,p}(\Omega') \cap L^p_{\mu_j}(\Omega'). \end{cases} \tag{1.11}$$

which satisfy

$$\int_{\Omega'} |u_j|^p d\mu_j < C \,, \tag{1.12}$$

where C is a positive constant independent of j.

We say that the pair (b_j, μ_j) γ_A*-converges* to the pair (b, μ) if, for every open set $\Omega' \subset \Omega$, for every $f \in W^{-1,p'}(\Omega')$ the cluster points in the weak topology of $W^{1,p}(\Omega')$ of any sequence (u_j) which satisfies (1.11) and (1.12) are solutions of the problem

$$\begin{cases} u \in W^{1,p}(\Omega') \cap L^p_{\mu}(\Omega'), \\ \displaystyle\int_{\Omega'} \big(a(x, Du), Dv\big)\, dx + \int_{\Omega'} b(x, u) v \, d\mu = \langle f, v \rangle \\ \forall v \in W_0^{1,p}(\Omega') \cap L^p_{\mu}(\Omega'). \end{cases} \tag{1.13}$$

Clearly if the sequence (b_j, μ_j) γ_A-converges to (b, μ) in Ω, then (b_j, μ_j) γ_A-converges to (b, μ) in Ω' for every open set $\Omega' \subset \Omega$.

The following proposition shows the strong convergence of the gradients of the solutions.

Proposition 1.15. *Let (μ_j) be a sequence in $\mathcal{M}_0^p(\Omega)$ and let (b_j) be a sequence in $\mathcal{F}(c_3, c_4, \sigma)$, and let (u_j) be a sequence of solutions of problems (1.11) which satisfies (1.12). Assume that (u_j) converges to some function u weakly in $W^{1,p}(\Omega')$, then (u_j) converges to u strongly in $W^{1,r}(\Omega')$ for every $r < p$ and $\big(a(x, Du_j)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega', \mathbf{R}^n)$ and strongly in $L^s(\Omega', \mathbf{R}^n)$ for every $s < q$.*

Proof. See [6], Proposition 4.4. □

The compactness of the γ_A-convergence is given by the following theorem.

Theorem 1.16. *Let (μ_j) be a sequence in $\mathcal{M}_0^p(\Omega)$ and let (b_j) be a sequence in $\mathcal{F}(c_3, c_4, \sigma)$. Then there exist a subsequence (b_{j_k}, μ_{j_k}) of (b_j, μ_j), a measure $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$, and a function $b \in \mathcal{F}(c_3', c_4', \sigma')$, with $c_3' = \min\{c_1, c_3\}$, $c_4' = C(c_1, c_2, c_3, c_4, p)$, and $\sigma' = \min\{\sigma, \frac{1}{p-\sigma}\}$, such that (μ_{j_k}) γ_p-converges to μ and (b_{j_k}, μ_{j_k}) γ_A-converges to (b, μ).*

Proof. See [12], Theorem 2.1, or [17], Theorem 6.5, for the γ_p-convergence, and [6], Theorem 5.4 and Remark 5.5, for the γ_A-convergence. □

By a penalization method it is possible to prove the following result.

Lemma 1.17. *Let (μ_j) be a sequence in $\mathcal{M}_0^p(\Omega)$ and let (b_j) be a sequence in $\mathcal{F}(c_3, c_4, \sigma)$. Suppose that (b_j, μ_j) γ_A-converges to (b, μ), with $\mu \in \mathcal{M}_0^p(\Omega)$ and $b \in \mathcal{F}(c_3, c_4, \sigma)$. Let $\mathcal{U}$ be the set of all functions u in $W_0^{1,p}(\Omega)$ with the following property: there exists $f \in W^{-1,q}(\Omega)$ such that sequence (u_j) of the solutions of the problems*

$$
(1.14)\qquad \begin{cases} u_j \in W_0^{1,p}(\Omega) \cap L^p_{\mu_j}(\Omega)\,, \\ \displaystyle\int_\Omega \big(a(x, Du_j), Dv\big)\, dx + \int_\Omega b_j(x, u_j) v \, d\mu_j \;=\; \langle f, v\rangle \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_{\mu_j}(\Omega) \end{cases}
$$

converges to u weakly in $W_0^{1,p}(\Omega)$. Then $\mathcal{U}$ is dense in $W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$.

Proof. Let us fix $u \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$, and for every integer k let u_k be the solution of the following problem

$$
\begin{cases} u_k \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \\ \displaystyle\int_\Omega \big(a(x, Du_k), Dv\big)\, dx + \int_\Omega b(x, u_k) v \, d\mu \;=\; k \int_\Omega \big(|u|^{p-2}u - |u_k|^{p-2}u_k\big) v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,. \end{cases}
$$

Let $f_k = k\,\big(|u|^{p-2}u - |u_k|^{p-2}u_k\big)$ and, for every j, let u_j^k be the solution of problem (1.14) with $f = f_k$. By Definition 1.14 the sequence (u_j^k) converges to u_k weakly in $W_0^{1,p}(\Omega)$ as $j \to \infty$. Therefore $u_k \in \mathcal{U}$ for every k. Arguing as in the proof of Lemma 5.2 in [17] we obtain that (u_k) converges to u both in $W_0^{1,p}(\Omega)$ and in $L^p_\mu(\Omega)$. □

Let us remark that the γ_A-limit of the sequence (b_j,μ_j) is not unique. For instance, if (b_j,μ_j) γ_A-converges to (b,μ), then (b_j,μ_j) γ_A-converges also to $(b,\tilde\mu)$, since for every u, $v\in W^{1,p}(\Omega)\cap L^p_\mu(\Omega)=W^{1,p}(\Omega)\cap L^p_{\tilde\mu}(\Omega)$ we have

$$\int_\Omega b(x,u)v\,d\mu \;=\; \int_\Omega b(x,u)v\,d\tilde\mu$$

by Lemma 1.9 and Remark 1.11. The following lemma shows the relationships between two different γ_A-limits (b,μ) and (g,λ) of the same sequence (b_j,μ_j).

Lemma 1.18. *Let (μ_j) be a sequence in $\mathcal{M}^p_0(\Omega)$ and let (b_j) be a sequence in $\mathcal{F}(c_3,c_4,\sigma)$. Suppose that (b_j,μ_j) γ_A-converges to (b,μ), with $\mu\in\mathcal{M}^p_0(\Omega)$ and $b\in\mathcal{F}(c_3,c_4,\sigma)$. Let $\lambda\in\mathcal{M}^p_0(\Omega)$ and let $g\in\mathcal{F}(c_3,c_4,\sigma)$.*

(a) *If for every Borel set $B\subset\Omega$ and for every $s\in\mathbf{R}$*

$$\int_B g(x,s)\,d\lambda \;=\; \int_B b(x,s)\,d\mu\,,$$

then $L^p_\lambda(\Omega)=L^p_\mu(\Omega)$, (b_j,μ_j) γ_A-converges to (g,λ), and

$$\int_B g(x,u)v\,d\lambda \;=\; \int_B b(x,u)v\,d\mu$$

for every u, $v\in L^p_\mu(\Omega)$ and for every Borel set $B\subset\Omega$.

(b) *If μ and λ belong to $\tilde{\mathcal{M}}^p_0(\Omega)$ and (b_j,μ_j) γ_A-converges to (g,λ), then*

$$\int_B g(x,s)\,d\lambda \;=\; \int_B b(x,s)\,d\mu$$

for every Borel set $B\subset\Omega$ and for every $s\in\mathbf{R}$.

Proof. Let us prove (a). Suppose that

$$\int_B g(x,s)\,d\lambda \;=\; \int_B b(x,s)\,d\mu \tag{1.15}$$

for every Borel set $B\subset\Omega$ and for every $s\in\mathbf{R}$. By (IV) and (V) we have that

$$\frac{c_3}{c_4}\mu \;\le\; \lambda \;\le\; \frac{c_4}{c_3}\mu\,, \tag{1.16}$$

hence $L^p_\lambda(\Omega) = L^p_\mu(\Omega)$. In order to prove that (b_j,μ_j) γ_A-converges to (g,λ) it is enough to show that

$$\int_B g(x,u)v\,d\lambda = \int_B b(x,u)v\,d\mu \tag{1.17}$$

for every u, $v\in L^p_\mu(\Omega)$ and for every Borel set $B\subset\Omega$. Let us fix u, $v\in L^p_\mu(\Omega)$ and let $E=\{v\neq 0\}$. Then μ is σ-finite on E. If we apply (1.15) with $s=1$ we obtain

$$\lambda(B) = \int_B \frac{b(x,1)}{g(x,1)}\,d\mu \tag{1.18}$$

for every Borel set $B\subset E$. Therefore (1.15) gives

$$b(x,s) = g(x,s)\frac{b(x,1)}{g(x,1)} \tag{1.19}$$

for μ-a.e. $x\in E$ and for every $s\in\mathbf{R}$. The continuity with respect to s, assumed in (II), implies that the μ-null set where (1.19) is not satisfied can be chosen independently of s. From (1.19) and (1.18) for every Borel set $B\subset\Omega$ we obtain

$$\begin{aligned}\int_B b(x,u)v\,d\mu &= \int_{B\cap E} b(x,u)v\,d\mu = \int_{B\cap E} g(x,u)v\frac{b(x,1)}{g(x,1)}\,d\mu = \\ &= \int_{B\cap E} g(x,u)v\,d\lambda = \int_B g(x,u)v\,d\lambda\,,\end{aligned}$$

which proves (1.17) and concludes the proof of (a).

Let us prove (b). Assume that μ and λ belong to $\tilde{\mathcal{M}}^p_0(\Omega)$ and that (b_j,μ_j) γ_A-converges to (b,μ) and (g,λ), and let $\mathcal{U}$ be the set of functions defined in Lemma 1.17. By the definition of γ_A-convergence we have

$$\int_\Omega g(x,u)v\,d\lambda = \int_\Omega b(x,u)v\,d\mu \tag{1.20}$$

for every u, $v\in\mathcal{U}$. By (IV) and (V) this implies that

$$\frac{c_3}{c_4}\int_\Omega |u|^p d\mu \le \int_\Omega |u|^p d\lambda \le \frac{c_4}{c_3}\int_\Omega |u|^p d\mu$$

for every $u\in\mathcal{U}$. Since $\mathcal{U}$ is dense in $W^{1,p}_0(\Omega)\cap L^p_\mu(\Omega)$ and in $W^{1,p}_0(\Omega)\cap L^p_\lambda(\Omega)$ (Lemma 1.17), we conclude that $W^{1,p}_0(\Omega)\cap L^p_\mu(\Omega) = W^{1,p}_0(\Omega)\cap L^p_\lambda(\Omega)$. From (II) and (V) we deduce that (1.20) holds for every $u, v \in W_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$. We have to prove that (1.15) holds. Let us prove that

$$\int_B sb(x,s)\,d\mu \le \int_B sg(x,s)\,d\lambda \tag{1.21}$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$. Since μ and λ belong to $\tilde{\mathcal{M}}_0^p(\Omega)$, it is enough to prove (1.21) when B is C_p-quasi open (Remark 1.5). Moreover it is not restrictive to suppose that $\lambda(B) < +\infty$, otherwise the inequality is trivial. By Lemma 1.1 there exists an increasing sequence (v_k) is in $W_0^{1,p}(\Omega)$ which converges to 1_B pointwise C_p-q.e. in Ω and satisfies the inequalities $0 \le v_k \le 1_B$ C_p-q.e. in Ω. As $\lambda(B) < +\infty$, the function sv_k belongs to $W_0^{1,p}(\Omega) \cap L^p_\lambda(\Omega)$. Then we can take $u = v = sv_k$ in (1.20), and passing to the limit as $k \to \infty$ we get (1.21) for every C_p-quasi open subset of Ω. In the same way we obtain the opposite inequality, and dividing by s we obtain (1.15).
$\square$

Remark 1.19. By Theorem 1.16 and Lemma 1.18 it is easy to see that to prove the γ_A-convergence of the sequence (b_j, μ_j) to (b, μ) it is enough to verify the weak convergence of the sequences of solutions with boundary value zero, i.e., that for every $f \in W^{-1,q}(\Omega)$ the sequence (u_j) of the solutions of the problems (1.14) converges weakly in $W_0^{1,p}(\Omega)$ to the solution u of the problem (1.9).

Our aim in the next sections is to determine the function b and the measure μ in the limit problem (1.12) by means of the asymptotic behaviour of some capacities related with the approximating problems (1.11).

2. A non-linear capacity

In this section we introduce a notion of intrinsic capacity related with problems of the form (1.9). Let us fix a function b in the class $\mathcal{F}(c_3, c_4, \sigma)$, with $0 < c_3 \le c_4$ and $0 < \sigma \le 1$. This function will remain fixed until the end of Section 6. Let $\mu \in \mathcal{M}_0^p(\Omega)$, let $B \subset\subset \Omega$ be a Borel set, and let $s \in \mathbf{R}$. Let us consider the problem

$$\begin{cases} u \in W_0^{1,p}(\Omega)\,, \qquad u - s \in L^p_\mu(B)\,, \\ \displaystyle\int_\Omega \big(a(x,Du), Dv\big)\,dx + \int_B b(x, u-s)v\,d\mu \;=\; 0 \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B)\,. \end{cases} \tag{2.1}$$

Since $B \subset\subset \Omega$, there exists a function $\varphi \in W_0^{1,p}(\Omega)$ such that $\varphi = 1$ C_p-q.e. in B. This implies that u is a solution of problem (2.1) if and only if $z = u - s\varphi$ is a solution of the problem

$$
\text{(2.2)} \qquad \begin{cases} z \in W_0^{1,p}(\Omega) \cap L^p_\mu(B)\,, \\ \displaystyle\int_\Omega \big(a(x, Dz + sD\varphi), Dv\big)\,dx + \int_B b(x,z)v\,d\mu \;=\; 0 \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B)\,, \end{cases}
$$

which admits a unique solution by the classical theory of monotone operators.

Definition 2.1. The solution u of problem (2.1) is called the $C_A^{b,\mu}$*-capacitary potential of* B *relative to* s. The quantity

$$
\text{(2.3)} \qquad C_A^{b,\mu}(B,s) \;=\; \int_\Omega \big(a(x,Du), Du\big)\,dx + \int_B b(x, u-s)(u-s)\,d\mu
$$

is called the $C_A^{b,\mu}$*-capacity of* B *relative to* s.

Remark 2.2. If E is a Borel subset of Ω and μ is the measure ∞_E introduced in (1.2), then by Remark 1.13 it is easy to see that $C_A^{b,\mu}(B,s)$ coincides with $C_A(B \cap E, s)$, where C_A is the capacity relative to the operator A studied in [18], Section 6 (with $F = \Omega$). If $B \cap E$ is compact, then $C_A(B \cap E, s)$ is defined by (0.3) and (0.4) with $K = B \cap E$.

Remark 2.3. It follows immediately from the definition that $C_A^{b,\mu}(B_1,s) = C_A^{b,\mu}(B_2,s)$ if $C_p(B_1 \triangle B_2) = 0$.

Remark 2.4. If φ is a function in $W_0^{1,p}(\Omega)$ such that $\varphi = 1$ μ-a.e. in B, then we can take $v = u - s\varphi$ as test function in (2.1) and we obtain

$$
\int_\Omega \big(a(x,Du), Du - sD\varphi\big)\,dx + \int_B b(x,u-s)(u-s)\,d\mu \;=\; 0\,.
$$

This implies that

$$
C_A^{b,\mu}(B,s) \;=\; s\int_\Omega \big(a(x,Du), D\varphi\big)\,dx\,.
$$

Let us prove some basic properties of the $C_A^{b,\mu}$-capacitary potentials. In the sequel we shall always give the proofs of our statements only in the case $s > 0$, the proof in the other case being analogous.

Proposition 2.5. *Let $B \subset\subset \Omega$ be a Borel set, let $s \in \mathbf{R}$, and let u be the corresponding $C_A^{b,\mu}$-capacitary potential. Then $0 \le us \le s^2$ C_p-q.e. in Ω.*

Proof. Assume that $s > 0$. First we prove that $u \le s$ C_p-q.e. in Ω. Let us consider the function $v = (u - s)^+$. Since $v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B)$ we can take it as test function in problem (2.1) and we obtain

$$\int_{\{u>s\}} \big(a(x, Dv), Dv\big)\, dx + \int_{B\cap\{u>s\}} b(x, v)v\, d\mu \;=\; 0\,.$$

By assumptions (i) and (I) we get $\int_\Omega |Dv|^p dx \le 0$ and this implies that $u \le s$ C_p-q.e. in Ω. Similarly, since $|u^-| \le |u - s|$, we can take $v = u^-$ as test function in (2.1), and we obtain that $u \ge 0$ C_p-q.e. in Ω. ☐

In the sequel we shall consider the $C_A^{b,\mu}$-capacitary potentials as functions defined in $\mathbf{R}^n$ by setting them equal to zero in $\Omega^c = \mathbf{R}^n \setminus \Omega$.

Theorem 2.6. *Let $B \subset\subset \Omega$ be a Borel set, let $s \in \mathbf{R}$, and let u be the corresponding $C_A^{b,\mu}$-capacitary potential. Then there exist two Radon measures λ and ν in $W^{-1,q}(\mathbf{R}^n)$, with* $\mathrm{supp}(\lambda) \subset \overline{B}$ *and* $\mathrm{supp}(\nu) \subset \partial\Omega$, *such that*

$$Au \;=\; \lambda - \nu \quad \textit{in } W^{-1,q}(\mathbf{R}^n)\,.$$

Moreover the measures $s\lambda$ and $s\nu$ are non-negative.

Proof. Assume that $s > 0$. Given $v \in W_0^{1,p}(\Omega)$, with $v \ge 0$ C_p-q.e. in Ω, and $\varepsilon > 0$, let us consider the function $w = \varepsilon v \wedge (s - u)$. Since $w \in W_0^{1,p}(\Omega) \cap L^p_\mu(B)$, using w as test function in problem (2.1) we obtain

$$\varepsilon \int_{\{\varepsilon v<s-u\}} \big(a(x, Du), Dv\big)\, dx - \int_{\{\varepsilon v\ge s-u\}} \big(a(x, Du), Du\big)\, dx + \int_\Omega b(x, u - s)w\, d\mu \;=\; 0\,.$$

Since $w(u - s) \le 0$ C_p-q.e. in Ω, we obtain

$$\int_{\{\varepsilon v<s-u\}} \big(a(x, Du), Dv\big)\, dx \;\ge\; 0$$

for every $\varepsilon > 0$. As $s - u \ge 0$ and $a(x, 0) = 0$ a.e. in Ω, taking the limit as $\varepsilon \to 0$ we obtain $\int_\Omega \big(a(x, Du), Dv\big)\, dx \;\ge\; 0$ for every $v \in W_0^{1,p}(\Omega)$ with $v \ge 0$ C_p-q.e. in Ω, and hence $Au \geq 0$ in $W^{-1,q}(\Omega)$. Then there exists a non-negative Radon measure $\lambda \in W^{-1,q}(\Omega)$ such that $Au = \lambda$ in $W^{-1,q}(\Omega)$, i.e.,

$$\int_\Omega \big(a(x, Du), Dv\big)\, dx = \int_\Omega v\, d\lambda \tag{2.4}$$

for every $v \in W_0^{1,p}(\Omega)$. Let us prove now that $\mathrm{supp}(\lambda) \subset \overline{B}$. Let $\varphi \in C_0^\infty(\Omega)$, with $\varphi = 0$ in $\overline{B}$. As $\varphi \in W_0^{1,p}(\Omega) \cap L^p_\mu(B)$, by (2.1) and (2.4) we get

$$\int_{\mathbf{R}^n} \varphi\, d\lambda = 0\,,$$

hence $\mathrm{supp}(\lambda) \subset \overline{B}$. This shows that λ is a non-negative bounded measure on $\mathbf{R}^n$ and that $\lambda \in W^{-1,q}(\mathbf{R}^n)$.

Given $z \in W^{1,p}(\mathbf{R}^n)$, with $z \geq 0$ C_p-q.e. in Ω, and $\varepsilon > 0$, we can take $\varepsilon z \wedge u$ as test function in (2.4) and we get

$$\varepsilon \int_{\{\varepsilon z < u\}} \big(a(x, Du), Dz\big)\, dx + \int_{\{\varepsilon z \geq u\}} \big(a(x, Du), Du\big)\, dx - \int_{\mathbf{R}^n} (\varepsilon z \wedge u)\, d\lambda = 0\,,$$

and hence

$$\varepsilon \int_{\{\varepsilon z < u\}} \big(a(x, Du), Dz\big)\, dx - \varepsilon \int_{\mathbf{R}^n} z\, d\lambda \leq 0\,.$$

Since, by Proposition 2.5, $u \geq 0$ C_p-q.e. in Ω, and since $a(x, 0) = 0$ a.e. in Ω, taking the limit as $\varepsilon \to 0$ we obtain

$$\int_{\mathbf{R}^n} \big(a(x, Du), Dz\big)\, dx - \int_{\mathbf{R}^n} z\, d\lambda \leq 0$$

for every $z \in W^{1,p}(\mathbf{R}^n)$ with $z \geq 0$ C_p-q.e. in Ω. This implies that there exists a non-negative Radon measure ν in $W^{-1,q}(\mathbf{R}^n)$ such that $Au = \lambda - \nu$ in $W^{-1,q}(\mathbf{R}^n)$.

Since $a(x, 0) = 0$ a.e. in $\mathbf{R}^n$ and $\mathrm{supp}(\lambda) \subset \overline{B}$, we have $Au = 0 = \lambda$ in $W^{-1,q}(\mathbf{R}^n \setminus \overline{\Omega})$. This implies that $\mathrm{supp}(\nu) \subset \overline{\Omega}$. As $Au = \lambda$ in $W^{-1,q}(\Omega)$, we conclude that $\mathrm{supp}(\nu) \subset \partial\Omega$. $\square$

Definition 2.7. The measures λ and ν in Theorem 2.6 are called the *inner* and the *outer* $C_A^{b,\mu}$*-capacitary distributions of* B *relative to* s.

Proposition 2.8. *Let* $s \in \mathbf{R}$*, let* $B \subset\subset \Omega$ *be a Borel set, and let* λ *and* ν *be the inner and the outer distributions of* B *relative to* s*. Then*

$$C_A^{b,\mu}(B,s) \;=\; s\nu(\mathbf{R}^n) \;=\; s\lambda(\mathbf{R}^n)\,. \tag{2.5}$$

Proof. Assume that $s > 0$. Let u be the $C_A^{b,\mu}$-capacitary potential of B relative to s. By Theorem 2.6 we obtain

$$\int_{\mathbf{R}^n} v\,d\lambda \;=\; \int_\Omega \big(a(x,Du),Dv\big)\,dx \;=\; -\int_B b(x,u-s)v\,d\mu$$

for every $v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B)$. Since $\mathrm{supp}(\lambda) \subset \overline{B} \subset \Omega$, by using a cut-off function, it is easy to see that

$$\int_{\mathbf{R}^n} v\,d\lambda \;=\; -\int_B b(x,u-s)v\,d\mu \tag{2.6}$$

for every $v \in W^{1,p}(\Omega) \cap L^p_\mu(B)$. Taking $v = u - s$ in (2.6) we obtain

$$\int_{\mathbf{R}^n} u\,d\lambda - s\lambda(\mathbf{R}^n) \;=\; -\int_B b(x,u-s)(u-s)\,d\mu\,.$$

Since by Theorem 2.6

$$\int_\Omega \big(a(x,Du),Du\big)\,dx \;=\; \int_{\mathbf{R}^n} u\,d\lambda\,,$$

we conclude that $C_A^{b,\mu}(B,s) \;=\; s\lambda(\mathbf{R}^n)$.

Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ with $\varphi = 1$ in $\overline{\Omega}$. As $a(x,0) = 0$ a.e. in $\mathbf{R}^n$, we have

$$\int_{\mathbf{R}^n} \big(a(x,Du),D\varphi\big)\,dx \;=\; 0\,.$$

Since $\mathrm{supp}(\lambda)$ and $\mathrm{supp}(\nu)$ are contained in $\overline{\Omega}$, from Theorem 2.6 we obtain

$$\lambda(\mathbf{R}^n) - \nu(\mathbf{R}^n) \;=\; \int_{\mathbf{R}^n} \varphi\,d\lambda - \int_{\mathbf{R}^n} \varphi\,d\nu \;=\; \int_{\mathbf{R}^n} \big(a(x,Du),D\varphi\big)\,dx \;=\; 0\,,$$

which gives $\lambda(\mathbf{R}^n) = \nu(\mathbf{R}^n)$ and concludes the proof of the proposition. □

3. Monotonicity properties

In this section we shall prove the main monotonicity properties of the $C_A^{b,\mu}$-capacity and some comparison result that will be useful in the sequel.

Lemma 3.1. *Let $B \subset\subset \Omega$ be a Borel set and let μ_1 and μ_2 be two measures in $\mathcal{M}_0^p(\Omega)$ such that $\mu_1 \llcorner B \le \mu_2 \llcorner B$. Let u_1 (resp. u_2) be the C_A^{b,μ_1}-capacitary (resp. C_A^{b,μ_2}-capacitary) potential of B relative to a constant $s \in \mathbf{R}$. Then $|u_1| \le |u_2|$ C_p-q.e. in Ω.*

Proof. Assume that $s > 0$. Since, by Proposition 2.5, $u_i \ge 0$ C_p-q.e. in Ω, we have to prove that $u_1 \le u_2$ C_p-q.e. in Ω. Let $v = (u_1 - u_2)^+$. Since $0 \le v \le s - u_2$, we have that $v \in W_0^{1,p}(\Omega) \cap L^p_{\mu_2}(B) \subset W_0^{1,p}(\Omega) \cap L^p_{\mu_1}(B)$. As $u_1 - s \le 0$ (Proposition 2.5), by (2.1) we get

$$\begin{aligned} 0 \,=\, &\int_\Omega \big(a(x,Du_1),Dv\big)\,dx + \int_B b(x,u_1-s)v\,d\mu_1 \,\ge \\ \ge\, &\int_\Omega \big(a(x,Du_1),Dv\big)\,dx + \int_B b(x,u_1-s)v\,d\mu_2 \end{aligned}$$

and

$$\int_\Omega \big(a(x,Du_2),Dv\big)\,dx + \int_B b(x,u_2-s)v\,d\mu_2 \,=\, 0\,.$$

By taking the difference we obtain

$$\begin{aligned} &\int_{\{u_1>u_2\}} \big(a(x,Du_1)-a(x,Du_2),Du_1-Du_2)\big)\,dx\,+ \\ +&\int_{\{u_1>u_2\}} \big(b(x,u_1-s)-b(x,u_2-s)\big)(u_1-u_2)\,d\mu_2 \,\le\, 0\,. \end{aligned}$$

By assumptions (i) and (I) we have that $\int_\Omega |Dv|^p dx \le 0$, and hence $u_1 \le u_2$ C_p-q.e. in Ω. ☐

Proposition 3.2. *Under the same assumptions of Lemma 3.1, let ν_1 (resp. ν_2) be the C_A^{b,μ_1}-capacitary (resp. C_A^{b,μ_2}-capacitary) outer distribution of B relative to a constant $s \in \mathbf{R}$. Then $|\nu_1| \le |\nu_2|$ in $\mathbf{R}^n$.*

Proof. Assume that $s > 0$. Let us apply Lemma 1.12 with $u_1 = w_1$, $u_2 = w_2$, $E = \Omega^c$, and $U = \overline{B}^c$. Since $Au_1 = Au_2 = 0$ in $W^{-1,q}(\Omega \setminus \overline{B})$ and $u_1 \le u_2$ C_p-q.e. in $\mathbf{R}^n$ (Lemma 3.1), we obtain $Au_1 \ge Au_2$ in $W^{-1,q}(\overline{B}^c)$. Thus for every $\varphi \in W^{1,p}(\mathbf{R}^n)$, with $\varphi = 0$ C_p-q.e. in $\overline{B}$ and $\varphi \ge 0$ C_p-q.e. in $\overline{B}^c$, we have

$$\int_{\mathbf{R}^n} \varphi\,d\nu_1 \;=\; -\int_\Omega \big(a(x,Du_1),D\varphi\big)\,dx \;\le\; -\int_\Omega \big(a(x,Du_2),D\varphi\big)\,dx \;=\; \int_{\mathbf{R}^n} \varphi\,d\nu_2\,.$$

Since $\operatorname{supp}(\nu_1) \subset \partial\Omega$ and $\operatorname{supp}(\nu_2) \subset \partial\Omega$, this inequality implies that $\nu_1 \le \nu_2$ in $\mathbf{R}^n$. ☐

Theorem 3.3. *Under the same assumptions of Lemma 3.1, we have* $C_A^{b,\mu_1}(B,s) \leq C_A^{b,\mu_2}(B,s)$ *for every* $s \in \mathbf{R}$.

Proof. The conclusion follows directly from Propositions 2.8 and 3.2. □

We are now in a position to prove that $C_A^{b,\mu}(\cdot,s)$ is an increasing set function.

Theorem 3.4. *Let* $\mu \in \mathcal{M}_0^p(\Omega)$ *and let* $s \in \mathbf{R}$. *Then*

$$C_A^{b,\mu}(B_1,s) \;\leq\; C_A^{b,\mu}(B_2,s)$$

for every pair B_1, B_2 *of Borel sets such that* $B_1 \subset B_2 \subset\subset \Omega$.

Proof. Since $C_A^{b,\mu}(B_1,s) = C_A^{b,\mu \llcorner B_1}(B_2,s)$, to prove the result it is enough to apply Theorem 3.3 with $B = B_2$, $\mu_1 = \mu \llcorner B_1$, and $\mu_2 = \mu$. □

Proposition 3.5. *There exists a constant* $k > 0$, *depending only on* p, c_1, c_2, c_3, c_4, *and* $\mathrm{diam}(\Omega)$, *such that*

$$C_A^{b,\mu}(B,s) \leq C_A(B,s) \leq k\,(|s| + |s|^p)\, C_p(B)\,, \tag{3.1}$$

$$C_A^{b,\mu}(B,s) \leq -s \int_B b(x,-s)\, d\mu \tag{3.2}$$

for every $\mu \in \mathcal{M}_0^p(\Omega)$, *for every Borel set* $B \subset\subset \Omega$, *and for every* $s \in \mathbf{R}$.

Proof. Let us prove (3.1). Let ν be the measure ∞_B introduced in (1.2). By Remark 2.2 we have that $C_A^{b,\nu}(B,s) = C_A(B,s)$. The first inequality in (3.1) follows from Theorem 3.3 and from the fact that $\mu \llcorner B \leq \nu \llcorner B$ for every $\mu \in \mathcal{M}_0^p(\Omega)$. The last inequality in (3.1) is proved in [18], Proposition 6.6.

To prove (3.2) we can suppose that $s > 0$ and $\mu(B) < +\infty$. In this case the constant functions belong to the space $L^p_\mu(B)$, so by (2.6), taking $v = s$, we get

$$s\lambda(\mathbf{R}^n) \;=\; -\int_B b(x, u-s)s\, d\mu\,,$$

where λ is the inner $C_A^{b,\mu}$-capacitary distribution of B relative to s and u is the $C_A^{b,\mu}$-capacitary potential of B relative to s. By Proposition 2.5 we have that $u \geq 0$ C_p-q.e. in Ω, so that by the monotonicity of $b(x,\cdot)$ and by Proposition 2.8 we obtain

$$C_A^{b,\mu}(B,s) \;=\; s\lambda(\mathbf{R}^n) \;\leq\; -s\int_B b(x,-s)\, d\mu\,,$$

which concludes the proof of (3.2). □

In the sequel we shall need to compare the $C_A^{b,\mu}$-capacity relative to a constant s with the $C_A^{b,\mu}$-capacity relative to the constant 1. To this aim we prove the following proposition, which shows the relationships between the $C_A^{b,\mu}$-capacity and the C_p^{μ}-capacity introduced in [11] and defined for every Borel set $B \subset\subset \Omega$ by

$$C_p^{\mu}(B) \,=\, \min_{u\in W_0^{1,p}(\Omega)} \Big(\int_\Omega |Du|^p dx + \int_B |u-1|^p d\mu \Big)\,. \tag{3.3}$$

By using the direct methods of the calculus of variations it is easy to prove that the minimum problem in the definition of $C_p^{\mu}(B)$ has a unique minimum point v, called the C_p^{μ}-capacitary potential of B, and that $v-1$ belongs to $L^p_\mu(B)$. By looking at the Euler condition satisfied by v we can prove that $C_p^{\mu}(B) = C_{A_p}^{b_p,\mu}(B,1)$, where A_p is the p-Laplacian, defined by $A_p u = -\mathrm{div}\big(|Du|^{p-2}Du\big)$, and $b_p(x,s)=|s|^{p-2}s$.

Proposition 3.6. *There exist two positive constants* k_1 *and* k_2*, depending only on* p*,* c_1*,* c_2*,* c_3*,* c_4*, and* $\mathrm{diam}(\Omega)$*, such that*

$$k_1|s|^p C_p^{\mu}(B) \,\le\, C_A^{b,\mu}(B,s) \,\le\, k_2(|s|+|s|^p)\, C_p^{\mu}(B) \tag{3.4}$$

for every $\mu \in \mathcal{M}_0^p(\Omega)$*, for every Borel set* $B\subset\subset\Omega$*, and for every* $s\in\mathbf{R}$*.*

Proof. Let us prove the result for $s>0$, the proof for the other case being analogous. The first inequality in (3.4) follows immediately from the definition of C_p^{μ} given in (3.3) and from inequalities (iv) and (IV).

Let us prove the estimate from above. Let v be the C_p^{μ}-capacitary potential of B, i.e., the minimum point in (3.3), and let $z=(2v-1)^+$. Clearly $z\in W_0^{1,p}(\Omega)$. Moreover it is easy to see that $|z-1|\le 2|v-1|$, hence $z-1\in L^p_\mu(B)$.

Let u be the $C_A^{b,\mu}$-capacitary potential of B relative to s. As $u-sz$ belongs to $W_0^{1,p}(\Omega)\cap L^p_\mu(B)$, we can take it as test function in problem (2.1) and we get

$$\begin{aligned} &\int_\Omega \big(a(x,Du),Du\big)\,dx + \int_B b(x,u-s)(u-s)\,d\mu \,= \\ &= s\int_\Omega \big(a(x,Du),Dz\big)\,dx \,+\, s\int_B b(x,u-s)(z-1)\,d\mu\,, \end{aligned}$$

hence

$$C_A^{b,\mu}(B,s) \,=\, s\int_\Omega \big(a(x,Du),Dz\big)\,dx \,+\, s\int_B b(x,u-s)(z-1)\,d\mu\,. \tag{3.5}$$

Let $U = \{v > 1/2\} = \{z > 0\}$. By the definition of $C_A^{b,\mu}(B,s)$, by inequalities (v) and (V), and by Hölder's inequality we get

$$\begin{aligned} C_A^{b,\mu}(B,s) &\le c\,s\int_U (1+|Du|^{p-1})|Dv|\,dx + c\,s\int_B |u-s|^{p-1}|z-1|\,d\mu \le \\ &\le c\,s\,\mathrm{meas}(U)^{1/q}\Big(\int_\Omega |Dv|^p dx\Big)^{1/p} + c\,s\Big(\int_\Omega |Du|^p dx\Big)^{1/q}\Big(\int_\Omega |Dv|^p dx\Big)^{1/p} + \\ &\qquad + c\,s\Big(\int_B |u-s|^p d\mu\Big)^{1/q}\Big(\int_B |z-1|^p d\mu\Big)^{1/p}, \end{aligned} \tag{3.6}$$

where c denotes a positive constant, depending only on p, c_1, c_2, c_3, c_4, and $\mathrm{diam}(\Omega)$, whose value can change from line to line. By Poincaré's inequality we have

$$\mathrm{meas}(U) \le 2^p\int_\Omega |v|^p dx \le c\int_\Omega |Dv|^p dx\,. \tag{3.7}$$

Applying Young's inequality and taking the inequality $|z-1| \le 2|v-1|$ into account, from (3.6) and (3.7) we obtain for every $\varepsilon > 0$

$$\begin{aligned} C_A^{b,\mu}(B,s) &\le c\,s\int_\Omega |Dv|^p dx + \varepsilon\int_\Omega |Du|^p dx + c(\varepsilon)\,s^p\int_\Omega |Dv|^p dx + \\ &\qquad + \varepsilon\int_B |u-s|^p d\mu + c(\varepsilon)\,s^p\int_B |v-1|^p d\mu\,, \end{aligned}$$

where $c(\varepsilon)$ is a positive constant which depends only on p, c_1, c_2, c_3, c_4, $\mathrm{diam}(\Omega)$, and ε. If we choose $\varepsilon < \min\{c_1/2, c_3/2\}$ and if we take (iv) and (IV) into account, we obtain that there exists a positive constant k_2 such that

$$C_A^{b,\mu}(B,s) \le k_2(s+s^p)\,C_p^\mu(B)\,,$$

and this concludes the proof. □

Remark 3.7. By Proposition 3.6 we have

$$k_1(2k_2)^{-1}|s|^p C_A^{b,\mu}(B,1) \le C_A^{b,\mu}(B,s) \le k_2 k_1^{-1}(|s|+|s|^p)\,C_A^{b,\mu}(B,1) \tag{3.8}$$

for every $\mu \in \mathcal{M}_0^p(\Omega)$, for every Borel set $B \subset\subset \Omega$, and for every $s \in \mathbf{R}$.

We prove now the continuity of $C_A^{b,\mu}(B,s)$ with respect to s.

Proposition 3.8. *Let $\tau = \sigma/(p-\sigma)$, where σ is the exponent which appears in condition (II). Then there exists a positive constant k, depending only on p, c_1, c_2, c_3, c_4, σ, and* $\mathrm{diam}(\Omega)$, *such that*

$$\left|C_A^{b,\mu}(B,s_1) - C_A^{b,\mu}(B,s_2)\right| \leq k\, C_p^{\mu}(B)\, (1+|s_1|+|s_2|)^{p-\tau}|s_1-s_2|^{\tau} \tag{3.9}$$

for every $\mu \in \mathcal{M}_0^p(\Omega)$, for every Borel set $B \subset\subset \Omega$, and for every s_1, $s_2 \in \mathbf{R}$.

Proof. Since by (3.4) $C_A^{b,\mu}(B,s)$ tends to $C_A^{b,\mu}(B,0) = 0$ as $s \to 0$, it is not restrictive to assume that $s_1 \neq 0$ and $s_2 \neq 0$. Let u_1 and u_2 be the $C_A^{b,\mu}$-capacitary potentials of B relative to the constants s_1 and s_2. Let v and z be the functions introduced in the proof of Proposition 3.6. Since the function $u_1 - u_2 - (s_1 - s_2)z$ belongs to $W_0^{1,p}(\Omega) \cap L_\mu^p(B)$, we can take it as test function in the problems satisfied by u_1 and u_2. Subtracting the two equations we obtain

$$\begin{aligned}
&\int_\Omega \big(a(x,Du_1) - a(x,Du_2), Du_1 - Du_2\big)\, dx + \\
&+ \int_B \big(b(x,u_1-s_1) - b(x,u_2-s_2)\big)\big((u_1-s_1)-(u_2-s_2)\big)\, d\mu = \\
&= (s_1-s_2)\int_\Omega \big(a(x,Du_1) - a(x,Du_2), Dz\big)\, dx + \\
&+ (s_1-s_2)\int_B \big(b(x,u_1-s_1) - b(x,u_2-s_2)\big)(z-1)\, d\mu\,.
\end{aligned}$$

From (i), (ii), (I), (II) it follows that

$$\begin{aligned}
&c_1\int_\Omega |Du_1 - Du_2|^p dx + c_3\int_B |(u_1-s_1)-(u_2-s_2)|^p d\mu \leq \\
&\leq c_2|s_1-s_2|\,\mathcal{J}_1 + c_4|s_1-s_2|\,\mathcal{J}_2\,,
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
\mathcal{J}_1 &= \int_\Omega (1+|Du_1|+|Du_2|)^{p-2}|Du_1-Du_2|\,|Dz|\, dx\,, \\
\mathcal{J}_2 &= \int_B (|u_1-s_1|-|u_2-s_2|)^{p-1-\sigma}|(u_1-s_1)-(u_2-s_2)|^{\sigma}|z-1|\, d\mu\,.
\end{aligned} \tag{3.11}$$

In the rest of the proof the letter c will denote various positive constants, depending only on p, c_1, c_2, c_3, c_4, σ, and $\mathrm{diam}(\Omega)$, whose value can change from line to line.

Let $U = \{v > 1/2\} = \{z > 0\}$. As $|z-1| \le 2|v-1|$ and $|Dz| \le 2|Dv|$, by Hölder's inequality we have

$$
\begin{aligned}
\mathcal{J}_1 \,&\le\, c\Big(\mathrm{meas}(U) + \int_\Omega |Du_1|^p dx + \int_\Omega |Du_2|^p dx\Big)^{(p-2)/p} \cdot \\
&\qquad \cdot \Big(\int_\Omega |Du_1 - Du_2|^p dx\Big)^{1/p}\Big(\int_\Omega |Dv|^p dx\Big)^{1/p}, \\
\mathcal{J}_2 \,&\le\, c\Big(\int_B |u_1 - s_1|^p d\mu + \int_B |u_2 - s_2|^p d\mu\Big)^{(p-1-\sigma)/p} \cdot \\
&\qquad \cdot \Big(\int_B |(u_1 - s_1) - (u_2 - s_2)|^p d\mu\Big)^{\sigma/p}\Big(\int_B |v-1|^p d\mu\Big)^{1/p}.
\end{aligned}
\tag{3.12}
$$

Let

$$
\mathcal{A} \,=\, C_p^\mu(B) + C_A^{b,\mu}(B, s_1) + C_A^{b,\mu}(B, s_2)\,.
\tag{3.13}
$$

By (3.7) and by (i) and (I) we have

$$
\begin{aligned}
\mathrm{meas}(U) \le c\int_\Omega |Dv|^p dx \,&\le\, c\,\mathcal{A}\,, \\
\int_\Omega |Du_1|^p dx + \int_\Omega |Du_2|^p dx \,&\le\, c\,\mathcal{A}\,, \\
\int_B |u_1 - s_1|^p d\mu + \int_B |u_2 - s_2|^p d\mu \,&\le\, c\,\mathcal{A}\,.
\end{aligned}
\tag{3.14}
$$

Taking the definition of v into account, from (3.12) and (3.14) we obtain

$$
\begin{aligned}
\mathcal{J}_1 \,&\le\, c\,\mathcal{A}^{(p-2)/p} C_p^\mu(B)^{1/p}\Big(\int_\Omega |Du_1 - Du_2|^p dx\Big)^{1/p}, \\
\mathcal{J}_2 \,&\le\, c\,\mathcal{A}^{(p-1-\sigma)/p} C_p^\mu(B)^{1/p}\Big(\int_B |(u_1 - s_1) - (u_2 - s_2)|^p d\mu\Big)^{\sigma/p}.
\end{aligned}
\tag{3.15}
$$

By Young's inequality from (3.10) and (3.15) we obtain

$$
\begin{aligned}
&\int_\Omega |Du_1 - Du_2|^p dx + \int_B |(u_1 - s_1) - (u_2 - s_2)|^p d\mu \,\le \\
&\quad \le\, c\,|s_1 - s_2|^{p/(p-1)} \mathcal{A}^{(p-2)/(p-1)} C_p^\mu(B)^{1/(p-1)} + \\
&\quad + c\,|s_1 - s_2|^{p/(p-\sigma)} \mathcal{A}^{(p-1-\sigma)/(p-\sigma)} C_p^\mu(B)^{1/(p-\sigma)}\,,
\end{aligned}
$$

and by (3.4) this implies

$$
\begin{aligned}
&\int_\Omega |Du_1 - Du_2|^p dx + \int_B |(u_1 - s_1) - (u_2 - s_2)|^p d\mu \,\le \\
&\quad \le\, c\,|s_1 - s_2|^{p/(p-\sigma)} \mathcal{A}^{(p-1-\sigma)/(p-\sigma)} C_p^\mu(B)^{1/(p-\sigma)}\,.
\end{aligned}
\tag{3.16}
$$

By (3.5) and by (ii) and (II) we have

$$\Big|\frac{1}{s_1}C_A^{b,\mu}(B,s_1)-\frac{1}{s_2}C_A^{b,\mu}(B,s_2)\Big| \,\le\, c\,\mathcal{J}_1+c\,\mathcal{J}_2\,.$$

Therefore (3.15) and (3.16) yield

$$\begin{gathered}\Big|\frac{1}{s_1}C_A^{b,\mu}(B,s_1)-\frac{1}{s_2}C_A^{b,\mu}(B,s_2)\Big| \,\le \\ \le\, c\,\mathcal{A}^{(p-2)/p}C_p^\mu(B)^{1/p}|s_1-s_2|^{1/(p-\sigma)}\mathcal{A}^{(p-1-\sigma)/p(p-\sigma)}C_p^\mu(B)^{1/p(p-\sigma)}+ \\ +\,c\,\mathcal{A}^{(p-1-\sigma)/p}C_p^\mu(B)^{1/p}|s_1-s_2|^{\sigma/(p-\sigma)}\mathcal{A}^{\sigma(p-1-\sigma)/p(p-\sigma)}C_p^\mu(B)^{\sigma/p(p-\sigma)}\,.\end{gathered}\tag{3.17}$$

From (3.4), (3.17), and (3.13) we obtain

$$\Big|\frac{1}{s_1}C_A^{b,\mu}(B,s_1)-\frac{1}{s_2}C_A^{b,\mu}(B,s_2)\Big| \,\le\, c\,C_p^\mu(B)\,(1+|s_1|+|s_2|)^{p-1-\tau}|s_1-s_2|^\tau\,,$$

which, together with (3.4), gives (3.9). □

4. Continuity properties

In this section we prove the continuity properties of $C_A^{b,\mu}(\cdot,s)$ along monotone sequences of Borel sets.

Theorem 4.1. *Let $\mu\in\mathcal{M}_0^p(\Omega)$ and $s\in\mathbf{R}$. If $B\subset\subset\Omega$ is the union of an increasing sequence (B_j) of Borel subsets of Ω, then*

$$C_A^{b,\mu}(B,s) \,=\, \lim_{j\to\infty} C_A^{b,\mu}(B_j,s) \,=\, \sup_j\, C_A^{b,\mu}(B_j,s)\,.$$

Proof. Let us consider the case $s>0$, the other case being analogous.

Let $S=\sup_j C_A^{b,\mu}(B_j,s)$. By monotonicity (Theorem 3.4) we have $S\le C_A^{b,\mu}(B,s)$. It remains to prove the opposite inequality. For every j let u_j be the $C_A^{b,\mu}$-capacitary potential of B_j relative to s. Since

$$c_1\int_\Omega|Du_j|^p dx\,+\,c_3\int_{B_j}|u_j-s|^p d\mu\,\le\, C_A^{b,\mu}(B_j,s)\,\le\, C_A^{b,\mu}(B,s)<+\infty\,,\tag{4.1}$$

and since, by Lemma 3.1, (u_j) is increasing, we have that the sequence (u_j) converges weakly in $W_0^{1,p}(\Omega)$ to some function $u\in W_0^{1,p}(\Omega)$. Since $|u-s|=s-u\le s-u_j=|u_j-s|$, from (4.1) we obtain that $u-s\in L^p_\mu(B)$. Let us prove that u is the $C^{b,\mu}_A$-capacitary potential of B relative to s. Since $W^{1,p}_0(\Omega)\cap L^p_\mu(B)\subset W^{1,p}_0(\Omega)\cap L^p_\mu(B_j)$, for every j we have

$$\int_\Omega \big(a(x,Du_j),Dv\big)\,dx+\int_{B_j} b(x,u_j-s)v\,d\mu \;=\;0 \qquad \forall\, v\in W^{1,p}_0(\Omega)\cap L^p_\mu(B)\,. \tag{4.2}$$

Since (u_j) is increasing, (u_j) converges to u μ-a.e. in Ω (see [14], Lemma 1.2). Together with (4.1) and (V), this implies that $\big(1_{B_j}b(x,u_j-s)\big)$ converges to $1_B b(x,u-s)$ weakly in $L^q_\mu(B)$. Moreover, if we apply Proposition 1.15 to the sequence (u_j-s), we obtain that $\big(a(x,Du_j)\big)$ converges to $a(x,Du)$ weakly in $L^q(\Omega,\mathbf{R}^n)$. Therefore, taking the limit in (4.2) as $j\to\infty$ we obtain

$$\int_\Omega \big(a(x,Du),Dv\big)\,dx+\int_{B} b(x,u-s)v\,d\mu \;=\;0 \qquad \forall\, v\in W^{1,p}_0(\Omega)\cap L^p_\mu(B)\,,$$

so that u is the $C^{b,\mu}_A$-capacitary potential of B relative to s.

If $\varphi\in W^{1,p}_0(\Omega)$ and $\varphi=1$ C_p-q.e. in $\overline{B}$, then by Remark 2.4 we have

$$C^{b,\mu}_A(B,s)\;=\;s\int_\Omega\big(a(x,Du),D\varphi\big)\,dx\;=$$
$$=\;\lim_{j\to\infty} s\int_\Omega\big(a(x,Du_j),D\varphi\big)\,dx\;=\;\lim_{j\to\infty} C^{b,\mu}_A(B_j,s)\,,$$

which concludes the proof of the theorem. □

To prove the continuity along decreasing sequences we need an additional assumption on the sequence and the regularity of the measure μ. Let us prove first the following lemma which gives the γ_p-convergence of the restrictions of a measure $\mu\in\tilde{\mathcal{M}}^p_0(\Omega)$ to a decreasing sequence of C_p-quasi closed sets.

Lemma 4.2. *Let $\mu\in\tilde{\mathcal{M}}^p_0(\Omega)$ and let K be the intersection of a decreasing sequence (K_j) of C_p-quasi closed subsets of Ω. Then for every Borel subset B of Ω the sequence of measures $\big(\mu\llcorner(B\cup K_j)\big)$ γ_p-converges to $\mu\llcorner(B\cup K)$.*

Proof. Let us consider first the case $B=\emptyset$. By the compactness of the γ_p-convergence (Theorem 1.16), we have that, up to a subsequence, $(\mu\llcorner K_j)$ γ_p-converges to some measure $\mu_0\in\tilde{\mathcal{M}}^p_0(\Omega)$. Let us prove that $\mu_0=\mu\llcorner K$.

By Remark 1.7(a) we have $\int_\Omega |u|^p d\mu_0=0$ for every $u\in W^{1,p}_0(\Omega)$ such that $u=0$ C_p-q.e. on K_j for some j. By Lemma 1.1 this implies that $\mu_0=0$ on $\Omega\setminus K$. Let us prove now that $\mu_0 \le \mu \llcorner K$. Let w be the solution of problem (1.8) and let $\varphi \in C_0^\infty(\Omega)$. By Remark 1.7(a) we have

$$\int_\Omega |\varphi w|^p d\mu_0 \le \lim_{j\to\infty} \int_{K_j} |\varphi w|^p d\mu = \int_K |\varphi w|^p d\mu . \tag{4.3}$$

This implies that

$$\int_B |w|^p d\mu_0 \le \int_{K\cap B} |w|^p d\mu$$

for every Borel set $B \subset \Omega$, hence $\mu_0 \le \mu \llcorner K$ on $\{w > 0\}$. Since $(\mu \llcorner K)(B) = +\infty$ if $C_p(B \cap K \cap \{w = 0\}) > 0$ (Lemma 1.9) and since $\mu_0 = 0$ on $\Omega \setminus K$, we conclude that $\mu_0 \le \mu \llcorner K$.

Let us prove finally that $\mu \llcorner K \le \mu_0$. Let z be the solution of the problem

$$\begin{cases} z \in W_0^{1,p}(\Omega) \cap L^p_{\mu_0}(\Omega) , \\ \displaystyle\int_\Omega |Dz|^{p-2} Dz Dv \, dx + \int_\Omega |z|^{p-2} zv \, d\mu_0 = \int_\Omega v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_{\mu_0}(\Omega) , \end{cases}$$

and, for every j, let z_j be the solution of the problem

$$\begin{cases} z_j \in W_0^{1,p}(\Omega) \cap L^p_\mu(K_j) , \\ \displaystyle\int_\Omega |Dz_j|^{p-2} Dz_j Dv \, dx + \int_{K_j} |z_j|^{p-2} z_j v \, d\mu = \int_\Omega v \, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(K_j) . \end{cases}$$

Since $(\mu \llcorner K_j)$ γ_p-converges to μ_0, the sequence (z_j) converges to z weakly in $W_0^{1,p}(\Omega)$ and strongly in $W_0^{1,r}(\Omega)$ for every $r < p$ (Proposition 1.15). Moreover, since by Lemma 3.1 (z_j) is decreasing, we have also that (z_j) converges to (the C_p-quasi continuous representative of) z pointwise C_p-q.e. in Ω (see [14], Lemma 1.2). Then, by Fatou's lemma, for every $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega |Dz|^p \varphi \, dx + \int_K |z|^p \varphi \, d\mu \le$$

$$\le \liminf_{j\to\infty} \Big(\int_\Omega |Dz_j|^p \varphi \, dx + \int_K |z_j|^p \varphi \, d\mu \Big) \le \liminf_{j\to\infty} \Big(\int_\Omega |Dz_j|^p \varphi \, dx + \int_{K_j} |z_j|^p \varphi \, d\mu \Big) =$$

$$= \liminf_{j\to\infty} \Big(- \int_\Omega |Dz_j|^{p-2} Dz_j D\varphi \, z_j \, dx + \int_\Omega \varphi z_j \, dx \Big) =$$

$$= - \int_\Omega |Dz|^{p-2} Dz D\varphi \, z \, dx + \int_\Omega \varphi z \, dx = \int_\Omega |Dz|^p \varphi \, dx + \int_\Omega |z|^p \varphi \, d\mu_0 .$$

This implies that

$$\int_K |z|^p \varphi \, d\mu \le \int_\Omega |z|^p \varphi \, d\mu_0$$

for every $\varphi \in C_0^\infty(\Omega)$, and hence $\mu \llcorner K \le \mu_0$ in $\{z > 0\}$. Since $\mu_0(B) = +\infty$ if $C_p(B \cap \{z = 0\}) > 0$ (Lemma 1.9), we have proved that $\mu \llcorner K \le \mu_0$ in Ω, which, together with the opposite inequality, gives $\mu_0 = \mu \llcorner K$.

Let us fix now a Borel set $B \subset \Omega$ and let us prove that $\big(\mu \llcorner (B \cup K_j)\big)$ γ_p-converges to $\mu \llcorner (B \cup K)$. By Remark 1.7 it is enough to prove that for every sequence (u_j) which converges to a function u weakly in $W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |Du|^p dx + \int_{B\cup K} |u|^p d\mu \le \liminf_{j\to\infty} \Big(\int_\Omega |Du_j|^p dx + \int_{B \cup K_j} |u_j|^p d\mu \Big) \tag{4.4}$$

and that for every $u \in W_0^{1,p}(\Omega)$ there exists a sequence (u_j) which converges to u weakly in $W_0^{1,p}(\Omega)$ such that

$$\limsup_{j\to\infty} \Big(\int_\Omega |Du_j|^p dx + \int_{B\cup K_j} |u_j|^p d\mu \Big) \le \int_\Omega |Du|^p dx + \int_{B \cup K} |u|^p d\mu \,. \tag{4.5}$$

Let (u_j) be a sequence which converges to a function u weakly in $W_0^{1,p}(\Omega)$. By the lower semicontinuity of the norm we get

$$\int_\Omega |Du|^p dx \le \liminf_{j\to\infty} \int_\Omega |Du_j|^p dx \,. \tag{4.6}$$

Since μ vanishes on all sets of C_p-null sets, by Fatou's lemma the functional $u \mapsto \int_{B\cup K} |u|^p d\mu$ is lower semicontinuous in the strong topology of $W_0^{1,p}(\Omega)$. As this functional is convex, it is lower semicontinuous in the weak topology of $W_0^{1,p}(\Omega)$. Therefore

$$\int_{B\cup K} |u|^p d\mu \le \liminf_{j\to\infty} \int_{B\cup K} |u_j|^p d\mu \le \liminf_{j\to\infty} \int_{B\cup K_j} |u_j|^p d\mu \tag{4.7}$$

and inequality (4.4) follows from (4.6) and (4.7).

Let us fix now a function $u \in W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K)$ and let us construct a sequence (u_j) which satisfies (4.5). Since, by the previous step, $(\mu \llcorner K_j)$ γ_p-converges to $\mu \llcorner K$, there exists a sequence (v_j) which converges to u weakly in $W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |Du|^p dx + \int_K |u|^p d\mu = \lim_{j\to\infty} \Big(\int_\Omega |Dv_j|^p dx + \int_{K_j} |v_j|^p d\mu \Big) \,. \tag{4.8}$$

Since the sequence (v_j) satisfies (4.6) and (4.7) with $B = \emptyset$, by (4.8) we obtain

$$(4.9) \qquad \int_\Omega |Du|^p dx = \lim_{j\to\infty} \int_\Omega |Dv_j|^p dx \qquad \text{and} \qquad \int_K |u|^p d\mu = \lim_{j\to\infty} \int_{K_j} |v_j|^p d\mu\,,$$

hence (v_j) converges to u strongly in $W_0^{1,p}(\Omega)$.

We are now in a position to construct a sequence (u_j) such that (4.5) holds. Let $u_j = (v_j \wedge |u|) \vee (-|u|)$, i.e.,

$$u_j = \begin{cases} |u|\,, & \text{if } v_j > |u|, \\ v_j\,, & \text{if } |v_j| \le |u|, \\ -|u|\,, & \text{if } v_j < -|u|. \end{cases}$$

It is easy to see that (u_j) converges to u strongly in $W_0^{1,p}(\Omega)$. Since $|u_j| = |v_j| \wedge |u|$, with $|v_j| \in L^p_\mu(K_j)$ and $|u| \in L^p_\mu(B)$, we conclude that $|u_j| \in L^p_\mu(K_j \cup B)$. Every subsequence of $(|u_j|^p 1_{B\setminus K_j})$ has a further subsequence which converges to $|u|^p 1_{B\setminus K}$ C_p-q.e. in Ω. As $|u_j|^p 1_{B\setminus K_j} \le |u|^p 1_B$, by the Dominated Convergence Theorem we have

$$\lim_{j\to\infty} \int_{B\setminus K_j} |u_j|^p d\mu = \int_{B\setminus K} |u|^p d\mu\,.$$

Thus, by (4.9) and taking into account that $|u_j| \le |v_j|$, we get

$$\begin{aligned} \limsup_{j\to\infty} \int_{B\cup K_j} |u_j|^p d\mu &\le \limsup_{j\to\infty} \int_{K_j} |u_j|^p d\mu + \limsup_{j\to\infty} \int_{B\setminus K_j} |u_j|^p d\mu \le \\ &\le \lim_{j\to\infty} \int_{K_j} |v_j|^p d\mu + \int_{B\setminus K} |u|^p d\mu = \int_{B\cup K} |u|^p d\mu\,. \end{aligned}$$

This fact, together with the strong convergence of (u_j) in $W_0^{1,p}(\Omega)$, implies that (4.5) holds and concludes the proof of the lemma. □

Theorem 4.3. *Let $\mu \in \tilde{\mathcal{M}}^p_0(\Omega)$ and let K be the intersection of a decreasing sequence (K_j) of C_p-quasi closed sets such that $K_j \subset\subset \Omega$ for every j. Then*

$$C_A^{b,\mu}(B\cup K, s) = \lim_{j\to\infty} C_A^{b,\mu}(B\cup K_j, s) = \inf_j C_A^{b,\mu}(B\cup K_j, s)$$

for every Borel set $B \subset\subset \Omega$ and for every $s \in \mathbf{R}$.

Proof. For every j let u_j be the $C_A^{b,\mu}$-capacitary potential of $B \cup K_j$ relative to s. As in the proof of Theorem 4.1 we have that the sequence (u_j) is decreasing and converges weakly in $W_0^{1,p}(\Omega)$ and C_p-quasi everywhere in Ω to some function u in $W_0^{1,p}(\Omega)$. Since $W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K_j) \subset W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K_i)$ for every $i \geq j$, following the lines of the proof of Theorem 4.1 we obtain that $u - s \in L^p_\mu(B \cup K)$ and

$$\int_\Omega \big(a(x, Du), Dv\big)\, dx + \int_{B\cup K} b(x, u - s) v\, d\mu \,=\, 0 \tag{4.10}$$

for every function v which belongs to $W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K_j)$ for some j. To conclude that u is the $C_A^{b,\mu}$-capacitary potential of $B \cup K$ relative to s it is enough to prove (4.10) for every $v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K)$.

Let z_j be the solution of the problem

$$\begin{cases} z_j \in W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K_j)\,, \\ \displaystyle\int_\Omega |Dz_j|^{p-2} Dz_j Dv\, dx + \int_{B\cup K_j} |z_j|^{p-2} z_j v\; d\mu \,=\, \int_\Omega v\, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K_j)\,, \end{cases} \tag{4.11}$$

and let z be the solution of the problem

$$\begin{cases} z \in W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K)\,, \\ \displaystyle\int_\Omega |Dz|^{p-2} Dz Dv\, dx + \int_{B\cup K} |z|^{p-2} z v\; d\mu \,=\, \int_\Omega v\, dx \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K)\,. \end{cases} \tag{4.12}$$

By Lemma 4.2 we have that $\big(\mu \llcorner (B \cup K_j)\big)$ γ_p-converges to $\mu \llcorner (B \cup K)$ and hence (z_j) converges to z weakly in $W_0^{1,p}(\Omega)$. By taking $v = z_j$ in (4.11) and $v = z$ in (4.12) we obtain

$$\int_\Omega |Dz|^p dx \,+\, \int_{B\cup K} |z|^p d\mu \,=\, \lim_{j\to\infty} \Big(\int_\Omega |Dz_j|^p dx \,+\, \int_{B\cup K_j} |z_j|^p d\mu \Big)\,. \tag{4.13}$$

Since the functional $z \mapsto \int_{B\cup K} |z|^p d\mu$ is convex and lower semicontinuous in the strong topology of $W_0^{1,p}(\Omega)$, it is lower semicontinuous in the weak topology. Therefore

$$\int_{B\cup K} |z|^p d\mu \,\leq\, \liminf_{j\to\infty} \int_{B\cup K} |z_j|^p d\mu \,\leq\, \liminf_{j\to\infty} \int_{B\cup K_j} |z_j|^p d\mu\,. \tag{4.14}$$

As (z_j) converges to z weakly in $W_0^{1,p}(\Omega)$, (4.13) and (4.14) imply that (z_j) converges to z strongly in $W_0^{1,p}(\Omega)$ and in $L^p_\mu(B \cup K)$.

Given $\varphi \in C_0^\infty(\Omega)$, we take now $v = z_j \varphi$ as test function in (4.10). If we pass to the limit as $j \to \infty$ we obtain (4.10) for every v of the form $v = z\varphi$, with $\varphi \in C_0^\infty(\Omega)$. Using Lemma 1.10 we obtain (4.10) for every $v \in W_0^{1,p}(\Omega) \cap L^p_\mu(B \cup K)$. This shows that u is the $C_A^{b,\mu}$-capacitary potential of $B \cup K$ relative to s. We can then conclude the proof as in Theorem 4.1. $\square$

5. Approximation properties and subadditivity

In this section we conclude the study of the properties of the $C_A^{b,\mu}$-capacity by proving some approximation result and the countable subadditivity of the $C_A^{b,\mu}$-capacity.

Theorem 5.1. *Let $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$. Then*

$$C_A^{b,\mu}(B,s) = \sup\{C_A^{b,\mu}(K,s) : K \text{ compact}, \ K \subset B\} \tag{5.1}$$

for every Borel set $B \subset\subset \Omega$ and for every $s \in \mathbf{R}$.

Proof. Let us fix $s \in \mathbf{R}$ and a Borel set $B \subset\subset \Omega$. Let α be the set function defined by $\alpha(E) = C_A^{b,\mu}(E \cap \overline{B}, s)$ for every Borel set $E \subset \mathbf{R}^n$. Since $\overline{B}$ is a compact set contained in Ω, by Theorems 3.4, 4.1, and 4.3 the function α satisfies the following properties:

(a) if $E \subset F$, then $\alpha(E) \le \alpha(F)$;

(b) if E is the union of an increasing sequence (E_j) of Borel sets in $\mathbf{R}^n$, then $\alpha(E) = \sup_j \alpha(E_j)$;

(c) if K is the intersection of a decreasing sequence (K_j) of compact sets in $\mathbf{R}^n$, then $\alpha(K) = \inf_j \alpha(K_j)$.

Therefore α is a capacity in the sense of Choquet. By the Capacitability Theorem ([7], Theorem 1) for every Borel set $E \subset \mathbf{R}^n$ we have

$$\alpha(E) = \sup\{\alpha(K) : K \text{ compact}, \ K \subset E\}. \tag{5.2}$$

The conclusion follows by taking $E = B$. □

In the following lemma we prove that $C_A^{b,\mu}(\cdot, s)$ is subadditive on the family $\mathcal{E}$ of all Borel subsets of Ω of the form $E = K \cap U$, with K C_p-quasi closed and U C_p-quasi open.

Lemma 5.2. *Let $\mu \in \mathcal{M}_0^p(\Omega)$ and let E_1 and E_2 be two Borel sets of the class $\mathcal{E}$. Then*

$$C_A^{b,\mu}(E_1 \cup E_2, s) \le C_A^{b,\mu}(E_1, s) + C_A^{b,\mu}(E_2, s)$$

for every $s \in \mathbf{R}$.

Proof. By Lemma 1.5 in [18] there exist two increasing sequences (K_1^j) and (K_2^j) of compact sets, contained in E_1 and $E_2 \setminus E_1$ respectively, whose unions cover C_p-quasi all of E_1 and $E_2 \setminus E_1$. By Theorem 4.1 and Remark 2.3 we have

$$C_A^{b,\mu}(E_1 \cup E_2, s) \;=\; \lim_{j\to\infty} C_A^{b,\mu}(K_1^j \cup K_2^j, s)\,.$$

Moreover, by monotonicity (Theorem 3.4), we have

$$C_A^{b,\mu}(K_1^j, s) + C_A^{b,\mu}(K_2^j, s) \;\le\; C_A^{b,\mu}(E_1, s) + C_A^{b,\mu}(E_2, s)\,.$$

Thus it is enough to prove that, given two arbitrary disjoint compact sets K_1 and K_2 contained in Ω, we have

$$C_A^{b,\mu}(K_1 \cup K_2, s) \;\le\; C_A^{b,\mu}(K_1, s) + C_A^{b,\mu}(K_2, s)\,.$$

Since K_1 and K_2 are disjoint, there exist two disjoint open sets V_1 and V_2 such that $K_1 \subset V_1 \subset \Omega$ and $K_2 \subset V_2 \subset \Omega$. It is not restrictive to assume that $s > 0$. Let u, u_1, and u_2 (resp. λ, λ_1, and λ_2) be the $C_A^{b,\mu}$-capacitary potentials (resp. inner $C_A^{b,\mu}$-capacitary distributions) of $K_1 \cup K_2$, K_1, and K_2 relative to s. We want to prove that

$$\lambda(B \cap K_1) \;\le\; \lambda_1(B) \qquad \text{and} \qquad \lambda(B \cap K_2) \;\le\; \lambda_2(B) \tag{5.3}$$

for every Borel set $B \subset \Omega$. By Lemma 3.1 and by Proposition 2.5 we have $u_1 \le u$ and $u_2 \le u$ C_p-q.e. in $\mathbf{R}^n$. Let $\varphi \in C_0^\infty(V_1)$, with $\varphi \ge 0$, and let $\varepsilon > 0$. The function $v = \varepsilon\varphi \wedge (u - u_1)$ belongs to $W_0^{1,p}(\Omega) \cap L^p_\mu(K_1 \cup K_2)$, thus we can take it as test function in the problems solved by u_1 and u, and using the argument of the proof of Theorem 2.6 we find that

$$\int_\Omega \big(a(x, Du) - a(x, Du_1), D\varphi\big)\, dx \;\le\; 0$$

for every $\varphi \in C_0^\infty(V_1)$ with $\varphi \ge 0$. By Theorem 2.6 this implies $\lambda(B) \le \lambda_1(B)$ for every Borel set $B \subset V_1$. Since $\mathrm{supp}(\lambda_1) \subset K_1 \subset V_1$ we obtain

$$\lambda(B \cap K_1) \;\le\; \lambda_1(B \cap K_1) \;=\; \lambda_1(B)$$

for every Borel set $B \subset \mathbf{R}^n$. Similarly we obtain the second inequality of (5.3).

Finally, since $\mathrm{supp}(\lambda) \subset K_1 \cup K_2$, we get

$$\lambda(\mathbf{R}^n) \;=\; \lambda(K_1) + \lambda(K_2) \;\le\; \lambda_1(K_1) + \lambda_2(K_2) \;=\; \lambda_1(\mathbf{R}^n) + \lambda_2(\mathbf{R}^n)\,,$$

which concludes the proof by Proposition 2.8. □

Theorem 5.3. *Let* $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$. *Then*

$$C_A^{b,\mu}(B,s) \,=\, \inf\{C_A^{b,\mu}(U,s) \,:\, U \;\; C_p\text{-quasi open}\,,\;\; B \subset U \subset\subset \Omega\} \tag{5.4}$$

for every Borel set $B \subset\subset \Omega$ *and for every* $s \in \mathbf{R}$.

Proof. Let us denote the right hand side of (5.4) by I. By monotonicity we have $C_A^{b,\mu}(B,s) \,\leq\, I$. Let us prove the opposite inequality in the case $s > 0$. Let u be the $C_A^{b,\mu}$-capacitary potential of B relative to s. For every integer j let $B_j = \{x \in B \,:\, u(x) \leq s - 1/j\}$. Since $u - s \in L^p_\mu(B)$, it is easy to see that $\mu(B_j) < +\infty$ for every j. This implies that there exist a compact set K_j and a C_p-quasi open set U_j such that $K_j \subset B_j \subset U_j \subset\subset \Omega$ and $\mu(U_j \setminus K_j) < 1/j$.

Let Ω' be an open set such that $B \subset \Omega' \subset\subset \Omega$. For every j we define let $V_j = \{x \in \Omega' \,:\, u(x) > s - 1/j\}$ and $H_j = \{x \in \overline{\Omega}' \,:\, u(x) \geq s - 1/j\}$. Then the set $A_j = U_j \cup V_j$ is C_p-quasi open and $B \subset A_j \subset U_j \cup H_j \subset\subset \Omega$. Thus, by Lemma 5.2, we have

$$I \,\leq\, C_A^{b,\mu}(A_j,s) \,\leq\, C_A^{b,\mu}(K_j \cup H_j, s) + C_A^{b,\mu}(U_j \setminus K_j, s)\,.$$

Since $\mu(U_j \setminus K_j) < 1/j$, by Theorem 3.4, Proposition 3.5, and inequality (V) we have that

$$I \,\leq\, C_A^{b,\mu}(B \cup H_j, s) + \frac{1}{j} c_4 |s|^p\,. \tag{5.5}$$

As (H_j) is a decreasing sequence of C_p-quasi closed sets contained in $\overline{\Omega}'$, whose intersection (up to C_p-null sets) is the set $H = \{x \in \overline{\Omega}' \,:\, u(x) = s\}$, by Proposition 4.3 we have that

$$C_A^{b,\mu}(B \cup H, s) \,=\, \lim_{j\to\infty} C_A^{b,\mu}(B \cup H_j, s)\,.$$

Then, taking the limit as $j \to \infty$, from (5.5) we get $I \,\leq\, C_A^{b,\mu}(B \cup H, s)$. Finally it is easy to see that $C_A^{b,\mu}(B \cup H, s) = C_A^{b,\mu}(B,s)$, hence $I \,\leq\, C_A^{b,\mu}(B,s)$. □

We are now in a position to complete the proof of the subadditivity of $C_A^{b,\mu}(\cdot,s)$.

Theorem 5.4. *Let* $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$. *Then*

$$C_A^{b,\mu}(B_1 \cup B_2, s) \,\leq\, C_A^{b,\mu}(B_1,s) + C_A^{b,\mu}(B_2,s)\,.$$

for every $s \in \mathbf{R}$ *and for every pair* B_1, B_2 *of Borel sets such that* $B_1 \subset\subset \Omega$ *and* $B_2 \subset\subset \Omega$.

Proof. By Theorem 5.3 for every $\varepsilon > 0$ there exist two C_p-quasi open sets U_1 and U_2 such that $B_1 \subset U_1 \subset\subset \Omega$, $B_2 \subset U_2 \subset\subset \Omega$, and

$$C_A^{b,\mu}(U_1, s) + C_A^{b,\mu}(U_2, s) \le C_A^{b,\mu}(B_1, s) + C_A^{b,\mu}(B_2, s) + \varepsilon .$$

Since $C_A^{b,\mu}(U_1 \cup U_2, s) \le C_A^{b,\mu}(U_1, s) + C_A^{b,\mu}(U_2, s)$ (Lemma 5.2), the conclusion follows from the monotonicity of $C_A^{b,\mu}$ (Theorem 3.4). □

Now we prove that the $C_A^{b,\mu}$-capacity is countably subadditive.

Theorem 5.5. *Let* $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$. *Then*

$$C_A^{b,\mu}(B, s) \le \sum_{j=1}^{\infty} C_A^{b,\mu}(B_j, s)$$

for every sequence (B_j) *of Borel sets whose union* B *is relatively compact in* Ω.

Proof. The result follows immediately from Theorems 4.1 and 5.4. □

Finally, we prove that the $C_A^{b,\mu}$-capacity of any Borel set can be approximated from above by the $C_A^{b,\mu}$-capacity of open sets.

Theorem 5.6. *Let* $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$. *Then*

$$C_A^{b,\mu}(B, s) = \inf\{C_A^{b,\mu}(U, s) : U \text{ open}, \ B \subset U \subset\subset \Omega\} . \tag{5.6}$$

for every Borel set $B \subset\subset \Omega$ *and for every* $s \in \mathbf{R}$.

Proof. Let us denote the right hand side of (5.6) by I. By monotonicity (Theorem 3.4) we have that $C_A^{b,\mu}(B, s) \le I$. Let us prove the opposite inequality. Thanks to Theorem 5.3 it is enough to prove (5.6) when B is C_p-quasi open. In this case for every $\varepsilon > 0$ there exists an open set $U \subset\subset \Omega$ such that $C_p(U \triangle B) < \varepsilon$. This implies that there exists an open set $V \subset\subset \Omega$ such that $U \triangle B \subset V$ and $C_p(V) < \varepsilon$. Thus by Theorems 5.4 and 3.5 we have

$$\begin{aligned} I &\le C_A^{b,\mu}(U \cup V, s) = C_A^{b,\mu}(B \cup V, s) \le C_A^{b,\mu}(B, s) + C_A^{b,\mu}(V, s) \le \\ &\le C_A^{b,\mu}(B, s) + k\,(|s| + |s|^p)\, C_p(V) \le C_A^{b,\mu}(B, s) + k\,(|s| + |s|^p)\, \varepsilon , \end{aligned}$$

hence $I \le C_A^{b,\mu}(B, s)$. □

6. Measures and capacities

In this section we prove a formula which allows us to construct, for every $s \in \mathbf{R}$, the measure $b(x,s)\mu$ once we know $C_A^{b,\mu}(B,-s)$ for every Borel set $B \subset \Omega$.

Theorem 6.1. *Suppose that* $2 \le p \le n$. *Let* $\mu \in \mathcal{M}_0^p(\Omega)$, *let* $s \in \mathbf{R}$, *and let* $B \subset\subset \Omega$ *be a Borel set. Then*

$$\int_B sb(x,s)d\mu \,=\, \sup \sum_{i\in I} C_A^{b,\mu}(B_i,-s)\,, \tag{6.1}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ *of* B.

Remark 6.2. Theorem 6.1 characterizes the measure $\lambda(B) = \int_B sb(x,s)\,d\mu$ as the least among the Borel measures ν such that $\nu(B) \ge C_A^{b,\mu}(B,-s)$ for every Borel set $B \subset\subset \Omega$ (see, e.g., [9], Lemma 4.1).

Proof of Theorem 6.1. Let us fix $s > 0$. For every Borel set $B \subset\subset \Omega$ let $\lambda(B)$ and $\nu(B)$ be the left and the right hand side of (6.1). We want to prove that $\nu(B) = \lambda(B)$.

By Proposition 3.5 we have that $C_A^{b,\mu}(B,-s) \le \lambda(B)$ for every $B \subset\subset \Omega$. Since λ is additive, by the definition of ν we have

$$C_A^{b,\mu}(B,-s) \,\le\, \nu(B) \,\le\, \lambda(B) \tag{6.2}$$

for every Borel set $B \subset\subset \Omega$. It remains to prove that $\lambda(B) \le \nu(B)$. This will be done in three steps.

Step 1. Assume that $\mu \in W^{-1,q}(\Omega)$. As $b(\cdot,s)$ is bounded, we have also $\lambda \in W^{-1,q}(\Omega)$. Since $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$, by Theorem 5.5 the set function $C_A^{b,\mu}(\cdot,s)$ is countably subadditive, and, consequently, ν is a non-negative Borel measure (see, e.g., [9], Lemma 4.1). By the Radon–Nikodym Theorem there exists a Borel function $g\colon \Omega \to [0,1]$ such that

$$\nu(B) \,=\, \int_B g\, d\lambda \tag{6.3}$$

for every Borel set $B \subset\subset \Omega$.

In order to prove that $\lambda \le \nu$, we shall show that $g = 1$ λ-a.e. in Ω. We argue by contradiction. Suppose that $\lambda(\{x \in \Omega :\ g(x) < 1\}) > 0$. Then there exists $\varepsilon > 0$ such that

$$\lambda(\{x \in \Omega :\ g(x) < 1-\varepsilon\}) \,>\, 0\,. \tag{6.4}$$

Let $E = \{x \in \Omega :\ g(x) < 1 - \varepsilon\}$, $\mu_E = \mu \llcorner E$, and $\lambda_E = \lambda \llcorner E = sb(x,s)\mu_E$. Since $0 \le \lambda_E \le \lambda$, we have that $\lambda_E \in W^{-1,q}(\Omega)$. By (6.2) and (6.3) for every Borel set $B \subset\subset \Omega$ we obtain

$$
\begin{aligned}
&C_A^{b,\mu_E}(B,-s) \,=\, C_A^{b,\mu}(B \cap E, -s) \,\le\, \nu(B \cap E) \,= \\
&= \int_{B\cap E} g\, d\lambda \,\le\, (1-\varepsilon)\lambda_E(B) \,=\, (1-\varepsilon)\int_B sb(x,s)\, d\mu_E\,.
\end{aligned}
\tag{6.5}
$$

For every Borel set $B \subset\subset \Omega$ let u_B be the corresponding C_A^{b,μ_E}-capacitary potential relative to the constant $-s$. Since $0 \le u_B + s \le s$, by (II) and (V) we have

$$
b(x, u_B + s)(u_B + s) \,\ge\, b(x,s)s - k_1|u_B|^\sigma\,,
$$

where $k_1 = c_4(1 + 2^{p-1-\sigma})\,|s|^{p-\sigma}$. Thus, by the definition of the C_A^{b,μ_E}-capacity, we obtain

$$
C_A^{b,\mu_E}(B,-s) \,\ge\, \int_B b(x, u_B+s)(u_B+s)\, d\mu_E \,\ge\, \int_B b(x,s)s\, d\mu_E \,-\, k_1\int_B |u_B|^\sigma d\mu_E\,.
$$

Therefore, by (6.5) and (IV) we have

$$
\int_B |u_B|^\sigma d\mu_E \,\ge\, \frac{\varepsilon}{k_1}\int_B b(x,s)s\, d\mu_E \,\ge\, k_2\mu_E(B)\,,
\tag{6.6}
$$

where $k_2 = c_3 s^p \varepsilon / k_1$. Now let U be an open set such that $U \subset\subset \Omega$ and let u_U be the corresponding C_A^{b,μ_E}-capacitary potential relative to the constant $-s$. By Lemma 3.1, if $B \subset U$, then $|u_B| \le |u_U|$ C_p-q.e. in Ω. Thus by (6.6) we obtain

$$
\int_B |u_U|^\sigma d\mu_E \,\ge\, k_2\mu_E(B)
$$

for every Borel set B such that $B \subset U \subset\subset \Omega$. Therefore for every open set $U \subset\subset \Omega$ we get

$$
|u_U| \,\ge\, k_2 \qquad \lambda_E\text{-a.e. in } U\,.
$$

Now let F be the C_p-quasi support of λ_E, i.e., the smallest C_p-quasi closed set F such that λ_E is identically zero on the complement of F (see [11], Definition 2.5). By applying Theorem 2.6 of [11] we obtain

$$
|u_U| \,\ge\, k_2 \qquad C_p\text{-q.e. in } U \cap F
$$

for every open set $U \subset\subset \Omega$.

Then, by the definition of the C_p-capacity and by assumption (i), we get

$$C_A^{b,\mu_E}(U,-s) = \int_\Omega \big(a(x,Du_U),Du_U\big)\,dx + \int_U b(x,u_U+s)(u_U+s)\,d\mu_E \geq$$
$$\geq c_1 \int_\Omega |Du_U|^p dx \geq c_1 k_2^p C_p(U\cap F);$$

taking (6.5) into account, we have

$$C_p(U\cap F) \leq k_3 \lambda_E(U),$$

where $k_3 = (1-\varepsilon)/(c_1 k_2^p)$, and hence

$$\int_0^r \Big(\frac{C_p(F\cap B_\rho(x))}{\rho^{n-p}}\Big)^{1/(p-1)} \frac{d\rho}{\rho} \leq k_3^{1/(p-1)} \int_0^r \Big(\frac{\lambda_E(B_\rho(x))}{\rho^{n-p}}\Big)^{1/(p-1)} \frac{d\rho}{\rho} \tag{6.7}$$

whenever $B_r(x) \subset\subset \Omega$. Now, since $\lambda_E \in W^{-1,q}(\Omega)$, the right hand side of (6.7) is finite λ_E-a.e. in F (see [23], Corollary to Theorem 1, or [34], Theorem 4.7.5), while the left hand side is infinite C_p-q.e. in F (hence λ_E-a.e. in F) by the Kellog property for non-linear potentials (see [23], Theorem 2). This implies $\lambda_E(F) = 0$, hence $\lambda(E) = 0$, which contradicts (6.4). This concludes the proof in the case $\mu \in W^{-1,q}(\Omega)$.

Step 2. Assume that $\mu \in \mathcal{M}_0^p(\Omega)$ and that $\nu(B) < +\infty$ for every Borel set $B \subset\subset \Omega$. By Proposition 3.6 there exists a constant $k > 0$ such that

$$k\,|s|^p C_p^\mu(B) \leq C_A^{b,\mu}(B,-s) \tag{6.8}$$

for every Borel set $B \subset\subset \Omega$. We define

$$\nu_p(B) = \sup \sum_{i\in I} C_p^\mu(B_i), \tag{6.9}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B. By (6.8) we have

$$k\,|s|^p \nu_p(B) \leq \nu(B) < +\infty \tag{6.10}$$

for every Borel set $B \subset\subset \Omega$. By applying Theorem 5.7 of [8] and Lemma 2.3 of [12] we obtain that there exists a non-negative Radon measure $\rho \in W^{-1,q}(\Omega)$ and a non-negative Borel function $\psi\colon \Omega \to [0,+\infty]$ such that

$$\int_U (u^+)^p d\mu = \int_U (u^+)^p \psi\, d\rho \tag{6.11}$$

for every open set $U \subset \Omega$ and for every $u \in W^{1,p}(U)$. For every integer k let $\psi_k = \psi \wedge k$, and let ω_k and ω be the measures defined by

$$\omega_k(B) = \int_B \psi_k d\rho, \qquad \omega(B) = \int_B \psi \, d\rho. \tag{6.12}$$

By (6.11) and (3.3) we obtain

$$C_p^{\omega_k}(U) \le C_p^{\omega}(U) = C_p^{\mu}(U) \le \nu_p(U) \tag{6.13}$$

for every open set $U \subset\subset \Omega$. Since C_p^μ is countably subadditive ([11], Theorem 3.2), by (6.10) the set function ν_p is a Radon measure (see, e.g., [9], Lemma 4.1). Consequently from (6.13) we obtain $C_p^{\omega_k}(B) \le \nu_p(B)$ for every Borel set $B \subset\subset \Omega$. Since $\omega_k \in W^{-1,q}(\Omega)$, from Step 1 we get $\omega_k(B) \le \nu_p(B)$ for every Borel set $B \subset\subset \Omega$. By the Monotone Convergence Theorem and by (6.12) this implies $\omega(B) \le \nu_p(B) < +\infty$ for every Borel set $B \subset\subset \Omega$. As ω is a Radon measure we deduce from (6.11) that μ is a Radon measure too and that $\omega = \mu$. From Theorem 3.3 we obtain

$$C_A^{b,\omega_k}(B, -s) \le C_A^{b,\mu}(B, -s) \le \nu(B)$$

for every k and for every Borel set $B \subset\subset \Omega$. From Step 1 and from (6.12) we obtain

$$\int_B sb(x,s)\psi_k d\rho \le \nu(B).$$

As $k \to \infty$ we get

$$\lambda(B) = \int_B sb(x,s) d\mu = \int_B sb(x,s)\psi d\rho \le \nu(B)$$

for every Borel set $B \subset\subset \Omega$. This proves that $\lambda(B) \le \nu(B)$ whenever $\nu(B) < +\infty$ for every Borel set $B \subset\subset \Omega$.

Step 3. Let us consider now the general case. We want to prove that $\lambda \le \nu$. Let us fix a Borel set $E \subset\subset \Omega$. If $\nu(E) = +\infty$, the inequality $\lambda(E) \le \nu(E)$ is trivial. If $\nu(E) < +\infty$, we consider the measure $\mu \llcorner E$. Since $C_A^{b,\mu \llcorner E}(B, -s) = C_A^{b,\mu}(E \cap B, -s)$, for every Borel set $B \subset\subset \Omega$ we have

$$\sup \sum_{i \in I} C_A^{b,\mu \llcorner E}(B_i, -s) = (\nu \llcorner E)(B) < +\infty,$$

where the supremum is taken over all finite partitions $(B_i)_{i\in I}$ of B. This shows that the pair $(b, \mu\llcorner E)$ satisfies the assumptions of Step 2. Therefore

$$\int_{E\cap B} sb(x,s)\,d\mu \;=\; \int_B sb(x,s)\,d(\mu\llcorner E)\;\leq\;(\nu\llcorner E)(B)\;=\;\nu(E\cap B)$$

for every Borel set $B \subset\subset \Omega$. By taking $B = E$ we obtain

$$\lambda(E)\;=\;\int_E sb(x,s)\,d\mu\;\leq\;\nu(E)\,,$$

which concludes the proof. □

Remark 6.3. The assumption $p \leq n$ in Theorem 6.1 is used to prove that (6.7) leads to a contradiction. The conclusion of Theorem 6.1 is false, in general, when $p > n$, as shown in Example 4.3 of [11].

7. Sequences of Dirichlet problems and sequences of capacities

In this section we consider an arbitrary sequence of measures (μ_j) in $\mathcal{M}_0^p(\Omega)$ and an arbitrary sequence of functions (b_j) in $\mathcal{F}(c_3, c_4, \sigma)$, with $0 < c_3 \leq c_4$ and $0 < \sigma \leq 1$, and we study the relationships between the γ_A-convergence of the sequence (b_j, μ_j) and the convergence of the corresponding $C_A^{b_j,\mu_j}$-capacities.

Let us start with a preliminary result which concerns the convergence properties of the restrictions of the sequence (b_j, μ_j). We shall use the notion of rich family of open subsets of Ω and some results related with the theory of increasing set functions. We recall here the definition of rich family, while we refer to Chapters 14 and 15 of [10] for a general treatment of this subject.

Definition 7.1. We say that a family $\mathcal{D}$ of open sets $U \subset\subset \Omega$ is *dense* if for every pair (K, V), with K compact, V open, and $K \subset V \subset\subset \Omega$, there exist $U \in \mathcal{D}$ such that $K \subset U \subset V$. We say that a family $\mathcal{R}$ of open sets $U \subset\subset \Omega$ is *rich* if, for every chain $(U_t)_{t\in\mathbf{R}}$ of open sets in Ω, the set $\{t \in \mathbf{R} : U_t \notin \mathcal{R}\}$ is at most countable. By a *chain* we mean a family $(U_t)_{t\in\mathbf{R}}$ of open sets such that $U_s \subset\subset U_t \subset\subset \Omega$ for every $s,\ t \in \mathbf{R}$ with $s < t$.

Proposition 7.2. *Let (μ_j) be a sequence in $\mathcal{M}_0^p(\Omega)$, let (b_j) be a sequence in $\mathcal{F}(c_3, c_4, \sigma)$, let $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$, and let $b \in \mathcal{F}(c_3', c_4', \sigma')$, where c_3', c_4', σ' are the constants which appear in Theorem 1.16. Suppose that (μ_j) γ_p-converges to μ and that (b_j, μ_j) γ_A-converges to (b, μ) in Ω. Then there exists a rich family $\mathcal{R}$ of open sets of $U \subset\subset \Omega$ such that the sequence $(b_j, \mu_j \llcorner U)$ γ_A-converges to $(b, \mu \llcorner U)$ for every $U \in \mathcal{R}$.*

Proof. By Theorem 4.4 of [12] there exists a rich family $\mathcal{R}$ of open sets $U \subset\subset \Omega$ such that $(\mu_j \llcorner U)$ γ_p-converges to $\mu \llcorner U$ for every $U \in \mathcal{R}$. Then, by Theorem 1.16, for every $U \in \mathcal{R}$ there exists a function $b_U \in \mathcal{F}(c_3', c_4', \sigma')$ such that a subsequence of $(b_j, \mu_j \llcorner U)$ γ_A-converges to $(b_U, \mu \llcorner U)$. Moreover, by a localization argument we have that the same subsequence of $(b_j, \mu_j \llcorner U)$ γ_A-converges in U to (b, μ) and to $(b_U, \mu \llcorner U)$. By Lemma 1.18(b) this implies that

$$\int_B b(x,s)\,d\mu = \int_B b_U(x,s)\,d\mu$$

for every Borel set $B \subset U$ and for every $s \in \mathbf{R}$. Therefore

$$\int_B b(x,s)\,d(\mu \llcorner U) = \int_B b_U(x,s)\,d(\mu \llcorner U)$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$. Applying Lemma 1.18(a), we obtain that a subsequence of $(b_j, \mu_j \llcorner U)$ γ_A-converges to $(b, \mu \llcorner U)$. Since this result does not depend on the choice of the γ_A-convergent subsequence, we conclude that the whole sequence $(b_j, \mu_j \llcorner U)$ γ_A-converges to $(b, \mu \llcorner U)$ for every $U \in \mathcal{R}$. ☐

Theorem 7.3. *Let (μ_j) be a sequence in $\mathcal{M}_0^p(\Omega)$, let (b_j) be a sequence in $\mathcal{F}(c_3, c_4, \sigma)$, let $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$, and let $b \in \mathcal{F}(c_3', c_4', \sigma')$, where c_3', c_4', σ' are the constants which appear in Theorem 1.16. Assume that $2 \le p \le n$. Then the following conditions are equivalent:*

(a) *(b_j, μ_j) γ_A-converges to (b, μ);*

(b) *for every $s \in \mathbf{R}$ we have*

$$\limsup_{j\to\infty} C_A^{b_j,\mu_j}(U,s) \le C_A^{b,\mu}(V,s) \le \liminf_{j\to\infty} C_A^{b_j,\mu_j}(W,s)$$

whenever U, V, W are open sets with $U \subset\subset V \subset\subset W \subset\subset \Omega$;

(c) *there exists a rich family $\mathcal{R}$ of open sets $U \subset\subset \Omega$ such that*

$$\lim_{j\to\infty} C_A^{b_j,\mu_j}(U,s) = C_A^{b,\mu}(U,s)$$

for every $U \in \mathcal{R}$ *and for every* $s \in \mathbf{R}$*.*

Proof. (a)$\Rightarrow$(b). Assume (a). Let us prove the first inequality in (b) arguing by contradiction. Suppose that there exist $s \in \mathbf{R}$ and two open sets U and V, with $U \subset\subset V \subset\subset \Omega$, such that

$$\limsup_{j\to\infty} C_A^{b_j,\mu_j}(U,s) \; > \; C_A^{b,\mu}(V,s)\,.$$

Passing, if necessary, to a subsequence, still denoted by (b_j,μ_j), we may assume that

$$\lim_{j\to\infty} C_A^{b_j,\mu_j}(U,s) \; > \; C_A^{b,\mu}(V,s)\,. \tag{7.1}$$

By Theorem 1.16 there exist a further subsequence, still denoted by (b_j,μ_j), a measure $\lambda \in \tilde{\mathcal{M}}_0^p(\Omega)$, and a function $g \in \mathcal{F}(c_3', c_4', \sigma')$, such that (μ_j) γ_p-converges to λ and (b_j,μ_j) γ_A-converges to (g,λ). By Lemma 1.18 we have

$$C_A^{g,\lambda}(B,s) = C_A^{b,\mu}(B,s) \tag{7.2}$$

for every Borel set $B \subset\subset \Omega$. Since every rich family is dense ([10], Remark 14.13), by Proposition 7.2 there exists a Borel set B such that $U \subset B \subset V$ and $(b_j, \mu_j \llcorner B)$ γ_A-converges to $(g, \lambda \llcorner B)$. Let u_j be the $C_A^{b_j,\mu_j}$-capacitary potential of B relative to the constant s, and let $z_j = u_j - s$. Then z_j is a solution of the problem

$$\begin{cases} z_j \in W^{1,p}(\Omega) \cap L^p_{\mu_j \llcorner B}(\Omega)\,, \\ \displaystyle\int_\Omega \big(a(x,Dz_j), Dv\big)\,dx + \int_\Omega b(x,z_j) v\, d(\mu_j \llcorner B) \; = \; 0 \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_{\mu_j \llcorner B}(\Omega)\,. \end{cases} \tag{7.3}$$

Let ψ be a function in $W^{1,p}(\Omega)$ such that $\psi = 0$ C_p-q.e. in B and $\psi + s \in W_0^{1,p}(\Omega)$. Taking $v = z_j - \psi$ as test function in (7.3), it is easy to see that

$$\int_\Omega |Dz_j|^p dx + \int_\Omega |z_j|^p d(\mu_j \llcorner B) \le C\,.$$

Then, up to a subsequence, (z_j) converges weakly in $W^{1,p}(\Omega)$ to some function z. Since $(b_j, \mu_j \llcorner B)$ γ_A-converges to $(g, \lambda \llcorner B)$ we have that z is a solution of the problem

$$\begin{cases} z \in W^{1,p}(\Omega) \cap L^p_{\lambda \llcorner B}(\Omega)\,, \\ \displaystyle\int_\Omega \big(a(x,Dz), Dv\big)\,dx + \int_\Omega b(x,z) v\, d(\lambda \llcorner B) \; = \; 0 \\ \forall v \in W_0^{1,p}(\Omega) \cap L^p_{\lambda \llcorner B}(\Omega)\,. \end{cases} \tag{7.4}$$

As $z_j + s \in W_0^{1,p}(\Omega)$ for every j, we have $z + s \in W_0^{1,p}(\Omega)$, so that $z + s$ coincides with the $C_A^{g,\lambda}$-capacitary potential u of B relative to the constant s.

Taking $v = z_j - \psi$ as test function in (7.3), and taking into account that $\psi = 0$ C_p-q.e. in B, we get

$$
\begin{aligned}
C_A^{b_j,\mu_j}(B,s) \;&=\; \int_\Omega \big(a(x,Dz_j),Dz_j\big)\,dx + \int_B b(x,z_j)z_j\,d\mu_j \;= \\
&=\; \int_\Omega \big(a(x,Dz_j),D\psi\big)\,dx\,.
\end{aligned}
\tag{7.5}
$$

Similarly, taking $v = u - \psi$ as test function in (7.4), and using (7.2), we get

$$
C_A^{b,\mu}(B,s) \;=\; C_A^{g,\lambda}(B,s) \;=\; \int_\Omega \big(a(x,Dz),D\psi\big)\,dx\,.
$$

Since by Proposition 1.15 the sequence $\big(a(x,Dz_j)\big)$ converges to $a(x,Dz)$ weakly in $L^q(\Omega,\mathbf{R}^n)$, passing to the limit in (7.5) as $j\to\infty$ we obtain

$$
\lim_{j\to\infty} C_A^{b_j,\mu_j}(B,s) \;=\; C_A^{b,\mu}(B,s)\,.
$$

By monotonicity (Theorem 3.4) we have

$$
\lim_{j\to\infty} C_A^{b_j,\mu_j}(U,s) \;\le\; \lim_{j\to\infty} C_A^{b_j,\mu_j}(B,s) \;=\; C_A^{b,\mu}(B,s) \;\le\; C_A^{b,\mu}(V,s)\,,
$$

which contradicts (7.1). Therefore the first inequality in (b) is proved. The second inequality in (b) can be obtained in the same way.

(b)$\Rightarrow$(c). For every Borel set $B \subset\subset \Omega$ and for every $s\in\mathbf{R}$ let

$$
\alpha'(B,s) \;=\; \liminf_{j\to\infty} C_A^{b_j,\mu_j}(B,s)\,, \qquad \alpha''(B,s) \;=\; \limsup_{j\to\infty} C_A^{b_j,\mu_j}(B,s)\,,
$$
$$
\alpha(B,s) = C_A^{b,\mu}(B,s)\,.
$$

By Theorem 3.4 the functions $\alpha'(B,s)$, $\alpha''(B,s)$, $\alpha(B,s)$ are increasing with respect to B, and by Proposition 3.8 they are continuous with respect to s. If (b) holds, then

$$
\alpha''(U,s) \le \alpha(V,s) \le \alpha'(W,s) \le \alpha''(W,s)
$$

whenever U, V, W are open sets with $U\subset\subset V\subset\subset W\subset\subset\Omega$. By a general property of increasing set functions (see [10], Theorem 15.18) these inequalities imply that there exists a rich family $\mathcal{R}$ of open sets $U\subset\subset\Omega$ such that

$$
\alpha(U,s) = \alpha'(U,s) = \alpha''(U,s)
$$

for every $U \in \mathcal{R}$ and for every $s \in \mathbf{R}$. By the definition of α, α', α'' this is equivalent to (c).

(c) $\Rightarrow$ (a). Assume (c). By Theorem 1.16 there exist a subsequence, still denoted by (b_j, μ_j), a measure $\lambda \in \tilde{\mathcal{M}}_0^p(\Omega)$, and a function $g \in \mathcal{F}(c_3', c_4', \sigma')$ such that (b_j, μ_j) γ_A-converges to (g, λ). Since (a) $\Rightarrow$ (c), there exists a rich family $\mathcal{R}'$ of open sets $U \subset\subset \Omega$ such that

$$\lim_{j\to\infty} C_A^{b_j,\mu_j}(U,s) = C_A^{g,\lambda}(U,s)$$

for every $U \in \mathcal{R}'$ and for every $s \in \mathbf{R}$. By Remark 14.13 of [10] the family $\mathcal{R}'' = \mathcal{R} \cap \mathcal{R}'$ is rich and

$$C_A^{b,\mu}(U,s) = C_A^{g,\lambda}(U,s) \tag{7.6}$$

for every $U \in \mathcal{R}''$ for every $s \in \mathbf{R}$. We want to prove that $C_A^{b,\mu}(\cdot,s)$ and $C_A^{g,\lambda}(\cdot,s)$ coincide on every Borel set $B \subset\subset \Omega$. Let us fix an open set $U \subset\subset \Omega$ and $s \in \mathbf{R}$. By Theorem 5.1 for every $\varepsilon > 0$ there exists a compact set $K \subset U$ such that

$$C_A^{b,\mu}(U,s) - \varepsilon \le C_A^{b,\mu}(K,s).$$

Since by Remark 14.13 of [10] $\mathcal{R}''$ is dense there exists an open set $V \in \mathcal{R}''$ such that $K \subset V \subset U$. By monotonicity (Theorem 3.4) and by (7.6), we have

$$C_A^{b,\mu}(U,s) - \varepsilon \le C_A^{b,\mu}(K,s) \le C_A^{b,\mu}(V,s) = C_A^{g,\lambda}(V,s) \le C_A^{g,\lambda}(U,s),$$

so that $C_A^{b,\mu}(U,s) \le C_A^{g,\lambda}(U,s)$. Since the opposite inequality can be obtained in the same way, we have proved that (7.6) holds for every open set $U \subset\subset \Omega$ and for every $s \in \mathbf{R}$. By Theorem 5.6 the same equality holds on Borel sets. Thus Theorem 6.1 implies that

$$\int_B s b(x,s)\, d\mu = \int_B s g(x,s)\, d\lambda \tag{7.7}$$

for every Borel set $B \subset \Omega$ for every $s \in \mathbf{R}$, and from Lemma 1.18 we obtain that (b_j, μ_j) γ_A-converges to (b, μ). Since the result does not depend on the subsequence, we have proved the convergence of the whole sequence. $\square$

Proof of Theorem 0.1. For every j let $E_j = \Omega \setminus \Omega_j$, let μ_j be the measure ∞_{E_j} introduced in (1.2), and let b_j be an arbitrary function in $\mathcal{F}(c_3, c_4, \sigma)$. By Remark 1.13 for every $f \in W^{-1,q}(\Omega)$ the solution u_j of problem (0.1) coincides with the solution of problem (1.14). By assumption for every $f \in W^{-1,q}(\Omega)$ the sequence (u_j) of the solutions of problems (0.1) converges weakly in $W_0^{1,p}(\Omega)$ to some function u. By the compactness of the γ^A-convergence (Theorem 1.16) there exist a measure $\mu \in \tilde{\mathcal{M}}_0^p(\Omega)$ and a function $b \in \mathcal{F}(c_3', c_4', \sigma')$ such that a subsequence (b_{j_k}, μ_{j_k}) of (b_j, μ_j) γ_A-converges to (b, μ). This means that, for every $f \in W_0^{1,p}(\Omega)$, the limit u of the sequence (u_{j_k}) is the solution of problem (1.9). Since, by assumption, the whole sequence (u_j) converges, by the definition of γ_A-convergence (Definition 1.14) the whole sequence (b_j, μ_j) γ_A-converges to (b, μ). By Remark 2.2 we have

$$C_A^{b_j,\mu_j}(U, s) = C_A(U \setminus \Omega_j, s) \tag{7.8}$$

for every j, for every $s \in \mathbf{R}$, and for every open set $U \subset\subset \Omega$. Let H and K be two compact sets such that $H \subset \mathring{K} \subset K \subset \Omega$, and let U and V be two open sets such that $H \subset U \subset\subset V \subset\subset \mathring{K}$. By the monotonicity of the A-capacity (see [18], Theorem 4.3), by (7.8), and by Theorem 7.3 we have

$$\begin{aligned}\limsup_{j\to\infty} C_A(H \setminus \Omega_j, s) \le \limsup_{j\to\infty} C_A^{b_j,\mu_j}(U, s) \le C_A^{b,\mu}(V, s) \le \\ \le \liminf_{j\to\infty} C_A^{b_j,\mu_j}(\mathring{K}, s) \le \liminf_{j\to\infty} C_A(K \setminus \Omega_j, s),\end{aligned}$$

which concludes the proof of the theorem. □

Theorem 7.4. *Let (μ_j) be a sequence in $\mathcal{M}_0^p(\Omega)$ and let (b_j) be a sequence in $\mathcal{F}(c_3, c_4, \sigma)$. Suppose that $2 \le p \le n$ and that*

$$\limsup_{j\to\infty} C_A^{b_j,\mu_j}(U, s) \le \liminf_{j\to\infty} C_A^{b_j,\mu_j}(V, s)$$

for every $s \in \mathbf{R}$ and for every pair U, V of open sets such that $U \subset\subset V \subset\subset \Omega$. For every $s \in \mathbf{R}$ let $\alpha(\cdot, s)$ be an increasing set function such that

$$\limsup_{j\to\infty} C_A^{b_j,\mu_j}(U, s) \le \alpha(V, s) \le \liminf_{j\to\infty} C_A^{b_j,\mu_j}(W, s) \tag{7.9}$$

whenever U, V, W are open sets with $U \subset\subset V \subset\subset W \subset\subset \Omega$, and let $\beta(\cdot, s)$ be the regularized version of $\alpha(\cdot, s)$ defined by

$$\begin{aligned}\beta(U, s) &= \sup\{\alpha(V, s) : V \text{ open}, V \subset\subset U\}, &&\text{if } U \text{ is an open set in } \Omega, \\ \beta(B, s) &= \inf\{\beta(U, s) : U \text{ open}, B \subset U \subset \Omega\}, &&\text{if } B \text{ is a Borel set in } \Omega.\end{aligned} \tag{7.10}$$

Then $\beta(\cdot,s)$ is countably subadditive. For every $s\in\mathbf{R}$ let $\nu(\cdot,s)$ be the measure defined for every Borel set $B\subset\Omega$ by

$$\nu(B,s) \,=\, \sup\sum_{i\in I}\beta(B_i,-s)\,, \tag{7.11}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B.

Then the measure $\mu(B)=\nu(B,1)$ belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$ and there exists a function $b\colon\Omega\times\mathbf{R}\to\mathbf{R}$, which belongs to $\mathcal{F}(c_3',c_4',\sigma')$ for suitable constants $0<c_3'\le c_4'$ and $0<\sigma'\le 1$, such that

$$\int_B b(x,s)\,d\mu \,=\, \frac{1}{s}\nu(B,s) \tag{7.12}$$

for every $s\in\mathbf{R}$ and for every Borel set $B\subset\Omega$. Finally, the sequence (b_j,μ_j) γ_A-converges to (b,μ) and $\beta(B,s)=C_A^{b,\mu}(B,s)$ for every Borel set $B\subset\subset\Omega$.

Proof. By the compactness of the γ_A-convergence (Theorem 1.16) there exist a measure $\lambda\in\tilde{\mathcal{M}}_0^p(\Omega)$ and a function $g\in\mathcal{F}(c_3',c_4',\sigma')$ such that a subsequence (b_{j_k},μ_{j_k}) of (b_j,μ_j) γ_A-converges to (g,λ). By Theorem 7.3 for every $s\in\mathbf{R}$ we have

$$\limsup_{k\to\infty} C_A^{b_{j_k},\mu_{j_k}}(U,s) \,\le\, C_A^{g,\lambda}(V,s) \,\le\, \liminf_{k\to\infty} C_A^{b_{j_k},\mu_{j_k}}(W,s)$$

whenever U, V, W are open sets with $U\subset\subset V\subset\subset W\subset\subset\Omega$.

Let us prove that $\beta(B,s)=C_A^{g,\lambda}(B,s)$ for every $s\in\mathbf{R}$ and for every Borel set $B\subset\subset\Omega$. Let U, V, W be open sets such that $U\subset\subset V\subset\subset W\subset\subset\Omega$. By our assumption on α and by the monotonicity of the $C_A^{b,\mu}$-capacity (Theorem 3.4) we have

$$\alpha(U,s) \,\le\, \liminf_{k\to\infty} C_A^{b_{j_k},\mu_{j_k}}(V,s) \,\le\, C_A^{g,\lambda}(W,s)\,,$$
$$C_A^{g,\lambda}(U,s) \,\le\, \liminf_{k\to\infty} C_A^{b_{j_k},\mu_{j_k}}(V,s) \,\le\, \alpha(W,s)\,.$$

This gives $\alpha(U,s)\le C_A^{g,\lambda}(V,s)$ and $C_A^{g,\lambda}(U,s)\le\alpha(V,s)$ for every pair of open sets U, V with $U\subset\subset V\subset\subset\Omega$. By Theorem 5.1 and by the definition of β this implies that $\beta(U,s)=C_A^{g,\lambda}(U,s)$ for every open set $U\subset\subset\Omega$ and for every $s\in\mathbf{R}$. By Theorem 5.6 and by (7.10) we have $\beta(B,s)=C_A^{g,\lambda}(B,s)$ for every Borel set $B\subset\subset\Omega$ and for every $s\in\mathbf{R}$. Therefore $\beta(\cdot,s)$ is countably subadditive by Theorem 5.5.

As $\beta=C_A^{g,\lambda}$, by Theorem 6.1 we have that

$$\nu(B,s) \,=\, \int_B sg(x,s)\,d\lambda \tag{7.13}$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$. Therefore

$$\mu(B) = \nu(B,1) = \int_B g(x,1)\,d\lambda\,.$$

Since $c_3' \leq g(x,1) \leq c_4'$, by Remark 1.5 it is clear that μ belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$ and that

$$c_3'\lambda \leq \mu \leq c_4'\lambda\,.$$

Let w be the solution of problem (1.8). As μ is σ-finite on $\{w > 0\}$, by the Radon–Nikodym Theorem there exists a Borel function $\psi\colon \{w > 0\} \to [c_3', c_4']$ such that $\lambda = \psi\mu$ in $\{w > 0\}$. Let us extend ψ to Ω by setting $\psi = c_3'$ in $\{w = 0\}$. Since $\lambda(B) = \mu(B) = +\infty$ for every Borel set $B \subset \Omega$ with $C_p(B \cap \{w = 0\}) > 0$ (Lemma 1.9), we obtain that

$$\lambda = \psi\mu \quad \text{in } \Omega\,. \tag{7.14}$$

Let $b\colon \Omega\times\mathbf{R} \to \mathbf{R}$ be the function defined by $b(x,s) = g(x,s)\psi(x)$. Then g belongs to $\mathcal{F}((c_3')^2, (c_4')^2, \sigma)$ and, by (7.13) and (7.14), we have

$$\nu(B,s) = \int_B sg(x,s)\,d\lambda = \int_B sb(x,s)\,d\mu$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$. Thus by Lemma 1.18 the subsequence (b_{j_k}, μ_{j_k}) γ_A-converges also to (b, μ).

If $(b_{j_k'}, \mu_{j_k'})$ is another subsequence which γ_A-converges to (g', λ'), with $g' \in \mathcal{F}(c_3', c_4', \sigma')$ and $\lambda' \in \tilde{\mathcal{M}}_0^p(\Omega)$, then

$$\int_B sg'(x,s)\,d\lambda' = \nu(B,s) = \int_B sb(x,s)\,d\mu$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$. Thus by Lemma 1.18 the subsequence $(b_{j_k'}, \mu_{j_k'})$ γ_A-converges also to (b, μ). Since the result does not depend on the subsequence, we have the convergence of the whole sequence. □

Proof of Theorem 0.2. For every j let $E_j = \Omega \setminus \Omega_j$, let μ_j be the measure ∞_{E_j} introduced in (1.2), and let b_j be an arbitrary function in $\mathcal{F}(c_3, c_4, \sigma)$. By Remark 1.13 for every $f \in W^{-1,q}(\Omega)$ the solution u_j of problem (0.1) coincides with the solution of problem (1.14). By Remark 2.2 we have

$$C_A^{b_j,\mu_j}(B,s) = C_A(B \setminus \Omega_j, s) \tag{7.15}$$

for every j, for every $s \in \mathbf{R}$, and for every Borel set $B \subset\subset \Omega$. By (0.6) the set function $\beta(\cdot, s)$ defined by (0.7) satisfies (7.9). This implies that $\beta(\cdot, s)$ coincides with the set function defined by (7.10). The conclusion follows then from Theorem 7.4. □

We consider now the special case where, for every $s \in \mathbf{R}$, the set function $\alpha(\cdot, s)$ which appears in Theorem 7.4 is bounded by a Radon measure. By Proposition 3.6 it is enough to assume that there exists a non-negative Radon measure λ on Ω such that

$$\limsup_{j\to\infty} C_p^{\mu_j}(U) \le \lambda(V), \tag{7.16}$$

for every pair U, V of open sets such that $U \subset\subset V \subset\subset \Omega$. In the following theorem we prove that, in this case, the measure μ and the function b which appear in the limit problem (1.9) can be obtained by a derivation argument with respect to the measure λ.

Theorem 7.5. *In addition to the hypotheses of Theorem 7.4, assume that there exists a non-negative Radon measure λ on Ω such that (7.16) holds for every open set $U \subset\subset \Omega$. Then for λ-a.e. $x \in \Omega$ the limit*

$$\lim_{\rho\to 0} \frac{\beta(B_\rho(x), -s)}{\lambda(B_\rho(x))} = \psi(x, s) \tag{7.17}$$

exists for every $s \in \mathbf{R}$. Let μ be the Radon measure defined by

$$\mu(B) = \int_B \psi(x, 1)\, d\lambda$$

for every Borel set $B \subset \Omega$. Then μ belongs to $\tilde{\mathcal{M}}_0^p(\Omega)$ and there exists a function $b\colon \Omega\times\mathbf{R} \to \mathbf{R}$, which belongs to $\mathcal{F}(c_3', c_4', \sigma')$ for suitable constants $0 < c_3' \le c_4'$ and $0 < \sigma' \le 1$, such that

$$b(x, s) = \frac{1}{s}\frac{\psi(x, s)}{\psi(x, 1)} \tag{7.18}$$

for μ-a.e. $x \in \Omega$ and for every $s \in \mathbf{R}$. Finally, the sequence (b_j, μ_j) γ_A-converges to (b, μ).

Proof. It follows easily from (7.16) and (3.4) that the set function $\beta(\cdot, s)$ defined by (7.9) and (7.10) satisfies the inequality

$$\beta(B, s) \le k_2(|s| + |s|^p)\, \lambda(B)$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$. Since $\beta(\cdot, s)$ is countably subadditive (Theorem 7.4), by Theorem 1.1 of [1] there exists a Borel set $N \subset \Omega$, with $\lambda(N) = 0$, such that the limit in (7.17) exists for every $x \in \Omega \setminus N$ and for every rational number s. By (3.9) and (7.16) for every Borel set $B \subset\subset \Omega$ and for every $s_1,\ s_2 \in \mathbf{R}$ we have

$$\big|\beta(B,-s_1) - \beta(B,-s_2)\big| \ \le\ k\,\lambda(B)\,(1+|s_1|+|s_2|)^{p-\tau}|s_1-s_2|^\tau\,,$$

which gives

$$\Big|\frac{\beta(B_\rho(x),-s_1)}{\lambda(B_\rho(x))} - \frac{\beta(B_\rho(x),-s_2)}{\lambda(B_\rho(x))}\Big| \ \le\ k\,(1+|s_1|+|s_2|)^{p-\tau}|s_1-s_2|^\tau$$

for every $x \in \Omega$ and for every $\rho > 0$ such that $B_\rho(x) \subset\subset \Omega$. This implies that the limit in (7.17) exists for every $x \in \Omega \setminus N$ and for every $s \in \mathbf{R}$. Moreover, Theorem 1.1 of [1] guarantees that the measure $\nu(\cdot, s)$ defined by (7.11) satisfies

$$\nu(B,s) \ =\ \int_B \psi(x,s)\,d\lambda \tag{7.19}$$

for every Borel set $B \subset \Omega$ and for every $s \in \mathbf{R}$. Since

$$\mu(B) \ =\ \int_B \psi(x,1)\,d\lambda \ =\ \nu(B,1) \tag{7.20}$$

for every Borel set $B \subset \Omega$, by Theorem 7.4 the measure μ belongs to $\tilde{\mathcal{M}}^p_0(\Omega)$. Moreover there exists a function $b\colon \Omega\times\mathbf{R} \to \mathbf{R}$, which belongs to $\mathcal{F}(c_3', c_4', \sigma)$ for suitable constants $0 < c_3' \le c_4'$ and $0 < \sigma \le 1$, such that (b_j, μ_j) γ_A-converges to (b, μ) and

$$\int_B sb(x,s)\,d\mu \ =\ \nu(B,s) \tag{7.21}$$

for every $s \in \mathbf{R}$ and for every Borel set $B \subset \Omega$. From (7.19), (7.20), and (7.21) it follows that for every $s \in \mathbf{R}$ we have $\psi(x,s) = sb(x,s)\psi(x,1)$ for λ-a.e. $x \in \Omega$, which implies (7.18) and concludes the proof of the theorem. □

Proof of Theorem 0.3. Let E_j, μ_j, and b_j be as in the proof of Theorem 0.2. By (0.11), (7.15), and (3.4) condition (7.16) is satisfied, with λ replaced by λ/k_1. The conclusion follows then from Theorem 7.5. □

Acknowledgments

This work is part of the following projects: "EURHomogenization", Contract SC1-CT91-0732 of the Program SCIENCE of the Commission of the European Communities; "Mathematical Problems in Homogenization", Contract INTAS-93-2716; "Relaxation and Homogenization Methods in the Study of Composite Materials", Progetto Strategico CNR, 1995, "Matematica per la Tecnologia e la Società".

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