THE CAPACITY METHOD FOR ASYMPTOTIC DIRICHLET PROBLEMS

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Abstract

We prove that the asymptotic behaviour of the solutions of Dirichlet problems for linear elliptic equations in perforated domains of the form $\Omega_h = \Omega \setminus E_h$ is uniquely determined by the asymptotic behaviour, as $h \to \infty$, of suitable capacities of the sets $B \cap E_h$, where B runs in a conveniently large class of subsets of Ω.

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Introduction

Let L be a linear elliptic operator on a bounded open set Ω of $\mathbf{R}^n$, $n \geq 2$, and let (Ω_h) be a sequence of open sets contained in Ω. In this paper we prove that the asymptotic behaviour, as $h \to \infty$, of the solutions u_h of the Dirichlet problems

$$(0.1) \qquad \begin{cases} u_h \in H_0^1(\Omega_h)\,, \\ Lu_h = f \qquad \text{in } \Omega_h\,, \end{cases}$$

for $f \in H^{-1}(\Omega)$, is uniquely determined by the asymptotic behaviour, for a suitable class of sets $E \subset\subset \Omega$, of the capacities $\mathrm{cap}^L(E \setminus \Omega_h)$ associated with the operator L according to Stampacchia [13]. In particular we prove (Theorem 6.1) that, if

$$(0.2) \qquad \lim_{h\to\infty} \mathrm{cap}^L(E \setminus \Omega_h) = \alpha(E)$$

for all sets E in a sufficiently large class $\mathcal{E}$ of subsets of Ω, then for every $f \in H^{-1}(\Omega)$ the solutions u_h of (0.1), extended by 0 in $\Omega \setminus \Omega_h$, converge weakly in $H_0^1(\Omega)$ to the solution u of the "relaxed Dirichlet problem"

$$(0.3) \qquad \begin{cases} u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,, \\ \langle Lu, v\rangle + \displaystyle\int_\Omega uv\, d\mu = \langle f, v\rangle \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,, \end{cases}$$

where μ is a non-negative Borel measure on Ω, which is uniquely determined by the set function α defined by (0.2). More precisely, let β be the regularization of α defined by

$$\beta(U) = \sup\{\alpha(E) \,:\, E \in \mathcal{E}\,,\ E \subset\subset U\}\,, \qquad \text{if } U \text{ is open in } \Omega\,,$$
$$\beta(B) = \inf\{\beta(U) \,:\, U \text{ open}\,,\ B \subseteq U \subseteq \Omega\}\,, \qquad \text{if } B \subseteq \Omega\,.$$

Then the measure μ which appears in (0.3) is the smallest Borel measure on Ω which satisfies $\mu(B) \geq \beta(B)$ for every Borel set $B \subseteq \Omega$: it is given by the formula

$$\mu(B) = \sup \sum_{i\in I} \beta(B_i)\,,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B.

If there exists a Radon measure ν on Ω such that $\beta(B) \leq \nu(B)$ for every Borel set $B \subseteq \Omega$, then μ can be obtained also by a derivation argument: we prove (Theorem 5.11 and Remark 5.12) that the limit

$$\lim_{r\to 0} \frac{\beta(B_r(x))}{\nu(B_r(x))} = g(x)$$

exists for ν-almost every $x \in \Omega$ and that

$$\mu(B) = \int_B g\, d\nu$$

for every Borel set $B \subseteq \Omega$.

In the paper we consider, more in general, the asymptotic behaviour of the solutions u_h of the "relaxed Dirichlet problems"

$$\begin{cases} u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)\,, \\ \langle Lu_h, v\rangle + \int_\Omega u_h v\, d\mu_h = \langle f, v\rangle \qquad \forall v \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)\,, \end{cases} \tag{0.4}$$

where (μ_h) is a sequence of measures of the class $\mathcal{M}_0(\Omega)$ defined in Section 1. In this case the behaviour of the solutions u_h is determined by the behaviour of the μ_h-capacities (introduced in [9] and [10]) on a "sufficiently large" class of Borel subsets of Ω. We will show explicitly that all problems of the form (0.1) can be written in the form (0.4) for a suitable choice of the measures μ_h (Remark 1.4), and that, in this case, the corresponding μ_h-capacities coincide with the set functions $E \mapsto \mathrm{cap}^L(E \setminus \Omega_h)$ considered above (Remark 2.3).

When the operator L is symmetric, these results were obtained in [2] and [4] by using Γ-convergence techniques and the variational properties of cap^L. The results of the present paper are valid also in the non-symmetric case. This fact forces to deep changes in the proofs, because now cap^L is not characterized by a minimum problem, and the relevant properties of cap^L have been proved only recently in [7]. Our results are based on the new compactness theorem proved in [6] and on a careful study of the properties of the μ-capacity for possibly non-symmetric elliptic operators introduced in [9].

1. Notation and preliminary results

Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 2$. We denote by $H^1(\Omega)$ and $H_0^1(\Omega)$ the usual Sobolev spaces, and by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.

For every subset E of Ω the (*harmonic*) *capacity* of E in Ω, denoted by $\mathrm{cap}(E, \Omega)$, is defined as the infimum of $\int_\Omega |Du|^2\, dx$ over the set of all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E. We use the notation $\mathrm{cap}(E)$ when Ω is clear from the context. We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (abbreviated as *q.e.*) in a set E if it holds for all $x \in E$ except for a subset N of E with $\mathrm{cap}(N) = 0$. The expression *almost everywhere* (abbreviated as *a.e.*) refers, as usual, to the Lebesgue measure. A function $u: \Omega \to \mathbf{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists a set $A \subseteq \Omega$, with $\mathrm{cap}(A) < \varepsilon$, such that the restriction of u to $\Omega \setminus A$ is continuous.

It is well known that every $u \in H^1(\Omega)$ has a quasi continuous representative, which is uniquely defined up to a set of capacity zero. In the sequel we shall always identify u with its quasi continuous representative, so that the pointwise values of a function $u \in H^1(\Omega)$ are defined quasi everywhere. We recall that, if a sequence (u_h) converges to u in $H^1_0(\Omega)$, then a subsequence of (u_h) converges to u q.e. in Ω. For all these properties of quasi continuous representatives of Sobolev functions we refer to [14], Section 3.

A subset A of Ω is said to be a *quasi open* if for every $\varepsilon > 0$ there exists an open subset U_ε of Ω, with $\mathrm{cap}(U_\varepsilon) < \varepsilon$, such that $A \cup U_\varepsilon$ is open. It is clear that, if u is quasi continuous, then the level sets $\{u > t\} = \{x \in \Omega : u(x) > t\}$ are quasi open for every $t \in \mathbf{R}$. This is true, in particular, when $u \in H^1(\Omega)$.

Lemma 1.1. *For every quasi open subset A of Ω there exists an increasing sequence (v_h) of non-negative functions of $H^1_0(\Omega)$ which converges to 1_A pointwise q.e. in Ω.*

Proof. See [3], Lemma 1.5. ☐

Lemma 1.2. *Let (u_h) be a bounded sequence of $H^1_0(\Omega)$ which converges to a function u pointwise q.e. in Ω. Then u is (the quasi continuous representative of) a function of $H^1_0(\Omega)$ and (u_h) converges to u weakly in $H^1_0(\Omega)$.*

Proof. Let $\varphi_h = \inf_{k \geq h} u_k$ and $\psi_h = \sup_{k \geq h} u_k$. It is easy to see that $\varphi_h \nearrow u$ q.e. in Ω and $\psi_h \searrow u$ q.e. in Ω. Moreover $\varphi_h \leq u_k \leq \psi_h$ for every $h \leq k$. Now for every h the set $K_h = \{v \in H^1_0(\Omega) : \varphi_h \leq v \leq \psi_h \text{ q.e. in } \Omega\}$ is convex and closed in $H^1_0(\Omega)$, thus it is weakly closed. Since (u_h) is bounded in $H^1_0(\Omega)$, a subsequence of (u_h) converges weakly in $H^1_0(\Omega)$ to a function v. Then $v \in K_h$, so that $\varphi_h \leq v \leq \psi_h$ q.e. in Ω for every h. This implies $u = v$ q.e. in Ω and concludes the proof of the lemma. ☐

By a non-negative *Borel measure* in Ω we mean a countably additive set function defined in the Borel σ-field of Ω and with values in $[0, +\infty]$. By a non-negative *Radon measure* in Ω we mean a non-negative Borel measure which is finite on every compact subset of Ω. We shall always identify a non-negative Borel measure with its completion. If μ is a non-negative Borel measure, by $\mathrm{supp}\,\mu$ we denote the support of μ, i.e., the smallest closed set whose complement has measure zero under μ. If E is μ-measurable in Ω, the Borel measure $\mu\llcorner E$ is defined by $(\mu\llcorner E)(B)=\mu(E\cap B)$ for every Borel set $B\subseteq\Omega$. By $L^p_\mu(\Omega)$, $1\le p\le+\infty$, we denote the usual Lebesgue space with respect to the measure μ. If μ is the Lebesgue measure, we use the standard notation $L^p(\Omega)$.

Definition 1.3. We denote by $\mathcal{M}_0(\Omega)$ the set of all non-negative Borel measures μ in Ω such that $\mu(B)=0$ for every Borel set $B\subseteq\Omega$ with $\mathrm{cap}(B)=0$.

Let $L\colon H^1(\Omega)\to H^{-1}(\Omega)$ be an elliptic operator of the form

$$Lu=-\sum_{i,j=1}^n D_i(a_{ij}D_ju)\,, \tag{1.1}$$

where (a_{ij}) is an $n{\times}n$ matrix of functions of $L^\infty(\Omega)$ satisfying, for a suitable constant $\alpha>0$, the ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x)\xi_j\xi_i\ge\alpha|\xi|^2$$

for a.e. $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$. By $a(u,v)$ we denote the corresponding bilinear form in $H^1(\Omega)$. The adjoint operator, related to the matrix a_{ji}, is denoted by L^*, and the corresponding bilinear form by $a^*(u,v)$.

Let $\mu\in\mathcal{M}_0(\Omega)$, $g\in H^1(\Omega)$, and $f\in H^{-1}(\Omega)$. We shall consider the following *relaxed Dirichlet problem* (see [9] and [10]): find u such that

$$\begin{cases} u\in H^1(\Omega)\cap L^2_\mu(\Omega)\,,\quad u-g\in H^1_0(\Omega)\,,\\[2mm] a(u,v)+\displaystyle\int_\Omega uv\,d\mu\;=\;\langle f,v\rangle\qquad\forall v\in H^1_0(\Omega)\cap L^2_\mu(\Omega)\,,\end{cases} \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. If there exists $z\in H^1(\Omega)\cap L^2_\mu(\Omega)$ such that $z-g\in H^1_0(\Omega)$, then problem (1.2) has a unique solution (see [9], Theorem 2.4). In this case we say that g is μ-admissible. Note that, if $\mathrm{supp}\,\mu$ is compact in Ω, then every $g\in H^1(\Omega)$ is μ-admissible.

Remark 1.4. For every subset E of Ω let ∞_E be the measure in $\mathcal{M}_0(\Omega)$ defined by

$$\infty_E(B)=\begin{cases}0\,, & \text{if } \mathrm{cap}(B\cap E)=0,\\ +\infty\,, & \text{otherwise}\end{cases} \tag{1.3}$$

for every Borel set $B \subseteq \Omega$. It is easy to see that, if E is closed in the relative topology of Ω and there exists a function $\psi \in H^1(\Omega)$ such that $\psi - g \in H^1_0(\Omega)$ and $\psi = 0$ q.e. in E, then g is ∞_E-admissible and the solution u of problem (1.2) coincides in $\Omega \setminus E$ with the solution v of the classical boundary value problem

$$\begin{cases} v - \psi \in H^1_0(\Omega \setminus E)\,, \\ Lv = f \qquad \text{in } \Omega \setminus E\,, \end{cases}$$

while $u = 0$ q.e. in E.

Proposition 1.5. *(Comparison principle) Let f_1, $f_2 \in H^{-1}(\Omega)$, let μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$, and let g_1, $g_2 \in H^1(\Omega)$. Suppose that u_1 and u_2 are the solutions of problem (1.2) corresponding to f_1, μ_1, g_1 and to f_2, μ_2, g_2. If $0 \le f_1 \le f_2$, $\mu_2 \le \mu_1$, and $0 \le g_1 \le g_2$ in Ω, then $0 \le u_1 \le u_2$ q.e. in Ω.*

Proof. See [9], Proposition 2.10. □

Proposition 1.6. *Let $\mu \in \mathcal{M}_0(\Omega)$, let g be a non-negative μ-admissible function of $H^1(\Omega)$, and let u be the solution of the relaxed Dirichlet problem (1.2) corresponding to $f = 0$. Then $a(u,v) \le 0$ for every $v \in H^1_0(\Omega)$ with $v \ge 0$ q.e. in Ω.*

Proof. See [9], Proposition 2.6. □

Definition 1.7. Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. We say that (μ_h) *γ^L-converges to μ (in Ω)* if for every $f \in H^{-1}(\Omega)$ the solutions u_h of the problems

$$(1.4) \qquad \begin{cases} u_h \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)\,, \\ a(u_h, v) + \int_\Omega u_h v \, d\mu_h \;=\; \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega) \end{cases}$$

converge weakly in $H^1_0(\Omega)$, as $h \to \infty$, to the solution u of the problem

$$\begin{cases} u \in H^1_0(\Omega) \cap L^2_{\mu}(\Omega)\,, \\ a(u, v) + \int_\Omega u v \, d\mu \;=\; \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_{\mu}(\Omega)\,. \end{cases}$$

The definition of γ^L-convergence is expressed in terms of the solutions of problem (1.2) with $g = 0$. The case $g \neq 0$ is considered in the following proposition.

Proposition 1.8. *Let* (μ_h) *be a sequence of measures of* $\mathcal{M}_0(\Omega)$ *which* γ^L*-converges to a measure* $\mu_0 \in \mathcal{M}_0(\Omega)$. *Suppose that there exists a compact subset* K *of* Ω *such that* $\operatorname{supp}\mu_h \subseteq K$ *for every* h. *Then* $\operatorname{supp}\mu_0 \subseteq K$. *Moreover for every function* $g \in H^1(\Omega)$ *and for every* $f \in H^{-1}(\Omega)$ *the solution* u_h *of problem (1.2) corresponding to* $\mu = \mu_h$ *converges weakly in* $H^1(\Omega)$ *to the solution* u_0 *of the same problem with* $\mu = \mu_0$.

Proof. If the operator L is symmetric one can adapt the proof of Proposition 5.12 of [10]. For the general case we refer to Theorem 4.9 of [6]. ☐

Theorem 1.9. *(Compactness of the* γ^L*-convergence) Every sequence of measures of* $\mathcal{M}_0(\Omega)$ *contains a* γ^L*-convergent subsequence.*

Proof. See [10], Theorem 4.14, for the symmetric case, and [6], Theorem 4.5, for the general case. ☐

Theorem 1.10. *(Localization of the* γ^L*-convergence) Let* (μ_h) *be a sequence of measures of* $\mathcal{M}_0(\Omega)$ *which* γ^L*-converges in* Ω *to a measure* $\mu \in \mathcal{M}_0(\Omega)$, *and let* $\hat{\Omega}$ *be an open subset of* Ω. *Then* (μ_h) γ^L*-converges to* μ *in* $\hat{\Omega}$.

Proof. See [6], Theorem 4.10. ☐

We introduce now an equivalence relation on $\mathcal{M}_0(\Omega)$, suggested by the role of the measure μ in problem (1.2).

Definition 1.11. We say that two measures μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$ are *equivalent* if $\int_\Omega u^2 d\mu_1 = \int_\Omega u^2 d\mu_2$ for every $u \in H^1_0(\Omega)$.

Remark 1.12. Since every quasi open set differs from a Borel set by a set of capacity zero, all quasi open sets are μ-measurable for every $\mu \in \mathcal{M}_0(\Omega)$. It is easy to see that μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$ are equivalent if and only if they agree on all quasi open subsets of Ω (see [4], Theorem 2.6). Moreover, if this condition is satisfied, then $H^1_0(\Omega) \cap L^2_{\mu_1}(A) = H^1_0(\Omega) \cap L^2_{\mu_2}(A)$ for every quasi open set $A \subseteq \Omega$ and $\int_A uv d\mu_1 = \int_A uv d\mu_2$ for every u, $v \in H^1_0(\Omega) \cap L^2_{\mu_1}(A)$.

Remark 1.13. By the previous remark the solution of the relaxed Dirichlet problem (1.2) does not change when the measure μ varies in its equivalence class. Therefore the γ^L-convergence of the sequence (μ_h) to μ in $\mathcal{M}_0(\Omega)$ does not depend on the choice of μ_h and μ in their equivalence classes in $\mathcal{M}_0(\Omega)$.

Definition 1.14. We denote by $\tilde{\mathcal{M}}_0(\Omega)$ the class of measures $\mu \in \mathcal{M}_0(\Omega)$ such that

$$\mu(B) = \inf\{\mu(A) \,:\, A \text{ quasi open}\,,\ B \subseteq A \subseteq \Omega\} \tag{1.5}$$

for every Borel set $B \subseteq \Omega$. For every $\mu \in \mathcal{M}_0(\Omega)$ we define

$$\tilde{\mu}(B) = \inf\{\mu(A) \,:\, A \text{ quasi open}\,,\ B \subseteq A \subseteq \Omega\} \tag{1.6}$$

for every Borel set $B \subseteq \Omega$.

Remark 1.15. For every measure $\mu \in \mathcal{M}_0(\Omega)$ the set function $\tilde{\mu}$ defined by (1.6) is a measure and belongs to $\tilde{\mathcal{M}}_0(\Omega)$. It is the unique measure in $\tilde{\mathcal{M}}_0(\Omega)$ equivalent to μ and $\tilde{\mu} \geq \lambda$ for every $\lambda \in \mathcal{M}_0(\Omega)$ in the equivalence class of μ (see [4], Section 3). It is easy to see that, if μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$ and $\mu_1 \leq \mu_2$, then $\tilde{\mu}_1 \leq \tilde{\mu}_2$. Finally, if $\mu \in \mathcal{M}_0(\Omega)$ is a Radon measure, then $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and no other measure is equivalent to μ.

Remark 1.16. It is easy to see that, if μ belongs to $\tilde{\mathcal{M}}_0(\Omega)$ and E is a closed subset of Ω, then the measures $\mu \llcorner E$ and ∞_E belong to $\tilde{\mathcal{M}}_0(\Omega)$. This is not true, in general, when E is not closed.

Many properties of the measure $\mu \in \mathcal{M}_0(\Omega)$ can be studied by means of the solutions w and w^* of the problems

$$\begin{cases} w \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,, \\ a(w, v) + \displaystyle\int_\Omega wv\,d\mu = \int_\Omega v\,dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,, \end{cases} \tag{1.7}$$

$$\begin{cases} w^* \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,, \\ a^*(w^*, v) + \displaystyle\int_\Omega w^* v\,d\mu = \int_\Omega v\,dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,. \end{cases} \tag{1.8}$$

Note that $w \geq 0$ and $w^* \geq 0$ q.e. in Ω by the comparison principle (Proposition 1.5).

These functions have been introduced in [6], where the γ^L-convergence is defined only for measures of the class $\tilde{\mathcal{M}}_0(\Omega)$ (denoted by $\mathcal{M}_0(\Omega)$ in that paper). The advantage of that choice is that in the class $\tilde{\mathcal{M}}_0(\Omega)$ there is a one to one correspondence between the measure μ and the solution w of problem (1.7), and it is possible to construct explicitly μ from w (Theorem 1.20). In the present paper we are forced to consider also measures of $\mathcal{M}_0(\Omega)$ that are not in $\tilde{\mathcal{M}}_0(\Omega)$, since we need to use the restriction $\mu \llcorner E$ of a measure μ to non-closed sets E (see Remark 1.16).

Lemma 1.17. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let w be the solution of problem (1.7). Then $\tilde{\mu}(B) = +\infty$ for every Borel set $B \subseteq \Omega$ with* $\mathrm{cap}(B \cap \{w = 0\}) > 0$.

Proof. See [6], Lemma 3.2. □

Lemma 1.18. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let w be the solution of problem (1.7). Then the set $\{w\varphi : \varphi \in C_0^\infty(\Omega)\}$ is dense in the space $H_0^1(\Omega) \cap L^2_\mu(\Omega)$.*

Proof. When $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ the result is proved in [11], Proposition 5.5. The general case follows from Remarks 1.12 and 1.13. □

Lemma 1.19. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let w (resp. w^*) be the solution of problem (1.7) (resp. (1.8)). Then* $\mathrm{cap}(\{w > 0\} \triangle \{w^* > 0\}) = 0$, *where $\triangle$ denotes the symmetric difference of sets.*

Proof. Since $w^* \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$, by Lemma 1.18 there exists a sequence of functions $\varphi_h \in C_0^\infty(\Omega)$ such that $(w\varphi_h)$ converges to w^* in $H_0^1(\Omega) \cap L^2_\mu(\Omega)$ and q.e. in Ω. This implies $w^* = 0$ q.e. in $\{w = 0\}$. Similarly we obtain that $w = 0$ q.e. in $\{w^* = 0\}$. □

Theorem 1.20. *Let $\mu \in \mathcal{M}_0(\Omega)$, let w be the solution of problem (1.7), and let $\nu = 1 - Lw$. Then ν is a non-negative Radon measure of $H^{-1}(\Omega)$ and for every Borel set $B \subseteq \Omega$ we have*

$$\tilde{\mu}(B) = \begin{cases} \displaystyle\int_B \frac{d\nu}{w}\,, & \text{if } \mathrm{cap}(B \cap \{w = 0\}) = 0, \\ +\infty\,, & \text{if } \mathrm{cap}(B \cap \{w = 0\}) > 0. \end{cases}$$

Moreover $\nu(B \cap \{w > 0\}) = \int_B w \, d\tilde{\mu}$ *for every Borel set $B \subseteq \Omega$. In particular*

$$\int_\Omega vw \, d\mu \le \langle 1 - Lw, v \rangle \tag{1.9}$$

for every $v \in H_0^1(\Omega)$ with $v \ge 0$.

Proof. See [6], Proposition 3.4, with obvious modifications. □

Finally, the solutions of problems (1.7) are useful to characterize the γ^L-convergence of measures in $\mathcal{M}_0(\Omega)$. Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let w_h (resp. w_h^*) be the solution of problem (1.7) (resp. (1.8)) corresponding to $\mu = \mu_h$. The following result characterizes the γ^L-convergence in terms of convergence of the functions w_h or w_h^*.

Theorem 1.21. *The following conditions are equivalent:*

(a) (w_h) *converges to* w *weakly in* $H^1_0(\Omega)$*;*

(b) (w^*_h) *converges to* w^* *weakly in* $H^1_0(\Omega)$*;*

(c) (μ_h) γ^L*-converges to* μ*;*

(d) (μ_h) γ^{L^*}*-converges to* μ*.*

Proof. See [6], Theorem 4.3. □

2. The μ-capacity with respect to the operator L

Let A and B be two arbitrary sets with $A \subseteq B \subseteq \Omega$. Suppose that there exists a function $v \in H^1_0(\Omega)$ such that $v = 1$ q.e. in A and $v = 0$ q.e. in $\Omega \setminus B$. Then the *capacity* of A in B with respect to L is defined as $\mathrm{cap}^L(A,B) = a(u,u)$, where u is the solution of the following problem

$$(2.1)\qquad \begin{cases} u \in H^1_0(\Omega)\,, \;\; u = 1 \text{ q.e. in } A\,, \;\; u = 0 \text{ q.e. in } \Omega \setminus B\,, \\ a(u,v) = 0 \qquad \forall v \in H^1_0(\Omega)\,, \;\; v = 0 \text{ q.e. in } A \cup (\Omega \setminus B)\,. \end{cases}$$

The function u is called the *capacitary potential* of A in B with respect to L. When A is closed and B is open this definition of capacity coincides with the definition given by Stampacchia (see [13]). The general case was studied in [7]. When $B = \Omega$, we shall write simply $\mathrm{cap}^L(A)$. For technical reasons we have to consider also situations where A is not closed and B is not open.

The capacity relative to L is increasing, strongly subadditive, and countably subadditive with respect to A, and decreasing with respect to B. These properties are well known when the operator L is symmetric and were proved in [7] when L is not symmetric.

In this section we shall study the main properties of the μ-capacity with respect to the operator L, defined in [9]. These properties will be the basic tools to describe, in Section 5, the γ^L-limit of a sequence of measures in $\mathcal{M}_0(\Omega)$.

Let $\mu \in \mathcal{M}_0(\Omega)$ and let E be a Borel subset of Ω such that $E \subset\subset \Omega$. Then there exists a unique solution v_E of the problem

$$(2.2)\qquad \begin{cases} v_E \in H^1(\Omega) \cap L^2_\mu(E)\,, \;\; v_E - 1 \in H^1_0(\Omega)\,, \\ a(v_E, v) + \displaystyle\int_E v_E v \, d\mu = 0 \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(E)\,. \end{cases}$$

Definition 2.1. The solution v_E of problem (2.2) is called the μ*-capacitary potential* of E in Ω, with respect to the operator L, and the μ*-capacity* of E in Ω, with respect to L, is defined by

$$\mathrm{cap}_\mu^L(E,\Omega) \,=\, a(v_E,v_E) + \int_E v_E^2 d\mu\,.$$

We shall write simply $\mathrm{cap}_\mu^L(E)$ when no ambiguity can arise.

Remark 2.2. By Remark 1.12 it is easy to see that, if μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$ are two equivalent measures, then $\mathrm{cap}_{\mu_1}^L$ and $\mathrm{cap}_{\mu_2}^L$ agree on all quasi open subsets of Ω. In particular, by Remark 1.15, $\mathrm{cap}_\mu^L(A) = \mathrm{cap}_{\tilde\mu}^L(A)$ for every $\mu \in \mathcal{M}_0(\Omega)$ and for every quasi open set $A \subseteq \Omega$.

Remark 2.3. It is easy to see that, if F is a subset of Ω and μ is the measure ∞_F defined by (1.3), then $\mathrm{cap}_\mu^L(E) = \mathrm{cap}^L(E\cap F)$.

Remark 2.4. By the comparison principle (Proposition 1.5) we have $0 \le v_E \le 1$ q.e. in Ω.

Lemma 2.5. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $E \subset\subset \Omega$ be a Borel set, and let v_E be the μ-capacitary potential of E relative to L. Let us extend v_E to $\mathbf{R}^n$ by setting $v_E = 1$ q.e. on $\mathbf{R}^n \setminus \Omega$. Then there exist two non-negative Radon measures λ_E and ν_E in $H^{-1}(\mathbf{R}^n)$ such that $Lv_E = \lambda_E - \nu_E$ in the sense of distributions in $\mathbf{R}^n$, with $\mathrm{supp}\,\lambda_E \subseteq \partial\Omega$ and $\mathrm{supp}\,\nu_E \subseteq \overline{E}$. In particular we have*

$$a(v_E,v) \,=\, \lambda_E(\partial\Omega) \,-\, \int_\Omega v\, d\nu_E \tag{2.3}$$

for every $v \in H^1(\Omega)$ with $v-1 \in H_0^1(\Omega)$.

Proof. By Proposition 1.6 we have that $a(v_E,v) \le 0$ for every $v \in H_0^1(\Omega)$ with $v \ge 0$ q.e. in Ω. By the Riesz representation theorem, there is a non-negative Radon measure $\nu_E \in H^{-1}(\Omega)$ such that

$$a(v_E,v) \,=\, -\int_\Omega v\, d\nu_E$$

for every $v \in H_0^1(\Omega)$. Moreover, for every $v \in H_0^1(\Omega)$ with $v = 0$ q.e. in $\overline{E}$, by (2.2) we have

$$0 \,=\, a(v_E,v) \,=\, -\int_\Omega v\, d\nu_E\,,$$

and this implies that $\operatorname{supp}\nu_E \subseteq \overline{E}$. In order to prove the existence of the measure λ_E we follow the lines of the proof of Lemma 2.1 in [8]. Let Ω' be a bounded open set such that $\Omega \subset\subset \Omega'$ and let z be the solution of the obstacle problem

$$\begin{cases} z \in H^1_0(\Omega')\,, \;\; z \geq 0 \text{ q.e. in } \Omega' \setminus \Omega\,, \\ \langle Lz + \nu_E, v - z\rangle \geq 0 \qquad \forall v \in H^1_0(\Omega')\,,\; v \geq 0 \text{ q.e. in } \Omega' \setminus \Omega\,, \end{cases}$$

where, in this case, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega')$ and $H^1_0(\Omega')$. It is well known that there exists a unique solution z of this problem, and that z is a supersolution of the equation $Lu = -\nu_E$, i.e., $Lz + \nu_E = \lambda_E$ in the sense of $H^{-1}(\Omega')$ for some non-negative Radon measure $\lambda_E \in H^{-1}(\Omega')$. Moreover $z \leq \zeta$ for every supersolution $\zeta \in H^1(\Omega')$ of the equation $Lu = -\nu_E$ with $\zeta \geq 0$ q.e. in $\Omega' \setminus \Omega$ (see [12], Section II.6). In particular $z \leq 0$ q.e. in Ω and this implies that $z = 0$ q.e. in $\Omega' \setminus \Omega$, hence $z \in H^1_0(\Omega)$. Since $Lz + \nu_E = 0$ and $Lv_E + \nu_E = 0$ in the sense of $H^{-1}(\Omega)$, by uniqueness we obtain $z = v_E - 1$. This implies that $Lv_E = \lambda_E - \nu_E$ in Ω'. As $Lv_E = -\nu_E$ in Ω, $\operatorname{supp}\nu_E \subseteq \overline{E}$, and $v_E = 1$ q.e. in $\mathbf{R}^n \setminus \overline{\Omega}$, we conclude that $\operatorname{supp}\lambda_E \subseteq \partial\Omega$. This implies that λ_E is a bounded Radon measure on $\mathbf{R}^n$ and that $Lv_E = \lambda_E - \nu_E$ in $\mathbf{R}^n$. Finally, in order to prove (2.3), let $\varphi \in C^\infty_0(\Omega)$ be a function such that $\varphi = 1$ in $\overline{\Omega}$, and let $v \in H^1(\Omega)$ with $v - 1 \in H^1_0(\Omega)$. Let us extend v to $\mathbf{R}^n$ by setting $v = 1$ q.e. in $\mathbf{R}^n \setminus \Omega$. Then $\varphi v \in H^1(\mathbf{R}^n)$. As $Lv_E = \lambda_E - \nu_E$ in $\mathbf{R}^n$, we obtain

$$a(v_E, v) \;=\; a(v_E, \varphi v) \;=\; \int_{\partial\Omega} \varphi v \, d\lambda_E \;-\; \int_\Omega \varphi v \, d\nu_E\,. \tag{2.4}$$

Since $\varphi = 1$ in $\overline{\Omega}$ and $v = 1$ q.e. in $\partial\Omega$, we have that $\varphi v = v$ in Ω and $\varphi v = 1$ q.e. in $\partial\Omega$. Thus (2.3) follows from (2.4). □

The measures ν_E and λ_E, defined in Lemma 2.5, are called the *inner* and the *outer* *μ-capacitary distribution* of E in Ω relative to L.

Lemma 2.6. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $E \subset\subset \Omega$ be a Borel set, let v_E be the μ-capacitary potential of E in Ω with respect to the operator L, and let ν_E be the corresponding inner μ-capacitary distribution. Then*

$$\int_\Omega v \, d\nu_E \;=\; \int_E v v_E \, d\mu \tag{2.5}$$

for every $v \in H^1(\Omega) \cap L^2_\mu(E)$.

Proof. It is enough to prove (2.5) for every $v \in H^1(\Omega) \cap L^2_\mu(E)$ with $v \geq 0$ q.e. in Ω. Since every function v with these properties can be approximated pointwise q.e. in Ω by an increasing sequence of functions of $H^1_0(\Omega) \cap L^2_\mu(E)$, it suffices to prove (2.5) for every $v \in H^1_0(\Omega) \cap L^2_\mu(E)$. From the definitions of ν_E and v_E it follows that

$$\int_\Omega v \, d\nu_E \;=\; -a(v_E, v) \;=\; \int_E v v_E \, d\mu$$

for every $v \in H^1_0(\Omega) \cap L^2_\mu(E)$, and the lemma is proved. □

Lemma 2.7. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $E \subset\subset \Omega$ be a Borel set, let v_E be the μ-capacitary potential of E in Ω with respect to L, and let ν_E and λ_E be the corresponding inner and outer μ-capacitary distributions. Then $\mathrm{cap}^L_\mu(E, \Omega) = \nu_E(\Omega) = \lambda_E(\partial\Omega)$.*

Proof. By taking $v = 1$ in (2.3) we obtain $\nu_E(\Omega) = \lambda_E(\partial\Omega)$. If we take $v = v_E$ in (2.3), by (2.5) we obtain also

$$a(v_E, v_E) \;=\; \lambda_E(\partial\Omega) \;-\; \int_\Omega v_E \, d\nu_E \;=\; \lambda_E(\partial\Omega) \;-\; \int_\Omega v_E^2 d\mu \,,$$

which, by the definition of μ-capacity, implies $\mathrm{cap}^L_\mu(E, \Omega) = \lambda_E(\partial\Omega)$. □

The following result will be fundamental in the proof of the main properties of the μ-capacity.

Theorem 2.8. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $E \subset\subset \Omega$ be a Borel set. Then $\mathrm{cap}^L_\mu(E) = \mathrm{cap}^{L^*}_\mu(E)$.*

Proof. Let v_E and v^*_E be the μ-capacitary potentials of E relative to L and L^*, and let ν_E and ν^*_E (resp. λ_E and λ^*_E) be the corresponding inner (resp. outer) μ-capacitary distributions. By (2.5) we have

$$\int_\Omega v^*_E \, d\nu_E \;=\; \int_E v_E v^*_E \, d\mu \;=\; \int_\Omega v_E \, d\nu^*_E \,.$$

Therefore by Lemma 2.7 and (2.3)

$$\begin{aligned} \mathrm{cap}^L_\mu(E) \;&=\; \lambda_E(\partial\Omega) \;=\; a(v_E, v^*_E) \;+\; \int_\Omega v^*_E \, d\nu_E \;= \\ &=\; a^*(v^*_E, v_E) \;+\; \int_\Omega v_E \, d\nu^*_E \;=\; \lambda^*_E(\partial\Omega) \;=\; \mathrm{cap}^{L^*}_\mu(E) \,, \end{aligned}$$

which concludes the proof of the theorem. □

We are now in a position to study the monotonicity properties of $\mathrm{cap}^L_\mu(E,\Omega)$ with respect to μ (Theorem 2.10), E (Theorem 2.11), and Ω (Theorem 2.12). We begin with an auxiliary lemma.

Lemma 2.9. *Let μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$, with $\mu_1 \le \mu_2$, and let $E \subset\subset \Omega$ be a Borel set. Let v_1 (resp. v_2^*) be the μ_1-capacitary (resp. μ_2-capacitary) potential of E relative to L (resp. L^*) and let ν_1 (resp. ν_2^*) be the corresponding inner μ_1-capacitary (resp. μ_2-capacitary) distribution. Then*

$$\int_\Omega v_2^*\, d\nu_1 \le \int_\Omega v_1\, d\nu_2^* .$$

Proof. For every $h \in \mathbf{N}$ let $U_h = \{v_2^* > 1/h\}$. Since U_h is quasi open, by Lemma 1.1 for every h there exists an increasing sequence (z_h^k) in $H_0^1(\Omega)$ converging to 1_{U_h} pointwise q.e. in Ω as $k \to \infty$ and such that $0 \le z_h^k \le 1_{U_h}$ q.e. in Ω for every h and k. As $v_2^* \in L^2_{\mu_2}(E)$, we have $\mu_2(E \cap U_h) < +\infty$ and hence $z_h^k v_1 \in H^1(\Omega) \cap L^2_{\mu_2}(E)$. Thus by (2.5) we have

$$\int_E z_h^k v_1 v_2^*\, d\mu_1 \le \int_E z_h^k v_1 v_2^*\, d\mu_2 = \int_\Omega z_h^k v_1 d\nu_2^* \le \int_\Omega v_1 d\nu_2^*$$

for every h and k. Taking the limit as $k \to \infty$ we obtain

$$\int_{E\cap U_h} v_1 v_2^*\, d\mu_1 \le \int_\Omega v_1 d\nu_2^*$$

for every h. Since $v_2^* \in L^2_{\mu_2}(E) \subseteq L^2_{\mu_1}(E)$, taking the limit as $h \to \infty$, by (2.5) we get

$$\int_\Omega v_2^*\, d\nu_1 = \int_{E\cap\{v_2^*>0\}} v_2^* v_1\, d\mu_1 \le \int_\Omega v_1 d\nu_2^* ,$$

and this concludes the proof. □

Theorem 2.10. *Let μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$, with $\mu_1 \le \mu_2$, and let $E \subset\subset \Omega$ be a Borel set. Then $\mathrm{cap}^L_{\mu_1}(E) \le \mathrm{cap}^L_{\mu_2}(E)$.*

Proof. Let v_1 (resp. v_2^*) be the μ_1-capacitary (resp. μ_2-capacitary) potential of E relative to L (resp. L^*) and let ν_1 and λ_1 (resp. ν_2^* and λ_2^*) be the corresponding inner and outer μ_1-capacitary (resp. μ_2-capacitary) distributions. By Lemmas 2.5, 2.7, and 2.9 we have

$$\begin{aligned}\operatorname{cap}^L_{\mu_1}(E) \;=\; \lambda_1(\partial\Omega) \;=\; a(v_1, v_2^*) \;+\; \int_\Omega v_2^*\, d\nu_1 \;\leq \\ \leq\; a^*(v_2^*, v_1) \;+\; \int_\Omega v_1\, d\nu_2^* \;=\; \lambda_2^*(\partial\Omega) \;=\; \operatorname{cap}^{L^*}_{\mu_2}(E)\,.\end{aligned}$$

The conclusion follows now from Theorem 2.8. □

Theorem 2.11. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let E and F be two Borel sets such that $E \subseteq F \subset\subset \Omega$. Then $\operatorname{cap}^L_\mu(E) \leq \operatorname{cap}^L_\mu(F)$.*

Proof. It is enough to apply Theorem 2.10 to the measures $\mu_1 = \mu \llcorner E$ and $\mu_2 = \mu$, noticing that $\operatorname{cap}^L_\mu(E) = \operatorname{cap}^L_{\mu\llcorner E}(F) \leq \operatorname{cap}^L_\mu(F)$. □

Theorem 2.12. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $\hat\Omega$ be an open subset of Ω, and let E be a Borel set such that $E \subset\subset \hat\Omega \subseteq \Omega$. Then $\operatorname{cap}^L_\mu(E, \Omega) \leq \operatorname{cap}^L_\mu(E, \hat\Omega)$.*

Proof. Let v_E be the μ-capacitary potential of E relative to L in Ω and let $\hat v_E^*$ be the μ-capacitary potential of E relative to L^* in $\hat\Omega$. We extend v_E and $\hat v_E^*$ to $\mathbf{R}^n$ by setting $v_E = 1$ q.e. in $\mathbf{R}^n \setminus \Omega$ and $\hat v_E^* = 1$ q.e. in $\mathbf{R}^n \setminus \hat\Omega$. Let ν_E and λ_E be the inner and the outer μ-capacitary distributions of E relative to L in Ω, and let $\hat\nu_E^*$ and $\hat\lambda_E^*$ be the inner and the outer μ-capacitary distributions of E in $\hat\Omega$ relative to L^*. Now from (2.5) we have that

$$\int_\Omega \hat v_E^*\, d\nu_E \;=\; \int_E \hat v_E^* v_E\, d\mu \;=\; \int_{\hat\Omega} v_E\, d\hat\nu_E^*\,.$$

Since $0 \leq v_E \leq 1$ q.e. in $\mathbf{R}^n$ (Remark 2.4), by Lemmas 2.5 and 2.7 we get

$$\begin{aligned}\operatorname{cap}^L_\mu(E, \Omega) \;=\; \lambda_E(\partial\Omega) \;=\; a(v_E, \hat v_E^*) \;+\; \int_\Omega \hat v_E^*\, d\nu_E \;= \\ =\; a^*(\hat v_E^*, v_E) \;+\; \int_{\hat\Omega} v_E\, d\hat\nu_E^* \;= \\ =\; \int_{\partial\hat\Omega} v_E\, d\hat\lambda_E^* \;\leq\; \hat\lambda_E^*(\partial\hat\Omega) \;=\; \operatorname{cap}^{L^*}_\mu(E, \hat\Omega)\,.\end{aligned}$$

The conclusion follows now from Theorem 2.8. □

The following theorem shows the subadditivity of $\operatorname{cap}^L_\mu(\cdot)$.

Theorem 2.13. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let E_1 and E_2 be two Borel set such that $E_1 \subset\subset \Omega$ and $E_2 \subset\subset \Omega$. Then*

$$\mathrm{cap}^L_\mu(E_1 \cup E_2) \,\le\, \mathrm{cap}^L_\mu(E_1) \,+\, \mathrm{cap}^L_\mu(E_2)\,.$$

Proof. Let $v_{E_1\cup E_2}$ and $\nu_{E_1\cup E_2}$ (resp. $\lambda_{E_1\cup E_2}$) be the μ-capacitary potential and the inner (resp. outer) μ-capacitary distribution of $E_1 \cup E_2$ relative to L and let $v^*_{E_1}$, $v^*_{E_2}$ and $\lambda^*_{E_1}$, $\lambda^*_{E_2}$ be the μ-capacitary potentials and the outer μ-capacitary distributions of E_1 and E_2 relative to L^*. We note that $v^*_{E_1}\wedge v^*_{E_2} = v^*_{E_1} + v^*_{E_2} - v^*_{E_1}\vee v^*_{E_2}$ and that $v^*_{E_1}\wedge v^*_{E_2} \in L^2_\mu(E_1\cup E_2)$. Since $v^*_{E_1}\wedge v^*_{E_2} - 1 \in H^1_0(\Omega)$, from (2.5) and (2.3) we obtain

$$\begin{aligned}\lambda_{E_1\cup E_2}(\partial\Omega) \,&=\, a(v_{E_1\cup E_2}, v^*_{E_1}\wedge v^*_{E_2}) \,+\int_{E_1\cup E_2}(v^*_{E_1}\wedge v^*_{E_2})v_{E_1\cup E_2}\,d\mu \,=\\ &=\, a^*(v^*_{E_1}, v_{E_1\cup E_2}) \,+\, a^*(v^*_{E_2}, v_{E_1\cup E_2}) \,-\, a(v_{E_1\cup E_2}, v^*_{E_1}\vee v^*_{E_2}) +\\ &+\int_{E_1\cup E_2} v^*_{E_1}v_{E_1\cup E_2}\,d\mu \,+\int_{E_1\cup E_2} v^*_{E_2}v_{E_1\cup E_2}\,d\mu \,-\int_{E_1\cup E_2}(v^*_{E_1}\vee v^*_{E_2})v_{E_1\cup E_2}\,d\mu\,.\end{aligned}$$

We note that by (2.3) and (2.5)

$$a^*(v^*_{E_i}, v_{E_1\cup E_2}) + \int_{E_i} v^*_{E_i}v_{E_1\cup E_2}\,d\mu = \lambda^*_{E_i}(\partial\Omega)\,, \qquad i = 1,2\,.$$

Moreover, as $\lambda_{E_1\cup E_2}(\partial\Omega) = \nu_{E_1\cup E_2}(\Omega)$ (Lemma 2.7) and $v^*_{E_1}\vee v^*_{E_2} - 1 \in H^1_0(\Omega)$, by (2.3) we have

$$a(v_{E_1\cup E_2}, v^*_{E_1}\vee v^*_{E_2}) \,=\, \nu_{E_1\cup E_2}(\Omega) \,-\int_\Omega v^*_{E_1}\vee v^*_{E_2}\,d\nu_{E_1\cup E_2} \,\ge\, 0\,.$$

Thus we obtain

$$\begin{aligned}\lambda_{E_1\cup E_2}(\partial\Omega) \,&\le\, \lambda^*_{E_1}(\partial\Omega) \,+\, \lambda^*_{E_2}(\partial\Omega) \,+\int_{E_2\setminus E_1} v^*_{E_1}v_{E_1\cup E_2}\,d\mu +\\ &+\int_{E_1\setminus E_2} v^*_{E_2}v_{E_1\cup E_2}\,d\mu \,-\int_{E_1\cup E_2}(v^*_{E_1}\vee v^*_{E_2})v_{E_1\cup E_2}\,d\mu\,.\end{aligned}$$

Since

$$\int_{E_2\setminus E_1} v^*_{E_1}v_{E_1\cup E_2}\,d\mu \,+\int_{E_1\setminus E_2} v^*_{E_2}v_{E_1\cup E_2}\,d\mu \,\le\int_{E_1\cup E_2}(v^*_{E_1}\vee v^*_{E_2})v_{E_1\cup E_2}\,d\mu\,,$$

we get $\lambda_{E_1\cup E_2}(\partial\Omega) \,\le\, \lambda^*_{E_1}(\partial\Omega) + \lambda^*_{E_2}(\partial\Omega)$, and the conclusion follows from Lemma 2.7 and Theorem 2.8. □

Finally, we give a bound from above for the μ-capacity in terms of the harmonic capacity and of the measure μ.

Proposition 2.14. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let E be a Borel set such that $E \subset\subset \Omega$. Then*
(a) $\mathrm{cap}^L_\mu(E) \leq \mu(E)$,
(b) $\mathrm{cap}^L_\mu(E) \leq \mathrm{cap}^L(E) \leq k\,\mathrm{cap}(E)$,
where the constant k depends only on the ellipticity constant α and on the L^∞ bounds of the coefficients a_{ij} of L.

Proof. Property (a) is trivial if $\mu(E) = +\infty$. If $\mu(E) < +\infty$, let v_E be the μ-capacitary potential of E relative to the operator L and let ν_E be the inner μ-capacitary distribution. Since $1 \in L^2_\mu(E)$, by Lemma 2.7 and by (2.5) we get

$$\mathrm{cap}^L_\mu(E) = \nu_E(\Omega) = \int_\Omega d\nu_E = \int_E v_E d\mu \leq \mu(E),$$

and (a) is proved.

Let us prove (b). Since for every $\mu \in \mathcal{M}_0(\Omega)$ we have $\mu \leq \infty_\Omega$ (Remark 1.4), by Theorem 2.10 and Remark 2.3 we obtain that $\mathrm{cap}^L_\mu(E) \leq \mathrm{cap}^L(E)$. The inequality $\mathrm{cap}^L(E) \leq k\,\mathrm{cap}(E)$ is proved in [13], Theorem 3.11. □

3. Continuity properties of the μ-capacity

In this section we prove the continuity of the μ-capacity along increasing sequences of sets and study the approximation properties by means of compact and open sets.

Lemma 3.1. *Let $\mu \in \mathcal{M}_0(\Omega)$. If (E_h) is an increasing sequence of Borel subsets of Ω and $E = \cup_h E_h$, then the sequence $(\mu \llcorner E_h)$ γ^L-converges to the measure $\mu \llcorner E$.*

Proof. Let w_h be the solutions of the problems

$$\text{(3.1)} \qquad \begin{cases} w_h \in H^1_0(\Omega) \cap L^2_\mu(E_h), \\ a(w_h, v) + \displaystyle\int_{E_h} w_h v\, d\mu = \int_\Omega v\, dx \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(E_h). \end{cases}$$

By the ellipticity condition it is easy to see that (w_h) is bounded in $H^1_0(\Omega)$. Therefore we may assume that (w_h) converges weakly in $H^1_0(\Omega)$ to a function w. By Proposition 1.5

the sequence (w_h) is decreasing and hence, by Lemma 1.2, it converges to w pointwise q.e. in Ω. Therefore $(1_{E_h} w_h)$ converges to $1_E w$ pointwise μ-a.e. in Ω. Since

$$\int_\Omega 1_{E_h}^2 w_h^2 \, d\mu \;=\; \int_{E_h} w_h^2 \, d\mu \;=\; \int_\Omega w_h \, dx - a(w_h, w_h) \;\leq\; \int_\Omega w_h \, dx \,,$$

the sequence $(1_{E_h} w_h)$ is bounded in $L^2_\mu(\Omega)$. This implies that $w \in L^2_\mu(E)$ and that $(1_{E_h} w_h)$ converges to $1_E w$ weakly in $L^2_\mu(\Omega)$. For every h we can take any function $v \in H^1_0(\Omega) \cap L^2_\mu(E)$ as test function in (3.1) and, passing to the limit, we obtain that w is the solution of the problem

$$\begin{cases} w \in H^1_0(\Omega) \cap L^2_\mu(E) \,, \\ a(w, v) + \displaystyle\int_E wv \, d\mu \;=\; \int_\Omega v \, dx \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(E) \,. \end{cases}$$

The conclusion follows from the characterization of the γ^L-convergence (Theorem 1.21). □

Theorem 3.2. *Let $\mu \in \mathcal{M}_0(\Omega)$. If (E_h) is an increasing sequence of Borel subsets of Ω and $E = \cup_h E_h \subset\subset \Omega$, then*

$$\operatorname{cap}^L_\mu(E) \;=\; \sup_h \operatorname{cap}^L_\mu(E_h) \,.$$

Proof. Since $\operatorname{cap}^L_\mu(\cdot)$ is increasing (Theorem 2.11), we have only to prove that $\operatorname{cap}^L_\mu(E) \leq \sup_h \operatorname{cap}^L_\mu(E_h)$. If v_{E_h} is the μ-capacitary potential of E_h, by Lemma 3.1 and Proposition 1.8 the sequence (v_{E_h}) converges weakly in $H^1(\Omega)$ to the μ-capacitary potential v_E of E. Now, since $v_E \leq v_{E_h}$ q.e. in Ω (Proposition 1.5) and the quadratic form $a(v, v)$ is lower semicontinuous in the weak topology of $H^1(\Omega)$, for every $k \in \mathbf{N}$ we have

$$\begin{aligned} a(v_E, v_E) + \int_{E_k} v_E^2 \, d\mu \leq \liminf_{h\to\infty} \Big(a(v_{E_h}, v_{E_h}) + \int_{E_k} v_{E_h}^2 \, d\mu \Big) \leq \\ \leq \liminf_{h\to\infty} \Big(a(v_{E_h}, v_{E_h}) + \int_{E_h} v_{E_h}^2 \, d\mu \Big) \,. \end{aligned}$$

As $k \to \infty$ we conclude the proof. □

As a consequence of Theorem 3.2 we obtain the countable subadditivity of the μ-capacity.

Theorem 3.3. *Let $\mu \in \mathcal{M}_0(\Omega)$. If (E_h) is a sequence of Borel sets, with $E_h \subset\subset \Omega$, and $E \subseteq \cup_h E_h$ is a Borel set, with $E \subset\subset \Omega$, then*

$$\mathrm{cap}_\mu^L(E) \leq \sum_h \mathrm{cap}_\mu^L(E_h).$$

Proof. The result follows easily from Theorems 2.11, 2.13, and 3.2. □

Theorem 3.4. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then*

$$\mathrm{cap}_\mu^L(A) = \sup\{\mathrm{cap}_\mu^L(K) : K \text{ compact}, K \subseteq A\},$$
$$\mathrm{cap}_\mu^L(A) = \inf\{\mathrm{cap}_\mu^L(U) : U \text{ open}, A \subseteq U \subset\subset \Omega\}$$

for every quasi open set $A \subset\subset \Omega$.

Proof. Once we have proved Theorems 2.11, 2.13, 2.14(a), 3.2, we can follow the lines of the proof given in [4], Theorem 2.9(i) and (j). □

Finally we prove the outer regularity of the μ-capacity when the measure μ belongs to $\tilde{\mathcal{M}}_0(\Omega)$.

Theorem 3.5. *Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Then*

$$\mathrm{cap}_\mu^L(B) = \inf\{\mathrm{cap}_\mu^L(U) : U \text{ open}, B \subseteq U \subset\subset \Omega\}$$

for every Borel set $B \subset\subset \Omega$.

Proof. By Theorem 3.4 it is enough to prove that

$$\mathrm{cap}_\mu^L(B) = \inf\{\mathrm{cap}_\mu^L(A) : A \text{ quasi open}, B \subseteq A \subset\subset \Omega\} \tag{3.2}$$

for every Borel set $B \subset\subset \Omega$. Let us fix B and let us denote by I the right hand side of (3.2). By monotonicity (Theorem 2.11) we have $\mathrm{cap}_\mu^L(B) \leq I$. It remains to prove the opposite inequality.

Let v_B be the μ-capacitary potential of B in Ω. Since $v_B \in L^2_\mu(B)$ we have that $\mu(B \cap \{v_B \geq \varepsilon\}) < +\infty$ for every $\varepsilon > 0$. Thus, by the definition of $\tilde{\mathcal{M}}_0(\Omega)$, there exists a quasi open set U_ε such that $B \cap \{v_B \geq \varepsilon\} \subseteq U_\varepsilon \subset\subset \Omega$ and $\mu(U_\varepsilon \setminus (B \cap \{v_B \geq \varepsilon\})) < \varepsilon$. Let us consider the quasi open set $\{v_B < \varepsilon\}$. In order to prove that $\{v_B < \varepsilon\} \subset\subset \Omega$

for ε small enough, let us choose two open sets B_0 and Ω_0 with smooth boundary such that $B \subseteq B_0 \subset\subset \Omega \subseteq \Omega_0$, and let z be the solution of the problem

$$\begin{cases} Lz = 0 & \text{in } \Omega_0 \setminus \overline{B}_0, \\ z = 0 & \text{in } \overline{B}_0, \\ z = 1 & \text{in } \partial\Omega_0. \end{cases}$$

Since $v_B - 1 \in H^1_0(\Omega)$ and $Lv_B = 0$ on $\Omega \setminus \overline{B}$, by the maximum principle we have $v_B \geq z$ q.e. in Ω, so that $\{v_B < \varepsilon\} \subseteq \{z < \varepsilon\}$. As z is continuous in $\overline{\Omega}_0$ by De Giorgi's Theorem and $\{z = 0\} = \overline{B}_0 \subset\subset \Omega$ by the strong maximum principle, for ε small enough we have $\{v_B < \varepsilon\} \subseteq \{z < \varepsilon\} \subset\subset \Omega$.

Let us fix $\varepsilon > 0$ such that $\{v_B < \varepsilon\} \subset\subset \Omega$ and let us define $v_\varepsilon = \max\{0, \frac{v_B - \varepsilon}{1-\varepsilon}\}$. We have $v_\varepsilon - 1 \in H^1_0(\Omega)$, $0 \leq v_\varepsilon \leq \frac{v_B}{1-\varepsilon}$ q.e. in Ω, $v_\varepsilon \in L^2_\mu(B)$, $v_\varepsilon = 0$ q.e. in $\{v_B \leq \varepsilon\}$, and $v_\varepsilon = \frac{v_B-\varepsilon}{1-\varepsilon}$ q.e. in $\{v_B \geq \varepsilon\}$. By the definition of v_ε and v_B for every $v \in H^1_0(\Omega) \cap L^2_\mu(B)$, with $v = 0$ q.e. in $\{v_B \leq \varepsilon\}$, we obtain

$$\begin{aligned} a(v_\varepsilon, v) \;&=\; \frac{1}{1-\varepsilon} a(v_B, v) \;=\; -\frac{1}{1-\varepsilon}\int_B v_B v\, d\mu \;= \\ &=\; -\int_{B\cap\{v_B>\varepsilon\}} v_\varepsilon v\, d\mu \;-\; \frac{\varepsilon}{1-\varepsilon}\int_{B\cap\{v_B>\varepsilon\}} v\, d\mu\,. \end{aligned} \tag{3.3}$$

Let us define the Borel measure ρ by

$$\rho(E) \;=\; \begin{cases} \mu(E) + \dfrac{\varepsilon}{1-\varepsilon}\displaystyle\int_E \frac{d\mu}{v_\varepsilon}\,, & \text{if } \operatorname{cap}(E \setminus (B \cap \{v_B > \varepsilon\})) = 0, \\ +\infty\,, & \text{otherwise.} \end{cases}$$

Note that ρ belongs to $\mathcal{M}_0(\Omega)$ and that

$$\int_{B\cup\{v_B\leq\varepsilon\}} v_\varepsilon v\, d\rho \;=\; \int_{B\cap\{v_B>\varepsilon\}} v_\varepsilon v\, d\mu \;+\; \frac{\varepsilon}{1-\varepsilon}\int_{B\cap\{v_B>\varepsilon\}} v\, d\mu \tag{3.4}$$

for every Borel function $v \geq 0$. By taking $v = v_\varepsilon$ we obtain $v_\varepsilon \in L^2_\rho(B \cup \{v_B \leq \varepsilon\})$, using the fact that v_ε is bounded and $\mu(B \cap \{v_B > \varepsilon\}) < +\infty$. Since $\mu \leq \rho$, every function in $H^1_0(\Omega) \cap L^2_\rho(B \cup \{v_B \leq \varepsilon\})$ belongs to $H^1_0(\Omega) \cap L^2_\mu(B)$ and is zero q.e. in $\{v_B \leq \varepsilon\}$. Then, by (3.3) and (3.4), it is easy to check that v_ε is the solution of the problem

$$\begin{cases} v_\varepsilon \in H^1_0(\Omega) \cap L^2_\rho(B \cup \{v_B \leq \varepsilon\})\,, \quad v_\varepsilon - 1 \in H^1_0(\Omega)\,, \\ a(v_\varepsilon, v) + \displaystyle\int_{B\cup\{v_B\leq\varepsilon\}} v_\varepsilon v\, d\rho = 0 \qquad \forall v \in H^1_0(\Omega) \cap L^2_\rho(B \cup \{v_B \leq \varepsilon\})\,, \end{cases}$$

and hence v_ε is the ρ-capacitary potential of the set $B \cup \{v_B \le \varepsilon\}$ in Ω. Moreover by Theorem 2.10 we have

$$\mathrm{cap}^L_\mu(B \cup \{v_B \le \varepsilon\}) \le \mathrm{cap}^L_\rho(B \cup \{v_B \le \varepsilon\}) . \tag{3.5}$$

Finally let us define $A_\varepsilon = U_\varepsilon \cup \{v_B < \varepsilon\}$; the set A_ε is quasi open, contains B, and $A_\varepsilon \subset\subset \Omega$. Then, by (3.4), (3.5), and Theorems 2.13 and 2.14(a), we get

$$\begin{aligned} I &\le \mathrm{cap}^L_\mu(A_\varepsilon) \le \mathrm{cap}^L_\mu(B \cup \{v_B \le \varepsilon\}) + \mathrm{cap}^L_\mu(U_\varepsilon \setminus B) \le \\ &\le \mathrm{cap}^L_\rho(B \cup \{v_B \le \varepsilon\}) + \mu(U_\varepsilon \setminus (B \cap \{v_B \ge \varepsilon\})) \le \\ &\le a(v_\varepsilon, v_\varepsilon) + \int_{B\cap\{v_B>\varepsilon\}} v_\varepsilon^2 d\mu + \frac{\varepsilon}{1-\varepsilon}\int_{B\cap\{v_B>\varepsilon\}} v_\varepsilon d\mu + \varepsilon \le \\ &\le \frac{1}{(1-\varepsilon)^2} a(v_B, v_B) + \frac{1}{1-\varepsilon}\int_{B\cap\{v_B>\varepsilon\}} v_B v_\varepsilon d\mu + \varepsilon \le \frac{1}{(1-\varepsilon)^2}\mathrm{cap}^L_\mu(B) + \varepsilon . \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ we conclude the proof. □

Remark 3.6. For every measure $\mu \in \mathcal{M}_0(\Omega)$, by Theorem 3.5 and Remark 2.2, we have

$$\mathrm{cap}^L_{\tilde\mu}(B) = \inf\{\mathrm{cap}^L_\mu(U) : U \text{ open}, \ B \subseteq U \subset\subset \Omega\}$$

for every Borel set $B \subset\subset \Omega$.

4. Getting μ from its μ-capacity

In this section we state a derivation theorem for the μ-capacity and a theorem which allows us to reconstruct the measure μ from the knowledge of its μ-capacity. The proofs are omitted, since they are identical to those given in [2] and [4] when the operator L is symmetric. Indeed in the previous sections we have proved that all relevant properties of the μ-capacity in the symmetric case can be extended to the case of non-symmetric operators.

We begin with the derivation theorem, which will be used in the proof of Theorem 5.11. The open ball in $\mathbf{R}^n$ of center x and radius r is denoted by $B_r(x)$.

Theorem 4.1. *Let $\mu \in \mathcal{M}_0(\Omega)$, let ν be a Radon measure of the class $\mathcal{M}_0(\Omega)$, and for every $x \in \Omega$ let*

$$g(x) = \liminf_{r \to 0} \frac{\mathrm{cap}_\mu^L(B_r(x))}{\nu(B_r(x))} . \tag{4.1}$$

Assume that $g \in L^1_\nu(\Omega)$ and $g(x) < +\infty$ for q.e. $x \in \Omega$. Then μ is a Radon measure and $\mu(E) = \int_E g\, d\nu$ for every Borel set $E \subseteq \Omega$. Moreover the lower limit in (4.1) is a limit for ν-a.e. $x \in \Omega$.

Proof. When L is symmetric this result was proved in [2], Theorem 2.3, by using some properties of the μ-capacity and of the Green's function of the operator L. Since these properties are still true when L is non-symmetric, the proof remains valid also in the general case. □

The following theorem characterizes μ as the least measure which is greater than or equal to cap_μ^L.

Theorem 4.2. *Let $\mu \in \mathcal{M}_0(\Omega)$. Then for every Borel set $B \subset\subset \Omega$ we have*

$$\mu(B) = \sup \sum_{i \in I} \mathrm{cap}_\mu^L(B_i) ,$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i \in I}$ of B.

Proof. As in [4], Theorem 4.3, this result can be obtained as consequence of the derivation theorem (Theorem 4.1). □

5. μ-capacity and γ^L-convergence

In this section we shall study the connection between the γ^L-convergence of a sequence of measures (μ_h) and the convergence of the corresponding μ_h-capacities relative to the operator L.

First of all we prove that inequalities between measures in $\tilde{\mathcal{M}}_0(\Omega)$ are preserved by γ^L-convergence. To this aim let us establish some preliminary lemmas.

Lemma 5.1. *Let μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$ be two measures such that $\mu_1 \leq \mu_2$. Let w_1 (resp. w_2^*) be the solution of problem (1.7) (resp. (1.8)) corresponding to $\mu = \mu_1$ (resp. $\mu = \mu_2$). Then for every $\varphi \in C_0^\infty(\Omega)$, with $\varphi \geq 0$, we have*

$$\langle 1 - Lw_1, \varphi w_2^* \rangle \leq \langle 1 - L^* w_2^*, \varphi w_1 \rangle .$$

Proof. First note that, since w_1 and w_2^* are non-negative, we have

$$\int_\Omega \varphi w_1 w_2^* \, d\mu_1 \leq \int_\Omega \varphi w_1 w_2^* \, d\mu_2 . \tag{5.1}$$

Since $L^2_{\mu_2}(\Omega) \subseteq L^2_{\mu_1}(\Omega)$, we have $w_2^* \in L^2_{\mu_1}(\Omega)$ and hence

$$\int_\Omega \varphi w_1 w_2^* \, d\mu_1 = \langle 1 - Lw_1, \varphi w_2^* \rangle . \tag{5.2}$$

Moreover by (1.9) we have

$$\int_\Omega \varphi w_1 w_2^* \, d\mu_2 \leq \langle 1 - L^* w_2^*, \varphi w_1 \rangle . \tag{5.3}$$

The conclusion follows from (5.1), (5.2), and (5.3). ☐

Lemma 5.2. *Fix $\varphi \in C_0^\infty(\Omega)$. Then the bilinear form defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by*

$$b(u, v) = \langle Lu, \varphi v \rangle - \langle L^* v, \varphi u \rangle$$

is sequentially weakly continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$, i.e., if (u_h) and (v_h) are two sequences in $H_0^1(\Omega)$ which converge weakly to some functions u and v, then $b(u_h, v_h)$ converges to $b(u, v)$.

Proof. It is enough to note that

$$\langle Lu, \varphi v \rangle - \langle L^* v, \varphi u \rangle = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi \Big) v \, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j \varphi D_i v \Big) u \, dx .$$

☐

Theorem 5.3. *Let* (μ_1^h) *and* (μ_2^h) *be two sequences of measures of* $\mathcal{M}_0(\Omega)$ *which* γ^L*-converge to* μ_1 *and* μ_2 *respectively. If* $\tilde{\mu}_1^h \le \tilde{\mu}_2^h$ *for every* h*, then* $\tilde{\mu}_1 \le \tilde{\mu}_2$*.*

Proof. Let w_1^h be the solution of problem (1.7) corresponding to $\mu = \tilde{\mu}_1^h$ and let $(w_2^h)^*$ be the solution of problem (1.8) corresponding to $\mu = \tilde{\mu}_2^h$. If $\tilde{\mu}_1^h \le \tilde{\mu}_2^h$, then by Lemma 5.1 we have

$$\langle 1 - Lw_1^h, \varphi(w_2^h)^* \rangle \le \langle 1 - L^*(w_2^h)^*, \varphi w_1^h \rangle \tag{5.4}$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$. By Theorem 1.21 and by Remark 1.13 the functions w_1^h (resp. $(w_2^h)^*$) converge weakly in $H_0^1(\Omega)$ to the solution w_1 (resp. w_2^*) of problem (1.7) (resp. (1.8)) corresponding to $\mu = \tilde{\mu}_1$ (resp. $\mu = \tilde{\mu}_2$). By Lemma 5.2 we can pass to the limit in (5.4) and we obtain

$$\langle 1 - Lw_1, \varphi w_2^* \rangle \le \langle 1 - L^* w_2^*, \varphi w_1 \rangle \tag{5.5}$$

for every $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$. By approximation (5.5) holds for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $\varphi \ge 0$. Let w_1^* (resp. $(w_1^h)^*$) be the solution of problem (1.8) corresponding to $\mu = \tilde{\mu}_1$ (resp. $\mu = \tilde{\mu}_1^h$). By the comparison principle (Proposition 1.5) we have that $(w_2^h)^* \le (w_1^h)^*$ q.e. in Ω. Taking the limit as $h \to \infty$, we obtain $w_2^* \le w_1^*$ q.e. in Ω. Hence $w_2^* \in L^2_{\tilde{\mu}_1}(\Omega)$. By Lemma 1.17, $\tilde{\mu}_2(B) = +\infty$ for every Borel set B such that $\mathrm{cap}(B \cap \{w_2^* = 0\}) > 0$. Then it is sufficient to prove that $\tilde{\mu}_1 \le \tilde{\mu}_2$ in $\{w_2^* > 0\}$. Now let $W_k = \{w_2^* > \frac{1}{k}\} \cap \{w_1 > \frac{1}{k}\}$, so that $\tilde{\mu}_2(W_k) < +\infty$. If B is a quasi open subset of W_k, then by Lemma 1.1 there exists an increasing sequence (φ_h) in $H_0^1(\Omega)$ which converges to 1_B q.e. in Ω and such that $0 \le \varphi_h \le 1_B$. As w_1 is bounded (see [6], Section 3) and $\tilde{\mu}_2(B) < +\infty$, we have $w_1\varphi_h \in L^2_{\tilde{\mu}_2}(\Omega)$. Therefore (5.5) and the equations satisfied by w_1 and w_2^* imply that

$$\int_\Omega w_1 w_2^* \varphi_h \, d\tilde{\mu}_1 \le \int_\Omega w_1 w_2^* \varphi_h \, d\tilde{\mu}_2 \,.$$

Passing to the limit as $h \to \infty$ we obtain

$$\int_B w_1 w_2^* \, d\tilde{\mu}_1 \le \int_B w_1 w_2^* \, d\tilde{\mu}_2$$

for every quasi open set $B \subseteq W_k$. Since the measures $w_1 w_2^* \tilde{\mu}_1$ and $w_1 w_2^* \tilde{\mu}_2$ are finite on W_k, this relation holds for every Borel set of W_k. Finally, if B is a Borel set in $\{w_2^* > 0\}$, then

$$\tilde{\mu}_1(B \cap W_k) = \int_{B\cap W_k} \frac{1}{w_1 w_2^*} w_1 w_2^* \, d\tilde{\mu}_1 \le \int_{B\cap W_k} \frac{1}{w_1 w_2^*} w_1 w_2^* \, d\tilde{\mu}_2 = \tilde{\mu}_2(B \cap W_k) \,.$$

Passing to the limit we obtain

$$\tilde{\mu}_1(B \cap \{w_1 > 0\}) \leq \tilde{\mu}_2(B \cap \{w_1 > 0\}) .$$

Since $B \subseteq \{w_2^* > 0\} \subseteq \{w_1^* > 0\}$ and by Lemma 1.19 $\mathrm{cap}(\{w_1^* > 0\} \triangle \{w_1 > 0\}) = 0$, we have that $\tilde{\mu}_1(B) = \tilde{\mu}_1(B \cap \{w_1 > 0\}) \leq \tilde{\mu}_2(B \cap \{w_1 > 0\}) = \tilde{\mu}_2(B)$. □

Let us recall now some notions related to the general theory of increasing set functions, for which we refer to [5], Chapters 14 and 15. As usual the family of all Borel subsets of Ω is denoted by $\mathcal{B}(\Omega)$.

Definition 5.4. We say that a family $\mathcal{E}$ of Borel sets $E \subset\subset \Omega$ is *dense* (in $\mathcal{B}(\Omega)$) if for every pair (K, U), with K compact, U open, and $K \subseteq U \subset\subset \Omega$, there exist $E \in \mathcal{E}$ such that $K \subseteq E \subseteq U$. We say that $\mathcal{E}$ is *rich* (in $\mathcal{B}(\Omega)$) if, for every chain $(E_t)_{t \in \mathbf{R}}$ in $\mathcal{B}(\Omega)$, the set $\{t \in \mathbf{R} : E_t \notin \mathcal{E}\}$ is at most countable. By a *chain* in $\mathcal{B}(\Omega)$ we mean a family $(E_t)_{t \in \mathbf{R}}$ of Borel subsets of Ω, such that $\overline{E}_s \subseteq \mathring{E}_t$ for every s, $t \in \mathbf{R}$ with $s < t$.

Remark 5.5. It is easy to check that any countable intersection of rich families is rich. Moreover it is possible to prove that every rich family is dense (see [5], Chapter 14).

We say that a function $\alpha \colon \mathcal{B}(\Omega) \to \overline{\mathbf{R}}$ is increasing if $\alpha(E) \leq \alpha(F)$ whenever $E \subseteq F$.

Proposition 5.6. *Let α, $\beta : \mathcal{B}(\Omega) \to \overline{\mathbf{R}}$ be two increasing functions. Then the following conditions are equivalent:*

(i) *α and β coincide in a dense subset of $\mathcal{B}(\Omega)$;*

(ii) *α and β coincide in a rich subset of $\mathcal{B}(\Omega)$.*

Proof. See [5], Proposition 14.15. □

Proposition 5.7. *Let α, $\beta : H_0^1(\Omega) \times \mathcal{B}(\Omega) \to \overline{\mathbf{R}}$ be two functionals such that $\alpha(u, \cdot)$ and $\beta(u, \cdot)$ are increasing for every $u \in H_0^1(\Omega)$. Assume, in addition, that for every $E \in \mathcal{B}(\Omega)$ the functionals $\alpha(\cdot, E)$ and $\beta(\cdot, E)$ are lower semicontinuous with respect to the strong topology of $H_0^1(\Omega)$. If $\beta(u, E) \leq \alpha(u, F) \leq \beta(u, G)$ for every E, F, $G \in \mathcal{B}(\Omega)$ with $\overline{E} \subseteq \mathring{F} \subseteq \overline{F} \subseteq \mathring{G}$ and for every $u \in H_0^1(\Omega)$, then there exists a rich subset $\mathcal{R}$ of $\mathcal{B}(\Omega)$ such that $\alpha(u, E) = \beta(u, E)$ for every $u \in H_0^1(\Omega)$ and for every $E \in \mathcal{R}$.*

Proof. See [5], Proposition 15.18. □

In order to study the convergence of the μ_h-capacities when the sequence (μ_h) γ^L-converges to $\mu \in \mathcal{M}_0(\Omega)$, we need to know the convergence properties of the restriction $(\mu_h \llcorner E)$ of the sequence (μ_h) to an arbitrary Borel set E. By the compactness theorem we can assume that $(\mu_h \llcorner E)$ γ^L-converges to some $\lambda \in \mathcal{M}_0(\Omega)$, but, in general, we cannot say that λ is equivalent to $\mu \llcorner E$. Indeed by the localization property (Theorem 1.10) we obtain that λ is equivalent to $\mu \llcorner E$ in $\mathring{E}$ and in $\Omega \setminus \overline{E}$, but it is possible to construct easy examples where λ and $\mu \llcorner E$ are so different in ∂E that λ is not equivalent to $\mu \llcorner E$ (see [10], Example 5.5). Nevertheless the class of Borel sets $E \subset\subset \Omega$ such that $(\mu_h \llcorner E)$ γ^L-converges to $\mu \llcorner E$ is large enough, as stated in the following theorem.

Theorem 5.8. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ which γ^L-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$. Then the family of Borel subsets E of Ω such that $(\mu_h \llcorner E)$ γ^L-converges to $\mu \llcorner E$ is rich.*

Proof. For every Borel subset E of Ω let us denote by $\mathcal{M}^E$ the class of all measures $\lambda \in \mathcal{M}_0(\Omega)$ for which there exists a subsequence (μ_{h_k}) of (μ_h) such that $(\mu_{h_k} \llcorner E)$ γ^L-converges to λ. Let us define the following functionals on $H^1_0(\Omega) \times \mathcal{B}(\Omega)$:

$$\begin{aligned}
\alpha(u, E) &= \int_E u^2 d\mu\,, \\
\beta(u, E) &= \sup_{\lambda \in \mathcal{M}^E} \int_\Omega u^2 d\lambda\,, \\
\delta(u, E) &= \inf\,\{\liminf_{h\to\infty} \hat{\delta}(u_h, E) \;:\; u_h \stackrel{H^1_0(\Omega)}{\longrightarrow} u\}\,,
\end{aligned}$$

where $\hat{\delta}(u, E) = \inf_{\lambda \in \mathcal{M}^E} \int_\Omega u^2 d\lambda$. Since μ vanishes on all sets of capacity zero, the functional $\alpha(\cdot, E)$ is lower semicontinuous in the strong topology of $H^1_0(\Omega)$. Moreover $\alpha(u, \cdot)$ is increasing. The same properties hold for the functionals $\beta(u, E)$ and $\delta(u, E)$. The first one is lower semicontinuous since it is the supremum of a family of lower semicontinuous functionals and the second one by construction. Let us prove that $\beta(u, \cdot)$ and $\delta(u, \cdot)$ are increasing for every $u \in H^1_0(\Omega)$. Let us fix two Borel sets E and F, with $E \subseteq F \subseteq \Omega$, and a function $u \in H^1_0(\Omega)$. Let $t < \beta(u, E)$ and let $\lambda \in \mathcal{M}^E$ be a measure such that $t < \int_\Omega u^2 d\lambda$. Since $\lambda \in \mathcal{M}^E$ there exists a subsequence (μ_{h_k}) of (μ_h) such that $(\mu_{h_k} \llcorner E)$ γ^L-converges to λ. By the compactness theorem (Theorem 1.9) a subsequence of $(\mu_{h_k} \llcorner F)$ γ^L-converges to some measure $\nu \in \mathcal{M}^F$. By Theorem 5.3 and

Remark 1.15 we have $\tilde{\lambda} \leq \tilde{\nu}$ and hence

$$t < \int_\Omega u^2 d\lambda = \int_\Omega u^2 d\tilde{\lambda} \leq \int_\Omega u^2 d\tilde{\nu} = \int_\Omega u^2 d\nu \leq \beta(u,F)\,.$$

By the arbitrariness of $t < \beta(u,E)$ we obtain that $\beta(u,E) \leq \beta(u,F)$. Similarly we can prove that $\hat{\delta}(u,\cdot)$ is increasing, and the same property holds for $\delta(u,\cdot)$.

We want to apply Proposition 5.7 to the functionals α, β, and δ. To this aim let us fix a Borel set $E \subseteq \Omega$ and let us consider a measure $\lambda \in \mathcal{M}^E$. By the localization theorem (Theorem 1.10) applied to $\hat{\Omega} = \mathring{E}$ and $\hat{\Omega} = \Omega \setminus \overline{E}$ we obtain $\tilde{\lambda} = \tilde{\mu}$ in $\mathring{E}$ and $\lambda = 0$ in $\Omega \setminus \overline{E}$. Moreover, by Theorem 5.3 and Remark 1.15, we have $\lambda \leq \tilde{\lambda} \leq \tilde{\mu}$ in Ω. Thus, if E, F, and G are three Borel subsets of Ω such that $\overline{E} \subseteq \mathring{F} \subseteq \overline{F} \subseteq \mathring{G}$, for every $\lambda \in \mathcal{M}^E$ and $\nu \in \mathcal{M}^G$ we get $\lambda \leq \tilde{\mu} \llcorner \mathring{F} \leq \tilde{\mu} \llcorner \mathring{G} \leq \tilde{\nu}$. By Remarks 1.12 and 1.15 this implies that

$$\begin{aligned} \int_\Omega u^2 d\lambda \leq \int_{\mathring{F}} u^2 d\tilde{\mu} = \int_{\mathring{F}} u^2 d\mu \leq \int_F u^2 d\mu \leq \\ \leq \int_{\mathring{G}} u^2 d\mu = \int_{\mathring{G}} u^2 d\tilde{\mu} \leq \int_\Omega u^2 d\tilde{\nu} = \int_\Omega u^2 d\nu\,. \end{aligned}$$

Therefore $\beta(u,E) \leq \alpha(u,F) \leq \beta(u,G)$ and $\delta(u,E) \leq \alpha(u,F) \leq \delta(u,G)$ whenever $u \in H_0^1(\Omega)$ and $\overline{E} \subseteq \mathring{F} \subseteq \overline{F} \subseteq \mathring{G}$. Consequently, by Proposition 5.7, there exists a rich subset $\mathcal{R}$ of $\mathcal{B}(\Omega)$ such that

$$\beta(u,E) = \delta(u,E) = \alpha(u,E) = \int_\Omega u^2 d(\mu \llcorner E) \tag{5.6}$$

for every $u \in H_0^1(\Omega)$ and $E \in \mathcal{R}$.

Let us prove that $(\mu_h \llcorner E)$ γ^L-converges to $\mu \llcorner E$ for every $E \in \mathcal{R}$. Let us fix $E \in \mathcal{R}$ and $\lambda \in \mathcal{M}^E$. By the definition of β and δ we have $\delta(u,E) \leq \int_\Omega u^2 d\lambda \leq \beta(u,E)$ for every $u \in H_0^1(\Omega)$; so that, by (5.6), we get

$$\int_\Omega u^2 d(\mu \llcorner E) = \int_\Omega u^2 d\lambda$$

for every $u \in H_0^1(\Omega)$, hence $\mu \llcorner E$ and λ are equivalent. By Remark 1.13 this implies that every convergent subsequence of $(\mu_h \llcorner E)$ γ^L-converges to $\mu \llcorner E$. Since γ^L-convergence is compact (Theorem 1.9), we conclude that the whole sequence $(\mu_h \llcorner E)$ γ^L-converges to $\mu \llcorner E$. □

We are now in a position to prove the main result of this section.

Theorem 5.9. *Let* (μ_h) *be a sequence in* $\mathcal{M}_0(\Omega)$ *and let* $\mu \in \mathcal{M}_0(\Omega)$. *Then the following conditions are equivalent:*

(a) (μ_h) γ^L*-converges to* μ*;*

(b) $\lim\limits_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E) = \mathrm{cap}^L_\mu(E)$ *for every* E *in a dense subset of* $\mathcal{B}(\Omega)$*;*

(c) $\lim\limits_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E) = \mathrm{cap}^L_\mu(E)$ *for every* E *in a rich subset of* $\mathcal{B}(\Omega)$.

Proof. (c) $\Rightarrow$ (b). See Remark 5.5.

(b) $\Rightarrow$ (c). For every Borel set $E \subset\subset \Omega$ let $\alpha'(E) = \liminf\limits_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E)$, $\alpha''(E) = \limsup\limits_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E)$, and $\alpha(E) = \mathrm{cap}^L_\mu(E)$. By Proposition 5.6 condition (b) implies that $\alpha' = \alpha''$ in a rich subset $\mathcal{R}_1$ of $\mathcal{B}(\Omega)$ and $\alpha' = \alpha$ in a rich subset $\mathcal{R}_2$ of $\mathcal{B}(\Omega)$. By Remark 5.5 the class $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ is rich in $\mathcal{B}(\Omega)$ and we have

$$\liminf_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E) \,=\, \limsup_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E) \,=\, \mathrm{cap}^L_\mu(E)$$

for every $E \in \mathcal{R}$.

(a) $\Rightarrow$ (c). If (μ_h) γ^L-converges to μ, then there exists a rich subset $\mathcal{R}$ of $\mathcal{B}(\Omega)$ such that $(\mu_h \llcorner E)$ γ^L-converges to $\mu \llcorner E$ for every $E \in \mathcal{R}$ (Theorem 5.8). Let $E \in \mathcal{R}$ and let v^h_E and v_E be the μ_h-capacitary potential and the μ-capacitary potential of E relative to L. Then (v^h_E) converges to v_E weakly in $H^1_0(\Omega)$ (Proposition 1.8). Moreover, if ν^h_E and ν_E are the inner μ_h-capacitary distribution and the inner μ-capacitary distribution of E relative to L, then (ν^h_E) converges to ν_E weakly in $H^{-1}(\Omega)$ (Lemma 2.5). Since $E \subset\subset \Omega$, it is possible to find $\varphi \in C^\infty_0(\Omega)$ such that $\varphi = 1$ in $\overline{E}$ and, since $\mathrm{supp}\,\nu^h_E \subseteq \overline{E}$ and $\mathrm{supp}\,\nu_E \subseteq \overline{E}$, by Lemma 2.7 we have

$$\lim_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E) \,=\, \lim_{h\to\infty} \int_\Omega \varphi\, d\nu^h_E \,=\, \int_\Omega \varphi\, d\nu_E \,=\, \mathrm{cap}^L_\mu(E)\,.$$

(c) $\Rightarrow$ (a). By the compactness of the γ^L-convergence there exists a subsequence of (μ_h) which γ^L-converges to some measure $\lambda \in \mathcal{M}_0(\Omega)$. It is enough to prove that μ and λ are equivalent. By the previous step we have that $\mathrm{cap}^L_{\mu_h}(E)$ converges to $\mathrm{cap}^L_\lambda(E)$ for every E in a rich subset of $\mathcal{B}(\Omega)$. Since the intersection of two rich sets is rich (Remark 5.5), (c) implies that $\mathrm{cap}^L_\lambda(E) = \mathrm{cap}^L_\mu(E)$ for every E in a rich subset $\mathcal{R}$ of $\mathcal{B}(\Omega)$. Let $U \subset\subset \Omega$ be an arbitrary open set and let $\varepsilon > 0$. By Theorem 3.4 there exists a compact set K contained in U such that $\mathrm{cap}^L_\mu(U) \le \mathrm{cap}^L_\mu(K) + \varepsilon$. Since $\mathcal{R}$ is dense, there exists $E \in \mathcal{R}$ such that $K \subseteq E \subseteq U$. By monotonicity (Theorem 2.11) we have that $\mathrm{cap}^L_\mu(U) \le \mathrm{cap}^L_\mu(E)+\varepsilon = \mathrm{cap}^L_\lambda(E)+\varepsilon \le \mathrm{cap}^L_\lambda(U)+\varepsilon$. Since $\varepsilon>0$ is arbitrary, we obtain $\mathrm{cap}^L_\mu(U) \le \mathrm{cap}^L_\lambda(U)$. By exchanging the roles of λ and μ we prove the opposite inequality, hence $\mathrm{cap}^L_\mu(U) = \mathrm{cap}^L_\lambda(U)$. By Remark 3.6 this implies that $\mathrm{cap}^L_{\tilde\mu}(B) = \mathrm{cap}^L_{\tilde\lambda}(B)$ for every Borel set $B \subset\subset \Omega$. Therefore $\tilde\mu = \tilde\lambda$ by Theorem 4.2, so that μ and λ are equivalent by Remark 1.15. □

Theorem 5.10. *Let (μ_h) be a sequence in $\mathcal{M}_0(\Omega)$. Suppose that there exists a dense subset $\mathcal{D}$ of $\mathcal{B}(\Omega)$ such that*

$$\lim_{h\to\infty} \mathrm{cap}^L_{\mu_h}(E) = \alpha(E)$$

for every $E\in\mathcal{D}$. Let β be the increasing set function defined by

$$\begin{aligned} &\beta(U) = \sup\{\alpha(E) : E\in\mathcal{D},\ E\subset\subset U\}, \qquad \text{if } U \text{ is open in } \Omega, \\ &\beta(B) = \inf\{\beta(U) : U \text{ open},\ B\subseteq U\subseteq\Omega\}, \qquad \text{if } B\subseteq\Omega. \end{aligned} \tag{5.7}$$

Finally, let μ be the measure defined for every Borel set $B\subseteq\Omega$ by

$$\mu(B) = \sup\sum_{i\in I}\beta(B_i), \tag{5.8}$$

where the supremum is taken over all finite Borel partitions $(B_i)_{i\in I}$ of B.

Then $\mu\in\tilde{\mathcal{M}}_0(\Omega)$, the sequence (μ_h) γ^L-converges to μ, and $\beta(B) = \mathrm{cap}^L_\mu(B)$ for every Borel set $B\subset\subset\Omega$.

Proof. By compactness of the γ^L-convergence we can assume that the sequence (μ_h) γ^L-converges to a measure λ in $\tilde{\mathcal{M}}_0(\Omega)$ and, by Theorem 5.9, that $\mathrm{cap}^L_{\mu_h}(E)$ converges to $\mathrm{cap}^L_\lambda(E)$ for every E in a rich subset $\mathcal{R}$ of $\mathcal{B}(\Omega)$. We have to prove that $\lambda=\mu$.

Let us consider an open set $U\subseteq\Omega$ and a set $E\in\mathcal{D}$ with $E\subset\subset U$. Since $\mathcal{R}$ is dense (Remark 5.5), there exists $F\in\mathcal{R}$ such that $E\subseteq F\subseteq U$. This implies that

$$\alpha(E) = \lim_{h\to\infty}\mathrm{cap}^L_{\mu_h}(E) \le \lim_{h\to\infty}\mathrm{cap}^L_{\mu_h}(F) = \mathrm{cap}^L_\lambda(F) \le \mathrm{cap}^L_\lambda(U).$$

By the definition of β this implies $\beta(U)\le\mathrm{cap}^L_\lambda(U)$, and from Theorem 3.5 we obtain $\beta(B)\le\mathrm{cap}^L_\lambda(B)$ for every Borel set $B\subset\subset\Omega$.

To prove the opposite inequality, let us consider an open set $U\subseteq\Omega$ and a compact set $K\subseteq U$. Since $\mathcal{D}$ and $\mathcal{R}$ are dense, there exist $E\in\mathcal{D}$ and $F\in\mathcal{R}$ such that $K\subseteq F\subseteq E\subset\subset U$. Then

$$\mathrm{cap}^L_\lambda(K) \le \mathrm{cap}^L_\lambda(F) = \lim_{h\to\infty}\mathrm{cap}^L_{\mu_h}(F) \le \lim_{h\to\infty}\mathrm{cap}^L_{\mu_h}(E) = \alpha(E) \le \beta(U).$$

By Theorem 3.4 this implies $\mathrm{cap}^L_\lambda(U)\le\beta(U)$, and from Theorem 3.5 we obtain $\mathrm{cap}^L_\lambda(B)\le\beta(B)$, and hence $\mathrm{cap}^L_\lambda(B)=\beta(B)$, for every Borel set $B\subset\subset\Omega$. Then the conclusion follows from (5.8) and Theorem 4.2. □

As consequence of Theorems 4.1 and 5.9 we obtain the following characterization of the limit measure by means of a derivation argument.

Theorem 5.11. *Let* (μ_h) *be a sequence measures of the class* $\mathcal{M}_0(\Omega)$ *and let* ν *be a Radon measure of the class* $\mathcal{M}_0(\Omega)$*. Assume that*

$$\liminf_{r\to 0}\liminf_{h\to\infty}\frac{\mathrm{cap}^L_{\mu_h}(B_r(x))}{\nu(B_r(x))} = \liminf_{r\to 0}\limsup_{h\to\infty}\frac{\mathrm{cap}^L_{\mu_h}(B_r(x))}{\nu(B_r(x))} = g(x) \tag{5.9}$$

for q.e. $x\in\Omega$*, and that* $\int_\Omega g\,d\nu < +\infty$*. Then* (μ_h) γ^L*-converges to* $\mu = g\nu$ *and the* $\liminf\limits_{r\to 0}$ *is actually a* $\lim\limits_{r\to 0}$ *for* ν*-a.e.* $x\in\Omega$*.*

Proof. The result follows from Theorem 5.9 and 4.1, as in the proof of Theorem 5.2 in [2]. □

Remark 5.12. Under the hypotheses of Theorem 5.10, condition (5.9) is satisfied, for instance, when $\beta(B)\le\nu(B)$ for every Borel set $B\subseteq\Omega$.

6. Dirichlet problems in perforated domains

The asymptotic behaviour of Dirichlet problems in varying domains can be obtained as a particular case of the previous results. We consider only the consequence of Theorem 5.10. Similar results can be obtained also from Theorems 5.9 and 5.11.

Theorem 6.1. *Let* (Ω_h) *be a sequence of open subsets of* Ω*. Suppose that there exists a dense subset* $\mathcal{D}$ *of* $\mathcal{B}(\Omega)$ *such that*

$$\lim_{h\to\infty}\mathrm{cap}^L(E\cap\Omega_h) = \alpha(E)$$

for every $E\in\mathcal{D}$*. Let* β *be the increasing set function defined by (5.7) and let* μ *be the measure defined by (5.8). Then for every* $f\in H^{-1}(\Omega)$ *the solution* u_h *of the Dirichlet problem*

$$\begin{cases} u_h\in H^1_0(\Omega_h)\,,\\ Lu_h = f \qquad \text{in } \Omega_h\,, \end{cases} \tag{6.1}$$

extended by 0 *in* $\Omega \setminus \Omega_h$ *, converges weakly in* $H_0^1(\Omega)$ *to the solution* u *of the relaxed Dirichlet problem*

$$\begin{cases} u \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \,, \\ a(u,v) \,+\, \displaystyle\int_\Omega uv\, d\mu \,=\, \langle f, v\rangle \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega) \,. \end{cases}$$

Moreover $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ *and* $\beta(B) = \mathrm{cap}^L_\mu(B)$ *for every Borel set* $B \subset\subset \Omega$ *.*

Proof. Let $E_h = \Omega \setminus \Omega_h$ and let $\mu_h = \infty_{E_h}$. By Remark 1.4 the solution of (6.1), extended by 0 in $\Omega \setminus \Omega_h$, coincides with the solution of (1.4). By Remark 2.3 we have $\mathrm{cap}^L_{\mu_h}(B) = \mathrm{cap}^L(B \cap E_h)$ for every Borel set $B \subset\subset \Omega$. The conclusion follows now from Theorem 5.10 and from the definition of γ^L-convergence. ☐

In the rest of this section we shall use the previous result to prove that, if μ_0 is a Radon measure in $\mathcal{M}_0(\Omega)$, then there exists a sequence Ω_h of open subset of Ω such that the conclusion of Theorem 6.1 holds with $\mu = \mu_0$. This approximation result is obtained by an explicit construction of the sets Ω_h, which are obtained from Ω by removing a suitable disjoint family of "small" closed sets, whose size depends on the local value of μ.

For every $h \in \mathbf{N}$ we consider the partition of $\mathbf{R}^n$ composed of the semi-open cubes of side $1/h$

$$Q_h^i = \{x \in \mathbf{R}^n \colon\ i_k/h \le x_k < (i_k+1)/h \text{ for } \ k = 1, \ldots, n\} \,, \qquad i = (i_1, \ldots, i_n) \in \mathbf{Z}^n \,,$$

and we denote by N_h the set of all indices i such that $Q_h^i \subset\subset \Omega$.

We fix a Radon measure μ_0 in $\mathcal{M}_0(\Omega)$ and for every $h \in \mathbf{N}$ and $i \in N_h$ we consider a closed set $E_h^i \subseteq Q_h^i$ such that $\mathrm{cap}^L(E_h^i, Q_h^i) = \mu_0(Q_h^i)$. Let E_h be the union of the sets E_h^i for $i \in N_h$ and let $\Omega_h = \Omega \setminus E_h$. We shall prove that, in this case, the conclusion of Theorem 6.1 holds with $\mu = \mu_0$. More generally, for every $i \in N_h$ we fix a constant $c_h^i \ge 0$ and we choose the closed sets $E_h^i \subseteq Q_h^i$ so that $\mathrm{cap}^L(E_h^i, Q_h^i) = c_h^i \mu_0(Q_h^i)$. Then the asymptotic behaviour of the solutions of problems (6.1) is uniquely determined by the weak* limit in $L^\infty_{\mu_0}(\Omega)$ of the sequence (ψ_h) defined by

$$\psi_h(x) = \sum_{i \in N_h} c_h^i 1_{Q_h^i}(x) \,. \tag{6.2}$$

The following theorem is a generalization, to the case of non-symmetric operators, of the approximation result given in [8], Theorem 2.5, and in [1], Theorem 2.2.

Theorem 6.2. *Let μ_0 be a Radon measure belonging to $\mathcal{M}_0(\Omega)$ and let $(c_h^i)_{h\in\mathbf{N},i\in N_h}$ be a family of non-negative real numbers. For every $h \in \mathbf{N}$ let $E_h = \bigcup_{i\in N_h} E_h^i$, where E_h^i are closed sets contained in Q_h^i with $\operatorname{cap}^L(E_h^i, Q_h^i) = c_h^i \mu_0(Q_h^i)$. Suppose that the sequence (ψ_h) defined by (6.2) converges to some function ψ in the weak* topology of $L^\infty_{\mu_0}(\Omega)$. Then for every $f \in H^{-1}(\Omega)$ the solution u_h of problem (6.1) converges weakly in $H^1_0(\Omega)$ to the solution u of the relaxed Dirichlet problem*

$$\begin{cases} u \in H^1_0(\Omega) \cap L^2_\lambda(\Omega)\,, \\ a(u,v) + \displaystyle\int_\Omega uv\,d\lambda \;=\; \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega)\cap L^2_\lambda(\Omega)\,, \end{cases}$$

where $\lambda = \psi\mu_0$.

Proof. We just give an outline of the proof, since it follows closely the one given in [1], Theorem 2.2. We know that problem (6.1) can be rewritten as a relaxed Dirichlet problem in Ω by choosing $\mu_h = \infty_{E_h}$ (Remark 1.4). Then by the compactness of the γ^L-convergence (Theorem 1.9) we can suppose that (∞_{E_h}) γ^L-converges to a measure $\lambda \in \mathcal{M}_0(\Omega)$. We have to prove that $\lambda = \psi\mu_0$.

Step 1. We prove that $\lambda \le \psi\mu_0$. Since $\operatorname{cap}^L_{\mu_h}$ is subadditive and, by Theorem 5.9,

$$\lim_{h\to\infty} \operatorname{cap}^L(E_h \cap E) \;=\; \lim_{h\to\infty} \operatorname{cap}^L_{\mu_h}(E) \;=\; \operatorname{cap}^L_\lambda(E)$$

for every E belonging to a rich subset of $\mathcal{B}(\Omega)$, we can repeat the proof of Proposition 2.3 of [1] and we obtain $\operatorname{cap}^L_\lambda(E) \le \int_E \psi d\mu_0$ for every Borel set $E \subset\subset \Omega$. The conclusion follows now from Theorem 4.2.

Step 2. We prove that for every open set $U \subset\subset \Omega$ and for every $\delta > 0$ the following estimate holds

$$\lambda(\overline{U}) \;\ge\; (1-c\delta)^2 \int_U \psi(x)\,d\mu_0(x) \;-\; \frac{c}{\delta}\iint_{\overline{U}\times\overline{U}} G(x-y)\,d\mu_0(x)d\mu_0(y)\,, \tag{6.3}$$

where G is the fundamental solution for the Laplace operator in $\mathbf{R}^n$ and c is a positive constant independent of U and δ. This estimate can be obtained as in [1], Lemmas 2.6 and 2.7. The only difference is in the proof of the "local almost-superadditivity" of the capacity of the sets E_h (see Lemma 6.3 below), that in [1] relies heavily on the symmetry of the operator L.

Step 3. If $\mu_0 \in H^{-1}(\Omega)$, estimate (6.3) implies that $\lambda \ge (1-c\delta)^2\psi\mu_0$ by Lemma 2.5 of [1]. Since $\delta > 0$ is arbitrary, we get $\lambda \ge \psi\mu_0$. To extend this result to any Radon measure of $\mathcal{M}_0(\Omega)$ we use the truncation argument of Theorem 2.2 in [1], which in our case is based on Theorem 5.3. □

We conclude by proving the "local almost-superadditivity" used in Step 2 of Theorem 6.2.

Lemma 6.3. *Let U be an open set, with $U \subset\subset \Omega$, and let $0 < \delta < 1$. Let u be the capacitary potential of $E_h \cap U$ in Ω with respect to the operator L. For every $h \in \mathbf{N}$ we denote by I_h the set of all indices $i \in N_h$ such that $Q_h^i \cap U \neq \emptyset$ and $u \leq \delta$ q.e. in ∂Q_h^i. Then*

$$\sum_{i \in I_h} \mathrm{cap}^L(E_h^i, Q_h^i) \leq \frac{1}{(1-\delta)^2} \mathrm{cap}^L(E_h \cap U, \Omega) .$$

Proof. Let us consider the function $v = \max\{0, \frac{u-\delta}{1-\delta}\}$ and for every $h \in \mathbf{N}$ and $i \in I_h$ let v_h^i be the function such that $v_h^i = v$ q.e. in $\{u > \delta\} \cap Q_h^i$ and $v_h^i = 0$ q.e. in $\Omega \setminus (\{u > \delta\} \cap Q_h^i)$. It is easy to see that v_h^i is the capacitary potential of E_h^i in $\{u > \delta\} \cap Q_h^i$ according to (2.1), hence

$$\mathrm{cap}^L(E_h^i, \{u > \delta\} \cap Q_h^i) = \int_{\{u>\delta\} \cap Q_h^i} \big(\sum_{i,j=1}^n a_{ij} D_j v D_i v \big) \, dx .$$

Then, by the monotonicity properties of cap^L (see [7], Theorem 3.3), we get

$$\sum_{i \in I_h} \mathrm{cap}^L(E_h^i, Q_h^i) \leq \sum_{i \in I_h} \mathrm{cap}^L(E_h^i, \{u > \delta\} \cap Q_h^i) =$$

$$= \sum_{i \in I_h} \int_{\{u>\delta\} \cap Q_h^i} \big(\sum_{i,j=1}^n a_{ij} D_j v D_i v \big) \, dx \leq \frac{1}{(1-\delta)^2} \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j u D_i u \big) \, dx ,$$

which, by the definition of u, concludes the proof. □

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