ON THE RELAXED FORMULATION
OF SOME SHAPE OPTIMIZATION PROBLEMS

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Abstract

We study the relaxed formulation of the shape optimization problem with constraints

$$\min_A \left\{ \int_A j(x,u_A)\,d\lambda \ :\ A \text{ open} \subseteq \Omega\,,\ \lambda(A)\in T\,,\ Lu_A = f \text{ in } A\,,\ u_A\in H^1_0(A)\right\},$$

where Ω is a bounded open set in $\mathbf{R}^n$, $n\geq 2$, $j\colon\Omega\times\mathbf{R}\to\mathbf{R}$ is a Carathéodory function, λ is a nonnegative Radon measure on Ω vanishing on all sets with harmonic capacity zero, T is a closed subinterval of $[0,\lambda(\Omega)]$, L is an elliptic operator on Ω, and $f\in H^{-1}(\Omega)$.

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0. Introduction

A large class of shape optimization problems can be studied by considering cost functionals of the form

$$J(A) = F(u_A)\,, \tag{0.1}$$

where A varies in a suitable class $\mathcal{A}$ of open subsets of a given bounded domain $\Omega \subseteq \mathbf{R}^n$, and u_A is the solution of a partial differential equation in A (see, for instance, [21], [22], [27], [23], [25], [10], [3], [4], [5]). The papers [9], [7], [12], [8], [19] deal in particular with the case where $\mathcal{A}$ is the family of all open sets A contained in Ω, and u_A is the solution of the problem

$$Lu_A = f \quad \text{in } A\,, \qquad u_A = 0 \quad \text{in } \overline{\Omega}\backslash A\,, \tag{0.2}$$

where L is a given elliptic operator, $f \in H^{-1}(\Omega)$, and $F(u)$ in (0.1) is an integral functional, continuous on $L^2(\Omega)$, of the form

$$F(u) = \int_\Omega j(x,u)\,dx.$$

In these papers it was pointed out that in general the corresponding minimization problem

$$\min_A \left\{ \int_\Omega j(x,u_A)\,dx \;:\; A \text{ open} \subseteq \Omega\,,\; Lu_A = f \;\text{ in } A\,,\; u_A = 0 \;\text{ in } \overline{\Omega}\backslash A \right\} \tag{0.3}$$

does not admit any solution. An explanation of this fact is that, if $\{A_h\}$ is a sequence of open subsets of Ω, in particular a minimizing sequence of (0.3), then the corresponding sequence $\{u_{A_h}\}$ of solutions of (0.2) has a subsequence converging to a function u in $L^2(\Omega)$, but, in general, there is no open set A such that $u = u_A$. Nevertheless there exists a nonnegative, possibly unbounded, Borel measure μ such that u is the solution (in the sense (1.4) below) of the problem

$$Lu_\mu + \mu u_\mu = f \quad \text{in } \Omega\,, \qquad u_\mu = 0 \quad \text{on } \partial\Omega\,. \tag{0.4}$$

The relaxed form of (0.3) is studied in [9] and it is given by

$$\min_\mu \left\{ \int_\Omega j(x,u_\mu)\,dx \;:\; \mu \in \mathcal{M}_0(\Omega)\,,\; Lu_\mu + \mu u_\mu = f \;\text{ in } \Omega\,,\; u_\mu = 0 \;\text{ on } \partial\Omega \right\}\,,$$

where $\mathcal{M}_0(\Omega)$ is the class of all nonnegative Borel measures on Ω which vanish on all Borel sets of harmonic capacity zero. Moreover, if we identify each open set A with the measure μ_A defined in (1.1) below, the functional $\tilde{J}(\mu) = \int_\Omega j(x, u_\mu)\, dx$ which appears in the relaxed problem turns out to be the lower semicontinuous envelope of the functional $J(A) = \int_\Omega j(x, u_A)\, dx$ with respect to a suitable notion of convergence in $\mathcal{M}_0(\Omega)$, called γ^L-convergence, introduced in [18], [11], [15]. We recall that a sequence of measures $\{\mu_h\}$ of $\mathcal{M}_0(\Omega)$ γ^L-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$ if for every $f \in H^{-1}(\Omega)$ the sequence $\{u_{\mu_h}\}$ of the solutions to the problems

$$Lu_{\mu_h} + \mu_h u_{\mu_h} = f \quad \text{in } \Omega\,, \qquad u_{\mu_h} = 0 \quad \text{on } \partial\Omega\,,$$

converge strongly in $L^2(\Omega)$ to the solution u_μ of (0.4). It is possible to prove that the topological space $\big(\mathcal{M}_0(\Omega), \gamma^L\big)$ is actually a compact metric space (see [17], [15]), and hence the relaxation of problem (0.3) studied in [9] can be considered in the general framework of [1].

In this paper we treat a case where the cost functionals J depend on the unknown domain A not only through the solution u_A as in (0.1). More precisely, we consider functionals J of the form

$$J(A) = \begin{cases} \int_A j(x, u_A)\, d\lambda, & \text{if } \lambda(A) \in T, \\ +\infty, & \text{otherwise,} \end{cases}$$

where λ is a bounded measure in $\mathcal{M}_0(\Omega)$, and $T = [m, M]$ is a subinterval of $[0, \lambda(\Omega)]$ possibly degenerating to a point. Notice that the functional J depends on A through the domain of integration, through the solution u_A of the differential equation, and through the constraint $\lambda(A) \in T$. We shall prove that also in this case the minimum problem

$$\min\{J(A)\colon\ A \text{ open}\,,\ A \subseteq \Omega\}$$

has, in general, no solution, and that the relaxed problem can be written as

$$\min\left\{\overline{J}(\mu)\colon\ \mu \in \mathcal{M}_0(\Omega)\right\}\,,$$

where $\overline{J}$ is the lower semicontinuous envelope of J in $\mathcal{M}_0(\Omega)$ with respect to the γ^L-convergence (Remark 1.12).

The main result of the paper is an explicit integral representation of $\overline{J}$ in terms of the integrand j and of the constraint T (Theorem 3.1). The relevant new difficulty with respect to [9] lies in the fact that A appears in J also in the domain of integration. This requires a substantial change in the proof, which is based on some new measure theoretical arguments.

The paper is organized as it follows:

– in Section 1 we recall the main properties of the class $\mathcal{M}_0(\Omega)$ together with the preliminary results we shall use in the sequel;

– in Section 2 we prove a lower semicontinuity result with respect to the γ^L-convergence for some functionals defined on the class of open subsets of Ω, or more generally, for their extension to $\mathcal{M}_0(\Omega)$;

– Section 3 is devoted to the representation of the functionals $\overline{J}$.

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1. Preliminaries

Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 2$. We shall denote by $H^1(\Omega)$ and $H_0^1(\Omega)$ the usual Sobolev spaces and by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. By $L^p_\mu(\Omega)$, $1 \leq p \leq +\infty$, we denote the Lebesgue space with respect to a nonnegative measure μ. If μ is the Lebesgue measure, we shall use the standard notation $L^p(\Omega)$.

In the sequel $B(x,r)$ will denote the open ball with center x and radius r.

For every subset E of Ω the *(harmonic) capacity* of E in Ω, denoted by $\mathrm{cap}(E,\Omega)$, is defined as the infimum of

$$\int_\Omega |Du|^2\,dx$$

over the set of all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E.

We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (shortly q.e.) in a set E if it holds for all $x \in E$ except for a subset N of E with $\mathrm{cap}(N,\Omega) = 0$. A function $u\colon \Omega \to \mathbf{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$, with $\mathrm{cap}(E,\Omega) < \varepsilon$, such that the restriction of u to $\Omega \setminus E$ is continuous.

It is well known that every $u \in H^1(\Omega)$ has a quasi continuous representative, which is uniquely defined up to a set of capacity zero. In the sequel we shall always identify u with its quasi continuous representative, so that the pointwise values of a function $u \in H^1(\Omega)$ are defined quasi everywhere. We recall that, if a sequence $\{u_h\}$ converges to u in $H^1_0(\Omega)$, then a subsequence of $\{u_h\}$ converges to u q.e. in Ω. Moreover if $u, v \in H^1_0(\Omega)$ and $u \le v$ a.e. in Ω, then $u \le v$ q.e. in Ω. For all these properties of quasi continuous representatives of Sobolev functions we refer to [26], Section 3.

A subset A of Ω is said to be *quasi open* if for every $\varepsilon > 0$ there exists an open subset U_ε of Ω, with $\mathrm{cap}(U_\varepsilon, \Omega) < \varepsilon$, such that $A \cup U_\varepsilon$ is open.

Let us denote by $\mathcal{B}(\Omega)$ the σ-field of all Borel subsets of Ω. By a *nonnegative Borel measure* on Ω we mean a countably additive set function $\mu: \mathcal{B}(\Omega) \to [0, +\infty]$. By a *nonnegative Radon measure* on Ω we mean a nonnegative Borel measure which is bounded on every compact subset of Ω. Given a nonnegative Borel measure μ, its completion is still denoted by μ. If μ is a nonnegative Borel measure and h is a nonnegative Borel measurable function, we shall denote by $h\mu$ the nonnegative Borel measure defined by $(h\mu)(B) = \int_B h\, d\mu$ for every $B \in \mathcal{B}(\Omega)$.

We say that a nonnegative Borel measure μ is nonatomic if $\mu(\{x\}) = 0$ for every $x \in \Omega$. It is well known that, if μ is nonnegative and nonatomic, then for every $B \in \mathcal{B}(\Omega)$ there exists a Borel subset B_1 of B such that $0 < \mu(B_1) \le \frac{1}{2}\mu(B)$, and, by induction, for every $k \in \mathbf{N}$ there exists $B_k \subseteq B$ such that

$$0 < \mu(B_k) \le \frac{1}{2^k}\mu(B).$$

It is also well known that a nonnegative, nonatomic bounded measure has the following continuity property: for every choice of $A, B \in \mathcal{B}(\Omega)$, with $A \subseteq B$, and for every α in the interval $[\mu(A), \mu(B)]$ there exists a set $C \in \mathcal{B}(\Omega)$ such that $A \subseteq C \subseteq B$ and $\mu(C) = \alpha$ (see, e.g., [20]). In the sequel we shall need the following continuity result, involving only open sets.

Lemma 1.1. *Let μ be a nonnegative nonatomic bounded measure in Ω and let A, B be two open subsets of Ω such that $A \subseteq B$. Then for every α such that $\mu(A) \le \alpha \le \mu(B)$ there exists an open set C such that $A \subseteq C \subseteq B$ and $\mu(C) = \alpha$.*

Proof. Let $\{C_k\}$ be an increasing sequence of open subsets of B such that $C_0 = A$ and $\alpha \ge \mu(C_{k+1}) \ge s_k - 1/k$, where

$$s_k = \sup\{\mu(U) : U \text{ open}, \ C_k \subseteq U \subseteq B, \ \mu(U) \le \alpha\}.$$

If we define $C = \bigcup_k C_k$ and

$$s = \sup\{\mu(U) : U \text{ open}, \ C \subseteq U \subseteq B, \ \mu(U) \le \alpha\},$$

then C is open, $A \subseteq C \subseteq B$, and $0 \le s \le s_k$ for every k. As $\{C_k\}$ is an increasing sequence, we have

$$\mu(C) = \lim_{k\to\infty} \mu(C_k) \ \ge\ \lim_{k\to\infty} s_k \ \ge\ s\,,$$

hence $\mu(C) = s$ and $\mu(U) = s$ for every open set U such that $C \subseteq U \subseteq B$ and $\mu(U) \le \alpha$.

It remains to prove that $s = \alpha$. By contradiction, if $s < \alpha$, then $\mu(B \setminus C) > 0$. Let us fix $0 < \beta < \alpha - s$. Since μ is nonatomic, there exists a set $E \in \mathcal{B}(\Omega)$ such that $E \subseteq B\setminus C$ and $0 < \mu(E) < \beta$. Moreover, since μ is a Radon measure, there exists an open set V such that $E \subseteq V \subseteq B$ and $\mu(E) \le \mu(V) \le \beta$. If we denote $U = C \cup V$, then $C \subseteq U \subseteq B$, $\mu(U) \le s$, and

$$\mu(U) \ \ge\ \mu(C \cup E) \ =\ \mu(C) + \mu(E) \ >\ \mu(C) \ =\ s\,,$$

which gives a contradiction. Thus $\mu(C) = \alpha$, and this concludes the proof. □

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We say that a nonnegative Radon measure ν on Ω belongs to $H^{-1}(\Omega)$ if there exists $f \in H^{-1}(\Omega)$ such that

$$\langle f, \varphi\rangle = \int_\Omega \varphi\, d\nu \qquad \forall \varphi \in C_0^\infty(\Omega)\,,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We shall always identify f and ν. Having identified each function u in $H_0^1(\Omega)$ with its quasi continuous representative, for every nonnegative Radon measure $\nu \in H^{-1}(\Omega)$ we have $H_0^1(\Omega) \subseteq L^1_\nu(\Omega)$, and $\langle f, u\rangle = \int_\Omega u\, d\nu$ for every $u \in H_0^1(\Omega)$. Moreover the injection of $H_0^1(\Omega)$ into $L^1_\nu(\Omega)$ is compact. Indeed, if $\{v_h\}$ is a sequence of functions which converges to a function v weakly in $H_0^1(\Omega)$, then $|v_h - v|$ converges to 0 weakly in $H_0^1(\Omega)$, so that $\int_\Omega |v_h - v|\, d\nu = \langle \nu, |v_h - v|\rangle$ tends to 0.

We denote by $\mathcal{M}_0(\Omega)$ the set of all nonnegative Borel measures μ on Ω such that

(i) $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B, \Omega) = 0$,

(ii) $\mu(B) = \inf\{\mu(A) : A \text{ quasi open}\,,\ B \subseteq A\}$ for every Borel set $B \subseteq \Omega$.

Since all quasi open sets differs from a Borel set by a set of capacity zero, all quasi open sets are μ–measurable for every nonnegative Borel measure μ which satisfies (i). Therefore $\mu(A)$ is well defined when A is quasi open, and condition (ii) makes sense.

It is well known that every nonnegative Radon measure which belongs to $H^{-1}(\Omega)$ belongs also to $\mathcal{M}_0(\Omega)$ (see [26], Section 4.7). For every quasi open set $A\subseteq\Omega$ we denote by μ_A the measure defined by

$$\mu_A(B)=\begin{cases}0, & \text{if } \operatorname{cap}(B\setminus A,\Omega)=0,\\ \infty, & \text{otherwise.}\end{cases} \tag{1.1}$$

Notice that μ_A belongs to $\mathcal{M}_0(\Omega)$. Indeed condition (i) is clearly satisfied and (ii) is trivial whenever $\mu_A(B)=+\infty$. Moreover, if $\mu_A(B)=0$, then $\operatorname{cap}(B\setminus A,\Omega)=0$, so that $B\cup A$ is a quasi open set containing B with $\mu_A(A\cup B)=\mu_A(B)$. This implies (ii).

In the sequel we shall use the following result.

Proposition 1.2. *For every measure $\mu\in\mathcal{M}_0(\Omega)$ there exist a nonnegative Borel measurable function h and a nonnegative measure $\nu\in H^{-1}(\Omega)$ such that $\mu(A)=(h\nu)(A)$ for every quasi open subset A of Ω.*

Proof. See [13], Theorem 2.2. □

Let $L\colon H^1_0(\Omega)\to H^{-1}(\Omega)$ be an elliptic operator of the form

$$Lu=-\sum_{i,j=1}^{n} D_i(a_{ij}D_ju)\,, \tag{1.2}$$

where (a_{ij}) is an $n\times n$ matrix of functions of $L^\infty(\Omega)$ satisfying, for a suitable constant $\alpha>0$, the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i\geq\alpha|\xi|^2 \tag{1.3}$$

for a.e. $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$.

Let $\mu\in\mathcal{M}_0(\Omega)$ and $f\in H^{-1}(\Omega)$. We shall consider the following *relaxed Dirichlet problem* (see [17] and [18]): find $u\in H^1_0(\Omega)\cap L^2_\mu(\Omega)$ such that

$$\langle Lu,v\rangle+\int_\Omega uv\,d\mu \;=\;\langle f,v\rangle\qquad\forall v\in H^1_0(\Omega)\cap L^2_\mu(\Omega)\,. \tag{1.4}$$

In [18] it was proved that there exists a unique solution of problem (1.4). For the sake of simplicity in the sequel we formally write problem (1.4) as

$$Lu+\mu u=f\quad\text{in }\Omega\,,\qquad u\in H^1_0(\Omega)\cap L^2_\mu(\Omega)\,,$$

although the equality can not be interpreted in the usual distributional sense, because, in general, $H^1_0(\Omega)\cap L^2_\mu(\Omega)$ does not contain $C^\infty_0(\Omega)$.

Remark 1.3. It is easy to see that if A is an open set and $\mu = \mu_A$, then $u \in H_0^1(\Omega)$ is the solution of problem (1.4) if and only if $u = 0$ q.e. in $\Omega \setminus A$ and u is the solution in A of the classical boundary value problem

$$u \in H_0^1(A)\,, \qquad Lu = f \quad \text{in } A\,.$$

For every $\mu \in \mathcal{M}_0(\Omega)$ we define w_μ to be the unique solution in the sense of (1.4) of the problem

$$Lw_\mu + \mu w_\mu \;=\; 1 \quad \text{in } \Omega\,, \qquad w_\mu \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,. \tag{1.5}$$

The functions w_μ are bounded in $L^\infty(\Omega)$ uniformly with respect to μ, $w_\mu \geq 0$ q.e. in Ω, and for every solution of problem (1.4), with $f \in L^\infty(\Omega)$, we have $|u| \leq \|f\|_\infty w_\mu$ q.e. in Ω (see [15], Section 3). In the sequel we shall use the following comparison principle.

Lemma 1.4. *If $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ and $\mu_1 \geq \mu_2$, then $w_{\mu_1} \leq w_{\mu_2}$ q.e. in Ω.*

Proof. The result follows from a more general comparison principle proved in [17], Theorem 2.10. □

By A_μ we shall denote the set $\{x \in \Omega : \; w_\mu(x) > 0\}$. Notice that A_μ is defined only up to a set of capacity zero, hence all the equalities or inclusions involving A_μ are intended up to sets of capacity zero. Since w_μ is quasi continuous, A_μ is quasi open.

Lemma 1.5. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let w_μ be the solution of problem (1.5). Then $\mu(B) = +\infty$ for every Borel subset B of Ω with $\mathrm{cap}(B \setminus A_\mu, \Omega) > 0$.*

Proof. See [15], Lemma 3.2. □

Remark 1.6. It is easy to see that, if μ is a Radon measure of $\mathcal{M}_0(\Omega)$, then $A_\mu = \Omega$. If $\mu = \mu_A$ and A is open, then $A_\mu = A$ by Remark 1.3 and by the strong maximum principle.

Definition 1.7. Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. We say that $\{\mu_h\}$ *γ^L-converges to* μ (in Ω) if the sequence $\{w_{\mu_h}\}$ converges to w_μ weakly in $H_0^1(\Omega)$.

Remark 1.8. This convergence of measures is equivalent to that one given in [15] (Theorem 4.3) and is the natural extension of the notion of γ^L-convergence introduced in [18], when L is the Laplace operator, and in [11] when L is symmetric. More precisely, the sequence of measures $\{\mu_h\}$ γ^L-converges to the measure μ in the sense of Definition 1.7 if and only if, for every $f \in H^{-1}(\Omega)$, the sequence $\{u_h\}$ of the solutions of the relaxed Dirichlet problems

$$Lu_h + \mu_h u_h = f \quad \text{in } \Omega\,, \qquad u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$$

converges weakly in $H_0^1(\Omega)$ to the solution u of the relaxed Dirichlet problem

$$Lu + \mu u = f \quad \text{in } \Omega\,, \qquad u \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega)\,.$$

Then main properties of γ^L-convergence are stated in the following propositions.

Proposition 1.9. *(Compactness). Every sequence of measures of $\mathcal{M}_0(\Omega)$ contains a γ^L-convergent subsequence.*

Proof. See [15], Theorem 4.5. □

Proposition 1.10. *(Density). Let us fix a nonnegative Radon measure λ. Then for every $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence $\{E_h\}$ of compact subsets of Ω such that $\{\mu_{\Omega\setminus E_h}\}$ γ^L-converges to μ, and $\lambda(E_h) = 0$ for every $h \in \mathbf{N}$.*

Proof. Since by Proposition 3.7 of [15] every measure of $\mathcal{M}_0(\Omega)$ can be approximated in γ^L-convergence by a sequence of Radon measures of $\mathcal{M}_0(\Omega)$, here it is not restrictive to suppose that μ is a Radon measure. When L is symmetric, an explicit approximation for every Radon measure is given in [2], Theorem 2.2, by means of a sequence $\mu_{\Omega\setminus E_h}$, where E_h is the union of closed balls with centers on a periodic lattice Z_h. For the extension of this result to the nonsymmetric case see [16]. It is easy to see that the construction of [2], Definition 2.1, can be carried over, with minor changes, taking E_h as the disjoint union of $(n-1)$-dimensional balls with centers near the lattice points of Z_h and lying on hyperplanes parallel to a fixed hyperplane. Since the measure λ is σ–finite it is possible to choose these hyperplanes with measure zero for λ. □

Finally, let us consider a real valued functional J defined on the class of all open subsets of Ω. With every open subset A of Ω we can associate the measure μ_A. Thus the functional J can be considered as a functional defined on the subclass $\{\mu_A :\ A \text{ open},\ A \subseteq \Omega\}$ of $\mathcal{M}_0(\Omega)$.

Definition 1.11. We shall call *relaxation* of J in $\mathcal{M}_0(\Omega)$ with respect to the γ^L-convergence, and we shall denote it by $\overline{J}$, the greatest γ^L-lower semicontinuous functional defined on $\mathcal{M}_0(\Omega)$ such that $\overline{J}(\mu_A) \leq J(A)$ for every open set $A \subseteq \Omega$.

Remark 1.12. One can check that

$$\overline{J}(\mu) = \inf\bigl\{\liminf_{h\to\infty} J(A_h) \,:\, A_h \text{ open, } \{\mu_{A_h}\}\ \gamma^L\text{-converging to } \mu\bigr\}\,, \tag{1.6}$$

for every $\mu \in \mathcal{M}_0(\Omega)$. The previous formula characterizes the relaxation $\overline{J}$ as the unique functional which satisfies the following properties for every $\mu \in \mathcal{M}_0(\Omega)$:

(i) for every sequence $\{A_h\}$ of open sets with $\{\mu_{A_h}\}$ γ^L-converging to μ in $\mathcal{M}_0(\Omega)$

$$\overline{J}(\mu) \leq \liminf_{h\to\infty} J(A_h)\,;$$

(ii) there exists a sequence $\{A_h\}$ of open sets such that $\{\mu_{A_h}\}$ γ^L-converges to μ in $\mathcal{M}_0(\Omega)$ and

$$\overline{J}(\mu) \geq \limsup_{h\to\infty} J(A_h)\,.$$

The relaxation $\overline{J}$ describes the behaviour of the minimizing sequences of J. More precisely $\overline{J}$ is γ^L-lower semicontinuous and so, by the direct method of calculus of variations, $\overline{J}$ has a minimum point on the γ^L-compact set $\mathcal{M}_0(\Omega)$. Moreover

$$\min_{\mathcal{M}_0(\Omega)} \overline{J}(\mu) = \inf_{\substack{\text{A open}\\ A\subseteq\Omega}} J(A)\,,$$

every cluster point of a minimizing sequence for J is a minimum point for $\overline{J}$ in $\mathcal{M}_0(\Omega)$, and every minimum point for $\overline{J}$ in $\mathcal{M}_0(\Omega)$ is the limit of a minimizing sequence for J (in the last statements we identify every open set $A \subseteq \Omega$ with the corresponding measure μ_A).

For a more general treatment of this subject see, e.g., [6] and [14].

2. Lower Semicontinuity

In this section we study the lower semicontinuity, with respect to the γ^L-convergence, of some functionals defined on $\mathcal{M}_0(\Omega)$. More precisely, fixed a bounded measure λ in

$\mathcal{M}_0(\Omega)$, a function g in $L^1_\lambda(\Omega)$, and a closed (possibly degenerating to a point) subinterval $T=[m,M]$ of $[0,\lambda(\Omega)]$, we consider the functional

$$G_T(\mu)=\inf\left\{\int_B g\,d\lambda\ :B\in\mathcal{B}(\Omega),\,A_\mu\subseteq B,\,\lambda(B)\in T\right\},\tag{2.1}$$

where A_μ is the quasi open set introduced in Section 1. We shall always use the convention $\inf\emptyset=+\infty$.

Theorem 2.1. *The functional G_T defined in (2.1) is lower semicontinuous in $\mathcal{M}_0(\Omega)$ with respect to the γ^L-convergence.*

Remark 2.2. Let us fix a constant c such that $0\le c\le\lambda(\Omega)$. If the set T takes the form $T=\{c\}$, $T=[0,c]$, or $T=[c,\lambda(\Omega)]$, then the functional G_T becomes respectively

$$G_c(\mu)=\inf\left\{\int_B g\,d\lambda\ :\ B\in\mathcal{B}(\Omega)\,,\ A_\mu\subseteq B\,,\ \lambda(B)\,=\,c\right\},$$
$$G_{[0,c]}(\mu)=\inf\left\{\int_B g\,d\lambda\ :\ B\in\mathcal{B}(\Omega)\,,\ A_\mu\subseteq B\,,\ \lambda(B)\le c\right\},$$
$$G_{[c,\lambda(\Omega)]}(\mu)=\inf\left\{\int_B g\,d\lambda\ :\ B\in\mathcal{B}(\Omega)\,,\ A_\mu\subseteq B\,,\ \lambda(B)\ge c\right\}.$$

Notice that, in general, G_c, $G_{[0,c]}$, and $G_{[c,\lambda(\Omega)]}$ are different as it can be easily seen by choosing $g\equiv 1$ in each functional. Indeed in this case we have

$$G_c(\mu)=\begin{cases} c, & \text{if } \lambda(A_\mu)\le c,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$G_{[0,c]}(\mu)=\begin{cases} \lambda(A_\mu), & \text{if } \lambda(A_\mu)\le c,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$G_{[c,\lambda(\Omega)]}(\mu)=\sup\{c,\lambda(A_\mu)\}.$$

Moreover, for every $g\in L^1_\lambda(\Omega)$ the functional $G_{[0,c]}$ can be rewritten as

$$G_{[0,c]}(\mu)=\int_{A_\mu} g^+\,d\lambda-\sup\int_B g^-\,d\lambda\,,$$

where g^+ and g^- are the positive and the negative part of g respectively, and the supremum is taken over all the Borel subset B of Ω such that $A_\mu \subseteq B$, and $\lambda(B) = c$. When $c = \lambda(\Omega)$, it reduces to the functional

$$G_{[0,\lambda(\Omega)]}(\mu) = \int_{A_\mu} g^+ \, d\lambda - \int_\Omega g^- \, d\lambda = \int_{A_\mu} g \, d\lambda - \int_{\Omega \setminus A_\mu} g^- \, d\lambda \,,$$

which then turns out to be γ^L-lower semicontinuous on $\mathcal{M}_0(\Omega)$.

In order to prove Theorem 2.1 we need a result of measure theory.

Lemma 2.3. *Let λ be a nonnegative, nonatomic bounded Borel measure on Ω, and let $g \in L^1_\lambda(\Omega)$. Fixed a closed subinterval $T = [m, M]$ of $[0, \lambda(\Omega)]$, let us consider the functional*

$$\mathcal{G}(A) = \inf \left\{ \int_B g \, d\lambda \ : B \in \mathcal{B}(\Omega), \, A \subseteq B, \, \lambda(B) \in T \right\} \tag{2.2}$$

defined on the class of all Borel subsets A of Ω. Then, for every A_1, A_2 such that $\lambda(A_1) \le M$ and $\lambda(A_2) \le M$, we have

$$\mathcal{G}(A_1) \le \mathcal{G}(A_2) + 2\omega(\lambda(A_1 \setminus A_2)) \,, \tag{2.3}$$

where $\omega(\delta) = \sup \left\{ \int_B |g| \, d\lambda \ : \lambda(B) \le \delta \right\}$, and hence $\omega(\delta) \to 0$ as $\delta \to 0$.

Proof. Let B_2 be any Borel set such that $A_2 \subseteq B_2$ and $\lambda(B_2) \in T$. Since λ is nonatomic and $\lambda(B_2 \setminus A_1) = \lambda(B_2) - \lambda(A_1) + \lambda(A_1 \setminus B_2)$, if $\lambda(B_2) - \lambda(A_1) \ge 0$ one can find a Borel set $E \subseteq B_2 \setminus A_1$ such that $\lambda(E) = \lambda(A_1 \setminus B_2)$. Thus, setting $F = B_2 \setminus (A_1 \cup E) = (B_2 \setminus A_1) \setminus E$ and $B_1 = A_1 \cup F = (A_1 \cup B_2) \setminus E$, we have $A_1 \subseteq B_1$, and

$$\lambda(B_1) = \lambda(A_1) + \lambda(F) = \lambda(A_1) + \lambda(B_2 \setminus A_1) - \lambda(A_1 \setminus B_2) = \lambda(B_2) \,,$$

and hence $\lambda(B_1)$ belongs to T. Moreover, as $A_1 \cup B_2 = B_1 \cup E$ and $B_1 \cap E = \emptyset$, we have

$$\int_{B_1} g \, d\lambda = \int_{B_2} g \, d\lambda + \int_{A_1 \setminus B_2} g \, d\lambda - \int_E g \, d\lambda \,.$$

Since $\lambda(A_1 \setminus B_2) = \lambda(E) \le \lambda(A_1 \setminus A_2)$ we get

$$\mathcal{G}(A_1) \le \int_{B_1} g \, d\lambda \le \int_{B_2} g \, d\lambda + 2\omega(\lambda(A_1 \setminus A_2)) \,.$$

If $\lambda(B_2)-\lambda(A_1)\le 0$, then $\lambda(B_2)\le\lambda(A_1)\le M$, so that $\lambda(A_1)\in T$ and

$$\mathcal{G}(A_1)\ \le\ \int_{A_1} g\,d\lambda\,.$$

Since in this case $\lambda(B_2\backslash A_1)\le\lambda(A_1\backslash B_2)$, we have

$$\mathcal{G}(A_1)\le\int_{B_2} g\,d\lambda+\int_{A_1\backslash B_2} g\,d\lambda-\int_{B_2\backslash A_1} g\,d\lambda\ \le\ \int_{B_2} g\,d\lambda+2\omega(\lambda(A_1\backslash A_2))\,.$$

Therefore (2.3) follows by taking the infimum over all admissible B_2. The fact that $\omega(\delta)\to 0$ as $\delta\to 0$ is a consequence of the absolute continuity of the integral. □

We consider now a first lower semicontinuity result for functionals defined on $\mathcal{M}_0(\Omega)$.

Lemma 2.4. *Let $\lambda\in\mathcal{M}_0(\Omega)$ and let $\{\mu_h\}$ be a sequence in $\mathcal{M}_0(\Omega)$ γ^L-converging to $\mu\in\mathcal{M}_0(\Omega)$. Then*

$$\lambda(A_\mu)\ \le\ \liminf_{h\to\infty}\lambda(A_{\mu_h})\,.$$

If, in addition, $\lambda(A_\mu)<+\infty$, then $\lim\limits_{h\to\infty}\lambda(A_\mu\backslash A_{\mu_h})=0$.

Proof. By Proposition 1.2 there exist a nonnegative measure $\nu\in H^{-1}(\Omega)$ and a nonnegative Borel function h such that $\lambda=h\nu$ on the class of all quasi open subsets of Ω. In particular $\lambda(A_\mu)=\int_{A_\mu} h\,d\nu$ for every $\mu\in\mathcal{M}_0(\Omega)$. Let $f:\Omega\times\mathbf{R}\to[0,+\infty]$ be the Borel function defined by

$$f(x,s)=\begin{cases} h(x)\,, & \text{if } s>0,\\ 0\,, & \text{if } s\ \le\ 0.\end{cases}$$

Then $f(x,\cdot)$ is lower semicontinuous, and by definition of A_μ we have

$$\lambda(A_\mu)=\int_\Omega f(x,w_\mu)\,d\nu\qquad\forall\mu\in\mathcal{M}_0(\Omega)\,,$$

where w_μ is the solution of problem (1.5). Let now $\{\mu_h\}$ be a sequence in $\mathcal{M}_0(\Omega)$ which γ^L-converges to a measure $\mu\in\mathcal{M}_0(\Omega)$, i.e., the sequence $\{w_{\mu_h}\}$ converges to w_μ weakly in $H^1_0(\Omega)$ (see Definition 1.7). Since $\nu\in H^{-1}(\Omega)$, $\{w_{\mu_h}\}$ converges to w_μ in the strong topology of $L^1_\nu(\Omega)$. Thus, possibly passing to a subsequence, $\{w_{\mu_h}\}$ converge to w_μ ν-a.e. in Ω. Therefore, by Fatou's lemma and the lower semicontinuity of $f(x,\cdot)$, we obtain

$$\lambda(A_\mu)=\int_\Omega f(x,w_\mu)\,d\nu\le\liminf_{h\to\infty}\int_\Omega f(x,w_{\mu_h})\,d\nu=\liminf_{h\to\infty}\lambda(A_{\mu_h})\,,$$

and the proof of the first statement is complete. Let us suppose now that $\lambda(A_\mu) < +\infty$. As we have shown before, the γ^L-convergence of $\{\mu_h\}$ to μ implies the strong convergence in $L^1_\nu(\Omega)$ of $\{w_{\mu_h}\}$ to w_μ. Hence, using Fatou's lemma again, we get

$$\limsup_{h\to\infty} \lambda(A_\mu \setminus A_{\mu_h}) = \limsup_{h\to\infty} [\lambda(A_\mu) - \lambda(A_{\mu_h} \cap A_\mu)] =$$

$$= \int_{A_\mu} f(x, w_\mu)\, d\nu - \liminf_{h\to\infty} \int_{A_\mu} f(x, w_{\mu_h})\, d\nu \leq 0\,,$$

which concludes the proof. □

Remark 2.5. If the measure λ does not belong to $\mathcal{M}_0(\Omega)$, the conclusion of Lemma 2.4 may be false. In fact, take $n \geq 2$, $x_0 \in \Omega$ and let λ be the Dirac measure δ_{x_0}. It is easy to see that the measures $\mu_h = \mu_{\Omega\setminus B(x_0,1/h)}$ γ^L-converge to the measure μ which is identically zero on Ω. On the other hand we have $A_{\mu_h} = \Omega \setminus B(x_0, 1/h)$, $A_\mu = \Omega$, so that $\lambda(A_\mu) = 1 > 0 = \liminf_{h\to\infty} \lambda(A_{\mu_h})$.

Proof of Theorem 2.1. Let μ be a fixed measure in $\mathcal{M}_0(\Omega)$, and let $\{\mu_h\}$ be a sequence in $\mathcal{M}_0(\Omega)$ which γ^L-converges to μ. It is not restrictive to suppose that $G_T(\mu_h) < +\infty$ for every $h \in \mathbf{N}$. This implies that $\lambda(A_{\mu_h}) \leq M$ for every $h \in \mathbf{N}$, and then, by Lemma 2.4, we have also $\lambda(A_\mu) \leq M$. Since for every quasi open set A there exists a Borel set B containing A, with $\mathrm{cap}(B \setminus A, \Omega) = 0$, we can apply Lemma 2.3 with $A_1 = A_\mu$, and $A_2 = A_{\mu_h}$, and we get

$$G_T(\mu) \leq G_T(\mu_h) + 2\omega(\lambda(A_\mu \setminus A_{\mu_h}))\,.$$

The γ^L-lower semicontinuity of G_T follows now from the second part of Lemma 2.4. □

3. Relaxation

In this section we apply the previous lower semicontinuity results in order to obtain an explicit representation of the relaxation of some cost functionals in optimal shape design.

Let us fix a functional $f \in H^{-1}(\Omega)$, a bounded measure λ in $\mathcal{M}_0(\Omega)$, a closed interval $T = [m, M]$ of $[0, \lambda(\Omega)]$ with $M > 0$, and a Borel function $j\colon \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying the following conditions:

(i) for every $u \in H^1_0(\Omega)$ the function $j(x,u(x))$ belongs to $L^1_\lambda(\Omega)$;

(ii) the map $u \mapsto \int_\Omega j(x,u)\,d\lambda$ is sequentially continuous in the weak topology of $H^1_0(\Omega)$.

We shall consider the functional J defined on the class of all open subsets of Ω as

$$J(A) = \begin{cases} \int_A j(x,u_A)\,d\lambda, & \text{if } \lambda(A) \in T, \\ \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.1}$$

where u_A is the unique solution of the Dirichlet problem

$$\begin{cases} Lu_A = f & \text{on } A, \\ u_A = 0 & \text{on } \partial A. \end{cases} \tag{3.2}$$

Different hypotheses on λ and j can be made in order to fulfill (i) and (ii). For instance, if we assume that $j(x,s)$ is measurable in x and continuous in s, we can require one of the following properties:

(1) λ is the Lebesgue measure and $|j(x,s)| \leq c(1+|s|^p)$ for λ-a.e. x in Ω and for every $p < 2n/(n-2)$. Namely, in this case, by Sobolev imbedding (i) and (ii) are obviously fulfilled.

(2) λ is the $(n-1)$-dimensional Hausdorff measure restricted on a smooth $(n-1)$-dimensional hypersurface $S \subseteq \Omega$ and $|j(x,s)| \leq c(1+|s|^2)$ for λ-a.e. x in S. In this case (i) and (ii) follow from the compactness of the trace operator between $H^1_0(\Omega)$ and $L^2(S)$.

(3) λ belongs to $H^{-1}(\Omega)$ and j has linear growth in s. In this case, (i) and (ii) follow from the compactness of the injection of $H^1_0(\Omega)$ into $L^1_\lambda(\Omega)$.

It is well known that, under these very weak assumptions, the optimal design problem

$$\min_{\lambda(A)\in T} J(A)$$

in general has no solution (see Examples 3.11 and 3.12). Thus, in order to investigate the asymptotic behaviour of the minimizing sequences of J, we are interested in its relaxation. To this aim, for every $\mu \in \mathcal{M}_0(\Omega)$ we denote by u_μ the unique solution in the sense of (1.4) of the problem

$$Lu_\mu + \mu u_\mu = f \quad \text{in } \Omega, \qquad u_\mu \in H^1_0(\Omega) \cap L^2_\mu(\Omega). \tag{3.3}$$

The main result of this section is the following.

Theorem 3.1. *Let $f \in H^{-1}(\Omega)$, let λ be a bounded measure in $\mathcal{M}_0(\Omega)$, and let $j:\Omega \times \mathbf{R} \to \mathbf{R}$ be a Borel function satisfying (i) and (ii). Fixed a closed subinterval $T = [m, M]$ of $[0, \lambda(\Omega)]$ with $M > 0$, we consider the functional J defined by (3.1) for every open set $A \subseteq \Omega$, where u_A is the solution of problem (3.2). Then its relaxation $\overline{J}$ in $\mathcal{M}_0(\Omega)$ is given by*

$$\overline{J}(\mu) = \int_{A_\mu} j(x, u_\mu)\, d\lambda + \inf\left\{ \int_{B \setminus A_\mu} j(x, 0)\, d\lambda : B \in \mathcal{B}(\Omega),\ A_\mu \subseteq B,\ \lambda(B) \in T \right\}, \tag{3.4}$$

with the convention $\inf \emptyset = +\infty$.

Example 3.2. If there exists a constant k such that $j(x, 0) = k$ for every λ-a.e. $x \in \Omega$, then (3.4) can be simplified. Namely, if k is positive, then (3.4) becomes

$$\overline{J}(\mu) = \begin{cases} \int_{A_\mu} j(x, u_\mu)\, d\lambda \,+\, k(m - \lambda(A_\mu))^+, & \text{if } \lambda(A_\mu) \le M, \\ +\infty, & \text{otherwise,} \end{cases}$$

while, if k is negative, we get

$$\overline{J}(\mu) = \begin{cases} \int_{A_\mu} j(x, u_\mu)\, d\lambda \,+\, k(M - \lambda(A_\mu)), & \text{if } \lambda(A_\mu) \le M, \\ +\infty, & \text{otherwise.} \end{cases}$$

In particular, if $T = \{c\}$, then (3.4) takes the form

$$\overline{J}(\mu) = \begin{cases} \int_{\Omega} j(x, u_\mu)\, d\lambda \,+\, k(c - \lambda(\Omega)), & \text{if } \lambda(A_\mu) \le c, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 3.3. The functional J can be written as

$$J(A) = \int_\Omega j(x, u_A)\, d\lambda - \int_\Omega j(x, 0)\, d\lambda + \int_A j(x, 0)\, d\lambda\,,$$

where u_A is extended to 0 on $\Omega \setminus A$. For every fixed $\mu \in \mathcal{M}_0(\Omega)$ and for every sequence $\{A_h\}$ of open subsets of Ω such that $\{\mu_{A_h}\}$ γ^L-converges to μ, we have that the sequence $\{u_{A_h}\}$ of the solutions of the Dirichlet problems on A_h converges to the solution u_μ of the relaxed Dirichlet problem (3.3) in the weak topology of $H_0^1(\Omega)$ (see Remarks 1.3 and 1.8), so that, by (ii),

$$\lim_{h \to \infty} \int_\Omega j(x, u_{A_h})\, d\lambda - \int_\Omega j(x, 0)\, d\lambda \;=\; \int_\Omega j(x, u_\mu)\, d\lambda - \int_\Omega j(x, 0)\, d\lambda\,.$$

Hence, by (1.6), in order to exhibit the relaxation $\overline{J}$ of J, it is enough to relax the functional

$$J_0(A) = \begin{cases} \int_A j(x,0)\,d\lambda, & \text{if } \lambda(A) \in T, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.5}$$

Since the solution u_μ of (3.3) is zero q.e. in $\Omega \setminus A_\mu$ (Lemma 1.5), to conclude the proof it is enough to show that the relaxation $\overline{J}_0$ of J_0 coincides with the functional

$$\tilde{J}_0(\mu) = \inf\left\{ \int_B j(x,0)\,d\lambda : B \in \mathcal{B}(\Omega),\ A_\mu \subseteq B\,,\ \lambda(B) \in T \right\}. \tag{3.6}$$

By Theorem 2.1, $\tilde{J}_0$ is γ^L-lower semicontinuous. Since $\tilde{J}_0(\mu_A) \le J_0(A)$ for every open set $A \subseteq \Omega$, by definition of $\overline{J}_0$, we have $\tilde{J}_0 \le \overline{J}_0$.

Suppose that, for every $\mu \in \mathcal{M}_0(\Omega)$ with $\tilde{J}_0(\mu) < +\infty$ and for every Borel set B containing A_μ with $\lambda(B) \in T$, we are able to find a sequence $\{A_h\}$ of open subsets of Ω such that $\lambda(A_h) \in T$, $\{\mu_{A_h}\}$ γ^L-converges to μ, and

$$\lim_{h\to\infty} \int_{A_h} j(x,0)\,d\lambda = \int_B j(x,0)\,d\lambda\,.$$

Then by (1.6) we get

$$\overline{J}_0(\mu) \le \lim_{h\to\infty} J_0(A_h) = \int_B j(x,0)\,d\lambda\,.$$

Taking the infimum over all admissible B we obtain $\overline{J}_0(\mu) \le \tilde{J}_0(\mu)$, and hence $\overline{J}_0(\mu) = \tilde{J}_0(\mu)$.

In order to construct the sequence $\{A_h\}$ we shall require that the set B is open. As shown in the following lemma, this condition is not restrictive for a large class of measures in $\mathcal{M}_0(\Omega)$.

Lemma 3.4. *Let g, λ, and G be as in Theorem 2.1 and let $T = [m, M] \subseteq [0, \lambda(\Omega)]$. Then*

$$G(\mu) = \inf\left\{ \int_B g\,d\lambda\ : B \text{ open}\ ,\ A_\mu \subseteq B,\ \lambda(B) \in T \right\} \tag{3.7}$$

for every measure $\mu \in \mathcal{M}_0(\Omega)$ such that $\lambda(A_\mu) < M$.

Proof. Let $\mu \in \mathcal{M}_0(\Omega)$ with $\lambda(A_\mu) < M$. Let us denote by $H(\mu)$ the right–hand side of (3.7). It is enough to prove that $G(\mu) \ge H(\mu)$, since the opposite inequality is trivial. Since $\lambda(A_\mu) < M$, λ is nonatomic, and T is an interval, $G(\mu)$ coincides with the infimum taken over all Borel sets B such that $\lambda(B) \in T$, $A_\mu \subseteq B$, and $\lambda(B) > \lambda(A_\mu)$.

Given one of these sets B, we shall exhibit a sequence $\{A_h\}$ of open subsets of Ω, with $\lambda(A_h) = \lambda(B)$ and $A_\mu \subseteq A_h$, such that

$$\lim_{h\to\infty} \int_{A_h} g\, d\lambda \;=\; \int_B g\, d\lambda\,. \tag{3.8}$$

Since $H(\mu) \le \int_{A_h} g\, d\lambda$, taking the limit as $h \to \infty$ we obtain

$$H(\mu) \;\le\; \int_B g\, d\lambda\,, \tag{3.9}$$

and taking the infimum with respect to B we get $H(\mu) \le G(\mu)$. It remains to construct the sequence A_h. Since λ is bounded, we can find a sequence $\{U_h\}$ of open subsets of Ω such that $A_\mu \subseteq U_h$ for every $h \in \mathbf{N}$, and $\lambda(U_h \backslash A_\mu) \le 1/h$. Moreover for every $h \in \mathbf{N}$ there exists an open set $B_h \supseteq B$ such that $\lambda(B_h \backslash B) \le 1/h$. It is not restrictive to suppose $U_h \subseteq B_h$, since one can always replace B_h with $B_h \cup U_h$, and

$$\lambda((B_h \cup U_h)\backslash B) \;\le\; \lambda(B_h\backslash B) + \lambda(U_h\backslash B) \;\le\; \lambda(B_h\backslash B) + \lambda(U_h\backslash A_\mu) \;\le\; 2/h\,.$$

Thus, for h large enough, $\lambda(U_h) \;\le\; \lambda(A_\mu) + 1/h < \lambda(B) \;\le\; \lambda(B_h)$, so that, by Lemma 1.1 we can find an open set A_h such that $U_h \subseteq A_h \subseteq B_h$ and $\lambda(A_h) = \lambda(B)$. Moreover we have that $A_\mu \subseteq U_h \subseteq A_h$ and

$$\int_{A_h} g\, d\lambda \;=\; \int_B g\, d\lambda - \int_{B\backslash A_h} g\, d\lambda + \int_{A_h\backslash B} g\, d\lambda\,.$$

Since $\lambda(A_h) = \lambda(B)$, we have $\lambda(B\backslash A_h) = \lambda(A_h\backslash B) \le \lambda(B_h\backslash B) \le 1/h$. Then, as $g \in L^1_\lambda(\Omega)$, (3.8) follows from the absolute continuity of the integral. □

Lemma 3.5. *Let λ and T be as in Theorem 3.1, and let $\mu \in \mathcal{M}_0(\Omega)$. Then for every open set B containing A_μ, with $\lambda(B) \in T$, there exists a sequence $\{A_h\}$ of open subsets of B, with $\lambda(B\backslash A_h) = 0$, such that the sequence $\{\mu_{A_h}\}$ γ^L-converges to μ.*

Proof. Applying Proposition 1.10 with Ω replaced by B, it is possible to find a sequence $\{A_h\}$ of open subsets of B with $\lambda(B\backslash A_h) = 0$, such that the sequence $\{\mu_{A_h}\}$ γ^L-converges to μ in B, i.e., replacing Ω by B in Definition 1.7, the sequence $\{w_h\}$ of the solutions to the problems

$$Lw_h = 1 \quad \text{in } A_h\,, \qquad w_h = 0 \quad \text{on } \partial A_h\,, \tag{3.10}$$

extended to zero outside A_h weakly converges in $H_0^1(B)$ to the solution w in the sense of (1.4) of the problem

$$Lw + \mu w = 1 \quad \text{in } B\,, \qquad w \in H_0^1(B) \cap L^2_\mu(B) \tag{3.11}$$

extended to zero outside B. Since $A_\mu \subseteq B$, the solution w_μ of (1.5) in Ω equals zero q.e. on $\Omega \backslash B$, so that $w_\mu \in H_0^1(B) \cap L^2_\mu(B)$ and satisfies (3.11). As $A_h \subseteq B$, this shows that $\{\mu_{A_h}\}$ γ^L-converges to μ in Ω. □

We are now in a position to prove (3.4) for every $\mu \in \mathcal{M}_0(\Omega)$ such that $\lambda(A_\mu) < M$.

Proposition 3.6. *Let j, λ, and T be as in Theorem 3.1 and let $\tilde{J}_0$ be the functional defined in (3.6). Then $\overline{J}_0(\mu) = \tilde{J}_0(\mu)$ for every $\mu \in \mathcal{M}_0(\Omega)$ with $\lambda(A_\mu) < M$.*

Proof. Let us fix $\mu \in \mathcal{M}_0(\Omega)$ such that $\lambda(A_\mu) < M$. As pointed out in Remark 3.3, it is enough to show that $\overline{J}_0(\mu) \le \tilde{J}_0(\mu)$. Let us consider an open set B, containing A_μ, with $\lambda(B) \in T$. By Lemma 3.5 there exists a sequence $\{A_h\}$ of open subsets of B, with $\lambda(A_h) = \lambda(B)$, such that $\{\mu_{A_h}\}$ γ^L-converges to μ. Thus, as in Remark 3.3, we obtain $\overline{J}_0(\mu) \le \int_B j(x,0)\,d\lambda$. Taking the infimum over all admissible open sets B, by Lemma 3.4 applied to $g(x) = j(x,0)$, we obtain $\overline{J}_0(\mu) \le \tilde{J}_0(\mu)$. □

In order to extend the equality $\overline{J}_0(\mu) = \tilde{J}_0(\mu)$ to every $\mu \in \mathcal{M}_0(\Omega)$, we need the following lemma.

Lemma 3.7. *Let $\{A_h\}$ and $\{\tilde{A}_h\}$ be two sequences of quasi open subsets of Ω such that $A_h \subseteq \tilde{A}_h$ for every h and $\operatorname{cap}(\tilde{A}_h \backslash A_h, \Omega) \to 0$. Let w_h and $\tilde{w}_h$ be the solutions of problem (1.5) with $\mu = \mu_{A_h}$ and $\mu = \mu_{\tilde{A}_h}$. If $\{w_h\}$ and $\{\tilde{w}_h\}$ converge weakly to w and $\tilde{w}$, then $w = \tilde{w}$ q.e. in Ω.*

Proof. Since $A_h \subseteq \tilde{A}_h$, by Lemma 1.4 we have $w_h \le \tilde{w}_h$ q.e. in Ω for every $h \in \mathbf{N}$, and then $w \le \tilde{w}$ q.e. in Ω. Let us prove the opposite inequality. To this aim, following the ideas of Stampacchia (see [24]), for every subset E of Ω we denote with K_E the set of all functions $v \in H_0^1(\Omega)$ such that $v \ge 1$ q.e. in E and, if K_E is nonempty, we consider the unique solution z of the variational inequality

$$\begin{cases} z \in K_E\,, \\ \langle Lz, v - z\rangle \ge 0 \qquad \forall v \in K_E. \end{cases}$$

The function z is called the L–capacitary potential of E in Ω and the L–capacity of E in Ω is defined by $\mathrm{cap}^L(E,\Omega) = \langle Lz, z\rangle$. We set $\mathrm{cap}^L(E,\Omega) = +\infty$ if $K_E = \emptyset$. It is easy to see that

$$\langle Lz, \psi\rangle = 0 \qquad \forall\psi \in H^1_0(\Omega)\,, \text{ with } \psi = 0 \text{ q.e. in } E\,.$$

Moreover, by the maximum principle, one can check that $z = 1$ q.e. in E. Finally $\mathrm{cap}^L(E,\Omega) \le c\,\mathrm{cap}(E,\Omega)$ so that, by hypothesis, $\mathrm{cap}^L(\tilde A_h \setminus A_h, \Omega) \to 0$. Then by the ellipticity assumption on L the sequence $\{z_h\}$ of the L-capacitary potentials of the sets $\tilde A_h \setminus A_h$ in Ω converges to zero strongly in $H^1_0(\Omega)$. Let c be a positive constant such that $\tilde w_h \le c$ q.e. in Ω for every $h \in \mathbf{N}$. We claim that for every $h \in \mathbf{N}$

$$\tilde w_h \le w_h + c z_h \qquad \text{q.e. in } \Omega\,. \tag{3.12}$$

As $z_h = 1$ q.e. in $\tilde A_h \setminus A_h$, $w_h \ge 0$, and $\tilde w_h = 0$ q.e. in $\Omega \setminus \tilde A_h$, (3.12) is trivially satisfied in $\Omega \setminus A_h$. Since $\langle L z_h, \psi\rangle = 0$ for every $\psi \in H^1_0(\Omega)$ with $\psi = 0$ q.e. in $\tilde A_h \setminus A_h$, in particular, we have

$$\langle L(\tilde w_h - w_h - c z_h), \psi\rangle = 0 \tag{3.13}$$

for every $\psi \in H^1_0(\Omega)$ with $\psi = 0$ q.e. in $\Omega \setminus A_h$. Taking in (3.13) $\psi = (\tilde w_h - w_h - c\,z_h)^+$, by the ellipticity assumption on L, we obtain that $(\tilde w_h - w_h - c\,z_h)^+ = 0$ q.e. in Ω, which proves (3.12). □

Proof of Theorem 3.1. By Remark 3.3 it is enough to characterize the relaxation of the functional J_0 defined by (3.5). By Proposition 3.6 $\tilde J_0(\mu) = \overline{J}_0(\mu)$ for every $\mu \in \mathcal{M}_0(\Omega)$ with $\lambda(A_\mu) < M$. Let us consider now a measure $\mu \in \mathcal{M}_0(\Omega)$ with $\lambda(A_\mu) = M$. Let $\lambda_{|A_\mu}$ be the measure on Ω defined by $(\lambda_{|A_\mu})(B) = \lambda(A_\mu \cap B)$. Since $M > 0$, there exists a point $x \in \mathrm{supp}\,\lambda_{|A_\mu}$, that is $\lambda(B(x,r) \cap A_\mu) > 0$ for every $r > 0$. Setting $B_h = B(x, 1/h)$, $A_h = A_\mu \setminus B_h$, and $\mu_h = \mu + \mu_{\Omega\setminus B_h}$ we have that $A_{\mu_h} \subseteq A_h$ and $\lambda(A_{\mu_h}) < M$. By Lemma 3.7 applied to $\tilde A_h = A_\mu$, the sequence $\{\mu_h\}$ γ^L-converges to μ. Thus by the γ^L-lower semicontinuity of $\overline{J}_0$ we have

$$\overline{J}_0(\mu) \le \liminf_{h\to\infty} \overline{J}_0(\mu_h)\,.$$

As $\lambda(A_{\mu_h}) < M$, by Proposition 3.6 $\overline{J}_0(\mu_h) = \tilde J_0(\mu_h)$ for every $h \in \mathbf{N}$. Moreover, since $A_{\mu_h} \subseteq A_\mu$ and $\lambda(A_\mu) = M$, we have $\tilde J_0(\mu_h) \le \int_{A_\mu} j(x,0)\,d\lambda = \tilde J_0(\mu)$ for every h. Therefore

$$\overline{J}_0(\mu) \le \liminf_{h\to\infty} \tilde J_0(\mu_h) \le \tilde J_0(\mu)\,,$$

and hence $\overline{J}_0(\mu) = \tilde J_0(\mu)$ for every $\mu \in \mathcal{M}_0(\Omega)$. □

Remark 3.8. If $T=\{0\}$ and λ is the Lebesgue measure, then the functional J_0 defined in (3.5) takes the form

$$J_0(A) = \begin{cases} 0, & \text{if } A=\emptyset, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.14}$$

Thus its relaxation $\overline{J}_0$ is the functional defined by

$$\overline{J}_0(\mu) = \begin{cases} 0, & \text{if } \mu=\mu_\emptyset, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.15}$$

It turns out that $\overline{J}_0$ coincides with the functional $\tilde{J}_0$ defined in (3.6). Indeed $\tilde{J}_0$ is finite and, more precisely, takes the value zero, only for $\mu\in\mathcal{M}_0(\Omega)$ with $\lambda(A_\mu)=0$. It is well known that every quasi open set with Lebesgue measure zero has capacity zero, so that $\lambda(A_\mu)=0$ if and only if $\mu=\mu_\emptyset$.

The following counterexample shows that, for a general λ, the functional

$$\tilde{J}_0(\mu) = \begin{cases} 0, & \text{if } \lambda(A_\mu)=0, \\ +\infty, & \text{otherwise,} \end{cases}$$

does not coincide with the relaxation of the functional J_0 when $T=\{0\}$.

Counterexample 3.9. Let $\{q_h\}=\Omega\cap\mathbf{Q}^n$ and let $\{r_h\}$ a sequence of positive number such that $\mathrm{cap}(B(q_h,r_h),\Omega)<1/2^h$. Let $V=\bigcup_h B(q_h,r_h)$ and let u be the function in $H_0^1(\Omega)$ such that $u=1$ q.e. in V and $\int_\Omega |Du|^2dx=\mathrm{cap}(V,\Omega)$. Let us consider the measure λ defined as

$$\lambda(B)=\mathcal{L}^n(V\cap B)$$

for every $B\in\mathcal{B}(\Omega)$, where $\mathcal{L}^n$ denotes the n–dimensional Lebesgue measure. Since for every open subset A of Ω with $A\neq\emptyset$ we have $\lambda(A)>0$, the functional J_0 defined in (3.5) corresponding to this choice of λ takes the form (3.14) and its relaxation $\overline{J}_0$ is given by (3.15). On the other hand, if we consider the quasi open set $A=\{x\in\Omega : u(x)<1/2\}$, since $u=1$ q.e. in V we have $\mathrm{cap}(A\cap V,\Omega)=0$ and hence $\lambda(A)=0$. Finally A has positive Lebesgue measure, and then positive capacity, so that $\mu_A(A)=0\neq+\infty=\mu_\emptyset(A)$. Thus $\tilde{J}_0(\mu_A)=0$, but $\overline{J}_0(\mu_A)=+\infty$.

Remark 3.10. If $T = [0, \lambda(\Omega)]$, then by Remarks 2.2 and 3.3 the relaxed functional $\overline{J}$ can be written as

$$\overline{J}(\mu) \;=\; \int_{A_\mu} j(x, u_\mu)\, d\lambda - \int_{\Omega \setminus A_\mu} j^-(x, 0)\, d\lambda \,.$$

The following example shows that for every fixed $\nu \in \mathcal{M}_0(\Omega)$ there exists a functional J_ν as in Theorem 3.1, with $f = 1$ and $T = [0, \lambda(\Omega)]$, such that ν is the unique minimum point of $\overline{J}_\nu$. If $\nu \neq \mu_A$ for every open set $A \subseteq \Omega$, then the minimum problem

$$\min_{\substack{A \text{ open} \\ A \subseteq \Omega}} \; J_\nu(A)$$

has no solution. Indeed $J_\nu(A) \geq \overline{J}_\nu(\mu_A) > \overline{J}_\nu(\nu)$ for every open set $A \subseteq \Omega$, and

$$\inf_{\substack{A \text{ open} \\ A \subseteq \Omega}} \; J_\nu(A) = \min_{\mu \in \mathcal{M}_0(\Omega)} \overline{J}_\nu(\mu) = \overline{J}_\nu(\nu)$$

by Remark 1.12.

Example 3.11. Let w_0 be the solution of the Dirichlet problem

$$w_0 \in H^1_0(\Omega) \,, \qquad L w_0 = 1 \quad \text{in } \Omega \,.$$

Then $w_0 \in L^\infty(\Omega)$ and $w_\mu \leq w_0$ q.e. in Ω by the comparison principle (Lemma 1.4). Let us fix $\nu \in \mathcal{M}_0(\Omega)$ and let $g_\nu \in L^\infty(\Omega)$ be the function defined by

$$g_\nu = \begin{cases} w_\nu & \text{in } A_\nu, \\ 3k & \text{in } \Omega \setminus A_\nu, \end{cases}$$

where $k \in \mathbf{R}$ and $k \geq \|w_0\|_{L^\infty(\Omega)}$. Let $j_\nu \colon \Omega \times \mathbf{R} \to \mathbf{R}$ be the function defined by

$$j_\nu(x, s) = |s - g_\nu(x)|^2 - k^2 \,.$$

Finally let $f = 1$, let λ be the Lebesgue measure, and let $T = [0, \lambda(\Omega)]$. Then the functional defined by (3.1) is given by

$$J_\nu(A) = \int_A (|w_A - g_\nu|^2 - k^2)\, dx \,,$$

for every open set $A \subseteq \Omega$. By Remark 3.10 the relaxation $\overline{J}_\nu$ of J_ν takes the form

$$\overline{J}_\nu(\mu) = \int_{A_\mu} (|w_\mu - g_\nu|^2 - k^2)\, dx - \int_{\Omega \setminus A_\mu} (g_\nu^2 - k^2)^-\, dx \tag{3.16}$$

for every $\mu \in \mathcal{M}_0(\Omega)$. We want to prove that $\overline{J}_\nu(\mu) > \overline{J}_\nu(\nu) = -k^2\lambda(A_\nu)$ for every $\mu \in \mathcal{M}_0(\Omega)$ with $\mu \neq \nu$. By (3.16) we can write

$$\begin{aligned}\overline{J}_\nu(\mu) = &\int_{A_\mu \cap A_\nu} (|w_\mu - w_\nu|^2 - k^2)\, dx + \int_{A_\mu \setminus A_\nu} (w_\mu^2 - 6k w_\mu + 8k^2)\, dx \\ &- \int_{\Omega \setminus (A_\mu \cup A_\nu)} (9k^2 - k^2)^-\, dx - \int_{A_\nu \setminus A_\mu} (w_\nu^2 - k^2)^-\, dx\,,\end{aligned}$$

Since $w_\nu = 0$ q.e. in $A_\mu \setminus A_\nu$, $w_\mu = 0$ q.e. in $A_\nu \setminus A_\mu$, and $0 \leq w_\mu \leq k$ q.e. in Ω, we have $w_\mu^2 = |w_\mu - w_\nu|^2$ q.e. in $A_\mu \setminus A_\nu$, $-6k w_\mu + 8k^2 \geq 0$ q.e. in Ω, and $-(w_\nu^2 - k^2)^- = w_\nu^2 - k^2 = |w_\mu - w_\nu|^2 - k^2$ q.e. in $A_\nu \setminus A_\mu$. Hence

$$\begin{aligned}\overline{J}_\nu(\mu) \geq &\int_{A_\mu \cap A_\nu} |w_\mu - w_\nu|^2\, dx - k^2\lambda(A_\mu \cap A_\nu) + \int_{A_\mu \setminus A_\nu} |w_\mu - w_\nu|^2\, dx \\ &+ \int_{A_\nu \setminus A_\mu} |w_\mu - w_\nu|^2\, dx - k^2\lambda(A_\nu \setminus A_\mu) = \int_\Omega |w_\mu - w_\nu|^2 - k^2\lambda(A_\nu)\,.\end{aligned}$$

This shows that $\overline{J}_\nu(\mu) \geq \overline{J}_\nu(\nu) = -k^2\lambda(A_\nu)$ for every $\mu \in \mathcal{M}_0(\Omega)$. If $\overline{J}_\nu(\mu) = \overline{J}_\nu(\nu)$, then $\int_\Omega |w_\mu - w_\nu|^2 = 0$, hence $w_\mu = w_\nu$ a.e. in Ω, and this implies $\mu = \nu$ by Lemma 3.3 of [15].

In the following example, which is a particular case of the previous one, the function j is continuous, and even $C^\infty(\Omega)$ if the coefficients of the operator L are $C^\infty(\Omega)$.

Example 3.12. Let w_0, k, λ, f, T be as in Example 3.9, let $j(x,s) = |s - \frac{1}{2}w_0|^2 - k^2$, and let

$$J(A) = \int_A (|w_A - \frac{1}{2}w_0|^2 - k^2)\, dx$$

be the corresponding functional defined for every open set $A \subseteq \Omega$. Then

$$\inf_{\substack{A \text{ open} \\ A \subseteq \Omega}} J(A) = -k^2\lambda(\Omega)\,, \tag{3.17}$$

where λ is the Lebesgue measure, but the minimum in (3.17) is not achieved. To prove this fact it is enough to notice that $w_0 > 0$ in Ω by the strong maximum principle, and that

$$L(\frac{1}{2}w_0) + \frac{1}{2}\frac{w_0}{w_0} = 1 \quad \text{in } \Omega\,,$$

hence $\frac{1}{2}w_0$ is the solution of (1.5) corresponding to the measure $\nu = \lambda/w_0$. Therefore $\frac{1}{2}w_0 = w_\nu$. As $A_\nu = \Omega$, using the notation of the Example 3.11 we have $g_\nu = \frac{1}{2}w_0$, $j_\nu = j$, $J_\nu = J$. Therefore $\overline{J}(\mu) > \overline{J}(\nu) = -k^2\lambda(\Omega)$ for every $\mu \in \mathcal{M}_0(\Omega)$ with $\mu \neq \nu$. The conclusion follows from (3.16) and (3.17).

References

[1] M. BELLONI, G. BUTTAZZO & L. FREDDI: *Completion by* Γ*-convergence for optimal control problems.* Ann. Fac. Sci. Toulouse Math., to appear.

[2] A. BRAIDES & A. MALUSA: *Approximation of relaxed Dirichlet problems.* Proceedings of the Colloquium “Calculus of Variations, Homogenization and Continuum Mechanics”(Marseille, June 1993), Advances in Mathematics for Applied Science, World Scientific, Singapore, to appear.

[3] D. BUCUR: *Capacitary extension and two dimensional shape optimization.* Preprint 93.24, Institut Non Lináire de Nice, 1993.

[4] D. BUCUR & J.P. ZOLESIO: *N-dimensional shape optimization under capacitary constraint.* Preprint 93.22, Institut Non Lináire de Nice, 1993.

[5] D. BUCUR & J.P. ZOLESIO: *Flat cone condition and shape analysis.* Control of Partial Differential Equations, G. Da Prato and L. Tubaro eds., Marcel Dekker Pubs., to appear.

[6] G. BUTTAZZO: *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman Res. Notes Math. Ser. **207**, Longman, Harlow (1989).

[7] G. BUTTAZZO: *Relaxed formulation for a class of shape optimization problems.* Proceedings of “Boundary Control and Boundary Variations”, Sophia-Antipolis, October 15–17, 1990, Lecture Notes in Control and Inf. Sci. **178**, Springer-Verlag, Berlin (1992), 50–59.

[8] G. BUTTAZZO: *Existence via relaxation for some domain optimization problems.* Proceedings of “Topology Design of Structures”, Sesimbra 20–26 June 1992, NATO ASI Series E: Applied Sciences **227**, Kluwer Academic Publishers, Dordrecht (1993), 337–343.

[9] G. BUTTAZZO & G. DAL MASO: *Shape optimization for Dirichlet problems: relaxed formulation and optimality conditions.* Appl. Math. Optim., **23** (1991), 17–49.

[10] G. BUTTAZZO & G. DAL MASO: *An existence result for a class of shape optimization problems.* Arch. Rational Mech. Anal., **122** (1993), 183–195.

[11] G. BUTTAZZO, G. DAL MASO & U. MOSCO: *A derivation theorem for capacities with respect to a Radon measure.* J. Funct. Anal., **71** (1987), 263–278.

[12] M. CHIPOT & G. DAL MASO: *Relaxed shape optimization: the case of nonnegative data for the Dirichlet problem.* Adv. Math. Sci. Appl., **1** (1992), 47–81.

[13] G. DAL MASO: *On the integral representation of certain local functionals.* Ricerche Mat. **32** (1983), 85-113.

[14] G. DAL MASO: *An Introduction to* Γ*-convergence.* Birkhäuser, Boston (1993).

[15] G. DAL MASO & A. GARRONI: *New results on the asymptotic behaviour of Dirichlet problems in perforated domains.* Math. Mod. Meth. Appl. Sci., to appear.

[16] G. DAL MASO & A. GARRONI: *Capacitary methods for the study of asymptotic Dirichlet problems.* In preparation.

[17] G. DAL MASO & U. MOSCO: *Wiener's criteria and energy decay for relaxed Dirichlet problems.* Arch. Rational. Mech. Anal., **95** (1986), 345–387.

[18] G. DAL MASO & U. MOSCO: *Wiener's criterion and* Γ*-convergence.* Appl. Math. Optim., **15** (1987), 15–63.

[19] S. FINZI VITA: *Numerical shape optimization for relaxed Dirichlet problems.* Math. Mod. Meth. Appl. Sci., **3** (1993), 19–34.

[20] P. R. HALMOS: *Measure theory.* Springer–Verlag, Berlin (1974).

[21] F. MURAT & J. SIMON: *Etude de problèmes d'optimal design.* Optimization Techniques. Modelling and Optimization in the Service of Man (Nice, 1975). Lecture Notes in Computer Science, Vol 41, Springer–Verlag, Berlin (1976), 54–62.

[22] F. MURAT & J. SIMON: *Sur le contrôle par un domaine géometrique.* Preprint 76015, Univ. Paris VI, 1976.

[23] O. PIRONNEAU: *Optimal Shape Design for Elliptic Systems.* Springer-Verlag, Berlin (1984).

[24] G. STAMPACCHIA, *Le problème de Dirichlet pour les èquations elliptiques du second ordre à coefficients discontinus.* Ann. Inst. Fourier (Grenoble), **15** (1965), 189–258.

[25] W. ŠVERÁK: *On optimal shape design.* J. Math. Pures Appl., **72** (1993), 537–551.

[26] W. P. ZIEMER: *Weakly Differentiable Functions.* Springer–Verlag, Berlin (1989).

[27] J.P. ZOLESIO: *The material derivative (or speed) method for shape optimization.* Optimization of Distributed Parameter Structures (Iowa City, 1980), Sijthoff and Noordhoff eds., Rockville (1981), 1089–1151.