

**METASTABILITY AND DYNAMICS OF DISCRETE  
TOPOLOGICAL SINGULARITIES IN TWO DIMENSIONS:  
A  $\Gamma$ -CONVERGENCE APPROACH**

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ABSTRACT. This paper aims at building a variational approach to the dynamics of discrete topological singularities in two dimensions, based on  $\Gamma$ -convergence.

We consider discrete systems, described by scalar functions defined on a square lattice and governed by periodic interaction potentials. Our main motivation comes from  $XY$  spin systems, described by the phase parameter, and screw dislocations, described by the displacement function. For these systems, we introduce a discrete notion of vorticity. As the lattice spacing tends to zero we derive the first order  $\Gamma$ -limit of the free energy which is referred to as renormalized energy and describes the interaction of vortices.

As a byproduct of this analysis, we show that such systems exhibit increasingly many metastable configurations of singularities. Therefore, we propose a variational approach to depinning and dynamics of discrete vortices, based on minimizing movements. We show that, letting first the lattice spacing and then the time step of the minimizing movements tend to zero, the vortices move according with the gradient flow of the renormalized energy, as in the continuous Ginzburg-Landau framework.

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## 1. INTRODUCTION

Phase transitions mediated by the formation of topological singularities characterize many physical phenomena such as superconductivity, superfluidity and plasticity. For its central role in Materials Science, this subject has attracted much attention in the last decades ([9], [30], [31], see also [41]), and has brought new interest on fascinating research fields in the mathematical community, as in the theory of harmonic maps on manifolds ([4], [15], [10]). In particular, new variational methods have been developed to describe and predict the relevant phenomena, such as the formation of topological singularities and the corresponding concentration of energy. Two paradigmatic examples of the appearance of topological singularities are given by screw dislocations in crystals and vortices in superconductors. We now introduce two basic discrete models to describe these phenomena.

Given an open set  $\Omega \subset \mathbb{R}^2$ , consider the square lattice  $\varepsilon\mathbb{Z}^2 \cap \Omega$ , representing the reference configuration of our physical system. In the case of screw dislocations we consider the elastic energy defined on scalar functions  $u : \varepsilon\mathbb{Z}^2 \cap \Omega \rightarrow \mathbb{R}$  given by

$$(1.1) \quad SD_\varepsilon(u) := \frac{1}{2} \sum_{i,j \in \varepsilon\mathbb{Z}^2 \cap \Omega, |i-j|=\varepsilon} \text{dist}^2(u(i) - u(j), \mathbb{Z}).$$

Here  $\varepsilon$  represents the lattice spacing of a cubic lattice casted in a cylindrical crystal,  $\varepsilon\mathbb{Z}^2 \cap \Omega$  is a reference planar section of the crystal, and  $u$  represents the vertical displacement (scaled by  $1/\varepsilon$ ). The periodicity of the energy is consistent with the fact that plastic deformations, corresponding to integer jumps of  $u$ , do not store elastic energy, according with Nabarro Peierls and Frenkel Kontorova theories [24]. Potentials as in (1.1) are commonly used in models for dislocations (see e.g. [17], [26], [21], [34]; see also [8] for more general discrete lattice energies accounting for defects).

A celebrated discrete model which allows to describe the formation of topological singularities, as vortices in superconductors, is the so-called XY spin model. Here, the order parameter is a vectorial spin field  $v : \varepsilon\mathbb{Z}^2 \cap \Omega \rightarrow \mathcal{S}^1$  and the corresponding energy is given by

$$XY_\varepsilon(v) := \frac{1}{2} \sum_{i,j \in \varepsilon\mathbb{Z}^2 \cap \Omega, |i-j|=\varepsilon} |v(i) - v(j)|^2.$$

Notice that  $XY_\varepsilon(v)$  can be written in terms of a representative of the phase of  $v$ , defined as a scalar field  $u$  such that  $v = e^{2\pi i u}$ . In this respect, both models can be regarded as specific examples of scalar systems governed by periodic potentials  $f$  acting on first neighbors, whose energy is of the type

$$F_\varepsilon(u) := \sum_{i,j \in \varepsilon\mathbb{Z}^2 \cap \Omega, |i-j|=\varepsilon} f(u(i) - u(j)).$$

How do dislocations or vortices enter in this description? Loosely speaking, they are defined through a discrete notion of topological degree of the field  $v = e^{2\pi i u}$ ; they are point singularities, and can be identified by the discrete vorticity measure  $\mu(u)$ . This is a finite sum of Dirac masses centered in the squares of the lattice, and with multiplicities equal to either  $+1$  or  $-1$ . This notion in the case of dislocations corresponds to the discrete circulation of the plastic strain, and  $\mu(u)$  represents the Nye dislocation density.

This paper aims at studying the statics and the dynamics of such topological singularities, by variational principles.

The first step is the asymptotic analysis by  $\Gamma$ -convergence of the discrete energies  $F_\varepsilon$ , as  $\varepsilon \rightarrow 0$ . This analysis relies on the powerful machinery developed in the recent past for the analysis of Ginzburg-Landau functionals, which can be somehow considered the continuous counterpart of the energies  $F_\varepsilon$ . We recall that, for a given  $\varepsilon > 0$ , the Ginzburg-Landau energy  $GL_\varepsilon : H^1(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}$  is defined by

$$(1.2) \quad GL_\varepsilon(w) = \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx + \frac{1}{\varepsilon^2} \int_{\Omega} (1 - |w|^2)^2 \, dx.$$

Starting from the pioneering book [10], the variational analysis as  $\varepsilon \rightarrow 0$  of  $GL_\varepsilon$  has been the subject of a vast literature. The analysis in [10] shows that, as  $\varepsilon$  tends to zero, vortex-like singularities appear by energy minimization (induced for instance by the boundary conditions), and each singularity carries a quantum of energy of order  $|\log \varepsilon|$ . Removing this leading term from the energy, a finite quantity remains, called renormalized energy, depending on the positions of the singularities. This asymptotic analysis has been also developed through the solid formalism of  $\Gamma$ -convergence ([28], [29], [36], [38], [3]). It turns out that the relevant object to deal with is the distributional Jacobian  $Jw$ , which, in the continuous setting, plays the role of the discrete vorticity measure. A remarkable fact is that these results also contain a compactness statement. Indeed, for sequences with bounded energy the vorticity measure is not in general bounded in mass; this is due to the fact that many dipoles are compatible with a logarithmic energy bound. Therefore, the compactness of the vorticity measures fails in the usual sense of weak star convergence. Nevertheless, compactness holds in the flat topology, i.e., in the dual of Lipschitz continuous functions with compact support.

Recently, part of this  $\Gamma$ -convergence analysis has been exported to two-dimensional discrete systems. In [33], [1], [2] it has been proved that the functionals  $\frac{1}{|\log \varepsilon|} F_\varepsilon$   $\Gamma$ -converge to  $\pi|\mu(\Omega)|$ , where  $\mu$  is the limiting vorticity measure and is given by a finite sum of Dirac masses. This  $\Gamma$ -limit is not affected by the position of the singularities and hence does not account for their interaction, which is an essential ingredient in order to study the dynamics. In this paper, we make a further step in this direction, deriving the renormalized energy for our discrete systems by  $\Gamma$ -convergence, using the notion of  $\Gamma$ -convergence expansion introduced by [7] (see also [14]). Precisely, in Theorem 4.2 we prove that, given  $M \in \mathbb{N}$ , the functionals  $F_\varepsilon(u) - M\pi|\log \varepsilon|$   $\Gamma$ -converge to  $\mathbb{W}(\mu) + M\gamma$ , where  $\mu$  is a sum of  $M$  singularities  $x_i$  with degrees  $d_i = \pm 1$ . Here  $\mathbb{W}$  is the renormalized energy as in the Ginzburg-Landau setting, defined by

$$\mathbb{W}(\mu) := -\pi \sum_{i \neq j} d_i d_j \log |x_i - x_j| - \pi \sum_i d_i R_0(x_i),$$

where  $R_0$  is a suitable harmonic function (see (4.1)), and  $\gamma$  can be viewed as a core energy, depending on the specific discrete interaction energy (see (4.6)).

An intermediate step to prove Theorem 4.2 is Theorem 3.1 (ii), which establishes a localized lower bound of the energy around the limiting vortices. This result is obtained using a tool introduced by Sandier [35] and Jerrard [28] for the functionals  $GL_\varepsilon$ , referred to as *ball construction*; it consists in providing suitable pairwise disjoint annuli, where much of the energy is stored, and estimating from below the energy on each of such annuli. In the continuous case, the lower bound on each annulus is the straightforward estimate

$$\frac{1}{2} \int_{B_R \setminus B_r} |\nabla w|^2 \, dx \geq \pi |\deg(w, \partial B_R)| \log \frac{R}{r}, \quad w \in H^1(B_R \setminus B_r; \mathcal{S}^1).$$

In Proposition 3.2 we prove a similar lower bound for  $F_\varepsilon$ , with  $R/r$  replaced by  $R/(r + C\varepsilon |\log \varepsilon|)$ , the error being due to the discrete structure of our energies. This weaker estimate, inserted in the ball construction machinery, is refined enough to prove the lower bound in Theorem 3.1 (ii).

The second part of the paper is devoted to the analysis of metastable configurations for  $F_\varepsilon$  and to our variational approach to the dynamics of discrete topological singularities.

We now draw a parallel between the continuous Ginzburg-Landau model and our discrete systems, stressing out the peculiarities of our framework.

In [32], [27], [37], it has been proved that the parabolic flow of  $GL_\varepsilon$  can be described, as  $\varepsilon \rightarrow 0$ , by the gradient flow of the renormalized energy  $\mathbb{W}(\mu)$ . Precisely the limiting flow is a measure  $\mu(t) = \sum_{i=1}^M d_{i,0} \delta_{x_i(t)}$ , where  $x(t) = (x_1(t), \dots, x_M(t))$  solves

$$(1.3) \quad \begin{cases} \dot{x}(t) = -\frac{1}{\pi} \nabla W(x(t)) \\ x(0) = x_0, \end{cases}$$

with  $W(x(t)) = \mathbb{W}(\mu(t))$ . The advantage of this description is that the effective dynamics is described by an ODE involving only the positions of the singularities. This result has been derived through a purely variational approach in [37], based on the idea that the gradient flow structure is consistent with  $\Gamma$ -convergence, under some assumptions which imply that the slope of the approximating functionals converges to the slope of their  $\Gamma$ -limit. The gradient flow approach to dynamics used in the Ginzburg-Landau context fails for our discrete systems. In fact, the free energy of discrete systems is often characterized by the presence of many energy barriers, which affect the dynamics and are responsible for pinning effects (for a variational description of pinning effects in discrete systems see [13] and the references therein). As a consequence of our  $\Gamma$ -convergence analysis, we show that  $F_\varepsilon$  has many local minimizers. Precisely, in Theorem 5.5 and Theorem 5.6 we show that, under suitable assumptions on the potential  $f$ , given any configuration of singularities  $x \in \Omega^{2M}$ , there exists a stable configuration  $\tilde{x}$  at a distance of order  $\varepsilon$  from  $x$ . Starting from these configurations, the gradient flow of  $F_\varepsilon$  is clearly stuck. Moreover, these stable configurations are somehow attractive wells for the dynamics. These results are proven for a general class of energies, including  $SD_\varepsilon$ , while the case of the  $XY_\varepsilon$  energy, to our knowledge, is still open. A similar analysis of stable configurations in the triangular lattice has been recently carried on in

[25], combining PDEs techniques with variational arguments, while our approach is purely variational and based on  $\Gamma$ -convergence.

On one hand, our analysis is consistent with the well-known pinning effects due to energy barriers in discrete systems; on the other hand, it is also well understood that dislocations are able to overcome the energetic barriers to minimize their interaction energy (see [16], [21], [26], [34]). The mechanism governing these phenomena is still matter of intense research. Certainly, thermal effects and statistical fluctuations play a fundamental role. Such analysis is beyond the purposes of this paper. Instead, we raise the question whether there is a simple variational mechanism allowing singularities to overcome the barriers, and then which would be the effective dynamics. We face these questions, following the minimizing movements approach à la De Giorgi ([5], [6], [9]). More precisely, we discretize time by introducing a time scale  $\tau > 0$ , and at each time step we minimize a total energy, which is given by the sum of the free energy plus a dissipation. For any fixed  $\tau$ , we refer to this process as discrete gradient flow. This terminology is due to the fact that, as  $\tau$  tends to zero, the discrete gradient flow is nothing but the Euler implicit approximation of the continuous gradient flow of  $F_\varepsilon$ . Therefore, as  $\tau \rightarrow 0$  it inherits the degeneracy of  $F_\varepsilon$ , and pinning effects are dominant. The scenario changes completely if instead we keep  $\tau$  fixed, and send  $\varepsilon \rightarrow 0$ . In this case, it turns out that, during the step by step energy minimization, the singularities are able to overcome the energy barriers, that are of order  $\varepsilon$ . Finally, sending  $\tau \rightarrow 0$  the solutions of the discrete gradient flows converge to a solution of (1.3). In our opinion, this purely variational approach based on minimizing movements, mimics in a realistic way more complex mechanisms, providing an efficient and simple view point on the dynamics of discrete topological singularities in two dimensions.

Summarizing, in order to observe an effective dynamics of the vortices we are naturally led to let  $\varepsilon \rightarrow 0$  for a fixed time step  $\tau$ , obtaining a discrete gradient flow of the renormalized energy. A technical issue is that the renormalized energy is not bounded from below, and therefore, in the step by step minimization we are led to consider local rather than global minimizers. Precisely, we minimize the energy in a  $\delta$  neighborhood of the minimizer at the previous step. Without this care, already at the first step we would have the trivial solution  $\mu = 0$ , corresponding to the fact that dipoles annihilate and the remaining singularities reach the boundary of the domain. Nevertheless, for  $\tau$  small the minimizers do not touch the constraint, so that they are in fact true local minimizers.

In order to discuss some mathematical aspects of the asymptotic analysis of discrete gradient flows as  $\varepsilon, \tau \rightarrow 0$ , we need to clarify the specific choice of the dissipations we deal with. The canonical dissipation corresponding to continuous parabolic flow is clearly the  $L^2$  dissipation. On the other hand, once  $\varepsilon$  is sent to zero, we have a finite dimensional gradient flow of the renormalized energy, for which it is more natural to consider as dissipation the Euclidean distance between the singularities. This choice, for  $\varepsilon > 0$ , corresponds to the flat distance of the vorticity measures. We are then led to consider also the discrete gradient flow with flat dissipation. With this choice the analysis is indeed simpler and somehow instructive in order to face the more complex case of  $L^2$  dissipation.

We first discuss in details the discrete gradient flows with flat dissipation. To this purpose, it is convenient to introduce the functional  $\mathcal{F}_\varepsilon(\mu)$ , defined as the minimum of  $F_\varepsilon(u)$  among all  $u$  whose vorticity measure  $\mu(u)$  is equal to  $\mu$ . We fix an initial

condition  $\mu_0 := \sum_{i=1}^M d_{i,0} \delta_{x_{i,0}}$  with  $|d_{i,0}| = 1$  and a sequence  $\mu_{\varepsilon,0} \in X_\varepsilon$  satisfying

$$\mu_{\varepsilon,0} \xrightarrow{\text{flat}} \mu_0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})}{|\log \varepsilon|} = \pi |\mu_0|(\Omega).$$

Then given  $\delta > 0$  and let  $\varepsilon, \tau > 0$  we define  $\mu_{\varepsilon,k}^\tau$  by the following minimization problem

$$(1.4) \quad \mu_{\varepsilon,k}^\tau \in \operatorname{argmin} \left\{ \mathcal{F}_\varepsilon(\mu) + \frac{\pi \|\mu - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}}^2}{2\tau} : \mu \in X_\varepsilon, \|\mu - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}} \leq \delta \right\}$$

with  $\mu_{\varepsilon,0}^\tau = \mu_{\varepsilon,0}$ . As a direct consequence of our  $\Gamma$ -convergence analysis, in Theorem 6.7 we will show that, as  $\varepsilon \rightarrow 0$ ,  $\mu_{\varepsilon,k}^\tau$  converges, up to a subsequence, to a solution  $\mu_k^\tau \in X$  to

$$\mu_k^\tau \in \operatorname{argmin} \left\{ \mathbb{W}(\mu) + \frac{\pi \|\mu - \mu_{k-1}^\tau\|_{\text{flat}}^2}{2\tau} : \mu = \sum_{i=1}^M d_{i,0} \delta_{x_i}, \|\mu - \mu_{k-1}^\tau\|_{\text{flat}} \leq \delta \right\}.$$

After identifying the vorticity measure with the positions of its singularities, we get that the vortices  $x_k^\tau$  of  $\mu_k^\tau$  satisfy the following finite-dimensional problem

$$x_k^\tau \in \operatorname{argmin} \left\{ W(x) + \frac{\pi |x - x_{k-1}^\tau|^2}{2\tau} : x \in \Omega^M, |x - x_{k-1}^\tau| \leq \delta \right\}.$$

In Theorem 6.6 we show that this constrained scheme converges, as  $\tau \rightarrow 0$ , to the gradient flow of the renormalized energy (1.3), until a maximal time  $\tilde{T}_\delta$ . The proof of this fact follows the standard Euler implicit method, with some care to handle local rather than global minimization. Moreover, as  $\delta \rightarrow 0$ ,  $\tilde{T}_\delta$  converges to the critical time  $T^*$  (see Definition 6.3), at which either a vortex touches the boundary or two vortices collapse.

We now discuss the discrete gradient flow with the  $L^2$  dissipation. Once again, we consider a step by step minimization problem as in (1.4), with  $\|\mu - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}}^2$  replaced by  $\|v - v_{\varepsilon,k-1}^\tau\|_{L^2}^2 / |\log \tau|$ . More precisely,

$$u_{\varepsilon,k}^\tau \in \operatorname{argmin} \left\{ F_\varepsilon(u) + \frac{\|e^{2\pi i u} - e^{2\pi i u_{\varepsilon,k-1}^\tau}\|_{L^2}^2}{2\tau |\log \tau|} : \|\mu(u) - \mu(u_{\varepsilon,k-1}^\tau)\|_{\text{flat}} \leq \delta \right\}.$$

The prefactor  $1/|\log \tau|$  in front of the dissipation can be viewed as a time reparametrization, on which we will comment later.

The asymptotics of these discrete gradient flow as  $\varepsilon \rightarrow 0$  relies again on a  $\Gamma$ -convergence analysis, which keeps memory also of the  $L^2$  limit  $v$  of the variable  $e^{2\pi i u_\varepsilon}$ . Under suitable assumptions on the initial data, in Theorem 7.13 we show that, as  $\varepsilon \rightarrow 0$ , the solutions  $u_{\varepsilon,k}^\tau$  converge, up to a subsequence, to a solution to

$$v_k^\tau \in \operatorname{argmin} \left\{ \mathcal{W}(v) + \frac{\|v - v_{k-1}^\tau\|_{L^2}^2}{2\tau |\log \tau|} : v \in H_{\text{loc}}^1(\Omega \setminus \cup_{i=1}^M \{y_{i,k}\}; \mathcal{S}^1), \right. \\ \left. Jv = \sum_{i=1}^M d_{i,0} \delta_{y_{i,k}}, \|Jv - Jv_{k-1}^\tau\|_{\text{flat}} \leq \delta \right\},$$

where  $Jv$  is the distributional Jacobian of  $v$  and  $\mathcal{W}$  is the renormalized energy in terms of  $v$  (see Theorem 4.5); namely,  $\min_{Jv=\mu} \mathcal{W}(v) = \mathbb{W}(\mu)$ . Now, we wish to send  $\tau$  to zero. This step is much more delicate than in the case of flat dissipation. Indeed, it is at this stage that we adopt the abstract method [37], and exploit it in the context of minimizing movements instead of gradient flows. This method

relies on the proof of two energetic inequalities; the first relates the slope of the approximating functionals with the slope of the renormalized energy; the second one relates the scaled  $L^2$  norm underlying the parabolic flow of  $GL_\varepsilon$  with the Euclidean norm of the time derivative of the limit singularities. In our discrete in time framework, we adapt the arguments in [37] by replacing derivatives by finite differences. A heuristic argument to justify the prefactor  $1/|\log \tau|$  is that it is the correct scaling for the canonical vortex  $x/|x|$ . Indeed, given  $V \in \mathbb{R}^2$  representing the vortex velocity, a direct computation shows that

$$(1.5) \quad \lim_{\tau \rightarrow 0} \frac{1}{\tau |\log \tau|} \left\| \frac{x}{|x|} - \frac{x - \tau V}{|x - \tau V|} \right\|_2^2 = \pi |V|^2.$$

As a matter of fact, in order to get a non trivial dynamics in the limit, we have to accelerate the time scale as  $\tau$  tends to zero. This feature is well known also in the parabolic flow of Ginzburg-Landau functionals. In this respect, we observe that our time scaling is expressed directly as a function of the time step  $\tau$ , while for the functionals  $GL_\varepsilon$  it depends on the only scale parameter of the problem, which is the length scale  $\varepsilon$ . The explicit computation in (1.5) has not an easy counterpart for general solutions  $v_k^r$ , and (1.5) has to be replaced by more sophisticated estimates (see (7.17) and (7.56)). This point is indeed quite technical, and makes use of a lot of analysis developed in [37], [38].

In conclusion, we believe that this paper provides a better understanding of equilibria of discrete systems characterized by energy concentration, and contributes to the debate in the mathematical community over the microscopic mechanisms governing the dynamics of discrete topological singularities, as vortices in XY spin systems and dislocations in crystals. For the latter, richer models could be considered, with more realistic energy densities and dissipations, taking into account the specific material properties and the kinematic constraints of the crystal lattice. Our variational approach, rather than giving a complete analysis of a specific model, aims to be simple and robust, with possible applications to a wide class of discrete systems.

## 2. THE DISCRETE MODEL FOR TOPOLOGICAL SINGULARITIES

In this Section we introduce the discrete formalism used in the analysis of the problem we deal with. We will follow the approach of [8]; specifically, we will use the formalism and the notations in [2] (see also [33]).

Let  $\Omega \subset \mathbb{R}^2$  be a bounded open set with Lipschitz boundary, representing the domain of definition of the relevant fields in the models we deal with.

**The discrete lattice.** For every  $\varepsilon > 0$ , we define  $\Omega_\varepsilon \subset \Omega$  as follows

$$\Omega_\varepsilon := \bigcup_{i \in \varepsilon \mathbb{Z}^2: i + \varepsilon Q \subset \bar{\Omega}} (i + \varepsilon Q),$$

where  $Q = [0, 1]^2$  is the unit square. Moreover we set  $\Omega_\varepsilon^0 := \varepsilon \mathbb{Z}^2 \cap \Omega_\varepsilon$ , and  $\Omega_\varepsilon^1 := \{(i, j) \in \Omega_\varepsilon^0 \times \Omega_\varepsilon^0 : |i - j| = \varepsilon, i \leq j\}$  (where  $i \leq j$  means that  $i_l \leq j_l$  for  $l \in \{1, 2\}$ ). These objects represent the reference lattice and the class of nearest neighbors, respectively. The cells contained in  $\Omega_\varepsilon$  are labeled by the set of indices  $\Omega_\varepsilon^2 = \{i \in \Omega_\varepsilon^0 : i + \varepsilon Q \subset \Omega_\varepsilon\}$ . Finally, we define the *discrete boundary* of  $\Omega$  as

$$(2.1) \quad \partial_\varepsilon \Omega := \partial \Omega_\varepsilon \cap \varepsilon \mathbb{Z}^2.$$

In the following, we will extend the use of these notations to any given open subset  $A$  of  $\mathbb{R}^2$ .

**2.1. Discrete functions and discrete topological singularities.** Here we introduce the classes of discrete functions on  $\Omega_\varepsilon^0$ , and a notion of discrete topological singularities. To this purpose, we first set

$$\mathcal{AF}_\varepsilon(\Omega) := \{u : \Omega_\varepsilon^0 \rightarrow \mathbb{R}\},$$

which represents the class of admissible scalar functions on  $\Omega_\varepsilon^0$ .

Moreover, we introduce the class of admissible fields from  $\Omega_\varepsilon^0$  to the set  $\mathcal{S}^1$  of unit vectors in  $\mathbb{R}^2$

$$(2.2) \quad \mathcal{AXY}_\varepsilon(\Omega) := \{v : \Omega_\varepsilon^0 \rightarrow \mathcal{S}^1\},$$

Notice that, to any function  $u \in \mathcal{AF}_\varepsilon(\Omega)$ , we can associate a function  $v \in \mathcal{AXY}_\varepsilon(\Omega)$  setting

$$v = v(u) := e^{2\pi i u}.$$

With a little abuse of notation for every  $v : \Omega_\varepsilon^0 \rightarrow \mathbb{R}^2$  we denote

$$(2.3) \quad \|v\|_{L^2}^2 = \sum_{j \in \Omega_\varepsilon^0} \varepsilon^2 |v(j)|^2.$$

Now we can introduce a notion of discrete vorticity corresponding to both scalar and  $\mathcal{S}^1$  valued functions. To this purpose, let  $P : \mathbb{R} \rightarrow \mathbb{Z}$  be defined as follows

$$(2.4) \quad P(t) = \operatorname{argmin} \{|t - s| : s \in \mathbb{Z}\},$$

with the convention that, if the argmin is not unique, then we choose the minimal one.

Let  $u \in \mathcal{AF}_\varepsilon(\Omega)$  be fixed. For every  $i \in \Omega_\varepsilon^2$  we introduce the vorticity

$$(2.5) \quad \begin{aligned} \alpha_u(i) := & P(u(i + \varepsilon e_1) - u(i)) + P(u(i + \varepsilon e_1 + \varepsilon e_2) - u(i + \varepsilon e_1)) \\ & - P(u(i + \varepsilon e_1 + \varepsilon e_2) - u(i + \varepsilon e_2)) - P(u(i + \varepsilon e_2) - u(i)). \end{aligned}$$

One can easily see that the vorticity  $\alpha_u$  takes values in  $\{-1, 0, 1\}$ . Finally, we define the vorticity measure  $\mu(u)$  as follows

$$(2.6) \quad \mu(u) := \sum_{i \in \Omega_\varepsilon^2} \alpha_u(i) \delta_{i + \frac{\varepsilon}{2}(e_1 + e_2)}.$$

this definition of vorticity extends to  $\mathcal{S}^1$  valued fields in the obvious way, by setting  $\mu(v) = \mu(u)$  where  $u$  is any function in  $\mathcal{AF}_\varepsilon(\Omega)$  such that  $v(u) = v$ .

Let  $\mathcal{M}(\Omega)$  be the space of Radon measures in  $\Omega$  and set

$$(2.7) \quad \begin{aligned} X := & \left\{ \mu \in \mathcal{M}(\Omega) : \mu = \sum_{i=1}^N d_i \delta_{x_i}, N \in \mathbb{N}, d_i \in \mathbb{Z} \setminus \{0\}, x_i \in \Omega \right\}, \\ X_\varepsilon := & \left\{ \mu \in X : \mu = \sum_{i \in \Omega_\varepsilon^2} \alpha(i) \delta_{i + \frac{\varepsilon}{2}(e_1 + e_2)}, \alpha(i) \in \{-1, 0, 1\} \right\}. \end{aligned}$$

We will denote by  $\mu_n \xrightarrow{\text{flat}} \mu$  the flat convergence of  $\mu_n$  to  $\mu$ , i.e., in the dual  $W^{-1,1}$  of  $W_0^{1,\infty}$ .



**2.2. The discrete energy.** Here we introduce a class of energy functionals defined on  $\mathcal{AF}_\varepsilon(\Omega)$ . We will consider periodic potentials  $f : \mathbb{R} \rightarrow \mathbb{R}$  which satisfy the following assumptions: For any  $a \in \mathbb{R}$

- (1)  $f(a + z) = f(a)$  for any  $z \in \mathbb{Z}$ ,
- (2)  $f(a) \geq \frac{1}{2}|e^{2\pi ia} - 1|^2 = 1 - \cos 2\pi a$ ,
- (3)  $f(a) = 2\pi^2(a - z)^2 + O(|a - z|^3)$  for any  $z \in \mathbb{Z}$ .

For any  $u \in \mathcal{AF}_\varepsilon(\Omega)$ , we define

$$(2.8) \quad F_\varepsilon(u) := \sum_{(i,j) \in \Omega_\varepsilon^1} f(u(i) - u(j)).$$

As explained in the Introduction, the main motivation for our analysis comes from the study discrete screw dislocations in crystals and  $XY$  spin systems. We introduce the basic energies for these two models as in [2].

Regarding the screw dislocations, for any  $u : \Omega_\varepsilon^0 \rightarrow \mathbb{R}$ , we define

$$(2.9) \quad SD_\varepsilon(u) := \frac{1}{2} \sum_{(i,j) \in \Omega_\varepsilon^1} \text{dist}^2(u(i) - u(j), \mathbb{Z}).$$

It is easy to see that this potential fits (up to the prefactor  $4\pi^2$ ) with our general assumptions.

As for the  $XY$  model, for any  $v : \Omega_\varepsilon^0 \rightarrow \mathcal{S}^1$ , we define

$$(2.10) \quad XY_\varepsilon(v) := \frac{1}{2} \sum_{(i,j) \in \Omega_\varepsilon^1} |v(i) - v(j)|^2.$$

Also this potential fits our framework, once we rewrite it in terms of the phase  $u$  of  $v$ . Indeed, setting  $f(a) = 1 - \cos(2\pi a)$ , we have

$$(2.11) \quad XY_\varepsilon(v) = \sum_{(i,j) \in \Omega_\varepsilon^1} f(u(i) - u(j)) \quad \text{with } v = e^{2\pi i u}.$$

We notice that assumption (2) on  $F_\varepsilon$  reads as

$$(2.12) \quad F_\varepsilon(u) \geq XY_\varepsilon(e^{2\pi i u}).$$

Let  $\{T_i^\pm\}$  be the family of the  $\varepsilon$ -simplices of  $\mathbb{R}^2$  whose vertices are of the form  $\{i, i \pm \varepsilon e_1, i \pm \varepsilon e_2\}$ , with  $i \in \varepsilon\mathbb{Z}^2$ . For any  $v : \Omega_\varepsilon^0 \rightarrow \mathcal{S}^1$ , we denote by  $\tilde{v} : \Omega_\varepsilon \rightarrow \mathcal{S}^1$  the piecewise affine interpolation of  $v$ , according with the triangulation  $\{T_i^\pm\}$ . It is easy to see that, up to boundary terms,  $XY_\varepsilon(v)$  corresponds to the Dirichlet energy of  $\tilde{v}$  in  $\Omega_\varepsilon$ ; more precisely

$$(2.13) \quad \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla \tilde{v}|^2 dx + C\varepsilon \geq XY_\varepsilon(v) \geq \frac{1}{2} \int_{\Omega_\varepsilon} |\nabla \tilde{v}|^2 dx,$$

where  $C$  depends only on  $\Omega$ .

**Remark 2.1.** Let  $v : \Omega_\varepsilon^0 \rightarrow \mathcal{S}^1$ . One can easily verify that if  $A$  is an open subset of  $\Omega$  and if  $|\tilde{v}| > c > 0$  on  $\partial A_\varepsilon$ , then

$$(2.14) \quad \mu(v)(A_\varepsilon) = \text{deg}(\tilde{v}, \partial A_\varepsilon),$$

where the *degree* of a function  $w \in H^{\frac{1}{2}}(\partial A; \mathbb{R}^2)$  with  $|w| \geq c > 0$ , is defined by

$$(2.15) \quad \text{deg}(w, \partial A) := \frac{1}{2\pi} \int_{\partial A} \left( \frac{w}{|w|} \times \nabla \frac{w}{|w|} \right) \cdot \tau ds,$$

with  $v \times \nabla v := v_1 \nabla v_2 - v_2 \nabla v_1$ , for  $v \in H^1(A; \mathbb{R}^2)$ .

In particular, whenever  $|\tilde{v}| > 0$  on  $i + \varepsilon Q$  we have  $\mu(v)(i + \varepsilon Q) = 0$ .

### 3. LOCALIZED LOWER BOUNDS

In this section we will prove a lower bound for the energies  $F_\varepsilon$  localized on open subsets  $A \subset \Omega$ . We will use the standard notation  $F_\varepsilon(\cdot, A)$  (and as well  $XY_\varepsilon(\cdot, A)$ ) to denote the functional  $F_\varepsilon$  defined in (2.8) with  $\Omega$  replaced by  $A$ .

To this purpose, thanks to assumption (2) on the energy density  $f$ , it will be enough to prove a lower bound for the  $XY_\varepsilon$  energy. As a consequence of this lower bound, we obtain a sharp zero-order  $\Gamma$ -convergence result for the functionals  $F_\varepsilon$ .

**3.1. The zero-order  $\Gamma$ -convergence.** We recall that the space of finite sums of weighted Dirac masses has been denoted in (2.7) by  $X$ .

**Theorem 3.1.** *The following  $\Gamma$ -convergence result holds.*

- (i) (*Compactness*) Let  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  be such that  $F_\varepsilon(u_\varepsilon) \leq C|\log \varepsilon|$  for some positive  $C$ . Then, up to a subsequence,  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu$ , for some  $\mu \in X$ .
- (ii) (*Localized  $\Gamma$ -liminf inequality*) Let  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  be such that  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu = \sum_{i=1}^M d_i \delta_{x_i}$  with  $d_i \in \mathbb{Z} \setminus \{0\}$  and  $x_i \in \Omega$ . Then, there exists a constant  $C \in \mathbb{R}$  such that, for any  $i = 1, \dots, M$  and for every  $\sigma < \frac{1}{2} \text{dist}(x_i, \partial\Omega \cup \bigcup_{j \neq i} x_j)$ , we have

$$(3.1) \quad \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, B_\sigma(x_i)) - \pi |d_i| \log \frac{\sigma}{\varepsilon} \geq C.$$

*In particular*

$$(3.2) \quad \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) - \pi |\mu|(\Omega) \log \frac{\sigma}{\varepsilon} \geq C.$$

- (iii) ( *$\Gamma$ -limsup inequality*) For every  $\mu \in X$ , there exists a sequence  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  such that  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu$  and

$$\pi |\mu|(\Omega) \geq \limsup_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|}.$$

The above theorem has been proved in [33] for  $F_\varepsilon = SD_\varepsilon$  and in [2] for  $F_\varepsilon = XY_\varepsilon$ , with (ii) replaced by the standard global  $\Gamma$ -liminf inequality

$$(3.3) \quad \pi |\mu|(\Omega) \leq \liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon)}{|\log \varepsilon|},$$

which is clearly implied by (3.2).

By (2.12), the compactness property (i) follows directly from the zero-order  $\Gamma$ -convergence result for the  $XY_\varepsilon$  energies, while the proof of (ii) requires a specific analysis. For the convenience of the reader we will give a self contained proof of both (i) and (ii) of Theorem 3.1. We will omit the proof of the  $\Gamma$ -lim sup inequality (iii) which is standard and identical to the  $XY_\varepsilon$  case.

Before giving the proof of Theorem 3.1, we need to revisit a construction referred to as ball construction and introduced in the continuous framework in [35], [28].

**3.2. Lower bound on annuli.** Let  $w \in H^1(B_R \setminus B_r; \mathcal{S}^1)$  with  $\deg(w, \partial B_R) = d$ . By Jensen's inequality, the following lower bound holds

$$(3.4) \quad \begin{aligned} \frac{1}{2} \int_{B_R \setminus B_r} |\nabla w|^2 dx &\geq \frac{1}{2} \int_r^R \int_{\partial B_\rho} |(w \times \nabla w) \cdot \tau|^2 ds d\rho \\ &\geq \int_r^R \frac{1}{\rho} \pi d^2 d\rho \geq \pi |d| \log \frac{R}{r}. \end{aligned}$$

The latter is a key estimate in the context of continuous Ginzburg-Landau. In the following we will prove an analogous lower bound for the energy  $XY_\varepsilon(v, \cdot)$  in an annulus in which the piecewise affine interpolation  $\tilde{v}$  satisfies  $|\tilde{v}| \geq \frac{1}{2}$ . In view of (2.12) such a lower bound will hold also for the energy  $F_\varepsilon$ .

**Proposition 3.2.** *Fix  $\varepsilon > 0$  and let  $\sqrt{2}\varepsilon < r < R - 2\sqrt{2}\varepsilon$ . For any function  $v : (B_R \setminus B_r) \cap \varepsilon\mathbb{Z}^2 \rightarrow \mathcal{S}^1$  with  $|\tilde{v}| \geq \frac{1}{2}$  in  $B_{R-\sqrt{2}\varepsilon} \setminus B_{r+\sqrt{2}\varepsilon}$ , it holds*

$$(3.5) \quad XY_\varepsilon(v, B_R \setminus B_r) \geq \pi |\mu(v)(B_r)| \log \frac{R}{r + \varepsilon(\alpha |\mu(v)(B_r)| + \sqrt{2})},$$

where  $\alpha > 0$  is a universal constant.

*Proof.* By (2.13), using Fubini's theorem, we have that

$$(3.6) \quad XY_\varepsilon(v, B_R \setminus B_r) \geq \frac{1}{2} \int_{r+\sqrt{2}\varepsilon}^{R-\sqrt{2}\varepsilon} \int_{\partial B_\rho} |\nabla \tilde{v}|^2 ds d\rho.$$

Fix  $r + \sqrt{2}\varepsilon < \rho < R - \sqrt{2}\varepsilon$  and let  $T$  be a simplex of the triangulation of the  $\varepsilon$ -lattice. Set  $\gamma_T(\rho) := \partial B_\rho \cap T$ , let  $\tilde{\gamma}_T(\rho)$  be the segment joining the two extreme points of  $\gamma_T(\rho)$  and let  $\bar{\gamma}(\rho) = \bigcup_T \tilde{\gamma}_T(\rho)$ ; then

$$(3.7) \quad \begin{aligned} \frac{1}{2} \int_{\partial B_\rho} |\nabla \tilde{v}|^2 ds &= \frac{1}{2} \int_{\bigcup_T \gamma_T(\rho)} |\nabla \tilde{v}|^2 ds = \frac{1}{2} \sum_T |\nabla \tilde{v}|_{|T}^2 \mathcal{H}^1(\gamma_T(\rho)) \\ &\geq \frac{1}{2} \sum_T |\nabla \tilde{v}|_{|T}^2 \mathcal{H}^1(\tilde{\gamma}_T(\rho)) = \frac{1}{2} \int_{\bar{\gamma}(\rho)} |\nabla \tilde{v}|^2 ds. \end{aligned}$$

Set  $m(\rho) := \min_{\bar{\gamma}(\rho)} |\tilde{v}|$ ; using Jensen's inequality and the fact that  $\mathcal{H}^1(\bar{\gamma}(\rho)) \leq \mathcal{H}^1(\partial B_\rho)$  we get

$$(3.8) \quad \begin{aligned} \frac{1}{2} \int_{\bar{\gamma}(\rho)} |\nabla \tilde{v}|^2 ds &\geq \frac{1}{2} \int_{\bar{\gamma}(\rho)} m^2(\rho) \left| \left( \frac{\tilde{v}}{|\tilde{v}|} \times \nabla \frac{\tilde{v}}{|\tilde{v}|} \right) \cdot \tau \right|^2 ds \\ &\geq \frac{1}{2} \frac{m^2(\rho)}{\mathcal{H}^1(\bar{\gamma}(\rho))} \left| \int_{\bar{\gamma}(\rho)} \left( \frac{\tilde{v}}{|\tilde{v}|} \times \nabla \frac{\tilde{v}}{|\tilde{v}|} \right) \cdot \tau ds \right|^2 \\ (3.9) \quad &\geq \frac{m^2(\rho)}{\rho} \pi |d|^2 \end{aligned}$$

where we have set  $d := \deg(\tilde{v}, \partial B_\rho) = \mu(v)(B_r)$ , which does not depend on  $\rho$  since  $|\tilde{v}| \geq 1/2$ .

Now, let  $T(\rho)$  be the simplex in which the minimum  $m(\rho)$  is attained. Without loss of generality we assume that  $T(\rho) = T_{\bar{i}}^-$  for some  $\bar{i} \in \varepsilon\mathbb{Z}^2$ . Let  $P$  one of the

points of  $\bar{\gamma}(\rho)$  for which  $|\tilde{v}(P)| = m(\rho)$ . By elementary geometric arguments, one can show that

$$(3.10) \quad \frac{1}{2} \int_{\partial B_\rho(0)} |\nabla \tilde{v}|^2 ds \geq \tilde{\alpha} \frac{1 - m^2(\rho)}{\varepsilon},$$

for some universal positive constant  $\tilde{\alpha}$ .

In view of (3.7), (3.9) and (3.10), for any  $r + \sqrt{2}\varepsilon < \rho < R - \sqrt{2}\varepsilon$  we have

$$\frac{1}{2} \int_{\partial B_\rho} |\nabla \tilde{v}|^2 ds \geq \frac{m^2(\rho)}{\rho} \pi |d| \vee \tilde{\alpha} \frac{1 - m^2(\rho)}{\varepsilon} \geq \frac{\pi |d| \tilde{\alpha}}{\varepsilon \pi |d| + \tilde{\alpha} \rho}.$$

By this last estimate and (3.6) we get

$$(3.11) \quad XY_\varepsilon(v, B_R \setminus B_r) \geq \pi |\mu(v)(B_r)| \log \frac{\varepsilon(\frac{\pi}{\tilde{\alpha}} |\mu(v)(B_r)| - \sqrt{2}) + R}{\varepsilon(\frac{\pi}{\tilde{\alpha}} |\mu(v)(B_r)| + \sqrt{2}) + r}.$$

Assuming, without loss of generality,  $\tilde{\alpha} < 1$ , we immediately get (3.5) for  $\alpha = \frac{\pi}{\tilde{\alpha}}$ .  $\square$

**3.3. Ball Construction.** Here we introduce a construction referred to as *ball construction*, introduced in [35], [28]. Let  $\mathcal{B} = \{B_{R_1}(x_1), \dots, B_{R_N}(x_N)\}$  be a finite family of pairwise disjoint balls in  $\mathbb{R}^2$  and let  $\mu = \sum_{i=1}^N d_i \delta_{x_i}$  with  $d_i \in \mathbb{Z} \setminus \{0\}$ . Let  $F$  be a positive superadditive set function on the open subsets of  $\mathbb{R}^2$ , i.e., such that  $F(A \cup B) \geq F(A) + F(B)$ , whenever  $A$  and  $B$  are open and disjoint. We will assume that there exists  $c > 0$  such that

$$(3.12) \quad F(A_{r,R}(x)) \geq \pi |\mu(B_r(x))| \log \frac{R}{c+r},$$

for any annulus  $A_{r,R}(x) = B_R(x) \setminus B_r(x)$ , with  $A_{r,R}(x) \subset \Omega \setminus \bigcup_i B_{R_i}(x_i)$ .

The purpose of this construction is to select a family of larger and larger annuli in which the main part of the energy  $F$  concentrates. Let  $t$  be a parameter which represents an artificial time. For any  $t > 0$  we want to construct a finite family of balls  $\mathcal{B}(t)$  which satisfies the following properties

- (1)  $\bigcup_{i=1}^N B_{R_i}(x_i) \subset \bigcup_{B \in \mathcal{B}(t)} B$ ,
- (2) the balls in  $\mathcal{B}(t)$  are pairwise disjoint,
- (3)  $F(B) \geq \pi |\mu(B)| \log(1+t)$  for any  $B \in \mathcal{B}(t)$ ,
- (4)  $\sum_{B \in \mathcal{B}(t)} R(B) \leq (1+t) \sum_i R_i + (1+t) cN(N^2 + N + 1)$ , where  $R(B)$  denotes the radius of the ball  $B$ .

We construct the family  $\mathcal{B}(t)$ , closely following the strategy of the ball construction due to Sandier and Jerrard, that we need to slightly revise in order to include our case: The only difference in our discrete setting is the appearance of the error term  $c > 0$  in (3.12) and in (4), while in the continuous setting  $c = 0$ .

The ball construction consists in letting the balls alternatively expand and merge each other as follows. It starts with an expansion phase if  $\text{dist}(B_{R_i}(x_i), B_{R_j}(x_j)) > 2c$  for all  $i \neq j$ , and with a merging phase otherwise. Assume that the first phase is an expansion. It consists in letting the balls expand, without changing their centers, in such a way that, at each (artificial) time, the ratio  $\theta(t) := \frac{R_i(t)}{c+R_i}$  is independent of  $i$ . We will parametrize the time enforcing  $\theta(t) = 1+t$ . Note that with this choice  $R_i(0) = R_i + c$  so that the balls  $\{B_{R_i(0)}(x_i)\}$  are pairwise disjoint. The first expansion phase stops at the first time  $T_1$  when two balls bump into each other. Then the merging phase begins. It consists in identifying a suitable partition  $\{S_j^1\}_{j=1, \dots, N_n}$  of the family  $\{B_{R_i(T_1)}(x_i)\}$ , and, for each subclass  $S_j^1$ , in finding a

ball  $B_{R_j^1}(x_j^1)$  which contains all the balls in  $S_j^1$  such that the following properties hold:

- i) for every  $j \neq k$ ,  $\text{dist}(B_{R_j^1}(x_j^1), B_{R_k^1}(x_k^1)) > 2c$ ;
- ii)  $R_j^1 - Nc$  is not larger than the sum of all the radii of the balls  $B_{R_i(T_1)}(x_i) \in S_j^1$ , i.e., contained in  $B_{R_j^1}(x_j^1)$ .

After the merging, another expansion phase begins, during which we let the balls  $\{B_{R_j^1}(x_j^1)\}$  expand in such a way that, for  $t \geq T_1$ , for every  $j$  we have

$$(3.13) \quad \frac{R_j^1(t)}{c + R_j^1} = \frac{1 + t}{1 + T_1}.$$

Again note that  $R_j^1(T_1) = R_j^1 + c$ . We iterate this process obtaining a set of merging times  $\{T_1, \dots, T_n\}$ , and a family  $\mathcal{B}(t) = \{B_{R_j^k(t)}(x_j^k)\}_j$  for  $t \in [T_k, T_{k+1})$ , for all  $k = 1, \dots, n-1$ . Notice that  $n \leq N$ . If the condition  $\text{dist}(B_{R_i}(x_i), B_{R_j}(x_j)) > 2c$  for all  $i \neq j$ , is not satisfied we clearly can start this process with a merging phase (in this case  $T_1 = 0$ ).

By construction, we clearly have (1) and (2). We now prove (4). Set  $N(t) = \#\{B \in \mathcal{B}(t)\}$  and  $I(t) = \{1, \dots, N(t)\}$ . Moreover, for every merging time  $T_k$  and  $1 \leq j \leq N(T_k)$ , set

$$I_j(T_k) := \left\{ i \in I(T_{k-1}) : B_{R_i^{k-1}}(x_i^{k-1}) \subset B_{R_j^k}(x_j^k) \right\}.$$

By ii) and (3.13) it follows that for any  $1 \leq k \leq n$

$$(3.14) \quad \begin{aligned} \sum_{j=1}^{N(T_k)} (R_j^k - Nc) &\leq \sum_{j=1}^{N(T_k)} \sum_{l \in I_j(T_k)} R_l^{k-1}(T_k) \\ &= \sum_{j=1}^{N(T_{k-1})} \left( \frac{1 + T_k}{1 + T_{k-1}} R_j^{k-1} + \frac{1 + T_k}{1 + T_{k-1}} c \right) \\ &= \frac{1 + T_k}{1 + T_{k-1}} \sum_{j=1}^{N(T_{k-1})} R_j^{k-1} + \frac{1 + T_k}{1 + T_{k-1}} c N(T_{k-1}) \\ &\leq \frac{1 + T_k}{1 + T_{k-1}} \sum_{j=1}^{N(T_{k-1})} R_j^{k-1} + (1 + T_k) c N. \end{aligned}$$

Let  $T_k \leq t < T_{k+1}$  for some  $1 \leq k \leq n$ ; by (3.13) and iterating (3.14) we get

$$(3.15) \quad \begin{aligned} \sum_{j=1}^{N(T_k)} R_j^k(t) &= \frac{1 + t}{1 + T_k} \sum_{j=1}^{N(T_k)} R_j^k + \frac{1 + t}{1 + T_k} c N(T_k) \\ &\leq (1 + t) \sum_{i=1}^N R_i + (1 + t) c N(N^2 + N + 1), \end{aligned}$$

and this concludes the proof of (4).

It remains to prove (3). For  $t = 0$  it is trivially satisfied. We will show that it is preserved during the merging and the expansion times. Let  $T_k$  be a merging time

and assume that (3) holds for all  $t < T_k$ . Then for every  $j \in I(T_k)$

$$\begin{aligned} F(B_{R_j^k}(x_j^k)) &\geq \sum_{l \in I_j(T_k)} F(B_{R_l^{k-1}(T_k)}(x_l^{k-1})) \\ &\geq \pi \log(1 + T_k) \sum_{l=1}^j |\mu(B_{R_l^{k-1}(T_k)}(x_l^{k-1}))| \\ &\geq \pi \log(1 + T_k) |\mu(B_{R_j^k}(x_j^k))|. \end{aligned}$$

Finally, for a given  $t \in [T_k, T_{k+1})$  and for any ball  $B_{R_i^k(t)}(x_i^k(t)) \in \mathcal{B}(t)$  we have

$$\begin{aligned} F(B_{R_i^k(t)}(x_i^k)) &\geq F(B_{R_i^k(t)}(x_i^k) \setminus B_{R_i^k}(x_i^k)) + F(B_{R_i^k}(x_i^k)) \\ &\geq \pi |\mu(B_{R_i^k(t)}(x_i^k))| \log \frac{1+t}{1+T_k} + \pi |\mu(B_{R_i^k}(x_i^k))| \log(1+T_k) \\ &= \pi |\mu(B_{R_i^k(t)}(x_i^k))| \log(1+t), \end{aligned}$$

where we have used that  $\frac{R_i^k(t)}{c+R_i^k} = \frac{1+t}{1+T_k}$ .

**3.4. Proof of Theorem 3.1.** First, we give an elementary lower bound of the energy localized on a single square of the lattice, whose proof is immediate.

**Proposition 3.3.** *There exists a positive constant  $\beta$  such that for any  $\varepsilon > 0$ , for any function  $v \in \mathcal{AXY}_\varepsilon(\Omega)$  and for any  $i \in \Omega_\varepsilon^2$  such that the piecewise affine interpolation  $\tilde{v}$  of  $v$  satisfies  $\min_{i+\varepsilon Q} |\tilde{v}| < \frac{1}{2}$ , it holds  $XY_\varepsilon(v, i + \varepsilon Q) \geq \beta$ .*

*Proof of Theorem 3.1.* By (2.12), it is enough to prove (i) and (ii) for  $F_\varepsilon = XY_\varepsilon$ , using as a variable  $v_\varepsilon = e^{2\pi i u_\varepsilon}$ . The proof of (iii) is standard and left to the reader. *Proof of (i).* For every  $\varepsilon > 0$ , set  $I_\varepsilon := \{i \in \Omega_\varepsilon^2 : \min_{i+\varepsilon Q} |\tilde{v}_\varepsilon| \leq \frac{1}{2}\}$ . Notice that, in view of Remark 2.1,  $\mu(v_\varepsilon)$  is supported in  $I_\varepsilon + \frac{\varepsilon}{2}(e_1 + e_2)$ .

Starting from the family of balls  $B_{\frac{\sqrt{2}\varepsilon}{2}}(i + \frac{\varepsilon}{2}(e_1 + e_2))$ , and eventually passing through a merging procedure we can construct a family of pairwise disjoint balls

$$\mathcal{B}_\varepsilon := \{B_{R_{i,\varepsilon}}(x_{i,\varepsilon})\}_{i=1,\dots,N_\varepsilon},$$

with  $\sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq \varepsilon \sharp I_\varepsilon$ . Then, by Proposition 3.3 and by the energy bound, we immediately have that  $\sharp I_\varepsilon \leq C |\log \varepsilon|$  and hence

$$(3.16) \quad \sum_{i=1}^{N_\varepsilon} R_{i,\varepsilon} \leq \varepsilon C |\log \varepsilon|.$$

We define the sequence of measures

$$\mu_\varepsilon := \sum_{i=1}^{N_\varepsilon} \mu(v_\varepsilon)(B_{R_{i,\varepsilon}}(x_{i,\varepsilon})) \delta_{x_{i,\varepsilon}}.$$

Since  $|\mu_\varepsilon(B)| \leq \sharp I_\varepsilon$  for each ball  $B \in \mathcal{B}_\varepsilon$ , by (3.5) we deduce that (3.12) holds with  $F(\cdot) = XY_\varepsilon(v_\varepsilon, \cdot \setminus \cup_{B \in \mathcal{B}_\varepsilon} B)$  and  $c = \varepsilon(\alpha \sharp I_\varepsilon + 2\sqrt{2})$ .

We let the balls in the families  $\mathcal{B}_\varepsilon$  grow and merge as described in Subsection 3.3, and let  $\mathcal{B}_\varepsilon(t) := \{B_{R_{i,\varepsilon}(t)}(x_{i,\varepsilon}(t))\}$  be the corresponding family of balls at time  $t$ .

Set moreover  $t_\varepsilon := \frac{1}{\sqrt{\varepsilon}} - 1$ ,  $N_\varepsilon(t_\varepsilon) := \#\mathcal{B}_\varepsilon(t_\varepsilon)$  and define

$$(3.17) \quad \nu_\varepsilon := \sum_{\substack{i=1, \dots, N_\varepsilon(t_\varepsilon) \\ B_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon)) \subset \Omega}} \mu_\varepsilon(B_{R_{i,\varepsilon}(t_\varepsilon)}(x_{i,\varepsilon}(t_\varepsilon))) \delta_{x_{i,\varepsilon}(t_\varepsilon)}.$$

By (3) in Subsection 3.3, for any  $B \in \mathcal{B}_\varepsilon(t_\varepsilon)$ , with  $B \subseteq \Omega$ , we have

$$XY_\varepsilon(v_\varepsilon, B) \geq \pi |\mu_\varepsilon(B)| \log(1 + t_\varepsilon) = \pi |\nu_\varepsilon(B)|^{\frac{1}{2}} |\log \varepsilon|;$$

by the energy bound, we have immediately that  $|\nu_\varepsilon|(\Omega) \leq M$  and hence  $\{\nu_\varepsilon\}$  is precompact in the weak\* topology. By (4) in Subsection 3.3, it follows that

$$\sum_{j=1}^{N_\varepsilon(t_\varepsilon)} R_j(t_\varepsilon) \leq C \sqrt{\varepsilon} (\#\mathcal{I}_\varepsilon)^4,$$

which easily implies that  $\|\nu_\varepsilon - \mu_\varepsilon\|_{\text{flat}} \rightarrow 0$ ; moreover, using (3.16), it is easy to show that  $\|\mu_\varepsilon - \mu(v_\varepsilon)\|_{\text{flat}} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  (see [3] for more details). We conclude that also  $\mu(v_\varepsilon)$  is precompact in the flat topology.

*Proof of (ii).* Fix  $i \in \{1, \dots, M\}$ . Without loss of generality, and possibly extracting a subsequence, we can assume that

$$(3.18) \quad \liminf_{\varepsilon \rightarrow 0} XY_\varepsilon(v_\varepsilon, B_\sigma(x_i)) - \pi |d_i| |\log \varepsilon| \\ = \lim_{\varepsilon \rightarrow 0} XY_\varepsilon(v_\varepsilon, B_\sigma(x_i)) - \pi |d_i| |\log \varepsilon| < +\infty.$$

We consider the restriction  $\bar{v}_\varepsilon \in \mathcal{AX}\mathcal{Y}_\varepsilon(B_\sigma(x_i))$  of  $v_\varepsilon$  to  $B_\sigma(x_i)$ . Notice that  $\text{supp}(\mu(\bar{v}_\varepsilon) - \mu(v_\varepsilon) \llcorner B_\sigma(x_i)) \subseteq B_\sigma(x_i) \setminus B_{\sigma-\varepsilon}(x_i)$ . On the other hand, by (3.18) and Proposition 3.3 it follows that

$$(3.19) \quad |\mu(v_\varepsilon)|(B_\sigma(x_i) \setminus B_{\sigma-\varepsilon}(x_i)) \leq C |\log \varepsilon|.$$

Then, using (3.19) one can easily get

$$(3.20) \quad \|\mu(\bar{v}_\varepsilon) - \mu(v_\varepsilon) \llcorner B_\sigma(x_i)\|_{\text{flat}} \rightarrow 0,$$

and hence

$$(3.21) \quad \|\mu(\bar{v}_\varepsilon) - d_i \delta_{x_i}\|_{\text{flat}} \rightarrow 0.$$

We repeat the ball construction procedure used in the proof of (i) with  $\Omega$  replaced by  $B_\sigma(x_i)$ ,  $v_\varepsilon$  by  $\bar{v}_\varepsilon$  and  $I_\varepsilon$  by

$$I_{i,\varepsilon} := \left\{ j \in (B_\sigma(x_i))_\varepsilon^2 : \min_{j+\varepsilon Q} |\bar{v}_\varepsilon| \leq \frac{1}{2} \right\}.$$

We denote by  $\mathcal{B}_{i,\varepsilon}$  the corresponding family of balls and by  $\mathcal{B}_{i,\varepsilon}(t)$  the family of balls constructed at time  $t$ .

Fix  $0 < \gamma < 1$  such that

$$(3.22) \quad (1 - \gamma)(|d_i| + 1) > |d_i|.$$

Let  $t_{\varepsilon,\gamma} = \varepsilon^{\gamma-1} - 1$  and let  $\nu_{\varepsilon,\gamma}$  be the measure defined as in (3.17) with  $\Omega$  replaced by  $B_\sigma(x_i)$  and  $t_\varepsilon$  replaced by  $t_{\varepsilon,\gamma}$ . As in the previous step, since  $\gamma > 0$  we deduce that  $\|\nu_{\varepsilon,\gamma} - d_i \delta_{x_i}\|_{\text{flat}} \rightarrow 0$ ; moreover, for any  $B \in \mathcal{B}_{i,\varepsilon}(t_{\varepsilon,\gamma})$  we have

$$(3.23) \quad XY_\varepsilon(v_\varepsilon, B) \geq \pi |\nu_{\varepsilon,\gamma}(B)| (1 - \gamma) |\log \varepsilon|.$$

Now, if  $\liminf_{\varepsilon \rightarrow 0} |\nu_{\varepsilon,\gamma}|(B_\sigma(x_i)) > |d_i|$ , then, thanks to (3.22), (3.1) holds true. Otherwise we can assume that  $|\nu_{\varepsilon,\gamma}|(B_\sigma(x_i)) = |d_i|$  for  $\varepsilon$  small enough. Then

$\nu_{\varepsilon,\gamma}$  is a sum of Dirac masses concentrated on points which converge to  $x_i$ , with weights all having the same sign and summing to  $d_i$ . Let  $C_1 > 0$  be given and set  $\bar{t}_\varepsilon := \frac{\sigma}{C_1(\#\mathcal{I}_{i,\varepsilon})^4\varepsilon} - 1$ . By (3.15), we have that any ball  $B \in \mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon)$  satisfies

$$\text{diam}(B) \leq \frac{C_2}{C_1}\sigma,$$

where  $C_2 > 0$  is a universal constant. We fix  $C_1 > 2C_2$  so that  $\text{diam}(B) < \frac{\sigma}{2}$ . Recall that, for  $\varepsilon$  small enough,  $\text{supp}(\nu_{\varepsilon,\gamma}) \subseteq B_{\sigma/2}(x_i)$ ; hence if  $B \in \mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon)$  with  $\text{supp}(\nu_{\varepsilon,\gamma}) \cap B \neq \emptyset$ , then  $B \subseteq B_\sigma(x_i)$  and one can easily show that

$$\mu(\bar{\nu}_\varepsilon)\left(\bigcup_{\substack{B \in \mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon) \\ B \subset B_\sigma(x_i)}} B\right) = d_i.$$

We have immediately that

$$XY_\varepsilon(\bar{\nu}_\varepsilon, B_\sigma(x_i) \setminus \bigcup_{\substack{B \in \mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon) \\ B \subset B_\sigma(x_i)}} B) \geq \pi \sum_{\substack{B \in \mathcal{B}_{i,\varepsilon}(\bar{t}_\varepsilon) \\ B \subset B_\sigma(x_i)}} |\mu(\bar{\nu}_\varepsilon)(B)| \log(1 + \bar{t}_\varepsilon) \geq \pi|d_i| \log \frac{\sigma}{C_1(\#\mathcal{I}_{i,\varepsilon})^4\varepsilon}.$$

On the other hand, by Proposition 3.3 there exists a positive constant  $\beta$  such that

$$XY_\varepsilon(\bar{\nu}_\varepsilon, j + \varepsilon Q) \geq \beta \quad \text{for any } j \in \mathcal{I}_{i,\varepsilon};$$

therefore,  $XY_\varepsilon(\bar{\nu}_\varepsilon, \bigcup_{B \in \mathcal{B}_{i,\varepsilon}} B) \geq \beta \#\mathcal{I}_{i,\varepsilon}$ . Finally, we get

$$\begin{aligned} XY_\varepsilon(\bar{\nu}_\varepsilon, B_\sigma(x_i)) &\geq XY_\varepsilon(\bar{\nu}_\varepsilon, B_\sigma(x_i) \setminus \bigcup_{B \in \mathcal{B}_{i,\varepsilon}} B) + XY_\varepsilon(\bar{\nu}_\varepsilon, \bigcup_{B \in \mathcal{B}_{i,\varepsilon}} B) \\ &\geq \pi|d_i| \log \frac{\sigma}{\varepsilon} - \log(C_1(\#\mathcal{I}_{i,\varepsilon})^4) + \#\mathcal{I}_{i,\varepsilon}\beta \geq \pi|d_i| \log \frac{\sigma}{\varepsilon} + C \end{aligned}$$

and (3.1) follows sending  $\varepsilon \rightarrow 0$ .  $\square$

#### 4. THE RENORMALIZED ENERGY AND THE FIRST ORDER $\Gamma$ -CONVERGENCE.

In this section we will prove the first order  $\Gamma$ -convergence of  $F_\varepsilon$  to the renormalized energy, introduced in the continuous framework of Ginzburg-Landau energies in [10]. To this purpose we begin by recalling the many definitions and results of [10] we need.

**4.1. Revisiting the analysis of Bethuel-Brezis-Hélein.** Fix  $\mu = \sum_{i=1}^M d_i \delta_{x_i}$  with  $d_i \in \{-1, +1\}$  and  $x_i \in \Omega$ . In order to define the renormalized energy, consider the following problem

$$\begin{cases} \Delta \Phi = 2\pi\mu & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and let  $R_0(x) = \Phi(x) - \sum_{i=1}^M d_i \log|x - x_i|$ . Notice that  $R_0$  is harmonic in  $\Omega$  and  $R_0(x) = -\sum_{i=1}^M d_i \log|x - x_i|$  for any  $x \in \partial\Omega$ . The *renormalized energy* corresponding to the configuration  $\mu$  is then defined by

$$(4.1) \quad \mathbb{W}(\mu) := -\pi \sum_{i \neq j} d_i d_j \log|x_i - x_j| - \pi \sum_i d_i R_0(x_i).$$

Let  $\sigma > 0$  be such that the balls  $B_\sigma(x_i)$  are pairwise disjoint and contained in  $\Omega$  and set  $\Omega^\sigma := \Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i)$ . A straightforward computation shows that

$$(4.2) \quad \mathbb{W}(\mu) = \lim_{\sigma \rightarrow 0} \frac{1}{2} \int_{\Omega^\sigma} |\nabla \Phi|^2 dx - M\pi |\log \sigma|,$$



In this respect the renormalized energy represents the finite energy induced by  $\mu$  once the leading logarithmic term has been removed.

It is convenient to consider (as done in [10]) suitable cell problems and auxiliary minimum problems. Set

$$(4.3) \quad \begin{aligned} m(\sigma, \mu) &:= \min_{w \in H^1(\Omega^\sigma; \mathcal{S}^1)} \left\{ \frac{1}{2} \int_{\Omega^\sigma} |\nabla w|^2 dx : \deg(w, \partial B_\sigma(x_i)) = d_i \right\}, \\ \tilde{m}(\sigma, \mu) &:= \min_{w \in H^1(\Omega^\sigma; \mathcal{S}^1)} \left\{ \frac{1}{2} \int_{\Omega^\sigma} |\nabla w|^2 dx : \right. \\ &\quad \left. w(\cdot) = \frac{\alpha_i}{\sigma^{d_i}} (\cdot - x_i)^{d_i} \text{ on } \partial B_\sigma(x_i), |\alpha_i| = 1 \right\}. \end{aligned}$$

For any  $x \in \mathbb{R}^2 \setminus \{0\}$ , we define  $\theta(x)$  as the polar coordinate  $\arctan x_2/x_1$ , also referred to as the lifting of the function  $\frac{x}{|x|}$ . Given  $\varepsilon > 0$  we introduce a discrete minimization problem in the ball  $B_\sigma$

$$(4.4) \quad \gamma(\varepsilon, \sigma) := \min_{u \in \mathcal{AF}_\varepsilon(B_\sigma)} \{F_\varepsilon(u, B_\sigma) : 2\pi u(\cdot) = \theta(\cdot) \text{ on } \partial_\varepsilon B_\sigma\},$$

where the discrete boundary  $\partial_\varepsilon$  is defined in (2.1).

**Theorem 4.1.** *It holds*

$$(4.5) \quad \lim_{\sigma \rightarrow 0} m(\sigma, \mu) - \pi|\mu|(\Omega)|\log \sigma| = \lim_{\sigma \rightarrow 0} \tilde{m}(\sigma, \mu) - \pi|\mu|(\Omega)|\log \sigma| = \mathbb{W}(\mu).$$

Moreover, for any fixed  $\sigma > 0$ , the following limit exists finite

$$(4.6) \quad \lim_{\varepsilon \rightarrow 0} (\gamma(\varepsilon, \sigma) - \pi|\log \frac{\varepsilon}{\sigma}|) =: \gamma \in \mathbb{R}.$$

The proof of (4.5) is contained in [10], whereas the statement in (4.6) is a discrete version of Lemma III.1 in [10] and can be proved similarly. We give the details of the proof of (4.6) for the convenience of the reader.

*Proof of (4.6).* First, by scaling, it is easy to see that  $\gamma(\varepsilon, \sigma) = I(\frac{\varepsilon}{\sigma})$  where  $I(t)$  is defined by

$$I(t) := \min \left\{ F_1(\theta, B_{\frac{1}{t}}) \mid 2\pi u = \theta \text{ on } \partial_\varepsilon B_{\frac{1}{t}} \right\}.$$

We aim to prove that

$$(4.7) \quad 0 < t_1 \leq t_2 \Rightarrow I(t_1) \leq \pi \log \frac{t_2}{t_1} + I(t_2) + O(t_2).$$

Notice that by (4.7) it easily follows that  $\lim_{t \rightarrow 0^+} (I(t) - \pi|\log t|)$  exists and is not  $+\infty$ . Moreover, by Theorem 3.1, there exists a universal constant  $C$  such that

$$I(t) \geq \pi|\log t| + C \quad \forall t \in (0, 1].$$

We conclude that  $\lim_{t \rightarrow 0^+} (I(t) - \pi|\log t|)$  is not  $-\infty$ .

In order to complete the proof we have to show that (4.7) holds. To this end, set  $A_{r,R} = B_R \setminus B_r$ , and let  $\theta$  be the lifting of the function  $\frac{x}{|x|}$ . Since  $|\nabla \theta(x)| \leq c/r$  for every  $x \in A_{r,R}$ , by standard interpolation estimates (see for instance [19]) and using assumption (3) on  $f$ , we have that, as  $r < R \rightarrow \infty$ ,

$$(4.8) \quad F_1(\theta/2\pi, A_{r,R}) \leq \pi \log \frac{R}{r} + O(1/r).$$

Let  $u_2$  be a minimizer for  $I(t_2)$  and for any  $i \in \mathbb{Z}^2$  define

$$u_1(i) := \begin{cases} u_2(i) & \text{if } |i| \leq \frac{1}{t_2} \\ \frac{\theta(i)}{2\pi} & \text{if } \frac{1}{t_2} \leq |i| \leq \frac{1}{t_1}, \end{cases}$$

By (4.8) we have

$$\begin{aligned} I(1/R) &\leq \sum_{\substack{(i,j) \in (B_r)_1^1 \\ i,j \in (B_r)_1}} f(u_1(i) - u_1(j)) + \sum_{\substack{(i,j) \in (A_{r-\sqrt{2},R})_1^1 \\ i,j \in (A_{r-\sqrt{2},R})_1}} f(u_1(i) - u_1(j)) \\ &\leq I(1/r) + \pi \log \frac{r}{R} + O(1/r), \end{aligned}$$

which yields (4.7) for  $r = \frac{1}{t_2}$  and  $R = \frac{1}{t_1}$ .  $\square$

**4.2. The main  $\Gamma$ -convergence result.** We are now in a position to state the first-order  $\Gamma$ -convergence theorem for the functionals  $F_\varepsilon$ .

**Theorem 4.2.** *The following  $\Gamma$ -convergence result holds.*

- (i) (*Compactness*) Let  $M \in \mathbb{N}$  and let  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  be a sequence satisfying  $F_\varepsilon(u_\varepsilon) - M\pi|\log \varepsilon| \leq C$ . Then, up to a subsequence,  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu$  for some  $\mu = \sum_{i=1}^N d_i \delta_{x_i}$  with  $d_i \in \mathbb{Z} \setminus \{0\}$ ,  $x_i \in \Omega$  and  $\sum_i |d_i| \leq M$ . Moreover, if  $\sum_i |d_i| = M$ , then  $\sum_i |d_i| = N = M$ , namely  $|d_i| = 1$  for any  $i$ .
- (ii) ( $\Gamma$ -lim inf inequality) Let  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  be such that  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu$ , with  $\mu = \sum_{i=1}^M d_i \delta_{x_i}$  with  $|d_i| = 1$  and  $x_i \in \Omega$  for every  $i$ . Then,

$$(4.9) \quad \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) - M\pi|\log \varepsilon| \geq \mathbb{W}(\mu) + M\gamma.$$

- (iii) ( $\Gamma$ -lim sup inequality) Given  $\mu = \sum_{i=1}^M d_i \delta_{x_i}$  with  $|d_i| = 1$  and  $x_i \in \Omega$  for every  $i$ , there exists  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  with  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu$  such that

$$F_\varepsilon(u_\varepsilon) - M\pi|\log \varepsilon| \rightarrow \mathbb{W}(\mu) + M\gamma.$$

In our analysis it will be convenient to introduce the energy functionals  $F_\varepsilon$  in term of the variable  $\mu$ , i.e., by minimizing  $F_\varepsilon$  with respect to all  $u \in \mathcal{AF}_\varepsilon(\Omega)$  with  $\mu(u) = \mu$ . Precisely, let  $\mathcal{F}_\varepsilon : X \rightarrow [0, +\infty]$  be defined by

$$(4.10) \quad \mathcal{F}_\varepsilon(\mu) := \inf \{F_\varepsilon(u) : u \in \mathcal{AF}_\varepsilon(\Omega), \mu(u) = \mu\}.$$

Theorem 4.2 can be rewritten in terms of  $\mathcal{F}_\varepsilon$  as follows.

**Theorem 4.3.** *The following  $\Gamma$ -convergence result holds.*

- (i) (*Compactness*) Let  $M \in \mathbb{N}$  and let  $\{\mu_\varepsilon\} \subset X$  be a sequence satisfying  $\mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log \varepsilon| \leq C$ . Then, up to a subsequence,  $\mu_\varepsilon \xrightarrow{\text{flat}} \mu = \sum_{i=1}^N d_i \delta_{x_i}$  with  $d_i \in \mathbb{Z} \setminus \{0\}$ ,  $x_i \in \Omega$  and  $\sum_i |d_i| \leq M$ . Moreover, if  $\sum_i |d_i| = M$ , then  $\sum_i |d_i| = N = M$ , namely  $|d_i| = 1$  for every  $i$ .
- (ii) ( $\Gamma$ -lim inf inequality) Let  $\{\mu_\varepsilon\} \subset X$  be such that  $\mu_\varepsilon \xrightarrow{\text{flat}} \mu = \sum_{i=1}^M d_i \delta_{x_i}$  with  $|d_i| = 1$  and  $x_i \in \Omega$  for every  $i$ . Then,

$$(4.11) \quad \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi|\log \varepsilon| \geq \mathbb{W}(\mu) + M\gamma.$$

(iii) ( $\Gamma$ -lim sup inequality) Given  $\mu = \sum_{i=1}^M d_i \delta_{x_i}$  with  $|d_i| = 1$  and  $x_i \in \Omega$  for every  $i$ , there exists  $\{\mu_\varepsilon\} \subset X$  with  $\mu_\varepsilon \xrightarrow{\text{flat}} \mu$  such that

$$(4.12) \quad \mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi |\log \varepsilon| \rightarrow \mathbb{W}(\mu) + M\gamma.$$

**4.3. The proof of Theorem 4.2.** Recalling that  $F_\varepsilon(u) \geq XY_\varepsilon(e^{2\pi i u})$ , the proof of the compactness property (i) will be done for  $F_\varepsilon = XY_\varepsilon$ , and will be deduced by Theorem 3.1. On the other hand, the constant  $\gamma$  in the definition of the  $\Gamma$ -limit depends on the details of the discrete energy  $F_\varepsilon$ , and its derivation requires a specific proof.

*Proof of (i): Compactness.* The fact that, up to a subsequence,  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu = \sum_{i=1}^N d_i \delta_{x_i}$  with  $\sum_{i=1}^N |d_i| \leq M$  is a direct consequence of the zero order  $\Gamma$ -convergence result stated in Theorem 3.1 (i). Assume now  $\sum_{i=1}^N |d_i| = M$  and let us prove that  $|d_i| = 1$ . Let  $0 < \sigma_1 < \sigma_2$  be such that  $B_{\sigma_2}(x_i)$  are pairwise disjoint and contained in  $\Omega$  and let  $\varepsilon$  be small enough so that  $B_{\sigma_2}(x_i)$  are contained in  $\Omega_\varepsilon$ . For any  $0 < r < R$  and  $x \in \mathbb{R}^2$ , set  $A_{r,R}(x) := B_R(x) \setminus B_r(x)$ . Since  $F_\varepsilon(u_\varepsilon) \geq XY_\varepsilon(e^{2\pi i u_\varepsilon})$ ,

$$(4.13) \quad F_\varepsilon(u_\varepsilon) \geq \sum_{i=1}^N XY_\varepsilon(e^{2\pi i u_\varepsilon}, B_{\sigma_1}(x_i)) + \sum_{i=1}^N XY_\varepsilon(e^{2\pi i u_\varepsilon}, A_{\sigma_1, \sigma_2}(x_i)).$$

To ease notation we set  $v_\varepsilon = e^{2\pi i u_\varepsilon}$  and we indicate with  $\tilde{v}_\varepsilon$  the piecewise affine interpolation of  $v_\varepsilon$ . Moreover let  $t$  be a positive number and let  $\varepsilon$  be small enough so that  $t > \sqrt{2}\varepsilon$ . Then, by (3.1) and (2.13), we get

$$(4.14) \quad F_\varepsilon(u_\varepsilon) \geq \pi \sum_{i=1}^N |d_i| \log \frac{\sigma_1}{\varepsilon} + \frac{1}{2} \sum_{i=1}^N \int_{A_{\sigma_1+t, \sigma_2-t}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx + C.$$

By the energy bound, we deduce that  $\int_{A_{\sigma_1+t, \sigma_2-t}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx \leq C$  and hence, up to a subsequence,  $\tilde{v}_\varepsilon \rightharpoonup v_i$  in  $H^1(A_{\sigma_1+t, \sigma_2-t}(x_i); \mathbb{R}^2)$  for some field  $v_i$ . Moreover, since

$$\frac{1}{\varepsilon^2} \int_{A_{\sigma_1+t, \sigma_2-t}(x_i)} (1 - |\tilde{v}_\varepsilon|^2)^2 dx \leq CXY_\varepsilon(v_\varepsilon) \leq C \log \frac{1}{\varepsilon},$$

(see Lemma 2 in [1] for more details), we deduce that  $|v_i| = 1$  a.e.

Furthermore, by standard Fubini's arguments, for a.e.  $\sigma_1 + t < \sigma < \sigma_2 - t$ , up to a subsequence the trace of  $\tilde{v}_\varepsilon$  is bounded in  $H^1(\partial B_\sigma(x_i); \mathbb{R}^2)$ , and hence it converges uniformly to the trace of  $v_i$ . By the very definition of degree it follows that  $\deg(v_i, \partial B_\sigma(x_i)) = d_i$ .

Hence, by (3.4), for every  $i$  we have

$$(4.15) \quad \frac{1}{2} \int_{A_{\sigma_1+t, \sigma_2-t}(x_i)} |\nabla v_i|^2 dx \geq |d_i|^2 \pi \log \frac{\sigma_2 - t}{\sigma_1 + t}.$$

By (4.14) and (4.15), we conclude that for  $\varepsilon$  sufficiently small

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\geq \pi \sum_{i=1}^N \left( |d_i| \log \frac{\sigma_1}{\varepsilon} + |d_i|^2 \log \frac{\sigma_2 - t}{\sigma_1 + t} \right) + C \\ &\geq M\pi |\log \varepsilon| + \pi \sum_{i=1}^N (|d_i|^2 - |d_i|) \log \frac{\sigma_2}{\sigma_1} + \pi \sum_{i=1}^N |d_i|^2 \log \frac{\sigma_1(\sigma_2 - t)}{\sigma_2(\sigma_1 + t)} + C. \end{aligned}$$

The energy bound yields that the sum of the last two terms is bounded; letting  $t \rightarrow 0$  and  $\sigma_1 \rightarrow 0$ , we conclude  $|d_i| = 1$ .

*Proof of (ii):  $\Gamma$ -liminf inequality.* Fix  $r > 0$  so that the balls  $B_r(x_i)$  are pairwise disjoint and compactly contained in  $\Omega$ . Let moreover  $\{\Omega^h\}$  be an increasing sequence of open smooth sets compactly contained in  $\Omega$  such that  $\cup_{h \in \mathbb{N}} \Omega^h = \Omega$ . Without loss of generality we can assume that  $F_\varepsilon(u_\varepsilon) \leq M\pi|\log \varepsilon| + C$ , which together with Theorem 3.1 yields

$$(4.16) \quad F_\varepsilon(u_\varepsilon, \Omega \setminus \bigcup_{i=1}^M B_r(x_i)) \leq C.$$

We set  $v_\varepsilon := e^{2\pi i u_\varepsilon}$  and we denote by  $\tilde{v}_\varepsilon$  the piecewise affine interpolation of  $v_\varepsilon$ . For every  $r > 0$ , by (4.16) and by (2.12) we deduce  $XY_\varepsilon(v_\varepsilon \setminus \bigcup_{i=1}^N B_r(x_i)) \leq C$ . Fix  $h \in \mathbb{N}$  and let  $\varepsilon$  be small enough so that  $\Omega^h \subset \Omega_\varepsilon$ . Then,

$$\frac{1}{2} \int_{\Omega^h \setminus \bigcup_{i=1}^N B_{2r}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx \leq C;$$

therefore, by a diagonalization argument, there exists a unitary field  $v$  with  $v \in H^1(\Omega \setminus \bigcup_{i=1}^M B_\rho(x_i); \mathcal{S}^1)$  for any  $\rho > 0$  and a subsequence  $\{\tilde{v}_\varepsilon\}$  such that  $\tilde{v}_\varepsilon \rightarrow v$  in  $H_{\text{loc}}^1(\Omega \setminus \bigcup_{i=1}^M \{x_i\}; \mathbb{R}^2)$ .

Let  $\sigma > 0$  be such that  $B_\sigma(x_i)$  are pairwise disjoint and contained in  $\Omega^h$ . For any  $0 < r < R < +\infty$  and for any  $x \in \mathbb{R}^2$ , set  $A_{r,R}(x) := B_R(x) \setminus B_r(x)$ ,  $A_{r,R} := A_{r,R}(0)$ . Let  $t \leq \sigma$ , and consider the minimization problem

$$\min_{w \in H^1(A_{t/2,t}; \mathcal{S}^1)} \left\{ \frac{1}{2} \int_{A_{t/2,t}} |\nabla w|^2 dx : \deg(w, \partial B_{\frac{t}{2}}) = 1 \right\}.$$

It is easy to see that the minimum is  $\pi \log 2$  and that the set of minimizers is given by (the restriction at  $A_{t/2,t}$  of the functions in)

$$(4.17) \quad \mathcal{K} := \left\{ \alpha \frac{z}{|z|} : \alpha \in \mathbb{C}, |\alpha| = 1 \right\}.$$

Set

$$(4.18) \quad d_t(w, \mathcal{K}) := \min \left\{ \|\nabla w - \nabla v\|_{L^2(A_{t/2,t}; \mathbb{R}^2)} : v \in \mathcal{K} \right\}.$$

It is easy to see that for any given  $\delta > 0$  there exists a positive  $\omega(\delta)$  (independent of  $t$ ) such that if  $d_t(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K}) \geq \delta$ , then

$$(4.19) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{A_{\frac{t}{2} + \sqrt{2}\varepsilon, t - \sqrt{2}\varepsilon}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx \geq \pi \log 2 + \omega(\delta).$$

By a scaling argument we can assume  $t = 1$ . Then, arguing by contradiction, if there exists a subsequence  $\{\tilde{v}_\varepsilon\}$  such that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{A_{\frac{1}{2} + \sqrt{2}\varepsilon, 1 - \sqrt{2}\varepsilon}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx = \pi \log 2,$$

then, by the lower semicontinuity of the  $L^2$  norm, we get

$$(4.20) \quad \pi \log 2 \leq \frac{1}{2} \int_{A_{1/2,1}(x_i)} |\nabla v|^2 dx \leq \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{A_{\frac{1}{2} + \sqrt{2}\varepsilon, 1 - \sqrt{2}\varepsilon}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx = \pi \log 2.$$

It follows that  $v(\cdot + x_i) \in \mathcal{K}$ , and that  $\tilde{v}_\varepsilon \rightarrow v$  strongly in  $H^1(A_{1/2,1}(x_i); \mathbb{R}^2)$ , which yields the contradiction  $\text{dist}(v(\cdot + x_i), \mathcal{K}) \geq \delta$ .

Let  $L \in \mathbb{N}$  be such that  $L\omega(\delta) \geq \mathbb{W}(\mu) + M(\gamma - \pi \log \sigma - C)$  where  $C$  is the constant in (3.1). For  $l = 1, \dots, L$ , set  $C_l(x_i) := B_{2^{l-1}\sigma}(x_i) \setminus B_{2^{l-2}\sigma}(x_i)$ .

We distinguish among two cases.

*First case:* for  $\varepsilon$  small enough and for every fixed  $1 \leq l \leq L$ , there exists at least one  $i$  such that  $d_{2^{l-1}\sigma}(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K}) \geq \delta$ . Then, by (3.1), (4.19) and the lower semicontinuity of the  $L^2$  norm, we conclude

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \Omega^h) &\geq \sum_{i=1}^M XY_\varepsilon(v_\varepsilon, B_{2^{-L}\sigma}(x_i)) + \sum_{l=1}^L \sum_{i=1}^M XY_\varepsilon(v_\varepsilon, C_l(x_i)) \\ &\geq M(\pi \log \frac{\sigma}{2L} + \pi |\log \varepsilon| + C) + L(M\pi \log 2 + \omega(\delta)) + o(\varepsilon) \\ &\geq M\pi |\log \varepsilon| + M\gamma + \mathbb{W}(\mu) + o(\varepsilon). \end{aligned}$$

*Second case:* Up to a subsequence, there exists  $1 \leq \bar{l} \leq L$  such that for every  $i$  we have  $d_{\bar{\sigma}}(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K}) \leq \delta$ , where  $\bar{\sigma} := 2^{1-\bar{l}}\sigma$ . Let  $\alpha_{\varepsilon,i}$  be the unitary vector such that  $\|\tilde{v}_\varepsilon - \alpha_{\varepsilon,i} \frac{x-x_i}{|x-x_i|}\|_{H^1(C_{\bar{l}}(x_i); \mathbb{R}^2)} = d_{\bar{\sigma}}(\tilde{v}_\varepsilon(\cdot + x_i), \mathcal{K})$ .

One can construct a function  $\bar{u}_\varepsilon \in \mathcal{AF}_\varepsilon(\Omega)$  such that

- (i)  $\bar{u}_\varepsilon = u_\varepsilon$  on  $\partial_\varepsilon(\mathbb{R}^2 \setminus B_{2^{-\bar{l}}\sigma}(x_i))$ ;
- (ii)  $e^{2\pi i \bar{u}_\varepsilon} = \alpha_{\varepsilon,i} e^{i\theta}$  on  $\partial_\varepsilon B_{2^{1-\bar{l}}\sigma}(x_i)$
- (iii)  $F_\varepsilon(u_\varepsilon, B_\sigma(x_i)) \geq F_\varepsilon(\bar{u}_\varepsilon, B_{\bar{\sigma}}(x_i)) + r(\varepsilon, \delta)$  with  $\lim_{\delta \rightarrow 0} \lim_{\varepsilon \rightarrow 0} r(\varepsilon, \delta) = 0$ .

The proof of (i)-(iii) is quite technical, and consists in adapting standard cut-off arguments to our discrete setting. For the reader convenience we skip the details of the proof, and assuming (i)-(iii) we conclude the proof of the lower bound.

By Theorem (4.1), we have that

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\geq XY_\varepsilon(v_\varepsilon, \Omega^h \setminus \bigcup_{i=1}^M B_{\bar{\sigma}}(x_i)) + \sum_{i=1}^M F_\varepsilon(u_\varepsilon, B_{\bar{\sigma}}(x_i)) \\ &\geq \frac{1}{2} \int_{\Omega^h \setminus \bigcup_{i=1}^M B_{\bar{\sigma}}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx + \sum_{i=1}^M F_\varepsilon(\bar{u}_\varepsilon, B_{\bar{\sigma}}(x_i)) + r(\varepsilon, \delta) + o(\varepsilon) \\ &\geq \frac{1}{2} \int_{\Omega^h \setminus \bigcup_{i=1}^M B_{\bar{\sigma}}(x_i)} |\nabla \tilde{v}_\varepsilon|^2 dx + M(\gamma - \pi \log \frac{\varepsilon}{\bar{\sigma}}) + r(\varepsilon, \delta) + o(\varepsilon) \\ &\geq \frac{1}{2} \int_{\Omega \setminus \bigcup_{i=1}^M B_{\bar{\sigma}}(x_i)} |\nabla v|^2 dx + M(\gamma - \pi \log \frac{\varepsilon}{\bar{\sigma}}) + r(\varepsilon, \delta) + o(\varepsilon) + o(1/h) \\ &\geq M\pi |\log \varepsilon| + M\gamma + \mathbb{W}(\mu) + r(\varepsilon, \delta) + o(\varepsilon) + o(\bar{\sigma}) + o(1/h). \end{aligned}$$

The proof follows sending  $\varepsilon \rightarrow 0$ ,  $\delta \rightarrow 0$ ,  $\sigma \rightarrow 0$  and  $h \rightarrow \infty$ .

*Proof of (iii):  $\Gamma$ -limsup inequality.* This proof is standard in the continuous case, and we only sketch its discrete counterpart. Let  $w_\sigma$  be a function that agrees with a minimizer of (4.3) in  $\Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i) =: \Omega^\sigma$ . Then,  $w_\sigma = \alpha_i \frac{x-x_i}{\sigma}$  on  $\partial B_\sigma(x_i)$  for some  $|\alpha_i| = 1$ .

For every  $\rho > 0$  we can always find a function  $w_{\sigma,\rho} \in C^\infty(\bar{\Omega}^\sigma; \mathcal{S}^1)$  such that  $w_{\sigma,\rho} = \alpha_i \frac{x-x_i}{\sigma}$  on  $\partial B_\sigma(x_i)$ , and

$$\frac{1}{2} \int_{\Omega^\sigma} |\nabla w_{\sigma,\rho}|^2 dx - \frac{1}{2} \int_{\Omega^\sigma} |\nabla w_\sigma|^2 dx \leq \rho.$$

Moreover, for every  $i$  let  $w_i \in \mathcal{AXY}_\varepsilon(B_\sigma(x_i))$  be a function which agrees with  $\alpha_i \frac{x-x_i}{|x-x_i|}$  on  $\partial_\varepsilon B_\sigma(x_i)$  and such that its phase minimizes problem (4.4). If necessary, we extend  $w_i$  to  $(\overline{B_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2) \setminus (B_\sigma(x_i))_\varepsilon^0$  to be equal to  $\alpha_i \frac{x-x_i}{|x-x_i|}$ . Finally, define the function  $w_{\varepsilon,\sigma,\rho} \in \mathcal{AXY}_\varepsilon(\Omega)$  which coincides  $w_{\sigma,\rho}$  on  $\Omega^\sigma \cap \varepsilon\mathbb{Z}^2$  and with  $w_i$  on  $\overline{B_\sigma(x_i)} \cap \varepsilon\mathbb{Z}^2$ . Then, in view of assumption (3) on  $f$ , a straightforward computation shows that any phase  $u_{\varepsilon,\sigma,\rho}$  of  $w_{\varepsilon,\sigma,\rho}$  is a recovery sequence, i.e.,

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_{\varepsilon,\sigma,\rho}) - M\pi |\log \varepsilon| = M\gamma + \mathbb{W}(\mu) + o(\rho, \sigma),$$

with  $\lim_{\sigma \rightarrow 0} \lim_{\rho \rightarrow 0} o(\rho, \sigma) = 0$ .

**4.4.  $\Gamma$ -convergence analysis in the  $L^2$  topology.** Here we prove a  $\Gamma$ -convergence result for  $F_\varepsilon(u_\varepsilon) - M\pi |\log \varepsilon|$ , where  $M$  is fixed positive integer, with respect to the flat convergence of  $\mu(u_\varepsilon)$  and the  $L^2$ -convergence of  $\tilde{v}_\varepsilon$ , where  $\tilde{v}_\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{R}^2$  is the piecewise affine interpolation of  $e^{2\pi i u_\varepsilon}$ .

To this purpose, for  $N \in \mathbb{N}$  let us first introduce the set

$$(4.21) \quad \mathcal{D}_N := \left\{ v \in L^2(\Omega; \mathcal{S}^1) : Jv = \pi \sum_{i=1}^N d_i \delta_{x_i} \text{ with } |d_i| = 1, x_i \in \Omega, \right. \\ \left. v \in H_{\text{loc}}^1(\Omega \setminus \text{supp}(Jv); \mathcal{S}^1) \right\}.$$

Notice that, if  $v \in \mathcal{D}_M$ , then the function

$$\frac{1}{2} \int_{\Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i)} |\nabla v|^2 dx - M\pi |\log \sigma|,$$

is monotonically decreasing with respect to  $\sigma$ . Therefore, it is well defined the functional  $\mathcal{W} : L^2(\Omega; \mathcal{S}^1) \rightarrow \overline{\mathbb{R}}$  given by

$$(4.22) \quad \mathcal{W}(v) = \begin{cases} \lim_{\sigma \rightarrow 0} \frac{1}{2} \int_{\Omega \setminus \bigcup_{i=1}^M B_\sigma(x_i)} |\nabla v|^2 dx - M\pi |\log \sigma| & \text{if } v \in \mathcal{D}_M; \\ -\infty & \text{if } v \in \mathcal{D}_N \text{ for some } N < M; \\ +\infty & \text{otherwise} \end{cases}$$

Notice that, by (4.5) we have that, for every  $\mu = \sum_{i=1}^M d_i \delta_{x_i}$  with  $|d_i| = 1$

$$(4.23) \quad \mathbb{W}(\mu) = \min_{\substack{v \in H_{\text{loc}}^1(\Omega \setminus \text{supp}(\mu); \mathcal{S}^1) \\ Jv = \mu}} \mathcal{W}(v).$$

**Remark 4.4.** We can rewrite  $\mathcal{W}(v)$  as follows

$$\mathcal{W}(v) = \frac{1}{2} \int_{\Omega \setminus \bigcup_i B_\rho(x_i)} |\nabla v|^2 dx + M\pi \log \rho + \sum_{i=1}^M \sum_{j=0}^{+\infty} \left( \frac{1}{2} \int_{C_{i,j}} |\nabla v|^2 dx - \pi \log 2 \right),$$

where  $C_{i,j}$  denotes the annulus  $B_{2^{-j}\rho}(x_i) \setminus B_{2^{-(j+1)}\rho}(x_i)$ . In particular, for the lower bound (3.4) we deduce that

$$(4.24) \quad \sup_{i,j} \frac{1}{2} \int_{C_{i,j}} |\nabla v|^2 dx \leq \pi \log 2 + \mathcal{W}(v) - M\pi \log \rho.$$

**Theorem 4.5.** *Let  $M \in \mathbb{N}$  be fixed. The following  $\Gamma$ -convergence result holds.*

(i) (Compactness) *Let  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  be such that  $F_\varepsilon(u_\varepsilon) \leq M\pi|\log \varepsilon| + C$ . Then, up to a subsequence,  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} \mu = \sum_{i=1}^N d_i \delta_{x_i}$  with  $d_i \in \mathbb{Z} \setminus \{0\}$ ,  $x_i \in \Omega$  and  $\sum_{i=1}^N |d_i| \leq M$ . Moreover, if  $\sum_{i=1}^N |d_i| = M$ , then  $|d_i| = 1$  and up to a further subsequence  $\tilde{v}_\varepsilon \rightarrow v$  in  $H_{\text{loc}}^1(\Omega \setminus \text{supp}(\mu); \mathbb{R}^2)$  for some  $v \in \mathcal{D}_M$ .*

(ii) ( $\Gamma$ -liminf inequality) *Let  $v \in \mathcal{D}_M$  and let  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  be such that  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} Jv$  and  $\tilde{v}_\varepsilon \rightarrow v$  in  $L^2(\Omega; \mathbb{R}^2)$ . Then,*

$$(4.25) \quad \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) - M\pi|\log \varepsilon| \geq \mathcal{W}(v) + M\gamma.$$

(iii) ( $\Gamma$ -limsup inequality) *Given  $v \in \mathcal{D}_M$ , there exists  $\{u_\varepsilon\} \subset \mathcal{AF}_\varepsilon(\Omega)$  such that  $\mu(u_\varepsilon) \xrightarrow{\text{flat}} Jv$ ,  $\tilde{v}_\varepsilon \rightarrow v$  in  $H_{\text{loc}}^1(\Omega \setminus \text{supp}(Jv); \mathbb{R}^2)$  and*

$$(4.26) \quad \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) - M\pi|\log \varepsilon| = \mathcal{W}(v) + M\gamma.$$

*Proof. Proof of (i).* The compactness properties concerning the sequence  $\{\mu(u_\varepsilon)\}$  are given in Theorem 4.2 (i) while the weak convergence up to a subsequence of  $\{\tilde{v}_\varepsilon\}$  to a unitary field  $v$  such that  $v \in \mathcal{D}_M$  has been shown in the first lines of the proof of Theorem 4.2 (ii).

*Proof of (ii).* The proof of  $\Gamma$ -liminf inequality follows strictly the one of Theorem 4.2 (ii) and we leave it to the reader.

*Proof of (iii).* Let  $Jv = \pi \sum_{i=1}^M d_i \delta_{x_i}$ , with  $x_i \in \Omega$ ,  $|d_i| = 1$ . Fix  $\sigma > 0$  and  $\Omega^\sigma := \Omega \setminus \cup_{i=1}^M B_\sigma(x_i)$ . Without loss of generality we can assume that  $\mathcal{W}(v) < +\infty$  and hence for some fixed constant  $C > 0$  and for every  $\sigma$

$$\frac{1}{2} \int_{\Omega^\sigma} |\nabla v|^2 dx \leq M\pi|\log \sigma| + C.$$

Now, fix  $\sigma > 0$ , and let  $C_{i,j}$  denote the annulus  $B_{2^{-j}\sigma}(x_i) \setminus B_{2^{-(j+1)}\sigma}(x_i)$ . By Remark 4.4, it follows that for every  $i = 1, \dots, M$

$$(4.27) \quad \lim_{j \rightarrow \infty} \frac{1}{2} \int_{C_{i,j}} |\nabla v|^2 dx = \pi \log 2.$$

z Recall that  $\pi \log 2$  is the minimal possible energy in each annulus, and that the class of minimizers is given by the set  $\mathcal{K}$  defined in (4.17). Using standard scaling arguments and (4.27), one can show (see (4.20)) that for any  $j \in \mathbb{N}$  there exists a unitary vector  $\alpha_{i,j}$  such that

$$(4.28) \quad \frac{1}{2} \int_{C_{i,j}} \left| \nabla \left( v - \alpha_{i,j} \frac{x - x_i}{|x - x_i|} \right) \right|^2 dx = r(i, j),$$

with  $\lim_{j \rightarrow \infty} r(i, j) = 0$ . Moreover, we can find a function  $w_j \in C^\infty(\overline{\Omega^{2^{-j}\sigma}}; \mathcal{S}^1)$  such that

$$(4.29) \quad \frac{1}{2} \int_{\Omega^{2^{-j}\sigma}} |\nabla w_j - \nabla v|^2 dx \leq \frac{1}{j}.$$

Let  $\varphi \in C^1([\frac{1}{2}, 1]; [0, 1])$  be such that  $\varphi(\frac{1}{2}) = 1$  and  $\varphi(1) = 0$ , and let define the function  $v_{i,j}$  in  $C_{i,j}$ , with

$$v_{i,j}(x) := \varphi(2^j \sigma^{-1} |x - x_i|) \alpha_{i,j} \frac{x - x_i}{|x - x_i|} + (1 - \varphi(2^j \sigma^{-1} |x - x_i|)) w_j(x).$$

Then define the function  $v_j$  as follows

$$(4.30) \quad v_j = \begin{cases} w_j & \text{in } \Omega^{2^{-j}\sigma} \\ v_{i,j} & \text{in } C_{i,j}. \end{cases}$$

Finally for every  $i$  we denote by  $\bar{v}_{i,j}^\varepsilon \in \mathcal{A}\mathcal{X}\mathcal{Y}_\varepsilon(B_{2^{-j-1}\sigma}(x_i))$  a function which agrees with  $\alpha_{i,j} \frac{x-x_i}{|x-x_i|}$  on  $\partial_\varepsilon B_{2^{-j-1}\sigma}(x_i)$  and such that its phase (up to an additive constant) minimizes problem (4.4). If necessary, we extend  $\bar{v}_{i,j}$  to  $(\overline{B_{2^{-j-1}\sigma}(x_i)} \cap \varepsilon\mathbb{Z}^2) \setminus (B_{2^{-j-1}\sigma}(x_i))_\varepsilon^0$  to be equal to  $\alpha_{i,j} \frac{x-x_i}{|x-x_i|}$ . Finally, consider the field the  $v_{\varepsilon,j}$  which coincides with  $v_j$  on the nodes of  $\Omega^{2^{-j-1}\sigma}$  and with  $\bar{v}_{i,j}^\varepsilon$  on  $\overline{B_{2^{-j}\sigma}(x_i)} \cap \varepsilon\mathbb{Z}^2$ . In view of assumption (3) on  $f$ , a straightforward computation shows that any phase  $u_{\varepsilon,j}$  of  $v_{\varepsilon,j}$  satisfies

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_{\varepsilon,j}) - M\pi |\log \varepsilon| = M\gamma + \frac{1}{2} \int_{\Omega^{2^{-j}\sigma}} |\nabla v|^2 dx - M\pi |\log(2^{-j}\sigma)| + o(j),$$

with  $\lim_{j \rightarrow \infty} o(j) = 0$ . A standard diagonal argument yields that there exists  $j(\varepsilon) \rightarrow 0$  such that  $u_{\varepsilon,j(\varepsilon)}$  is a recovery sequence in the sense of (4.26).  $\square$

## 5. ANALYSIS OF LOCAL MINIMIZERS

In this section we will prove the existence of many local minimizers for a large class of interaction potentials. We will assume some further hypotheses for the energy density  $f$  in addition to (1), (2) and (3):

- (4)  $f \in C^0([-\frac{1}{2}, \frac{1}{2}]) \cap C^2((-\frac{1}{2}, \frac{1}{2}))$ ;
- (5) There exists a neighborhood  $I$  of  $\frac{1}{2}$  such that for every  $x \in I$  we have  $C_1(\frac{1}{2} - x)^2 < f(\frac{1}{2}) - f(x)$  for some  $C_1 > 0$  and  $\sup_{t \in (-\frac{1}{2}, \frac{1}{2})} f''(t) < \frac{1}{9}C_1$ ;
- (6)  $f$  is increasing in  $[0, \frac{1}{2}]$  and even.

Notice that these conditions are satisfied by the energy density of the screw dislocations functionals,  $f(a) = \text{dist}^2(a, \mathbb{Z})$ , while they are not satisfied by the energy density of the XY model.

**5.1. Antipodal configurations and energy barriers.** When a discrete singularity of  $\mu(v)$  moves to a neighboring cell, then  $v$  has to pass through an antipodal configuration  $v(i) = -v(j)$  (i.e., such that the corresponding phase  $u$  satisfies  $\text{dist}(u(i) - u(j), \mathbb{Z}) = \frac{1}{2}$ ). We will show that such configurations are energy barriers.

**Lemma 5.1.** *There exist  $\alpha > 0$  and  $E > 0$  such that the following holds: Let  $u \in \mathcal{A}\mathcal{F}_\varepsilon(\Omega)$  such that  $\text{dist}(u(i) - u(j), \mathbb{Z}) > \frac{1}{2} - \alpha$  for some  $(i, j) \in \Omega_\varepsilon^1$ . Then there exists a function  $w$ , with  $w = u$  in  $\Omega_\varepsilon^0 \setminus \{i\}$  such that  $F_\varepsilon(w) \leq F_\varepsilon(u) - E$ .*

*Proof.* As a consequence of assumption (5), it is easy to see that there exists  $\gamma > 0$  and a positive constant  $C_2$  such that

$$(5.1) \quad f(\frac{1}{2}) - f(\gamma) - f(\frac{1}{2} - \gamma) > C_2.$$

First, we prove the statement assuming  $f \in C^2(\mathbb{R})$ . In this case, assumption (5) implies that  $f'(\frac{1}{2}) = 0$  and  $|f''(\frac{1}{2})| > C_1$ .

Without loss of generality we can assume that  $u(i) = 0$ . For sake of notation we set

$$(5.2) \quad E^i(u) = \sum_{|l-i|=\varepsilon} f(u(l)).$$



We will assume that  $i \notin \partial_\varepsilon \Omega$ , so that  $i$  has exactly four nearest neighbors, denoted by  $j, k_1, k_2$  and  $k_3$ . The case  $i \in \partial_\varepsilon \Omega$  is fully analogous (some explicit computations are indeed shorter), and left to the reader. By assumption

$$(5.3) \quad E^i(u) \geq f(\tfrac{1}{2} + \alpha) + \sum_{l=1}^3 f(u(k_l)).$$

We will distinguish two cases.

*Case 1:* There exists at least a nearest neighbor, say  $k_1$ , such that  $\text{dist}(u(k_1), \mathbb{Z}) \geq \frac{1}{2} - \alpha$ . In this case we have that

$$(5.4) \quad E^i(u) \geq 2f(\tfrac{1}{2} + \alpha) + f(u(k_2)) + f(u(k_3)).$$

Now there are two possibilities. In fact we may have either that  $\text{dist}(u(k_2), \mathbb{Z}) \vee \text{dist}(u(k_3), \mathbb{Z}) < 3\alpha$ , or that  $\text{dist}(u(k_2), \mathbb{Z}) \vee \text{dist}(u(k_3), \mathbb{Z}) \geq 3\alpha$ .

In the first case, set  $w(i) = \gamma$  with  $\gamma$  as in (5.1). Then, by continuity we have

$$E^i(w) = 2f(\tfrac{1}{2} - \gamma) + 2f(\gamma) + o(1),$$

where  $o(1) \rightarrow 0$  as  $\alpha \rightarrow 0$ . From (5.4) we have  $E^i(u) \geq 2f(\frac{1}{2} + \alpha)$ , which together with (5.1) yields

$$(5.5) \quad E^i(u) - E^i(w) \geq 2(f(\tfrac{1}{2} + \alpha) - f(\tfrac{1}{2})) + C_2 + o(1) = C_2 + o(1)$$

as  $\alpha \rightarrow 0$ . Suppose now that  $\text{dist}(u(k_2), \mathbb{Z}) \vee \text{dist}(u(k_3), \mathbb{Z}) \geq 3\alpha$ . Then we define  $w(i) = \frac{1}{2}$  and we get

$$E^i(w) \leq 2f(\alpha) + f(\tfrac{1}{2}) + f(\tfrac{1}{2} + 3\alpha).$$

Moreover, thanks to assumption (6) of  $f$  we have  $E^i(u) \geq 2f(\frac{1}{2} + \alpha) + f(3\alpha)$ . We conclude that

$$(5.6) \quad E^i(u) - E^i(w) \geq \frac{7}{2}\alpha^2(f''(0) - f''(\tfrac{1}{2})) \geq \frac{7}{2}\alpha^2 C_1$$

*Case 2:* For every  $i$  it holds  $\text{dist}(u(k_i), \mathbb{Z}) < \frac{1}{2} - \alpha$ . Set  $w(i) = \eta$  with  $|\eta| = 3\alpha$  and  $\eta \sum_{l=1}^3 f'(u(k_l)) \geq 0$ . Then

$$(5.7) \quad \begin{aligned} E^i(u) - E^i(w) &\geq f(\tfrac{1}{2} + \alpha) - f(\tfrac{1}{2} + \alpha - |\eta|) + \sum_{l=1}^3 f(u(k_l)) - f(u(k_l) - \eta) \\ &= \frac{1}{2}|f''(\tfrac{1}{2})||\eta|(|\eta| - 2\alpha) + \eta \sum_{l=1}^3 f'(u(k_l)) - \frac{1}{2}\eta^2 \sum_{l=1}^3 f''(u(k_l)) + o(\eta^2) \\ &\geq \frac{1}{2}|f''(\tfrac{1}{2})|3\alpha^2 - \frac{9}{2}\alpha^2 \sum_{l=1}^3 f''(u(k_l)) + o(\alpha^2) \geq \frac{3}{2}(C_1 - 9 \sup_t f''(t))\alpha^2 + o(\alpha^2) \end{aligned}$$

The combination of Step 1 and Step 2 concludes the proof in the case of  $f \in C^2(\mathbb{R})$ , by choosing  $\alpha$  small enough and  $E = (7C_1 \wedge 3(C_1 - 9 \sup_t f''(t)))\alpha^2/2$ . The general case can be recover by approximating  $f$  in a neighborhood of  $\frac{1}{2}$  with  $C^2$  functions still satisfying assumptions (4)-(6).  $\square$

Note that in the case of  $f(a) = \text{dist}^2(a, \mathbb{Z})$  the proof of the above Lemma can be obtained by a direct computation without the regularization.

**Remark 5.2.** Note that the function  $w$  constructed in Lemma 5.1 has a discrete vorticity that can be different from the one of  $u$  only in the four  $\varepsilon$ -squares sharing  $i$  as a vertex, and hence  $\|\mu(u) - \mu(w)\|_{\text{flat}} \leq 2\varepsilon$ .

**Definition 5.3.** We say that a function  $u \in \mathcal{AF}_\varepsilon(\Omega)$  satisfies the  $\alpha$ -cone condition if

$$\text{dist}(u(i) - u(j), \mathbb{Z}) \leq \frac{1}{2} - \alpha \quad \text{for every } (i, j) \in \Omega_\varepsilon^1.$$

**Remark 5.4.** Note that if  $u \in \mathcal{AF}_\varepsilon(\Omega)$  satisfies the  $\alpha$ -cone condition for some  $\alpha > 0$ , then for every  $w \in \mathcal{AF}_\varepsilon(\Omega)$  such that  $\sum_{i \in \Omega_\varepsilon^0} |w(i) - u(i)|^2 < \frac{\alpha^2}{16}$  we have  $\mu(w) = \mu(u)$ . In other words, the vorticity measure  $\mu(u)$  is stable with respect to small variations of  $u$ .

**5.2. Metastable configurations and pinning.** As a consequence of Lemma 5.1 we prove the existence of a minimizer for the energy  $F_\varepsilon$ , under assumptions (1)-(6) with singularities close to prescribed positions.

**Theorem 5.5.** Given  $\mu_0 = \sum_{i=1}^M d_i \delta_{x_i}$  with  $x_i \in \Omega$  and  $d_i \in \{1, -1\}$  for  $i = 1, \dots, M$ , there exists a constant  $K \in \mathbb{N}$  such that, for  $\varepsilon$  small enough, there exists  $k_\varepsilon \in \{1, \dots, K\}$  such that the following minimum problem is well-posed

$$(5.8) \quad \min\{F_\varepsilon(u) : \|\mu(u) - \mu_0\|_{\text{flat}} \leq k_\varepsilon \varepsilon\}.$$

Moreover, let  $\alpha$  be given by Lemma 5.1; any minimizer  $u_\varepsilon$  of the problem in (5.8) satisfies the  $\alpha$ -cone condition and it is a local minimizer for  $F_\varepsilon$ .

*Proof.* For any  $k \in \mathbb{N} \cup \{0\}$ , we set

$$(5.9) \quad I_\varepsilon^k := \inf\{F_\varepsilon(u) : \|\mu(u) - \mu_0\|_{\text{flat}} \leq (M + 2k)\varepsilon\},$$

By constructing explicit competitors one can show that

$$(5.10) \quad I_\varepsilon^0 \leq M\pi |\log \varepsilon| + C.$$

Then, we consider a minimizing sequence  $\{u_\varepsilon^{k,n}\}$  for  $I_\varepsilon^k$ . It is not restrictive to assume that  $0 \leq u_\varepsilon^{k,n}(i) \leq 1$  for any  $i \in \Omega_\varepsilon^0$ ; therefore, up to a subsequence,  $u_\varepsilon^{k,n} \rightarrow u_\varepsilon^k$  as  $n \rightarrow \infty$  for some  $u_\varepsilon^k \in \mathcal{AF}_\varepsilon(\Omega)$ . Note that if  $u_\varepsilon^k$  satisfies the  $\alpha$ -cone condition, then it is a minimizer for  $I_\varepsilon^k$ .

Set  $\bar{k} := \lceil \frac{C - \mathbb{W}(\mu_0) - M\gamma}{E} \rceil + 1$  and assume by contradiction that there exists a subsequence, still labelled with  $\varepsilon$ , such that for every  $k \in \{0, 1, \dots, \bar{k}\}$ , there exists a bond  $(i_\varepsilon, j_\varepsilon) \in \Omega_\varepsilon^1$ , with  $\text{dist}(u_\varepsilon^k(i_\varepsilon) - u_\varepsilon^k(j_\varepsilon), \mathbb{Z}) > \frac{1}{2} - \alpha$ . Thus, for  $n$  large enough, we have

$$\text{dist}(u_\varepsilon^{k,n}(i_\varepsilon) - u_\varepsilon^{k,n}(j_\varepsilon), \mathbb{Z}) > \frac{1}{2} - \alpha.$$

By Lemma 5.1, there exists a function  $w_\varepsilon^{k,n} \in \mathcal{AF}_\varepsilon(\Omega)$  such that  $w_\varepsilon^{k,n} \equiv u_\varepsilon^{k,n}$  in  $\Omega_\varepsilon^0 \setminus \{i\}$  and  $F_\varepsilon(w_\varepsilon^{k,n}) \leq F_\varepsilon(u_\varepsilon^{k,n}) - E$  for some  $E > 0$ . By construction (see Remark 5.2) we have that  $\|\mu(w_\varepsilon^{k,n}) - \mu(u_\varepsilon^{k,n})\|_{\text{flat}} \leq 2\varepsilon$ . It follows that

$$I_\varepsilon^{k+1} \leq F_\varepsilon(w_\varepsilon^{k,n}) \leq I_\varepsilon^k - E.$$

By an easy induction argument on  $k$  and by (5.10), we have immediately that

$$(5.11) \quad I_\varepsilon^k \leq I_\varepsilon^0 - kE \leq M\pi |\log \varepsilon| + C - kE.$$

By the lower bound (4.9) in Theorem 4.2, (5.11), and the definition of  $\bar{k}$  we get

$$\mathbb{W}(\mu_0) + M\gamma \leq \liminf_{\varepsilon \rightarrow 0} I_\varepsilon^{\bar{k}} - M\pi |\log \varepsilon| \leq C - \bar{k}E \leq \mathbb{W}(\mu_0) + M\gamma - E,$$

and so the contradiction. Then the statement holds for  $K = M + 2\bar{k}$ .  $\square$

Let  $\varepsilon > 0$  and let  $u_\varepsilon^0 \in \mathcal{AF}_\varepsilon(\Omega)$ . We say that  $u_\varepsilon = u_\varepsilon(t)$  is a gradient flow of  $F_\varepsilon$  from  $u_\varepsilon^0$  if  $u_\varepsilon$  is a solution of

$$\begin{cases} \frac{1}{|\log \varepsilon|} \dot{u}_\varepsilon = -\nabla F_\varepsilon(u_\varepsilon) & \text{in } (0, +\infty) \times \Omega_\varepsilon^0 \\ u_\varepsilon(0) = u_\varepsilon^0 & \text{in } \Omega_\varepsilon^0. \end{cases}$$

Clearly  $u_\varepsilon(t) \in \mathcal{AF}_\varepsilon(\Omega)$ , and we will write  $u_\varepsilon(t, i)$  in place of  $u_\varepsilon(t)(i)$ .

**Theorem 5.6.** *Let  $\mu_0 = \sum_{i=1}^M d_i \delta_{x_i}$  with  $x_i \in \Omega$  and  $d_i \in \{1, -1\}$  for  $i = 1, \dots, M$ . Let  $\{u_\varepsilon^0\} \subset \mathcal{AF}_\varepsilon(\Omega)$  be such that*

$$(5.12) \quad \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon^0) - M\pi |\log \varepsilon| = \mathbb{W}(\mu_0) + M\gamma.$$

*Let  $\alpha$  be given by Lemma 5.1. Then, for  $\varepsilon$  small enough, the following facts hold.*

- (i)  $u_\varepsilon^0$  satisfy the  $\alpha$ -cone condition.
- (ii) The gradient flow  $u_\varepsilon(t)$  of  $F_\varepsilon$  from  $u_\varepsilon^0$  satisfies  $\mu(u_\varepsilon(t)) = \mu(u_\varepsilon^0)$  for every  $t > 0$ .
- (iii) There exists  $\bar{u}_\varepsilon^0$  such that  $\bar{u}_\varepsilon^0 \in \operatorname{argmin}\{F_\varepsilon(u) : \mu(u) = \mu(u_\varepsilon^0)\}$ . Moreover  $\bar{u}_\varepsilon^0$  satisfies the  $\alpha$ -cone condition and it is a local minimizer for  $F_\varepsilon$ .

*Proof. Proof of (i).* Assume, by contradiction, that there exists a sequence  $\varepsilon_k \rightarrow 0$  such that  $u_{\varepsilon_k}^0$  does not satisfy the  $\alpha$ -cone condition, namely for every  $k \in \mathbb{N}$  there exists a bond  $(i_k, j_k) \in \Omega_{\varepsilon_k}^1$  with

$$\operatorname{dist}(u_{\varepsilon_k}^0(i_k) - u_{\varepsilon_k}^0(j_k), \mathbb{Z}) > \frac{1}{2} - \alpha.$$

By Lemma 5.1, for any  $k$  there exists a function  $w_{\varepsilon_k} \in \mathcal{AF}_{\varepsilon_k}(\Omega)$  such that  $w_{\varepsilon_k} \equiv u_{\varepsilon_k}^0$  in  $\Omega_{\varepsilon_k}^0 \setminus \{i_k\}$  and

$$(5.13) \quad F_{\varepsilon_k}(w_{\varepsilon_k}) \leq F_{\varepsilon_k}(u_{\varepsilon_k}^0) - E \leq F_{\varepsilon_k}(u_{\varepsilon_k}^0) - E.$$

Moreover, by construction (see Remark 5.2) we have that  $\|\mu(w_{\varepsilon_k}) - \mu(u_{\varepsilon_k}^0)\|_{\text{flat}} \leq 2\varepsilon_k$  and so  $\mu(w_{\varepsilon_k}) \xrightarrow{\text{flat}} \mu_0$ . By the lower bound (4.9) in Theorem 4.2, we get

$$(5.14) \quad \begin{aligned} \mathbb{W}(\mu_0) + M\gamma &\leq \liminf_{\varepsilon_k \rightarrow 0} F_{\varepsilon_k}(w_{\varepsilon_k}) - M\pi |\log \varepsilon_k| \\ &\leq \lim_{\varepsilon_k \rightarrow 0} F_{\varepsilon_k}(u_{\varepsilon_k}^0) - M\pi |\log \varepsilon_k| - E = \mathbb{W}(\mu_0) + M\gamma - E, \end{aligned}$$

and so the contradiction.

*Proof of (ii).* Assume, by contradiction, that there exists a sequence  $\varepsilon_k \rightarrow 0$  such that the gradient flows  $u_{\varepsilon_k}(t)$  of  $F_{\varepsilon_k}$  from  $u_{\varepsilon_k}^0$  does not satisfy (ii). Let  $t_k$  be the first time (in fact, the infimum) for which  $\mu(u_{\varepsilon_k}(t_k)) \neq \mu(u_{\varepsilon_k}^0)$ ; then, there exists  $(i_k, j_k) \in \Omega_{\varepsilon_k}^1$  such that  $\operatorname{dist}(u_{\varepsilon_k}(t_k, i_k) - u_{\varepsilon_k}(t_k, j_k), \mathbb{Z}) > \frac{1}{2} - \alpha$ . By Lemma 5.1 there exists  $w_{\varepsilon_k}(t_k) \in \mathcal{AF}_{\varepsilon_k}(\Omega)$  such that  $w_{\varepsilon_k}(t_k) \equiv u_{\varepsilon_k}(t_k)$  in  $\Omega_{\varepsilon_k}^0 \setminus \{i_k\}$  and  $F_{\varepsilon_k}(w_{\varepsilon_k}(t_k)) \leq F_{\varepsilon_k}(u_{\varepsilon_k}(t_k)) - E$ , for some positive constant  $E$  independent of  $k$ . Moreover, by (5.2), we have that

$$\|\mu(u_{\varepsilon_k}^0) - \mu(w_{\varepsilon_k}(t_k))\|_{\text{flat}} = \|\mu(u_{\varepsilon_k}(t_k)) - \mu(w_{\varepsilon_k}(t_k))\|_{\text{flat}} \leq 2\varepsilon_k;$$

Therefore, by the lower bound (4.9) in Theorem 4.2, arguing as in (5.14), we get a contradiction.

*Proof of (iii).* Let  $\{u_\varepsilon^n\}$  be a minimizing sequence for the minimum problem in (iii). We can always assume that  $0 \leq u_\varepsilon^n(i) \leq 1$  for any  $i \in \Omega_\varepsilon^0$ ; therefore, up to a subsequence,  $u_\varepsilon^n \rightarrow \bar{u}_\varepsilon^0$  as  $n \rightarrow \infty$  for some  $\bar{u}_\varepsilon^0 \in \mathcal{AF}_\varepsilon(\Omega)$ . To prove that  $\bar{u}_\varepsilon^0$  (for  $\varepsilon$  small enough) is a minimizer, it is enough to show that  $\mu(\bar{u}_\varepsilon^0) = \mu(u_\varepsilon)$ ; this follows once we have proved that  $\bar{u}_\varepsilon^0$  satisfies the  $\alpha$ -cone condition (see Remark 5.4). Assume by contradiction that there exists a sequence  $\varepsilon_k \rightarrow 0$  such that  $\text{dist}(\bar{u}_{\varepsilon_k}^0(i_k) - \bar{u}_{\varepsilon_k}^0(j_k), \mathbb{Z}) > \frac{1}{2} - \alpha$  for some bond  $(i_k, j_k) \in \Omega_{\varepsilon_k}^1$ . Then, for  $n$  large enough, we have

$$(5.15) \quad F_{\varepsilon_k}(u_{\varepsilon_k}^n) \leq F_{\varepsilon_k}(u_{\varepsilon_k}^0) + \varepsilon_k, \quad \text{dist}(u_{\varepsilon_k}^n(i) - u_{\varepsilon_k}^n(j), \mathbb{Z}) > \frac{1}{2} - \alpha.$$

Let  $\bar{n}$  so that (5.15) holds. By Lemma 5.1, there exists a function  $w_{\varepsilon_k} \in \mathcal{AF}_{\varepsilon_k}(\Omega)$  such that  $w_{\varepsilon_k} \equiv u_{\varepsilon_k}^{\bar{n}}$  in  $\Omega_{\varepsilon_k}^0 \setminus \{i\}$  and

$$(5.16) \quad F_{\varepsilon_k}(w_{\varepsilon_k}) \leq F_{\varepsilon_k}(u_{\varepsilon_k}^{\bar{n}}) - E \leq F_{\varepsilon_k}(u_{\varepsilon_k}^0) - E + \varepsilon_k.$$

By construction (see Remark 5.2), we have that  $\|\mu(w_{\varepsilon_k}) - \mu(u_{\varepsilon_k}^0)\|_{\text{flat}} = \|\mu(w_{\varepsilon_k}) - \mu(u_{\varepsilon_k}^{\bar{n}})\|_{\text{flat}} \leq 2\varepsilon_k$ . Therefore, by the lower bound (4.9) in Theorem 4.2, arguing as in (5.14), we get a contradiction.

Finally, by the  $\alpha$ -cone condition and Remark 5.4, we have immediately that  $F_\varepsilon(\bar{u}_\varepsilon^0) \leq F_\varepsilon(w)$  for any function  $w \in \mathcal{AF}_\varepsilon(\Omega)$  with  $\|w - u\|_{L^2} \leq \frac{\alpha}{4}$ , and hence  $\bar{u}_\varepsilon^0$  is a local minimizer of  $F_\varepsilon$ .  $\square$

**Remark 5.7.** By Theorem 5.6 it easily follows that there exists  $t_n \rightarrow \infty$  such that  $u_\varepsilon^\infty := \lim_{t_n \rightarrow \infty} u_\varepsilon(t_n)$  is a critical point of  $F_\varepsilon$ .

## 6. DISCRETE GRADIENT FLOW OF $\mathcal{F}_\varepsilon$ WITH FLAT DISSIPATION

In Section 5 we have seen that the energy  $\mathcal{F}_\varepsilon$  has many local minimizers. In particular, Theorem 5.5 shows that the length-scale of metastable configurations of singularities is of order  $\varepsilon$ . In this section we introduce and analyze an effective dynamics of vortices, which overcome the pinning effect due to the presence of these local minima. This is done considering a discrete in time gradient flow, following the minimizing movements method. It turns out that, for  $\varepsilon$  smaller than the time step  $\tau$ , the vortices overcome the energetic barriers and the dynamics is described (as  $\varepsilon, \tau \rightarrow 0$ ) by the gradient flow of the renormalized energy (see Definition 6.3).

We introduce the discrete gradient flow of  $F_\varepsilon$  with respect to the flat dissipation.

**Definition 6.1.** Fix  $\delta > 0$  and let  $\varepsilon, \tau > 0$ . Given  $\mu_{\varepsilon,0} \in X_\varepsilon$ , we say that  $\{\mu_{\varepsilon,k}^\tau\}$ , with  $k \in \mathbb{N} \cup \{0\}$ , is a solution of the flat discrete gradient flow of  $\mathcal{F}_\varepsilon$  from  $\mu_{\varepsilon,0}$  if  $\mu_{\varepsilon,0}^\tau = \mu_{\varepsilon,0}$ , and for any  $k \in \mathbb{N}$ ,  $\mu_{\varepsilon,k}^\tau$  satisfies

$$(6.1) \quad \mu_{\varepsilon,k}^\tau \in \operatorname{argmin} \left\{ \mathcal{F}_\varepsilon(\mu) + \frac{\pi \|\mu - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}}^2}{2\tau} : \mu \in X_\varepsilon, \|\mu - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}} \leq \delta \right\}.$$

Notice that the existence of a minimizer is obvious, since  $\mu$  lies in  $X_\varepsilon$  which is a finite set.

We want to analyze the limit as  $\varepsilon \rightarrow 0$  of the flat discrete gradient flow. To this purpose, let  $\mu_0 := \sum_{i=1}^M d_{i,0} \delta_{x_{i,0}}$  with  $|d_{i,0}| = 1$ , and let  $\mu_{\varepsilon,0} \in X_\varepsilon$  be such that

$$\mu_{\varepsilon,0} \xrightarrow{\text{flat}} \mu_0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})}{|\log \varepsilon|} = \pi |\mu_0|(\Omega).$$

In Theorem 6.7 we will show that, as  $\varepsilon \rightarrow 0$ , the sequence  $\mu_{\varepsilon, k}^\tau$  converges to some  $\mu_k^\tau \in X$ , whose singularities have the same degrees of those of the initial datum. Therefore, it is convenient to regard the renormalized energy as a function only of the positions of  $M$  singularities. To this end we introduce the following notation

$$W(x) := \mathbb{W}(\mu) \quad \text{where } \mu = \sum_{i=1}^M d_{i,0} \delta_{x_i} \quad \text{for every } x \in \Omega^M.$$

The right notion for the limit as  $\varepsilon \rightarrow 0$  of flat discrete gradient flows of  $\mathcal{F}_\varepsilon$  is given by the following definition of discrete gradient flow of the renormalized energy.

**Definition 6.2.** *Let  $\delta > 0$ ,  $K \in \mathbb{N} \cup \{0\}$ , and  $\tau > 0$ . Fix  $x_0 \in \Omega^M$ . We say that  $\{x_k^\tau\}$  with  $k = 0, 1, \dots, K$ , is a solution of the discrete gradient flow of  $W$  from  $x_0$  if  $x_0^\tau = x_0$  and, for any  $k = 1, \dots, K$ ,  $x_k^\tau \in \Omega^M$  satisfies*

$$(6.2) \quad x_k^\tau \in \operatorname{argmin} \left\{ W(x) + \frac{\pi |x - x_{k-1}^\tau|^2}{2\tau} : x \in \Omega^M, |x - x_{k-1}^\tau| \leq \delta \right\}.$$

In Theorem 6.6 we show that, as  $\tau \rightarrow 0$ , this discrete time evolution converges, until a maximal time  $\tilde{T}_\delta$ , to the gradient flow of the renormalized energy given by the following definition.

**Definition 6.3.** *Let  $M \in \mathbb{N}$  and  $x_0 \in \Omega^M$ . The gradient flow of the renormalized energy from  $x_0$  is given by*

$$(6.3) \quad \begin{cases} \dot{x}(t) = -\frac{1}{\pi} \nabla W(x(t)) \\ x(0) = x_0. \end{cases}$$

We denote by  $T^*$  the maximal time of existence of the solution, and we notice that until the time  $T^*$  the solution is unique, and that  $T^*$  is the minimal critical time among the first collision time and the exit time from  $\Omega$ .

As  $\delta \rightarrow 0$ ,  $\tilde{T}_\delta$  converges to the critical time  $T^*$ . Notice that the renormalized energy is not bounded from below and it blows up to  $-\infty$  whenever one of these critical events occur. This justifies the introduction of the parameter  $\delta$ , in order to explore local minima. Nevertheless, the solutions of flat discrete gradient flows defined above do not touch the constraint and hence, they satisfy the corresponding unconstrained Euler-Lagrange equations.

### 6.1. Flat discrete gradient flow of $W$ . Fix initial conditions

$$x_0 = (x_{1,0}, \dots, x_{M,0}) \in \Omega^M, \quad d_{1,0}, \dots, d_{M,0} \in \{-1, 1\},$$

and fix  $\delta > 0$  such that

$$(6.4) \quad \min \left\{ \frac{1}{2} \operatorname{dist}_{i \neq j} (x_{i,0}, x_{j,0}), \operatorname{dist}(x_{i,0}, \partial\Omega) \right\} - 2\delta =: c_\delta > 0.$$

**Definition 6.4.** *We say that a solution of the discrete gradient flow  $\{x_k^\tau\}$  of  $W$  from  $x_0$  is maximal if the minimum problem in (6.2) does not admit a solution for  $k = K + 1$ .*

Let  $\{x_k^\tau\}$  be a maximal solution of the flat discrete gradient flow of  $W$  from  $x_0$ , according with Definitions 6.2, 6.4; we set

$$(6.5) \quad k_\delta^\tau = k_\delta^\tau(\{x_k^\tau\}) := \min \{k \in \{1, \dots, K\} : \min \left\{ \frac{1}{2} \operatorname{dist}_{i \neq j} (x_{i,k}^\tau, x_{j,k}^\tau), \operatorname{dist}(x_{i,k}^\tau, \partial\Omega) \right\} \leq 2\delta \}.$$

We notice that, since  $|x_{k_\delta^\tau}^\tau - x_{k_\delta^\tau - 1}^\tau| \leq \delta$  and

$$\min\left\{\frac{1}{2}\text{dist}_{i \neq j}(x_{i, k_\delta^\tau}^\tau, x_{j, k_\delta^\tau}^\tau), \text{dist}(x_{i, k_\delta^\tau}^\tau, \partial\Omega)\right\} > 2\delta,$$

then

$$\min\left\{\frac{1}{2}\text{dist}_{i \neq j}(x_{i, k_\delta^\tau}^\tau, x_{j, k_\delta^\tau}^\tau), \text{dist}(x_{i, k_\delta^\tau}^\tau, \partial\Omega)\right\} > \delta,$$

i.e.,  $k_\delta^\tau < K$ . It follows that, for any  $k = 0, 1, \dots, k_\delta^\tau$ , we have

$$(6.6) \quad x_k^\tau \in K_\delta,$$

where  $K_\delta$  is the compact set given by

$$(6.7) \quad K_\delta := \left\{x \in \Omega^M : \min\left\{\frac{1}{2}\text{dist}_{i \neq j}(x_i, x_j), \text{dist}(x_i, \partial\Omega)\right\} \geq \delta\right\}.$$

Notice that  $W$  is smooth on  $K_\delta$ . In particular, we can set

$$(6.8) \quad C_\delta := \max_{x \in K_\delta} (W(x_0) - W(x)).$$

**Proposition 6.5.** *For  $\tau$  small enough the following holds. For every  $k = 1, \dots, k_\delta^\tau$ , we have that  $|x_k^\tau - x_{k-1}^\tau| < \delta$  and*

$$(6.9) \quad \partial_{x_i} W(x_k^\tau) + \pi \frac{x_{i, k}^\tau - x_{i, k-1}^\tau}{\tau} = 0 \quad \text{for } i = 1, \dots, M.$$

In particular, for every  $k = 1, \dots, k_\delta^\tau$

$$(6.10) \quad |x_k^\tau - x_{k-1}^\tau| \leq \max_{x \in K_\delta} |\nabla W(x)| \tau.$$

*Proof.* Since the energy  $W$  is clearly decreasing in  $k$ , for every  $k = 1, \dots, k_\delta^\tau$  we have

$$(6.11) \quad \frac{|x_k^\tau - x_{k-1}^\tau|^2}{2\tau} \leq \frac{1}{\pi} (W(x_{k-1}^\tau) - W(x_k^\tau)) \leq W(x_0) - W(x_k^\tau) \leq C_\delta.$$

It follows that for  $\tau$  small enough  $|x_k^\tau - x_{k-1}^\tau| < \delta$ . Therefore, the minimality of  $x_k^\tau$  clearly implies (6.9), as well as (6.10).  $\square$

Let  $x(t)$  be the solution of the gradient flow of  $W$  with initial datum  $x_0$  (see (6.3)) and let  $T^*$  be its maximal existence time. We set

$$(6.12) \quad T_\delta := \inf \left\{ t \in [0, T^*] : \min\left\{\frac{1}{2}\text{dist}_{i \neq j}(x_i(t), x_j(t)), \text{dist}(x_i(t), \partial\Omega)\right\} \leq 2\delta \right\}.$$

Notice that by definition we have

$$(6.13) \quad \lim_{\delta \rightarrow 0} T_\delta = T^*.$$

For  $0 \leq t \leq k_\delta^\tau \tau$ , we denote by  $x^\tau(t) = (x_1^\tau(t), \dots, x_M^\tau(t))$  the piecewise affine in time interpolation of  $\{x_k^\tau\}$ .

**Theorem 6.6.** *Let  $\{x_k^\tau\}_{\tau > 0}$  be a family of maximal solutions of the flat discrete gradient flow of  $W$  from  $x_0$ . Then,*

$$(6.14) \quad \tilde{T}_\delta := \liminf_{\tau \rightarrow 0} k_\delta^\tau \tau \geq T_\delta,$$

where  $k_\delta^\tau$  is defined in (6.5) and  $T_\delta$  is defined in (6.12).

Moreover, for every  $0 < T < \tilde{T}_\delta$ ,  $x^\tau \rightarrow x$  uniformly on  $[0, T]$ . Finally,  $\tilde{T}_\delta \rightarrow T^*$  as  $\delta \rightarrow 0$ .

*Proof.* By the very definition of  $k_\delta^\tau$ , it is easy to prove that

$$|x_{k_\delta^\tau}^\tau - x_0^\tau| > c_\delta,$$

where  $c_\delta$  is defined in (6.4). Moreover, by (6.10), for  $\tau$  small enough we get

$$|x_{k_\delta^\tau}^\tau - x_0^\tau| \leq \sum_{k=1}^{k_\delta^\tau} |x_k^\tau - x_{k-1}^\tau| \leq \max_{x \in K_\delta} |\nabla W(x)| k_\delta^\tau \tau,$$

and hence

$$k_\delta^\tau \tau \geq \frac{c_\delta}{\max_{x \in K_\delta} |\nabla W(x)|} > 0.$$

From (6.10) it is easy to see that  $x^\tau$  are equibounded and equicontinuous in  $[0, \tau k_\delta^\tau]$ , and hence by Ascoli Arzelà Theorem, they uniformly converge, up to a subsequence, to a function  $x$  on  $[0, T]$ , for every  $T < \tilde{T}_\delta$ . Let  $t \in (0, \tilde{T}_\delta)$  and let  $h > 0$ , by (6.9) we get

$$(6.15) \quad x^\tau(\tau \lfloor (t+h)/\tau \rfloor) - x^\tau(\tau \lfloor t/\tau \rfloor) = \sum_{k=\lfloor t/\tau \rfloor}^{\lfloor (t+h)/\tau \rfloor - 1} x_{k+1}^\tau - x_k^\tau = -\frac{\tau}{\pi} \sum_{k=\lfloor t/\tau \rfloor}^{\lfloor (t+h)/\tau \rfloor - 1} \nabla W(x_k^\tau).$$

Taking the limit as  $\tau \rightarrow 0$ , and then  $h \rightarrow 0$ , we obtain that the limit  $x$  is the unique solution of (6.3).

Moreover, it is easy to see that  $x^\tau(\tau k_\delta^\tau) \rightarrow x(\tilde{T}_\delta)$  and hence by definition of  $k_\delta^\tau$ , it is easy to see that (6.14) holds true and  $\tilde{T}_\delta < T^*$ . Since  $T_\delta \rightarrow T^*$  (see (6.13)) we conclude that  $\tilde{T}_\delta \rightarrow T^*$  as  $\delta \rightarrow 0$ .  $\square$

**6.2. Flat discrete gradient flow of  $\mathcal{F}_\varepsilon$ .** We are now in a position to state and prove the convergence of discrete gradient flows as  $\varepsilon \rightarrow 0$ .

**Theorem 6.7.** *Let  $\mu_0 := \sum_{i=1}^M d_{i,0} \delta_{x_{i,0}}$  with  $|d_{i,0}| = 1$ . Let  $\mu_{\varepsilon,0} \in X_\varepsilon$  be such that*

$$\mu_{\varepsilon,0} \xrightarrow{\text{flat}} \mu_0, \quad \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{F}_\varepsilon(\mu_{\varepsilon,0})}{|\log \varepsilon|} = \pi |\mu_0|(\Omega).$$

*Let  $\delta > 0$  be fixed such that  $\min \left\{ \frac{1}{2} \text{dist}_{i \neq j}(x_{i,0}, x_{j,0}), \text{dist}(x_{i,0}, \partial\Omega) \right\} > 2\delta$ . Given  $\tau > 0$ , let  $\mu_{\varepsilon,k}^\tau$  be a solution of the flat discrete gradient flow of  $\mathcal{F}_\varepsilon$  from  $\mu_{\varepsilon,0}$ .*

*Then, up to a subsequence, for any  $k \in \mathbb{N}$  we have  $\mu_{\varepsilon,k}^\tau \xrightarrow{\text{flat}} \mu_k^\tau$ , for some  $\mu_k^\tau \in X$  with  $|\mu_k^\tau|(\Omega) \leq M$ .*

*Moreover there exists a maximal solution of the discrete gradient flow,  $x_k^\tau = (x_{1,k}^\tau, \dots, x_{M,k}^\tau)$ , of  $W$  from  $x_0 = (x_{1,0}, \dots, x_{M,0})$ , according with Definition 6.2, such that*

$$\mu_k^\tau = \sum_{i=1}^M d_{i,0} \delta_{x_{i,k}^\tau} \quad \text{for every } k = 1, \dots, k_\delta^\tau,$$

*where  $k_\delta^\tau$  is defined in (6.5).*

*Proof.* Since  $\mathcal{F}_\varepsilon(\mu_{\varepsilon,k}^\tau)$  is not increasing in  $k$ , we have

$$\mathcal{F}_\varepsilon(\mu_{\varepsilon,k}^\tau) \leq \mathcal{F}_\varepsilon(\mu_{\varepsilon,0}) \leq M\pi |\log \varepsilon| + o(|\log \varepsilon|).$$

By Theorem 4.3(i), we have that, up to a subsequence,  $\mu_{\varepsilon,k}^\tau \xrightarrow{\text{flat}} \mu_k^\tau \in X$ , with  $|\mu_k^\tau|(\Omega) \leq M$  and  $\|\mu_k^\tau - \mu_{k-1}^\tau\|_{\text{flat}} \leq \delta$ . Let  $\tilde{k}_\delta^\tau$  be defined by

$$(6.16) \quad \begin{aligned} \tilde{k}_\delta^\tau &:= \sup\{k \in \mathbb{N} : \mu_l^\tau = \sum_{i=1}^M d_{i,0} \delta_{x_{i,l}^\tau}, \\ &\quad \min\{\frac{1}{2} \text{dist}_{i \neq j}(x_{i,l}^\tau, x_{j,l}^\tau), \text{dist}(x_{i,l}^\tau, \partial\Omega)\} > 2\delta, l = 0, \dots, k\}. \end{aligned}$$

Since  $|\mu_{\tilde{k}_\delta^\tau+1}^\tau|(\Omega) \leq M$  and  $\|\mu_{\tilde{k}_\delta^\tau+1}^\tau - \mu_{\tilde{k}_\delta^\tau}^\tau\|_{\text{flat}} \leq \delta$ , we deduce that  $\mu_{\tilde{k}_\delta^\tau+1}^\tau = \sum_{i=1}^M d_{i,0} \delta_{x_{i,\tilde{k}_\delta^\tau+1}^\tau}$ , while

$$(6.17) \quad \min\{\frac{1}{2} \text{dist}_{i \neq j}(x_{i,\tilde{k}_\delta^\tau+1}^\tau, x_{j,\tilde{k}_\delta^\tau+1}^\tau), \text{dist}(x_{i,\tilde{k}_\delta^\tau+1}^\tau, \partial\Omega)\} \leq 2\delta.$$

Moreover, since  $\|\mu_k^\tau - \mu_{k-1}^\tau\|_{\text{flat}} \leq \delta$ , it is easy to see that at each step  $k = 1, \dots, \tilde{k}_\delta^\tau + 1$  and for every singularity  $x_{i,k-1}^\tau$  of  $\mu_{k-1}^\tau$ , there is exactly one singularity of  $\mu_k^\tau$  at distance at most  $\delta$  from  $x_{i,k-1}^\tau$ ; we relabel it  $x_{i,k}^\tau$ . Therefore, we have that for  $k = 1, \dots, \tilde{k}_\delta^\tau + 1$

$$(6.18) \quad \|\mu_k^\tau - \mu_{k-1}^\tau\|_{\text{flat}} = |x_k^\tau - x_{k-1}^\tau|.$$

We now show that for  $k = 0, 1, \dots, \tilde{k}_\delta^\tau + 1$ ,  $x_k^\tau$  satisfies (6.2). For any measure  $\mu = \sum_{i=1}^M d_{i,0} \delta_{y_i}$  with  $\|\mu - \mu_{k-1}^\tau\| \leq \delta$ , by Theorem 4.3 (iii) there exists a recovery sequence  $\{\mu_\varepsilon\}$  such that  $\mathcal{F}_\varepsilon(\mu_\varepsilon) - M\pi |\log \varepsilon| \rightarrow \mathbb{W}(\mu) + M\gamma$  as  $\varepsilon \rightarrow 0$ . By a standard density argument we can assume that  $\|\mu_\varepsilon - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}} \leq \delta$ . Therefore by (ii) of Theorem 4.3, using the fact that  $\mu_{\varepsilon,k}^\tau$  satisfies (6.1), we get

$$\begin{aligned} \mathbb{W}(\mu_k^\tau) + M\gamma &+ \frac{\pi \|\mu_k^\tau - \mu_{k-1}^\tau\|_{\text{flat}}^2}{2\tau} \\ &\leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\mu_\varepsilon) - \pi M |\log \varepsilon| + \frac{\pi \|\mu_{\varepsilon,k}^\tau - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}}^2}{2\tau} \\ &\leq \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(\mu_\varepsilon) - \pi M |\log \varepsilon| + \frac{\pi \|\mu_\varepsilon - \mu_{\varepsilon,k-1}^\tau\|_{\text{flat}}^2}{2\tau} \\ &= \mathbb{W}(\mu) + M\gamma + \frac{\pi \|\mu - \mu_{k-1}^\tau\|_{\text{flat}}^2}{2\tau}, \end{aligned}$$

i.e.,  $\mu_k^\tau$  satisfies

$$\mu_k^\tau \in \operatorname{argmin} \left\{ \mathbb{W}(\mu) + \frac{\pi \|\mu - \mu_{k-1}^\tau\|_{\text{flat}}^2}{2\tau} : \mu = \sum_{i=1}^M d_{i,0} \delta_{x_i}, \|\mu - \mu_{k-1}^\tau\|_{\text{flat}} \leq \delta \right\}.$$

By (6.18) we have that  $x_k^\tau$  is a solution of the discrete gradient flow of  $W$  from  $x_0 = (x_{1,0}, \dots, x_{M,0})$  and by (6.17) that  $\tilde{k}_\delta^\tau + 1 = k_\delta^\tau$ .  $\square$

## 7. DISCRETE GRADIENT FLOW OF $F_\varepsilon$ WITH $L^2$ DISSIPATION

In this section we introduce the discrete gradient flows of  $F_\varepsilon$  with  $L^2$  dissipation (for the  $L^2$  norm, we will use the notation introduced in (2.3)).



**Definition 7.1.** Fix  $\delta > 0$  and let  $\varepsilon, \tau > 0$ . Given  $u_{\varepsilon,0} \in \mathcal{AF}_\varepsilon(\Omega)$ , we say that  $\{u_{\varepsilon,k}^\tau\}$ , with  $k \in \mathbb{N} \cup \{0\}$ , is a solution of the  $L^2$  discrete gradient flow of  $F_\varepsilon$  from  $u_{\varepsilon,0}$  if  $u_{\varepsilon,0}^\tau = u_{\varepsilon,0}$ , and for any  $k \in \mathbb{N}$ ,  $u_{\varepsilon,k}^\tau$  satisfies

$$(7.1) \quad u_{\varepsilon,k}^\tau \in \operatorname{argmin} \left\{ F_\varepsilon(u) + \frac{\|e^{2\pi i u} - e^{2\pi i u_{\varepsilon,k-1}^\tau}\|_{L^2}^2}{2\tau |\log \tau|} : u \in \mathcal{AF}_\varepsilon(\Omega), \right. \\ \left. \|\mu(u) - \mu(u_{\varepsilon,k-1}^\tau)\|_{\text{flat}} \leq \delta \right\}.$$

The constraint  $\|\mu(u) - \mu(u_{\varepsilon,k-1}^\tau)\|_{\text{flat}} \leq \delta$  is not closed in the  $L^2$  topology. Nevertheless, in Subsection 7.2 we prove an existence result for such a discrete gradient flow.

In the parabolic flow of Ginzburg-Landau functionals it is well known that, as  $\varepsilon \rightarrow 0$ , the dynamics becomes slower and slower, and in order to capture a non trivial dynamics it is needed to scale the time by  $|\log \varepsilon|$  (see for instance [37]). In our discrete in time evolution, with  $\tau \gg \varepsilon$ , it turns out that the natural scaling involves the time step  $\tau$  instead of the length scale  $\varepsilon$ . Such a time-scaling is plugged into the discrete dynamics through the  $1/|\log \tau|$  pre-factor in front of the  $L^2$  dissipation.

As in Section 6, we want to consider the limit as  $\varepsilon \rightarrow 0$  of such a discrete gradient flow. To this purpose, we will exploit the  $\Gamma$ -convergence analysis developed in Section 4.4. The limit dynamics will be described by a discrete gradient flow (that we shall define in the following) of the functional  $\mathcal{W}$  (defined in (4.22)).

Let  $v_0 \in \mathcal{D}_M$  (see (4.21)) be an initial condition with  $\mathcal{W}(v_0) < +\infty$ , and let  $u_{\varepsilon,0}$  be a recovery sequence for  $v_0$  in the sense of (4.26). We will show that the solutions  $u_{\varepsilon,k}^\tau$  of the  $L^2$  discrete gradient flow of  $F_\varepsilon$  from  $u_{\varepsilon,0}$  converge (according with the topology of our  $\Gamma$ -convergence analysis in Subsection 4.4) to some limit  $v_k^\tau$ . Moreover, at each time step  $k$ ,  $v_k^\tau \in \mathcal{D}_M$ , the  $\Gamma$ -limit  $\mathcal{W}$  is finite, and the degrees of the singularities coincide with the degrees  $d_{i,0}$  of the initial datum. Finally,  $\{v_k^\tau\}$  is a solution of the  $L^2$  discrete gradient flow according with the following definition.

**Definition 7.2.** Let  $\delta, \tau > 0$  and  $K \in \mathbb{N}$ . We say that  $\{v_k^\tau\}$ , with  $k = 0, 1, \dots, K$ , is a solution of the  $L^2$  discrete gradient flow of  $\mathcal{W}$  from  $v_0$  if  $v_0^\tau = v_0$  and, for any  $k = 1, \dots, K$ ,  $v_k^\tau$  satisfies

$$(7.2) \quad v_k^\tau \in \operatorname{argmin} \left\{ \mathcal{W}(v) + \frac{\|v - v_{k-1}^\tau\|_{L^2}^2}{2\tau |\log \tau|} : Jv = \sum_{i=1}^M d_{i,0} \delta_{y_{i,k}}, y_{i,k} \in \Omega, \right. \\ \left. v \in H_{\text{loc}}^1(\Omega \setminus \cup_{i=1}^M \{y_{i,k}\}; \mathcal{S}^1), \|Jv - Jv_{k-1}^\tau\|_{\text{flat}} \leq \delta \right\}.$$

As in Section 6, we first do the asymptotic analysis as  $\tau \rightarrow 0$ . In contrast with the flat gradient flow of  $W$ , such step will require a big effort, and will involve many ingredients developed in [37].

**7.1.  $L^2$  discrete gradient flow of  $\mathcal{W}$ .** Let  $v_0 \in \mathcal{D}_M$  with  $Jv_0 = \sum_{i=1}^M d_{i,0} \delta_{x_{i,0}}$ , and fix  $\delta > 0$  such that (6.4) holds true.

**Definition 7.3.** We say that a solution of the  $L^2$  discrete gradient flow  $\{v_k^\tau\}$  of  $\mathcal{W}$  from  $v_0$  is maximal if the minimum problem in (7.2) does not admit a solution for  $k = K + 1$ .

Let  $\{v_k^\tau\}$  be a maximal solution of the  $L^2$  discrete gradient flow of  $\mathcal{W}$  from  $v_0$ , let  $Jv_k^\tau := \sum_{i=1}^M d_{i,0} \delta_{x_{i,k}^\tau}$ , and let  $k_\delta^\tau$  be defined as in (6.5).

**Remark 7.4.** Since for any  $i = 1, \dots, M$ , we have that  $|d_{i,0}| = 1$  and thanks to the constraint  $\|Jv_k^\tau - Jv_{k-1}^\tau\|_{\text{flat}} \leq \delta$ , we get that at each step  $k = 1, \dots, k_\delta^\tau$  and for each singularity  $x_{i,k-1}^\tau$  of  $Jv_{k-1}^\tau$ , there is exactly one singularity of  $Jv_k^\tau$  whose distance from  $x_{i,k-1}^\tau$  is less than  $\delta$ . We label this singularity  $x_{i,k}^\tau$ .

The above remark guarantees that the following definition is well posed.

**Definition 7.5.** We set  $x_k^\tau := (x_{1,k}^\tau, \dots, x_{M,k}^\tau)$ , where  $x_{i,k}^\tau$  are labeled according with Remark 7.4. Moreover, we define  $x^\tau(t) := (x_1^\tau(t), \dots, x_M^\tau(t))$  as the piecewise affine in time interpolation of  $\{x_k^\tau\}$ .

As in Section 6 we have that  $x_k^\tau \in K_\delta$ , where  $K_\delta$  is defined in (6.7). Moreover, the energy  $\mathcal{W}$  is clearly decreasing in  $k$ . Since, for every  $k = 1, \dots, k_\delta^\tau$  we have

$$\frac{\|v_k^\tau - v_{k-1}^\tau\|_{L^2}^2}{2\tau|\log \tau|} \leq \mathcal{W}(v_{k-1}^\tau) - \mathcal{W}(v_k^\tau),$$

then

$$(7.3) \quad \sum_{k=1}^{k_\delta^\tau} \frac{\|v_k^\tau - v_{k-1}^\tau\|_{L^2}^2}{2\tau|\log \tau|} \leq \mathcal{W}(v_0) - \mathcal{W}(v_{k_\delta^\tau}^\tau) \leq \mathcal{W}(v_0) - W(x_{k_\delta^\tau}^\tau) \leq \mathcal{W}(v_0) - W(x_0) + C_\delta,$$

where  $C_\delta$  is defined in (6.8).

**Proposition 7.6.** For every  $k = 0, 1, \dots, k_\delta^\tau$  we have that  $\|Jv_k^\tau - Jv_{k-1}^\tau\|_{\text{flat}} < C\sqrt{\tau|\log \tau|}$ , where  $C > 0$  depends only on  $\delta$  (and on the initial condition  $v_0$ ).

*Proof.* Fix  $1 \leq k \leq k_\delta^\tau$  and  $1 \leq i \leq M$ . Set  $\rho_{i,k}^\tau := \frac{1}{4} \text{dist}(x_{i,k}^\tau, x_{i,k-1}^\tau)$ . Note that

$$(7.4) \quad \deg(v_k^\tau, \partial B_{\rho_{i,k}^\tau}(x_{i,k}^\tau)) \neq 0 = \deg(v_{k-1}^\tau, \partial B_{\rho_{i,k}^\tau}(x_{i,k}^\tau)).$$

Moreover, since  $\mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0)$ , from (4.24) we have that

$$(7.5) \quad \int_{B_{2\rho_{i,k}^\tau}(x_{i,k}^\tau) \setminus B_{\rho_{i,k}^\tau}(x_{i,k}^\tau)} (|\nabla v_k^\tau|^2 + |\nabla v_{k-1}^\tau|^2) dx \leq 2\mathcal{W}(v_0) + C.$$

As a consequence of (7.4) and (7.5), we have that

$$(7.6) \quad (\text{dist}(x_{i,k}^\tau, x_{i,k-1}^\tau))^2 \leq C \int_{B_{2\rho_{i,k}^\tau}(x_{i,k}^\tau) \setminus B_{\rho_{i,k}^\tau}(x_{i,k}^\tau)} |v_k^\tau - v_{k-1}^\tau|^2 dx.$$

Indeed, if by contradiction (7.6) does not hold, by a scaling argument we could find two sequences  $\{w_1^n\}$  and  $\{w_2^n\}$  of functions in  $H^1(B_2 \setminus B_1; \mathcal{S}^1)$  such that

$$\int_{B_2 \setminus B_1} (|\nabla w_1^n|^2 + |\nabla w_2^n|^2) dx \leq 2\mathcal{W}(v_0) + C, \quad \int_{B_2 \setminus B_1} |w_1^n - w_2^n|^2 dx \rightarrow 0,$$

and such that  $\deg(w_1^n, \partial B_\rho) \neq \deg(w_2^n, \partial B_\rho)$  for almost every  $\rho \in [1, 2]$ . This is impossible in view of the stability of the degree with respect to uniform convergence for continuous maps from  $\mathcal{S}^1$  to  $\mathcal{S}^1$ .

Now, from (7.3) we have that

$$\int_{B_{2\rho_{i,k}^\tau}(x_{i,k}^\tau) \setminus B_{\rho_{i,k}^\tau}(x_{i,k}^\tau)} |v_k^\tau - v_{k-1}^\tau|^2 dx \leq C\tau|\log \tau|,$$

which together with (7.6) yields

$$(7.7) \quad \|Jv_k^\tau - Jv_{k-1}^\tau\|_{\text{flat}} \leq C\sqrt{\tau|\log \tau|}.$$

□

For every  $k = 0, 1, \dots, k_\delta^\tau$  we set

$$(7.8) \quad D_k^\tau := \mathcal{W}(v_k^\tau) - W(x_k^\tau).$$

Moreover, set  $\tilde{T}_\delta := \liminf_{\tau \rightarrow 0} k_\delta^\tau \tau$ , and define for any  $t \in [0, \tilde{T}_\delta)$ , the *energy excess*

$$(7.9) \quad D(t) = \limsup_{\tau \rightarrow 0} D_{\lfloor t/\tau \rfloor}^\tau \geq 0.$$

Since  $\mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0)$ , by (6.6) we have

$$(7.10) \quad D_k^\tau = \mathcal{W}(v_k^\tau) - W(x_k^\tau) \leq \mathcal{W}(v_0) - W(x_k^\tau) \leq D(0) + C_\delta,$$

where  $C_\delta$  is defined in (6.8). From now on we will say that an initial condition  $v_0$  is *well prepared* if  $W(x_0) = \mathcal{W}(v_0)$ , i.e.,  $D(0) = 0$ .

We are in a position to state the main theorem of this section.

**Theorem 7.7.** *Let  $v_0$  be a well prepared initial condition. Let  $\{v_k^\tau\}_{\tau > 0}$  be a family of maximal solutions of the  $L^2$  discrete gradient flow of  $\mathcal{W}$  from  $v_0$ . Then,*

$$(7.11) \quad \tilde{T}_\delta := \liminf_{\tau \rightarrow 0} k_\delta^\tau \tau \geq T_\delta,$$

where  $k_\delta^\tau$  is defined in (6.5) and  $T_\delta$  is defined in (6.12).

Moreover, for every  $0 < T < \tilde{T}_\delta$ ,  $x^\tau \rightarrow x$  uniformly on  $[0, T]$ , where  $x^\tau$  is defined in Definition 7.5, and  $x$  is the solution of the gradient flow of  $W$  from  $x_0$  according with Definition 6.3. Finally,  $D(t) = 0$  for every  $0 \leq t < \tilde{T}_\delta$  and  $\tilde{T}_\delta \rightarrow T^*$  as  $\delta \rightarrow 0$ .

**Remark 7.8.** As a consequence of the uniform convergence of  $x^\tau$  and the estimate (7.7), one can prove that the 1-current associated to the polygonal  $x^\tau$  (with the natural orientation and multiplicity given by the integers  $d_{i,0}$ ), converges to the current associated to the limit  $x$  in the flat norm.

The proof of Theorem 7.7 is postponed at the end of the section, and will be obtained as a consequence of Theorem 7.9 below, which can be regarded as the discrete in time counterpart of Theorem 1.4 in [38].

**Theorem 7.9.** *Let  $v_0$  be a well prepared initial datum, i.e., with  $W(x_0) = \mathcal{W}(v_0)$ . Let  $\{v_k^\tau\}_{\tau > 0}$  be solutions of the  $L^2$  discrete gradient flow for  $\mathcal{W}$  from  $v_0$ , let  $T > 0$  be such that  $k_\delta^\tau \geq \lfloor T/\tau \rfloor$  for every  $\tau$ , and assume that  $x^\tau \rightarrow x$  uniformly in  $[0, T]$  for some  $x(t) \in H^1([0, T]; \Omega^M)$ . Moreover, assume that (i) and (ii) below are satisfied:*

(i) (Lower bound) For any  $s \in [0, T]$

$$\liminf_{\tau \rightarrow 0} \frac{\tau}{|\log \tau|} \sum_{k=1}^{\lfloor \frac{s}{\tau} \rfloor} \left\| \frac{v_k^\tau - v_{k-1}^\tau}{\tau} \right\|_{L^2}^2 \geq \pi \int_0^s |\dot{x}(t)|^2 dt.$$

(ii) (Construction) For any  $k = 0, 1, \dots, \lfloor T/\tau \rfloor - 1$ , there exists a field  $w_{k+1}^\tau \in H_{\text{loc}}^1(\Omega \setminus \cup_{i=1}^M \{x_{i,k}^\tau - \frac{\tau}{\pi} \partial_{x_i} W(x_k^\tau)\}; \mathcal{S}^1)$  and a constant  $M_\delta > 0$  such that

$$\begin{aligned} \mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) &\geq \frac{\tau}{\pi} |\nabla W(x_k^\tau)|^2 - \tau M_\delta D_k^\tau + o(\tau), \\ \frac{1}{|\log \tau|} \left\| \frac{w_{k+1}^\tau - v_k^\tau}{\tau} \right\|_{L^2}^2 &\leq \frac{1}{\pi} |\nabla W(x_k^\tau)|^2 + o(1). \end{aligned}$$

Then,  $D(t) = 0$  for every  $t \in [0, T]$ , and  $x(t)$  is a solution of the gradient flow (6.3) of  $W$  from  $x_0$  on  $[0, T]$ .

*Proof.* By (ii) and by the minimality of  $v_{k+1}^\tau$ , we have

$$\begin{aligned} \mathcal{W}(v_k^\tau) - \mathcal{W}(v_{k+1}^\tau) &= \mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) + \mathcal{W}(w_{k+1}^\tau) - \mathcal{W}(v_{k+1}^\tau) \\ &\geq \frac{\tau}{\pi} |\nabla W(x_k^\tau)|^2 - \frac{\tau}{2|\log \tau|} \left\| \frac{w_{k+1}^\tau - v_k^\tau}{\tau} \right\|_{L^2}^2 + \frac{\tau}{2|\log \tau|} \left\| \frac{v_{k+1}^\tau - v_k^\tau}{\tau} \right\|_{L^2}^2 - \tau M_\delta D_k^\tau + o(\tau) \\ &\geq \frac{\tau}{2\pi} |\nabla W(x_k^\tau)|^2 + \frac{\tau}{2|\log \tau|} \left\| \frac{v_{k+1}^\tau - v_k^\tau}{\tau} \right\|_{L^2}^2 - \tau M_\delta D_k^\tau + o(\tau). \end{aligned}$$

Now, let  $s \in [0, T]$ . Summing over  $k = 0, 1, \dots, \lfloor s/\tau \rfloor - 1$ , we have

$$\begin{aligned} \mathcal{W}(v_0^\tau) - \mathcal{W}(v_{\lfloor s/\tau \rfloor}^\tau) &\geq \frac{1}{2\pi} \int_0^{\tau \lfloor s/\tau \rfloor - \tau} |\nabla W(x_{\lfloor t/\tau \rfloor}^\tau)|^2 dt \\ &+ \frac{\tau}{2|\log \tau|} \sum_{k=0}^{\lfloor s/\tau \rfloor - 1} \left\| \frac{v_{k+1}^\tau - v_k^\tau}{\tau} \right\|_{L^2}^2 - M_\delta \int_0^{\tau \lfloor s/\tau \rfloor - \tau} D_{\lfloor t/\tau \rfloor}^\tau dt + o(1). \end{aligned}$$

By the uniform convergence of  $x^\tau$  to  $x$  in  $[0, T]$  and the fact that  $x \in H^1$ , we have that also  $x_{\lfloor \cdot/\tau \rfloor}^\tau \rightarrow x$  uniformly in  $[0, T]$ . Hence, passing to the lim inf as  $\tau \rightarrow 0$ , using (i) and (7.10), we get

$$(7.12) \quad \liminf_{\tau \rightarrow 0} (\mathcal{W}(v_0^\tau) - \mathcal{W}(v_{\lfloor s/\tau \rfloor}^\tau)) \geq \frac{1}{2} \int_0^s \frac{1}{\pi} |\nabla W(x(t))|^2 + \pi |\dot{x}(t)|^2 dt - M_\delta \int_0^s D(t) dt,$$

where  $D(t)$  is defined in (7.9).

Since  $\mathcal{W}(v_0^\tau) = \mathcal{W}(v_0) = W(x_0) = W(x(0))$ , we have immediately that

$$(7.13) \quad \liminf_{\tau \rightarrow 0} (\mathcal{W}(v_0^\tau) - \mathcal{W}(v_{\lfloor s/\tau \rfloor}^\tau)) = W(x(0)) - W(x(s)) - D(s).$$

Combining this with (7.12) yields

$$(7.14) \quad \begin{aligned} W(x(0)) - W(x(s)) - D(s) &\geq \frac{1}{2} \int_0^s \frac{1}{\pi} |\nabla W(x(t))|^2 + \pi |\dot{x}(t)|^2 dt \\ &- M_\delta \int_0^s D(t) dt. \end{aligned}$$

Since

$$(7.15) \quad \begin{aligned} W(x(0)) - W(x(s)) &= \int_0^s \langle -\nabla W(x(t)), \dot{x}(t) \rangle dt \\ &\leq \frac{1}{2} \int_0^s \frac{1}{\pi} |\nabla W(x(t))|^2 + \pi |\dot{x}(t)|^2 dt, \end{aligned}$$

then,

$$D(s) \leq M_\delta \int_0^s D(t) dt.$$

Since  $D(0) = 0$  by assumption, from Gronwall's lemma we find that  $D(s) = 0$  for all  $s \in [0, T]$ .

Using that  $D(s) = 0$ , by (7.14) and (7.15) we obtain

$$\int_0^s \left| \frac{1}{\sqrt{\pi}} \nabla W(x(t)) + \sqrt{\pi} \dot{x}(t) \right|^2 dt \leq 0,$$

and hence  $\dot{x}(t) = -\frac{1}{\pi} \nabla W(x(t))$  a.e. in  $[0, T]$ .  $\square$

The following propositions are devoted to show that the hypothesis of Theorem 7.9 are satisfied by the  $L^2$  discrete gradient flow defined in Definition 7.2.

**Proposition 7.10.** *Let  $\{v_k^\tau\}_{\tau>0}$  be a family of maximal solutions of the  $L^2$  discrete gradient flow of  $\mathcal{W}$  from  $v_0$ , let  $k_\delta^\tau$  be as in (6.5), and let  $x^\tau$  be defined as in Definition 7.5. Then*

$$(7.16) \quad \tilde{T}_\delta = \liminf_{\tau \rightarrow 0} k_\delta^\tau \tau \geq \pi \frac{c_\delta^2}{C_\delta},$$

where  $C_\delta$  and  $c_\delta$  are defined in (6.8) and (6.4) respectively.

Moreover, there exists a map  $x \in H^1([0, \tilde{T}_\delta]; \Omega^M)$  such that, up to a subsequence,  $x^\tau \rightarrow x$  uniformly on  $[0, T]$  for every  $0 < T < \tilde{T}_\delta$  and

$$(7.17) \quad \liminf_{\tau \rightarrow 0} \frac{\tau}{|\log \tau|} \sum_{k=1}^{\lfloor \frac{T}{\tau} \rfloor} \left\| \frac{v_k^\tau - v_{k-1}^\tau}{\tau} \right\|_{L^2}^2 \geq \pi \int_0^T |\dot{x}(t)|^2 dt.$$

*Proof.* The starting point of the proof consists in applying Theorem A.1 to piecewise affine interpolations in time of suitable regularizations of  $v_k^\tau$ . Clearly, the Ginzburg-Landau energy of  $v_k^\tau$  is not bounded. By the very definition of  $\mathcal{W}$ , we have

$$\frac{1}{2} \int_{\Omega \setminus \cup_i B_\tau(x_{i,k}^\tau)} |\nabla v_k^\tau|^2 dx - M\pi |\log \tau| \leq \mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0).$$

Moreover, the Dirichlet energy stored in  $B_\tau(x_{i,k}^\tau) \setminus B_{\tau/2}(x_{i,k}^\tau)$  is bounded. Therefore, by standard cut off arguments, we can easily construct fields  $\hat{v}_k^\tau$  which coincide with  $v_k^\tau$  in  $\Omega \setminus \cup_i B_\tau(x_{i,k}^\tau)$ , are equal to zero in  $B_{\tau/2}(x_{i,k}^\tau)$  and satisfy

$$(7.18) \quad \frac{1}{2} \int_{\Omega} |\nabla \hat{v}_k^\tau|^2 dx \leq M\pi |\log \tau| + C.$$

Then, we consider the piecewise affine in time interpolation  $\hat{v}^\tau : [0, +\infty) \times \Omega \rightarrow \mathbb{R}^2$  of  $\hat{v}_k^\tau$  defined by

$$\hat{v}^\tau(t, x) := \begin{cases} \left(1 - \frac{t - k\tau}{\tau}\right) \hat{v}_k^\tau(x) + \frac{t - k\tau}{\tau} \hat{v}_{k+1}^\tau(x) & \text{if } k\tau \leq t \leq (k+1)\tau \leq k_\delta^\tau \tau, \\ \hat{v}_{k_\delta^\tau}^\tau(x) & \text{if } t > k_\delta^\tau \tau. \end{cases}$$

For every fixed  $t > 0$ , we denote by  $\hat{\mu}^\tau(t)$  the (space) Jacobian of  $\hat{v}^\tau$ .

We will prove the theorem in several steps.

*Step 1.* *There exists a map  $x \in C^{0, \frac{1}{2}}([0, +\infty); \Omega^M)$  such that up to a subsequence, for every  $T > 0$  we have*

$$(7.19) \quad \hat{\mu}^\tau(t) \xrightarrow{\text{flat}} \mu(t) := \pi \sum_{i=1}^M d_{i,0} \delta_{x_i(t)} \quad \text{for every } t \in [0, T].$$

Fix  $T > 0$ . By the convexity of the Dirichlet energy and by (7.18), it follows that for any  $t \in [0, T]$

$$(7.20) \quad \frac{1}{2} \int_{\Omega} |\nabla \hat{v}^\tau(t, x)|^2 dx \leq M\pi |\log \tau| + C.$$

Moreover, by the definition of  $\hat{v}_k^\tau$ , it follows that for any  $k = 0, \dots, k_\delta^\tau - 1$

$$\|\hat{v}_{k+1}^\tau - \hat{v}_k^\tau\|_{L^2}^2 \leq \|v_{k+1}^\tau - v_k^\tau\|_{L^2}^2 + C\tau^2;$$

therefore, by (7.3), we get

$$\int_{[0, T] \times \Omega} |\partial_t \hat{v}^\tau|^2 dt dx = \sum_{k=0}^{k_\delta^\tau - 1} \tau \left\| \frac{\hat{v}_{k+1}^\tau - \hat{v}_k^\tau}{\tau} \right\|_{L^2}^2 \leq C |\log \tau|.$$

It is easy to see that for every  $t \in [k\tau, (k+1)\tau]$

$$(7.21) \quad \frac{1}{\tau} \int_{\Omega} (1 - |\hat{v}^\tau(t, x)|^2)^2 dx \leq \frac{C}{\tau} \|\hat{v}_{k+1}^\tau - \hat{v}_k^\tau\|_{L^2}^2 \leq C |\log \tau|.$$

In conclusion, for every  $t \in [0, T]$  we have

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla \hat{v}^\tau|^2 + \frac{1}{\tau} (1 - |\hat{v}^\tau|^2)^2 dx &\leq C |\log \tau| \\ \int_{[0, T] \times \Omega} |\partial_t \hat{v}^\tau|^2 dt dx &\leq C |\log \tau|. \end{aligned}$$

By Theorem A.1 applied with  $\varepsilon = \sqrt{\tau}$  and recalling that  $\mu(0) = \mu_0 = \sum_{i=1}^M d_{i,0} \delta_{x_{i,0}}$ , we deduce that

$$\mu(t) = \pi \sum_{i=1}^{M(t)} d_i \delta_{x_i(t)}, \quad \text{for all } t \in [0, T]$$

for some  $x_i(t) \in C^{0, \frac{1}{2}}([0, T_i]; \Omega)$  with  $T_i \leq T$ . Here  $T_i$  represents the first time when  $x_i(t)$  reaches  $\partial\Omega$ . Finally, by construction  $x_i(t)$  are defined on  $[0, T]$ , distinct, and contained in  $\Omega$ . The conclusion follows by a standard diagonalization argument.

*Step 2. Set*

$$\hat{T}_\delta := \inf \left\{ t \in [0, +\infty) : \min \left\{ \frac{1}{2} \text{dist}_{i \neq j} (x_i(t), x_j(t)), \text{dist}(x_i(t), \partial\Omega) \right\} \leq 2\delta \right\}.$$

Then,  $\hat{T}_\delta \geq \tilde{T}_\delta > 0$ .

Since  $x \in C^{0, \frac{1}{2}}$  and  $x(0) = x_0$  satisfies

$$\min \left\{ \frac{1}{2} \text{dist}_{i \neq j} (x_{i,0}, x_{j,0}), \text{dist}(x_{i,0}, \partial\Omega) \right\} > 2\delta,$$

we have  $\hat{T}_\delta > 0$ . Fixed  $t > \tilde{T}_\delta$ , by construction and Step 1 we have that

$$(7.22) \quad \hat{\mu}^\tau(t) = \hat{\mu}_{k_\delta^\tau}^\tau \xrightarrow{\text{flat}} \pi \sum_{i=1}^M d_{i,0} \delta_{x_i(\tilde{T}_\delta)}.$$

Set  $\mu^\tau(t) := \pi \sum_{i=1}^M d_{i,0} \delta_{x_i^\tau(t)}$  for  $t \leq k_\delta^\tau \tau$ , where  $x_i^\tau$  are defined in Definition 7.5. Let  $0 \leq k \leq k_\delta^\tau$ . Since  $\text{supp}(\hat{\mu}^\tau(k\tau)), \text{supp}(\mu^\tau(k\tau)) \subseteq \cup_i B_\tau(x_{i,k}^\tau)$  and

$\hat{\mu}^\tau(k\tau)(B_\tau(x_{i,k}^\tau)) = \mu^\tau(k\tau)(B_\tau(x_{i,k}^\tau))$ , for any  $\varphi \in C_c^{0,1}(\Omega)$  we have

$$\begin{aligned} \langle \hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau), \varphi \rangle &= \sum_{i=1}^M \langle \hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau), \varphi - \bar{\varphi}_i \rangle \\ &\leq (|\hat{\mu}^\tau(k\tau)|(\Omega) + |\mu^\tau(k\tau)|(\Omega)) \tau \|\nabla \varphi\|_{L^\infty}, \end{aligned}$$

where  $\bar{\varphi}_i$  denotes the average of  $\varphi$  on  $B_\tau(x_{i,k}^\tau)$ . Since, by Remark 4.4, we have

$$|\hat{\mu}^\tau(k\tau)|(\Omega) \leq C \sum_{i=1}^M \int_{B_\tau(x_{i,k}^\tau) \setminus B_{\frac{\tau}{2}}(x_{i,k}^\tau)} |\nabla \hat{v}_k^\tau|^2 dx \leq C,$$

we deduce that

$$(7.23) \quad \max_{k=0,1,\dots,k_\delta^\tau} \|\hat{\mu}^\tau(k\tau) - \mu^\tau(k\tau)\|_{\text{flat}} \leq C\tau.$$

This fact together with (7.22) yields

$$\sum_{i=1}^{M_0} d_{i,0} \delta_{x_{i,k_\delta^\tau}^\tau} \xrightarrow{\text{flat}} \sum_{i=1}^M d_{i,0} \delta_{x_i(\tilde{T}_\delta)}.$$

Therefore, by the very definition of  $k_\delta^\tau$ , we have that for every  $t > \tilde{T}_\delta$

$$\min\left\{\frac{1}{2} \text{dist}_{i \neq j}(x_i(t), x_j(t)), \text{dist}(x_i(t), \partial\Omega)\right\} \leq 2\delta.$$

By continuity, the previous inequality holds also for  $t = \tilde{T}_\delta$ , so that we conclude that  $\tilde{T}_\delta \geq \hat{T}_\delta > 0$ .

*Step 3.*  $x^\tau \rightarrow x$  uniformly on the compact subsets of  $[0, \tilde{T}_\delta]$ .

Let us show that

$$(7.24) \quad \max_{k=0,1,\dots,k_\delta^\tau} \|\hat{\mu}^\tau(k\tau) - \mu(k\tau)\|_{\text{flat}} =: \|\hat{\mu}^\tau(\bar{k}^\tau\tau) - \mu(\bar{k}^\tau\tau)\|_{\text{flat}} \rightarrow 0.$$

Up to a subsequence we can assume that  $\bar{k}^\tau\tau$  converges to some  $t_0 \in [0, \tilde{T}_\delta]$ . The fields

$$(7.25) \quad \tilde{v}^\tau(t, x) := \begin{cases} \hat{v}^\tau(t, x) & \text{if } t \leq \bar{k}^\tau\tau \\ \hat{v}^\tau(\bar{k}^\tau\tau, x) & \text{if } t > \bar{k}^\tau\tau \end{cases}$$

satisfy the assumptions of Theorem A.1, applied with  $\varepsilon = \sqrt{\tau}$ ; therefore, denoting by  $\tilde{\mu}^\tau(t)$  the (space) Jacobian of  $\tilde{v}^\tau$ , we have that, up to a subsequence,

$$(7.26) \quad \tilde{\mu}^\tau(t) \xrightarrow{\text{flat}} \tilde{\mu}(t) := \begin{cases} \mu(t) & \text{if } t \leq t_0 \\ \mu(t_0) & \text{if } t > t_0, \end{cases}$$

where the structure of  $\tilde{\mu}$  is a consequence of the continuity guaranteed by Theorem A.1. From (7.26) one can easily prove that  $\hat{\mu}^\tau(\bar{k}^\tau\tau) - \mu(t_0)$  converges to zero in the flat norm and hence we get (7.24). Combining (7.23) with (7.24) we also deduce that

$$(7.27) \quad \max_{k=0,1,\dots,k_\delta^\tau} \|\mu^\tau(k\tau) - \mu(k\tau)\|_{\text{flat}} \rightarrow 0.$$

Moreover, by the construction of  $\mu^\tau$  and (7.6), we have that

$$(7.28) \quad \max_{t \in [0, k_\delta^\tau\tau]} \|\mu^\tau(t) - \mu^\tau(\lfloor t/\tau \rfloor \tau)\|_{\text{flat}} \rightarrow 0.$$

Using (7.28), (7.27) and that  $\max_{t \in [0, k_\delta^\tau \tau]} \|\mu(\lfloor t/\tau \rfloor \tau) - \mu(t)\|_{\text{flat}} \rightarrow 0$ , by the triangular inequality we conclude that

$$\max_{t \in [0, k_\delta^\tau \tau]} |x^\tau(t) - x(t)| = \max_{t \in [0, k_\delta^\tau \tau]} \|\mu^\tau(t) - \mu(t)\|_{\text{flat}} \rightarrow 0.$$

*Step 4.* The function  $x$  belongs to  $H^1([0, \tilde{T}_\delta]; \Omega^M)$ , and, for any  $T \in [0, \tilde{T}_\delta]$ , (7.17) holds true. In particular,  $\tilde{T}_\delta \geq \pi c_\delta^2 / C_\delta$ .

The proof of this step is obtained as a consequence of Proposition A.3 applied to the fields  $\hat{v}^\tau$ , with  $\varepsilon = \tau$  and  $\tilde{T} = \tilde{T}_\delta$ . By (7.3) and recalling (7.21), it easily follows that

$$\frac{1}{\tau^2} \int_0^{\tilde{T}_\delta} \int_\Omega (1 - |\hat{v}^\tau|^2)^2 \, dx \, dt \leq C \sum_{k=1}^{k_\delta^\tau} \tau \left\| \frac{\hat{v}_{k+1}^\tau - \hat{v}_k^\tau}{\tau} \right\|_{L^2}^2 \leq C |\log \tau|;$$

and hence (A.5) holds with  $\varepsilon = \tau$  and  $w_\varepsilon = v^\tau$ . This fact together with (7.19) and (7.20) guarantees that the hypothesis of Proposition A.5 are satisfied. Therefore, we deduce that (7.17) holds true with  $v_k^\tau$  replaced by  $\hat{v}_k^\tau$ . Since  $\|\hat{v}_k^\tau - v_k^\tau\|_{L^2} = O(\tau^2)$ , we deduce (7.17).

Finally, by (7.3) and recalling (6.8), we have

$$(7.29) \quad \begin{aligned} \pi \int_0^T |\dot{x}(t)|^2 \, dt &\leq \liminf_{\tau \rightarrow 0} \frac{\tau}{|\log \tau|} \sum_{k=1}^{\lfloor \frac{T}{\tau} \rfloor} \left\| \frac{v_k^\tau - v_{k-1}^\tau}{\tau} \right\|_{L^2}^2 \\ &\leq \liminf_{\tau \rightarrow 0} (\mathcal{W}(v_0) - \mathcal{W}(v_{\lfloor \frac{T}{\tau} \rfloor}^\tau)) \leq \liminf_{\tau \rightarrow 0} (W(x_0) - W(x_{\lfloor \frac{T}{\tau} \rfloor}^\tau)) \leq C_\delta. \end{aligned}$$

By Hölder inequality, and recalling (6.4), we conclude

$$(7.30) \quad c_\delta \leq |x(\tilde{T}_\delta) - x(0)| \leq \int_0^{\tilde{T}_\delta} |\dot{x}| \, dt \leq \|\dot{x}\|_{L^2([0, \tilde{T}_\delta]; \mathbb{R}^{2M})} \sqrt{\tilde{T}_\delta}.$$

By (7.29) and (7.30) we immediately get (7.16) □

Since we have proved assumption (i) in Theorem 7.10, it remains to prove only assumption (ii). To this aim, at each time step  $k = 0, 1, \dots, k_\delta^\tau$ , we construct, a field  $w_{k+1}^\tau$  whose vortices are obtained translating  $x_{i,k}^\tau$  in the direction of the renormalized energy  $\nabla W(x_k^\tau)$ . The variation of the energy  $\mathcal{W}$  associated to the fields  $v_k^\tau$  and  $w_{k+1}^\tau$  is proportional to the distance among the vortices of the two functions (i.e.  $|\nabla W(x_k^\tau)|$ ) up to an error given by the energy excess  $D_k^\tau$  defined in (7.10).

**Proposition 7.11.** *For any  $k = 0, 1, \dots, k_\delta^\tau - 1$ , there exists a field  $w_{k+1}^\tau \in H_{\text{loc}}^1(\Omega \setminus \bigcup_{i=1}^M \{x_{i,k}^\tau - \frac{\tau}{\pi} \partial_{x_i} W(x_k^\tau)\}; \mathcal{S}^1)$  such that*

$$(7.31) \quad \mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) \geq \frac{\tau}{\pi} |\nabla W(x_k^\tau)|^2 - M_\delta \tau D_k^\tau + o(\tau)$$

$$(7.32) \quad \frac{\|w_{k+1}^\tau - v_k^\tau\|_{L^2}^2}{\tau^2 |\log \tau|} \leq \frac{1}{\pi} |\nabla W(x_k^\tau)|^2 + o(1),$$

where  $M_\delta$  is a positive constant depending only on  $\delta$ .



*Proof.* Fix  $k \in \{0, 1, \dots, k_\delta^\tau - 1\}$ ; to ease notations we set

$$(7.33) \quad V_i = (V_{i1}, V_{i2}) := -\frac{1}{\pi} \partial_{x_i} W(x_k^\tau), \quad V := (V_1, \dots, V_M).$$

With a little abuse of notations, from now on we will set  $x_i := x_{i,k}^\tau$  and  $y_i = x_i + \tau V_i$  for every  $i = 1, \dots, M$ . By (6.5), the balls  $B_{\delta/2}(x_i)$  are pairwise disjoint and contained in  $\Omega$ .

In order to construct the field  $w_{k+1}^\tau$ , we wish to “push” the vortices  $x_i$  along the direction  $V_i$ . For every  $i = 1, \dots, M$ , we can find smooth, compactly supported vector fields in  $\Omega$ ,  $X_{i1}$  and  $X_{i2}$  such that

$$\begin{aligned} X_{i1}(x) &= (1, 0) & X_{i2}(x) &= (0, 1) & \text{for } x \in B_{\delta/2}(x_i), \\ X_{i1}(x) &= X_{i2}(x) & &= (0, 0) & \text{for } x \in B_{\delta/2}(x_j), j \neq i \end{aligned}$$

and such that  $\|\nabla X_{ij}\|_{L^\infty} \leq \frac{2}{\delta}$  for every  $i, j$ . Then, define  $X_V = \sum_{i=1}^M \sum_{j=1,2} V_{ij} X_{ij}$ . Since  $W$  is smooth in  $K_\delta$  (see (6.7)), there exists a constant  $M_\delta$  depending only on  $\delta$  such that

$$(7.34) \quad \|\det \nabla X_V\|_{L^\infty} \leq \frac{1}{2} M_\delta.$$

For any  $t \in [0, \tau]$ , we define  $\chi_t(x) := x + tX_V(x)$  for every  $x \in \Omega$ ; notice that  $\chi_t(x) = x + tV_i$  for  $x \in B_{\delta/2}(x_i)$ . For any  $t \in [0, \tau]$  let  $\Phi^t$  be the solution of

$$\begin{cases} \Delta \Phi^t = 2\pi \sum_{i=1}^M d_{i,0} \delta_{x_i + tV_i} & \text{in } \Omega \\ \Phi^t = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$(7.35) \quad R^t(x) := \Phi^t(x) - \sum_{i=1}^M d_{i,0} \log |x - x_i - tV_i|.$$

By definition  $R^t$  are smooth harmonic functions in  $\Omega$ ; we denote by  $\tilde{R}^t$  their harmonic conjugates with zero average in  $\Omega$ . Moreover, we denote by  $\theta_i^t$  the polar coordinates centered at  $x_i + tV_i$  and set  $\tilde{\Phi}^t := \sum_{i=1}^M d_{i,0} \theta_i^t + \tilde{R}^t$ . Notice that  $\nabla \tilde{\Phi}^t$  is nothing but the  $\pi/2$  rotation of  $\nabla \Phi^t$ . We define

$$(7.36) \quad \psi^t(\cdot) = \tilde{\Phi}^t(\chi_t(\cdot)) - \tilde{\Phi}^0(\cdot).$$

Notice that  $\psi^t$  is a smooth function in  $\Omega$ , the singularities at  $x_i$  canceling out, and that it is smooth in space-time. In particular, using (6.6) one can show that, for  $\tau$  small enough, there exists a constant  $C$  depending only on  $\delta$  such that

$$(7.37) \quad \sup_{t \in [0, \tau]} \left( \|\nabla \psi^t\|_{L^\infty(\Omega)} + \left\| \frac{d}{dt} \psi^t \right\|_{L^\infty(\Omega)} \right) \leq C.$$

For any  $0 < \sigma < \delta$ , we define  $\Omega_\sigma^t := \Omega \setminus \cup_{i=1}^M B_\sigma(x_i + tV_i)$ . By definition of  $\tilde{\Phi}^t$ , the renormalized energy associated to the configuration  $x_k^\tau + tV$  is given by

$$(7.38) \quad W(x_k^\tau + tV) = \lim_{\sigma \rightarrow 0} \frac{1}{2} \int_{\Omega_\sigma^t} |\nabla \tilde{\Phi}^t|^2 - M\pi |\log \sigma|.$$

Since  $v_k^\tau \in H^1(\Omega_\sigma^0; \mathcal{S}^1)$ , there exist a family  $\{L_i\}_{i=1, \dots, M}$  of cuts of the domain  $\Omega$  ( $L_i$  is a segment from  $x_i$  to  $\partial\Omega$ ) and a function  $\varphi^0 \in H^1(\Omega_\sigma^0 \setminus \cup_{i=1}^M \{L_i\}; \mathbb{R})$  such that  $v_k^\tau = e^{i\varphi^0}$ .

Recalling (7.36), we introduce the field  $w_{k+1}^\tau$  defined by the following identity (notice that  $\chi_\tau$  is invertible for  $\tau$  small enough)

$$(7.39) \quad w_{k+1}^\tau(\chi_\tau(x)) := v_k^\tau(x) e^{i\psi^\tau(x)} = e^{i(\varphi^0(x) + \psi^\tau(x))}.$$

By definition,  $w_{k+1}^\tau \in H^1(\Omega_\sigma^\tau; \mathcal{S}^1)$  and  $Jw_{k+1}^\tau = \sum_{i=1}^M d_{i,0} \delta_{y_i}$ .

We notice that if  $\varphi^0 = \tilde{\Phi}^0$ , then by (7.38) we get

$$(7.40) \quad \begin{aligned} \mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) &= \lim_{\sigma \rightarrow 0} \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla v_k^\tau|^2 dx - \frac{1}{2} \int_{\Omega_\sigma^\tau} |\nabla w_{k+1}^\tau|^2 dy \\ &= \lim_{\sigma \rightarrow 0} \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla \tilde{\Phi}^0|^2 dx - \frac{1}{2} \int_{\Omega_\sigma^\tau} |\nabla \tilde{\Phi}^\tau|^2 dy \\ (7.41) \quad &= W(x_k^\tau) - W(x_k^\tau + \tau V). \end{aligned}$$

Recalling (6.6) and (6.7), by Taylor expansion we conclude

$$(7.42) \quad W(x_k^\tau) - W(x_k^\tau - \frac{\tau}{\pi} \nabla W(x_k^\tau)) = \frac{\tau}{\pi} |\nabla W(x_k^\tau)|^2 + O(\tau^2).$$

We show now that  $w_{k+1}^\tau$  satisfies (7.31) even when  $v_k^\tau$  is not optimal in energy. To this purpose, we show that the difference  $\mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau)$  can be bounded from below by the variation of the renormalized energy up to an error given by the defect  $D_k^\tau$  defined in (7.8). More precisely, set

$$(7.43) \quad D_{\sigma,k}^\tau := \frac{1}{2} \int_{\Omega_\sigma^0} (|\nabla \varphi^0|^2 - |\nabla \tilde{\Phi}^0|^2) dx,$$

so that  $D_k^\tau = \lim_{\sigma \rightarrow 0} D_{\sigma,k}^\tau$ . We want to prove that, for  $0 < \sigma \ll \tau$ ,

$$(7.44) \quad \begin{aligned} \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla v_k^\tau|^2 dx - \frac{1}{2} \int_{\Omega_\sigma^\tau} |\nabla w_{k+1}^\tau|^2 dy &\geq \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla \tilde{\Phi}^0|^2 dx \\ &\quad - \frac{1}{2} \int_{\Omega_\sigma^\tau} |\nabla \tilde{\Phi}^\tau|^2 dy - M_\delta \tau D_{\sigma,k}^\tau + O(\sqrt{\sigma |\log \sigma|}). \end{aligned}$$

Notice that, taking the limit as  $\sigma \rightarrow 0$  in (7.44), we get

$$\mathcal{W}(v_k^\tau) - \mathcal{W}(w_{k+1}^\tau) \geq W(x_k^\tau) - W(x_k^\tau - \tau \nabla W(x_k^\tau)) - M_\delta \tau D_k^\tau,$$

which, in view of (7.42), concludes the proof of (7.31).

We now prove (7.44). By the change of variable  $y = \chi_\tau(x)$  and by definition of  $w_{k+1}^\tau$  in (7.39), we get

$$(7.45) \quad \begin{aligned} \frac{1}{2} \int_{\Omega_\sigma^\tau} |\nabla w_{k+1}^\tau|^2 dy &= \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla w_{k+1}^\tau(\chi_\tau)|^2 |J\chi_\tau| dx \\ &= \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla \varphi^0 + \nabla \psi^\tau|^2 |J\chi_\tau| dx \end{aligned}$$

We claim that the following two estimates hold.

$$(7.46) \quad \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla \varphi^0|^2 |J\chi_\tau| dx \leq \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla \tilde{\Phi}^0|^2 |J\chi_\tau| dx + (1 + M_\delta \tau) D_{\sigma,k}^\tau,$$

$$(7.47) \quad \int_{\Omega_\sigma^0} \langle \nabla \psi^\tau, \nabla \varphi^0 \rangle |J\chi_\tau| dx = \int_{\Omega_\sigma^0} \langle \nabla \psi^\tau, \nabla \tilde{\Phi}^0 \rangle |J\chi_\tau| dx + O(\sqrt{\sigma |\log \sigma|}).$$

By (7.46) and (7.47), we conclude the proof of (7.44) as follows: Using (7.36) and the change of variables  $y = \chi_\tau(x)$ , by (7.45) we get

$$\begin{aligned}
 (7.48) \quad & \frac{1}{2} \int_{\Omega_\tau^\sigma} |\nabla w_{k+1}^\tau|^2 \, dy \\
 & \leq \frac{1}{2} \int_{\Omega_\sigma^0} |\nabla \tilde{\Phi}^0 + \nabla \psi^\tau|^2 |J\chi_\tau| \, dx + (1 + M_\delta \tau) D_{\sigma,k}^\tau + O(\sqrt{\sigma |\log \sigma|}) \\
 & = \frac{1}{2} \int_{\Omega_\tau^\sigma} |\nabla \tilde{\Phi}^\tau|^2 \, dy + (1 + M_\delta \tau) D_{\sigma,k}^\tau + O(\sqrt{\sigma |\log \sigma|}).
 \end{aligned}$$

By (7.43) and straightforward algebraic manipulations we obtain (7.44).

Now, we will prove the claims (7.46) and (7.47). Claim (7.46) follows by

$$\begin{aligned}
 \frac{1}{2} \int_{\Omega_\sigma^0} (|\nabla \varphi^0|^2 - |\nabla \tilde{\Phi}^0|^2) |J\chi_\tau| \, dx & \leq \|J\chi_\tau\|_{L^\infty} D_{\sigma,k}^\tau \\
 & \leq (1 + \frac{1}{2} M_\delta \tau + O(\tau^2)) D_{\sigma,k}^\tau \leq (1 + M_\delta \tau) D_{\sigma,k}^\tau.
 \end{aligned}$$

We pass to the proof of (7.47). We have

$$\begin{aligned}
 (7.49) \quad & \int_{\Omega_\sigma^0} \langle \nabla \psi^\tau, \nabla \varphi^0 \rangle |J\chi_\tau| \, dx \\
 & = \int_{\Omega_\sigma^0} \langle \nabla \psi^\tau, \nabla \tilde{\Phi}^0 \rangle |J\chi_\tau| \, dx + \int_{\Omega_\sigma^0} \langle \nabla \psi^\tau, \nabla \varphi^0 - \nabla \tilde{\Phi}^0 \rangle |J\chi_\tau| \, dx.
 \end{aligned}$$

Using again that  $\|J\chi_\tau\|_{L^\infty} \leq 1 + M_\delta \tau$  and Hölder inequality, we get

$$\begin{aligned}
 (7.50) \quad & \int_{\Omega_\sigma^0} \langle \nabla \psi^\tau, \nabla \varphi^0 - \nabla \tilde{\Phi}^0 \rangle |J\chi_\tau| \, dx \\
 & \leq (1 + M_\delta \tau) \left( \int_{\Omega_\sigma^0} |\nabla \psi^\tau|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega_\sigma^0} |\nabla \varphi^0 - \nabla \tilde{\Phi}^0|^2 \, dx \right)^{\frac{1}{2}}.
 \end{aligned}$$

Moreover, since  $\mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0)$ , we have

$$\int_{\Omega_\sigma^0} |\nabla \varphi^0 - \nabla \tilde{\Phi}^0|^2 \, dx \leq 2 \int_{\Omega_\sigma^0} (|\nabla \varphi^0|^2 + |\nabla \tilde{\Phi}^0|^2) \, dx \leq 4\mathcal{W}(v_0) + 4M\pi |\log \sigma| + o_\sigma(1).$$

By (7.36), since  $X_V$  has compact support in  $\Omega$  we have

$$\frac{\partial \psi^t}{\partial \nu} = \frac{\partial \Phi^t}{\partial \nu^\perp} - \frac{\partial \Phi}{\partial \nu^\perp} = 0 \quad \text{on } \partial\Omega.$$

Therefore, in view of (7.37),

$$\int_{\Omega_\sigma^0} |\nabla \psi^\tau|^2 \, dx = \int_{\partial\Omega} \psi^\tau \frac{\partial \psi^\tau}{\partial \nu} \, ds - \sum_{i=1}^M \int_{B_\sigma(x_i)} |\nabla \psi^\tau|^2 \, ds \leq C\sigma^2.$$

Combining the above estimates with (7.49) and (7.50) we get (7.47).

To complete the proof it remains to show that (7.32) holds. By definition of  $w_{k+1}^\tau(x)$  (see (7.39)), we have immediately

$$\begin{aligned} \|v_k^\tau - w_{k+1}^\tau\|_{L^2}^2 &= \int_{\Omega} \left| v_k^\tau - v_k^\tau(\chi_\tau^{-1}) e^{i\psi^\tau(\chi_\tau^{-1})} \right|^2 dy \\ (7.51) \quad &= \int_{\Omega} |v_k^\tau - v_k^\tau(\chi_\tau^{-1})|^2 dy \end{aligned}$$

$$(7.52) \quad + \int_{\Omega} \left| v_k^\tau(\chi_\tau^{-1}) - v_k^\tau(\chi_\tau^{-1}) e^{i\psi^\tau(\chi_\tau^{-1})} \right|^2 dy$$

$$(7.53) \quad + 2 \int_{\Omega} \langle v_k^\tau - v_k^\tau(\chi_\tau^{-1}), v_k^\tau(\chi_\tau^{-1}) - v_k^\tau(\chi_\tau^{-1}) e^{i\psi^\tau(\chi_\tau^{-1})} \rangle dy.$$

In order to prove (7.32) it is enough to show that

$$(7.54) \quad \int_{\Omega} |v_k^\tau - v_k^\tau(\chi_\tau^{-1})|^2 dy \leq \pi\tau^2 |\log \tau| |V|^2 + o(\tau^2 |\log \tau|),$$

$$(7.55) \quad \int_{\Omega} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau(\chi_\tau^{-1}) e^{i\psi^\tau(\chi_\tau^{-1})}|^2 dy \leq C\tau^2;$$

indeed, once we got (7.54) and (7.55), by Hölder inequality, we have immediately that the integral in (7.53) is  $O(\tau^2 \sqrt{|\log \tau|})$ .

First, we prove (7.55). By the change of variable  $y = \chi_\tau(x)$  and the fact that  $\psi^0 = 0$ , in view of (7.37), we obtain

$$\begin{aligned} \int_{\Omega} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau(\chi_\tau^{-1}) e^{i\psi^\tau(\chi_\tau^{-1})}|^2 dy &= \int_{\Omega} |1 - e^{i\psi^\tau}|^2 |J\chi_\tau| dx \\ &\leq (1 + M_\delta\tau) \left\| \frac{d}{dt} \psi^t \right\|_{L^\infty(\Omega)}^2 \tau^2 |\Omega| \leq C\tau^2. \end{aligned}$$

Finally, to complete the proof of the Theorem it remains to show that (7.54) holds. By Hölder inequality, we have

$$\begin{aligned} &\int_{\Omega \cup_i B_{\delta/2}(y_i)} |v_k^\tau - v_k^\tau(\chi_\tau^{-1})|^2 dy \\ &\leq \int_{\Omega \cup_i B_{\delta/2}(y_i)} \tau \int_0^\tau |\nabla v_k^\tau(\chi_t^{-1})|^2 \left\| \frac{d}{dt} \chi_t^{-1} \right\|_{L^\infty}^2 dy dt \\ &\leq C(1 + M_\delta\tau) \tau^2 (\mathcal{W}(v_k^\tau) + M\pi |\log \frac{\delta}{2}|) \leq C\tau^2, \end{aligned}$$

where  $C$  depends only on  $\delta$  and we have used that  $\mathcal{W}(v_k^\tau) \leq \mathcal{W}(v_0)$ .

In order to complete the proof of (7.54), it is enough to show that

$$(7.56) \quad \int_{B_{\delta/2}(y_i)} |v_k^\tau - v_k^\tau(\chi_\tau^{-1})|^2 dy \leq \pi\tau^2 |\log \tau| |V_i|^2 + o(\tau^2 |\log \tau|).$$

Let  $N > 0$  be given; then, for any  $i = 1, \dots, M$ ,

$$(7.57) \quad \int_{B_{\delta/2}(y_i)} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau|^2 dy \leq \int_{B_{\delta/2}(y_i) \setminus B_{N\tau}(y_i)} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau|^2 dy + 4N^2\tau^2\pi.$$

Without loss of generality we can assume  $d_{i,0} = \deg(v_k^\tau, \partial B_{\delta/2}(x_i)) = 1$ .

We first show the estimate (7.56) in the case  $v_k^\tau = \frac{x-x_i}{|x-x_i|}$  in  $B_{\delta/2}(x_i)$ . Let  $(r, \theta)$  be the polar coordinates with respect to  $y_i$ ; denoting by  $\alpha = \alpha(r, \theta)$  the angle

between the vectors  $\frac{y-y_i}{|y-y_i|}$  and  $v_k^\tau(y) = \frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|}$ , we have

$$(7.58) \quad \int_{B_{\delta/2}(y_i) \setminus B_{N\tau}(y_i)} \left| \frac{y-y_i}{|y-y_i|} - \frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|} \right|^2 dy = \int_{N\tau}^{\delta/2} r dr \int_0^{2\pi} 4 \sin^2 \frac{\alpha}{2} d\theta.$$

Using elementary geometry identities and Taylor expansion, for  $N\tau \leq r \leq \delta/2$  we get

$$\sin \alpha = \frac{\tau |V_i| \sin \theta}{r} \frac{1}{\sqrt{1 + \frac{\tau^2 |V_i|^2}{r^2} - 2 \frac{\tau |V_i| \cos \theta}{r}}} = \frac{\tau |V_i| \sin \theta}{r} (1 + O(1/N)),$$

so that  $\sin^2 \frac{\alpha}{2} = \frac{\tau^2 |V_i|^2 \sin^2 \theta}{4r^2} + O(1/N)$ . Therefore, by (7.58) we get

$$(7.59) \quad \begin{aligned} \int_{B_{\delta/2}(y_i) \setminus B_{N\tau}(y_i)} \left| \frac{y-y_i}{|y-y_i|} - \frac{y-y_i+\tau V_i}{|y-y_i+\tau V_i|} \right|^2 dy \\ = \tau^2 |V_i|^2 \int_{N\tau}^{\delta/2} \frac{1}{r} dr \int_0^{2\pi} \sin^2 \theta d\theta + O(1/N) \\ = \pi \tau^2 |\log \tau| |V_i|^2 + \pi \tau^2 \log \frac{\delta}{2N} |V_i|^2 + O(1/N). \end{aligned}$$

Then, (7.56) follows (in the case  $v_k^\tau = \frac{x-x_i}{|x-x_i|}$ ) by choosing  $N = |\log \tau|$ .

Now, we prove (7.56) in the general case, i.e., without assuming  $v_k^\tau = \frac{x-x_i}{|x-x_i|}$ . Set  $L := \lfloor \log_2 \frac{\delta}{2N\tau} \rfloor$  and let  $\theta_i$  be the angle in polar coordinates with center in  $y_i$ , i.e., the phase of the function  $\frac{y-y_i}{|y-y_i|}$ . For every  $l = 1, \dots, L$ , we set

$$C_l(y_i) := B_{2^{-l}\delta}(y_i) \setminus B_{2^{-l-1}\delta}(y_i), \quad \tilde{C}_l(y_i) := B_{2^{-l+1}\delta}(y_i) \setminus B_{2^{-l-2}\delta}(y_i).$$

Set  $\tilde{\varphi}_{i,l}^0 = \frac{1}{|\tilde{C}_l(y_i)|} \int_{\tilde{C}_l(y_i)} \varphi^0(x) dy$  and notice that the average of  $\theta_i$  is equal to  $\pi$ . We have

$$\begin{aligned} \int_{B_{\delta/2}(y_i) \setminus B_{N\tau}(y_i)} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau|^2 dy &= \sum_{l=1}^L \int_{C_l(y_i)} |v_k^\tau(\chi_\tau^{-1}) - v_k^\tau|^2 dy \\ &= \sum_{l=1}^L \int_{C_l(y_i)} |e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}_{i,l}^0 + \pi)} - e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^0 + \pi)}|^2 dy \\ &= \sum_{l=1}^L \int_{C_l(y_i)} |e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i}|^2 dy \\ &\quad + \int_{C_l(y_i)} |e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}_{i,l}^0 + \pi)} - e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^0 + \pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i})|^2 dy \\ + 2 \int_{C_l(y_i)} \langle e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i}, e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}_{i,l}^0 + \pi)} - e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^0 + \pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i}) \rangle dy. \end{aligned}$$

Estimating the last term of the right hand side of the above formula by Hölder's inequality and recalling (7.59), in order to prove (7.56) it is enough to show the following estimate

$$\sum_{l=1}^L \int_{C_l(y_i)} \left| e^{i(\varphi^0(\chi_\tau^{-1}) - \tilde{\varphi}_{i,l}^0 + \pi)} - e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^0 + \pi)} - (e^{i\theta_i(\chi_\tau^{-1})} - e^{i\theta_i}) \right|^2 dy \leq C\tau^2.$$

By definition of  $\chi_\tau$ , for any  $y \in C_l(y_i)$ ,  $\chi_\tau^{-1}(y) = y - \tau V_i$  and then

$$\begin{aligned} e^{i(\varphi^0(\chi_\tau^{-1}(y)) - \tilde{\varphi}_{i,l}^0 + \pi)} - e^{i(\varphi^0(y) - \tilde{\varphi}_{i,l}^0 + \pi)} &= - \int_0^\tau \nabla e^{i(\varphi^0(y-tV_i) - \tilde{\varphi}_{i,l}^0 + \pi)} \cdot V_i \, dt, \\ e^{i\theta_i(\chi_\tau^{-1}(y))} - e^{i\theta_i(y)} &= - \int_0^\tau \nabla e^{i\theta_i(y-tV_i)} \cdot V_i \, dt; \end{aligned}$$

then, by Jensen and Cauchy inequalities,

$$\begin{aligned} & \int_{C_l(y_i)} \left| e^{i(\varphi^0(\chi_\tau^{-1}(y)) - \tilde{\varphi}_{i,l}^0 + \pi)} - e^{i(\varphi^0(y) - \tilde{\varphi}_{i,l}^0 + \pi)} - (e^{i\theta_i(\chi_\tau^{-1}(y))} - e^{i\theta_i(y)}) \right|^2 dy \\ &= \int_{C_l(y_i)} \left| \int_0^\tau (\nabla e^{i\theta_i(y-tV_i)} - \nabla e^{i(\varphi^0(y-tV_i) - \tilde{\varphi}_{i,l}^0 + \pi)}) \cdot V_i \, dt \right|^2 dy \\ &\leq \tau |V_i|^2 \int_{C_l(y_i)} dy \int_0^\tau \left| \nabla e^{i\theta_i(y-tV_i)} - \nabla e^{i(\varphi^0(y-tV_i) - \tilde{\varphi}_{i,l}^0 + \pi)} \right|^2 dt \\ (7.60) \quad &\leq \tau^2 |V_i|^2 \int_{\tilde{C}_l(y_i)} \left| \nabla e^{i\theta_i} - \nabla e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^0 + \pi)} \right|^2 dy. \end{aligned}$$

Furthermore

$$\begin{aligned} (7.61) \quad & \sum_{l=1}^L \int_{\tilde{C}_l(y_i)} \left| \nabla e^{i\theta_i} - \nabla e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^0 + \pi)} \right|^2 dy \\ &\leq 2 \sum_{l=1}^L \int_{\tilde{C}_l(y_i)} \left| \nabla e^{i\theta_i} \right|^2 \left| 1 - e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^0 + \pi - \theta_i)} \right|^2 dy + \int_{\tilde{C}_l(y_i)} \left| \nabla(\theta_i - \varphi^0) \right|^2 dy \\ &\leq 2 \sum_{l=1}^L \int_{\tilde{C}_l(y_i)} 2^{2l+4} \delta^{-2} \left| e^{i(\theta_i - \varphi^0)} - e^{i(-\tilde{\varphi}_{i,l}^0 + \pi)} \right|^2 dy + \int_{\tilde{C}_l(y_i)} \left| \nabla(\theta_i - \varphi^0) \right|^2 dy, \end{aligned}$$

where the last inequality follows from the fact that  $|\nabla e^{i\theta_i}(y)|^2 = \frac{1}{|y-y_i|^2}$  and that  $2^{-l-2}\delta \leq |y - y_i| \leq 2^{-l+1}\delta$  for  $y \in \tilde{C}_l(x_i)$ .

Finally, by Poincaré inequality, it follows that

$$\begin{aligned} (7.62) \quad & \int_{\tilde{C}_l(y_i)} \left| e^{i(\theta_i - \varphi^0)} - e^{i(-\tilde{\varphi}_{i,l}^0 + \pi)} \right|^2 dx \\ &\leq \int_{\tilde{C}_l(x_i)} \left| \theta_i - \varphi^0 - (\pi - \tilde{\varphi}_{i,l}^0) \right|^2 dx \leq C 2^{-2l} \delta^2 \int_{\tilde{C}_l(y_i)} \left| \nabla(\theta_i - \varphi^0) \right|^2 dy, \end{aligned}$$

where  $C$  is a positive constant. By the minimality of  $\theta_i$ , we have

$$\int_{\tilde{C}_l(y_i)} \left| \nabla(\varphi^0 - \theta_i) \right|^2 dy = \int_{\tilde{C}_l(y_i)} \left| \nabla \varphi^0 \right|^2 - \int_{\tilde{C}_l(y_i)} \left| \nabla \theta_i \right|^2 dy.$$

By (7.61) and Remark 4.4, we obtain

$$\begin{aligned} & \sum_{l=1}^L \int_{\tilde{C}_l(y_i)} |\nabla e^{i(\varphi^0 - \tilde{\varphi}_{i,l}^\tau + \pi)} - \nabla e^{i\theta_i}|^2 dy \\ & \leq C \sum_{l=1}^L \int_{\tilde{C}_l(y_i)} (|\nabla \varphi^0|^2 - |\nabla \theta_i|^2) dy \\ & = C \sum_{l=1}^L \left( \int_{\tilde{C}_l(y_i)} |\nabla \varphi^0|^2 dx - 6\pi \log 2 \right) \leq C. \end{aligned}$$

This together with (7.60) concludes the proof.  $\square$

We are now in a position to prove Theorem 7.7.

*Proof of Theorem 7.7.* By Theorems 7.10, 7.11, we can apply Theorem 7.9 for any  $T < \pi \frac{c_\delta^2}{C_\delta}$ , and in view of (7.16) we obtain that

- 1)  $x^\tau$  converges to the solution  $x$  of (6.3), uniformly on  $[0, T]$ ;
- 2)  $D(T) = 0$ .

Let  $T^{max} \leq \liminf_{\tau \rightarrow 0} k_\delta^\tau \tau$  be the maximal time such that 1) and 2) holds true on  $[0, T]$  for every  $T < T^{max}$ . Recalling (see (6.13)) that  $T_\delta \rightarrow T^*$  as  $\delta \rightarrow 0$ , it remains only to prove that  $T^{max} \geq T_\delta$ . This follows by a standard continuation argument: Assume by contradiction that  $T^{max} < T_\delta$ , and let  $T < T^{max}$ . Then we have

$$\min_{t \in [0, T^{max}]} \min \left\{ \frac{1}{2} \text{dist}_{i \neq j} (x_i(t), x_j(t)), \text{dist}(x_i(t), \partial\Omega) \right\} - 2\delta = c'_\delta > 0.$$

Consider now  $x_{[T/\tau]}^\tau, v_{[T/\tau]}^\tau$  as the initial condition of a new  $L^2$  discrete gradient flow. Notice that, in view of 2), these initial conditions are well prepared; the fact that the initial time is not zero is not relevant, since all the equations are autonomous. Moreover, even if the initial conditions depend on  $\tau$ , they converge as  $\tau \rightarrow 0$ . Therefore, Theorems 7.10, 7.11, and Theorem 7.9 still hold true with the obvious modifications, and we easily deduce that 1) and 2) holds true as long as  $0 \leq t - T \leq (c'_\delta)^2 / C_\delta$ . This time interval in which we can extend the solution is independent of  $T < T^{max}$ , which contradicts the maximality of  $T^{max}$ .  $\square$

**7.2.  $L^2$  discrete gradient flow of  $F_\varepsilon$ .** We conclude this section by analyzing the existence of the  $L^2$  discrete gradient flow of  $F_\varepsilon$  and studying its asymptotic behaviour as  $\varepsilon \rightarrow 0$ . The existence will be obtained for  $\varepsilon$  small enough by making use of the auxiliary problem studied in the previous section. To this aim it is convenient to introduce a relaxed version of such discrete evolution.

**Definition 7.12.** Fix  $\delta > 0$  and let  $\varepsilon, \tau > 0$ . Given  $u_{\varepsilon,0} \in \mathcal{AF}_\varepsilon(\Omega)$ , we say that  $\{\bar{u}_{\varepsilon,k}^\tau : k \in \mathbb{N}\}$ , is a relaxed  $L^2$  discrete gradient flow of  $F_\varepsilon$  from  $u_{\varepsilon,0}$  if  $\bar{u}_{\varepsilon,0}^\tau = u_{\varepsilon,0}$  and, for any  $k \in \mathbb{N}$ , there exists a sequence  $\{u_{\varepsilon,k,n}^\tau\}_n$  such that

$$(7.63) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \|e^{2\pi i u_{\varepsilon,k,n}^\tau} - e^{2\pi i \bar{u}_{\varepsilon,k}^\tau}\|_{L^2} = 0, \\ & \|\mu(u_{\varepsilon,k,n}^\tau) - \mu(\bar{u}_{\varepsilon,k-1}^\tau)\|_{\text{flat}} \leq \delta \quad \text{for every } n \in \mathbb{N}, \\ & \lim_{n \rightarrow \infty} F_\varepsilon(u_{\varepsilon,k,n}^\tau) + \frac{\|e^{2\pi i u_{\varepsilon,k,n}^\tau} - e^{2\pi i \bar{u}_{\varepsilon,k-1}^\tau}\|_{L^2}^2}{2\tau |\log \tau|} = I_{\varepsilon,k}^\tau, \end{aligned}$$

where

$$I_{\varepsilon,k}^{\tau} = \inf_{u \in \mathcal{AF}_{\varepsilon}(\Omega)} \left\{ F_{\varepsilon}(u) + \frac{\|e^{2\pi i u} - e^{2\pi i \bar{u}_{\varepsilon,k-1}^{\tau}}\|_{L^2}^2}{2\tau |\log \tau|} : \|\mu(u) - \mu(\bar{u}_{\varepsilon,k-1}^{\tau})\|_{\text{flat}} \leq \delta \right\}.$$

The existence of such relaxed discrete gradient flow is obvious. To show that it is actually a strong  $L^2$  discrete gradient flow it is enough to show that  $\|\mu(\bar{u}_{\varepsilon,k}^{\tau}) - \mu(\bar{u}_{\varepsilon,k-1}^{\tau})\|_{\text{flat}} \leq \delta$ . A key argument is given by the following estimate that one can easily check by contradiction

(7.64)

$$\limsup_{n \rightarrow +\infty} \|\mu(u_{\varepsilon,k,n}^{\tau}) - \mu(\bar{u}_{\varepsilon,k}^{\tau})\|_{\text{flat}} \leq C\varepsilon \#\{(i,j) \in \Omega_{\varepsilon}^1 : \text{dist}(\bar{u}_{\varepsilon,k}^{\tau}(i) - \bar{u}_{\varepsilon,k}^{\tau}(j), \mathbb{Z}) = \frac{1}{2}\}$$

**Theorem 7.13.** *Let  $v_0$  be such that  $\mathcal{W}(v_0) < +\infty$  and let  $Jv_0 = \sum_{i=1}^M d_{i,0} \delta_{x_{i,0}} =: \mu_0$  with  $|d_{i,0}| = 1$ . Let  $u_{\varepsilon,0} \in \mathcal{AF}_{\varepsilon}(\Omega)$  such that*

$$\mu(u_{\varepsilon,0}) \xrightarrow{\text{flat}} \mu_0, \quad F_{\varepsilon}(u_{\varepsilon,0}) \leq \pi|\mu_0|(\Omega) \log \varepsilon + C.$$

Let  $\delta > 0$  be fixed such that  $\min\{\frac{1}{2} \text{dist}_{i \neq j}(x_{i,0}, x_{j,0}), \text{dist}(x_{i,0}, \partial\Omega)\} > 2\delta$ . Given  $\tau > 0$ , let  $\bar{u}_{\varepsilon,k}^{\tau}$  be a relaxed  $L^2$  discrete gradient flow of  $F_{\varepsilon}$  from  $u_{\varepsilon,0}$ .

Then, up to a subsequence, for any  $k \in \mathbb{N}$  we have  $\mu(\bar{u}_{\varepsilon,k}^{\tau}) \xrightarrow{\text{flat}} \mu_k^{\tau}$ , for some  $\mu_k^{\tau} \in X$  with  $|\mu_k^{\tau}|(\Omega) \leq M$  and there exists a maximal  $L^2$  discrete gradient flow,  $v_k^{\tau}$ , of  $\mathcal{W}$  from  $v_0$ , according with Definition 7.2, such that

$$(7.65) \quad \mu_k^{\tau} = Jv_k^{\tau} = \sum_{i=1}^M d_{i,0} \delta_{x_{i,k}^{\tau}}, \quad \text{for every } k = 1, \dots, k_{\delta}^{\tau},$$

with  $k_{\delta}^{\tau}$  as defined in (6.5).

Moreover denoting by  $\tilde{v}_{\varepsilon,k}^{\tau}$  the piecewise affine interpolation of  $e^{2\pi i \bar{u}_{\varepsilon,k}^{\tau}}$ , we have

$$(7.66) \quad \tilde{v}_{\varepsilon,k}^{\tau} \rightharpoonup v_k^{\tau} \quad \text{in } H_{\text{loc}}^1(\Omega \setminus \cup_{i=1}^M \{x_{i,k}^{\tau}\}; \mathbb{R}^2), \quad \text{for every } k = 1, \dots, k_{\delta}^{\tau}.$$

Finally for  $\tau$  and  $\varepsilon$  small enough such  $\bar{u}_{\varepsilon,k}^{\tau}$  is indeed a minimizer of problem (7.1) and hence it is a (strong)  $L^2$  discrete gradient flow.

*Proof.* The proof of this result uses the first order  $\Gamma$ -convergence result (Theorem 4.5) and follows closely the proof of the analogous statement in Section 6 (see Theorem 6.7). Indeed by the definition of relaxed  $L^2$  discrete gradient flow we have that for any  $k \in \mathbb{N}$

$$F_{\varepsilon}(\bar{u}_{\varepsilon,k}^{\tau}) + \frac{\|e^{2\pi i \bar{u}_{\varepsilon,k}^{\tau}} - e^{2\pi i \bar{u}_{\varepsilon,k-1}^{\tau}}\|_{L^2}^2}{2\pi\tau |\log \tau|} \leq F_{\varepsilon}(\bar{u}_{\varepsilon,k-1}^{\tau}).$$

By induction on  $k$ , one can show that

$$F_{\varepsilon}(\bar{u}_{\varepsilon,k}^{\tau}) \leq F_{\varepsilon}(u_{\varepsilon,0}, \Omega) \leq M\pi |\log \varepsilon| + C.$$

This estimate together with (7.64) we obtain that  $\|\mu(\bar{u}_{\varepsilon,k}^{\tau}) - \mu(\bar{u}_{\varepsilon,k-1}^{\tau})\|_{\text{flat}} \leq \delta + C\varepsilon |\log \varepsilon|$ . Then using the Compactness result stated in Theorem 4.2(i), and arguing as in the proof of Theorem 6.7 we deduce (7.65) and (7.66).

In order to show that, for  $\varepsilon$  small enough,  $\bar{u}_{\varepsilon,k}^{\tau}$  is an  $L^2$  discrete gradient flow according with Definition 7.1, it is enough to recall that thanks to Proposition 7.6 we have that  $\|\mu_k^{\tau} - \mu_{k-1}^{\tau}\|_{\text{flat}} \leq C\sqrt{\tau |\log \tau|}$ . Then the conclusion follows by the convergence in the flat norm of  $\mu(\bar{u}_{\varepsilon,k}^{\tau})$  to  $\mu_k^{\tau}$ .  $\square$



## 8. CONCLUSIONS

We have obtained an asymptotic expansion by  $\Gamma$ -convergence for a large class of discrete energies accounting for defects (including the elastic energy in crystals with screw dislocations and the energy of XY spin systems with vortices). Based on this analysis, we have been able to show existence of metastable configurations, and we have introduced a discrete in time variational dynamics, which allows to overcome the energy barriers and mimics the effect of more complex mechanisms, as thermal effects. We have described the dynamics up to the first collision time; it would be interesting to model the collision of discrete vortices, and study the dynamics after the critical time as in the Ginzburg-Landau setting (see [11], [39], [40]). In all the paper we have focused on Neumann boundary conditions, but our analysis could be extended to the case of Dirichlet boundary conditions.

In the proof of our results we have made use of a new variational principle that allows to deduce the presence of local minimizers for  $\Gamma$ -converging sequences, also in the absence of local minimizers in their limit. This has been possible for a large class of interaction potentials, which includes the case of screw dislocations but not the XY model, for which this fact is still unclear.

In the discrete dynamics we have analyzed two different dissipations. This is motivated also by applications. Indeed, the flat dissipation is a natural choice in the study of screw dislocation dynamics, and can be viewed as a measure of the number of energy barriers to be overcome in order to move a dislocation. The  $L^2$  dissipation is a standard choice for parabolic flows and measures the variations in the spin variable. We note that, in the case of dislocations, one could also consider variants of the flat dissipation accounting for the glide directions of the crystal. This would lead to a different effective dynamics. We also believe that our approach could be generalized to anisotropic energies and to more general lattice structures. It is still open the case of edge dislocations, for which a complete  $\Gamma$ -expansion of the energy is not yet available ([20], [18]).

Having proved a pinning phenomenon, it remains open to characterize a critical  $\varepsilon$ - $\tau$  regime for the motion of dislocations, and an effective depinning threshold in this regime. This is a relevant issue and it might be worth facing it by using our variational approach.

The effective dynamics of our discrete systems agrees with the asymptotic parabolic flow of the Ginzburg-Landau functionals. In the latter, the time scaling needed to get a non-trivial effective dynamics depends on the space parameter  $\varepsilon$ . It is worth noticing that, in our discrete in time gradient flow with  $L^2$  dissipation, the time scaling is expressed only in terms of the time step  $\tau$ . In this respect, an analysis of critical  $\varepsilon$ - $\tau$  regimes would make an interesting bridge between these two approaches.

## APPENDIX A. PRODUCT-ESTIMATE

In this section we collect some results in [36] that are used in the proofs of Section 7.

We first introduce some notation. Let  $A$  be an open bounded subset of  $\mathbb{R}^3$ . Given  $w = (w_1, w_2) \in H^1(A; \mathbb{R}^2)$ , its Jacobian  $Jw$  can be regarded as a 2-form in

$\mathbb{R}^3$  given by

$$(A.1) \quad Jw = dw_1 \wedge dw_2 = \sum_{j < k} (\partial_j w_1 \partial_k w_2 - \partial_j w_2 \partial_k w_1) dx_j \wedge dx_k.$$

Thus  $Jw$  acts on vector fields  $X, Y \in C(A; \mathbb{R}^3)$  with the standard rule that

$$dx_j \wedge dx_k(X, Y) = \frac{1}{2} (X_j Y_k - X_k Y_j).$$

The Jacobian  $Jw$  can be also identified with a 1-dimensional current  $\star Jw$  which acts on 1 forms  $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3$  as

$$\langle \star Jw, \omega \rangle = \int_A Jw \wedge \omega.$$

In particular, for any  $X, Y \in C(A; \mathbb{R}^3)$

$$\langle \star Jw, X \wedge Y \rangle = \int_A Jw(X, Y) dx,$$

where, with a little abuse of notation, we identify 1 forms with vector fields.

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^2$  and  $T > 0$ . For a given  $w \in H^1([0, T] \times \Omega; \mathbb{R}^2)$ , we denote by  $\mu, V^1, V^2$  the  $L^1$  functions such that

$$(A.2) \quad Jw = \mu dx_1 \wedge dx_2 + V^1 dx_1 \wedge dt + V^2 dx_2 \wedge dt.$$

The theorem below collects the results of Theorem 1 and Theorem 3 in [36]. We remind that the definition of the functionals  $GL_\varepsilon$  is given in (1.2).

**Theorem A.1.** *Let  $w_\varepsilon \in H^1([0, T] \times \Omega; \mathbb{R}^2)$  be such that*

$$(A.3) \quad \int_0^T GL_\varepsilon(w_\varepsilon(t, \cdot)) dt + \int_0^T \int_\Omega |\partial_t w_\varepsilon(t, x)|^2 dx dt \leq C |\log \varepsilon|.$$

*Then, there exists a rectifiable integer 1-current  $J$  such that, up to a subsequence,*

$$\frac{\star J u_\varepsilon}{\pi} \rightarrow J \quad \text{in } (C_c^{0,\gamma}([0, T] \times \Omega; \mathbb{R}^3))', \forall \gamma \in (0, 1].$$

*Moreover, for any  $X, Y \in C_c^0([0, T] \times \Omega; \mathbb{R}^3)$*

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\pi |\log \varepsilon|} \left( \int_{[0, T] \times \Omega} |X \cdot \nabla w_\varepsilon|^2 dx dt \int_{[0, T] \times \Omega} |Y \cdot \nabla w_\varepsilon|^2 dx dt \right)^{\frac{1}{2}} \geq |\langle J, X \wedge Y \rangle|.$$

*If in addition we assume that*

$$(A.4) \quad \sup_{t \in [0, T]} GL_\varepsilon(w_\varepsilon(t, \cdot)) \leq C |\log \varepsilon|,$$

*then,  $J$  can be written as in (A.2) with  $\mu \in C^{0, \frac{1}{2}}([0, T]; (C_c^{0,\gamma}(\Omega))')$  for every  $\gamma \in (0, 1]$  and  $V^1, V^2 \in L^2([0, T]; \mathcal{M}(\Omega))$ .*

*Finally, up to a subsequence,*

$$\mu_\varepsilon(t) \xrightarrow{\text{flat}} \mu(t) \quad \text{for all } t \in [0, T].$$

We now state a variant of Corollary 4 in [36] which is a direct consequence of Theorem A.1.

**Corollary A.2.** *Let  $0 \leq t_1 < t_2$  and let  $w_\varepsilon \in H^1([t_1, t_2] \times \Omega; \mathbb{R}^2)$  be such that (A.3) holds true with  $[0, T]$  replaced by  $[t_1, t_2]$ , and such that for all  $t \in [t_1, t_2]$*

$$\frac{1}{2} \int_{\Omega} |\nabla w_\varepsilon(t, x)|^2 dx \leq M\pi |\log \varepsilon| + C$$

for some  $M \in \mathbb{N}$  and  $C \in \mathbb{R}$ . Assume moreover that

$$\mu_\varepsilon(t) \xrightarrow{\text{flat}} \mu(t) := \pi \sum_{i=1}^M d_i \delta_{x_i(t)},$$

with  $|d_i| = 1$  and  $x_i(t) \in C([t_1, t_2]; \Omega)$  for every  $i$  with  $x_i(t) \neq x_j(t)$  for  $i \neq j$ .

Then, for any  $X, Y \in C_c^0(\Omega; \mathbb{R}^3)$

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{[t_1, t_2] \times \Omega} \langle X \cdot \nabla w_\varepsilon, Y \cdot \nabla w_\varepsilon \rangle dx dt = \pi \int_{t_1}^{t_2} \sum_{i=1}^M \langle X(x_i(t)), Y(x_i(t)) \rangle dt.$$

Here we state a result analogous to Corollary 7 in [36].

**Proposition A.3.** *Let  $\tilde{T} > 0$  and let  $w_\varepsilon \in H^1([0, \tilde{T}] \times \Omega; \mathbb{R}^2)$  be such that (A.3) holds true, and such that for all  $t \in [0, \tilde{T}]$*

$$\frac{1}{2} \int_{\Omega} |\nabla w_\varepsilon(t, x)|^2 dx \leq M\pi |\log \varepsilon| + C$$

for some  $M \in \mathbb{N}$  and  $C \in \mathbb{R}$ . Assume moreover that

$$\mu_\varepsilon(t) \xrightarrow{\text{flat}} \mu(t) := \pi \sum_{i=1}^M d_i \delta_{x_i(t)},$$

with  $|d_i| = 1$  and  $x_i(t) \in C([0, \tilde{T}]; \Omega)$  for any  $i$  with  $x_i(t) \neq x_j(t)$  for  $i \neq j$ . Then

$$(A.5) \quad \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \int_{[0, \tilde{T}] \times \Omega} |\partial_t w_\varepsilon|^2 dx dt \geq \pi \sum_{i=1}^M \int_0^{\tilde{T}} |\dot{x}_i|^2 dt.$$

*Proof.* The proof of this result coincides with the one of Corollary 7 in [36], the only difference being that [36] assumes that for every  $t \in [0, \tilde{T}]$

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - |w_\varepsilon(x, t)|^2)^2 dx \leq C |\log \varepsilon|.$$

Here this assumption is replaced by (A.3), which is enough to apply Corollary A.2. Once the statement of Corollary A.2 holds true, the rest of the proof follows exactly as in in [36].  $\square$

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