HOMOGENIZATION OF PERIODIC NONLINEAR MEDIA WITH STIFF AND SOFT INCLUSIONS

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We study the asymptotic behaviour of highly oscillating periodic nonlinear functionals of the form $F_\varepsilon(u)=\int_\Omega f(\frac{x}{\varepsilon},Du(x))\,dx$. We characterize their variational limit under the hypotheses that there exists c such that the region where $f(x,\xi)\le c(1+|\xi|^p)$ does not hold for all matrices ξ is composed of well-separated sets, and the condition $f(x,\xi)\ge|\xi|^p$ for all matrices ξ is verified on a connected open set with Lipschitz boundary.

1. Introduction

In this paper we study the asymptotic behaviour of sequences of highly oscillating functionals which model various kinds of phenomena in media with a fine microstructure. We shall focus our analysis on different types of degenerate structures. As examples we can think of cellular frameworks of composite elastic and rigid materials, condenser with conducting or insulating small impurities, porous media or materials with holes, the torsion problem for a bar with small cavities, and so on. Many of these problems can be formulated by introducing suitable energy functionals of the form

$$F_\varepsilon(u)=\int_\Omega f(\frac{x}{\varepsilon},Du(x))\,dx,$$

where Ω is an open subset of the euclidean space $\mathbf{R}^n$, u is a function belonging to some space $X(\Omega)$ of $\mathbf{R}^m$-valued functions, ε represents the microscopic scale of the media, and f is an energy density characterizing the behaviour of the material at this lower scale. The problem of homogenization of the sequence (F_ε) consists in determining, usually by a limit procedure, a simpler "homogeneous" functional,

$$F_0(u)=\int_\Omega f_{hom}(Du(x))\,dx,$$

which describes the overall behaviour of the medium described by F_ε at a microscopic level. Another interesting issue is to find conditions that guarantee the non-degeneracy of this limit functional. In the case dealt with in this paper, we wish to ensure that the domain of F_0 is a Sobolev space; this condition translates into requiring that the homogenized energy density f_{hom} verify the so-called standard growth condition

$$c_1|\xi|^p \leq f_{hom}(\xi) \leq c_2(1+|\xi|^p)$$

with $p>1$ (here ξ is a $n\times m$ matrix). We shall show that this condition can be fulfilled even when the function f satisfies quite general growth assumption, with large zones of various types of degeneracy. We shall require that the function f is periodic and satisfies:

(i) there exists c such that the region where $f(x,\xi)\leq c(1+|\xi|^p)$ does not hold for all matrices ξ is composed of well-separated sets;

(ii) the condition $f(x,\xi)\geq|\xi|^p$ for all matrices ξ is verified on a connected open set with Lipschitz boundary.

Condition (i) must be imposed to avoid a "rigid" behaviour of the function f_{hom}, while (ii) prevents a "weakening" of the structure in the limit.

Let us remark that conditions (i) and (ii) allow a variety of different behaviours for the function f. In particular, we may have the so-called "stiff" inclusions modeled by a function of the form

$$f(x,\xi)=\begin{cases}0 & \text{if } \xi=0\\ +\infty & \text{if } \xi\neq 0,\end{cases}$$

for x in a region composed of well-separated components, and at the same time we can have a "soft" behaviour, or holes (in this case $f(x,\xi)\equiv 0$), provided that the complementary set of the 'void' region contains a connected periodic domain. The function f may also exhibit intermediate behaviours and satisfy globally a growth condition of the form

$$c_1|\xi|^{p_1}\leq f(\xi)\leq c_2(1+|\xi|^{p_2})$$

with $p_1\leq p\leq p_2$.

The main result of the paper is Theorem 3.1 where we show that, under some technical assumptions, hypotheses (i) and (ii) yield the existence of a function f_{hom}, given by an appropriate "asymptotic formula", satisfying the desired growth condition, such that the functional F_0 describes the limit of the functionals F_ε in a variational sense. In order to study this behaviour we make use of the theory of De Giorgi's Γ-convergence (see [16] [28], [3]). Since our problems do not fit the usual framework of functionals defined in Sobolev spaces, we adapt the localization method of Γ-convergence by proving a "fundamental estimate" for the sequence F_ε (see Proposition 3.3) that generalizes the method described in [16] Section 18 (see also [17]), taking into account the geometrical properties outlined in conditions (i) and (ii) in an essential way. Our result extends the previous ones on non convex media obtained by Braides [6] and Müller [24], in the case of functionals defined on vector-valued functions and satisfying standard growth conditions, and by Braides and Chiadò Piat [9], in the case of functionals verifying (ii) and a standard growth condition from above. Some results on convex functionals verifying (i) and (ii) have been obtained in [27] by duality methods (see also Acerbi, Percivale [2], Mortola, Profeti [23], Braides [5]). The corresponding Euler operators in domains with holes have been studied in detail by many authors, including Tartar, Cioranescu, Khrushlov, Saint Jean Paulin and others (see references).

The paper is organized as follows. In Section 2 we recall the notation, and make precise the requirements (i) and (ii) on the function f outlined above. Section 3 is devoted to the statement and proof of Theorem 3.1 following the scheme of the direct methods of Γ-convergence, and taking care in highlighting correspondences and differences with the existing theory. The main steps in the proof are Proposition 3.3 (the fundamental estimate for Γ-convergence), and Proposition 3.6 (the asymptotic formula for the homogenization). In Section 4 we apply the Γ-convergence results to obtain convergence of minima for certain Neumann and Dirichlet boundary value problems. Section 5 contains an example showing how the choice of the space $X(\Omega)$ may influence the form of f_{hom}. Finally in Section 6 a further example is exhibited of a polyconvex integrand f for which f_{hom} is not polyconvex.

2. Notations and preliminaries

Let n and m be two fixed positive integers. By $\mathbf{M}^{m\times n}$ we denote the space of $m\times n$ real matrices; if $\xi\in\mathbf{M}^{m\times n}$ and $x\in\mathbf{R}^n$ then $\xi{\cdot}x\in\mathbf{R}^m$ is defined by the usual product between matrices and vectors. $W^{1,p}(\Omega)=W^{1,p}(\Omega;\mathbf{R}^m)$ is the usual Sobolev space (of $\mathbf{R}^m$-valued functions), and $L^p(\Omega)=L^p(\Omega;\mathbf{R}^m)$. The family of all bounded open subsets of $\mathbf{R}^n$ is denoted by $\mathcal{A}$. A set $E\subset\mathbf{R}^n$ is said to be *periodic* if $E+e_i=E$ for every $i=1,\dots,n$, where (e_i) is the canonical basis of $\mathbf{R}^n$; a function $f:\mathbf{R}^n\to\mathbf{R}$ is said to be *periodic* if $f(x+e_i)=f(x)$ for every $i=1,\dots,n$, and for every $x\in E$. A function $g:\mathbf{R}^n\to\mathbf{R}$ is t*-periodic* if the function $x\mapsto g(\frac{x}{t})$ is periodic. Let A be an open set and $B\subset\subset A$; we say that a function φ is a *cut-off function* between B and A if $\varphi\in C^1_0(A)$ and $\varphi=1$ in B.

The set K will be a fixed subset of $\mathbf{R}^n$. We suppose that $(i+K)\cap(j+K)=\emptyset$ for all $i,j\in\mathbf{Z}^n$, $i\neq j$; more precisely, we denote by $\eta>0$ a positive number verifying the following property:

$$\operatorname{dist}(i+K,j+K)>3\eta\quad\forall i,j\in\mathbf{Z}^n\ i\neq j\,. \tag{2.1}$$

We will consider the periodic set $E=\mathbf{Z}^n+K$, and we will denote the set $\{x\in\mathbf{R}^n:\ \operatorname{dist}(x,K)<\eta\}$ by K_η and the set $\mathbf{Z}^n+K_\eta$ by E_η.

Remark 2.1. It is easy to see that assumption (2.1) implies that the set K is contained in the union of a finite number of cubes of the type $i+[0,1]^n$, $i\in \mathbf{Z}^n$, and in particular that its diameter is finite.

Consider now a Borel function $f(x,\xi)\colon \mathbf{R}^n\times \mathbf{M}^{m\times n}\to[0,+\infty]$ such that:

(i) $f(\cdot,\xi)$ is periodic;

(ii) there exists a periodic, connected open set U with Lipschitz boundary such that $|\xi|^p\le f(x,\xi)$, $1<p<+\infty$, for every $\xi\in\mathbf{M}^{m\times n}$ and $x\in U$;

(iii) $0\le f(x,\xi)\le c(1+|\xi|^p)$, $1<p<+\infty$, for every $\xi\in\mathbf{M}^{m\times n}$ and $x\in[0,1]^n\setminus E$;

(iv) there exists a Borel function $a\colon \mathbf{R}^n\times\mathbf{M}^{m\times n}\to[0,+\infty]$, convex in the second variable, such that

$$a(x,\xi)\le f(x,\xi)\le c(1+a(x,\xi))$$

for every $x\in\mathbf{R}^n$ and $\xi\in\mathbf{M}^{m\times n}$;

(v) there exists a positive constant α such that $a(x,2\xi)\le\alpha a(x,\xi)$ for every $\xi\in\mathbf{M}^{m\times n}$ and $x\in\mathbf{R}^n\setminus E$;

(vi) there exists a constant $b\ge 0$ such that $f(x,0)\le b$ for every $x\in E$.

For every bounded open subset Ω of $\mathbf{R}^n$, we will denote by $X(\Omega)$ a fixed subspace of $W^{1,1}(\Omega)$, with $C^1(\Omega)\subseteq X(\Omega)\subseteq W^{1,1}(\Omega)\cap L^p(\Omega)$, such that for every $u\in X(\Omega)$ and $\varphi\in C^1_0(\Omega)$ we have $u\varphi\in X(\Omega)$. It is understood that the meaning of the notation "X" does not change with Ω, i.e. if $u\in X(\Omega)$, and $\Omega'\subset\Omega$, then the restriction $u_{|\Omega'}$ belongs to $X(\Omega')$. Moreover, we will denote by $W^{1,1}_{per}(]0,1[^n)$ the closure in $W^{1,1}(\mathbf{R}^n)$ of the set of periodic C^1 functions and we will consider the space $X_{per}(]0,1[^n)=X_{loc}(\mathbf{R}^n)\cap W^{1,1}_{per}(]0,1[^n)$.

In the sequel we will study the limit behaviour of the following family of functionals $(F_\varepsilon)_{\varepsilon>0}$ defined in $L^p_{loc}(\mathbf{R}^n)\times\mathcal{A}$ by

$$F_\varepsilon(u,\Omega)=\begin{cases}\int_\Omega f(\frac{x}{\varepsilon},Du)\,dx & \forall u\in X(\Omega),\\ +\infty & \text{otherwise.}\end{cases}\tag{2.2}$$

In order to deal with such functionals it is convenient to use the following notion of variational convergence introduced by De Giorgi. For a comprehensive introduction to the theory of Γ-convergence we refer to the book by Attouch [3] and to the recent book by Dal Maso [16].

Definition 2.2. (De Giorgi & Franzoni [18]) Let (M,τ) be a metric space (throughout the paper we consider always L^p topologies), and let $(F_i)_{i\in I}$ be a family of real functions defined on M, $I\subset]0,+\infty[$ with $0\in\overline{I}$. Then, for $x_0\in M$, we define

$$\Gamma(\tau)\text{-}\liminf_{i\to0}F_i(x_0)=\inf\{\liminf_{i\to0}F_i(x_i)\ :\ x_i\xrightarrow{\tau}x_0\},\tag{2.3}$$

and

$$\Gamma(\tau)\text{-}\limsup_{i\to 0} F_i(x_0) = \inf\{\limsup_{i\to 0} F_i(x_i) \ : \ x_i \xrightarrow{\tau} x_0\}; \tag{2.4}$$

if these two quantities coincide, their common value will be called the Γ-*limit* of the family (F_i) in x_0, and will be denoted by $\Gamma(\tau)\text{-}\lim_{i\to 0} F_i(x_0)$. It is easy to check that the following statements are equivalent:

1) $l = \Gamma(\tau)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(x_0)$;
2) for every sequence of positive numbers (ε_h) converging to 0 there exists a subsequence (ε_{h_k}) for which we have
$$l = \Gamma(\tau)\text{-}\lim_{k\to\infty} F_{\varepsilon_{h_k}}(x_0);$$
3) for every sequence of positive numbers (ε_h) converging to 0 we have

   a) for every sequence (x_h) converging to x_0
   $$l \le \liminf_{h\to\infty} F_{\varepsilon_h}(x_h);$$
   b) there exists a sequence (x_h) converging to x_0 such that
   $$l \ge \limsup_{h\to\infty} F_{\varepsilon_h}(x_h).$$

In the notation for functionals of the form (2.2) we write $\Gamma(L^p)\text{-}\lim_{h\to\infty} F_{\varepsilon_h}(u, A)$ (and similar); it is understood that the limit is performed with respect to the topology $L^p(A)$.

Remark 2.3. The Γ-upper and lower limits defined above are τ-lower semicontinuous functions.

The importance of the Γ-convergence in the calculus of variations is clearly described by the following theorem about the convergence of minimum problems (see also Section 4).

Theorem 2.4. (De Giorgi & Franzoni [18]) *Let us suppose that the Γ-limit $\Gamma(\tau)\text{-}\lim_{i\to 0} F_i(x) = F_0(x)$ exists for every $x \in M$. If there exists a τ-compact subset K of M such that $\inf_M F_i = \inf_K F_i$ for all i, then we have $\lim_{i\to 0} \inf_M F_i = \min_M F_0$.*

In the sequel we will use the following homogenization result, which can be obtained as a particular case of [9] Theorem 1.4 (see also [1]).

Theorem 2.5. *Let U be a periodic, connected, open subset of $\mathbf{R}^n$ with Lipschitz boundary. Let $1 < p < +\infty$ and, for every $\varepsilon > 0$, let $F_\varepsilon : L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0, +\infty]$, be the functional defined by*

$$F_\varepsilon(u, A) = \begin{cases} \int_A f(\frac{x}{\varepsilon}, Du(x))dx & \text{if } u \in W^{1,1}(A) \cap L^p(A) \\ +\infty & \text{otherwise,} \end{cases}$$

where $f(x,\xi) = |\xi|^p$ *for every* $x \in U$ *and* $\xi \in \mathbf{M}^{m\times n}$, *and* $f(x,\xi) = 0$ *for every* $x \in \mathbf{R}^n \setminus U$. *Then there exist a constant* $k_1 > 0$ *depending only on* U, n, p, *and a continuous function* $f_0 : \mathbf{M}^{m\times n} \to \mathbf{R}$ *with*

$$k_1|\xi|^p \le f_0(\xi) \le \big|U \cap]0,1[^n\big|\,|\xi|^p$$

for all $\xi \in \mathbf{M}^{m\times n}$, *such that, setting* $F_0 : L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0,+\infty]$,

$$F_0(u,A) = \begin{cases} \int_A f_0(Du(x))dx & \text{if } u \in W^{1,p}(A) \\ \\ +\infty & \text{otherwise,} \end{cases}$$

we have $F_0(u,A) = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u,A)$ *for every* $A \in \mathcal{A}$ *and every* $u \in L^p_{loc}(\mathbf{R}^n)$. *Moreover, the function* f_0 *verifies*

$$f_0(\xi) = \min\Big\{ \int_{]0,1[^n\setminus U} |Du(x)+\xi|^p dx \;:\; u \in W^{1,p}_{per}(]0,1[^n) \Big\}$$

for every $\xi \in \mathbf{M}^{m\times n}$.

Finally, let us consider, for every $\varepsilon > 0$ and $i \in \mathbf{Z}^n$, the operator R_i^ε defined on $W^{1,\infty}_{loc}(\mathbf{R}^n)$ by

$$R_i^\varepsilon(u)(x) \;=\; \big(1-\psi\big(\frac{x}{\varepsilon}-i\big)\big)u(x) + \psi\big(\frac{x}{\varepsilon}-i\big) \fint_{\varepsilon i+\varepsilon K_\eta} u(y)\,dy\,,$$

where ψ is a fixed cut-off function between K and K_η such that $|D\psi| \le \frac{2}{\eta}$.

Remark 2.6. The operator R_i^ε acts over a function u in the following way

$$R_i^\varepsilon(u)(x) \;=\; \fint_{\varepsilon i+\varepsilon K_\eta} u(y)\,dy \qquad \text{if } x \in \varepsilon i + \varepsilon K\,,$$

in particular $R_i^\varepsilon(u)$ is constant on $\varepsilon i + \varepsilon K$, and

$$R_i^\varepsilon(u)(x) \;=\; u(x) \qquad \text{if } \operatorname{dist}(x, \varepsilon i+\varepsilon K) \ge \varepsilon\eta.$$

Thus we define the operator

$$\mathcal{R}^\varepsilon(u)(x) \;=\; \begin{cases} R_i^\varepsilon(u)(x) & \text{if } \operatorname{dist}(x,\varepsilon i+\varepsilon K) < \varepsilon\eta,\ i \in \mathbf{Z}^n \\ \\ u(x) & \text{otherwise.} \end{cases}$$

Remark 2.7. If $u \in W^{1,\infty}_{loc}(\mathbf{R}^n)$ it is easy to see that for every bounded open set Ω

$$\|D\mathcal{R}^\varepsilon u\|_{L^\infty(\Omega)} \le \frac{2}{\varepsilon\eta} \sup_{i\in\mathbf{Z}^n} \|u - ⨍_{\varepsilon i+\varepsilon K_\eta} u(y)\,dy\|_{L^\infty(\varepsilon i+\varepsilon K_\eta)} + \|Du\|_{L^\infty(\Omega)}.$$

Moreover, denoting by $d = \operatorname{diam} K_\eta$, we have

$$\|u - ⨍_{\varepsilon i+\varepsilon K_\eta} u(y)\,dy\|_{L^\infty(\varepsilon i+\varepsilon K_\eta)} \le \text{ess-}\sup_{\varepsilon i+\varepsilon K_\eta} u - \text{ess-}\inf_{\varepsilon i+\varepsilon K_\eta} u \le \varepsilon d\|Du\|_{L^\infty(\Omega)}\,,$$

thus we obtain

$$\|D\mathcal{R}^\varepsilon u\|_{L^\infty(\Omega)} \le \left(\frac{2}{\eta}d + 1\right)\|Du\|_{L^\infty(\Omega)}\,.$$

In the sequel we will denote $\mathcal{R}^1$ and R^1_i simply by $\mathcal{R}$ and R_i.

Finally we recall that a continuous function $\varphi : \mathbf{M}^{m\times n} \to \mathbf{R}$ is called *quasiconvex* if for every $\xi \in \mathbf{M}^{m\times n}$, $A \in \mathcal{A}$, and $u \in C^1_0(A)$ we have

$$|A|\varphi(\xi) \le \int_A \varphi(\xi + Du)dx.$$

This is a well-known necessary and sufficient condition for the lower semicontinuity of functionals defined on Sobolev spaces (see Ball [4], Morrey [22])

3. The Homogenization Theorem

In this section we will study the $\Gamma(L^p)$-limit of the family of functionals defined in (2.2).

Theorem 3.1. *Let E be the set defined above, verifying the assumption* (2.1). *Let $f(x,\xi)\colon \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty]$ be a Borel function satisfying* (i)–(vi)*, and, for every $\varepsilon > 0$, let $F_\varepsilon\colon L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0,+\infty]$ be the functional defined in* (2.2). *Then there exist two positive constants k_1 and k_2, and a quasiconvex function $f_{hom}\colon \mathbf{M}^{m\times n} \to [0,+\infty]$, with*

$$k_1|\xi|^p \le f_{hom}(\xi) \le k_2(1 + |\xi|^p) \tag{3.1}$$

for all $\xi \in \mathbf{M}^{m\times n}$, such that, defining $F_0\colon L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0,+\infty]$ by

$$F_0(u,\Omega) = \begin{cases} \int_\Omega f_{hom}(Du(x))\,dx & \forall u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.2}$$

we have

$$F_0(u,\Omega) = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u,\Omega)$$

for every $\Omega \in \mathcal{A}$ *and every* $u \in L^p(\Omega)$. *Moreover, the function* f_{hom} *verifies the following homogenization formula*

$$f_{hom}(\xi) \;=\; \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ \int_{]0,T[^n} f(x, \xi + Du)\, dx \;:\; u \in X_\xi^T \Big\} ,$$

where $X_\xi^T = \{u \in X(]0,T[^n) \;:\; u = \mathcal{R}(\xi{\cdot}x) - \xi{\cdot}x$ *in a neighbourhood of* $\partial]0,T[^n\}$.

To prove Theorem 3.1 it is not possible to apply directly the homogenization theorems as in [6] and [24]. Indeed the integrand of the functional

$$F(u,\Omega) \;=\; \begin{cases} \int_\Omega f(x, Du)\, dx & \text{if } u \in X(\Omega), \\ \\ +\infty & \text{otherwise} \end{cases}$$

does not verify the so called standard growth conditions

$$|\xi|^p \;\le\; f(x,\xi) \;\le\; c(1 + |\xi|^p)$$

in all the space. However the region where the growth condition from above is violated, thanks to assumption (2.1), is composed of well-isolated domains.

Definition 3.2. A family $(F_\varepsilon)_{\varepsilon>0}$ of non negative functionals satisfies *uniformly the fundamental estimate as* $\varepsilon \to 0$ if, for every A, B, A' bounded open sets such that $A' \subset\subset A$, and for every $\sigma > 0$ there exist two constants $M_\sigma > 0$ and $\varepsilon_\sigma > 0$ such that for any $u \in X(A)$, $v \in X(B)$, and $\varepsilon \le \varepsilon_\sigma$, there exists a cut-off function ϕ between A' and A, such that

$$F_\varepsilon(\phi u + (1-\phi)v, A' \cup B) \;\le\; (1+\sigma)(F_\varepsilon(u,A) + F_\varepsilon(v,B)) + M_\sigma \int_{(A\cup B)\setminus A'} |u-v|^p dx + \sigma \,.$$

This definition is similar to the one given in [16], Definition 18.2, and permits, as well as in [16] (see also [17]), to obtain results of regularity, as set functions, for the $\Gamma(L^p)$-liminf and the $\Gamma(L^p)$-limsup of the family $(F_\varepsilon)_{\varepsilon>0}$.

Proposition 3.3. *The family* $(F_\varepsilon)_{\varepsilon>0}$ *of functionals defined by* (2.2) *satisfies uniformly the fundamental estimate as* $\varepsilon \to 0$. *Proof.* We fix the sets A, A', and B, with $A' \subset\subset A$, and $\sigma > 0$. Let A'' be an open set such that $A' \subset\subset A'' \subset\subset A$. Given $N \in \mathbf{N}$ that we will fix later, let us consider the open set

$$A_l = \Big\{ x \;:\; \operatorname{dist}(x, A') < \frac{\operatorname{dist}(A', \partial A'')}{3N} l \Big\}$$

for every $l = 1, \ldots, 3N$, and, for every $k = 0, \ldots, N-1$, let φ_k be a cut-off function between the sets A_{3k+1} and A_{3k+2} such that $|D\varphi_k| < 4N$. Now we can fix ε_σ such that

$$\varepsilon_\sigma d < \frac{1}{3N} \,,$$

where d denotes the diameter of the set K_η that, by assumption (2.1), is finite (see Remark 2.1). Since $\mathrm{dist}(A_l, \partial A_{l+1}) = \frac{1}{3N}$, if, for some $i \in \mathbf{Z}^n$ and $\varepsilon < \varepsilon_\sigma$, $\varepsilon i + \varepsilon K$ intersects A_l, it does not intersect $\mathbf{R}^n \setminus A_{l+1}$. Then the functions $\phi_k = \mathcal{R}^\varepsilon \varphi_k$, $0 \le k \le N-1$, are cut-off functions between the sets A_{3k} and $A_{3(k+1)}$.
We will show that, for every $u \in X(A)$ and $v \in X(B)$ it is possible to prove the fundamental estimate choosing the cut-off function between this finite family $(\phi_k)_{k=0,\dots,N-1}$. Thus, fixed $u \in X(A)$ and $v \in X(B)$, for every $k = 0, \dots, N-1$ we have
(3.3)

$$\begin{aligned} F_\varepsilon(\phi_k u + (1-\phi_k)v, A' \cup B) \;&=\; F_\varepsilon(u, (A' \cup B) \cap \overline{A}_{3k})) + F_\varepsilon(v, B \setminus A_{3(k+1)}) + \\ &+ F_\varepsilon(\phi_k u + (1-\phi_k)v, B \cap (A_{3(k+1)} \setminus \overline{A}_{3k})) \;\le\; F_\varepsilon(u, A) + F_\varepsilon(v, B) + \\ &+ F_\varepsilon(\phi_k u + (1-\phi_k)v, B \cap (A_{3(k+1)} \setminus \overline{A}_{3k}))\,. \end{aligned}$$

Now we want to estimate the last term that we denote by I_k. Let $S_k = B \cap (A_{3(k+1)} \setminus \overline{A}_{3k})$. By construction we have $D\phi_k = 0$ on εE and by assumption (iv) we obtain

$$\begin{aligned} F_\varepsilon(\phi_k u + (1-\phi_k)v, S_k \cap \varepsilon E) \;&=\; \int_{S_k \cap \varepsilon E} f(\frac{x}{\varepsilon}, \phi_k Du + (1-\phi_k)Dv)\, dx \;\le \\ &\le\; c \int_{S_k \cap \varepsilon E} (1 + \phi_k a(\frac{x}{\varepsilon}, Du) + (1-\phi_k) a(\frac{x}{\varepsilon}, Dv))\, dx \;\le \\ &\le\; c|S_k \cap \varepsilon E| + cF_\varepsilon(u, S_k \cap \varepsilon E) + cF_\varepsilon(v, S_k \cap \varepsilon E)\,. \end{aligned} \tag{3.4}$$

Moreover by (iv) and (v) we have

$$\begin{aligned} &(3.5) F_\varepsilon(\phi_k u + (1-\phi_k)v, S_k \setminus \varepsilon E) \\ &\le\; c \int_{S_k \setminus \varepsilon E} (1 + a(\frac{x}{\varepsilon}, D\phi_k(u-v) + \phi_k Du + (1-\phi_k)Dv))\, dx \\ &\le\; c|S_k \setminus \varepsilon E| + c \int_{S_k \setminus \varepsilon E} \frac{\alpha}{2}[a(\frac{x}{\varepsilon}, D\phi_k(u-v)) + \phi_k a(\frac{x}{\varepsilon}, Du) + (1-\phi_k) a(\frac{x}{\varepsilon}, Dv)]\, dx \\ &\le\; c(1 + c\frac{\alpha}{2})|S_k \setminus \varepsilon E| + \frac{\alpha}{2} c F_\varepsilon(u, S_k \setminus \varepsilon E) + c\frac{\alpha}{2} F_\varepsilon(v, S_k \setminus \varepsilon E) \\ &\quad + c^2 \frac{\alpha}{2} \|D\phi_k\|^p_{L^\infty} \int_{S_k \setminus \varepsilon E} |u - v|^p dx\,, \end{aligned}$$

where in the last inequality we have used that, by (iii) and (iv), $a(x,\xi) \le c(1+|\xi|^p)$ for every $\xi \in \mathbf{M}^{m \times n}$ and $x \in [0,1]^n \setminus E$. Since $\|D\varphi_k\|_{L^\infty} \le 4N$ by Remark 2.7 we obtain $\|D\phi_k\|^p_{L^\infty} \le [4N(\frac{2}{\eta}d + 1)]^p$. Taking $M_\sigma = c^2 \frac{\alpha}{2}[4N(\frac{2}{\eta}d+1)]^p$, by (3.4) and (3.5) we get

$$\begin{aligned} \min_{0 \le k \le N-1} I_k \;&\le\; \frac{1}{N} \sum_{k=1}^{N-1} I_k \;\le\; \frac{c}{N}(1 + c\frac{\alpha}{2})|A'' \setminus \overline{A}'| + \\ &+ (1 + \frac{\alpha}{2})\frac{c}{N}(F_\varepsilon(u, A) + F_\varepsilon(v, B)) + M_\sigma \int_{(A \cup B) \setminus (A' \cup \varepsilon E)} |u - v|^p dx\,. \end{aligned} \tag{3.6}$$

Thus, choosing N such that $\max\{c(1+c\frac{\alpha}{2})|A''\setminus\overline{A}'|, (1+\frac{\alpha}{2})c\} < \sigma N$, by (3.3) and (3.6) we obtain that there exists k, $0 \le k \le N-1$, such that the fundamental estimate holds with ϕ_k as cut-off function. $\square$

Now let us fix a sequence (ε_h), and denote F_{ε_h} by F_h. We will consider the $\Gamma(L^p)$-limsup and the $\Gamma(L^p)$-liminf of the sequence F_h as defined by (2.4) and (2.3), that will be denoted by F'' and F' respectively. A first important step to prove the homogenization theorem is to check the growth conditions for F'' and F'.

Proposition 3.4. *There exist two constants $k_1 > 0$ and $k_2 > 0$ such that*

$$k_1 \int_\Omega |Du|^p dx \le F'(u,\Omega) \le F''(u,\Omega) \le k_2 \int_\Omega (1+|Du|^p)\, dx$$

for every bounded open set Ω and every $u \in W^{1,p}(\Omega)$.

Proof. In order to estimate F' from below, let us consider the sequence of functionals $F_h^0: L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0,+\infty]$ defined by

$$F_h^0(u,\Omega) = \begin{cases} \int_\Omega \widetilde{f}(\frac{x}{\varepsilon_h}, Du)dx & \text{if } u \in W^{1,1}(\Omega) \cap L^p(\Omega), \\ \\ +\infty & \text{otherwise,} \end{cases}$$

with $\widetilde{f}(x,\xi) = |\xi|^p$ for every $x \in U$ and $\xi \in \mathbf{M}^{m\times n}$, and $\widetilde{f}(x,\xi) = 0$ for every $x \in \mathbf{R}^n \setminus U$ and $\xi \in \mathbf{M}^{m\times n}$. Since by (ii) U is a periodic connected open set with Lipschitz boundary, by Theorem 2.5 we have that there exists a constant $k_1 > 0$ such that

$$k_1 \int_\Omega |Du|^p dx \le \Gamma(L^p)\text{-}\lim_h F_h^0(u,\Omega) \tag{3.7}$$

for every $u \in W^{1,p}(\Omega)$. Since, by (ii), $F_h^0(u,\Omega) \le F_h(u,\Omega)$ for every open set Ω and every $u \in L^p(\Omega)$, we have that $\Gamma(L^p)\text{-}\lim_h F_h^0 \le F'$ and then by (3.7) we obtain the estimate from below.

Now let us prove the estimate from above. Let us fix a bounded open set Ω and let us consider a piecewise affine function u. We can write $u = \sum_{j=1}^s (\xi_j \cdot x + b_j)\chi_{A_j}$, where $\xi_j \in \mathbf{M}^{m\times n}$, $b_j \in \mathbf{R}^n$, (A_j) is a finite family of measurable disjoint subset of Ω such that $\bigcup_j A_j = \Omega$ and χ_{A_j} denotes the characteristic function of the set A_j. For every $h \in \mathbf{N}$ let $u_h = \mathcal{R}^{\varepsilon_h}(u)$. Since $Du_h = 0$ on $\varepsilon_h E$ and $Du = \xi_j$ a.e. in A_j, by the growth condition (iii) and by Remark 2.7 we get

$$\begin{aligned} \int_\Omega f(\frac{x}{\varepsilon_h}, Du_h)\, dx &= \sum_j \int_{A_j} f(\frac{x}{\varepsilon_h}, Du_h)\, dx \le c \sum_j \int_{A_j} (1+|Du_h|^p)\, dx \le \\ &\le c \sum_j |A_j| \big(1 + \|Du_h\|^p_{L^\infty(A_j)}\big)\, dx \le c \sum_j |A_j| \big(1 + (\frac{2}{\eta}d+1)^p |\xi_j|^p\big)\, dx \le \\ &\le c(\frac{2}{\eta}d+1)^p \sum_j |A_j| \big(1+|\xi_j|^p\big)\, dx \le c(\frac{2}{\eta}d+1)^p \int_\Omega \big(1+|Du|^p\big)\, dx\,. \end{aligned} \tag{3.8}$$

Then taking the limsup as $h \to \infty$ we obtain

$$F''(u,\Omega) \le \limsup_{h\to\infty} \int_\Omega f(\frac{x}{\varepsilon_h}, Du_h)\,dx \le k_2 \int_\Omega \big(1+|Du|^p\big)\,dx\,,$$

where $k_2 = c\big(\frac{2}{\eta}d+1\big)^p$, so that we have proved the growth condition from above for every piecewise affine function u and every open bounded set Ω. Finally, for any $u \in W^{1,p}(\Omega)$, there exists a sequence (u_h) of piecewise affine functions, that converges strongly to u in $W^{1,p}(\Omega)$. Thus, by lower-semicontinuity of F'' we have

$$F''(u,\Omega) \le \liminf_{h\to\infty} F''(u_h,\Omega) \le \liminf_{h\to\infty} k_2 \int_\Omega (1+|Du_h|^p)\,dx = k_2 \int_\Omega (1+|Du|^p)\,dx\,.$$

This concludes the proof. □

Theorem 3.5. *For every sequence (ε_h) that converges to zero, there exists a subsequence (ε_{h_k}) such that, for every bounded open set Ω, there exists the $\Gamma(L^p)$-limit $F_0(\cdot,\Omega)$ of $F_{h_k}(\cdot,\Omega)$. Moreover there exists a quasiconvex function $\varphi: \mathbf{M}^{m\times n} \to \mathbf{R}$ such that*

$$F_0(u,\Omega) = \begin{cases} \int_\Omega \varphi(Du)\,dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. The proof of this theorem follows the arguments of well-known compactness and integral representation results in Γ-convergence.

Using the fundamental estimate defined in Definition 3.2, it is easy to see, as in Proposition 18.3 of [16], that

$$F''(u, A' \cup B) \le F''(u,A) + F''(u,B)$$

for every open sets A, A' and B, with $A' \subset\subset A$, and for every $u \in L^p_{loc}(\mathbf{R}^n)$. Then by Theorem 16.9 and Proposition 18.6 of [16] there exists a subsequence (ε_{h_k}) and a functional $F_0: L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0,+\infty]$, such that, for every open set Ω and $u \in L^p_{loc}(\mathbf{R}^n)$,

$$\Gamma(L^p)\text{-}\lim_{k\to\infty} F_{h_k}(u,\Omega) = F_0(u,\Omega)\,.$$

Notice that in Proposition 18.6 of [16] it is required, for the sake of simplicity, that there exists a non-negative increasing functional $G: L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0,+\infty]$ such that $F_h \le G$ for every $h \in \mathbf{N}$, but to obtain the result it is sufficient to know that $F'' \le G$. We have this growth condition by Proposition 3.4. From Theorem 18.5 of [16] we have that for every $u \in L^p_{loc}(\mathbf{R}^n)$ the set function $F_0(u,\cdot)$ is the restriction to $\mathcal{A}$ of a Borel measure. Moreover we have:

(a) for every $\Omega \in \mathcal{A}$, and every $u \in W^{1,p}(\Omega)$

$$k_1 \int_\Omega |Du|^p dx \le F_0(u,\Omega) \le k_2 \int_\Omega (1+|Du|^p)\,dx$$

(by Proposition 3.4);

(b) F_0 is local: i.e., $F_0(u,\Omega) = F_0(u,\Omega)$, whenever $\Omega \in \mathcal{A}$ and $u = v$ a.e. on Ω (by the local character of the Γ-convergence);

(c) for every $\Omega \in \mathcal{A}$, the functional $F_0(\cdot,\Omega)$ is L^p-lower semicontinuous (by Remark 2.3);

(d) $F_0(u+c,\Omega) = F_0(u,\Omega)$, for every $\Omega \in \mathcal{A}$, $u \in L^p(\Omega)$, and $c \in \mathbf{R}$ (since all the functionals F_ε satisfy the same condition);

(e) $F_0(u,\Omega) = +\infty$ if $u \in L^p_{loc}(\mathbf{R}^n)\backslash W^{1,p}(\Omega)$ (again by Proposition 3.4).

By (a)–(e) and the fact that $F_0(u,\cdot)$ is a measure, we can apply well-known integral representation theorems (see for instance Theorem 4.3.2 of [10], see also Theorem 20.1 of [16]) and obtain the existence of a quasiconvex function $\varphi\colon \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty]$ such that

$$F_0(u,\Omega) \;=\; \begin{cases} \int_\Omega \varphi(x, Du(x))\,dx & \text{if } u \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

for every $\Omega \in \mathcal{A}$. Finally, by condition (d), it can be easily proven that the function φ can be chosen independent of the variable x (see [21]). ☐

The proof of Theorem 3.1 will be completed if we show that the function φ is independent of the sequence (ε_{h_k}). To this aim let us introduce the function f_{hom} as in the following proposition.

Proposition 3.6. *For every $\xi \in \mathbf{M}^{m\times n}$ there exists the limit*

$$f_{hom}(\xi) \;=\; \lim_{T\to\infty} \frac{1}{T^n} \inf\Big\{ \int_{]0,T[^n} f(x, \xi + Du)\,dx \;:\; u \in X^T_\xi \Big\}, \tag{3.9}$$

where $X^T_\xi = \{v \in X(]0,T[^n) \;:\; v = \mathcal{R}(\xi\cdot x) - \xi\cdot x$ in a neighbourhood of $\partial]0,T[^n\}$ and $f_{hom}(\xi) = \varphi(\xi)$ for every $\xi \in \mathbf{M}^{m\times n}$.

Proof. Let us fix $\xi \in \mathbf{M}^{m\times n}$ and define for $T > 0$

$$g_T(\xi) \;=\; \inf\Big\{ \int_{]0,T[^n} f(x, \xi + Du)\,dx \;:\; u \in X^T_\xi \Big\}.$$

Let $u^\xi_T \in X^T_\xi$ be a function that satisfies

$$\int_{]0,T[^n} f(x, \xi + Du^\xi_T)\,dx \;\le\; g_T(\xi) + \frac{1}{T}. \tag{3.10}$$

In the rest of the proof we will denote by $[t]$ the integral part of $t \in \mathbf{R}$. Then we extend u^ξ_T to the function v^ξ_T defined in $[0,[T+1]]^n$ by

$$v^\xi_T(x) \;=\; \begin{cases} \mathcal{R}(\xi\cdot x) - \xi\cdot x & \text{if } x \in [0,[T+1]]^n\backslash[0,T]^n, \\ u^\xi_T(x) & \text{if } x \in [0,T]^n, \end{cases}$$

and then we extend it to a function defined on all $\mathbf{R}^n$, that we always denote v_T^ξ, by $[T+1]$-periodicity. Since the function $v_k(x) = \varepsilon_{h_k} v_T^\xi(\frac{x}{\varepsilon_{h_k}})$ converges to zero as $k \to \infty$ in $L^p_{loc}(\mathbf{R}^n)$, by L^p-lower semicontinuity and by periodicity, we get

$$F_0(\xi{\cdot}x,]0,1[^n) \le \liminf_{k\to\infty} F_{h_k}(v_k+\xi{\cdot}x,]0,1[^n) \le \frac{1}{[T+1]^n}\int_{]0,[T+1][^n} f(x, \xi+Dv_T^\xi)dx.$$

By definition of v_T^ξ, by (3.10) and by the growth condition (iii) for f, we have that

$$F_0(\xi{\cdot}x,]0,1[^n) \;\le\; \frac{1}{T^n}\Big(\frac{T^n}{[T+1]^n}(g_T(\xi) + \frac{1}{T})\Big) + c(1+|\xi|^p)\frac{[T+1]^n - T^n}{[T+1]^n}\,.$$

Taking the limit as $T \to +\infty$ we get

$$\varphi(\xi) \;=\; F_0(\xi{\cdot}x,]0,1[^n) \;\le\; \liminf_{T\to+\infty} \frac{1}{T^n}\, g_T(\xi)\,. \tag{3.11}$$

Vice versa by definition of $\Gamma(L^p)$-convergence, there exists a sequence (u_k) of functions of $X(]0,1[^n)$, converging to $\xi{\cdot}x$ in $L^p(]0,1[^n)$, such that

$$F_0(\xi{\cdot}x,]0,1[^n) \;=\; \lim_{k\to\infty} F_{h_k}(u_k,]0,1[^n)\,. \tag{3.12}$$

We can use the fundamental estimate to modify the functions u_k on a neighbourhood on $\partial]0,1[^n$. Let us fix a compact set C of $]0,1[^n$, a set A' such that $C \subset A' \subset\subset\,]0,1[^n$, $\sigma > 0$, and choose in the fundamental estimate (see Definition 3.2) $B =]0,1[^n \backslash C$, $A =]0,1[^n$, $u = u_k$, and $v = v_k = \mathcal{R}^{\varepsilon_{h_k}}(\xi{\cdot}x)$. Then for every k big enough there exists a cut-off function $\phi_k \in C^1_0(]0,1[^n)$, between A' and A, such that, if we define $w_k = \phi_k u_k + (1-\phi_k)v_k$, we have

$$\begin{aligned} F_{h_k}(w_k,]0,1[^n) \;\le\; &(1+\sigma)(F_{h_k}(u_k,]0,1[^n) \\ &+ F_{h_k}(v_k,]0,1[^n\backslash C)) + M_\sigma \int_{C\backslash A'} |u_k - v_k|^p dx + \sigma\,. \end{aligned}$$

Let us denote $T_k = \frac{1}{\varepsilon_{h_k}}$. Since $(u_k - v_k)$ converges to zero in $L^p(]0,1[^n)$, taking the limit as $k \to +\infty$, using the definition of g_T and v_k, by (3.12) and (3.8), we get

$$\begin{aligned} F_0(\xi{\cdot}x,]0,1[^n) \;\ge\; &\frac{1}{1+\sigma}\limsup_{k\to\infty} F_{h_k}(w_k,]0,1[^n) - \limsup_{k\to\infty} F_{h_k}(v_k,]0,1[^n\backslash C)) \;\ge \\ \ge\; &\frac{1}{1+\sigma}\limsup_{k\to\infty}\frac{1}{T_k^n} g_{T_k}(\xi) - k_2\big|]0,1[^n\backslash C\big|(1+|\xi|^p)\,. \end{aligned}$$

By the arbitrariness of C and σ we obtain

$$\varphi(\xi) \;\ge\; \limsup_{k\to\infty}\frac{1}{T_k^n}\, g_{T_k}(\xi)\,.$$

Then, by (3.11), we have that the limit $\lim\limits_{k\to\infty}\dfrac{1}{T_k^n}\, g_{T_k}(\xi)$ exists, and

$$\lim_{k\to\infty}\frac{1}{T_k^n}\, g_{T_k}(\xi) \;=\; \varphi(\xi) \;=\; \liminf_{T\to\infty}\frac{1}{T^n}\, g_(\xi)\,. \tag{3.13}$$

Since the last term of (3.13) does not depend on the choice of the sequence (ε_{h_k}), we conclude that the limit $\lim\limits_{T\to\infty}\dfrac{1}{T^n} g_T(\xi)$ exists and coincides with $\varphi(\xi)$. $\square$

Remark 3.7. It is easy to see, following the arguments in the proof of Proposition 3.6, that the homogenization formula (3.9) is equivalent to

$$f_{hom}(\xi) \;=\; \lim_{T\to\infty} \inf\Big\{\frac{1}{T^n}\int_{]0,T[^n} f(x,\xi+Du)\,dx \;:\; u\in X_{per}(]0,T[^n)\Big\}$$

This formula reduces to

$$f_{hom}(\xi) \;=\; \inf\Big\{\int_{]0,1[^n} f(x,\xi+Du)\,dx \;:\; u\in X_{per}(]0,1[^n)\Big\}$$

when $f(x,\cdot)$ is convex (see [24], [21]).

4. Convergence of minima

In this section we study in brief the cases where it is possible to obtain the convergence of the minima for problems with continuous perturbations and for Dirichlet boundary values problems.

Let Ω be a Lipschitz bounded open subset of $\mathbf{R}^n$ and let us denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω. Let (g_ε) be a sequence of functions of $L^p(\Omega)$ converging to a function g strongly in $L^p(\Omega)$. Let $G_0, G_\varepsilon\colon L^p(\Omega)\times\mathcal{A}(\Omega)\to[0,+\infty]$ be the functionals defined by

$$G_\varepsilon(u,A) \;=\; F_\varepsilon(u,A) + \int_A |g_\varepsilon - u|^p dx$$

and

$$G_0(u,A) \;=\; F_0(u,A) + \int_A |g-u|^p dx$$

for every $A\in\mathcal{A}(\Omega)$ and $u\in L^p(\Omega)$. Since for every sequence (ε_h) and for every sequence (u_h) of functions in $L^p(\Omega)$ converging strongly in $L^p(\Omega)$ to a function u, we have

$$\int_A |g_{\varepsilon_h} - u_h|^p dx \to \int_A |g-u|^p dx \qquad \forall A\in\mathcal{A}(\Omega)\,,$$

by the $\Gamma(L^p)$-convergence of F_ε to F_0 we get

$$G_0(\cdot,A) \;=\; \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} G_\varepsilon(\cdot,A)$$

for every $A\in\mathcal{A}(\Omega)$.

Suppose that there exists $1\le q\le+\infty$ such that

$$|\xi|^q \;\le\; a(x,\xi) \qquad \forall x\in\,]0,1[^n\setminus U \text{ and } \xi\neq 0\,, \tag{4.1}$$

where $a(\cdot,\cdot)$ is the convex function given in the definition of f (condition (iv)). In this case, by conditions (ii), (iii), and (iv), for every $u \in W^{1,1}(\Omega) \cap L^p(\Omega)$, we obtain

$$
\begin{aligned}
G_\varepsilon(u,\Omega) \;&=\; \int_{\Omega\setminus\varepsilon U} f(\frac{x}{\varepsilon}, Du)\,dx + \int_{\Omega\cap\varepsilon U} f(\frac{x}{\varepsilon}, Du)\,dx + \int_\Omega |g_\varepsilon - u|^p dx \;\geq \\
&\geq \int_{\Omega\setminus\varepsilon U} |Du|^q dx + \int_{\Omega\cap\varepsilon U} |Du|^p dx + \int_\Omega |g_\varepsilon - u|^p dx\,.
\end{aligned}
$$

Since (g_ε) converges in $L^p(\Omega)$, we get

$$
\|u\|_{W^{1,s}(\Omega)} \;\leq\; K(G_\varepsilon(u,\Omega)+1)\,, \tag{4.2}
$$

where $s = p \wedge q$ and K is a positive constant independent of ε and u.

Proposition 4.1. *If (4.1) holds and either $q \geq n$ or $q < n$ with $q^* = \frac{qn}{n-q} > p$, then*

$$
\lim_{\varepsilon\to 0} \inf_{X(\Omega)} G_\varepsilon(\cdot,\Omega) \;=\; \min_{W^{1,p}(\Omega)} G_0(\cdot,\Omega)
$$

and the ε-minimizing sequences for G_ε converge to the minimizers for G_0.

Proof. By definition of Γ-convergence it follows immediately that

$$
\limsup_{\varepsilon\to 0} \inf_{X(\Omega)} G_\varepsilon(\cdot,\Omega) \;\leq\; \min_{W^{1,p}(\Omega)} G_0(\cdot,\Omega)\,. \tag{4.3}
$$

Let us prove the opposite inequality for the liminf. Since $\min G_0(\cdot,\Omega) < +\infty$, by (4.3) we may assume that there exists a positive constant M such that

$$
\inf_{X(\Omega)} G_\varepsilon(\cdot,\Omega) < M \tag{4.4}
$$

for every $\varepsilon > 0$. Let $u_\varepsilon \in X(\Omega)$ be such that $G_\varepsilon(u_\varepsilon,\Omega) \leq \inf_{X(\Omega)} G_\varepsilon(\cdot,\Omega) + \varepsilon$. By (4.2) and (4.4) we obtain that u_ε is uniformly bounded in $W^{1,s}(\Omega)$, where $s = p \wedge q$. Let (u_{ε_h}) be a sequence contained in (u_ε). Thanks to our assumption on q, by Rellich's Theorem, we get that, up to a subsequence, (u_{ε_h}) converges strongly in $L^p(\Omega)$ to some function u_0. Then, by definition of $\Gamma(L^p)$-limit, we have

$$
G_0(u_0,\Omega) \;\leq\; \liminf_{h\to\infty} G_{\varepsilon_h}(u_{\varepsilon_h},\Omega) \;=\; \liminf_{h\to\infty} \inf_{X(\Omega)} G_{\varepsilon_h}(\cdot,\Omega)\,.
$$

By (4.4) $u_0 \in W^{1,p}(\Omega)$ and, by (4.3) we can conclude that

$$
\min_{W^{1,p}(\Omega)} G_0(\cdot,\Omega) \;=\; G_0(u_0,\Omega) \;=\; \lim_{\varepsilon\to 0} \inf_{X(\Omega)} G_\varepsilon(\cdot,\Omega)\,. \qquad \square
$$

Similarly we will show how to study the limit of Dirichlet boundary value problems, when (4.1) is satisfied.

Let $\varphi \in W^{1,p}_{loc}(\mathbf{R}^n)$ and let (φ_ε) be a sequence of $W^{1,p}_{loc}(\mathbf{R}^n)$ such that

(α) $\varphi_\varepsilon \to \varphi$ in $L^p_{loc}(\mathbf{R}^n)$, as $\varepsilon \to 0$;

(β) there exists a function $\omega(\rho)$ with $\omega(\rho) \to 0$ as $\rho \to 0$, such that

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\varphi_\varepsilon, B) \,\le\, \omega(|B|)$$

for every open set B.

Remark 4.2. If $\varphi \in W^{1,\infty}(\mathbf{R}^n)$, a sequence φ_ε that satisfies (α) and (β) is given, for instance, by $\varphi_\varepsilon = \mathcal{R}^\varepsilon(\varphi)$. Suppose that K is connected, let $\varphi \in W^{1,p}_{loc}(\mathbf{R}^n)$ and let Ω be a bounded open set. If we set $I_\varepsilon = \{i \in \mathbf{Z}^n :\ (\varepsilon i + \varepsilon K_\eta) \cap \Omega \neq \emptyset\}$ and $\Omega_t = \{x \in \mathbf{R}^n :\ \mathrm{dist}(x,\Omega) < t\}$ for $t > 0$, using Poincaré inequality, we obtain

$$\int_\Omega \|\mathcal{R}^\varepsilon\varphi - \varphi\|^p dx \,\le\, \sum_{i\in I_\varepsilon}\int_{\varepsilon i+\varepsilon K_\eta} \Big|⨍_{\varepsilon i+\varepsilon K_\eta}\varphi(y)dy - \varphi(x)\Big|^p dx \,\le$$
$$\le\, \varepsilon^p C \sum_{i\in I_\varepsilon}\int_{\varepsilon i+\varepsilon K_\eta} |D\varphi|^p dx \,\le\, \varepsilon^p C\|D\varphi\|_{L^p(\Omega_{\varepsilon d})}\,,$$

where C is a positive constant independent of φ, ε, and Ω. Thus the sequence $\varphi_\varepsilon = \mathcal{R}^\varepsilon(\varphi)$ satisfies assumption (α). Moreover, since $D\mathcal{R}^\varepsilon\varphi(x) = D\varphi(x)$ if $x \notin \bigcup_i(\varepsilon i + \varepsilon K_\eta)$, $D\mathcal{R}^\varepsilon\varphi(x) = \frac{1}{\varepsilon}D\psi(⨍_{\varepsilon i+\varepsilon K_\eta}\varphi(y)dy - \varphi(x)) + D\varphi\psi$ if $x \in \varepsilon i + \varepsilon(K_\eta\setminus K)$, and $D\mathcal{R}^\varepsilon\varphi(x) = 0$ if $x \in \varepsilon i + \varepsilon K_\eta$, always by Poincaré inequality we get

$$\int_\Omega |D\mathcal{R}^\varepsilon\varphi(x)|^p dx \,\le\, 2^p\Big(\frac{2^p}{\eta^p} + 1\Big)C\int_{\Omega_{\varepsilon d}} |D\varphi|^p dx\,,$$

and thus the sequence $\varphi_\varepsilon = \mathcal{R}^\varepsilon(\varphi)$ also verifies (β).

Let $\Phi_0, \Phi_\varepsilon \colon L^p_{loc}(\mathbf{R}^n) \times \mathcal{A} \to [0,+\infty]$ be the functionals defined by

$$\Phi_\varepsilon(u,\Omega) \,=\, \begin{cases} F_\varepsilon(u,\Omega) & \text{if } u - \varphi_\varepsilon \in W^{1,p}_0(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\Phi_0(u,\Omega) \,=\, \begin{cases} F_0(u,\Omega) & \text{if } u - \varphi \in W^{1,p}_0(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 4.3. *If* (4.1) *holds and either* $q < n$ *with* $q^* = \frac{qn}{n-q} > p$ *or* $q \geq n$, *then*

$$\lim_{\varepsilon \to 0} \inf_{W^{1,p}(\Omega)} \Phi_\varepsilon(\cdot, \Omega) = \min_{W^{1,p}(\Omega)} \Phi_0(\cdot, \Omega)$$

and the ε*-minimizing sequences for* Φ_ε *converge to the minimizers for* Φ_0.

Proof. Let us prove that

$$\limsup_{\varepsilon \to 0} \inf_{W^{1,p}(\Omega)} \Phi_\varepsilon(\cdot, \Omega) \leq \min_{W^{1,p}(\Omega)} \Phi_0(\cdot, \Omega). \tag{4.5}$$

Let $u_0 \in W^{1,p}(\Omega)$ such that $u_0 - \varphi \in W_0^{1,p}(\Omega)$ and $\Phi_0(u_0, \Omega) = \min \Phi_0(\cdot, \Omega)$. Then $\Phi_0(u_0, \Omega) = F_0(u_0, \Omega)$ and there exists a sequence (u_ε) converging to u_0 in $L^p(\Omega)$ such that $F_0(u_0, \Omega) = \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega)$. By the fundamental estimate, as in the proof of Proposition 3.6, with $v_\varepsilon = \varphi_\varepsilon$, for every compact subset C of Ω, we can find a function w_ε such that $w_\varepsilon - \varphi_\varepsilon \in W_0^{1,p}(\Omega)$ and

$$\min \Phi_0(u_0, \Omega) = \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \Omega) \geq$$
$$\geq \frac{1}{1+\sigma} \limsup_{\varepsilon \to 0} F_\varepsilon(w_\varepsilon, \Omega) - \limsup_{\varepsilon \to 0} F_\varepsilon(\varphi_\varepsilon, \Omega \backslash C) - M_\sigma \int_{\Omega \backslash C} |u - \varphi|^p dx\,.$$

Since $F_\varepsilon(w_\varepsilon, \Omega) = \Phi_\varepsilon(w_\varepsilon, \Omega)$, by condition (β) we get

$$\min \Phi_0(u_0, \Omega) \geq \frac{1}{1+\sigma} \limsup_{\varepsilon \to 0} \inf_{W^{1,p}(\Omega)} \Phi_\varepsilon(\cdot, \Omega) - o(|\Omega \backslash C|)\,,$$

and by the arbitrariness of C and σ we obtain (4.5). It remains to prove that, for every sequence (ε_h), we have

$$\liminf_{\varepsilon_h \to 0} \inf_{W^{1,p}(\Omega)} \Phi_{\varepsilon_h}(\cdot, \Omega) \geq \min_{W^{1,p}(\Omega)} \Phi_0(\cdot, \Omega). \tag{4.6}$$

The estimate (4.6) follows using the same method of the proof of Proposition 4.1, remarking that, since the boundary values of the ε-minimizing sequence are the restrictions of a converging sequence in $W^{1,p}(\Omega)$, to apply Rellich's Theorem it is sufficient to estimate uniformly the norm $L^s(\Omega)$, with $s = p \wedge q$, of the gradient of the ε-minimizing sequence. □

5. Lavrentiev Phenomenon

With the following example we will show that the homogenized functional may depend on the space $X(\Omega)$, even though conditions (i)–(vi) guarantee a growth condition for f_{hom} independent on the choice of $X(\Omega)$.

Let $r > 0$ and $K_r =]r, \frac{1}{2}[^n \cup]\frac{1}{2}, 1-r[^n$, $p < n$. We will show that at least if we choose r small enough the homogenization formula gives two different results if we take in the definition of the functional $F(\cdot, \Omega)$ the space $X(\Omega)$ equal to $C^1(\Omega)$ or $W^{1,p}(\Omega)$. Let $f\colon \mathbf{R}^n \times \mathbf{M}^{m\times n} \to [0,+\infty]$ be a function periodic in the first variable such that

$$f(x,\xi) = \begin{cases} |\xi|^p & \text{if } x \in]0,1[^n \setminus K_r, \\ 0 & \text{if } x \in K_r,\ \xi = 0, \\ +\infty & \text{if } x \in K_r,\ \xi \neq 0. \end{cases}$$

Since f is convex, by Remark 3.7, the homogenization formula, in the two cases (one with $X(\Omega) = C^1(\Omega)$ and the other one with $X(\Omega) = W^{1,p}(\Omega)$), gives, respectively,

$$f^1_{hom}(\xi) = \inf\Big\{ \int_{]0,1[^n} |\xi + Du(x)|^p \ :\ u \in W^{1,p}_{per}(]0,1[^n)\,,\ Du = -\xi \text{ on } K_r \Big\}$$

and

$$f^2_{hom}(\xi) = \inf\Big\{ \int_{]0,1[^n} |\xi + Du(x)|^p \ :\ u \in C^1_{per}(]0,1[^n)\,,\ Du = -\xi \text{ on } K_r \Big\}.$$

We want to choose $r > 0$ and $\xi \in \mathbf{M}^{m\times n}$, such that $f^2_{hom}(\xi) < f^1_{hom}(\xi)$. To this aim, let us consider the functionals F^1_r and F^2_r defined by

$$F^1_r(u,]0,1[^n) = \begin{cases} \int_{]0,1[^n} |\xi + Du(x)|^p dx & \text{if } u \in C^1_{per}(]0,1[^n),\ \xi = -Du \text{ on } K_r, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$F^2_r(u,]0,1[^n) = \begin{cases} \int_{]0,1[^n} |\xi + Du(x)|^p dx & \text{if } u \in W^{1,p}_{per}(]0,1[^n),\ \xi = -Du \text{ on } K_r, \\ +\infty & \text{otherwise.} \end{cases}$$

Since F^1_r and F^2_r are increasing in r, it is easy to see (Proposition 5.4 of [16]) that F^1_r $\Gamma(L^p)$-converge to

$$F^1_0(u,]0,1[^n) = \begin{cases} \int_{]0,1[^n} |\xi + Du(x)|^p dx & \text{if } u \in W^{1,p}_{per}(]0,1[^n),\ u + \xi\cdot x \text{ is equal to a constant a.e. on } K_0, \\ +\infty & \text{otherwise,} \end{cases}$$

and that F^2_r $\Gamma(L^p)$-converge to

$$F^2_0(u,]0,1[^n) = \begin{cases} \int_{]0,1[^n} |\xi + Du(x)|^p dx & \text{if } u \in W^{1,p}_{per}(]0,1[^n),\ \xi = -Du \text{ on } K_0, \\ +\infty & \text{otherwise.} \end{cases}$$

Let us consider the case where $n = 2$, $m = 1$, and $\xi = (-1, 0)$.

Let $u_0 \in W^{1,p}_{per}(]0,1[^2)$ be the minimum point of $F^1_0(\cdot,]0,1[^2)$. Let us consider a function $u_1 \in W^{1,p}(]0,\frac{1}{2}[\times]\frac{1}{2},1[)$ such that

$$\int_{]0,\frac{1}{2}[\times]\frac{1}{2},1[} |\xi + Du_1|^p dx = \min\Big\{ \int_{]0,\frac{1}{2}[\times]\frac{1}{2},1[} |\xi + Dv|^p dx : \; v = u_0 \text{ on } \partial(]0,\frac{1}{2}[\times]\frac{1}{2},1[) \Big\}$$

Consider the function $v_1(x_1,x_2) = \frac{1}{2}u_1(x_1,x_2) - \frac{1}{2}u_1(\frac{1}{2}-x_1, \frac{3}{2}-x_2)$. Since $\xi = (-1,0)$ and $Du_0 = (1,0)$ on K_0, we can assume, by periodicity of u_0 and by translation invariance of the functional F^1_0, that $u_0(x_1,x_2) = x_1$ if $x_1 \in]0,\frac{1}{2}[$ and $x_2 = \frac{1}{2}$ or $x_2 = 1$, $u_0(0,x_2) = 1$ if $x_2 \in]\frac{1}{2},1[$, and $u_0(\frac{1}{2},x_2) = \frac{1}{2}$ if $x_2 \in]\frac{1}{2},1[$; so that $v_1(x_1,x_2) = x_1 - \frac{1}{4}$ if $x_1 \in]0,\frac{1}{2}[$ and $x_2 = \frac{1}{2}$ or $x_2 = 1$, $v_1(0,x_2) = \frac{1}{4}$ if $x_2 \in]\frac{1}{2},1[$, and $v_1(\frac{1}{2},x_2) = -\frac{1}{4}$ if $x_2 \in]\frac{1}{2},1[$. Moreover, since $|Du|^p$ is strictly convex and $v_1 \neq u_1$, we have

(5.1)

$$\begin{aligned}\int_{]0,\frac{1}{2}[\times]\frac{1}{2},1[} |\xi + Dv_1|^p dx &< \frac{1}{2}\int_{]0,\frac{1}{2}[\times]\frac{1}{2},1[} |\xi + Du_1(x_1,x_2)|^p dx + \\ &\quad + \frac{1}{2}\int_{]0,\frac{1}{2}[\times]\frac{1}{2},1[} |\xi + Du_1(\tfrac{1}{2}-x_1, \tfrac{3}{2}-x_2)|^p dx \\ &= \int_{]0,\frac{1}{2}[\times]\frac{1}{2},1[} |\xi + Du_1|^p dx \;\le\; \int_{]0,\frac{1}{2}[\times]\frac{1}{2},1[} |\xi + Du_0|^p dx\,.\end{aligned}$$

With a similar construction we can find a function $v_2 \in W^{1,p}(]\frac{1}{2},1[\times]0,\frac{1}{2}[)$ such that $v_2(x_1,x_2) = x_1 - \frac{3}{4}$ if $x_1 \in]\frac{1}{2},1[$ and $x_2 = 0$ or $x_2 = \frac{1}{2}$, $v_2(\frac{1}{2},x_2) = \frac{1}{4}$ if $x_2 \in]0,\frac{1}{2}[$, $v_2(1,x_2) = -\frac{1}{4}$ if $x_2 \in]0,\frac{1}{2}[$, and

$$\int_{]\frac{1}{2},1[\times]0,\frac{1}{2}[} |\xi + Dv_2|^p dx < \int_{]\frac{1}{2},1[\times]0,\frac{1}{2}[} |\xi + Du_0|^p dx\,. \tag{5.2}$$

Then if we consider the function

$$v \;=\; \begin{cases} v_1 & \text{in }]0,\frac{1}{2}[\times]\frac{1}{2},1[, \\ v_2 & \text{in }]\frac{1}{2},1[\times]0,\frac{1}{2}[, \\ x_1 - \frac{1}{4} & \text{in }]0,\frac{1}{2}[^2, \\ x_1 - \frac{3}{4} & \text{in }]\frac{1}{2},1[^2. \end{cases}$$

We have that $v \in W^{1,p}_{per}(]0,1[^2)$, $Dv = -\xi$ on K_0, and, by (5.1) and (5.2),

$$F^2_0(v,]0,1[^2) < F^1_0(u_0,]0,1[^2) \;=\; \min F^1_0(u,]0,1[^2)\,;$$

so that $\min F^2_0 < \min F^1_0$. Since, by Theorem 2.4,

$$\lim_{r\to 0} \inf F^1_r \;=\; \min F^1_0 \qquad \text{and} \qquad \lim_{r\to 0} \inf F^2_r \;=\; \min F^2_0\,,$$

for r small enough, we have

$$f^2_{hom}(\xi) \;=\; \inf F^2_r < \inf F^1_r \;=\; f^1_{hom}(\xi)\,,$$

and this concludes our example.

6. Loss of Polyconvexity after Homogenization

We recall that a function $f\colon \mathbf{M}^{m\times n} \to \mathbf{R}$ is *polyconvex* if $\xi \mapsto f(\xi)$ can be written as a convex function of all the minors of ξ (see Morrey [22], Ball [4]). This condition implies quasiconvexity, and has been widely used to model hyperelastic materials (see Ball [4], Ciarlet [11], Giaquinta, Modica and Souček [19]). The homogenization of media with stiff inclusions allows us to exhibit simple examples underlining the phenomenon of loss of polyconvexity induced by the process of averaging: we will show that the homogenization of a composite material with two polyconvex phases may be non-polyconvex. An analogous example with everywhere finite integrands is dealt with in more detail in [8].

We take $n = m = 2$ and we define the two functions f_1, f_2 as follows:

$$f_1(\xi) \;=\; |\xi_{11} - \xi_{22}|^p + |\xi_{12}|^p + |\xi_{21}|^p + \Big((\xi_{11} - 1)^+\Big)^p$$

($t^+ = \max\{t, 0\}$ is the positive part of t),

$$f_2(\xi) \;=\; \begin{cases} (1 - \det \xi) & \text{if } \xi = tI, \text{ with } -1 \le t \le 1, \\ \\ +\infty & \text{otherwise} \end{cases}$$

(I is the identity matrix of $\mathbf{M}^{2\times 2}$). The function f_1 is convex (hence polyconvex); the function f_2 is polyconvex: it can be written as $f_2(\xi) = h(\xi, \det \xi)$, where $h\colon \mathbf{R}^4 \times \mathbf{R} \to \mathbf{R}$ is the convex function

$$h(\xi, r) \;=\; \begin{cases} 1 - r & \text{if } \xi = tI, \text{ with } -1 \le t \le 1, \\ \\ +\infty & \text{otherwise.} \end{cases}$$

If we take K as the ball of center 0 and radius $1/4$, then we can define the function $f\colon \mathbf{R}^2 \times \mathbf{M}^{2\times 2} \to [0, +\infty]$ by

$$f(x, \xi) \;=\; \begin{cases} f_2(\xi) & \text{if } x \in \mathbf{Z}^2 + K, \\ \\ f_1(\xi) & \text{otherwise.} \end{cases} \tag{6.1}$$

By Theorem 3.1 we obtain the following proposition.

Proposition 6.1. *Let $p > 1$, let f, F_ε and F_0 be defined by* (6.1), (2.2). *There exists a function f_{hom} verifying*

$$k_1|\xi|^p - k_2 \;\le\; f_{hom}(\xi) \;\le\; k_3(1 + |\xi|^p)\,,$$

for suitable positive constants k_i, for all $\xi \in \mathbf{M}^{2\times 2}$, such that, if F_0 is the functional defined by (3.2), *then $F_0 = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_\varepsilon$.*

Proof. It suffices to check that the hypotheses (i)–(vi) of Theorem 3.1 are satisfied by the function $\widetilde{f}(x, \xi) = f(x, \xi) + C$, where C is a suitable constant added to have

$\widetilde{f} \geq 0$. This is easily done by taking $E = \mathbf{Z}^2 + K$, $b = 1 + C$, and defining the function $a\colon \mathbf{R}^2 \times \mathbf{M}^{2\times 2} \to [0, +\infty]$ in (iv) by

$$a(x,\xi) = \begin{cases} 0 & \text{if } \xi = tI, \text{ with } -1 \leq t \leq 1, \\ \\ +\infty & \text{otherwise} \end{cases} \qquad \forall x \in E\,,$$

and $a(x,\xi) = |\xi|^p$ if $x \in \,]0,1[^n \setminus E$. Remark that condition (ii) is satisfied if we also take C large enough to have $\widetilde{f}(x,\xi) \geq |\xi|^p$. □

Proposition 6.2. *Let* $1 < p < 2$. *Then* f_{hom} *is not polyconvex.*
Proof. We can repeat the argument of [8]: were f_{hom} polyconvex then it should be convex, since a polyconvex non-convex function must grow at least as $|\xi|^2$ in some direction (it is a convex non-constant function of $\det \xi$). Hence we have to show that f_{hom} is not convex. Since $f_{hom}(I) = f_{hom}(-I) = 0$, it suffices to prove that $f_{hom}(0) > 0$, that is $F_0(0,]0,1[^2) > 0$. Now, by definition of Γ-limit we can find a sequence (ε_h) of positive numbers converging to 0, and a sequence (u_h) of functions in $X(]0,1[^2)$ converging to 0 in $L^p(\Omega)$ such that $F_0(0,]0,1[^2) = \lim_h F_{\varepsilon_h}(u_h,]0,1[^2)$. Since $a(x,\xi) \geq c(|\xi_{11} - \xi_{22}|^p + |\xi_{12} + \xi_{21}|^p)$ we deduce that Δu_h converge strongly to zero in $W^{-1,p}(]0,1[^2)$. It easy to see by elliptic regularity (we refer for details to [8]) that we must have $u_h \to 0$ strongly in $W^{1,p}_{loc}(]0,1[^2)$; in particular $|\{x \in]0,1[^2\colon\ |Du_h| > 1\}| \to 0$. This convergence implies that

$$\begin{aligned} \lim_{h\to\infty} F_{\varepsilon_h}(u_h,]0,1[^2) &\geq \liminf_{h\to\infty} F_{\varepsilon_h}(u_h,]0,1[^2 \cap \{|Du_h| \leq 1\}) \geq \\ &\geq \liminf_{h\to\infty} F_{\varepsilon_h}(u_h,]0,1[^2 \cap \{|Du_h| \leq 1\} \cap \varepsilon_h(\mathbf{Z}^2 + K)) \geq \\ &\geq \frac{1}{2} \lim_{h\to\infty} |(\varepsilon_h(\mathbf{Z}^2 + K)) \cap [0,1]^2| > 0\,, \end{aligned}$$

and the proof is concluded. □

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