

**GRADIENT BOUNDS FOR MINIMIZERS
OF FREE DISCONTINUITY PROBLEMS RELATED TO
COHESIVE ZONE MODELS IN FRACTURE MECHANICS**

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ABSTRACT. In this note we consider a free discontinuity problem for a scalar function, whose energy depends also on the size of the jump. We prove that the gradient of every smooth local minimizer never exceeds a constant, determined only by the data of the problem.

Keywords: fracture mechanics, cohesive zone models, functions of bounded variations, local minimizers, free discontinuity problems.

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1. INTRODUCTION

The study of cohesive zone models in fracture mechanics in the one dimensional case (see, e.g., [7] and [6]) leads to functionals of the form

$$\int_0^l F(|\dot{u}|) dx + \sum_{S(u)} G(|[u]|) \quad u \in SBV(0, l), \quad (1)$$

where $F: [0, +\infty) \rightarrow [0, +\infty)$ is C^1 , strictly convex, increasing, superlinear at infinity, and satisfies $F(0) = F'(0) = 0$, and $G: [0, +\infty) \rightarrow [0, +\infty)$ is C^1 , concave, and satisfies $G(0) = 0$ and $G'(0) > 0$. Here and in the rest of the paper SBV is the space of special functions with bounded variation, for which we refer to [1], $S(u)$ denotes the jump set of u , and $[u]$ denotes the jump of u .

To prove the existence of a minimizer of (1) with appropriate boundary conditions we can consider the corresponding relaxed functional in $L^1(0, l)$, which for every $u \in BV(0, l)$ can be written as

$$\int_0^l \bar{F}(|\dot{u}|) dx + \sum_{S(u)} G(|[u]|) + G'(0) |u'_c|(0, l), \quad (2)$$

where \dot{u} is the density of the absolutely continuous part of the distributional derivative u' and u'_c is its Cantor part. In (2) $\bar{F}(\xi) = F(\xi)$ for $\xi \leq e_M$ and $\bar{F}(\xi) = F(e_M) + F'(e_M)(\xi - e_M)$ if $\xi > e_M$, where e_M is the unique constant such that $F'(e_M) = G'(0)$. It is possible to prove that the minimum problem for the relaxed functional (2) with appropriate boundary conditions has a solution. Moreover in [3] it was proved, by using one dimensional arguments, that if G is strictly concave, then every local minimizer u of (2) satisfies

$$|\dot{u}| \leq e_M \quad \text{a.e. on } (0, l), \quad |u'_c|(0, l) = 0.$$

In particular this implies that $\bar{F}(\dot{u}) = F(\dot{u})$ a.e. on $(0, l)$, so that u is a local minimizer of (1). Moreover

$$F'(|\dot{u}|) \leq G'(0).$$

This justifies the interpretation of $G'(0)$ as the ultimate stress for the problem (see, e.g., [4]).

In this note we study the same problem in dimension $n \geq 1$. We consider functionals of the form

$$\int_{\Omega} F(|\nabla u|) dx + \int_{S(u)} G(|[u]|) d\mathcal{H}^{n-1} \quad u \in SBV(\Omega), \quad (3)$$

where ∇u is the density of the absolutely continuous part of the distributional gradient Du , and F and G satisfy the same properties considered for (1).

Also in this case the functional is not lower semicontinuous, so in order to prove existence results we consider its *relaxed functional* in $L^1(\Omega)$ (see [2]), which is represented on $BV(\Omega)$ by

$$\mathcal{E}(u) = \int_{\Omega} \bar{F}(|\nabla u|) dx + \int_{S(u)} G(|[u]|) d\mathcal{H}^{n-1} + G'(0) |D^c u|(\Omega), \quad (4)$$

where \bar{F} is defined as for (2) and $D^c u$ denote the Cantor part of Du . Under appropriate boundary conditions the minimum problems for (4) have a solution. A *local minimizer* of \mathcal{E} in Ω is a function $u \in BV(\Omega)$, with $\mathcal{E}(u) < +\infty$, for which there exists $\eta > 0$ such that $\mathcal{E}(u) \leq \mathcal{E}(v)$ for every $v \in BV(\Omega)$ with $\text{supp}(v - u) \subset\subset \Omega$ and $\|v - u\|_{BV(\Omega)} < \eta$.

Also in this case it is reasonable to expect that any local minimizer u satisfies

$$|\nabla u| \leq e_M \quad \text{a.e. on } \Omega, \quad |D^c u|(\Omega) = 0, \quad (5)$$

where e_M is defined as for (2). In fracture mechanics the functionals (3) and (4) are used to study cohesive zone models in the antiplane case. In this context the first inequality in (5) says that the norm of the deformation gradient of the solution cannot exceed the constant e_M , which is interpreted as the yield strain of the problem. Since (5) implies $F'(|\nabla u|) \leq G'(0)$ a.e. on Ω , the constant $G'(0)$ plays the role of the ultimate stress for the crack problem.

The aim of this note is to present a partial result in this direction. Namely, we prove that, if

$$\lim_{t \rightarrow 0^+} \frac{G(t) - G'(0)t}{t^2} < 0$$

and u is a local minimizer of (4) in Ω , then

$$|\nabla u| \leq e_M$$

in every open subset of Ω where u is of class C^1 . As a consequence we have that, if u is a C^1 local minimizer for (4) in Ω , then it is also a local minimizer for (3).

2. STATEMENT AND PROOF OF THE RESULT

Let Ω be an open subset of \mathbb{R}^n , $n \geq 1$. We assume that the functions F and G satisfy the following properties:

- (a) F is C^1 , strictly convex, increasing, and superlinear at infinity, and satisfies $F(0) = F'(0) = 0$;
- (b) G is C^1 , nonnegative, concave, and satisfies $G(0) = 0$, $G'(0) > 0$, and

$$\lim_{t \rightarrow 0^+} \frac{G(t) - G'(0)t}{t^2} < 0. \quad (6)$$

The function \bar{F} is defined as follows

$$\bar{F}(\xi) = \begin{cases} F(\xi) & \text{if } \xi \leq e_M, \\ F(e_M) + F'(e_M)(\xi - e_M) & \text{if } \xi > e_M, \end{cases} \quad (7)$$

where e_M is the unique solution of the equation $F'(e_M) = G'(0)$.

Theorem 1. *Assume that F and G satisfy conditions (a) and (b) and let u be a local minimizer of the functional \mathcal{E} defined by (4). Suppose that u is of class C^1 on an open subset U of Ω . Then $|\nabla u| \leq e_M$ in U .*

The result stated in Theorem 1 implies that, if u is a local minimizer of (4) satisfying $u \in C^1(\Omega \setminus K)$, with K closed and $\mathcal{H}^{n-1}(K) < +\infty$, then u is also a local minimizer of (3). Indeed in this case $D^c u = 0$, hence $u \in SBV(\Omega)$, and $\bar{F}(|\nabla u|) = F(|\nabla u|)$ a.e. in Ω by Theorem 1.

Proof of Theorem 1. Without loss of generality we consider only the case $e_M = F'(e_M) = G'(0) = 1$ and $U = \Omega$. We argue by contradiction and we assume that there exists a point $x_0 \in \Omega$ such that $|\nabla u(x_0)| = \lambda$, with $\lambda > 1$. By changing the coordinate system, it is not restrictive to assume that $x_0 = 0$, $u(0) = 0$, and $\nabla u(0) = \lambda e_n$, where $e_n := (0, \dots, 0, 1)$ is the last vector of the canonical basis of \mathbb{R}^n .

We want to construct a competitor w by modifying u in a small set $V \subset \subset \Omega$ with piecewise C^1 boundary in such a way that w is close to u in the BV norm and the energy of w is strictly below the energy of u , contradicting the local minimality. In all cases we will take w of the form

$$w = \begin{cases} \alpha u & \text{in } V, \\ u & \text{otherwise,} \end{cases} \quad (8)$$

for a suitable constant $\alpha < 1$. The problem is reduced to choose α and V such that

$$\|u - w\|_{BV(\Omega)} < \eta \quad \text{and} \quad \mathcal{E}(u) - \mathcal{E}(w) > 0, \quad (9)$$

where η is the constant in the definition of local minimality for u .

We consider three cases corresponding to different hypotheses on G and u with increasing level of difficulty.

Case 1: $G''(0) = -\infty$.

Let us first consider the case where G satisfies the following condition

$$\lim_{t \rightarrow 0^+} \frac{G(t) - t}{t^2} = -\infty. \quad (10)$$

Let us fix $\varepsilon \in (0, \frac{1}{2})$, with $\lambda - \varepsilon > 1$. By the continuity of ∇u we can find $R > 0$ small enough so that

$$|\nabla u - \lambda e_n| < \varepsilon \quad \text{in } B_R, \quad (11)$$

where B_R is the closed ball with center 0 and radius R . As a consequence we can show that $|\nabla u| > \lambda - \varepsilon$ in B_R and that there exists $\delta > 0$ such that

$$\begin{aligned} u(x) &> \delta \quad \text{for every } x \in B_R \text{ with } x_n = \varepsilon R, \\ u(x) &< -\delta \quad \text{for every } x \in B_R \text{ with } x_n = -\varepsilon R. \end{aligned}$$

This implies that for $0 < \sigma < \delta$ the projection of the set $\{x \in B_R : u(x) = \sigma\}$ onto the hyperplane $\{x_n = 0\}$ contains the projection of the set $\{x \in B_R : x_n = \varepsilon R\}$, and therefore

$$\mathcal{H}^{n-1}(B_R \cap \{u = \sigma\}) \geq K_{\varepsilon, R} := \omega_{n-1} R^{n-1} (1 - \varepsilon^2)^{(n-1)/2}, \quad (12)$$

where ω_{n-1} is the $(n-1)$ -dimensional measure of the unit ball in \mathbb{R}^{n-1} . Moreover (11) implies that there exists a constant $L < +\infty$ such that

$$\mathcal{H}^{n-1}(\{x \in \partial B_R : 0 < u(x) < \sigma\}) \leq L \sigma \quad (13)$$

for every $\sigma > 0$.

For $0 < \sigma < \delta$ we define

$$V_\sigma := \{x \in B_R : 0 < u(x) < \sigma\}.$$

Since u is C^1 , there exists a constant M such that

$$\mathcal{H}^{n-1}(\partial V_\sigma) \leq M$$

for $0 < \sigma < \delta$.

We now fix $\alpha < 1$ such that $\alpha(\lambda - \varepsilon) > 1$ and $(1 - \alpha)(\|u\|_{BV(\Omega)} + \delta M) < \eta$, and define w as in (8) with $V := V_\sigma$ for some $\sigma \in (0, \delta)$ to be chosen later. Since

$$\|w - u\|_{BV(\Omega)} \leq (1 - \alpha)\|u\|_{BV(\Omega)} + (1 - \alpha)\sigma \mathcal{H}^{n-1}(\partial V_\sigma),$$

we have $\|w - u\|_{BV(\Omega)} \leq \eta$ for $0 < \sigma < \delta$, so that the first inequality in (9) is satisfied.

Using the definition of \mathcal{E} and \bar{F} , we get

$$\begin{aligned} \mathcal{E}(u) - \mathcal{E}(w) &= (1 - \alpha) \int_{V_\sigma} |\nabla u| dx - \int_{B_R \cap \{u=\sigma\}} G((1 - \alpha)u) d\mathcal{H}^{n-1} \\ &\quad - \int_{\partial B_R \cap V_\sigma} G((1 - \alpha)u) d\mathcal{H}^{n-1}. \end{aligned} \quad (14)$$

Since u is a C^1 local minimum of \mathcal{E} and $|\nabla u| > 1$ in B_R , in particular u is a C^1 local minimum of

$$\int_{B_R} |\nabla u| dx$$

and then, it satisfies the Euler equation

$$\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0 \quad \text{in the sense of distributions on } B_R. \quad (15)$$

Thus, by the divergence theorem, we have

$$\begin{aligned} \int_{V_\sigma} |\nabla u| dx &= \int_{B_R \cap \{u=\sigma\}} u d\mathcal{H}^{n-1} + \int_{\partial B_R \cap V_\sigma} \frac{\nabla u}{|\nabla u|} \frac{x}{|x|} u d\mathcal{H}^{n-1} \\ &\geq \int_{B_R \cap \{u=\sigma\}} u d\mathcal{H}^{n-1} - \int_{\partial B_R \cap V_\sigma} u d\mathcal{H}^{n-1}. \end{aligned} \quad (16)$$

Moreover, by condition (10), for any given $c > 0$ we can choose σ small enough so that

$$G((1 - \alpha)u) < (1 - \alpha)u - c(1 - \alpha)^2 u^2 \quad \text{on } V_\sigma.$$

This, together with (16) and (14), implies

$$\begin{aligned} \mathcal{E}(u) - \mathcal{E}(w) &\geq (1 - \alpha)\sigma \mathcal{H}^{n-1}(B_R \cap \{u = \sigma\}) - (1 - \alpha) \int_{\partial B_R \cap V_\sigma} u d\mathcal{H}^{n-1} \\ &\quad - (1 - \alpha) \mathcal{H}^{n-1}(B_R \cap \{u = \sigma\}) [\sigma - c(1 - \alpha)\sigma^2] \\ &\quad - (1 - \alpha) \int_{\partial B_R \cap V_\sigma} [u - c(1 - \alpha)u^2] d\mathcal{H}^{n-1} \\ &\geq (1 - \alpha)\sigma [c(1 - \alpha)\sigma \mathcal{H}^{n-1}(B_R \cap \{u = \sigma\}) - 2 \mathcal{H}^{n-1}(\partial B_R \cap V_\sigma)]. \end{aligned}$$

From (12) and (13) we get

$$\mathcal{E}(u) - \mathcal{E}(w) \geq (1 - \alpha)\sigma^2 [c(1 - \alpha)K_{\varepsilon, R} - 2L],$$

which gives the second inequality in (9) when c is big enough.

Next we consider the general case where G does not necessarily satisfy (10). In this case we must choose the set V more carefully. In order to explain the new ideas of the proof without technicalities, we prove first the result in two dimensions in the simplest case: when u is an affine function.

Case 2: $-\infty < G''(0) < 0$, u affine, and $n = 2$.

We now consider the case $n = 2$ with u affine. We assume that G satisfies the following condition

$$-\infty < \lim_{t \rightarrow 0^+} \frac{G(t) - t}{t^2} < 0. \quad (17)$$

Then there exist two constants $c_2 > c_1 > 0$ such that

$$t - c_2 t^2 < G(t) < t - c_1 t^2 \quad (18)$$

for $t > 0$ small enough.

It is not restrictive to take $u(x) = \lambda x_2$ for every $x = (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2$. We assume by contradiction that $\lambda > 1$. It is easy to check that in general we may not choose V to be a rectangle. Indeed, if $V = \{(x_1, x_2) \in \Omega : 0 < x_1 < S, 0 < x_2 < \delta\}$, following the computation of Case 1 we get for $\delta > 0$ small enough

$$\begin{aligned} \mathcal{E}(u) - \mathcal{E}(w) &\leq (1 - \alpha)\lambda S\delta - \int_0^\delta [(1 - \alpha)\lambda x_2 - c_2(1 - \alpha)^2\lambda^2 x_2^2] dx_2 \\ &\quad - (1 - \alpha)\lambda\delta S + c_2 S(1 - \alpha)^2\lambda^2\delta^2 \\ &= -(1 - \alpha)\lambda\frac{\delta^2}{2} + c_2(1 - \alpha)^2\lambda^2\frac{\delta^3}{3} + c_2 S(1 - \alpha)^2\lambda^2\delta^2, \end{aligned}$$

and the right-hand side is positive for every $\delta > 0$ only if $S \geq [2(1 - \alpha)\lambda c_2]^{-1}$. This condition may be incompatible with the inclusion $V \subset\subset \Omega$. For the same reason we can not define V as in Case 1.

Since the previous computation shows that the problem is given by the short sides of the rectangle, we are led to overcome this difficulty by defining a special profile for the boundary of V . Let us fix r and R , with $r < R$, and let $\varphi: [0, R] \rightarrow [0, +\infty)$ be a nonincreasing function, to be chosen later, satisfying $\varphi(\rho) = 1$ in $0 \leq \rho \leq r$ and $\varphi(R) = 0$. We take V of the form

$$V := \{(x_1, x_2) : |x_1| < R, 0 < x_2 < \sigma\varphi(|x_1|)\},$$

with $0 < \sigma < 1$, and we consider the function w defined by (8). Let us compute the energy of w and show that (9) holds for a suitable choice of r , R , φ , σ , and α .

If $\alpha < 1$ and $\alpha\lambda > 1$, using the definition of w we get

$$\begin{aligned} \mathcal{E}(u) - \mathcal{E}(w) &= (1 - \alpha)\lambda \mathcal{L}^2(V) - \int_{\partial V \setminus \{x_2=0\}} G((1 - \alpha)\lambda x_2) d\mathcal{H}^1(x) \\ &= 2(1 - \alpha)\lambda r\sigma + 2(1 - \alpha)\lambda \int_r^R \sigma\varphi(\rho) d\rho - 2rG((1 - \alpha)\lambda\sigma) \\ &\quad - 2 \int_r^R G((1 - \alpha)\lambda\sigma\varphi(\rho))\sqrt{1 + (\sigma\varphi'(\rho))^2} d\rho. \end{aligned}$$

Using the fact that $\sqrt{1 + t^2} \leq 1 + \frac{1}{2}t^2$ and $0 \leq G(t) \leq t - c_1 t^2$ for small $t > 0$ we obtain

$$\begin{aligned} \frac{1}{(1 - \alpha)\lambda} (\mathcal{E}(u) - \mathcal{E}(w)) &\geq 2c_1 r(1 - \alpha)\lambda\sigma^2 - \int_r^R \sigma^3\varphi(\rho) (\varphi'(\rho))^2 d\rho \\ &\quad + 2c_1 \int_r^R (1 - \alpha)\lambda\sigma^2 (\varphi(\rho))^2 d\rho \\ &\geq \int_r^R [2c_1(1 - \alpha)\lambda\sigma^2 (\varphi(\rho))^2 - \sigma^3\varphi(\rho) (\varphi'(\rho))^2] d\rho. \end{aligned}$$

The inequality $\mathcal{E}(u) - \mathcal{E}(w) > 0$ can be obtained easily for σ small enough if

$$\varphi'(\rho)^2 \varphi(\rho) = k(\varphi(\rho))^2$$

for a suitable constant k independent of σ . It is easy to check that a solution of this equation on $[r, R]$, with $\varphi(r) = 1$ and $\varphi(R) = 0$, is given by

$$\varphi(\rho) = \frac{(\rho - R)^2}{(r - R)^2}, \quad (19)$$

with $k = 4(R - r)^{-2}$. With this choice of the profile φ we get

$$\mathcal{E}(u) - \mathcal{E}(w) \geq (1 - \alpha) \lambda \int_r^R [2c_1(1 - \alpha) \lambda \sigma^2 - 4\sigma^3(R - r)^{-2}] (\varphi(\rho))^2 d\rho. \quad (20)$$

Now we choose $\alpha < 1$ such that $\alpha \lambda > 1$ and

$$(1 - \alpha) \left[\|u\|_{BV(\Omega)} + 2r\lambda + 2\lambda \int_r^R \varphi(\rho) \sqrt{1 + (\varphi'(\rho))^2} d\rho \right] < \eta.$$

Since

$$\|w - u\|_{BV(\Omega)} \leq (1 - \alpha) \left[\|u\|_{BV(\Omega)} + 2\sigma r\lambda + 2\lambda \int_r^R \sigma \varphi(\rho) \sqrt{1 + (\sigma \varphi'(\rho))^2} d\rho \right],$$

the first inequality in (9) is satisfied for $0 < \sigma < 1$. By (20) the second inequality in (9) is satisfied for $0 < \sigma < c_1(1 - \alpha) \lambda (R - r)^2/2$. This concludes the proof of Case 2.

Case 3: General case.

We finally prove the result in the general case. As in Case 1, for a given $\varepsilon \in (0, \frac{1}{2})$ such that $\lambda - \varepsilon > 1$ we may select $R > 0$ so small that $|\nabla u - \lambda e_n| < \varepsilon$ and $|\nabla u| > \lambda - \varepsilon$ in B_R . Now, inspired by the calculation of Case 2, we fix $r > 0$, with $r < R$, and we consider the function $a(x)$ defined in B_R by $a(x) = \varphi(|x|)$; i.e.,

$$a(x) = \begin{cases} \frac{(|x| - R)^2}{(r - R)^2} & \text{if } r < |x| < R, \\ 1 & \text{if } |x| \leq r. \end{cases}$$

Let $v := u/a$ and $S_\sigma := \{x \in B_R : v(x) = \sigma\} = \{x \in B_R : u(x) = \sigma a(x)\}$. Since u is C^1 , there exist $\delta > 0$ and $M > 0$ such that

$$\mathcal{H}^{n-1}(S_\sigma) \leq M \quad (21)$$

for $0 < \sigma < \delta$.

We now fix $\alpha < 1$ such that $\alpha(\lambda - \varepsilon) > 1$ and $(1 - \alpha) [\|u\|_{BV(\Omega)} + \delta M] < \eta$, and define w as in (8) with $V := \{x \in B_R : 0 \leq v(x) \leq \sigma\}$. Since

$$\|w - u\|_{BV(\Omega)} \leq (1 - \alpha) \|u\|_{BV(\Omega)} + (1 - \alpha) \sigma \mathcal{H}^{n-1}(S_\sigma),$$

we have $\|w - u\|_{BV(\Omega)} \leq \eta$ for $0 < \sigma < \delta$, so that the first inequality in (9) is satisfied.

To conclude the proof we have to show that σ can be chosen in $(0, \delta)$ so that the second inequality in (9) holds, contradicting the local minimality of u . If δ is small enough, we may assume that G satisfies the second inequality of (18) for $0 < t < \delta$. Let $C_R^r := B_R \setminus B_r$. By the definition of w we have $|\nabla w| = \alpha |\nabla u| > 1$ a.e. on V and thus

$$\begin{aligned} \mathcal{E}(u) - \mathcal{E}(w) &= (1 - \alpha) \int_V |\nabla u| dx - \int_{B_R \cap S_\sigma} G((1 - \alpha)u) d\mathcal{H}^{n-1} \\ &\geq (1 - \alpha) \int_V |\nabla u| dx - (1 - \alpha) \int_{B_r \cap S_\sigma} u d\mathcal{H}^{n-1} - (1 - \alpha) \int_{C_R^r \cap S_\sigma} u d\mathcal{H}^{n-1} \\ &\quad + c_1(1 - \alpha)^2 \int_{C_R^r \cap S_\sigma} u^2 d\mathcal{H}^{n-1} + c_1(1 - \alpha)^2 \int_{B_r \cap S_\sigma} u^2 d\mathcal{H}^{n-1}. \end{aligned} \quad (22)$$

As in Case 1 we use the fact that u satisfies (15). Since $\frac{\nabla v}{|\nabla v|}$ is the outer unit normal to S_σ and $\nabla v = \nabla u$ on $B_r \cap S_\sigma$, by the divergence theorem we get

$$\int_V |\nabla u| dx = \int_{C_R^r \cap S_\sigma} \frac{\nabla u}{|\nabla u|} \frac{\nabla v}{|\nabla v|} u d\mathcal{H}^{n-1} + \int_{B_r \cap S_\sigma} u d\mathcal{H}^{n-1}. \quad (23)$$

Since $\nabla v = \nabla u/a - u\nabla a/a^2 = (1/a)(\nabla u - \sigma\nabla a)$ on $C_R^r \cap S_\sigma$, we have

$$\frac{\nabla u}{|\nabla u|} \frac{\nabla v}{|\nabla v|} = \frac{|\nabla u| - \sigma \frac{\nabla a \cdot \nabla u}{|\nabla u|}}{|\nabla u - \sigma\nabla a|}$$

on $C_R^r \cap S_\sigma$. Using Taylor's expansion of the right-hand side with respect to σ we obtain

$$\frac{\nabla u}{|\nabla u|} \frac{\nabla v}{|\nabla v|} = 1 + \frac{(\nabla a \cdot \nabla u)^2 - |\nabla a|^2 |\nabla u|^2}{2|\nabla u|^4} \sigma^2 + O(\sigma^3),$$

and hence

$$\frac{\nabla u}{|\nabla u|} \frac{\nabla v}{|\nabla v|} \geq 1 - \frac{|\nabla a|^2}{2(\lambda - \varepsilon)^2} \sigma^2 + O(\sigma^3) \quad (24)$$

on $C_R^r \cap S_\sigma$. Since $|\nabla a|^2 a = 4(R - r)^{-2} a^2$ on $C_R^r \cap S_\sigma$, by (22), (23), and (24) we have

$$\mathcal{E}(u) - \mathcal{E}(w) \geq \sigma^2(1 - \alpha) [c_1(1 - \alpha) - K_{\varepsilon, r, R} \sigma] \int_{C_R^r \cap S_\sigma} a^2 d\mathcal{H}^{n-1} + O(\sigma^4),$$

with $K_{\varepsilon, r, R} := 2(R - r)^{-2}(\lambda - \varepsilon)^{-2}$. Taking now $\sigma > 0$ small enough we obtain $\mathcal{E}(u) - \mathcal{E}(w) > 0$, which concludes the proof. \square

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