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A VARIATIONAL VIEW OF PARTIAL BRITTLE DAMAGE EVOLUTION

Gilles A. Francfort
Laboratoire P.M.T.M., Institut Galilée,
Université Paris 13
Avenue Jean-Baptiste Clément,
93430 Villetaneuse, France
francfor@galilee.univ-paris13.fr

Adriana Garroni
Dipartimento di Matematica,
Universita` di Roma "La Sapienza"
Piazzale A. Moro, 2,
00185 Roma, Italia
garroni@mat.uniroma1.it

Abstract

Under time-dependent loading, an elastic material is undergoing the simplest form of damage, that which consists in passing from its original state to a weaker elastic state. Elaborating on prior work [?], we establish existence of a relaxed variational evolution where, at each time, the two states of the material combine to form a fine mixture, optimal from the standpoint of the applied load at that time, yet preserving the irreversibility of the damaging process.

Keywords: Variational models, energy minimization, rate-independent processes, damage, quasiconvexity, brittleness, quasistatic evolution, irreversibility.

1 Introduction

The recognition that the elastic response of some materials deteriorates throughout loading history was born out of the use of new materials like concrete in civil works. In the initial loading phase of an elastic sample, damage is at first undistinguishable from plasticity; when unloading occurs however, the damaged sample will return to an unstrained configuration, in contrast to the plastified sample, which will have undergone irretrievable stretches. But then, the sample will be less stiff than it was to start with. This simple mechanism is the macroscopic expression of a slew of usually ill-defined micro-events. Thus, the initial modeling of damage could only be phenomenological [?].

The phenomenological approach is conceptually straightforward. The stiffness tensor (Hooke's law) of the material, $A(x,t)$, is assumed to be a given function of some damage variable, $\alpha(x,t)$. The evolution of α is then governed by a criterion, as in plasticity, or, "equivalently", by some differential equation.

More recently, this approach was deemed too simplistic and, encouraged by the development of the theory of mixtures, many mechanicians proposed to establish the relationship between α to A through a homogenization procedure. They postulated a microstructure parameterized by α (the micro-mechanism) and computed $A(\alpha)$ as the macroscopic resulting stiffness (see e.g. the references in [?]).

In this paper, we only consider the case of partial brittle damage. For us, brittle means that, at a given point, the material is either healthy or damaged, or, equivalently, that the damage variable is a marker (say the characteristic function of the healthy part of the material); partial means that the damaged state retains some amount of elastic stiffness. The starting phenomenology is thus minimal and arguably simplistic. The merits of the approach that we follow here were discussed at length in [?] and we refer the mechanically inclined reader to the introduction of that paper. The mechanical background is discussed in Section 2 below, as well as the ensuing methodology. For now, it suffices to say that we view damage evolution as a time-parameterized minimization problem, or, in other word, postulate that, at each time, the material seeks to minimize an energy in which the stored elastic energy competes with the damage induced dissipated energy. As will be clear in the next section, this is but a slight departure from the classical thermodynamical modeling of brittle damage, provided that the material behavior is independent of the rate at which the loads vary. The consequence of that departure are however drastic.

We will show (see Theorem 4.1) that the resulting evolution is that of a progressively damaging material; the damage variable is now the local volume fraction of healthy material; it decreases with time. The corresponding stiffness is not imposed, but the result – or results, for lack of uniqueness – of a minimization process. So, in spite of a paucity of initial ingredients, the obtained model has many riches. Once again, we refer to [?] for a more detailed discussion.

From a mathematical standpoint, the present paper goes beyond the material presented in [?] and should be seen as the outcome of a maturation process initiated in that paper. In effect, the cited reference carried the germs of an approach which has proved fruitful in other settings and there is by now a growing number of papers that view quasi-static evolution for rate-independent material behavior as time-parameterized minimization problems, be it in plasticity [?, ?, ?, ?], fracture [?, ?, ?, ?], or phase transition [?, ?, ?]. The methodology is always the same: perform a step by step minimization, in effect dicretizing the time variable, then let the time-step tend to 0. The resulting field(s) will satisfy the postulated evolution, which is thus mathematically well-posed. In the case of damage, an additional hurdle should be overcome. At the first time step, the minimization problem has no solution; we need to relax the original problem. In doing so, we lose the brittle character of the damaging process; the damage variable becomes a volume fraction of the strong material in a mixture of strong and weak materials. But then, the implementation of the subsequent steps becomes problematic, as explained in Subsection 5.1. This is precisely where the paper [?] stalled.

In this paper, that obstacle is removed and the relaxed evolution is derived. This is, to our knowledge, the first paper that combines quasi-static variational evolution with relaxation. The only other work we are aware of in this direction is [?], where, for single-slip plasticity, a relaxation process for the time-discrete evolution is obtained.

The paper is organized as follows: Section 2 is devoted to a derivation of the model under investigation and it has a mechanical bias. Section 3 briefly introduces the concepts of homogenization that will be needed in the subsequent analysis. The evolution result is stated and commented upon in Section 4. In Section 5, the time discretization is detailed, while Section 6 details the limit process as the time-step vanishes. Finally, Section 7 checks the relevance of the obtained well-posed progressive evolution to the original ill-posed brittle evolution, in essence discussing the optimality of the obtained evolution.

2 The mechanical model

We consider an elastic material with elastic energy W, and assume geometric, as well as constitutive linearity. In other words, the elastic energy is a quadratic function of the strain tensor e, that is

$$W(e) := \frac{1}{2}Ae.e,$$

where A is the Hooke's law, an element of

$$\mathcal{F}(\alpha,\beta) := \{B \text{ fourth order tensors with the symmetries: } B_{ijkh} = B_{khij} = B_{jikh},$$
$$\text{such that } Be.e \in [\alpha e.e, \beta e.e], \ \ e \text{ symmetric } \in \mathbb{R}^{N\times N}\}.$$

The material occupies a domain $\Omega \in \mathbb{R}^N$. The state of possible damage is characterized by an internal variable $\chi \in [0,1]$, which may vary from point to point, so that W is a function of both e and χ; the greater the damage, the weaker the material, so that $W(e,\chi) \searrow$ as $\chi \nearrow$, or still A is a function of χ with $A(\chi) \searrow$ as $\chi \nearrow$. Here, and in the remainder of the paper,

Definition 2.1 *We say that $A(\chi)$ is monotonically decreasing in χ, $A(\chi) \searrow$ as $\chi \nearrow$, if and only if $A(\chi)e.e \geq A(\chi')e.e$, $\forall e$ symmetric $\in \mathbb{R}^{N\times N}$, $\chi \leq \chi'$.*

In the above definition, if e and e' are in $\mathbb{R}^{N\times N}$, $e.e' := \mathrm{tr}((e')^T e)$.

Let us denote by $\mathbb{I}_{[0,1]}$ the indicatrix function of the interval $[0,1]$. It is "classical" in thermodynamics [**?**] to write a constitutive law that relates the thermodynamic force

$$\mathbf{F} := -\left\{\frac{\partial}{\partial\chi}W(e,\chi) + \partial\mathbb{I}_{[0,1]}(\chi)\right\},$$

to a dissipation potential $\mathcal{D}$, lower semi-continuous and convex in $\dot\chi$. Thus,

$$\mathbf{F}(t) \in \partial\mathcal{D}(\dot\chi(t))$$

at each point of Ω.

Traditionally, damage models are viewed as rate-independent [**?**]. Here, we yield to tradition and consequently assume that $\mathcal{D}$ is positively homogeneous of degree one in the variable $\dot\chi$.

Now, since damage is by essence irreversible, the expression for $\mathcal{D}$ should prohibit any decrease in χ. The simplest dissipation potential endowed with the two required features is

$$\mathcal{D}(s) := \begin{cases} ks, & s \geq 0 \\ \infty, & s < 0. \end{cases}$$

We are unfortunately unable to fathom dynamics in the context of damage, but find comfort in sharing our misery with other researchers in the field. Indeed, there is, to our knowledge, no consensual model of the impact of kinetic energy on the growth of the damaged zone. This is a conceptual challenge to continuum mechanics and begs for additional physical laws, whether postulated, or derived from microscopic investigations like lattice mechanics. In any case, such ambitious investigations are a task that far exceeds the scope of this study, so that quasi-static behavior is assumed hereon. In other words, at each time, the domain is in elastic equilibrium with the data. For the sake of simplicity, those will consist only of a time-dependent body load $f(t)$, while the boundary $\partial\Omega$ of the investigated sample will be kept clamped. The reader should be assured that this is no restriction, but merely convenience (see Remark 4.2).

Summarizing, from a P.D.E. standpoint the pair solution $(u(t), \chi(t))$ satisfies the following system:

$$\begin{cases} -\mathrm{div}\left(\dfrac{\partial W}{\partial e}(e(u(t)), \chi(t))\right) = f(t), \quad u(t) = 0 \text{ on } \partial\Omega, \\ -\dfrac{\partial W}{\partial \chi}(u(t), \chi(t)) \in \partial\mathcal{D}(\dot\chi(t)) + \partial\mathbb{I}_{[0,1]}(\chi(t)), \quad \chi(0) = \chi_0, \end{cases}$$

or still, for $t \geq 0$,

$$\begin{cases} -\mathrm{div}\,(A(\chi(t))e(u(t))) = f(t), \quad u(t) = 0 \text{ on } \partial\Omega, \\ -\dfrac{1}{2}\dfrac{dA}{d\chi}(\chi(t))e(u(t)).e(u(t)) \leq k,\ 0 \leq \chi(t) < 1, \\ \left(\dfrac{1}{2}\dfrac{dA}{d\chi}(\chi(t))e(u(t)).e(u(t)) + k\right)\dot\chi(t) = 0,\ 0 < \chi(t) < 1, \\ \dot\chi(t) \geq 0, \quad \chi(0) = \chi_0. \end{cases} \tag{2.1}$$

From a classical standpoint, this model is intractable, and this had led many to introduce a regularizing term in the form of a gradient of the internal variable (see e.g. [**?**, **?**, **?**, **?**]).

For our part, we propose a formally equivalent rewriting of this model. To that end, for a given pair (v, ζ) with $0 \leq \zeta \leq 1$, we define the potential energy at time t as

$$E(t, v, \zeta) := \int_\Omega W(e(v), \zeta)dx - \langle f(t), v\rangle,$$

and the dissipation as

$$D(\zeta) := k\int_\Omega \zeta dx.$$

Assuming for now that the evolution (2.1) makes sense and that $(u(t), \chi(t))$ do exist over the time of existence of the data, say $[0, T]$, and that they (and the loads) are smooth enough for all that follows to be meaningful, a straightforward computation demonstrates that $(u(t), \chi(t))$ satisfies (2.1) if, and only if, with obvious notation,

(ULM) $(u(t), \chi(t))$ satisfies a first order (unilateral) minimality condition for $E(t, v, \zeta) + D(\zeta)$ among all $v \in H_0^1(\Omega; \mathbb{R}^N)$ and $\zeta \geq \chi(t)$ (**unilateral local minimality**);

(IR) $\dot\chi(t) \geq 0$ (**irreversibility**);

(EB) $\dfrac{d}{dt}\Big(E(t, u(t), \chi(t)) + D(\chi(t))\Big) = -\langle \dot f(t), u(t)\rangle$. After integrating over $[0, t]$ this also reads as

$$E(t, u(t), z(t)) + D(\chi(t)) = E(0, u(0), \chi_0) + D(\chi_0) - \int_0^t \langle \dot f(s), u(s)\rangle ds,$$

which, through an elementary integration by parts, becomes a statement of what is sometimes referred to as the mechanical version of the second law of thermodynamics [**?**] (**energy balance**).

For the sake of simplicity, we then limit our analysis to the very special case where the material only lives in a healthy state, the strong, undamaged state (indexed henceforth with the subscript s), and a damaged state, the weak state (indexed henceforth with the subscript

w). We thus enforce brittleness in the form of a single damaged state. Specifically, we assume that

$$W(e,\chi) = \frac{1}{2}(\chi A_w + (1-\chi)A_s)e.e,\ \chi \in \{0,1\},\ A_s, A_w \in \mathcal{F}(\alpha,\beta).$$

In effect, χ can be thought of as the characteristic function of the damaged material.

Now, as was already explained at length in [?], we cannot let the material lose all of its stiffness; we thus impose positive definiteness of A_w, so that

$$A_s \geq A_w > 0 \tag{2.2}$$

as quadratic forms acting on symmetric $N \times N$ matrices.

In conclusion, the classical quasi-static evolution of a linearly elastic material undergoing brittle partial damage is described through items (ULM), (IR), (EB) above. Let us emphasize that, up to this point, the proposed model cannot be challenged by mechanicians on the ground of mathematical divagation, because it is precisely that which is used in a large segment of the mechanics community.

In [?], it was proposed to somewhat depart from that model, replacing item (ULM) above by the following global minimality statement:

(UGM) $(u(t), \chi(t))$ satisfies

$$E(t,u(t),\chi(t)) + D(\chi(t)) \leq E(t,v,\zeta) + D(\zeta)$$

among all $v \in H^1_0(\Omega;\mathbb{R}^N)$ and $\zeta \geq \chi(t)$ (**unilateral global minimality**).

Of course, the stability criterion (UGM) should not be construed as an effort to introduce a new thermodynamic principle, but rather as a first attempt to deal constructively with (ULM). It is in effect a selection criterion, which may certainly be deemed over-restrictive. A better statement would be local unilateral minimality; unfortunately, this immediately begs the question of the meaning of locality, a usually distance dependent notion. In any case, the mathematical toolbox for proving local existence results is pretty empty at present, and this independently of the adopted distance.

Summarizing, we propose to investigate, for smooth enough data f, the evolution problem (UGM), (IR), (EB). It seems natural, both from a numerical and a mathematical standpoint, to tackle this evolution through time discretization and this is the route that was suggested in [?]. Taking a partition of $[0,T]$ into $0 = t^n_0 \leq \ldots \leq t^n_{k(n)} = T$, and setting

$$\Delta_n := t^n_{i+1} - t^n_i, \quad f^n_i := f(t^n_i)\ (f_0 := f^n_0 = f(0)),$$

we propose to find, for $i \geq 0$: $(u^n_{i+1}, \chi^n_{i+1})$ minimizer for

$$\int_\Omega W(e(v),\zeta)dx - \langle f^n_{i+1}, v\rangle + k\int_\Omega \zeta dx$$

among all pairs (v,ζ) with $v = 0$ on $\partial\Omega$ and $\zeta(x) \geq \chi^n_i(x)$ a.e. in Ω. Henceforth, $\langle , \rangle$ denotes the duality pairing between $H^1_0(\Omega;\mathbb{R}^N)$ and $H^{-1}(\Omega;\mathbb{R}^N)$, unless otherwise stated.

Remark 2.2 Note that the irreversibility constraint (IR) is encoded in the unilateral constraint on the admissible ζ's, while energy balance (EB) seems to have been forgotten altogether.

This scheme has proved successful in a few settings that were already mentioned in the introduction; as the time-step tends to zero, the approximations are expected to converge to a time-continuous evolution, which provides a "weak" solution for (2.1). We will see in the next section that the current setting is less accommodating, as had been noted in [**?**]. This is because the incremental minimization problem is ill-posed from the start and relaxation is required.

Remark 2.3 In [**?**], it was assumed that, besides (2.2),

$$A_w \text{ is isotropic, i.e. } (A_w)_{ijkh} = \lambda_w \delta_{ij}\delta_{kh} + \mu_w(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}).$$

That restriction is not essential to the model and can be done away with. It is important if one desires more explicit expressions for the various energies that come into play. This issue was central to [**?**], and should remain so if numerical implementation is contemplated. It is peripheral here, because our main goal is to provide a mathematically well-posed evolution; we thus drop that assumption in the sequel, keeping in mind that it should be re-introduced when concrete examples are investigated.

3 Homogenization

Homogenization is the main tool in the relaxation of the discrete evolution. We recall, for the reader's convenience, the notions of homogenization and of H-convergence (see e.g. [**?**] or [**?**] for the notion of G-convergence which agrees with that of H-convergence in the case of symmetric tensors), and specialize them to the case of two-phase mixtures of linearly elastic materials.

Consider a sequence $A^n \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$; we recall that, as in Section 2,

$$\begin{aligned}\mathcal{F}(\alpha,\beta) := \{&B \text{ fourth order tensors with the symmetries: } B_{ijkh} = B_{khij} = B_{jikh},\\ &\text{such that } Be.e \in [\alpha e.e, \beta e.e], \;\; e \text{ symmetric } \in \mathbb{R}^{N\times N}\}.\end{aligned}$$

We say that $A^n \overset{H}{\rightharpoonup} A$, $A \in L^\infty(\Omega; \mathcal{F}(\alpha,\beta))$, iff, for any body force $f \in H^{-1}(\Omega;\mathbb{R}^N)$, the solutions of the equilibrium equations

$$-\text{div } [A^n(e(u^n))] = f, \quad u^n \in H^1_0(\Omega;\mathbb{R}^N),$$

where the linearized strain tensor $e(u^n)$ is given by $e(u^n) := 1/2(\nabla u^n + (\nabla u^n)^T)$, satisfy

$$\begin{cases} u^n \rightharpoonup u, \text{ weakly in } H^1_0(\Omega;\mathbb{R}^N) \\ A^n e(u^n) \rightharpoonup Ae(u), \text{ weakly in } L^2(\Omega;\mathbb{R}^{N\times N}), \end{cases}$$

where u is the solution of

$$-\text{div } [Ae(u)] = f, \quad u \in H^1_0(\Omega;\mathbb{R}^N),$$

Now, let B and C be the stiffness tensors (Hooke's laws) of each phase, that is elements of $\mathcal{F}(\alpha,\beta)$. We look, for any mixture of those two phases – that is for any characteristic function χ of, say, phase B – at a new elastic material with stiffness

$$B_\chi := \chi B + (1-\chi)C.$$

Considering a sequence of characteristic functions $\chi^n \rightharpoonup \theta$, weak-* in $L^\infty(\Omega)$, we then investigate the possible H-limits of B_{χ^n}.

The properties of H-convergence that will be needed are

- *Compactness*: for any sequence $A^n \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$, there exists a subsequence, $A^{k(n)}$ and $A \in L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$ such that $A^n \overset{H}{\rightharpoonup} A$;

- *Convergence of the energy*: if $A^n \overset{H}{\rightharpoonup} A$, then, with u^n and u defined as above,
$$\int_\Omega A^n e(u^n).e(u^n)\,dx \to \int_\Omega Ae(u).e(u)\,dx;$$

- *Metrizability*: H-convergence is associated to a metrizable topology on $L^\infty(\Omega;\mathcal{F}(\alpha,\beta))$;

- *Ordering*: if $B^n \le A^n$ and $B^n \overset{H}{\rightharpoonup} B$, $A^n \overset{H}{\rightharpoonup} A$, then $B \le A$ (the inequalities are in the sense of Definition 2.1);

- *Locality*: if $B^n \overset{H}{\rightharpoonup} B$, $A^n \overset{H}{\rightharpoonup} A$, and χ is a characteristic function on Ω, then $\chi B^n + (1-\chi)A^n \overset{H}{\rightharpoonup} \chi B + (1-\chi)A$;

- *Periodicity*: if $A^n(x) := A(nx)$, with $A \in L^\infty([0,1]^N;\mathcal{F}(\alpha,\beta))$, then the whole sequence A^n H-converges to A^0, which is the constant tensor given by
$$A^0 e.e = \inf_{\varphi \text{ periodic}} \int_{[0,1]^N} A(y)(e+e(\varphi)).(e+e(\varphi))\,dy. \tag{3.1}$$

  In the case of a two-phase material, $B_{\chi^n}(x) = \chi(nx)B + (1-\chi(nx))C$, with $\chi \in L^\infty([0,1]^N;\{0,1\})$ and we speak of periodic mixtures in volume fraction $\theta := \int_{[0,1]^N} \chi\,dy$ of material B.

For a given weak-* limit $\theta \in L^\infty(\Omega,[0,1])$ of χ^n, we introduce the set $\mathcal{G}_\theta(B,C) \subset \mathcal{F}(\alpha,\beta)$ as the set of all possible H-limits of B_{χ^n}, when χ^n is only restricted through its target θ. Then, a locality result of [**?**] asserts that
$$\mathcal{G}_\theta(B,C) = \{D \in \mathcal{F}(\alpha,\beta) : D(x) \in \overline{G}_{\theta(x)}(B,C), \text{ a.e. in } \Omega\}, \tag{3.2}$$
where the set $G_\theta(B,C)$ is the set of all H-limits resulting from the periodic mixture of B and C in respective volume fractions $\theta, 1-\theta$. The determination of the set $G_\theta(B,C)$, or of its closure in the set of fourth order tensors $\overline{G}_\theta(B,C)$, is a problem of paramount significance in the theory of mixtures, but it is merely a collateral issue here. Of course our ability to implement the proposed method in practice will be severely tested if we lack minimal information on that set. As will be seen later, the only required knowledge is that of the minimum, for any fixed e, of $Ae.e$ over $A \in \overline{G}_\theta(B,C)$, so that we do not need to know $\overline{G}_\theta(B,C)$, but only its tangent hyperplanes with normal vectors located in the first quadrant.

Let us elaborate on the already evoked metrizable character of H-convergence. The associated distance is easily constructed. Consider a countable subset $\{g_k\}$ of $L^2(\Omega,\mathbb{R}^N)$, dense in the unit ball. For any $B \in L^\infty(\Omega,\mathcal{F}(\alpha,\beta))$ and for any k, we denote by u_k^B the solution to the following Dirichlet problem
$$-\mathrm{div}\,[B(e(u_k^B)] = g_k \quad u_k^B \in H_0^1(\Omega,\mathbb{R}^N)\,. \tag{3.3}$$

We now consider, for any $A, B \in L^\infty(\Omega, \mathcal{F}(\alpha, \beta))$,

$$d_H(A, B) := \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \int_\Omega Ae(u_k^A).e(u_k^A) - Be(u_k^B).e(u_k^B)\, dx \right| .$$

It is easily checked that this expression defines a distance on $L^\infty(\Omega, \mathcal{F}(\alpha, \beta))$ and that a sequence of elements of $L^\infty(\Omega, \mathcal{F}(\alpha, \beta))$ H-converges if, and only if it converges for the distance d_H. Moreover, since $\|g_k\|_{L^2} = 1,\ k \in \mathbb{N}$, we immediately get that there exists a constant C such that for every $A, B \in L^\infty(\Omega, \mathcal{F}(\alpha, \beta))$

$$d_H(A, B) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \int_\Omega g_k(u_k^A - u_k^B)\, dx \right| \le \sum_{k=1}^{\infty} \frac{1}{2^k} \|u_k^A - u_k^B\|_{L^2} \le C \tag{3.4}$$

A generalization of Helly's theorem for functions with value in a metric space [?] now permits to prove the following

Theorem 3.1 *Assume that $A^n(t) \in L^\infty(\Omega, \mathcal{F}(\alpha, \beta)),\ t \in [0, T]$, is monotonically decreasing, that is that, if $t \le t'$,*

$$A^n(t)e.e \ge A^n(t')e.e, \ \ \forall e \text{ symmetric} \in \mathbb{R}^{N\times N},\ \ a.e.\ \ in\ \Omega.$$

Then, there exists a subsequence $\{p(n)\}$ of $\{n\}$ and a monotonically decreasing $A(t) \in L^\infty(\Omega, \mathcal{F}(\alpha, \beta)),\ t \in [0, T]$ such that

$$A^{p(n)}(t) \xrightarrow{H} A(t),\ t \in [0, T].$$

Proof. The total variation of A^n, that is

$$\mathrm{Var}_{d_H}(A^n, [0, T]) := \sup\{\sum_{i=1}^{h} d_H(A^n(t_i), A^n(t_{i+1}))\ :\ h \in \mathbb{N}\ 0 = t_1 < t_2 < ... < t_h = T\},$$

is uniformly bounded. Indeed, given $t' \le t$, set $u_k^n(t) := u_k^{A^n}(t)$ and $u_k^n(t') := u_k^{A^n}(t')$ (see (3.3)). Since A^n is monotone in t,

$$\begin{aligned}
&0 \le \int_\Omega A^n(t)(e(u_k^n(t)) - e(u_k^n(t'))).(e(u_k^n(t)) - e(u_k^n(t')))\, dx \le \\
&\int_\Omega A^n(t)e(u_k^n(t)).e(u_k^n(t))\, dx + \int_\Omega A^n(t')e(u_k^n(t')).e(u_k^n(t'))\, dx - 2\int_\Omega A^n(t)e(u_k^n(t)).e(u_k^n(t'))\, dx \\
&= \int_\Omega g_k u_k^n(t) dx + \int_\Omega g_k u_k^n(t') dx - 2\int_\Omega g_k u_k^n(t') dx = \int_\Omega g_k u_k^n(t) dx - \int_\Omega g_k u_k^n(t') dx,
\end{aligned}$$

so that

$$\int_\Omega g_k u_k^n(t') dx \le \int_\Omega g_k u_k^n(t) dx,$$

or still

$$\int_\Omega A^n(t')e(u_k^n(t')).e(u_k^n(t'))\, dx \le \int_\Omega A^n(t)e(u_k^n(t)).e(u_k^n(t))\, dx\,. \tag{3.5}$$

Consequently, in view of the definition of d_H,

$$\mathrm{Var}_{d_H}(A^n, [0, T]) = d_H(A^n(0), A^n(T)) \le C\,. \tag{3.6}$$

We are then in a position to apply Theorem 3.2 in [?], which we restate here in a form that is convenient in the current framework.

Theorem *Let $(\mathcal{Y}, d)$ be a compact metric space and let $Y_n : [0,T] \to \mathcal{Y}$ be a sequence with equibounded total variation $\mathrm{Var}_d(Y_n, [0,T])$ with respect to the distance d. Then, there exists a subsequence $\{p(n)\}$ of $\{n\}$ and a function $Y : [0,T] \to \mathcal{Y}$ such that*

$$d(Y_{p(n)}(t), Y(t)) \xrightarrow{n} 0, \ \forall\ t \in [0,T]\,.$$

In view of (3.6), we can apply the above theorem to the sequence A^n in $L^\infty(\Omega, \mathcal{F}(\alpha, \beta))$ equipped with the distance d_H and we obtain the existence of a subsequence $\{p(n)\}$ of $\{n\}$, and of $A(t) \in L^\infty(\Omega, \mathcal{F}(\alpha, \beta))$ such that $A^{p(n)}(t)$ H-converges to $A(t)$ for every $t \in [0,T]$. The monotonically decreasing character of $A^{p(n)}(t)$ in t is preserved by H-convergence, so that $A(t)$ is indeed monotonically decreasing. This completes the proof. □

Remark 3.2 Note that, by the definition of the distance d_H, if $A^n(t) \in L^\infty(\Omega, \mathcal{F}(\alpha, \beta)),\ t \in [0,T]$, is monotonically decreasing in t and H-converges to $A(t)$ (also monotonically decreasing in t), then $d_H(A^n(t), A(t))$ is a measurable function because it is a countable sum of absolute values of differences of monotone functions. Further, $d_H(A^n(t), A(t)) \xrightarrow{n} 0$, for any $t \in [0,T]$. Finally, it is uniformly bounded in t and n by virtue of (3.4). Thus, applying the dominated convergence theorem, we get

$$\int_0^T d_H(A^n(t), A(t))\, dt \xrightarrow{n} 0.$$

This is true in particular for the subsequence $A^{p(n)}(t)$ of Theorem 4.1.

The previous remark, and the analogous remark below concerning weak-* convergence, will prove useful in passing to the limit in the discretization increment (Section 6), as well as in discussing optimality of the obtained evolution (Section 7).

Remark 3.3 Theorem 3.2 in [?] also applies to the unit ball in $L^\infty(\Omega; [0,1])$, a metric and compact set for the weak-* topology. Thus, an argument identical to that leading to Theorem 3.1 would show that, if $\Theta^n(t) \in L^\infty(\Omega; [0,1]),\ t \in [0,T]$, is monotonically decreasing in t, there exists a subsequence $\{p(n)\}$ of $\{n\}$ and a monotonically decreasing $\Theta(t) \in L^\infty(\Omega; [0,1]),\ t \in [0,T]$, such that

$$\Theta^{p(n)}(t) \overset{L^\infty}{\rightharpoonup} \Theta(t),\ t \in [0,T].$$

Furthermore, the analogue of Remark 3.2 also holds true in this latter setting with d_H replaced by the distance d_* associated to the weak-* topology on the unit ball.

4 The main result

As already mentioned, we will construct a quasi-static damage evolution by means of a discrete time approximation. The following theorem is the main result of the paper.

Theorem 4.1 *Consider two materials with respective stiffness tensors A_w and A_s, both in $\mathcal{F}(\alpha, \beta)$ and satisfying (2.2). Assume that Ω is Lipshitz and that*

$$f \in W^{1,1}(0,T; H^{-1}(\Omega; \mathbb{R}^N)). \tag{4.1}$$

There exist, for each $t \in [0,T]$, $u(t) \in H_0^1(\Omega; \mathbb{R}^N)$, $\Theta(t) \in L^\infty(\Omega)$, $A(t) \in \overline{G}_{1-\Theta(t)}(A_w, A_s)$, such that

- *Initial time:* $(u(0), A(0), (1-\Theta(0))$ *minimizes*

$$\int_\Omega \frac{1}{2} Ae(v).e(v)\, dx - \langle f(0), v\rangle + k\int_\Omega \theta\, dx,$$

among all $v \in H^1_0(\Omega;\mathbb{R}^N)$, $\theta \in L^\infty(\Omega;[0,1])$ *and* $A \in \overline{G}_\theta(A_w, A_s)$;

- *Monotonicity:* $A(t)$ *and* $\Theta(t)$ *are decreasing functions of* t, *as well as* $\overline{\Theta}(t) := \int_\Omega \Theta(t)\, dx$;

- *Continuity:* u *is continuous with values in* $H^1_0(\Omega;\mathbb{R}^N)$, *except at the (at most countable) discontinuity points of* $\overline{\Theta}$; *specifically, for every* $t, t' \in [0,T]$,

$$\|u(t') - u(t)\|_{H^1_0(\Omega;\mathbb{R}^N)} \leq C\|f(t') - f(t)\|_{H^{-1}(\Omega;\mathbb{R}^N)} + |\overline{\Theta}(t') - \overline{\Theta}(t)|^{\frac{1}{2}}; \tag{4.2}$$

- *One-sided minimality:* $(u(t), A(t), 0)$ *minimizes*

$$\int_\Omega \frac{1}{2} Ae(v).e(v)\, dx - \langle f(t), v\rangle + k\int_\Omega \Theta(t)\theta\, dx,$$

among all $v \in H^1_0(\Omega;\mathbb{R}^N)$, $\theta \in L^\infty(\Omega;[0,1])$ *and* $A \in \overline{G}_\theta(A_w, A(t))$;

- *Energy balance: The total energy*

$$\mathcal{T}(t) := \int_\Omega \frac{1}{2} A(t)e(u(t)).e(u(t))\, dx - \langle f(t), u(t)\rangle + k\int_\Omega (1-\Theta(t))\, dx$$

satisfies

$$\mathcal{T}(t) = \mathcal{T}(0) - \int_0^t \langle \dot{f}(\sigma), u(\sigma)\rangle\, d\sigma. \tag{4.3}$$

This theorem describes a well-posed evolution process for brittle damage in a linearly elastic material; the internal damage variable is the volume fraction Θ of the strong (undamaged) material, while the corresponding stiffness is well-defined as a function of Θ (although possibly non-unique). In doing so, we have in effect replaced a model of partial brittle damage with a richer one of partial progressive damage. As already noted in [**?**], we do not postulate the stiffness dependence upon Θ, which contrasts sharply with the phenomenological approach; nor do we a priori assume an underlying microstructure and an accompanying micro-mechanism for damage, which in turn contrasts with the micro-mechanical approach.

If, perchance, $\Theta(t) = 1 - \chi(t)$ with $\chi \in W^{1,1}((0,T); L^\infty(\Omega;\{0,1\}))$, then the pair $(u(t), \chi(t))$ is a solution of the original system (2.1).

Remark 4.2 Let us reiterate that, in the above theorem, the consideration of a force load $f \in W^{1,1}(0,T;H^{-1}(\Omega;\mathbb{R}^N))$ as only external load is unessential (not so however its absolutely continuous character). With nearly no modifications, we could also consider time-dependent surface tractions: to this effect, it would suffice to consider f as belonging to the space $W^{1,1}(0,T;[H^1]^*(\Omega;\mathbb{R}^N))$, where $[H^1]^*(\Omega;\mathbb{R}^N)$ is the dual of $H^1(\Omega;\mathbb{R}^N)$, identified with $H^{-1}(\Omega;\mathbb{R}^N) \times H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^N)$. In doing so, f becomes (f_{vol}, f_{surf}), with $f_{vol} \in W^{1,1}(0,T;H^{-1}(\Omega;\mathbb{R}^N))$ and $f_{surf} \in W^{1,1}(0,T;H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^N))$. Then the duality pairing $\langle f, u\rangle$ is replaced by the sum of the two duality pairings $\langle f_{vol}, u_0\rangle + \langle f_{surf}, u_{\partial\Omega}\rangle$ with $u = (u_0, u_{\partial\Omega})$, $u_0 \in H^1_0(\Omega;\mathbb{R}^N)$, $u_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^N)$.

As far as time-dependent displacement boundary conditions are concerned, a condition like $u = g(t)$ on $\partial\Omega$, with $g \in W^{1,1}(0,T;H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^N))$ could be incorporated into the statement of Theorem 4.1, at the expense of adding to the right hand-side of (4.3) the term

$$\int_0^t \left[\int_\Omega A(s)e(u(s)).e(\dot g(s))\; dx - \langle f(s), \dot g(s)\rangle \right] ds,$$

where g has been suitably extended to an element in $W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^N))$.

Remark 4.3 The estimate (4.2), and of the monotone character of $\overline{\Theta}$ indicate that both u and $\overline{\Theta}$ have at most a countable number of jumps as t increases from 0 to T. It would be reasonable to expect continuity of both u and $\overline{\Theta}$ in time. Whether this is true in the confines of the current problem remains undecided. The following one-dimensional example demonstrates that the evolution is brutal if surface loads are considered, in lieu of body loads.

Consider $\Omega := (0,L)$, $A_w = \alpha$ and $A_s = \beta > \alpha$. Apply an ever increasing force $f(t) = t$ at the extremity $x = L$, while maintaining the other extremity fixed. Thus, the ambient space is $H := \{v \in H^1(0,L); v(0) = 0\}$. For a given volume fraction $\theta = 1 - \Theta$ of weak material, the set $G_\theta(\alpha,\beta)$ is reduced to a single point, namely the harmonic mean

$$\underline{a}(\theta) := \frac{\alpha\beta}{\theta(\beta-\alpha)+\alpha}.$$

So, for a given $\theta(x,t)$, the minimal $u(x,t)$ satisfies

$$-\frac{d}{dx}\left(\underline{a}(\theta(x,t))\frac{du}{dx}(x,t)\right) = 0,\; u(0,t) = 0,\; a(\theta(L,t))\frac{du}{dx}(L,t) = t.$$

In this special context, one-sided minimality reduces to pointwise in x monotonicity of $\theta(x,t)$. A straightforward computation would establish that, for all but one t, the quasistatic evolution is unique, and that it is given, for a.e. $x \in (0,L)$, by

$$\theta(x,t) = \begin{cases} 0, & \text{if } t^2 < 2k\dfrac{\alpha\beta}{\beta-\alpha} \\ 1, & \text{if } t^2 > 2k\dfrac{\alpha\beta}{\beta-\alpha} \end{cases}.$$

Note however that the above brutal evolution of θ becomes smooth for different types of "loads", for example when a displacement $u(L,t) = t$ is imposed.

In terms of notation, in the remainder of the paper, C will denote a generic positive constant, so that e.g. $2C = C$.

5 Time discretization

In this section we proceed to analyze (and considerably modify) the discretization scheme proposed in [?]. To this effect, it is best to first focus on the initial time step.

5.1 The first time step

This subsection constitutes the bulk of the analysis presented in [?]. Here, we merely recall a few conclusions from that paper.

At the first time step, $t_0 = 0$, we wish to minimize, over $(v,\ \chi) \in H^1_0(\Omega;\mathbb{R}^N) \times L^\infty(\Omega;\{0,1\})$,

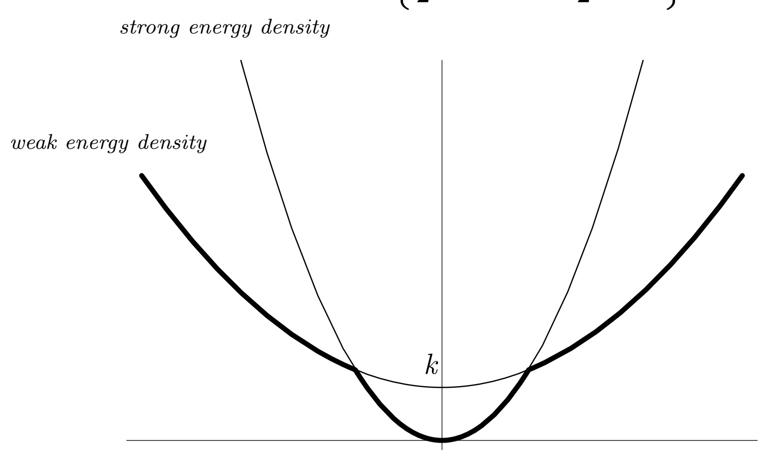
$$\int_\Omega \left[\frac{1}{2}(\chi A_w + (1-\chi)A_s)e(v).e(v) + k\chi\right]\, dx - \langle f_0, v\rangle,$$

with $f_0 := f(0)$. It is straightforward to eliminate χ in the minimization process, and we are thus left with the minimization over $v \in H^1_0(\Omega;\mathbb{R}^N)$ of

$$\int_\Omega W(t_0, e(v))\, dx - \langle f_0, v\rangle,$$

where

$$W(t_0, e) := \min\left\{\frac{1}{2}A_w e.e + k, \frac{1}{2}A_s e.e\right\}.$$



– Original energy density at first time step –

But the energy density $W(t_0,\cdot)$ is not convex, and the infimum

$$I(t_0) := \inf_v \int_\Omega W(t_0, e(v))\, dx - \langle f_0, v\rangle$$

is generically not attained. It is by now classical [?, ?] that

$$I(t_0) = \min_v \int_\Omega QW(t_0, e(v))\, dx - \langle f_0, v\rangle, \tag{5.1}$$

where the quasi-convex envelope $QW(t_0,\cdot)$ of $W(t_0,\cdot)$ is given through

$$QW(t_0, e) = \inf_\varphi \left\{\int_{[0,1]^N} W(t_0, e + e(\varphi))\, dx : \varphi \in W^{1,p}([0,1]^N;\mathbb{R}^N), \varphi \text{ periodic}\right\}.$$

Note that the perhaps more familiar Dirichlet boundary conditions may be imposed on the test φ's, in lieu of periodic boundary conditions [?].

In the particular case at hand, it is then immediately seen, with the definition of the G_θ-closure recalled in Section 3, that

$$QW(t_0, e) = \inf_{0\le\theta\le 1}\Big[\inf_{A\in G_\theta(A_w,A_s)}\{\frac{1}{2}Ae.e\} + k\theta\Big].$$

It is also true, although far from immediate, that the infima in the above formula can be replaced by minima (see [?]), at the expense of replacing $G_\theta(A_w, A_s)$ by its closure $\overline{G}_\theta(A_w, A_s)$ (in $\mathbb{R}^{N\times N\times N\times N}$), so that, finally

$$QW(t_0, e) = \min_{0\le\theta\le 1}\Big[\min_{A\in\overline{G}_\theta(A_w,A_s)}\{\frac{1}{2}Ae.e\} + k\theta\Big]. \tag{5.2}$$

Remark 5.1 It is one of the central tasks in [?] to derive as explicit an expression for $QW(t_0, e)$ as feasible. This is also part of the study of energy bounds to be found in [?].

Now, let u_0 be a minimizer for (5.1), and call θ_0 and A_0 measurable minimizers of the right hand-side of (5.2) for $e \equiv e(u_0)$. Denote by $\Theta_0 := 1 - \theta_0$ the volume fraction of strong material. Then

$$QW(t_0, e(u_0)) = \frac{1}{2}A_0e(u_0).e(u_0) + k(1 - \Theta_0).$$

We denote the total energy at time t_0 by

$$\mathcal{T}_0 := \int_\Omega \left\{\frac{1}{2}A_0e(u_0).e(u_0)\ dx + k(1 - \Theta_0)\right\}\ dx - \langle f_0, u_0\rangle, \tag{5.3}$$

so that

$$\mathcal{T}_0 = I(t_0).$$

This will not be true of the subsequent time steps; the correct relation is given in (5.11).

At this point, that is after the first time step, the incremental problem has produced, through relaxation, a local volume fraction of the weak material in lieu of a characteristic function of the weak material, and a homogenized stiffness in lieu of the original stiffnesses. From an equilibrium standpoint, this is perfectly satisfactory: microstructures will form to accommodate energy minimization, in the tradition of e.g. shape optimization [?] or phase transition [?]. From the standpoint of the evolution problem however, this is a source of embarrassment.

Indeed, since characteristic functions of the weak material have disappeared in the relaxation process, how should one implement (IR) at the next time step? The solution proposed in [?] was to relax the irreversibility constraint (IR), replacing it by the softer constraint that the volume fraction of the strong material at time t^n_i should decrease with i. This could be an acceptable compromise, but only provided there is no rotation in the direction of the body forces, because then, by the 2-homogeneity of $e \mapsto \min_{A\in\overline{G}_\theta} Ae.e$ and the fact that, for a given θ, the minimizing A is obtained through multiple layering in directions that are determined by the eigendirections of e (see [?]), a multiplication by a factor α of the force f will only increase (by some complicated function of α) the volume fraction θ, with no change in the eigendirections, hence no change in the layering directions. Thus, the monotonicity constraint will be enforced at the microstructural level.

A more general loading path will not interact so nicely with the underlying microstructures and those that are optimal at two different time steps will generically be mutually incompatible because the irreversibility constraint (IR) (in its discretized version) will not be respected.

We thus propose a different approach to the time stepping process after time $t = t_0$. This is the object of the next subsection.

5.2 The subsequent time steps

As we have seen in the previous subsection, whenever $\Theta_0(x) \neq 0, 1$, a mixture of A_w and A_s has formed at x, resulting in a stiffness tensor $A_0(x)$ at that point. The way out of the conundrum was suggested to us by J.J. Marigo; we wish to express our gratitude to him for providing such a fruitful lead. It goes as follows: at time t_i^n, look at all possible arrangements, within Ω, of the weak material A_w with material $A_{i-1}^n(x)$ (that, which was obtained at the point x at the previous time step). That material corresponds to a volume fraction $\Theta_{i-1}^n(x)$ of the strong material; thus $1 - \Theta_{i-1}^n(x)$ has already been "paid" in terms of "dissipated" energy at that point, up to time t_{i-1}^n. Therefore, the possible cost is either 0 if the material remains in the state it was at time t_{i-1}^n, or $\Theta_{i-1}^n(x)$ if the material becomes weak at x. In other words, at time t_i^n, compute

$$I(t_i^n) := \inf_{\chi \in L^\infty(\Omega;\{0,1\})} \left\{ \min_{v \in H_0^1(\Omega;\mathbb{R}^N)} \int_\Omega \left[\frac{1}{2}(\chi A_w + (1-\chi) A_{i-1}^n) e(v).e(v) + k\Theta_{i-1}^n \chi \right] dx - \langle f_i^n, v \rangle \right\}.$$

Once again, this problem, can be rephrased as

$$I(t_i^n) = \inf_{v \in H_0^1(\Omega;\mathbb{R}^N)} \int_\Omega W(t_i^n, e(v))\, dx - \langle f_i^n, v \rangle,$$

with

$$W(t_i^n, e) := \min \left\{ \frac{1}{2} A_w e.e + k\Theta_{i-1}^n, \frac{1}{2} A_{i-1}^n e.e \right\};$$

note that $W(t_i^n, \cdot)$ is actually a measurable function of x through the x-dependence of Θ_{i-1}^n and A_{i-1}^n upon x. Then, as before (see e.g. [?]),

$$I(t_i^n) = \min_v \int_\Omega QW(t_i^n, e(v))\, dx - \langle f_i^n, v \rangle, \tag{5.4}$$

with

$$QW(t_i^n, e) = \min_{0 \le \theta \le 1} \Big[\min_{A \in \overline{G}_\theta(A_w, A_{i-1}^n)} \{ \frac{1}{2} A e.e \} + k\Theta_{i-1}^n \theta \Big]. \tag{5.5}$$

Let u_i^n be a minimizer for (5.4), and θ_i^n and A_i^n be measurable minimizers of the right hand-side of (5.5) for $e \equiv e(u_i^n)$. The volume fraction of strong material is

$$\Theta_i^n := \Theta_{i-1}^n (1 - \theta_i^n), \quad \Theta_{-1}^0 := 1. \tag{5.6}$$

Then

$$QW(t_i^n, e(u_i^n)) = \frac{1}{2} A_i^n e(u_i^n).e(u_i^n) + k(\Theta_{i-1}^n - \Theta_i^n),$$

or still

$$I(t_i^n) = \int_\Omega \left\{ \frac{1}{2} A_i^n e(u_i^n).e(u_i^n)\, dx + k(\Theta_{i-1}^n - \Theta_i^n) \right\} dx - \langle f_i^n, u_i^n \rangle. \tag{5.7}$$

Remark 5.2 The proposed scheme should be distinguished from that originally suggested in [?] and briefly evoked at the end of the previous subsection. Indeed, the present scheme does not violate the discretized version of (IR), because at each time, the underlying microstructure at a point at that time is itself a mixture of that which existed at the previous time step with the weak material; thus monotonicity is enforced. This argument could be made rigorous upon appealing to the metrizable character of H-convergence (see the relevant remarks in Section 3).

The scheme is not entirely satisfactory because, in essence, each time step introduces a different scale for the underlying microstructure, whereas one should reasonably expect a single scale microstructure with the creation of finer and finer "bands" of the weak material with time. This objection is mitigated by the result of Proposition 7.1 below that demonstrates that the obtained time-continuous evolution can indeed be viewed as a limit of a unique microstructure which uses more and more weak material as time goes.

5.3 A few properties of the discrete evolution

We now establish the properties of u_i^n, A_i^n as $i \nearrow$ that will be used in passing to the time-continuous limit.

Minimality : Since u_i^n minimizes (5.4) and $A_i^n \in \overline{G}_{\theta_i^n}(A_w, A_{i-1}^n)$, it is obvious, in view of the expression (5.5) for $QW(t_i^n, e)$, that

$$u_i^n \text{ minimizes } \int_\Omega \frac{1}{2} A_i^n e(v).e(v)\, dx - \langle f_i^n, v\rangle,\ v \in H_0^1(\Omega; \mathbb{R}^N). \tag{5.8}$$

Monotonicity: For any $x \in \Omega$ and $A \in G_{\theta_i^n}(A_w, A_{i-1}^n(x))$, in view of the formula (3.1), there exists a characteristic function χ such that

$$\begin{aligned} Ae.e = & \inf_{\varphi \text{ periodic}} \int_{[0,1]^N} (\chi(y)A_w + (1-\chi(y))A_{i-1}^n(x))(e + e(\varphi)(y)).(e + e(\varphi)(y))\, dy \\ \leq & \int_{[0,1]^N} (\chi(y)A_w + (1-\chi(y))A_{i-1}^n(x))e.e\, dy \quad \leq \quad A_{i-1}^n(x)e.e. \end{aligned}$$

Thus, since $A_i^n \in \overline{G}_{\theta_i^n}(A_w, A_{i-1}^n)$,

$$A_i^n \leq A_{i-1}^n. \tag{5.9}$$

Lower bound on the total energy increment: Since, for $l \geq 1$, $A_{i+l}^n \in \overline{G}_{\theta_{i+l}^n}(A_w, A_{i+l-1}^n)$, while $A_{i+l-1}^n \in \overline{G}_{\theta_{i+l-1}^n}(A_w, A_{i+l-2}^n)$, a straightforward argument based on the metrizable character of H-convergence and on (3.2) would yield that

$$A_{i+l}^n \in \overline{G}_{\theta_{i+l}^n + (1-\theta_{i+l}^n)\theta_{i+l-1}^n}(A_w, A_{i+l-2}^n),$$

or equivalently, thanks to (5.6),

$$A_{i+l}^n \in \overline{G}_{1-[\frac{\Theta_{i+l}^n}{\Theta_{i+l-2}^n}]}(A_w, A_{i+l-2}^n).$$

A simple induction, leads, for $j > i$, to

$$A_j^n \in \overline{G}_{1-[\frac{\Theta_j^n}{\Theta_i^n}]}(A_w, A_i^n). \tag{5.10}$$

This allows us to derive a lower bound on the total energy

$$\mathcal{T}_i^n := I(t_i^n) + k\int_\Omega (1-\Theta_{i-1}^n)\, dx = \int_\Omega \frac{1}{2} A_i^n e(u_i^n).e(u_i^n)\, dx - \langle f_i^n, u_i^n\rangle + k\int_\Omega (1-\Theta_i^n)\, dx. \tag{5.11}$$

Indeed, by virtue of (5.7), for any $j > i$,

$$\int_\Omega \frac{1}{2} A_i^n e(u_i^n).e(u_i^n)\, dx + k\int_\Omega (\Theta_{i-1}^n - \Theta_i^n)\, dx - \langle f_i^n, u_i^n\rangle \leq \int_\Omega QW(t_i^n, e(u_j^n))\, dx - \langle f_i^n, u_j^n\rangle.$$

We recall the definition (5.5) of $QW(t_i^n,\cdot)$ and remark that, thanks to (5.10),

$$A_j^n \in \overline{G}_{1-[\frac{\Theta_j^n}{\Theta_{i-1}^n}]}(A_w, A_{i-1}^n).$$

Hence,

$$\int_\Omega QW(t_i^n, e(u_j^n))\,dx \le \int_\Omega \frac{1}{2}A_j^n e(u_j^n).e(u_j^n)\,dx + k\int_\Omega \Theta_{i-1}^n\left(1-\frac{\Theta_j^n}{\Theta_{i-1}^n}\right)\,dx.$$

Thus,

$$\begin{aligned}&\int_\Omega \frac{1}{2}A_i^n e(u_i^n).e(u_i^n)\,dx - \langle f_i^n, u_i^n\rangle + k\int_\Omega(\Theta_{i-1}^n - \Theta_i^n)\,dx\\ \le\ &\int_\Omega \frac{1}{2}A_j^n e(u_j^n).e(u_j^n)\,dx - \langle f_i^n, u_j^n\rangle + k\int_\Omega(\Theta_{i-1}^n - \Theta_j^n)\,dx,\end{aligned}$$

or still

$$\mathcal{T}_j^n - \mathcal{T}_i^n \ge -\langle f_j^n - f_i^n, u_j^n\rangle,\ j>i. \tag{5.12}$$

Continuity estimate: Since u_j^n satisfies (5.8) for j,

$$\int_\Omega A_j^n e(u_j^n).(e(u_i^n)-e(u_j^n))\,dx = \langle f_j^n, u_i^n - u_j^n\rangle,$$

so that, for $j>i$,

$$\int_\Omega \frac{1}{2}A_j^n(e(u_i^n)-e(u_j^n)).(e(u_i^n)-e(u_j^n))\,dx\ = \int_\Omega \frac{1}{2}A_j^n e(u_i^n).e(u_i^n)\,dx + \langle f_j^n, u_j^n - u_i^n\rangle -$$

$$\int_\Omega \frac{1}{2}A_j^n e(u_j^n).e(u_j^n)\,dx \le \int_\Omega \frac{1}{2}A_i^n e(u_i^n).e(u_i^n)\,dx\ +\ \langle f_j^n, u_j^n - u_i^n\rangle - \int_\Omega \frac{1}{2}A_j^n e(u_j^n).e(u_j^n)\,dx$$

$$= \mathcal{T}_i^n - \mathcal{T}_j^n + k\int_\Omega(\Theta_i^n - \Theta_j^n)\,dx + \langle f_i^n - f_j^n, u_i^n\rangle\ \le\ k\int_\Omega(\Theta_i^n - \Theta_j^n)\,dx + \langle f_i^n - f_j^n, u_i^n - u_j^n\rangle,$$

where we have used (5.9) in deriving the first inequality in the string above, and (5.12) in deriving the second one in that string.

Application of Korn's inequality to the first term in the above string, and of Poincaré and Young's inequalities to the last term of that string finally yields, for some positive constant C,

$$\|u_j^n - u_i^n\|_{H_0^1(\Omega;\mathbb{R}^N)} \le C\left\{\|f_j^n - f_i^n\|_{H^{-1}(\Omega;\mathbb{R}^N)} + \|\Theta_j^n - \Theta_i^n\|_{L^1(\Omega)}^{\frac{1}{2}}\right\}. \tag{5.13}$$

Relations (5.9), (5.10), (5.12), (5.13) will all play an essential role in the derivation of the continuous-time model.

Upper bound on the total energy: We observe that, in view of the expression (5.5) for $QW(t_i^n,\cdot)$ and of the obvious fact that $A_{i-1}^n \in \overline{G}_0(A_w, A_{i-1}^n)$,

$$QW(t_i^n, e(u_{i-1}^n)) \le \frac{1}{2}A_{i-1}^n e(u_{i-1}^n).e(u_{i-1}^n).$$

Consequently, recalling (5.6), (5.7) (at time t_{i-1}^n), we obtain

$$
\begin{aligned}
I(t_i^n) &\leq \int_\Omega QW(t_i^n, e(u_{i-1}^n))\, dx - \langle f_i^n, u_{i-1}^n\rangle \leq \int_\Omega \frac{1}{2} A_{i-1}^n e(u_{i-1}^n).e(u_{i-1}^n)\, dx - \langle f_i^n, u_{i-1}^n\rangle \\
&= \int_\Omega \frac{1}{2} A_{i-1}^n e(u_{i-1}^n).e(u_{i-1}^n)\, dx - \langle f_{i-1}^n, u_{i-1}^n\rangle - \langle \int_{t_{i-1}^n}^{t_i^n} \dot f(\sigma)\, d\sigma, u_{i-1}^n\rangle \\
&= I(t_{i-1}^n) - k\int_\Omega (\Theta_{i-2}^n - \Theta_{i-1}^n)\, dx - \langle \int_{t_{i-1}^n}^{t_i^n} \dot f(\sigma)\, d\sigma, u_{i-1}^n\rangle.
\end{aligned}
$$

Invoking definition (5.11) of the total energy, we conclude that

$$
\mathcal{T}_i^n \leq \mathcal{T}_{i-1}^n - \langle \int_{t_{i-1}^n}^{t_i^n} \dot f(\sigma)\, d\sigma, u_{i-1}^n\rangle. \tag{5.14}
$$

Iterating (5.14) and recalling (5.3), we finally obtain,

$$
\mathcal{T}_i^n = \int_\Omega \frac{1}{2} A_i^n e(u_i^n).e(u_i^n)\, dx - \langle f_i^n, u_i^n\rangle + k\int_\Omega (1-\Theta_i^n)\, dx \leq \mathcal{T}_0 - \sum_{j=1}^{i} \langle \int_{t_{j-1}^n}^{t_j^n} \dot f(\sigma)\, d\sigma, u_{j-1}^n\rangle. \tag{5.15}
$$

It now remains to define the piecewise constant in time approximations of the relevant quantities, which we do by setting, for any sequence $\{g_i^n\}_{i=0,\dots,k(n)}$,

$$
g(t) := g_i^n,\ t \in [t_i^n, t_{i+1}^n).
$$

We recall that $\Delta_n = t_{i+1}^n - t_i^n$, for any $i \in \{0, ..., k(n)\}$ denote by $\tau^n(t)$ the largest time $t_i^n \leq t$.

Then, (5.4), (5.8) become minimality statements,

$$
I(\tau^n(t)) = \int_\Omega QW^n(t, e(u^n(t)))\, dx - \langle f^n(t), u^n(t)\rangle = \min_v \int_\Omega QW^n(t, e(v))\, dx - \langle f^n(t), v\rangle, \tag{5.16}
$$

with

$$
QW^n(t,e) := \min_{0\leq\theta\leq1} \Big[\min_{A\in \overline{G}_\theta(A_w, A^n(t-\Delta_n))} \{\frac{1}{2} Ae.e\} + k\Theta^n(t-\Delta_n)\theta \Big], \tag{5.17}
$$

so that, in view of (5.6),

$$
I(\tau^n(t)) = \int_\Omega \frac{1}{2} A^n(t) e(u^n(t)).e(u^n(t))\, dx - \langle f^n(t), u^n(t)\rangle + k\int_\Omega (\Theta^n(t-\Delta_n) - \Theta^n(t))\, dx. \tag{5.18}
$$

Also,

$$
u^n(t) \text{ minimizes } \int_\Omega \frac{1}{2} A^n(t) e(v).e(v)\, dx - \langle f^n(t), v\rangle,\ v \in H_0^1(\Omega; \mathbb{R}^N). \tag{5.19}
$$

Further, (5.9) becomes a monotonicity property,

$$
A^n(t) \overset{t}{\searrow} \quad \text{as a quadratic form ,} \qquad \Theta^n(t) \overset{t}{\searrow} \quad . \tag{5.20}
$$

Relation (5.11) becomes an expression for the total energy,

$$
\begin{aligned}
\mathcal{T}^n(t) &= I(\tau^n(t)) + k\int_\Omega (1 - \Theta^n(t-\Delta_n))\, dx \\
&= \int_\Omega \frac{1}{2} A^n(t) e(u^n(t)).e(u^n(t))\, dx - \langle f^n(t), u^n(t)\rangle + k\int_\Omega (1-\Theta^n(t))\, dx.
\end{aligned} \tag{5.21}
$$

Then, (5.10) implies an inclusion property for $A^n(t)$,

$$A^n(t') \in \overline{G}_{1-[\frac{\Theta^n(t')}{\Theta^n(t)}]}(A_w, A^n(t)),\ t' \geq t. \tag{5.22}$$

A lower bound on energy increments is obtained from (5.12), namely,

$$\mathcal{T}^n(t') - \mathcal{T}^n(t) \geq -\langle f^n(t') - f^n(t), u^n(t')\rangle,\ t' > t. \tag{5.23}$$

Inequality (5.13) implies the following continuity property,

$$\|u^n(t') - u^n(t)\|_{H_0^1(\Omega;\mathbb{R}^N)} \leq C \left\{ \|f^n(t') - f^n(t)\|_{H^{-1}(\Omega;\mathbb{R}^N)} + \|\Theta^n(t') - \Theta^n(t)\|_{L^1(\Omega)}^{\frac{1}{2}} \right\}; \tag{5.24}$$

In particular, since $\|f^n(t)\|_{H^{-1}(\Omega;\mathbb{R}^N)} \leq C$,

$$\|u^n(t)\|_{H_0^1(\Omega;\mathbb{R}^N)} \leq C. \tag{5.25}$$

Finally, an upper energy bound is deduced from (5.15), that is

$$\begin{aligned} \mathcal{T}^n(t) &= \int_\Omega \frac{1}{2} A^n(t) e(u^n(t)).e(u^n(t))\, dx - \langle f^n(t), u^n(t)\rangle + k \int_\Omega (1 - \Theta^n(t))\, dx \\ &\leq \mathcal{T}_0 - \int_0^{\tau^n(t)} \langle \dot{f}(\sigma), u^n(\sigma)\rangle\, d\sigma. \end{aligned} \tag{5.26}$$

In the next section, we propose to let Δ_n tend to 0, so as to obtain a time-continuous limit.

6 The time-continuous limit

In this section, we let Δ_n tend to 0 with n (and $k(n)$ correspondingly tend to ∞). In view of (5.20), we can apply Theorem 3.1 to $A^n(t)$, and, since $u^n(t)$ satisfies (5.19), while $f^n \to f$ in $C^0([0,T]; H^{-1}(\Omega;\mathbb{R}^N))$, we obtain the existence of a subsequence of $\{n\}$, still denoted by $\{n\}$, such that

$$\begin{cases} A^n(t) \overset{H}{\rightharpoonup} A(t) \\ u^n(t) \overset{H_0^1}{\rightharpoonup} u(t) \end{cases} \quad t \in [0,T], \tag{6.1}$$

where further, by the very definition of H-convergence,

$$u(t) \text{ minimizes } \int_\Omega \frac{1}{2} A(t) e(v).e(v)\, dx - \langle f(t), v\rangle,\ v \in H_0^1(\Omega;\mathbb{R}^N). \tag{6.2}$$

Moreover, since $\Theta^n(t) \searrow$ with t, we apply Remark 3.3 and conclude that the chosen subsequence may also be assumed to satisfy

$$\Theta^n(t) \overset{L^\infty}{\rightharpoonup} \Theta(t),\ t \in [0,T], \tag{6.3}$$

for some monotonically decreasing $\Theta(t) \in L^\infty(\Omega; [0,1])$. We set

$$\overline{\Theta}^n(t) := \int_\Omega \Theta^n(t)\, dx, \quad \overline{\Theta}(t) := \int_\Omega \Theta(t)\, dx, \tag{6.4}$$

so that

$$\overline{\Theta}^n(t) \to \overline{\Theta}(t) \overset{t}{\searrow}, \ \forall t \in [0,T]. \tag{6.5}$$

Since $A^n(-\Delta_n) = A_s$ and $\Theta^n(-\Delta_n) = 1$, the inclusion property (5.22), applied to $t' = t$ and $t = -\Delta_n$, states that

$$A^n(t) \in \overline{G}_{1-\Theta^n(t)}(A_w, A_s).$$

In view of (6.1), together with metrizability for both H and L^∞-weak-* convergence, this easily implies that

$$A(t) \in \overline{G}_{1-\Theta(t)}(A_w, A_s). \tag{6.6}$$

Finally, weak lower semi-continuity of the norm implies that the continuity property (5.24) is preserved in the limit, i.e. that, for all $t, t' \in [0,T]$,

$$\begin{aligned} \|u(t') - u(t)\|_{H_0^1(\Omega;\mathbb{R}^N)} &\leq C\left\{\|f(t') - f(t)\|_{H^{-1}(\Omega;\mathbb{R}^N)} + \|\Theta(t') - \Theta(t)\|_{L^1(\Omega)}^{\frac{1}{2}}\right\} \\ &= C\left\{\|f(t') - f(t)\|_{H^{-1}(\Omega;\mathbb{R}^N)} + |\overline{\Theta}(t') - \overline{\Theta}(t)|^{\frac{1}{2}}\right\}. \end{aligned} \tag{6.7}$$

In particular,

$$\|u(t)\|_{H_0^1(\Omega;\mathbb{R}^N)} \leq C. \tag{6.8}$$

6.1 One-sided Minimality

We now establish a one-sided minimality property for the pair $u(t), A(t)$, which is reminiscent of similar properties in the study of brittle fracture evolution [?, ?, ?, ?]. Let θ be an arbitrary element of $L^\infty(\Omega;[0,1])$ and A be an arbitrary element of $\overline{G}_\theta(A_w, A(t))$ (really, $A(x) \in \overline{G}_{\theta(x)}(A_w, A(t,x))$, a.e. in Ω). Then, there exists a sequence χ_p of characteristic functions with

$$\begin{cases} \chi_p \overset{L^\infty}{\rightharpoonup} \theta \\ \chi_p A_w + (1-\chi_p)A(t) \overset{H}{\rightharpoonup} A, \ p \nearrow \infty. \end{cases} \tag{6.9}$$

Now, $A^n(t) \overset{H}{\rightharpoonup} A(t)$, hence, by the locality of H-convergence,

$$\chi_p A_w + (1-\chi_p)A^n(t) \overset{H}{\rightharpoonup} \chi_p A_w + (1-\chi_p)A(t), \ n \nearrow \infty.$$

Note that

$$\chi_p A_w + (1-\chi_p)A^n(t) \in \overline{G}_{[\theta^n(t)(1-\chi_p)+\chi_p]}(A_w, A^n(t-\Delta_n)),$$

with $\theta^n(t) := \dfrac{\Theta^n(t-\Delta_n) - \Theta^n(t)}{\Theta^n(t-\Delta_n)}$.

Then, in view of (5.16), (5.18), (5.17), for any $v \in H_0^1(\Omega;\mathbb{R}^N)$,

$$\begin{aligned}
\int_\Omega \frac{1}{2}A^n(t)e(u^n(t)).e(u^n(t))\,dx \quad &- \quad \langle f^n(t),u^n(t)\rangle + k\int_\Omega (\Theta^n(t-\Delta_n)-\Theta^n(t))\,dx \le \\
\int_\Omega QW^n(t,e(v))\,dx - \langle f^n(t),v\rangle \;\; &\le \;\; \int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A^n(t))e(v).e(v)\,dx - \langle f^n(t),v\rangle \\
&+ \quad k\int_\Omega \Theta^n(t-\Delta_n)(\theta^n(t)(1-\chi_p)+\chi_p)\,dx \\
&= \quad \int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A^n(t))e(v).e(v)\,dx - \\
\langle f^n(t),v\rangle + k\int_\Omega [\;(\;\; &\Theta^n(t-\Delta_n)-\Theta^n(t))(1-\chi_p)+\Theta^n(t-\Delta_n)\chi_p]\;dx.
\end{aligned} \tag{6.10}$$

Denote by v_p^n the minimizer of

$$\int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A^n(t))e(v).e(v)\,dx - \langle f^n(t),v\rangle$$

over $H_0^1(\Omega;\mathbb{R}^N)$, and remark that, by H-convergence, $v_p^n \overset{H_0^1}{\rightharpoonup} v_p$, where v_p is the minimizer of

$$\int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A(t))e(v).e(v)\,dx - \langle f(t),v\rangle$$

over $H_0^1(\Omega;\mathbb{R}^N)$. Also, possibly at the expense of extracting a (t-dependent) subsequence $\{n_t\}$, assume that

$$\Theta^{n_t}(t-\Delta_{n_t}) \overset{L^\infty}{\rightharpoonup} \Psi.$$

Then, passing to the limit in n_t in (6.10) with $v = v_p^{n_t}$, we get

$$\begin{aligned}
\int_\Omega \frac{1}{2}A(t)e(u(t)).e(u(t))\,dx - \langle f(t),u(t)\rangle + k\int_\Omega (\Psi-\Theta(t))\,dx \le \\
\int_\Omega \frac{1}{2}(\chi_p A_w + (1-\chi_p)A(t))e(v_p).e(v_p)\,dx - \langle f(t),v_p\rangle \\
+ k\textstyle\int_\Omega [(\Psi-\Theta(t))(1-\chi_p)+\Psi\chi_p]\;dx.
\end{aligned} \tag{6.11}$$

Now remark that, by H-convergence, together with (6.9), $v_p \overset{H_0^1}{\rightharpoonup} \overline{v}$, where

$$\overline{v} \text{ is the minimizer of } \int_\Omega \frac{1}{2}Ae(v).e(v)\,dx - \langle f(t),v\rangle \text{ over } H_0^1(\Omega;\mathbb{R}^N). \tag{6.12}$$

Thus, from (6.11), we finally conclude that

$$\begin{aligned}
\int_\Omega \frac{1}{2}A(t)e(u(t)).e(u(t))\,dx - \langle f(t),u(t)\rangle + k\int_\Omega (\Psi-\Theta(t))\,dx \le \\
\int_\Omega \frac{1}{2}Ae(\overline{v}).e(\overline{v})\,dx - \langle f(t),\overline{v}\rangle + k\int_\Omega [(\Psi-\Theta(t))(1-\theta)+\Psi\theta]\;dx.
\end{aligned}$$

By virtue of the minimality property (6.12) of $\overline{v}$, we obtain the following one-sided minimality property (where the one-sidedness refers to the admissible A's):

$$\int_\Omega \frac{1}{2}A(t)e(u(t)).e(u(t))\,dx - \langle f(t),u(t)\rangle \le \int_\Omega \frac{1}{2}Ae(v).e(v)\,dx - \langle f(t),v\rangle + k\int_\Omega \Theta(t)\theta\,dx, \tag{6.13}$$

and this, for any $v \in H_0^1(\Omega;\mathbb{R}^N)$, any $\theta \in L^\infty(\Omega;[0,1])$ and any $A \in \overline{G}_\theta(A_w, A(t))$.

6.2 Energy balance

We propose to derive a balance of energy for the evolution of the triplet $u(t), A(t), \Theta(t)$. To this effect, we recall (5.26) and pass to the limit in n. Thanks to (5.25), (6.1), (6.3), and the fact that $\tau^n(t) \to t$, the limit inequality is immediate; we get

$$\begin{aligned} \mathcal{T}(t) &:= \int_\Omega \frac{1}{2} A(t)e(u(t)).e(u(t))\, dx - \langle f(t), u(t)\rangle + k\int_\Omega (1-\Theta(t))\, dx \\ &\leq \mathcal{T}_0 - \int_0^t \langle \dot f(\sigma), u(\sigma)\rangle\, d\sigma, \end{aligned} \tag{6.14}$$

where we recall that $\mathcal{T}_0$ is given by (5.3).

To derive a lower bound, we appeal to (5.23) and, once again, immediately pass to the limit in n, recalling that, since $f \in C^0([0,T], H^{-1}(\Omega;\mathbb{R}^N)$, for all $t \in [0,T]$, $f^n(t) \to f(t)$, strongly in $H^{-1}(\Omega;\mathbb{R}^N)$. We obtain

$$\mathcal{T}(t') - \mathcal{T}(t) \geq -\langle f(t') - f(t), u(t')\rangle,\ t' > t. \tag{6.15}$$

Take the subdivision $J_p := \{0, \frac{t}{p}, \frac{2t}{p} ...., t\}$ of $[0,t]$. Fix $\varepsilon > 0$. Take E finite, such that it encapsulates nearly all the jumps of $\overline{\Theta}$, i.e. such that

$$\sum_{s\in[0,t]\setminus E} |\overline{\Theta}^+(s) - \overline{\Theta}^-(s)| < \varepsilon;$$

we will denote by $\dot{\overline{\Theta}}$ the approximate derivative of $\overline{\Theta}$. Now, take p large enough, so that

$$\sum_{j\in\{0,\dots,p\}:[\frac{jt}{p},\frac{(j+1)t}{p}]\cap E\neq\emptyset} \int_{[\frac{jt}{p},\frac{(j+1)t}{p}]} \|\dot f(s)\|_{H^{-1}(\Omega;\mathbb{R}^N)}\, ds \leq \varepsilon,$$

which is always possible because of the regularity of f.

In view of (6.7), if $s, s' \in [\frac{it}{p}, \frac{(i+1)t}{p}]$ with $[\frac{it}{p}, \frac{(i+1)t}{p}] \cap E = \emptyset$, then

$$\|u(s') - u(s)\|_{H^1_0(\Omega;\mathbb{R}^N)} \leq C\left\{\int_s^{s'} \|\dot f(\sigma)\|_{H^{-1}(\Omega;\mathbb{R}^N)}\, d\sigma + \left(\left|\int_s^{s'} (\dot{\overline{\Theta}}(\sigma))\, d\sigma\right| + \varepsilon\right)^{\frac{1}{2}}\right\},$$

while, if $[\frac{it}{p}, \frac{(i+1)t}{p}] \cap E \neq \emptyset$, then, by virtue of (6.8)

$$\|u(s') - u(s)\|_{H^1_0(\Omega;\mathbb{R}^N)} \leq C.$$

Thus,

$$\begin{aligned} &\left|\sum_{j=0}^{p} \langle f(\frac{(j+1)t}{p}) - f(\frac{jt}{p}), u(\frac{(j+1)t}{p})\rangle - \int_0^t \langle \dot f(\sigma), u(\sigma)\rangle\, d\sigma\right| \leq \\ &\sum_{[\frac{jt}{p},\frac{j+1}{p}]\cap E\neq\emptyset} \int_{\frac{jt}{p}}^{\frac{(j+1)t}{p}} \|u(\frac{(j+1)t}{p}) - u(s)\|_{H^1_0(\Omega;\mathbb{R}^N)} \|\dot f(s)\|_{H^{-1}(\Omega;\mathbb{R}^N)} ds + \\ &\sum_{[\frac{jt}{p},\frac{(j+1)t}{p}]\cap E=\emptyset} \int_{\frac{jt}{p}}^{\frac{(j+1)t}{p}} \|u(\frac{(j+1)t}{p}) - u(s)\|_{H^1_0(\Omega;\mathbb{R}^N)} \|\dot f(s)\|_{H^{-1}(\Omega;\mathbb{R}^N)} ds \leq \varepsilon C + \\ &\sup_{j=0,\dots,p}\left\{\int_{\frac{jt}{p}}^{\frac{(j+1)t}{p}} \|\dot f(\sigma)\|_{H^{-1}(\Omega;\mathbb{R}^N)}\, d\sigma + \left(\int_{\frac{jt}{p}}^{\frac{(j+1)t}{p}} (\dot{\overline{\Theta}}(s))\, ds + \varepsilon\right)^{\frac{1}{2}}\right\}\left[\int_0^t \|\dot f(\sigma)\|_{H^{-1}(\Omega;\mathbb{R}^N)} d\sigma\right]. \end{aligned}$$

Letting $p \nearrow \infty$, we infer, thanks to the integrable character of $\dot{f}$ and of $\dot{\overline{\Theta}}$, that

$$\limsup_p \left| \sum_{j=0}^{p} \langle f(\frac{j+1}{p}) - f(\frac{j}{p}), u(\frac{j+1}{p})\rangle - \int_0^t \langle \dot{f}(\sigma), u(\sigma)\rangle \, d\sigma \right|$$
$$\leq \varepsilon C + \varepsilon^{\frac{1}{2}} \int_0^t \|\dot{f}(\sigma)\|_{H^{-1}(\Omega;\mathbb{R}^N)} \, d\sigma.$$

Letting ε go to 0, we conclude that

$$\limsup_p \left| \sum_{j=0}^{p} \langle f(\frac{(j+1)t}{p}) - f(\frac{jt}{p}), u(\frac{(j+1)t}{p})\rangle - \int_0^t \langle \dot{f}(\sigma), u(\sigma)\rangle \, d\sigma \right| = 0. \tag{6.16}$$

Applying (6.15) to the partition J_p (with $t' = \frac{(j+1)}{p}, t = \frac{j}{p}$) yields

$$\mathcal{T}(t) - \mathcal{T}_0 \geq - \sum_{j=0}^{p} \langle f(\frac{j+1}{p}) - f(\frac{j}{p}), u(\frac{j+1}{p})\rangle,$$

hence, upon letting $p \nearrow \infty$, application of (6.16) yields in turn

$$\mathcal{T}(t) \geq \mathcal{T}_0 - \int_0^t \langle \dot{f}(\sigma), u(\sigma)\rangle \, d\sigma. \tag{6.17}$$

Energy balance, that is

$$\mathcal{T}(t) = \mathcal{T}_0 - \int_0^t \langle \dot{f}(\sigma), u(\sigma)\rangle \, d\sigma \tag{6.18}$$

is obtained by collecting (6.14) and (6.17).

Recalling (6.7), the monotonically decreasing character of both $A(t)$ and $\Theta(t)$, (6.6), (6.13), (6.18), the proof of Theorem 4.1 is now complete.

Remark 6.1 Note that we also obtain convergence of the energies for any time-stepping procedure which produces the same homogenized tensor $A(t)$; thus the time-discrete approximation may be viewed as a sound discrete approximation of the solution to the time-continuous problem.

Consider a piecewise constant triplet $u^n(t), \Theta^n(t), A^n(t)$ satisfying the incremental evolution defined through the minimization of (5.4). It is then immediate, by the definition of H-convergence, that, if $A^n(t) \overset{H}{\rightharpoonup} A(t)$, for all $t \in [0,T]$, where the triplet $u(t), \Theta(t), A(t)$ satisfies all the conclusions of Theorem 4.1, then, for all $t \in [0,T]$,

$$\begin{aligned} \int_\Omega \frac{1}{2} Ae(u^n(t)).e(u^n(t)) \, dx &\rightarrow \int_\Omega \frac{1}{2} Ae(u(t)).e(u(t)) \, dx \\ k \int_\Omega \Theta^n(t) \, dx &\rightarrow \int_\Omega \Theta(t) \, dx. \end{aligned}$$

Thus far, no claim has been made as to the relevance of the obtained well-posed evolution to the initial ill-posed problem. In other words, is the proposed evolution a reasonable lower bound on all possible evolutions of the damaged zone for the same loading history? This concern is partially addressed in the following section.

7 "Optimality" of the evolution

We first show that the evolution obtained in Theorem 4.1 is not too low, in the sense of the following

Proposition 7.1 *Given a time evolution $(u(t), A(t), \Theta(t))$ given by Theorem 4.1, there exists a time-parameterized sequence $\chi^n(t)$ of monotonically increasing characteristic functions of the weak material – that with stiffness tensor A_w – such that, for almost all t's in $[0,T]$,*

$$\begin{cases} \chi^n(t) \overset{L^\infty}{\rightharpoonup} \theta(t) := 1 - \Theta(t) \\ \chi^n(t)A_w + (1-\chi^n(t))A_s \overset{H}{\rightharpoonup} A(t). \end{cases}$$

In particular,

$$\int_\Omega (\chi^n(t)A_w + (1-\chi^n(t))A_s)e(v^n(t)).e(v^n(t))\, dx \to \int_\Omega A(t)e(u(t)).e(u(t))\, dx,$$

where $v^n(t)$ is the solution of

$$-div\,[(\chi^n(t)A_w + (1-\chi^n(t))A_s)(e(v^n(t)))] = f(t), \quad v^n(t) \in H^1_0(\Omega;\mathbb{R}^N).$$

Proof. Take $A^n(t)$ to be the piecewise constant in time sequence constructed in Section 5. Since $A^n(t) \in \overline{G}_{1-[\frac{\Theta^n(t)}{\Theta^n(t-\Delta_n)}]}(A_w, A^n(t-\Delta_n))$, a repeated diagonalization argument, based on the metrizable character of H-convergence and starting with $A^n(t) \equiv A^0,\ t \in [0,\Delta_n)$, would show the existence of a sequence of characteristic functions $\chi^n_k(t)$, monotonically increasing in t, such that

$$\begin{cases} \chi^n_k(t) \overset{L^\infty}{\rightharpoonup} \theta^n(t) := 1 - \Theta^n(t) \\ B^n_k(t) := \chi^n_k(t)A_w + (1-\chi^n_k(t))A_s \overset{H}{\rightharpoonup} A^n(t), \end{cases} \qquad k \nearrow \infty,\ t \in [0,T].$$

Recall Remark 3.2. Since both $B^n_k(t)$ and $A^n(t)$ are monotonically decreasing in t and respectively H-converge to $A^n(t)$ and $A(t)$, that remark applies; thus,

$$\int_0^T d_H(B^n_k(t), A^n(t))\, dt \overset{n}{\to} 0, \qquad \int_0^T d_H(A^n(t), A(t))\, dt \overset{n}{\to} 0.$$

Similarly, recalling the last part of Remark 3.3,

$$\int_0^T d_*(\chi^n_k(t), \theta^n(t))\, dt \overset{n}{\to} 0, \qquad \int_0^T d_*(\theta^n(t), \theta(t))\, dt \overset{n}{\to} 0.$$

Once again, an argument of diagonalization produces a subsequence $\{k(n)\}$ of $\{k\}$, such that

$$\begin{cases} \displaystyle\int_0^T d_*(\chi^n_{k(n)}(t), \theta(t))\, dt \to 0 \\ \displaystyle\int_0^T d_H(B^n_{k(n)}(t), A(t))\, dt \to 0\,, \end{cases}$$

which, in view of the elementary property of energy convergence for H-converging sequences, concludes the proof of the proposition upon setting $\chi^n(t) := \chi^n_{k(n)}(t),\ t \in [0,T]$.

Remark 7.2 Note that, in the previous result, we also have convergence of the total energy associated to $\chi^n(t)$ to that of the relaxed evolution at t, for a.e. $t \in [0, T]$.

It remains to demonstrate that the relaxed evolution in Theorem 4.1 is not too high. We state, without proof, a very weak version of this result in the following final

Remark 7.3 Consider a relaxed evolution in the sense of Theorem 4.1. If $\chi(t)$, the characteristic function of the weak material, is monotonically increasing in t, and if

$$\chi(t) \geq \theta(t) := 1 - \Theta(t), \ t > 0,$$

then, for all t's, the total energy associated to $\chi(t)$ is at least as large as $\mathcal{T}(t)$.

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