# Γ-limit of a phase-field model of dislocations

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**Abstract**

We study, by means of Γ-convergence, the asymptotic behaviour of a variational problem modeling a dislocation ensemble moving on a slip plane through a discrete array of obstacles. The variational problem is a two dimensional phase transition type energy given by a non local term and a non linear potential which penalizes non integer values. In this paper we consider a regime corresponding to a diluted distribution of obstacles. In this case the leading term of the energy can be described by means of a cell problem formula defining an appropriate notion of capacity (that we call dislocation capacity).

Keywords: Dislocations, Phase Transitions, Capacity, Γ-convergence

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# 1 Introduction

## 1.1 Formulation of the problem

In this paper we begin the study of the large body limit of a phase-field model for dislocations, recently introduced by Koslowski, Cuitino and Ortiz [5]. This model studies a dislocation ensemble moving within a single slip plane through an array of discrete obstacles (e.g. forest dislocations) under the action of an applied shear stress. In this theory, after a suitable rescaling (see the end of this subsection for the details), the slip (measured in units of the Burgers vectors) on the slip plane is represented by a scalar phase field $u$, which prefers to take integer values. We will consider a periodic setting, i.e. $u$ will be a periodic scalar valued function defined on the slip plane which is chosen as $T^2 \times \{0\}$. The first contribution to the energy, the so called Peierls potential, penalizes non-integer values of the slip distribution $u$ and is given by

$$\frac{1}{2\varepsilon} \int_{T^2} \mathrm{dist}^2(u, \mathbf{Z})\, dx \,. \tag{1.1}$$

Here $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ denotes the standard torus, i.e. functions on $T^2$ are periodic with period one. The small parameter $\varepsilon$ is proportional to the ratio between the Burgers vector (or equivalently the lattice spacing) and the physical size of the (periodic) domain under consideration. In particular the large body limit is characterized by the limit $\varepsilon \to 0$. The arguments in this paper do not require the special form of (1.1). Instead of the special integrand $\operatorname{dist}^2(u, \mathbf{Z})$ we could consider a general integrand $W(u)$, where $W$ is a $\mathbf{Z}$ periodic $C^1$ functions satisfying $W(u) \geq c \operatorname{dist}^2(u, \mathbf{Z})$, $c > 0$.

The second term in the energy represents the long-range elastic interaction induced by the slip. This can be obtained by considering a field $\widetilde{u}$ on $T^2 \times \mathbf{R}$ which has a jump of size $u$ across $\{x_3 = 0\}$, and its elastic energy

$$\frac{1}{2} \int_{T^2 \times \mathbf{R}} |\nabla \widetilde{u}|^2 \, dx \, .$$

One can easily verify that the optimal $\widetilde{u}$ (for a given jump $u$) is antisymmetric in $x_3$ (up to an irrelevant constant) and the elastic energy is given by the minimizer of the expression

$$\int_{T^2 \times (0,+\infty)} |\nabla \widetilde{u}|^2 \, dx$$

subject to the boundary condition $\widetilde{u}(x', 0) = \frac{1}{2} u(x')$. This energy is nothing but the square of the $H^{\frac{1}{2}}$ seminorm of $\frac{1}{2}u$ which in the Fourier representation is given by

$$\frac{1}{4} [u]^2_{H^{\frac{1}{2}}(T^2)} = \frac{1}{4} \sum_{k \in (2\pi \mathbf{Z})^2} |k| |\widehat{u}(k)|^2 \, , \tag{1.2}$$

where

$$\widehat{u}(k) = \int_{T^2} \mathrm{e}^{-ikx} u(x) \, dx \, .$$

In real space the energy can be written as

$$\frac{1}{4} [u]^2_{H^{\frac{1}{2}}(T^2)} = \frac{1}{2} \iint_{T^2 \times T^2} K(x-y) |u(x) - u(y)|^2 \, dx \, dy \, , \tag{1.3}$$

where the kernel $K(t)$ has the following properties:

i) $K(t) = O(|t|^{-3})$ as $|t| \to 0$

ii) $K(t)$ is periodic, i.e. is defined in $T^2$.

In fact the Fourier coefficients of $K$ are given by $\widehat{K}(k) = -\frac{1}{4}|k|$, so that $K(t) \sim \frac{1}{8\pi} t^{-3}$ as $t \to 0$.

The energy we are really looking at is the isotropic elastic bulk energy given in terms of the symmetrized displacement gradient $e(U) = \frac{1}{2}(\nabla U + \nabla U^T)$ as

$$\int_{T^2 \times \mathbf{R}} \mu \, |e(U)|^2 + \frac{\lambda}{2} \, |\operatorname{tr} e(U)|^2 \, dx' \, ,$$

where $U$ is the vector displacement and the jump of $U$ across $\{x_3 = 0\}$ is given by $u\mathbf{e_1}$. Using Fourier variables this leads to the following $H^{\frac{1}{2}}$ like energy

$$\frac{\mu}{4} \sum_{k \in (2\pi \mathbf{Z})^2} m_\nu(k) |\widehat{u}(k)|^2. \tag{1.4}$$

here the weight $m_\nu(k)$ is homogeneous of degree 1 and it is explicitly given by

$$m_\nu(k) = \frac{k_2^2}{\sqrt{k_1^2+k_2^2}} + \frac{1}{1-\nu}\frac{k_1^2}{\sqrt{k_1^2+k_2^2}}\,,$$

where $\nu < \frac{1}{2}$ is the Poisson ratio (see [5] for a detailed derivation of the above formula). If $\nu = 0$ then $m_\nu(k) = |k|$ and (1.4) reduces to (1.2); we call this the *isotropic case*.

One can also compute the real space version of the energy in (1.4) and this gives the following representation of the elastic energy

$$\frac{\mu}{2}\iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy\,, \tag{1.5}$$

where the kernel $K_\nu(t)$ satisfies conditions (i) and (ii). In fact $K_\nu$ is the Fourier series of $-\frac{1}{4}m_\nu(k)$ and a more explicit formula is given in (2.12) below. It is also clear that this non local energy is controlled from above and below by the $H^{\frac{1}{2}}$ periodic seminorm introduce above; more precisely,

$$[u]^2_{H^{\frac{1}{2}}(T^2)} \le \frac{1}{4}\sum_{k\in(2\pi\mathbf{Z})^2} m_\nu(k)|\widehat{u}(k)|^2 \le \frac{1}{1-\nu}[u]^2_{H^{\frac{1}{2}}(T^2)} \tag{1.6}$$

We now take $\mu = 1$. Then the total energy is thus given by

$$\frac{1}{2\varepsilon}\int_{T^2}\mathrm{dist}^2(u,\mathbf{Z})\,dx + \frac{1}{2}\iint_{T^2\times T^2}K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy - \int_{T^2}S^\varepsilon u\,dx. \tag{1.7}$$

The last term of the energy takes into account the interaction with the (non-dimensionalized) resolved shear stress $S^\varepsilon$. In this paper we will consider the case that $S^\varepsilon$ is of order one, more precisely that it converges weakly in $L^2$ as $\varepsilon \to 0$. To study the $\Gamma$-limit of the above energy it thus suffices to consider only the first two terms and to regard the third term as a continuous perturbation (see Remark 15). We shall study this energy subject to a pinning condition in order to model, e.g., a forest hardening mechanism by secondary dislocation. For definiteness we focus on the idealization of obstacles with infinitely strong pinning, i.e. we require that $u$ vanishes on the union of discs $B(x_i^\varepsilon, R\varepsilon) = B^i_{R\varepsilon}$ of radius $R\varepsilon$ and centers $x_i^\varepsilon$, $i = 1,\ldots,N_\varepsilon$ (the effect of a finite pinning strength are discussed in the appendix).

To summarize, we will study the asymptotic behaviour, in terms of $\Gamma$-convergence, of the following functional

$$E_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_{T^2}\mathrm{dist}^2(u,\mathbf{Z})\,dx + \iint_{T^2\times T^2}K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy & \text{if } u\in H^{\frac{1}{2}}(T^2)\,, \\ & u = 0 \text{ on } \bigcup_i B^i_{R\varepsilon} \\ +\infty & \text{otherwise}\,. \end{cases} \tag{1.8}$$

Before discussing in more detail the relevant limits let us briefly indicate how the above non-dimensionalized functional is related to the energy considered in [5]. Passing from the Fourier representation to the real space formulation and directly taking into account the periodicity of the phase field we can write the energy in [5] as follows

$$\frac{b^2}{2d}\int_{Q_L} \operatorname{dist}^2(v,\mathbf{Z})\,dx' + b^2 \iint_{Q_L\times Q_L} L^{-3}K_\nu\left(\frac{x'-y'}{L}\right)|v(x')-v(y')|^2\,dx'\,dy'$$

where $Q_L$ is the square in $\mathbf{R}^2$ of side $L$, where $b$ is the length of the Burgers vector and $d$ is the interplanar spacing (which is of the same order of $b$). Scaling with $x=\frac{x'}{L}$ and $y=\frac{y'}{L}$ and dividing the energy by $b^2L$ we get (1.8) with $\varepsilon$ of order $\frac{b}{L}$.

## 1.2 Mathematical context and scaling regimes for $N_\varepsilon$

From a mathematical point of view the functional $E_\varepsilon$ combines two features: a singular perturbation of Modica-Mortola type and a boundary condition on perforated domains, i.e. domains with many small holes.

If we ignore the boundary condition and also replace the $H^{\frac12}$ seminorm by the Dirichlet integral we obtain exactly a version of the Modica-Mortola problem for a potential with infinitely many wells [13, 14]. In this case the typical scaling of the energy is proportional to $1/\sqrt{\varepsilon}$. The situation for the $H^{\frac12}$ seminorm is more delicate and Alberti, Bouchitté and Seppecher [2] showed (for the case of two wells) that the natural scaling for the energy is $\ln 1/\varepsilon$ and after rescaling by $1/\ln(1/\varepsilon)$ the $\Gamma$-limit of the energy is proportional to the BV-norm $\int|\nabla u|$, i.e. to the length of the jump set of $u$ (the limiting energy is finite only on functions which take values in the wells).

If we ignore instead the singular Peierls energy then our functional falls in the class of variational problems in perforated domains. Again for the Dirichlet integral (and many other local functionals) a large literature is available (see e.g. [12] and [4], or [7] for a more general approach). The general idea is that in the limit a violation of the boundary condition carries no longer an infinite cost but only a finite cost computed by the integration of a suitable function of $u$ against a suitable measure which captures the local density of holes (in the sense of capacity). At least in terms of scaling, our problem (without the Peierls term) can be reduced to that setting by working with the harmonic extension $\widetilde{u}$ of $u$. This shows that without Peierls energy $E_\varepsilon$ should scale like the 'capacity density' $\varepsilon N_\varepsilon$. Combining these two results we expect two standard scaling regimes.

1. $\varepsilon N_\varepsilon \to 1$. In this case we expect that $E_\varepsilon$ is of order one and that the limiting energy can be obtained by solving a suitable cell problem which involves one obstacle in an infinite medium with boundary conditions at the obstacle and at infinity. In view of the results of [2] we expect also that the limit energy functional is finite only on constant, integer-valued functions, since a jump would result in an energy cost of order $\ln 1/\varepsilon$. A typical minimizer $u_\varepsilon$ of (1.8) looks almost constant with small perturbations (on a length scale $\varepsilon$) near the holes. The shape of these perturbations is essentially determined by the cell problem.

2. $\varepsilon N_\varepsilon/\ln(1/\varepsilon) \to 1$. In this case the contributions of the pinning energy discussed above and the Modica-Mortola like energy are of the same order. After rescaling by $1/\ln(1/\varepsilon)$ we expect a limiting energy of the form $\int|\nabla u| + \int D(u)$ (subject to the constraint $u(x)\in\mathbf{Z}$ almost everywhere) were, as before, the function $D(a)$ is computed from a cell problem with

boundary condition $a$ at infinity. In the physics literature this functional is referred to as a line-tension model because the first terms penalizes the length of the jump set of $u$. In fact, for the anisotropic kernel $K_\nu$ above, the term $|\nabla u|$ has to be replaced by an anisotropic line energy of the type $\gamma(\frac{\nabla u}{|\nabla u|})|\nabla u|$.

In this paper we investigate the first regime $\varepsilon N_\varepsilon \sim 1$ (see Theorem 14 below for a precise statement). In fact, it turns out that the regime $\varepsilon N_\varepsilon \to 0$ can be handled in exactly the same way if we scale the energy by $1/(\varepsilon N_\varepsilon)$ (see also Theorem 14). Going back to (1.8) the natural scaling of the resolved shear stress in this regime is $S_\varepsilon \sim \varepsilon N_\varepsilon$. The regimes $\varepsilon N_\varepsilon \sim \ln 1/\varepsilon$ and $\varepsilon N_\varepsilon \gg \ln 1/\varepsilon$ will be discussed in a forthcoming paper, [9].

We finally remark that while our analysis is phrased in terms of statics the same energy functional arises in the approximation of the evolution through a sequence of minimization problems at discrete times (see [5], Chapter 4). Actually in this case it is more natural to consider a "soft" pinning condition which can be treated exactly along the same lines (see Appendix). In this case the "pinning energy" represent the energy dissipated in crossing one of the pinning sites. A full understanding of dislocation dynamics and its macroscopic consequences such as hysteresis will of course require a much better understanding of local minimizers (and the energy barries between them). To do this rigorously seems currently out of reach. Nonetheless we can identify (in this paper and in [9]) in a rigorous way the relevant scaling regimes for the competition of elastic energy, applied stress, pinning energy, dissipation and line energy of the dislocations and we believe that this will be helpful for further studies.

# 2 The nonlocal energy

This section will be devoted to recalling some basic properties of the non local part of the energy. By minimizing an elastic energy on $\mathbf{R}^3_+$ we get, as in the periodic case, a non local energy equivalent to the $H^{\frac{1}{2}}(\mathbf{R}^2)$ seminorm. Indeed as above similar considerations give an energy of the form

$$\int_{\mathbf{R}^2}\left(\frac{\lambda_2^2}{|\lambda|}+\frac{1}{1-\nu}\frac{\lambda_1^2}{|\lambda|}\right)|\widehat{u}(\lambda)|^2 d\lambda$$

which can be written in spatial variables as

$$\frac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2}\Gamma_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy\,, \tag{2.9}$$

where the Fourie transform of the kernel $\Gamma_\nu(t)$, with $t\in\mathbf{R}^2$, is $-\frac{1}{4}\left(\frac{\lambda_2^2}{|\lambda|}+\frac{1}{1-\nu}\frac{\lambda_1^2}{|\lambda|}\right)$ and it can be computed explicitly, i.e.

$$\Gamma_\nu(t)=\frac{1}{2\pi(1-\nu)|t|^3}\left(\nu+1-3\nu\frac{t_2^2}{|t|^2}\right)\,. \tag{2.10}$$

In particular it is homogeneous of degree $-3$ and is positive if $\nu<\frac{1}{2}$. Clearly we also have

$$[u]^2_{H^{\frac{1}{2}}(\mathbf{R}^2)}\le\frac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2}\Gamma_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy\;\le\frac{1}{1-\nu}[u]^2_{H^{\frac{1}{2}}(\mathbf{R}^2)}\,, \tag{2.11}$$

where

$$[u]_{H^{\frac{1}{2}}(\mathbf{R}^2)} := \left(\frac{1}{4\pi}\iint_{\mathbf{R}^2\times\mathbf{R}^2}\frac{|u(x)-u(y)|^2}{|x-y|^3}\,dx\,dy\right)^{\frac{1}{2}} = \left(\int_{\mathbf{R}^2}|\lambda||\widehat{u}(\lambda)|^2\,d\lambda\right)^{\frac{1}{2}}.$$

**Proposition 1** *Let $K_\nu(t)$ be the anisotropic kernel defined above for the periodic case and let $\Gamma_\nu(t)$ be the corresponding kernel in $\mathbf{R}^2$. Then there exists a constant $C>0$ such that*

$$|\Gamma_\nu(t)-K_\nu(t)|\le C$$

*on* $\{t\in\mathbf{R}^2 \,:\, |t_i|\le 3/4\}$.

**Proof**. By the Poisson summation formula (see e.g. Stein and Weiss [15], Corollary 2.6) we have

$$K_\nu(t)=\sum_{z\in\mathbf{Z}^2}\Gamma_\nu(t+z)\,. \tag{2.12}$$

In particular for any $t\in\mathbf{R}^2$ such that $|t_i|\le 3/4$ we get

$$|K_\nu(t)-\Gamma_\nu(t)| = \sum_{z\in\mathbf{Z}^2\setminus\{0\}}\Gamma_\nu(t+z) \le \sum_{z\in\mathbf{Z}^2\setminus\{0\}}\frac{c}{|z|^3}\le C\,.$$

□

**Remark 2** By Proposition 1 using the homogeneity of $\Gamma_\nu$ we deduce that for every $\delta>0$

$$\lim_{\varepsilon\to 0}\varepsilon^3K_\nu(\varepsilon t)=\Gamma_\nu(t)$$

uniformly on $\{t\in\mathbf{R}^2 \,:\, |t|\le\delta\}$.

From the definition of $[\cdot]_{H^{\frac{1}{2}}}$ as a trace seminorm we can deduce a Poincaré type inequality for functions in $H^{\frac{1}{2}}(T^2)$. For a given bounded domain $D\subseteq\mathbf{R}^3$ a refinement of the classical Poincaré inequality permits to estimate the $L^2$ norm of a function in $H^1(D)$ with the $L^2$ norm of its gradient as long as the set where the function is zero is not too small. There exists a constant $C$ such that for every $w\in H^1(D)$

$$\int_D |w|^2dx' \le \frac{C}{\mathrm{Cap}(N(w))}\int_D|\nabla w|^2dx'\,, \tag{2.13}$$

where $\mathrm{Cap}(N(w))$ denote the harmonic capacity (with respect to $\mathbf{R}^3$) of the set $N(w)=\{x'\in D \,:\, w(x)=0\}$ (see [16], Corollary 4.5.2, or [8] Theorem 2.9) (note that in view of (2.13) the set $N(w)$ is well defined since the pointwise value of $w$ can be specified up to a set of zero harmonic capacity using its quasi continuous representative).

**Proposition 3** *There exists a constant $C_0$ such that for every $u\in H^{\frac{1}{2}}(T^2)$, with $u=0$ on $E\subseteq T^2$, we have*

$$\int_{T^2}|u|^2dx\le C_0\left(1+\frac{1}{\mathrm{Cap}(E\times\{0\})}\right)[u]^2_{H^{\frac{1}{2}}(T^2)}\,, \tag{2.14}$$

*where* $\mathrm{Cap}(E\times\{0\})$ *denote the harmonic capacity of* $E\times\{0\}$ *as a subset of* $\mathbf{R}^3$.

**Proof**. The proof follows immediately applying (2.13) to the harmonic extension $\widetilde{u}$ of $u$ in $D = (0,1)^3$ and by the fact that

$$\int_{T^2} |u|^2 dx \le c\, \|\widetilde{u}\|^2_{H^1(D)} .$$

$\square$

**Remark 4** Given an arbitrary $H^{\frac{1}{2}}(Q)$ function we can extend by reflection to a periodic function on $Q_2$ and applying the above inequality we get that there exists a constant $C_1$ such that

$$\int_Q |u|^2 dx \le C_1 \left(1 + \frac{1}{\mathrm{Cap}(E \times \{0\})}\right) \iint_{Q\times Q} \frac{|u(x)-u(y)|^2}{|x-y|^3} dx\, dy . \tag{2.15}$$

# 3 The cell problem

In this section we will define a suitable notion of capacity which will be the natural tool in order to study the asymptotics of our problem. We call the following set function the "$H^{\frac{1}{2}}$ dislocation capacity of an open set $E$ with respect to $\Omega$ at the integer level $a \in \mathbf{Z}$"

$$D^{\nu}_{\frac{1}{2}}(a, E, \Omega) := \inf \Bigg\{ \int_{\mathbf{R}^2} \mathrm{dist}^2(\zeta, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta(x)-\zeta(y)|^2\, dx\, dy : \tag{3.16}$$
$$\zeta = a \text{ on } E\,,\ \zeta = 0 \text{ on } \mathbf{R}^2 \setminus \Omega \Bigg\}$$

We denote by $D^{\nu}_{\frac{1}{2}}(a, E)$ the "$H^{\frac{1}{2}}$ dislocation capacity of an open set $E$ with respect to $\mathbf{R}^2$ at the integer level $a \in \mathbf{Z}$", namely

$$D^{\nu}_{\frac{1}{2}}(a, E) := \inf \Bigg\{ \int_{\mathbf{R}^2} \mathrm{dist}^2(\zeta, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta(x)-\zeta(y)|^2\, dx\, dy : \tag{3.17}$$
$$\zeta = a \text{ on } E\,,\ \zeta \in L^4(\mathbf{R}^2) \Bigg\}$$

The condition $\zeta \in L^4(\mathbf{R}^2)$ is the natural condition at $\infty$ in view of the following Sobolev inequality

$$\|u\|_{L^4(\mathbf{R}^2)} \le C^* [u]_{H^{\frac{1}{2}}(\mathbf{R}^2)} \qquad \forall\, u \in C^\infty_0(\mathbf{R}^2) . \tag{3.18}$$

**Remark 5** Denote

$$I(\zeta) := \int_{\mathbf{R}^2} \mathrm{dist}^2(\zeta, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta(x)-\zeta(y)|^2\, dx\, dy . \tag{3.19}$$

Using the fact that the kernel $\Gamma_\nu$ is positive (under the assumption $\nu < \frac{1}{2}$), it is easy to check that both terms in the energy are decreasing under truncations by integers. For every $a, b \in \mathbf{R}$ we set $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. Then for every $t \in \mathbf{Z}$, we have $I(\zeta \wedge t) \le I(\zeta)$ (and $I(\zeta \vee t) \le I(\zeta)$). Moreover $I(\zeta \wedge t) < I(\zeta)$ (and $I(\zeta \vee t) < I(\zeta)$) unless $\zeta \wedge t = \zeta$ a.e. (or $\zeta \vee t = \zeta$ a.e.).

**Proposition 6** *Let $\Omega$ be a bounded open subset of $\mathbf{R}^2$ and let $E \subseteq \Omega$ be an open set such that $D^\nu_{\frac{1}{2}}(a, E, \Omega) < +\infty$, with $a \in \mathbf{Z}$. Then there exists a minimum point $\zeta \in H^{\frac{1}{2}}(\Omega)$ for (3.16) and it satisfies $0 \le \zeta \le a$. We will call each minimum point a $D^\nu_{\frac{1}{2}}$-capacitary potential of $E$ with respect to $\Omega$.*

**Proof**. In order to obtain the existence let $\zeta_k$ be a minimizing sequence for (3.16). By Remark 5 we may assume that $0 \le \zeta_k \le a$. Now let $\xi_k : \mathbf{R}^2 \to \mathbf{Z}$ such that

$$\int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta_k, \mathbf{Z})\, dx = \int_{\mathbf{R}^2} |\zeta_k - \xi_k|^2 dx \,,$$

i.e. $\xi_k = \mathbf{P}_{\mathbf{Z}} \zeta_k$. Clearly $\xi_k = 0$ on $\mathbf{R}^2 \setminus \Omega$ and

$$\lim_{k\to\infty} \int_{\mathbf{R}^2} |\zeta_k - \xi_k|^2\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y) |\zeta_k(x) - \zeta_k(y)|^2\, dx\, dy = D^\nu_{\frac{1}{2}}(a, E, \Omega)\,.$$

By (2.11) and the assumption $D^\nu_{\frac{1}{2}}(a, E, \Omega) < +\infty$, the sequence $\{(\zeta_k, \xi_k)\}$ is bounded in $H^{\frac{1}{2}}(\mathbf{R}^2) \times L^2(\Omega, \mathbf{Z})$ and we may assume that $\zeta_k$ and $\xi_k$ converge weakly to a pair $(\zeta, \xi)$. By lower semicontinuity the limiting pair minimizes the energy and clearly satisfies $0 \le \zeta \le a$ and $\zeta = 0$ on $\mathbf{R}^2 \setminus \Omega$. □

**Remark 7** If $\Omega = B_R$, $E = B_r$ and $\nu = 0$ (in the "isotropic case") the capacitary potential is unique and radially symmetric. This follows immediately by the fact that both terms of the energy defining $D_{\frac{1}{2}}(a, B_r, B_R)$ are rotation invariant and the $H^{\frac{1}{2}}$ seminorm is strictly decreasing under radial rearrangements (for results on rearrangements for non local energies see e.g. [10] or the review paper [3]).

**Proposition 8** *There exists a minimizer for problem (3.17), the $D^\nu_{\frac{1}{2}}$-capacitary potential of $E$ with respect to $\mathbf{R}^2$. If, moreover, $E$ is bounded, then every minimizer converges to zero uniformly at infinity.*

**Proof**. As above the existence follows by considering a minimizing sequence and remarking that the "boundary condition" $\zeta - a \in L^4(\mathbf{R}^2)$ is preserved in view of (3.18).

Let now $\zeta$ be a potential of a bounded set $E$ with respect to $\mathbf{R}^2$. In the case $\nu = 0$ the decay at infinity follows by a comparison argument with the radially symmetric case (see Remark 7 above). Let us consider now the general case. Let $L$ be the linear operator representing the quadratic form defined by (2.9), so that

$$I(\zeta) = \langle L\zeta, \zeta \rangle + \int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta, \mathbf{Z})\, dx\,.$$

We will see that there exist a function $\psi \in L^4(\mathbf{R}^2)$ and a measure $\mu$ supported on $\overline{E}$ such that $\zeta$ is the solution in the sense of distribution of

$$L\zeta = \mu + \psi\,. \tag{3.20}$$

This would follow immediately from the Euler-Lagrange equation if $\mathrm{dist}^2(\cdot, \mathbf{Z}^2)$ was a $C^1$ function. For the case at hand we can argue as follows.

Fix $\eta \in C_0^\infty(\mathbf{R}^2)$, with $\eta \geq 0$ on $E$, and compute the variation of $I(\zeta)$ in direction $\eta$. We have

$$2\langle L\zeta, \eta\rangle + t\langle L\eta, \eta\rangle + \int_{\mathbf{R}^2} \frac{\mathrm{dist}^2(\zeta + t\eta, \mathbf{Z}) - \mathrm{dist}^2(\zeta, \mathbf{Z})}{t}\, dx \geq 0\,. \tag{3.21}$$

Since $\mathrm{dist}(\cdot, \mathbf{Z})$ is a Lipschitz function

$$\limsup_{t\to 0} \frac{\mathrm{dist}^2(\zeta(x) + t\eta(x), \mathbf{Z}) - \mathrm{dist}^2(\zeta(x), \mathbf{Z})}{t} \leq 2\mathrm{dist}(\zeta(x), \mathbf{Z})\, \eta(x) \qquad \text{a.e. } x \in \mathbf{R}^2$$

and hence, by Fatou's Lemma, we get

$$\langle L\zeta, \eta\rangle + \int_{\mathbf{R}^2} \mathrm{dist}(\zeta, \mathbf{Z})\, \eta\, dx \geq 0 \qquad \forall\, \eta \in C_0^\infty(\mathbf{R}^2)\,, \quad \eta \geq 0\,.$$

Thus there exists a positive measure $\widetilde{\mu}$ such that

$$L\zeta = \widetilde{\mu} - \mathrm{dist}(\zeta, \mathbf{Z}) \tag{3.22}$$

in the sense of distributions. Now consider $\eta \in C_0^\infty(\mathbf{R}^2)$ with $\eta = 0$ on $E$. We can apply the above argument to $\eta$ and $-\eta$ and we get

$$|\langle L\zeta, \eta\rangle| \leq \int_{\mathbf{R}^2} \mathrm{dist}(\zeta, \mathbf{Z})|\eta|\, dx \qquad \forall \eta \in C_0^\infty(\mathbf{R}^2 \setminus \overline{E})\,.$$

By density this holds also for $\eta \in C_0^0(\mathbf{R}^2 \setminus \overline{E})$ and since $\mathrm{dist}(\zeta, \mathbf{Z}) \in L^4(\mathbf{R}^2)$ we deduce that the restriction of $\widetilde{\mu}$ to $\mathbf{R}^2 \setminus \overline{E}$ is absolutely continuous with respect to the Lebesgue measure and its density is a $L^4$ function. Thus $\widetilde{\mu}$ can be written as the sum of a measure $\mu$ supported on $\overline{E}$ and a function belonging to $L^4(\mathbf{R}^2)$ which together with (3.22) gives (3.20).

Now let $G_\nu$ be the Green function of the operator $L + I$. We will need only rather mild decay properties of $G_\nu$ at $\infty$. To verify these it suffices to note that the Fourier transform of $G_\nu$ is given by

$$\widehat{G_\nu}(\lambda) = \frac{1}{1 + \frac{\lambda_2^2}{|\lambda|} + \frac{1}{1-\nu}\frac{\lambda_1^2}{|\lambda|}}\,,$$

since $\widehat{L\zeta}(\lambda) = \left(\frac{\lambda_2^2}{|\lambda|} + \frac{1}{1-\nu}\frac{\lambda_1^2}{|\lambda|}\right)\widehat{\zeta}(\lambda)$. One easily sees that $\widehat{G_\nu}$ and its first two derivatives are in $L^1(\mathbf{R}^2)$. Hence $G_\nu$ is continuous and $G_\nu(x) \leq C/(1+|x|^2)$. In particular $G_\nu \in L^{4/3}(\mathbf{R}^2)$.

Since $\psi$ and $\zeta$ belong to $L^4(\mathbf{R}^2)$, for every $\varepsilon > 0$ we can write $\psi + \zeta = \psi_1 + \psi_2$, where $\psi_1$ has compact support and $\|\psi_2\|_4 \leq \varepsilon$. Thus we have

$$\zeta(x) = G_\nu * \mu(x) + G_\nu * \psi_1(x) + G_\nu * \psi_2(x) \qquad \text{a.e. } \mathbf{R}^2\,.$$

We can estimate the $L^\infty$ norm of the last term of the right hand side using the Hölder inequality and we get

$$\|G_\nu * \psi_2\|_{L^\infty(\mathbf{R}^2)} \leq \|G_\nu\|_{L^{\frac{4}{3}}(\mathbf{R}^2)} \|\psi_2\|_{L^4(\mathbf{R}^2)} \leq C\varepsilon\,.$$

On the other hand the decay of $G_\nu$ and the fact that $\mu$ and $\psi_1$ have compact support guarantee that for $|x|$ big enough $\zeta$ is uniformly small. This concludes the proof. $\square$

The following proposition shows that as $a \to \infty$ the Peierls potential term becomes negligible and the dislocation capacity converges to the $H^{\frac{1}{2}}$-capacity, defined for any open set $E \subseteq \mathbf{R}^2$ as

$$\mathrm{Cap}^\nu_{H^{\frac12}}(E) = \inf\left\{ \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\eta(x)-\eta(y)|^2\,dx\,dy :\ \eta = 1 \text{ on } E\,,\ \eta \in L^4(\mathbf{R}^2) \right\} \tag{3.23}$$

(see for instance [1]).

**Proposition 9** *For any bounded open set $E \subseteq \mathbf{R}^2$ there exists a positive constant $C_E$ such that*

$$a^2\mathrm{Cap}^\nu_{H^{\frac12}}(E) \le D^\nu_{\frac12}(a,E) \le a^2\mathrm{Cap}^\nu_{H^{\frac12}}(E) + 2a^{3/2}\mathrm{Cap}^\nu_{H^{\frac12}}(E) + C_E a \tag{3.24}$$

*for every $a \in I\!N$. In particular*

$$\lim_{a\to\infty} \frac{D^\nu_{\frac12}(a,E)}{a^2} = \mathrm{Cap}^\nu_{H^{\frac12}}(E)\,. \tag{3.25}$$

**Proof**. The first inequality in (3.24) is trivial. In order to prove the estimate from above let $\eta_E$ be the $H^{\frac12}$-potential of $E$, i.e. the minimum point for (3.23). Using the fact that $\widehat{\Gamma_\nu}(\lambda)$ is homogeneous of degree 1, non-vanishing and smooth on the unit sphere, one can easily check that $\eta_E$ decays at infinity as $1/|x|$. Fix $a \in I\!N$ and define the function

$$v_a(x) = \begin{cases} \frac{a}{a-\sqrt{a}}(a\eta_E(x) - \sqrt{a}) & \text{if } x \in E_a = \{\eta_E > 1/\sqrt{a}\} \\ 0 & \text{otherwise} \end{cases}.$$

The function $v_a$ is admissible in the definition of $D^\nu_{\frac12}(a,E)$, thus

$$D^\nu_{\frac12}(a,E) \le \int_{E_a} \mathrm{dist}^2\big(\tfrac{a}{a-\sqrt{a}}(a\eta_E - \sqrt{a}), \mathbf{Z}\big)\,dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|v_a(x)-v_a(y)|^2 dx\,dy\,. \tag{3.26}$$

By the decay of $\eta_E$ at infinity we have that there exists a constant $C_E$ such that $|E_a| \le C_E a$. Thus

$$\int_{E_a} \mathrm{dist}^2\big(\tfrac{a}{a-\sqrt{a}}(a\eta_E - \sqrt{a}), \mathbf{Z}\big)\,dx \le C_E a\,.$$

Moreover

$$\begin{aligned} \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|v_a(x)-v_a(y)|^2 dx\,dy &\le \frac{a^2}{(1-\frac{1}{\sqrt{a}})^2} \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\eta_E(x)-\eta_E(y)|^2 dx\,dy \\ &= \frac{a^2}{(1-\frac{1}{\sqrt{a}})^2}\mathrm{Cap}^\nu_{H^{\frac12}}(E)\,. \end{aligned}$$

After possibly modifying the value of $C_E$ this yields (3.23). $\square$

We can extend the dislocation capacity to the class of all subset of $\Omega$ by setting

$$D^{\nu}_{\frac{1}{2}}(a, E, \Omega) = \inf\{D^{\nu}_{\frac{1}{2}}(a, A, \Omega) \,:\, A \text{ open}\,,\ A \supseteq E\} \tag{3.27}$$

for any set $E \subseteq \Omega$.

**Proposition 10** *The dislocation capacity satisfies the following properties:*

1) $D^{\nu}_{\frac{1}{2}}(a, E, \Omega) \leq D^{\nu}_{\frac{1}{2}}(a, F, \Omega)$ *if* $E \subseteq F \subseteq \Omega$*;*

2) $D^{\nu}_{\frac{1}{2}}(a, E, \Omega) \leq D^{\nu}_{\frac{1}{2}}(a, E, \Omega')$ *if* $E \subseteq \Omega' \subseteq \Omega$*;*

3) $D^{\nu}_{\frac{1}{2}}(a, E, \Omega) = D^{\nu}_{\frac{1}{2}}(-a, E, \Omega)$ *for every* $a \in \mathbf{Z}$*;*

4) *If* $0 \leq a \leq b$, $a, b \in \mathbf{Z}$*, then* $D^{\nu}_{\frac{1}{2}}(a, E, \Omega) \leq D^{\nu}_{\frac{1}{2}}(b, E, \Omega)$*;*

5) *(Sub-additivity) Given two open subsets of* $\Omega$*,* $E_1$ *and* $E_2$*,*

$$D^{\nu}_{\frac{1}{2}}(a, E_1 \cup E_2, \Omega) \leq D^{\nu}_{\frac{1}{2}}(a, E_1, \Omega) + D^{\nu}_{\frac{1}{2}}(a, E_2, \Omega)$$

*for every* $a \in \mathbf{Z}$*;*

6) *(Continuity on increasing sequences of sets) Given a sequence of subsets* $E_n \subseteq \Omega$*, such that* $E_n \subseteq E_{n+1}$*, and let* $E = \bigcup_n E_n$*, we have*

$$\lim_{n\to\infty} D^{\nu}_{\frac{1}{2}}(a, E_n, \Omega) = D^{\nu}_{\frac{1}{2}}(a, E, \Omega)$$

*for every* $a \in \mathbf{Z}$*;*

7) *(Continuity on decreasing sequences of compact sets) Given a sequence of compact subsets of* $\Omega$*,* $K_n$*, such that* $K_n \supseteq K_{n+1}$*, and let* $K = \bigcap_n K_n$*, we have*

$$\lim_{n\to\infty} D^{\nu}_{\frac{1}{2}}(a, K_n, \Omega) = D^{\nu}_{\frac{1}{2}}(a, K, \Omega)$$

*for every* $a \in \mathbf{Z}$*.*

**Proof**. The monotonicity properties 1)–4) can be checked directly by the definition. In order to prove property 5) let $\zeta_1$ and $\zeta_2$ be capacitary potentials of $E_1$ and $E_2$ respectively. Clearly the function $\zeta_1 \vee \zeta_2$ is a good competitor for the $D^{\nu}_{\frac{1}{2}}$-capacity of $E_1 \cup E_2$. The conclusion follows by the explicit computation of the energy remarking that

$$|\zeta_1(x) - \zeta_2(y)|^2 \leq |\zeta_1(x) - \zeta_1(y)|^2 \vee |\zeta_2(x) - \zeta_2(y)|^2$$

if $\zeta_1(x) \geq \zeta_2(x)$ and $\zeta_2(y) \geq \zeta_1(y)$.

Let us prove property 6). By 1) we have

$$\lim_{n\to\infty} D^{\nu}_{\frac{1}{2}}(a, E_n, \Omega) \leq D^{\nu}_{\frac{1}{2}}(a, E, \Omega)\,.$$

For the reverse inequality it is enough to consider the case of open sets. Let $\zeta_n \in H^{\frac{1}{2}}(\mathbf{R}^2)$ be a sequence of capacitary potentials, i.e. $\zeta_n = a$ a.e. on $E_n$ and

$$I(\zeta_n) = D^{\nu}_{\frac{1}{2}}(a, E_n, \Omega) \leq D^{\nu}_{\frac{1}{2}}(a, E, \Omega)\,.$$

Thus the $H^{\frac{1}{2}}$ seminorm of $\zeta_n$ is bounded. In view of (3.18) $\zeta_n$ is bounded in $L^4$ and hence in $H^{\frac{1}{2}}_{\text{loc}}(\mathbf{R}^2)$. Thus (a subsequence of) $\zeta_n$ converges strongly in $L^2_{\text{loc}}(\mathbf{R}^2)$. By the lower semicontinuity of $I(\cdot)$ we get

$$I(\zeta) \leq \liminf_{n\to\infty} I(\zeta_n) = \liminf_{n\to\infty} D^{\nu}_{\frac{1}{2}}(a, E_n, \Omega)\,.$$

Since $E = \bigcup_n E_n$ is also open, $\zeta = a$ a.e. in $E$ and hence is a good competitor for the definition of $D^{\nu}_{\frac{1}{2}}(a, E, \Omega)$. Thus

$$D^{\nu}_{\frac{1}{2}}(a, E, \Omega) \leq I(\zeta) \leq \liminf_{n\to\infty} D^{\nu}_{\frac{1}{2}}(a, E_n, \Omega)$$

which concludes the proof.

Finally, the proof of property 7) follows directly from the definition. In fact for any fixed $\varepsilon > 0$ there exists an open set $A \supseteq K$ such that $D^{\nu}_{\frac{1}{2}}(a, A, \Omega) \leq D^{\nu}_{\frac{1}{2}}(a, K, \Omega) + \varepsilon$. Since the sets $K_n$ are decreasing and compact, there exists an index $n_0$ such that $K_n \subseteq A$ for every $n \geq n_0$. The conclusion follows by the monotonicity of the dislocation capacity. □

**Remark 11** The properties proved above show that the dislocation capacity is a Choquet capacity (see for instance [1] or [11] for a general capacity theory). Note, however, that we define the dislocation capacity starting from open sets instead of compact sets, since this is more convenient for the present purpose.

In the following we will mostly consider the dislocation capacity of a ball with respect to either a concentric ball or $\mathbf{R}^2$. In order to simplify the notation we will state and prove some properties of the capacity in this particular case although most of them hold in a more general situation.

**Proposition 12** *Let $a \in \mathbf{Z}$ and $r > 0$. Then*

$$\lim_{R\to\infty} D^{\nu}_{\frac{1}{2}}(a, B_r, B_R) = D^{\nu}_{\frac{1}{2}}(a, B_r)\,.$$

**Proof**. By Proposition 10 we know that $D^{\nu}_{\frac{1}{2}}(a, B_r, B_R)$ is decreasing in $R$. Thus the limit always exists and

$$\lim_{R\to\infty} D^{\nu}_{\frac{1}{2}}(a, B_r, B_R) \geq D^{\nu}_{\frac{1}{2}}(a, B_r)\,.$$

In order to prove the reverse inequality fix $a \in \mathbf{Z}$ positive and let $\zeta$ be a capacitary potential of $B_r$ with respect to $\mathbf{R}^2$. We may assume that $0 \leq \zeta \leq a$ and by Proposition 8 we know that $\zeta$ decays to zero uniformly at infinity, i.e. for every $\varepsilon > 0$ there exists $R_0 > 0$ such that

$$|\zeta(x)| < \varepsilon \qquad \text{if} \quad |x| \geq R_0\,.$$

Fix $\varepsilon > 0$ and let us define the function $\zeta_\varepsilon$ as follows

$$\zeta_\varepsilon(x) = \begin{cases} \zeta(x) & \text{if } x \in \{\zeta \geq 1\} \\ \dfrac{\zeta(x)-\varepsilon}{1-\varepsilon} & \text{if } x \in \{\varepsilon \leq \zeta < 1\} \\ 0 & \text{if } x \in \{\zeta < \varepsilon\}, \end{cases}$$

We can write $\zeta_\varepsilon$ as $\zeta_\varepsilon(x) = \max\{\min\{\zeta(x), \frac{\zeta(x)-\varepsilon}{1-\varepsilon}\}, 0\}$ and hence $\zeta_\varepsilon \in H^{\frac{1}{2}}(\mathbf{R}^2)$. Moreover $\zeta_\varepsilon(x) = a$ on $B_r$ and $\zeta_\varepsilon(x) = 0$ in $\mathbf{R}^2 \setminus B_{R_0}$. This implies that $\zeta_\varepsilon$ is an admissible function in the definition of $D^\nu_{\frac{1}{2}}$ and

$$D^\nu_{\frac{1}{2}}(a, B_r, B_R) \leq \int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta_\varepsilon, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta_\varepsilon(x) - \zeta_\varepsilon(y)|^2\, dx\, dy = I(\zeta_\varepsilon)\,, \tag{3.28}$$

for every $R \geq R_0$. Let us compute $I(\zeta_\varepsilon)$ and show that $I(\zeta_\varepsilon) \leq I(\zeta) + o(1)$, as $\varepsilon \to 0$. Denote by $E_t$ the level set $\{\zeta \leq t\}$ and let us first estimate the dislocation part of the energy

$$\begin{aligned} &\int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta_\varepsilon, \mathbf{Z})\, dx = \int_{\mathbf{R}^2\setminus E_1} \operatorname{dist}^2(\zeta, \mathbf{Z})\, dx + \frac{1}{(1-\varepsilon)^2} \int_{E_1\setminus E_\varepsilon} \operatorname{dist}^2(\zeta-\varepsilon, (1-\varepsilon)\mathbf{Z})\, dx \\ &= \int_{\mathbf{R}^2\setminus E_1} \operatorname{dist}^2(\zeta, \mathbf{Z})\, dx + \frac{1}{(1-\varepsilon)^2} \int_{E_1\setminus E_{\frac{1+\varepsilon}{2}}} |\zeta-1|^2 dx + \frac{1}{(1-\varepsilon)^2} \int_{E_{\frac{1+\varepsilon}{2}}\setminus E_\varepsilon} |\zeta-\varepsilon|^2 dx\,. \end{aligned}$$

Since

$$\int_{E_{\frac{1+\varepsilon}{2}}\setminus E_\varepsilon} |\zeta-\varepsilon|^2 dx \leq \int_{E_{\frac{1+\varepsilon}{2}}\setminus E_\varepsilon} |\zeta|^2 dx$$

we have

$$\limsup_{\varepsilon\to 0} \int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta_\varepsilon, \mathbf{Z})\, dx \leq \int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta, \mathbf{Z})\, dx\,.$$

To estimate the non local term in $I(\zeta_\varepsilon)$ it suffices to note that $\zeta_\varepsilon = \psi_\varepsilon \circ \zeta$ with $\operatorname{Lip}\psi_\varepsilon \leq \frac{1}{1-\varepsilon}$. Hence $|\zeta_\varepsilon(x) - \zeta_\varepsilon(y)|^2 \leq (1-\varepsilon)^{-2}|\zeta(x) - \zeta(y)|^2$ and we get

$$\limsup_{\varepsilon\to 0} \iint_{\mathbf{R}^n\times\mathbf{R}^n} \Gamma_\nu(x-y)|\zeta_\varepsilon(x) - \zeta_\varepsilon(y)|^2\, dx\, dy \leq \iint_{\mathbf{R}^n\times\mathbf{R}^n} \Gamma_\nu(x-y)|\zeta(x) - \zeta(y)|^2\, dx\, dy\,.$$

Thus the conclusion follows from (3.28). □

# 4 Compactness and Γ-convergence result

In this section we will study the Γ-convergence of the functional $E_\varepsilon$ defined in (1.8) with a pinning condition on $N_\varepsilon$ balls of radius $\varepsilon R$ and centered in uniformly distributed and well separated points $x_i^\varepsilon$, $i \in I_\varepsilon \subset \mathbb{N}$, in the regime where $N_\varepsilon \varepsilon$ is bounded.

For every $i \in I_\varepsilon$ and $r > 0$ we denote by $B^i_r$ the ball in $\mathbf{R}^2$ of radius $r$ and center $x^\varepsilon_i$ ($B_r$ always denotes the ball of radius $r$ centered at 0).

In order to get a non trivial result we rescale the function $E_\varepsilon$ and we prove that the functional $F_\varepsilon(u) := E_\varepsilon(u)/N_\varepsilon\varepsilon$, i.e.

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{N_\varepsilon\varepsilon^2}\displaystyle\int_{T^2} \mathrm{dist}^2(u,\mathbf{Z})\,dx + \dfrac{1}{N_\varepsilon\varepsilon}\iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy & \text{if } u \in H^{\frac12}(T^2)\,, \\ & u = 0 \text{ on } \bigcup_i B^i_{R\varepsilon} \\ +\infty & \text{otherwise}\,, \end{cases}$$

$\Gamma$-converges, with respect to the strong $L^2$ topology, to the functional

$$F(u) = \begin{cases} D^\nu_{\frac12}(u, B_R) & \text{if } u = \text{const.} \in \mathbf{Z}\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{4.29}$$

For every subset $E$ of $(0,1)^2$ we denote by $I_\varepsilon(E) := \{i \in I_\varepsilon \ : \ x^\varepsilon_i \in E\}$. For the centers of the balls we assume the following conditions:

- (*Uniformly distributed*) There exists a constant $L > 0$ such that
$$|\#(I_\varepsilon(Q)) - N_\varepsilon|Q|| \le L \tag{4.30}$$
for every open square $Q \subset (0,1)^2$;
- (*Well separated*) There exists $\beta < 1$ such that
$$\mathrm{dist}(x^\varepsilon_i, x^\varepsilon_j) > 6\varepsilon^\beta \tag{4.31}$$
for every $i, j \in I_\varepsilon$, $i \ne j$, and for every $\varepsilon \in (0, \varepsilon_0)$;
- (*Finite capacity density*) There exists a constant $\Lambda \ge 0$ (possibly zero) such that $N_\varepsilon\varepsilon \to \Lambda$.

**Remark 13** For brevity we refer to the constant $\Lambda$ as the *capacity density* of the obstacles (more correctly $\Lambda\mathrm{Cap}^\nu_{H^{\frac12}}(B_R) \sim \Lambda R$ is the capacity density). Note that in order to get a $\Gamma$-convergence result the capacity density does not need to be constant. One could also consider either a case where the obstacles are not uniformly distributed in space or the case where the radii of the obstacles are varying (i.e. $B^i_\varepsilon = B(x^i_\varepsilon, R^i_\varepsilon\varepsilon)$). This would lead in general to a non constant capacity density $\Lambda(x)$. In this case condition (4.30) should be replaced by

$$\left| \sum_{x^i_\varepsilon \in Q} R^i_\varepsilon - \frac{1}{\varepsilon}\int_Q \Lambda(x)\,dx \right| \le L \tag{4.32}$$

with $\Lambda \in L^\infty$.

**Theorem 14** *Assume $N_\varepsilon \to +\infty$ and that the balls $B^i_{R\varepsilon}$ are uniformly distributed, well separated, with finite capacity density. Then*

*i) Every sequence $\{u_\varepsilon\}$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < \infty$ is pre-compact in $L^2$ and every cluster point is an integer constant;*

*ii) For every $u \in L^2(\mathbf{R}^2)$ there exists a sequence $\{u_\varepsilon\}$ strongly converging in $L^2$ to $u$ such that*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = F(u)\,;$$

*iii) Every sequence $\{u_\varepsilon\}$ strongly converging in $L^2$ to some function $u$ satisfies*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq F(u)\,.$$

**Remark 15** As mentioned in the Introduction, by general facts about Γ-convergence (see e.g. [6], Proposition 6.20), we can easily include an applied stress $S^\varepsilon$. If $S^\varepsilon$ converges to some $S$ strongly in $L^2$ then the functional

$$F_\varepsilon(u) - \int_{T^2} S^\varepsilon u\,dx$$

Γ-converges to $F(u) - \displaystyle\int_{T^2} S\,u\,dx$.

**Proof of (i) (Compactness).** Since $\sup_\varepsilon F_\varepsilon(u_\varepsilon) \leq C < +\infty$, by (1.6) we have

$$[u_\varepsilon]_{H^{\frac{1}{2}}(T^2)} \leq C N_\varepsilon \varepsilon\,. \tag{4.33}$$

Moreover $u_\varepsilon = 0$ on $\bigcup_i B^i_{R\varepsilon} = E_\varepsilon$. This obstacle condition is enough to deduce an $L^2$ estimate via Poincaré's inequality (2.15). Roughly speaking the idea is the standard fact that the capacity is almost additive on a union of small well separated sets and the 3-dimensional harmonic capacity of a disc is proportional to its radius, i.e.

$$\mathrm{Cap}(E_\varepsilon \times \{0\}) \approx \sum_i \mathrm{Cap}(B^i_{R\varepsilon} \times \{0\}) \approx N_\varepsilon \varepsilon\,.$$

In order to carry out this argument rigourously we cover the unit square with a lattice of small squares and apply the Poincaré inequality to each of them. The right estimate follows if we choose the side of each square small, but big enough in order to contain at least one obstacle.

Fix $r_\varepsilon = \sqrt{\frac{L+1}{N_\varepsilon}}$ ($L$ is the constant given by (4.30)). With a little abuse of notation we denote by $Q^{r_\varepsilon}_j$ the squares of a lattice on $(0,1)^2$ of size approximatively $r_\varepsilon$. Applying the Poincaré inequality (2.15), scaled to the square $Q^{r_\varepsilon}_j$, we get

$$\int_{Q^{r_\varepsilon}_j} |u_\varepsilon|^2 dx \leq C_0 r_\varepsilon \left(1 + \frac{r_\varepsilon}{\mathrm{Cap}((\{u_\varepsilon = 0\} \cap Q^{r_\varepsilon}_j) \times \{0\})}\right) [u]^2_{H^{\frac{1}{2}}(Q^{r_\varepsilon}_j)}\,. \tag{4.34}$$

By our choice of $r_\varepsilon$ and assumption (4.30) we have

$$1 \leq \#(I_\varepsilon(Q_j^{r_\varepsilon})) \leq 2L+1$$

and thus $\mathrm{Cap}((\{u_\varepsilon = 0\} \cap Q_j^{r_\varepsilon}) \times \{0\}) > CR\varepsilon$. Taking the sum over all $j$ in (4.34), by (4.33), we get

$$\int_{T^2} |u_\varepsilon|^2 dx \leq \sum_j C_0 r_\varepsilon \left(1 + \frac{r_\varepsilon}{CR\varepsilon}\right) [u]^2_{H^{\frac{1}{2}}(Q_j^{r_\varepsilon})} \leq C r_\varepsilon \left(1 + \frac{r_\varepsilon}{CR\varepsilon}\right) N_\varepsilon \varepsilon \leq C\,.$$

Thus $u_\varepsilon$ is precompact in the weak topology of $H^{\frac{1}{2}}$ and, by the compact embedding, in the strong topology of $L^2$.

Finally let $u$ be a cluster point. Assume for simplicity that the whole sequence $u_\varepsilon$ converges to $u$. In particular, since $\sup_\varepsilon F_\varepsilon(u_\varepsilon) \leq C$, we have

$$\lim_{\varepsilon \to 0} \int_{T^2} \mathrm{dist}^2(u_\varepsilon, \mathbf{Z})\, dx = 0\,.$$

This implies that $u \in H^{\frac{1}{2}}(T^2, \mathbf{Z})$. Then $u$ must be constant. This is obvious in the case when $N_\varepsilon \varepsilon \to 0$, but is true in general for any function in $H^{\frac{1}{2}}$ taking values in $\mathbf{Z}$ (this fact can be easily checked in one dimension, where jumps are not permitted, and then extended to any dimension by slicing). □

**Proof of (ii) (The upper bound).** It is clearly enough to prove the result for any constant function $u = a \in \mathbf{Z}$ (otherwise the upper bound is trivial). In order to construct a sequence $\{u_\varepsilon\}$ which converges strongly to $a$ in $L^2$ and satisfies

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = D^\nu_{\frac{1}{2}}(a, B_R)$$

fix $\rho_\varepsilon > 0$ such that $\varepsilon \leq \rho_\varepsilon \ll \varepsilon^{(\beta+1)/2}$ and let $\zeta_\varepsilon$ be a $H^{\frac{1}{2}}$-dislocation capacitary potential of $B_R$ with respect to $B_{\frac{\rho_\varepsilon}{\varepsilon}}$ at level $a$. Define

$$u^i_\varepsilon(x) = a - \zeta_\varepsilon\left(\frac{x - x^i_\varepsilon}{\varepsilon}\right)$$

and

$$u_\varepsilon(x) = \begin{cases} \sum_i u^i_\varepsilon(x) \chi_{B^i_{\varepsilon^\beta}}(x) & \text{if } x \in \bigcup_i B^i_{\varepsilon^\beta} \\ a & \text{otherwise.} \end{cases}$$

It is easy to check that the sequence $u_\varepsilon$ converges to $a$ in $L^2$. In order to control the non local term in the energy let us first show that long range interactions are negligible. Indeed using the fact that $u^i_\varepsilon = a$ outside $B^i_{\rho_\varepsilon}$ and the properties of the kernel $K_\nu$ we have

$$\begin{aligned} &\iint_{\substack{T^2 \times T^2 \\ |x-y| > \varepsilon^\beta}} K_\nu(x-y)\, |u_\varepsilon(x) - u_\varepsilon(y)|^2 dx\, dy \\ &\qquad \leq 2 \sum_i \int_{B^i_{\rho_\varepsilon}} \int_{T^2} \chi_{|x-y|>\varepsilon^\beta}\, K_\nu(x-y)\, |u^i_\varepsilon(x) - u_\varepsilon(y)|^2 dx\, dy \\ &\qquad \leq 2a^2 N_\varepsilon \int_{B_{\rho_\varepsilon}} \int_{T^2} \chi_{|y|>\varepsilon^\beta}\, K_\nu(y)\, dy\, dx \leq C \frac{\rho_\varepsilon^2 N_\varepsilon}{\varepsilon^\beta}\,. \end{aligned}$$

The constant $C$ depend on $a$, but since $a$ is fixed we suppress this dependence in the following. Since $(T^2 \setminus \cup_i B^i_{2\varepsilon^\beta}) \times T^2 \subseteq \left[(T^2 \setminus \cup_i B^i_{\varepsilon^\beta}) \times (T^2 \setminus \cup_i B^i_{\varepsilon^\beta})\right] \cup \{|x-y| > \varepsilon^\beta\}$ and $u_\varepsilon(x) = a$ outside $\cup_i B^i_{\varepsilon^\beta}$, by our choice of $\rho_\varepsilon$ and (4.31) we have

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\le \sum_i \frac{1}{N_\varepsilon \varepsilon}\left(\frac{1}{\varepsilon}\int_{B^i_{\varepsilon^\beta}} \mathrm{dist}^2(u^i_\varepsilon, \mathbf{Z})\,dx + \iint_{B^i_{3\varepsilon^\beta}\times B^i_{3\varepsilon^\beta}} K_\nu(x-y)\,|u^i_\varepsilon(x) - u^i_\varepsilon(y)|^2 dx\,dy\right) + o(1) \\ &= \frac{1}{\varepsilon^2}\int_{B_{\varepsilon^\beta}} \mathrm{dist}^2(\zeta_\varepsilon\left(\frac{x}{\varepsilon}\right), \mathbf{Z})\,dx + \frac{1}{\varepsilon}\iint_{B_{3\varepsilon^\beta}\times B_{3\varepsilon^\beta}} K_\nu(x-y)\,\left|\zeta_\varepsilon\left(\frac{x}{\varepsilon}\right) - \zeta_\varepsilon\left(\frac{y}{\varepsilon}\right)\right|^2 dx\,dy + o(1) \\ &= \int_{B_{\varepsilon^{\beta-1}}} \mathrm{dist}^2(\zeta_\varepsilon(x), \mathbf{Z})\,dx + \iint_{B_{3\varepsilon^{\beta-1}}\times B_{3\varepsilon^{\beta-1}}} \varepsilon^3 K_\nu(\varepsilon(x-y))\,|\zeta_\varepsilon(x) - \zeta_\varepsilon(y)|^2\,dx\,dy + o(1) \end{aligned} \tag{4.35}$$

Now, by Proposition 1, for $\varepsilon$ small enough we have

$$|\varepsilon^3 K_\nu(\varepsilon(x-y)) - \Gamma_\nu(x-y)| \le C\varepsilon^3 \qquad \forall\; x\,, y \in B_{3\varepsilon^{\beta-1}}$$

and hence

$$\begin{aligned} &\iint_{B_{3\varepsilon^{\beta-1}}\times B_{3\varepsilon^{\beta-1}}} \varepsilon^3\; K_\nu(\varepsilon(x-y))\,|\zeta_\varepsilon(x) - \zeta_\varepsilon(y)|^2\,dx\,dy \\ &\qquad\le \iint_{B_{3\varepsilon^{\beta-1}}\times B_{3\varepsilon^{\beta-1}}} \Gamma_\nu(x-y)\,|\zeta_\varepsilon(x) - \zeta_\varepsilon(y)|^2\,dx\,dy + C\varepsilon^3\varepsilon^{4(\beta-1)}\,. \end{aligned} \tag{4.36}$$

Thus by the definition of $\zeta_\varepsilon$ we have

$$F_\varepsilon(u_\varepsilon) \le D^{\frac{1}{2}}_\nu(a, B_R, B_{\frac{\rho_\varepsilon}{\varepsilon}}) + C\varepsilon^{4\beta-1}$$

which for the choice of $\beta > \frac{1}{4}$ together with Proposition 12 gives

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \le D^{\frac{1}{2}}_\nu(a, B_R)$$

(note that if (4.31) holds for some $\beta$ it also holds for all larger $\beta$). □

The proof of the lower bound is based on the following key lemma.

**Lemma 16** *Given $\mathcal{R} : \mathbf{R}_+ \to \mathbf{R}_+$, with $\mathcal{R}(\varepsilon) \to \infty$ as $\varepsilon \to 0$, there exists a function $\omega : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, with $\omega(\varepsilon, \delta) \to 0$ as $(\varepsilon, \delta) \to (0, 0)$, such that the following statement holds. Let $a \in \mathbf{Z}$. If $u \in H^{\frac{1}{2}}(B_{\mathcal{R}(\varepsilon)})$ satisfies*

$$\fint_{B_{\mathcal{R}(\varepsilon)}} |\zeta - a|dx \le \delta \tag{4.37}$$

*and $u = 0$ on $B_R$, then*

$$J_\varepsilon(\zeta) := \int_{B_{\mathcal{R}(\varepsilon)}} \mathrm{dist}^2(\zeta, \mathbf{Z})\,dx + \iint_{B_{\mathcal{R}(\varepsilon)}\times B_{\mathcal{R}(\varepsilon)}} K^\varepsilon(x-y)|\zeta(x) - \zeta(y)|^2\,dx\,dy \ge D^\nu_{\frac{1}{2}}(a, B_R) - \omega(\varepsilon, \delta)\,, \tag{4.38}$$

*where $K^\varepsilon(t) = \varepsilon^3 K_\nu(\varepsilon t)$.*

**Proof**. Assume for a contradiction that there exist $(\varepsilon_k, \delta_k) \to (0,0)$, $\eta > 0$, and $\zeta_k \in H^{\frac{1}{2}}(B_{\mathcal{R}(\varepsilon_k)})$, with $\zeta_k = 0$ on $B_R$ such that

$$J_{\varepsilon_k}(\zeta_k) \le D^{\nu}_{\frac{1}{2}}(a, B_R) - \eta$$

and

$$⨍_{B_{\mathcal{R}(\varepsilon_k)}} |\zeta_k - a| dx \to 0 \qquad \text{as } k \to \infty \,. \tag{4.39}$$

Denote $B^k = B_{\mathcal{R}(\varepsilon_k)}$. By the Sobolev embedding there exists a sequence of real numbers $a_k$ such that

$$\left( \int_{B^k} |\zeta_k - a_k|^4 dx \right)^{\frac{1}{2}} \le C \iint_{B^k \times B^k} K^{\varepsilon_k}(x-y) |\zeta_k(x) - \zeta_k(y)|^2 \, dx \, dy \le C \,.$$

Hence by Hölder inequality we have

$$⨍_{B^k} |\zeta_k - a_k| dx \le \left( ⨍_{B^k} |\zeta_k - a_k|^4 dx \right)^{\frac{1}{4}} \to 0$$

and thus we deduce that $a_k \to a$ as $k \to \infty$. In conclusion there exists a function $\zeta$ such that for every $r > 0$ we have that $\zeta_k$ converge weakly to $\zeta$ in $H^{\frac{1}{2}}(B_r)$ and in $L^4(B_r)$ and strongly in $L^2(B_r)$. Moreover

$$\int_{B_r} |\zeta - a|^4 dx \le \liminf_{k \to \infty} \int_{B_r} |\zeta_k - a_k|^4 dx \le \liminf_{k \to \infty} \int_{B^k} |\zeta_k - a_k|^4 dx \le C \,, \tag{4.40}$$

i.e. $\zeta$ is a good competitor for the definition of $D^{\nu}_{\frac{1}{2}}(a, B_R)$. In addition we have

$$\lim_{k \to \infty} \int_{B_r} \mathrm{dist}^2(\zeta_k, \mathbf{Z}) \, dx = \int_{B_r} \mathrm{dist}^2(\zeta, \mathbf{Z}) \, dx$$

and

$$\lim_{k \to \infty} \iint_{B_r \times B_r} \Gamma_\nu(x-y) |\zeta_k(x) - \zeta_k(y)|^2 \, dx \, dy = \iint_{B_r \times B_r} \Gamma_\nu(x-y) |\zeta(x) - \zeta(y)|^2 \, dx \, dy \,.$$

Finally by Proposition 1 and the homogeneity of $\Gamma_\nu$ we have

$$|K^{\varepsilon}(x-y) - \Gamma_\nu(x-y)| \le C_r \frac{\varepsilon^3}{|x-y|^3} \qquad \text{if } |x-y| \le \frac{3}{4\varepsilon}$$

and hence

$$\left| \iint_{B_r \times B_r} (K^{\varepsilon_k}(x-y) - \Gamma_\nu(x-y)) \, |\zeta_k(x) - \zeta_k(y)|^2 \, dx \, dy \right| \le C_r \varepsilon_k^3 \|\zeta_k\|_{H^{\frac{1}{2}}(B_r)} \,.$$

Thus for every $r > 0$ we get

$$D^{\nu}_{\frac{1}{2}}(a, B_R) - \eta \ge \limsup_{k \to \infty} J_{\varepsilon_k}(\zeta_k) \ge \int_{B_r} \mathrm{dist}^2(\zeta, \mathbf{Z}) \, dx + \iint_{B_r \times B_r} \Gamma_\nu(x-y) |\zeta(x) - \zeta(y)|^2 \, dx \, dy$$

so that

$$D^{\nu}_{\frac{1}{2}}(a, B_R) - \eta \geq \int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta(x) - \zeta(y)|^2\, dx\, dy\,.$$

This is a contradiction in view of the fact that $\zeta$ is a good test function in the definition of $D^{\nu}_{\frac{1}{2}}(a, B_R)$ and the proof is complete.

□

A second key point for the proof of the lower bound is to show that if the sequence $u_\varepsilon - a$ is close to zero in some ball $B_r$ then it is close to zero also at a smaller scale. This is a consequence of the following Proposition.

**Proposition 17** *There exists a positive constant* $C$ *such that for every* $0 < \rho < r$ *the following inequality holds*

$$⨍_{B_\rho} |u|\, dx \leq ⨍_{B_r} |u|\, dx + \frac{C}{\sqrt{\rho}} [u]_{H^{\frac{1}{2}}(B_r)} \tag{4.41}$$

*for all* $u \in H^{\frac{1}{2}}(B_r)$.

**Proof**. Let us first show that there exists a constant $C$ such that for any $u \in H^{\frac{1}{2}}(B_1)$

$$⨍_{B_\theta} |u|\, dx \leq ⨍_{B_1} |u|\, dx + C\, [u]_{H^{\frac{1}{2}}(B_1)} \tag{4.42}$$

for every $\theta \in [\frac{1}{2}, 1]$. By the Hölder inequality and the Sobolev embedding there exists a constant $c$ such that

$$\|u - c\|_{L^1(B_\theta)} \leq \|u - c\|_{L^1(B_1)} \leq C\, [u]_{H^{\frac{1}{2}}(B_1)}\,. \tag{4.43}$$

Moreover the constant $c$ can be estimate as follows

$$c = ⨍_{B_1} c \leq ⨍_{B_1} |u|\, dx + ⨍_{B_1} |u - c|\, dx \leq ⨍_{B_1} |u|\, dx + C\, [u]_{H^{\frac{1}{2}}(B_1)}\,,$$

and hence (4.42) follows by (4.43). By a scaling argument we obtain

$$⨍_{B_{\theta r}} |u|\, dx \leq ⨍_{B_r} |u|\, dx + \frac{C}{\sqrt{r}} [u]_{H^{\frac{1}{2}}(B_r)}\,,$$

for every $r > 0$, $\theta \in [\frac{1}{2}, 1]$ and $u \in H^{\frac{1}{2}}(B_r)$. Finally any $\rho < r$ can be written as $\rho = \theta 2^{-k} r$ for some $\theta \in (\frac{1}{2}, 1)$ and $k \in I\!N \cup \{0\}$, so that the conclusion follows by an iteration procedure, with a slightly modified constant $C$.

□

**Proof of (iii) (Lower bound).** Let $u_\varepsilon$ be a sequence in $H^{\frac{1}{2}}(T^2)$ and assume that $u_\varepsilon$ converges to $u$ strongly in $L^2$. In order to prove the lower bound we may assume that $\liminf_\varepsilon F_\varepsilon(u_\varepsilon) = \lim_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Thus by (i) (compactness) we have that $u = a \in \mathbf{Z}$. Since the energy decreases under truncation we may assume that $0 \le u_\varepsilon \le a$.

Consider a lattice of squares $Q_j^\varepsilon$ of size approximatively $1/\sqrt{N_\varepsilon}$. Let $\widehat{Q}_j^\varepsilon$ be concentric squares of twice times the size. Since each point is contained at most in 9 of the squares $\widehat{Q}_j^\varepsilon$ we have

$$\sum_j \iint_{\widehat{Q}_j^\varepsilon \times \widehat{Q}_j^\varepsilon} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^3}\, dx\, dy \le C N_\varepsilon \varepsilon$$

and

$$\sum_j \int_{\widehat{Q}_j^\varepsilon} |u_\varepsilon - a| dx \le \omega_\varepsilon\,,$$

where $\omega_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $\theta > 0$. Then there exist a set of indices $J^\varepsilon$ such that $\frac{1}{N_\varepsilon}\#(J^\varepsilon) \ge 1-\theta$ and a constant $C_\theta$ such that

$$\iint_{\widehat{Q}_j^\varepsilon \times \widehat{Q}_j^\varepsilon} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^3}\, dx\, dy \le C_\theta \varepsilon \tag{4.44}$$

and

$$\fint_{\widehat{Q}_j^\varepsilon} |u_\varepsilon - a| dx \le C_\theta \omega_\varepsilon \tag{4.45}$$

for all $j \in J^\varepsilon$. Let $0 < \delta < 1$. By applying Proposition 17 with $\rho = \varepsilon^\beta$, with $\frac{1}{2} < \beta < 1$, and $r = \frac{1}{\sqrt{N_\varepsilon}}$, for each $x_\varepsilon^i \in Q_j^\varepsilon$ we also have

$$\fint_{B^i_{\varepsilon^\beta}} |u_\varepsilon - a|^2 dx \le \delta \qquad \text{if } \varepsilon \le \varepsilon_0(\delta, \theta)\,. \tag{4.46}$$

Then by Lemma 16 and the assumption that $\operatorname{dist}(x_i^\varepsilon, x_j^\varepsilon) > 6\varepsilon^\beta$

$$F_\varepsilon(u_\varepsilon) \ge \frac{1}{N_\varepsilon}\left[\sum_{j \in J^\varepsilon} \#(I_\varepsilon(Q_j^\varepsilon))\right]\left(D_\nu^{\frac{1}{2}}(a, B_R) - \omega(\varepsilon,\delta)\right) \tag{4.47}$$

The uniform distribution of the obstacles (see condition (4.30)) implies that

$$\sum_{j \in J^\varepsilon} \#(I_\varepsilon(Q_j^\varepsilon)) = N_\varepsilon - \sum_{j \notin J^\varepsilon} \#(I_\varepsilon(Q_j^\varepsilon)) \ge N_\varepsilon - \sum_{j \notin J^\varepsilon} (N_\varepsilon |Q_j^\varepsilon| + L) = N_\varepsilon - (L+1)\#(\{j:\ j \notin J^\varepsilon\})\,.$$

Since $\#(\{j:\ j \notin J^\varepsilon\}) \le N_\varepsilon \theta$, we get

$$F_\varepsilon(u_\varepsilon) \ge (1 - \theta(L+1))\left(D_\nu^{\frac{1}{2}}(a, B_R) - \omega(\varepsilon,\delta)\right)$$

and this yields the required lower bound taking the limit as $\varepsilon \to 0$, then $\delta \to 0$ and finally $\theta \to 0$. $\square$

# Appendix: Finite pinning condition

We can model the hardening mechanism due to obstacles such as secondary dislocations by assuming a weaker pinning condition given by a concentrated force. Namely we assume that a crossing of an obstacle by a dislocation costs a finite amount of energy, i.e.

$$\lambda_0 \int_{B^i_{R\varepsilon}} \varepsilon \psi^i_\varepsilon |u|\, dx\,,$$

where $\psi^i_\varepsilon(x) = \varepsilon^{-2}\psi(\frac{x-x^i_\varepsilon}{\varepsilon})$, with $\operatorname{supp}\psi \subseteq B_R(0)$ and $\int_{B_R}\psi\, dx = 1$, and $\lambda_0\varepsilon\psi^i_\varepsilon$ is the force concentrated on each obstacle $B^i_\varepsilon$. Then we can consider the following functional

$$\widetilde{F}_\varepsilon(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_{T^2} \operatorname{dist}^2(u,\mathbf{Z})\, dx + \iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\, dx\, dy + \sum_i \lambda_0 \int_{B^i_{R\varepsilon}} \varepsilon\psi^i_\varepsilon |u|\, dx & \\ & \text{if } \ u \in H^{\frac12}(T^2)\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{.48}$$

With our scaling assumptions the total force due to the obstacles is finite and is given by

$$\sum_i \lambda_0 \int_{B^i_{R\varepsilon}} \varepsilon\psi^i_\varepsilon\, dx = N_\varepsilon \varepsilon \lambda_0 \approx \Lambda\lambda_0\,.$$

In order to study the $\Gamma$-limit of the functional $\widetilde{F}_\varepsilon$ another natural notion of capacity has to be defined, i.e.

$$\widetilde{D}^\nu_{\frac12}(a,\lambda_0,\psi) := \inf\Big\{ \int_{\mathbf{R}^2} \operatorname{dist}^2(\zeta,\mathbf{Z})\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta(x)-\zeta(y)|^2\, dx\, dy + \tag{.49}$$

$$+\lambda_0 \int_{\mathbf{R}^2} \psi|\zeta|\, dx : \quad \zeta - a \in L^4(\mathbf{R}^2) \Big\}\,.$$

Again one can check that this is a Choquet capacity and satisfies

$$\widetilde{D}^\nu_{\frac12}(a,\lambda_0,\psi) \le D^\nu_{\frac12}(a,B_R)\,.$$

Moreover

$$\lim_{\lambda_0\to\infty} \widetilde{D}^\nu_{\frac12}(a,\lambda_0,\psi) = D^\nu_{\frac12}(a,B_R)\,. \tag{.50}$$

Indeed $\widetilde{D}^\nu_{\frac12}(a,\lambda_0,\psi)$ is increasing in $\lambda_0$, thus the limit always exists. If $\widetilde{\zeta}_{\lambda_0}$ is a sequence of potentials for $\widetilde{D}^\nu_{\frac12}(a,\lambda_0,\psi)$, one can check that, up to a subsequence, it converges weakly in $H^{\frac12}$ to a function $\widetilde{\zeta}$ which is a good competitor for $D^\nu_{\frac12}(a,B_R)$.

Using this notion of capacity we can perform the same analysis as above and prove the following theorem.

**Theorem 18** *Assume $N_\varepsilon \to +\infty$ and that the balls $B^i_{R\varepsilon}$ are uniformly distributed, well separated, with finite capacity density. Denote by $\widetilde{F}$ the functional*

$$\widetilde{F}(u) = \begin{cases} \widetilde{D}^{\nu}_{\frac{1}{2}}(u, \lambda_0 \psi) & \text{if } u = \text{const.} \in \mathbf{Z}\,, \\ \\ +\infty & \text{otherwise}\,. \end{cases} \tag{.51}$$

*Then*

*i) Every sequence $\{u_\varepsilon\}$ such that $\sup_\varepsilon \widetilde{F}_\varepsilon(u_\varepsilon) < \infty$ is pre-compact in $L^2$ and every cluster point is an integer constant;*

*ii) For every $u \in L^2(\mathbf{R}^2)$ there exists a sequence $\{u_\varepsilon\}$ strongly converging in $L^2$ to $u$ such that*

$$\lim_{\varepsilon \to 0} \widetilde{F}_\varepsilon(u_\varepsilon) = \widetilde{F}(u)\,;$$

*iii) Every sequence $\{u_\varepsilon\}$ strongly converging in $L^2$ to some function $u$ satisfies*

$$\liminf_{\varepsilon \to 0} \widetilde{F}_\varepsilon(u_\varepsilon) \geq \widetilde{F}(u)\,.$$

**Remark 19** The new dislocation capacity for weak pinning condition is linear for $a$ big enough. In order to see this we can rewrite it for positive $a$ as follows

$$\widetilde{D}(a, \lambda_0, \psi) = \inf \Bigg\{ \int_{\mathbf{R}^2} \mathrm{dist}^2(w, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2 \times \mathbf{R}^2} \Gamma_\nu(x-y) |w(x) - w(y)|^2 \, dx\, dy + \tag{.52}$$
$$+ \lambda_0 \int_{\mathbf{R}^2} \psi(w + a)\, dx : \quad w \in L^4(\mathbf{R}^2) \text{ and } \quad -a \leq w \leq 0 \Bigg\}.$$

Then consider the following minimum problem

$$D_0(\lambda_0, \psi) := \inf \Bigg\{ \int_{\mathbf{R}^2} \mathrm{dist}^2(w, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2 \times \mathbf{R}^2} \Gamma_\nu(x-y) |w(x) - w(y)|^2 \, dx\, dy + \tag{.53}$$
$$+ \lambda_0 \int_{\mathbf{R}^2} \psi w \, dx : \quad w \in L^4(\mathbf{R}^2) \Bigg\}.$$

and let $w_0$ be a minimum point. As in the proof of Proposition 8 one can prove that there exists an $L^4$ function $f_0$ such that

$$L w_0 = f_0 - \frac{\lambda_0}{2} \psi$$

in the sense of distributions. Using the Green function $G_\nu$ of $L + I$ we show that $w_0$ is bounded. In fact

$$w_0(x) = G_\nu * f_0(x) + G_\nu * w_0(x) - \frac{\lambda_0}{2} G_\nu * \psi(x)$$

and the conclusion follows by Hölder inequality. Now let $a_0$ the smallest positive integer such that

$$w_0 \geq -a_0 \,.$$

Clearly we have

$$\widetilde{D}(a, \lambda_0, \psi) = D_0(\lambda_0, \psi) + \lambda_0 a \qquad \forall a \geq a_0 \,.$$

In particular if we minimize our energy subject to an external force $S$, i.e.

$$\min_{a \in \mathbf{Z}} \; \widetilde{F}(a) - a \int_{T^2} S \, dx \,,$$

then the minimum exists if and only if $|\int_{T^2} S \, dx| \leq \lambda_0$. If the force $S$ is greater than the total resistence of the obstacles, then no equilibrium states exist.

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