NEW RESULTS ON THE ASYMPTOTIC BEHAVIOUR OF DIRICHLET PROBLEMS IN PERFORATED DOMAINS

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Let A be a linear elliptic operator of the second order with bounded measurable coefficients on a bounded open set Ω of $\mathbf{R}^n$, and let (Ω_h) be an arbitrary sequence of open subsets of Ω. We prove the following compactness result: there exist a subsequence, still denoted by (Ω_h), and a positive Borel measure μ on Ω, not charging polar sets, such that, for every $f \in H^{-1}(\Omega)$, the solutions $u_h \in H^1_0(\Omega_h)$ of the equations $Au_h = f$ in Ω_h, extended to 0 on $\Omega \setminus \Omega_h$, converge weakly in $H^1_0(\Omega)$ to the unique solution $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ of the problem

$$\langle Au, v\rangle + \int_\Omega uv\, d\mu = \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(\Omega)\,.$$

When A is symmetric, this compactness result is already known and was obtained by Γ-convergence techniques.

Our new proof, based on the method of oscillating test functions, extends the result to the non-symmetric case. The new technique, which is completely independent of Γ-convergence, relies on the study of the behaviour of the solutions $w^*_h \in H^1_0(\Omega_h)$ of the equations $A^* w^*_h = 1$ in Ω_h, where A^* is the adjoint operator.

We prove also that the limit measure μ does not change if A is replaced by A^*. Moreover, we prove that μ depends only on the symmetric part of the operator A, if the coefficients of the skew-symmetric part are continuous, while an explicit example shows that μ may depend also on the skew-symmetric part of A, when the coefficients are discontinuous.

1. Introduction

In this paper we study the asymptotic behaviour of the solutions of elliptic equations with Dirichlet boundary conditions in perforated domains. Among the physical motivations of the problem we mention the applications to scattering theory (see [28], [46]), electrostatic screening (see [47]), and heat conduction in domains with a complicated boundary (see [46], [10]). A further motivation for the study of this problem in the most general case, without any geometric assumption on the domains, is given by the recent applications to a relaxed formulation of some optimal design problems (see [1], [6], [14], [5], [26]).

Our problem can be formulated as follows. Let A be a linear elliptic operator of the second order with bounded measurable coefficients on a bounded open set Ω of $\mathbf{R}^n$, and let (Ω_h) be an arbitrary sequence of open subsets of Ω. For every $f \in H^{-1}(\Omega)$ we consider the sequence (u_h) of the solutions of the Dirichlet problems

$$u_h \in H^1_0(\Omega_h)\,, \qquad Au_h = f \quad \text{in } \Omega_h\,. \tag{1.1}$$

If we extend u_h to Ω by setting $u_h = 0$ on $\Omega \backslash \Omega_h$, then (u_h) can be regarded as a sequence in $H^1_0(\Omega)$. The problem is to describe the asymptotic behaviour of (u_h) as $h \to \infty$.

The main result of the paper is the following compactness theorem (Theorem 4.6), which holds without any further hypothesis on the geometry of the sets Ω_h. For every sequence (Ω_h) of open subsets of Ω there exist a subsequence, still denoted by (Ω_h), and a positive Borel measure μ on Ω, not charging polar sets, such that, for every $f \in H^{-1}(\Omega)$, the solutions u_h of (1.1) converge weakly in $H^1_0(\Omega)$ to the unique solution $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ of the problem

$$\langle Au, v\rangle + \int_\Omega uv\, d\mu \;=\; \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(\Omega)\,, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

To prove this compactness theorem we observe (Remark 2.3) that all problems of the form (1.1) can be written as problems of the form (1.2) for a suitable choice of the measure μ in a special class of positive measures, denoted by $\mathcal{M}_0(\Omega)$, which includes also measures which take the value $+\infty$ on a large family of sets. We prove (Theorem 4.5) that, for every sequence (μ_h) of measures of the class $\mathcal{M}_0(\Omega)$, there exist a subsequence, still denoted by (μ_h), and a measure $\mu \in \mathcal{M}_0(\Omega)$ such that, for every $f \in H^{-1}(\Omega)$, the solutions $u_h \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$ of the problems

$$\langle Au_h, v\rangle + \int_\Omega u_h v\, d\mu_h \;=\; \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega) \tag{1.3}$$

converge weakly in $H^1_0(\Omega)$ to the unique solution $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ of (1.2). This more general formulation of the compactness theorem includes in our framework the problem of the asymptotic behaviour of the solutions of Schrödinger equations with positive oscillating potentials.

When A is symmetric, this compactness result is already known (see [2], [1], [23], [8], [38]), and the original proof is based on Γ-convergence techniques, for which we refer to [1] and [20]. In this case it is also possible to construct μ by using the limits of the capacities of the sets $U \backslash \Omega_h$, when U varies in the class of all relatively compact open subsets of Ω (see [7] and [19]).

Under some special hypotheses on the sequence (Ω_h), which imply, in particular, that the limit measure μ belongs to $H^{-1}(\Omega)$, the asymptotic behaviour of the solutions (u_h) of (1.1) was studied in [30], [34], [46], [47], [31], [35], [32] by an orthogonal projection method, in [46], [12], [13] by Brownian motion estimates, in [39], [40], [41] by Green's function estimates, in [17], [15], [16] by means of oscillating test functions, in [43], [25] by the point interaction approximation, in [4] by capacitary methods. These papers provide also a description of the limit measure μ in terms of the relevant properties of the sets Ω_h. The case of random sets Ω_h was studied in [28], [45], [42], [44], [24], [11], [3].

Our new proof of the compactness theorem holds in the general case, even if the operator A is not symmetric. The new method, which is more direct than the previous one, and is completely independent of Γ-convergence, is based on the original technique of the oscillating test functions, which was introduced by Tartar [50] in the study of homogenization problems for elliptic operators, and was adapted to the case of perforated domains by Cioranescu and Murat [16].

However, our choice of the test functions is new, and allows us to avoid any additional hypothesis on the sequence (Ω_h). Our proof relies on the study of the behaviour of the solutions w_h^* of the Dirichlet problems

$$w_h^* \in H_0^1(\Omega_h)\,, \qquad A^* w_h^* = 1 \quad \text{in } \Omega_h\,, \tag{1.4}$$

where A^* is the adjoint operator. For a complete study of the asymptotic behaviour of the solutions of (1.1) when A is symmetric and the sequence (w_h^*) converges strongly in $H_0^1(\Omega)$ we refer to [49]. The main difficulty of our result lies in the fact that (w_h^*) is compact only in the weak topology of $H_0^1(\Omega)$.

In the general case (1.3) we consider the solutions $w_h^* \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ of the problems

$$\langle A^* w_h^*, v\rangle + \int_\Omega w_h^* v\, d\mu_h = \int_\Omega v\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)\,. \tag{1.5}$$

By an elementary variational estimate the sequence (w_h^*) is bounded in $H_0^1(\Omega)$, and so we may assume that (w_h^*) converges weakly in $H_0^1(\Omega)$ to some function w^*. We prove (Section 3) that $\nu^* = 1 - A^* w^*$ is a positive Radon measure on Ω, which belongs to $H^{-1}(\Omega)$, and thus we can consider the measure $\mu \in \mathcal{M}_0(\Omega)$ defined by

$$\mu(B) = \begin{cases} \displaystyle\int_B \frac{d\nu^*}{w^*}\,, & \text{if } \operatorname{cap}(B \cap \{w^* = 0\}, \Omega) = 0, \\ +\infty\,, & \text{if } \operatorname{cap}(B \cap \{w^* = 0\}, \Omega) > 0. \end{cases}$$

This is the measure which appears in the limit problem (1.2). Since, by an elementary variational estimate, the sequence (u_h) of the solutions of (1.3) is bounded in $H_0^1(\Omega)$, we may assume also that (u_h) converges weakly in $H_0^1(\Omega)$ to a function u. Moreover, if $f \in L^\infty(\Omega)$, by the comparison principle (Proposition 2.5) the sequence (u_h) is bounded in $L^\infty(\Omega)$, and thus $u \in L^\infty(\Omega)$.

To prove that u is the solution of (1.2), we show that u_h satisfies the equation

$$\langle Au_h, w_h^*\varphi\rangle - \langle A^*w_h^*, u_h\varphi\rangle = \int_\Omega f w_h^*\varphi\,dx - \int_\Omega u_h\varphi\,dx \tag{1.6}$$

for every $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. As the difference of the first two terms is continuous with respect to the weak convergence of (u_h) and (w_h^*), it is easy to take the limit in (1.6) and to show that

$$\langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle = \int_\Omega f w^*\varphi\,dx - \int_\Omega u\varphi\,dx \tag{1.7}$$

for every $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. Then we prove (Lemma 3.5) that (1.7) has a unique solution $u \in H_0^1(\Omega)\cap L^\infty(\Omega)$, which coincides with the solution $u \in H_0^1(\Omega)\cap L^2_\mu(\Omega)$ of (1.2). This concludes the proof of our compactness result in the case $f \in L^\infty(\Omega)$. The case $f \in H^{-1}(\Omega)$ can be treated by an easy approximation argument. If we repeat the proof with A replaced by A^*, we obtain the same limit measure μ (Theorem 4.3).

So far we have considered only the problem of the weak convergence of (u_h) in $H_0^1(\Omega)$. In Section 5 we consider also the problem of the strong convergence of the gradients (Du_h) in $L^p(\Omega, \mathbf{R}^n)$. Using Meyers' estimate [36] and a general result due to Murat [37], we prove (Theorem 5.1) that, without any additional hypothesis, the sequence (Du_h) converges to Du strongly in $L^p(\Omega,\mathbf{R}^n)$ for every $1 \le p < 2$.

To obtain strong convergence of the gradients in $L^2(\Omega,\mathbf{R}^n)$ we construct a corrector term $P_h(x,s)$, $x\in\Omega$, $s\in\mathbf{R}$, which depends on the sequence (μ_h), but is independent of f, u, u_h. We prove (Theorem 5.2) that for every $f\in L^\infty(\Omega)$ we have

$$Du_h(x) = Du(x) + P_h(x,u(x)) + R_h(x) \qquad \text{a.e. in } \Omega\,,$$

where the remainders R_h tend to 0 strongly in $L^2(\Omega,\mathbf{R}^n)$. This improves the corrector results of [16] and [29], which assume that $\mu \in H^{-1}(\Omega)$, and those of [27], which assume that $w^* > 0$ a.e. in Ω. The corrector $P_h(x,s)$ is constructed explicitly in terms of the solutions of (1.4) or (1.5), with A^* replaced by A. If these functions converge strongly in $H_0^1(\Omega)$, we recover (Corollary 5.8) the result of [49].

In the last section we study the problem of the dependence of μ on the skew-symmetric part of the operator A. Extending a result of [16], we prove (Theorem 6.1) that the limit measure μ depends only on the symmetric part of the operator A, if the coefficients of the skew-symmetric part are continuous. Finally, we construct an explicit example, which shows that μ may depend also on the skew-symmetric part of A, when the coefficients are discontinuous.

2. Notation and Preliminaries

Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 2$. We denote by $H^{1,p}(\Omega)$ and $H_0^{1,p}(\Omega)$, $1 \leq p < +\infty$, the usual Sobolev spaces, and by $H^{-1,q}(\Omega)$, $1/q+1/p=1$, the dual of $H_0^{1,p}(\Omega)$. When $p=2$ we adopt the standard notation $H^1(\Omega)$, $H_0^1(\Omega)$, and $H^{-1}(\Omega)$. On $H_0^1(\Omega)$ we consider the norm

$$\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |Du|^2 dx\,.$$

By $L^p_\mu(\Omega)$, $1 \leq p \leq +\infty$, we denote the usual Lebesgue space with respect to the measure μ. If μ is the Lebesgue measure, we use the standard notation $L^p(\Omega)$.

For every subset E of Ω the (*harmonic*) *capacity* of E in Ω, denoted by $\mathrm{cap}(E,\Omega)$, is defined as the infimum of

$$\int_\Omega |Du|^2\, dx$$

over the set of all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E.

We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (abbreviated as q.e.) in a set E if it holds for all $x \in E$ except for a subset N of E with $\mathrm{cap}(N,\Omega)=0$. The expression *almost everywhere* (abbreviated as a.e.) refers, as usual, to the Lebesgue measure. A function $u\colon\Omega \to \mathbf{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists a set $A \subseteq \Omega$, with $\mathrm{cap}(A,\Omega) < \varepsilon$, such that the restriction of u to $\Omega\backslash A$ is continuous.

It is well known that every $u \in H^1(\Omega)$ has a quasi continuous representative, which is uniquely defined up to a set of capacity zero. In the sequel we shall always identify u with its quasi continuous representative, so that the pointwise values of a function $u \in H^1(\Omega)$ are defined quasi everywhere. We recall that, if a sequence (u_h) converges to u in $H_0^1(\Omega)$, then a subsequence of (u_h) converges to u q.e. in Ω. For all these properties of quasi continuous representatives of Sobolev functions we refer to [51], Section 3.

A subset A of Ω is said to be a *quasi open* if for every $\varepsilon > 0$ there exists an open subset U_ε of Ω, with $\mathrm{cap}(U_\varepsilon,\Omega) < \varepsilon$, such that $A \cup U_\varepsilon$ is open.

Lemma 2.1. *For every quasi open subset A of Ω there exists an increasing sequence (v_h) of non-negative functions of $H_0^1(\Omega)$ converging to 1_A pointwise q.e. in Ω.*

Proof. This lemma is an easy consequence of a more general result proved in [18], Lemma 1.5. For the reader's convenience, we give here the easy proof in this particular case. Let A be a quasi open subset of Ω. Then there exists a sequence (U_k) of open subsets of Ω, with $\mathrm{cap}(U_k,\Omega) < 1/k$, such that the sets $A_k = A \cup U_k$

are open. Therefore, for every $k \in \mathbf{N}$ there exists an increasing sequence $(\varphi_h^k)_h$ of non-negative functions of $C_0^\infty(\Omega)$ converging to 1_{A_k} pointwise q.e. in Ω. Since $\mathrm{cap}(U_k, \Omega) < 1/k$, for every $k \in \mathbf{N}$ there exists $u_k \in H_0^1(\Omega)$ such that $u_k \geq 1$ q.e. in U_k, $u_k \geq 0$ q.e. in Ω, and $\int_\Omega |Du_k|^2 dx < 1/k$. This implies that a subsequence of (u_k) converges to 0 q.e. in Ω. Moreover, as $\varphi_h^k \leq 1_{A_k}$, we have $(\varphi_h^k - u_k)^+ \leq 1_A$ q.e. in Ω. Let us define

$$v_h = \max_{1 \leq k \leq h} (\varphi_h^k - u_k)^+ , \qquad \psi = \sup_h v_h .$$

Then $v_h \in H_0^1(\Omega)$, $v_h \geq 0$ in Ω, the sequence (v_h) is increasing, and $\psi \leq 1_A$ q.e. in Ω. For every $h \geq k$ we have $v_h \geq \varphi_h^k - u_k$. As $A \subseteq A_k$, we get $\psi \geq 1 - u_k$ q.e. in A. Taking the limit as $k \to \infty$ along a suitable subsequence, we obtain $\psi \geq 1$ q.e. in A. This shows that $\psi = 1_A$ and concludes the proof of the lemma. $\square$

By a *Borel measure* on Ω we mean a positive, countably additive set function defined in the Borel σ-field of Ω and with values in $[0, +\infty]$. By a *Radon measure* on Ω we mean a Borel measure which is finite on every compact subset of Ω.

We denote by $\mathcal{M}_0(\Omega)$ the set of all Borel measures μ on Ω such that

(i) $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B, \Omega) = 0$,

(ii) $\mu(B) = \inf\{\mu(A) : A \text{ quasi open}, B \subseteq A\}$ for every Borel set $B \subseteq \Omega$.

This definition differs from the definition used in [22] and [23], where condition (ii) is not present. Our class $\mathcal{M}_0(\Omega)$ coincides with the class $\mathcal{M}_0^*(\Omega)$ introduced in [19] and used in [8]. We refer to [19] for a comparison between these definitions. It is well known that every Radon measure satisfies (ii), while there are examples of Borel measures which satisfy (i), but do not satisfy (ii).

For every closed set $E \subseteq \Omega$ we denote by ∞_E the measure of the class $\mathcal{M}_0(\Omega)$ defined by

$$\infty_E(B) = \begin{cases} 0, & \text{if } \mathrm{cap}(B \cap E, \Omega) = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

We shall see in Theorem 4.6 that the measures ∞_E will be useful in the study of the asymptotic behaviour of sequences of Dirichlet problems in varying domains.

Finally, we say that a Radon measure ν on Ω belongs to $H^{-1}(\Omega)$ if there exists $f \in H^{-1}(\Omega)$ such that

$$\langle f, \varphi \rangle = \int_\Omega \varphi \, d\nu \qquad \forall \varphi \in C_0^\infty(\Omega) , \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. We shall always identify f and ν. Note that, by the Riesz theorem, for every positive functional $f \in H^{-1}(\Omega)$, there exists a Radon measure ν such that (2.2) holds. It is well known that every Radon measure which belongs to $H^{-1}(\Omega)$ belongs also to $\mathcal{M}_0(\Omega)$ (see [51], Section 4.7).

Let $A: H^1(\Omega) \to H^{-1}(\Omega)$ be an elliptic operator of the form

$$Au = -\sum_{i,j=1}^{n} D_i(a_{ij} D_j u)\,, \tag{2.3}$$

where (a_{ij}) is an $n \times n$ matrix of functions of $L^\infty(\Omega)$ satisfying, for a suitable constant $\alpha > 0$, the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \ge \alpha|\xi|^2 \tag{2.4}$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^n$. Let $A^*: H^1(\Omega) \to H^{-1}(\Omega)$ be the adjoint operator, defined by

$$A^* u = -\sum_{i,j=1}^{n} D_i(a_{ji} D_j u)$$

for every $u \in H^1(\Omega)$. It is well known that $\langle A^* u, v\rangle = \langle Av, u\rangle$ for every u, $v \in H^1_0(\Omega)$.

Let $\mu \in \mathcal{M}_0(\Omega)$ and $f \in H^{-1}(\Omega)$. We shall consider the following *relaxed Dirichlet problem* (see [22] and [23]): find $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ such that

$$\langle Au, v\rangle + \int_\Omega uv\, d\mu \;=\; \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(\Omega). \tag{2.5}$$

The name is motivated by Theorem 4.6 and by the density results proved in [22] and [21].

Theorem 2.2. *For every $f \in H^{-1}(\Omega)$ there exists a unique solution of problem (2.5).*

Proof. The proof is a straightforward application of the Lax-Milgram lemma, see, e.g., [22], Theorem 2.4. □

By the ellipticity condition (2.4), if we take u as test function in (2.5), we obtain the following estimate

$$\|u\|_{H^1_0(\Omega)} \le \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{2.6}$$

A connection between classical Dirichlet problems and relaxed Dirichlet problems (2.5) is given by the following remark.

Remark 2.3. It is easy to see that, if E is a closed set and $\mu = \infty_E$, then $u \in H^1_0(\Omega)$ is the solution of problem (2.5) if and only if $u = 0$ q.e. in E and u is the solution in $\Omega \backslash E$ of the classical boundary value problem

$$u \in H^1_0(\Omega \backslash E)\,, \qquad Au = f \quad \text{in } \Omega \backslash E\,.$$

The solutions of relaxed Dirichlet problems satisfy a comparison principle.

Proposition 2.4. *Let $\mu \in \mathcal{M}_0(\Omega)$, let $f \in H^{-1}(\Omega)$, and let $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solution of problem (2.5). If $f \geq 0$ in Ω, then $u \geq 0$ q.e. in Ω.*

Proof. The proof is given in [22], Proposition 2.9, in a more general context. For the sake of completeness we give the proof in this simple case. Let $v = -(u \wedge 0)$. Then v is a non-negative function of $H_0^1(\Omega) \cap L^2_\mu(\Omega)$. Since $uv \leq 0$ q.e. in Ω and $\langle f, v\rangle \geq 0$, taking v as test function in (2.5) we obtain $\langle Au, v\rangle \geq 0$. As $Dv = -Du$ a.e. in $\{v > 0\}$ and $Dv = 0$ a.e. in $\{v = 0\}$, we have that $\langle Av, v\rangle = -\langle Au, v\rangle \leq 0$. By the ellipticity assumption we obtain $v = 0$ q.e. in Ω, hence $u \geq 0$ q.e. in Ω. $\square$

Proposition 2.5. *Let f_1, $f_2 \in H^{-1}(\Omega)$ and let μ_1, $\mu_2 \in \mathcal{M}_0(\Omega)$. Let u_1, $u_2 \in H_0^1(\Omega)$ be the solutions of problem (2.5) corresponding to f_1, μ_1 and to f_2, μ_2. If $0 \leq f_1 \leq f_2$ and $\mu_2 \leq \mu_1$ in Ω, then $0 \leq u_1 \leq u_2$ q.e. in Ω.*

Proof. This result is proved in [22], Proposition 2.10. For the reader's convenience we give here the complete proof. By Proposition 2.4 we have that $u_1 \geq 0$ q.e. in Ω and $u_2 \geq 0$ q.e. in Ω. Let $v = (u_1 - u_2)^+$. Since $0 \leq v \leq u_1$ and $\mu_2 \leq \mu_1$, we have $v \in L^2_{\mu_1}(\Omega) \cap L^2_{\mu_2}(\Omega)$. As $\int_\Omega u_2 v \, d\mu_2 \leq \int_\Omega u_2 v \, d\mu_1$, taking v as test function in the problems solved by u_1 and u_2 and subtracting the corresponding equations, we obtain

$$\langle A(u_1 - u_2), v\rangle + \int_\Omega (u_1 - u_2) v \, d\mu_1 \;\leq\; \langle f_1 - f_2, v\rangle \;\leq\; 0 \,.$$

Since $(u_1 - u_2)v \geq 0$ q.e. in Ω, by the ellipticity condition (2.4) we have

$$\alpha \|v\|^2_{H_0^1(\Omega)} \;\leq\; \langle Av, v\rangle \;=\; \langle A(u_1 - u_2), v\rangle \;\leq\; 0 \,.$$

Thus $v = 0$ q.e. in Ω and, consequently, $u_1 \leq u_2$ q.e. in Ω. $\square$

The following result will be useful in the sequel.

Proposition 2.6. *Let ν be a positive Radon measure on Ω which belongs to $H^{-1}(\Omega)$ and let u be the solution in $H_0^1(\Omega) \cap L^2_\mu(\Omega)$ of problem (2.5) corresponding to $f = \nu$. Then*

$$\langle Au, v\rangle \;\leq\; \int_\Omega v \, d\nu$$

for every $v \in H_0^1(\Omega)$ with $v \geq 0$ q.e. in Ω.

Proof. This proposition is proved in [22], Proposition 2.6, under more general hypotheses. Here we sketch the proof only in our particular case. Let $v \in H_0^1(\Omega)$ with $v \geq 0$ q.e. in Ω and let $v_h = (\frac{1}{h} v) \wedge u$. Since $u \geq 0$ (Proposition 2.4), we have that $v_h \geq 0$ q.e. in Ω and $v_h \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$. Then, taking v_h as test function in problem (2.5) with $f=\nu$, we obtain $\langle Au, v_h\rangle \le \int_\Omega v_h\,d\nu$. Since $Dv_h = \frac{1}{h}Dv$ in $\{v<hu\}$ and $Dv_h = Du$ in $\{v\ge hu\}$, we have

$$\frac{1}{h}\int_{\{v<hu\}} \Big(\sum_{i,j=1}^{n} a_{ij}D_juD_iv\Big)\,dx + \int_{\{v\ge hu\}} \Big(\sum_{i,j=1}^{n} a_{ij}D_juD_iu\Big)\,dx \le$$
$$\le \int_\Omega v_h\,d\nu \le \frac{1}{h}\int_\Omega v\,d\nu\,.$$

By neglecting the second term, which is non-negative by the ellipticity assumption, we get

$$\int_{\{v<hu\}} \Big(\sum_{i,j=1}^{n} a_{ij}D_juD_iv\Big)\,dx \le \int_\Omega v\,d\nu\,.$$

Taking the limit as $h\to\infty$, we obtain

$$\int_{\{u>0\}} \Big(\sum_{i,j=1}^{n} a_{ij}D_juD_iv\Big)\,dx \le \int_\Omega v\,d\nu\,.$$

Since $u\ge 0$ q.e. in Ω and $D_ju=0$ a.e. in $\{u=0\}$, the conclusion follows. $\square$

3. A Convex Set

In this section we shall study the properties of the set $\mathcal{K}(\Omega)$ of all functions $w\in H_0^1(\Omega)$ such that $w\ge 0$ q.e. in Ω and $Aw\le 1$ in Ω in the sense of $H^{-1}(\Omega)$. It is easy to see that $\mathcal{K}(\Omega)$ is a closed convex subset of $H_0^1(\Omega)$. Moreover, for every $w\in\mathcal{K}(\Omega)$ we have

$$\alpha\int_\Omega |Dw|^2dx \le \langle Aw,w\rangle \le \int_\Omega w\,dx\,.$$

This shows that $\mathcal{K}(\Omega)$ is bounded, and hence weakly compact, in $H_0^1(\Omega)$. Let w_0 be the solution of the Dirichlet problem

$$w_0\in H_0^1(\Omega)\,,\qquad Aw_0 = 1\,.$$

By the maximum principle we have $w\le w_0$ q.e. in Ω for every $w\in\mathcal{K}(\Omega)$. As $w_0\in L^\infty(\Omega)$ (see [48]), the set $\mathcal{K}(\Omega)$ is bounded in $L^\infty(\Omega)$.

Given $w\in\mathcal{K}(\Omega)$, let $\nu = 1-Aw$. By the definition of $\mathcal{K}(\Omega)$ we have $\nu\ge 0$ in Ω in the sense of distributions, hence ν is a positive Radon measure. As $Aw\in H^{-1}(\Omega)$, we have also $\nu\in H^{-1}(\Omega)$.

We shall see that, if $w\in\mathcal{K}(\Omega)$, then w can be characterized as the solution of a particular relaxed Dirichlet problem. To this aim we need some preliminary results.

Proposition 3.1. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$. For every $h \in \mathbf{N}$ let $u_h \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ be the solution of the problem*

$$\langle Au_h, v\rangle + \int_\Omega u_h v \, d\mu + h \int_\Omega (u_h - u) v \, dx = 0 \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(\Omega) . \tag{3.1}$$

Then (u_h) converges to u strongly in $H^1_0(\Omega)$ and in $L^2_\mu(\Omega)$. Moreover

$$\lim_{h\to\infty} \Big(\langle Au_h, u_h\rangle + \int_\Omega u_h^2 \, d\mu + h \int_\Omega (u_h - u)^2 dx \Big) = \langle Au, u\rangle + \int_\Omega u^2 \, d\mu . \tag{3.2}$$

Proof. Taking $v = u_h - u$ as test function in (3.1) we obtain

$$\langle Au_h, u_h - u\rangle + \int_\Omega u_h (u_h - u) \, d\mu + h \int_\Omega (u_h - u)^2 dx = 0 , \tag{3.3}$$

hence

$$\begin{aligned} \langle A(u_h - u), u_h - u\rangle + \int_\Omega (u_h - u)^2 d\mu + h \int_\Omega (u_h - u)^2 dx = \\ = -\langle Au, u_h - u\rangle - \int_\Omega u(u_h - u) \, d\mu . \end{aligned}$$

From the ellipticity condition (2.4) we get

$$\begin{aligned} \alpha \|u_h - u\|^2_{H^1_0(\Omega)} + \|u_h - u\|^2_{L^2_\mu(\Omega)} + h\|u_h - u\|^2_{L^2(\Omega)} \le \\ \le -\langle Au, u_h - u\rangle - \int_\Omega u(u_h - u) \, d\mu , \end{aligned} \tag{3.4}$$

hence

$$\begin{aligned} \alpha \|u_h - u\|^2_{H^1_0(\Omega)} + \|u_h - u\|^2_{L^2_\mu(\Omega)} + h\|u_h - u\|^2_{L^2(\Omega)} \le \\ \le \|Au\|_{H^{-1}(\Omega)} \|u_h - u\|_{H^1_0(\Omega)} + \|u\|_{L^2_\mu(\Omega)} \|u_h - u\|_{L^2_\mu(\Omega)} . \end{aligned}$$

By using the Cauchy inequality we obtain

$$\begin{aligned} \frac{\alpha}{2} \|u_h - u\|^2_{H^1_0(\Omega)} + \frac{1}{2} \|u_h - u\|^2_{L^2_\mu(\Omega)} + h\|u_h - u\|^2_{L^2(\Omega)} \le \\ \le \frac{1}{2\alpha} \|Au\|^2_{H^{-1}(\Omega)} + \frac{1}{2} \|u\|^2_{L^2_\mu(\Omega)} . \end{aligned}$$

This shows that (u_h) converges to u weakly in $H^1_0(\Omega)$ and in $L^2_\mu(\Omega)$. By (3.4) this implies that (u_h) converges to u strongly in $H^1_0(\Omega)$ and in $L^2_\mu(\Omega)$. Finally (3.3) gives

$$\langle Au_h, u_h\rangle + \int_\Omega u_h^2 \, d\mu + h \int_\Omega (u_h - u)^2 dx = \langle Au_h, u\rangle + \int_\Omega u_h u \, d\mu ,$$

which proves (3.2). ☐

Lemma 3.2. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $w \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solution of the problem*

$$\langle Aw, v\rangle + \int_\Omega wv\, d\mu = \int_\Omega v\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,.$$

Then $\mu(B) = +\infty$ for every Borel subset B of Ω with $\mathrm{cap}(B \cap \{w = 0\}, \Omega) > 0$.

Proof. Let $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$, with $0 \le u \le 1$ q.e. in Ω, and, for every $h \in \mathbf{N}$, let $u_h \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solution of the problem

$$\langle Au_h, v\rangle + \int_\Omega u_h v\, d\mu + h\int_\Omega u_h v\, dx = h\int_\Omega uv\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,.$$

By the comparison principle (Proposition 2.5) we have $0 \le u_h \le hw$ q.e. in Ω, hence $u_h = 0$ q.e. in $\{w = 0\}$. Since, by Proposition 3.1, (u_h) converges to u in $H_0^1(\Omega)$, we have $u = 0$ q.e. in $\{w = 0\}$.

Let U be a quasi open subset of Ω such that $\mu(U) < +\infty$. By Lemma 2.1 there exists an increasing sequence (z_h) in $H_0^1(\Omega)$ converging to 1_U pointwise q.e. in Ω and such that $0 \le z_h \le 1_U$ q.e. in Ω for every $h \in \mathbf{N}$. As $\mu(U) < +\infty$, each function z_h belongs to $L^2_\mu(\Omega)$, hence $z_h = 0$ q.e. on $\{w = 0\}$ by the previous step. This implies that $\mathrm{cap}(U \cap \{w = 0\}, \Omega) = 0$.

Let us consider a Borel set B with $\mathrm{cap}(B \cap \{w = 0\}, \Omega) > 0$. For every quasi open set U containing B we have $\mathrm{cap}(U \cap \{w = 0\}, \Omega) > 0$, hence $\mu(U) = +\infty$ by the previous step of the proof. By the definition of $\mathcal{M}_0(\Omega)$ we conclude that $\mu(B) = +\infty$. ☐

Lemma 3.3. *Let λ and μ be measures of $\mathcal{M}_0(\Omega)$. Assume that there exists a function $w \in H_0^1(\Omega) \cap L^2_\lambda(\Omega) \cap L^2_\mu(\Omega)$ such that*

$$\langle Aw, v\rangle + \int_\Omega wv\, d\lambda = \int_\Omega v\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\lambda(\Omega)\,, \tag{3.5}$$

$$\langle Aw, v\rangle + \int_\Omega wv\, d\mu = \int_\Omega v\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,. \tag{3.6}$$

Then $\lambda = \mu$.

Proof. Let us consider the measures λ_0 and μ_0 defined for every Borel set $B \subseteq \Omega$ by

$$\lambda_0(B) = \int_B w\, d\lambda\,, \qquad \mu_0(B) = \int_B w\, d\mu\,.$$

Let us prove that $\lambda_0 = \mu_0$. For every $\varepsilon > 0$ let λ_ε and μ_ε be the measures defined by

$$\lambda_\varepsilon(B) = \int_{B\cap\{w>\varepsilon\}} w\, d\lambda\,, \qquad \mu_\varepsilon(B) = \int_{B\cap\{w>\varepsilon\}} w\, d\mu\,.$$

To prove that $\lambda_0 = \mu_0$ it is enough to show that $\lambda_\varepsilon = \mu_\varepsilon$ for every $\varepsilon > 0$. Let us fix $\varepsilon > 0$. As $w \in L^2_\lambda(\Omega) \cap L^2_\mu(\Omega)$, λ_ε and μ_ε are bounded measures. Therefore it is enough to show that $\lambda_\varepsilon(U) = \mu_\varepsilon(U)$ for every open subset U of Ω. Let us fix U and let $U_\varepsilon = U \cap \{w > \varepsilon\}$. As U_ε is quasi open, by Lemma 2.1 there exists an increasing sequence (z_h) in $H^1_0(\Omega)$ converging to 1_{U_ε} pointwise q.e. in Ω and such that $0 \le z_h \le 1_{U_\varepsilon}$ q.e. in Ω. As $w \in L^2_\lambda(\Omega) \cap L^2_\mu(\Omega)$ and $w > \varepsilon$ q.e. in U_ε, we have $\lambda(U_\varepsilon) < +\infty$ and $\mu(U_\varepsilon) < +\infty$, hence $z_h \in L^2_\lambda(\Omega) \cap L^2_\mu(\Omega)$. From (3.5) and (3.6) we get

$$\int_\Omega w z_h \, d\lambda = \int_\Omega w z_h \, d\mu \,.$$

Taking the limit as $h \to \infty$ we obtain

$$\lambda_\varepsilon(U) = \int_{U_\varepsilon} w \, d\lambda = \int_{U_\varepsilon} w \, d\mu = \mu_\varepsilon(U) \,.$$

This shows that $\lambda_\varepsilon = \mu_\varepsilon$ for every $\varepsilon > 0$, hence $\lambda_0 = \mu_0$. For every Borel set $B \subseteq \{w > 0\}$ we have

$$\lambda(B) = \int_B \frac{1}{w} \, d\lambda_0 = \int_B \frac{1}{w} \, d\mu_0 = \mu(B) \,.$$

If B is Borel set contained in $\{w = 0\}$ and $\mathrm{cap}(B, \Omega) > 0$, then $\lambda(B) = \mu(B) = +\infty$ by Lemma 3.2. If $\mathrm{cap}(B, \Omega) = 0$, then $\lambda(B) = \mu(B) = 0$ by the definition of $\mathcal{M}_0(\Omega)$. Therefore $\lambda(B) = \lambda(B \cap \{w > 0\}) + \lambda(B \cap \{w = 0\}) = \mu(B \cap \{w > 0\}) + \mu(B \cap \{w = 0\}) = \mu(B)$ for every Borel set $B \subseteq \Omega$. $\square$

We give now the characterization of $\mathcal{K}(\Omega)$ in terms of relaxed Dirichlet problems.

Proposition 3.4. *A function $w \in H^1_0(\Omega)$ belongs to $\mathcal{K}(\Omega)$ if and only if there exists $\mu \in \mathcal{M}_0(\Omega)$ such that $w \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ and*

$$\langle Aw, v \rangle + \int_\Omega wv \, d\mu = \int_\Omega v \, dx \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(\Omega) \,. \tag{3.7}$$

The measure $\mu \in \mathcal{M}_0(\Omega)$ is uniquely determined by $w \in \mathcal{K}(\Omega)$. More precisely, for every $w \in \mathcal{K}(\Omega)$ and for every Borel set $B \subseteq \Omega$ we have

$$\mu(B) = \begin{cases} \displaystyle\int_B \frac{d\nu}{w} \,, & \text{if } \mathrm{cap}(B \cap \{w = 0\}, \Omega) = 0, \\ +\infty \,, & \text{if } \mathrm{cap}(B \cap \{w = 0\}, \Omega) > 0, \end{cases} \tag{3.8}$$

where ν is the measure of $H^{-1}(\Omega)$ defined by $\nu = 1 - Aw$. Moreover, we have

$$\nu(B \cap \{w > 0\}) = \int_B w \, d\mu \tag{3.9}$$

for every Borel set $B \subseteq \Omega$.

Proof. We follow the lines of the proof of Theorem 1 of [14]. Let $\mu \in \mathcal{M}_0(\Omega)$ and let $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be a solution of (3.7). Then $w \geq 0$ q.e. in Ω by Proposition 2.4 and $Aw \leq 1$ in Ω by Proposition 2.6, hence $w \in \mathcal{K}(\Omega)$.

Conversely, assume that $w \in \mathcal{K}(\Omega)$ and let μ be the measure defined by (3.8). Let us prove that $\mu \in \mathcal{M}_0(\Omega)$. Since $\nu \in H^{-1}(\Omega)$, we have $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B, \Omega) = 0$. It remains to prove that

$$\mu(B) = \inf\{\mu(A) : A \text{ quasi open}\,,\ B \subseteq A\} \tag{3.10}$$

for every Borel set $B \subseteq \Omega$ with $\mu(B) < +\infty$. For every $h \in \mathbf{N}$ let μ_h be the measure on Ω defined by $\mu_h(B) = \mu(B \cap \{w > \frac{1}{h}\})$. Note that

$$\mu_h(\Omega) = \mu(\{w > \tfrac{1}{h}\}) \leq h\,\nu(\{w > \tfrac{1}{h}\}) \leq h^2 \int_\Omega w\,d\nu = h^2\langle 1 - Aw, w\rangle < +\infty\,.$$

Let us fix a Borel set $B \subseteq \Omega$ with $\mu(B) < +\infty$. By the definition of μ we have $\mathrm{cap}(B \cap \{w = 0\}, \Omega) = 0$. For every $h \geq 2$ let $B_h = B \cap \{\frac{1}{h} < w \leq \frac{1}{h-1}\}$, and let $B_1 = \{1 < w\}$, so that $\mu(B) = \sum_h \mu(B_h)$. Since $\mu_h(\Omega) < +\infty$, for every $\varepsilon > 0$ and for every $h \in \mathbf{N}$ there exists an open set U_h, with $B_h \subseteq U_h \subseteq \Omega$, such that $\mu_h(U_h) < \mu_h(B_h) + \varepsilon 2^{-h} = \mu(B_h) + \varepsilon 2^{-h}$. Let $A_h = U_h \cap \{w > \frac{1}{h}\}$. As w is quasi continuous, the set A_h is quasi open. Moreover $B_h \subseteq A_h$ and $\mu(A_h) = \mu_h(U_h) < \mu(B_h) + \varepsilon 2^{-h}$. Let $A_0 = B \cap \{w = 0\}$ and let A be the union of all sets A_h for $h \geq 0$. Then A is quasi open, contains B, and $\mu(A) < \mu(B) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this proves (3.10).

Let us prove that w is a solution of (3.7). By (3.8) we have

$$\int_\Omega w^2 d\mu \;=\; \int_{\{w>0\}} w^2 d\mu \;=\; \int_{\{w>0\}} w\,d\nu \;=\; \langle 1 - Aw, w\rangle \;<\; +\infty\,,$$

hence $w \in L_\mu^2(\Omega)$. Let $v \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$. By (3.8) we have $v = 0$ q.e. in $\{w = 0\}$. By the definitions of μ and ν we have

$$\begin{aligned} \langle Aw, v\rangle \;+\; \int_\Omega wv\,d\mu \;&=\; \langle Aw, v\rangle \;+\; \int_{\{w>0\}} wv\,d\mu \;= \\ =\; \langle Aw, v\rangle \;+\; \int_{\{w>0\}} v\,d\nu \;&=\; \langle Aw, v\rangle \;+\; \int_\Omega v\,d\nu \;=\; \int_\Omega v\,dx\,, \end{aligned}$$

which proves (3.7). The uniqueness of μ follows from Lemma 3.3.

Property (3.9) is an easy consequence of (3.8). ☐

The following lemma will be crucial in the proof of Theorem 4.3.

Lemma 3.5. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $f \in L^\infty(\Omega)$. Let $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ and $w^* \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of the problems*

$$\langle Au, v\rangle + \int_\Omega uv\, d\mu = \int_\Omega fv\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega), \tag{3.11}$$

$$\langle A^*w^*, v\rangle + \int_\Omega w^*v\, d\mu = \int_\Omega v\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega), \tag{3.12}$$

Then u is the unique solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of the problem

$$\langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle = \int_\Omega fw^*\varphi\, dx - \int_\Omega u\varphi\, dx \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{3.13}$$

Proof. First of all, we note that (3.13) can be written as

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi\Big) w^* dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j \varphi D_i w^*\Big) u\, dx =$$

$$= \int_\Omega fw^*\varphi\, dx - \int_\Omega u\varphi\, dx \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{3.14}$$

Let $w \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solution of (3.7). By the comparison principle (Theorem 2.5) we have $|u| \le c\,w$ q.e. in Ω, with $c = \|f\|_{L^\infty(\Omega)}$. Since w is bounded, this implies that $u \in L^\infty(\Omega)$.

Let $\nu^* = 1 - A^*w^*$. By Proposition 2.6 ν^* is a non-negative Radon measure. By Lemma 3.4 (applied to A^*) we have that

$$\nu^*(B \cap \{w^* > 0\}) = \int_B w^* d\mu \tag{3.15}$$

for every Borel set $B \subseteq \Omega$. As $w^* \in L^2_\mu(\Omega)$, we have $w^*\varphi \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ for every $\varphi \in C_0^\infty(\Omega)$. As $u \in L^2_\mu(\Omega)$, by Lemma 3.2 (applied to A^*) we have $u = 0$ q.e. in $\{w^* = 0\}$. Therefore (3.15) implies that

$$\int_\Omega uw^*\varphi\, d\mu = \int_{\{w^*>0\}} u\varphi\, d\nu^* = \int_\Omega u\varphi\, d\nu^*.$$

Taking $v = w^*\varphi$ in (3.11) we obtain

$$\langle Au, w^*\varphi\rangle + \int_\Omega u\varphi\, d\nu^* = \int_\Omega fw^*\varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. As $\nu^* = 1 - A^*w^*$, we conclude that u is a solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.13).

Let us prove that the solution of (3.13) is unique. First of all we observe that, by an easy approximation argument, (3.13) holds for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Since the equation is linear in u, it is enough to consider the case $f = 0$. Let us fix a solution $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.13) with $f = 0$. By (3.14) we have that

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j z D_i v \Big) w^* dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j v D_i w^* \Big) z \, dx + \int_\Omega z v \, dx \;=\; 0$$

for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Taking $v = z$ we obtain

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j z D_i z \Big) w^* dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j z D_i w^* \Big) z \, dx + \int_\Omega z^2 dx \;=\; 0\,. \tag{3.16}$$

As $z D_j z = \frac{1}{2} D_j(z^2)$ and $\nu^* \geq 0$ we have

$$-\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j z D_i w^* \Big) z \, dx \;=\; -\frac{1}{2} \langle A^* w^*, z^2 \rangle \;\geq\; -\frac{1}{2} \int_\Omega z^2 dx\,,$$

and so (3.16) gives

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j z D_i z \Big) w^* dx + \frac{1}{2} \int_\Omega z^2 dx \;\leq\; 0\,. \tag{3.17}$$

Since $w^* \geq 0$ q.e. in Ω (Proposition 2.4), (3.17) and the ellipticity condition (2.4) imply that $z = 0$ a.e. in Ω. This concludes the proof of the uniqueness. □

4. The γ^A-Convergence and the Compactness Theorem

In this section we introduce the notion of γ^A-convergence in $\mathcal{M}_0(\Omega)$, related to the convergence of the solutions of the corresponding relaxed Dirichlet problems. When A is the Laplace operator $-\Delta$, this notion is defined in [23] in terms of the Γ-convergence of the functionals $\int_\Omega |Du|^2 dx + \int_\Omega u^2 d\mu$ associated with the relaxed Dirichlet problems. For the extension of this definition to the case of symmetric operators see [7] and [19]. The definition given here involves only the solutions of (2.5), and coincides with the previous ones in the symmetric cases.

Definition 4.1. Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. We say that (μ_h) γ^A-converges to μ (in Ω) if for every $f \in H^{-1}(\Omega)$ the solutions $u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ of the problems

$$\langle A u_h, v \rangle + \int_\Omega u_h v \, d\mu_h \;=\; \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega) \tag{4.1}$$

converge weakly in $H_0^1(\Omega)$, as $h \to \infty$, to the solution $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ of the problem

$$\langle A u, v \rangle + \int_\Omega u v \, d\mu \;=\; \langle f, v \rangle \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,. \tag{4.2}$$

We underline the fact that the γ^A-limit depends on the operator A. This fact will be discussed later in Section 6.

Remark 4.2. Since A is linear and the solutions of (4.1) depend continuously on the data, uniformly with respect to h (see the estimate (2.6)), a sequence (μ_h) γ^A-converges to μ if and only if the solutions of (4.1) converge weakly in $H_0^1(\Omega)$ to the solution of (4.2) for every f in a dense subset of $H^{-1}(\Omega)$.

Let (μ_h) be a sequence of measures of the class $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. Let $w_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $w \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega)$ be the solutions of the problems

$$\langle Aw_h, v\rangle + \int_\Omega w_h v \, d\mu_h = \int_\Omega v \, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)\,, \tag{4.3}$$

$$\langle Aw, v\rangle + \int_\Omega w v \, d\mu = \int_\Omega v \, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega)\,, \tag{4.4}$$

and let $w_h^* \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $w^* \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega)$ be the solutions of the corresponding problems for the adjoint operator A^*.

We are now in a position to characterize the γ^A-convergence of a sequence of measures (μ_h) in terms of the weak convergence in $H_0^1(\Omega)$ of the sequences (w_h) and (w_h^*).

Theorem 4.3. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. Let $w_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $w \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega)$ be the solutions of problems (4.3) and (4.4), and let $w_h^* \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$, $w^* \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega)$ be the solutions of the corresponding problems for A^*. The following conditions are equivalent:*

(a) (w_h) *converges to* w *weakly in* $H_0^1(\Omega)$;

(b) (w_h^*) *converges to* w^* *weakly in* $H_0^1(\Omega)$;

(c) (μ_h) γ^A*-converges to* μ;

(d) (μ_h) γ^{A^*}*-converges to* μ.

Proof. $(b) \Rightarrow (c)$. Given $f \in L^\infty(\Omega)$, let $u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $u \in H_0^1(\Omega) \cap L^2_{\mu}(\Omega)$ be the solutions of the problems (4.1) and (4.2). By Lemma 3.5 and by (3.14) we have

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i \varphi\Big) w_h^* \, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j \varphi D_i w_h^*\Big) u_h \, dx =$$
$$= \int_\Omega f w_h^* \varphi \, dx - \int_\Omega u_h \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega)\,. \tag{4.5}$$

By the estimate (2.6) the sequence (u_h) is bounded in $H_0^1(\Omega)$, so we may assume that (u_h) converges weakly in $H_0^1(\Omega)$ to some function $\tilde{u}$. By the comparison principle (Theorem 2.5) we have $|u_h| \le c\, w_h$ q.e. in Ω, with $c = \|f\|_{L^\infty(\Omega)}$. Taking the

limit as $h \to \infty$ we get $|\tilde{u}| \le c\,w$ q.e. in Ω, and hence $\tilde{u} \in L^\infty(\Omega)$. Moreover, taking the limit in (4.5) we obtain that $\tilde{u}$ satisfies (3.14), and so $\tilde{u} = u$ by Lemma 3.5. Therefore (μ_h) γ^A-convereges to μ by Remark 4.2.
$(c) \Rightarrow (a)$. It is enough to take $f = 1$ in the definition of γ^A-convergence.
$(a) \Rightarrow (d)$. It is enough to replace A by A^* in the proof of $(b) \Rightarrow (c)$.
$(d) \Rightarrow (b)$. It is enough to take $f = 1$ in the definition of γ^{A^*}-convergence. $\square$

Remark 4.4. The uniqueness of the γ^A-limit is an easy consequence of Theorem 4.3, which implies that, if (μ_h) γ^A-converges to λ and μ, then w satisfies (3.5) and (3.6), so that $\lambda = \mu$ by Lemma 3.3.

The following theorem proves the compactness of $\mathcal{M}_0(\Omega)$ with respect to γ^A-convergence.

Theorem 4.5. *Every sequence of measures of $\mathcal{M}_0(\Omega)$ contains a γ^A-convergent subsequence.*

Proof. Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ and, for every $h \in \mathbf{N}$, let $w_h \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$ be the solution of problem (4.3). By Proposition 3.4 we have $w_h \in \mathcal{K}(\Omega)$. Since $\mathcal{K}(\Omega)$ is compact in the weak topology of $H^1_0(\Omega)$, a subsequence of (w_h) converges weakly in $H^1_0(\Omega)$ to some function $w \in \mathcal{K}(\Omega)$. By Proposition 3.4 there exists a measure $\mu \in \mathcal{M}_0(\Omega)$ such that w is a solution in $H^1_0(\Omega) \cap L^2_\mu(\Omega)$ of problem (4.4). The conclusion follows now from Theorem 4.3. $\square$

The case of Dirichlet problems in perforated domains is considered in the following theorem.

Theorem 4.6. *Let (Ω_h) be an arbitrary sequence of open subsets of Ω. Then there exist a subsequence, still denoted by (Ω_h), and a measure $\mu \in \mathcal{M}_0(\Omega)$ such that, for every $f \in H^{-1}(\Omega)$, the solutions $u_h \in H^1_0(\Omega_h)$ of the equations $Au_h = f$ in Ω_h, extended to 0 on $\Omega \backslash \Omega_h$, converge weakly in $H^1_0(\Omega)$ to the unique solution $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ of problem (4.2).*

Proof. The conclusion follows easily from the compactness theorem (Theorem 4.5) and from the fact that each function u_h can be regarded as the solution of problem (4.1) with $\mu_h = \infty_{\Omega \backslash \Omega_h}$ (Remark 2.3). $\square$

Using Theorem 4.3 we can prove the following density result in $\mathcal{M}_0(\Omega)$. We shall see in Corollary 5.8 that the strong converegence in $H^1_0(\Omega)$ of the sequence (w_h) implies the strong converegence in $H^1_0(\Omega)$ of the sequence (u_h) of the solutions of (4.1) for every $f \in H^{-1}(\Omega)$.

Proposition 4.7. *Every measure $\mu \in \mathcal{M}_0(\Omega)$ is the γ^A-limit of a sequence (μ_h) of Radon measures of $\mathcal{M}_0(\Omega)$ such that the solutions w_h of (4.3) converge strongly in $H^1_0(\Omega)$ to the solution w of (4.4).*

Proof. By (3.8) a measure $\mu \in \mathcal{M}_0(\Omega)$ is a Radon measure if the solution w of (4.4) satisfies

$$\inf_K w > 0 \qquad \text{for every compact set } K \subseteq \Omega\,. \tag{4.6}$$

Now let $w_0 \in H^1_0(\Omega)$ be the solution of the equation $Aw_0 = 1$ in Ω. By the strong maximum principle (see [48]) we have that w_0 satisfies (4.6).

Let us fix $\mu \in \mathcal{M}_0(\Omega)$ and let $w \in \mathcal{K}(\Omega)$ be the solution of (4.4). For every $h \in \mathbf{N}$ let us define $w_h = (1 - \frac{1}{h})w + \frac{1}{h}w_0$. It is easy to see that w_h is a positive subsolution of the equation $Au = 1$, hence $w_h \in \mathcal{K}(\Omega)$. Moreover the functions w_h satisfy (4.6) and converge to w strongly in $H^1_0(\Omega)$. Therefore the measures $\mu_h \in \mathcal{M}_0(\Omega)$ associated with w_h by Proposition 3.4 are Radon measures and γ^A-converge to μ by Theorem 4.3. $\square$

The following proposition deals with the case where also f varies.

Proposition 4.8. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging to a measure $\mu \in \mathcal{M}_0(\Omega)$. Let (f_h) be a sequence in $H^{-1}(\Omega)$ converging strongly to $f \in H^{-1}(\Omega)$. For every $h \in \mathbf{N}$ let $v_h \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$ be the solution of the problem*

$$\langle Av_h, v\rangle + \int_\Omega v_h v \, d\mu_h = \langle f_h, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$$

and let $u \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ be the solution of problem (4.2). Then (v_h) converges to u weakly in $H^1_0(\Omega)$.

Proof. For every $h \in \mathbf{N}$, let u_h be the solution in $H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$ of problem (4.1). By the estimate (2.6) and by the linearity of the problem the sequence $(v_h - u_h)$ converges to 0 strongly in $H^1_0(\Omega)$. Moreover, by the definition of γ^A-convergence, (u_h) converges to u weakly in $H^1_0(\Omega)$. Therefore (v_h) converges to u weakly in $H^1_0(\Omega)$. $\square$

The following results (Theorem 4.9, Theorem 4.10, Corollary 4.11) show the local character of the γ^A-convergence. Let ω be an open subset of Ω. With a little abuse of notation we still denote by A the operator defined by (2.3) on $H^1(\omega)$, and by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\omega)$ and $H^1_0(\omega)$.

Theorem 4.9. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging in Ω to a measure $\mu \in \mathcal{M}_0(\Omega)$. Let ω be an open subset of Ω, let (f_h) be a sequence in $H^{-1}(\omega)$ converging to f strongly in $H^{-1}(\omega)$, and let (u_h) be a sequence in $H^1(\omega)$ converging to u weakly in $H^1(\omega)$. Suppose that $u_h \in L^2_{\mu_h}(\omega')$ for every $\omega' \subset\subset \omega$ and that*

$$\langle Au_h, v\rangle + \int_\omega u_h v \, d\mu_h \; = \; \langle f_h, v\rangle \tag{4.7}$$

for every $v \in H^1_0(\omega) \cap L^2_{\mu_h}(\omega)$ with $\operatorname{supp}(v) \subset\subset \omega$. Then $u \in L^2_\mu(\omega')$ for every $\omega' \subset\subset \omega$ and

$$\langle Au, v\rangle + \int_\omega uv \, d\mu \; = \; \langle f, v\rangle \tag{4.8}$$

for every $v \in H^1_0(\omega) \cap L^2_\mu(\omega)$ with $\operatorname{supp}(v) \subset\subset \omega$.

Proof. Let $\varphi \in C^\infty_0(\omega)$ and let $z_h = \varphi u_h$. Since for every $v \in H^1_0(\Omega)$ we have

$$\begin{aligned} \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j z_h D_i v\big)\, dx \; &= \; \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j \varphi D_i v\big) u_h \, dx \, + \\ + \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i v\big)\varphi \, dx \; &= \; \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j \varphi D_i v\big) u_h \, dx \, + \\ + \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i (v\varphi)\big)\, dx \; &- \; \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i \varphi\big) v \, dx \,, \end{aligned}$$

from (4.7) we obtain

$$\langle Az_h, v\rangle + \int_\Omega z_h v \, d\mu_h \; = \; \langle g_h, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)\,,$$

where

$$\langle g_h, v\rangle \; = \; \langle f_h, v\varphi\rangle \; + \; \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j \varphi D_i v\big) u_h \, dx \; - \; \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i \varphi\big) v \, dx$$

for every $v \in H^1_0(\Omega)$. Since (u_h) converges to u weakly in $H^1_0(\omega)$, (f_h) converges to f strongly in $H^{-1}(\omega)$, and φ has compact support in ω, it follows that (g_h) converges strongly in $H^{-1}(\Omega)$ to the functional $g \in H^{-1}(\Omega)$ defined by

$$\langle g, v\rangle \; = \; \langle f, v\varphi\rangle \; + \; \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j \varphi D_i v\big) u \, dx \; - \; \int_\Omega \big(\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi\big) v \, dx$$

for every $v \in H^1_0(\Omega)$. As (μ_h) γ^A-converges to μ and (z_h) converges to $z = \varphi u$ weakly in $H^1_0(\Omega)$, by Proposition 4.8 the function $z = \varphi u$ is the solution in $H^1_0(\Omega) \cap L^2_\mu(\Omega)$ of the problem

$$\langle Az, v\rangle + \int_\Omega zv \, d\mu \; = \; \langle g, v\rangle \qquad \forall v \in H^1_0(\Omega) \cap L^2_\mu(\Omega)\,. \tag{4.9}$$

Let us fix an open set ω' and a function $v \in H^1_0(\omega) \cap L^2_\mu(\omega)$ with $\operatorname{supp}(v) \subset\subset \omega' \subset\subset \omega$. If we choose $\varphi \in C^\infty_0(\omega)$ such that $\varphi = 1$ in ω', then $u = z$ q.e. in ω', hence $u \in L^2_\mu(\omega')$ and (4.9) implies (4.8). $\square$

Theorem 4.10. *Let (μ_h) a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging in Ω to a measure $\mu \in \mathcal{M}_0(\Omega)$, and let ω be an open subset of Ω. Then (μ_h) γ^A-converges to μ in ω.*

Proof. Let us fix $f \in H^{-1}(\omega)$. For every $h \in \mathbf{N}$ let u_h be the solution in $H_0^1(\omega) \cap L^2_{\mu_h}(\omega)$ of problem (4.1), with Ω replaced by ω. By the estimate (2.6) we know that a subsequence, still denoted by (u_h), converges weakly in $H_0^1(\omega)$ to a function $u \in H_0^1(\omega)$. Then, by Theorem 4.9, $u \in L^2_\mu(\omega')$ for every open set $\omega' \subset\subset \omega$ and u is a solution of problem (4.8).

It remains to prove that $u \in L^2_\mu(\omega)$. Since $u \in H_0^1(\omega)$ and $u \in L^2_\mu(\omega')$ for every open set $\omega' \subset\subset \omega$, there exists a sequence (v_h) in $H_0^1(\omega) \cap L^2_\mu(\omega)$, converging to u weakly in $H_0^1(\omega)$, with $\mathrm{supp}(v_h) \subset\subset \omega$ and $uv_h \geq 0$ q.e. in ω, such that the sequence (uv_h) is increasing and converges to u^2 pointwise q.e. in ω. Taking $v = v_h$ in (4.8) we get

$$\langle Au, v_h\rangle + \int_\omega uv_h \, d\mu \;=\; \langle f, v_h\rangle\,.$$

Taking the limit as $h \to \infty$ we obtain $\int_\omega u^2 d\mu = \langle f, u\rangle - \langle Au, u\rangle < +\infty$, and thus $u \in L^2_\mu(\omega)$. By an easy approximation argument we can prove that u is the unique solution in $H_0^1(\omega) \cap L^2_\mu(\omega)$ of the problem

$$\langle Au, v\rangle \;+ \int_\omega uv \, d\mu \;=\; \langle f, v\rangle \qquad \forall v \in H_0^1(\omega) \cap L^2_\mu(\omega)\,.$$

Since the limit does not depend on the subsequence, the proof is complete. $\square$

Corollary 4.11. *Let μ_h, $\mu \in \mathcal{M}_0(\Omega)$. Let $(\Omega_i)_{i \in I}$ be a family of open subsets of Ω which covers Ω. Then (μ_h) γ^A-converges to μ in Ω if and only if (μ_h) γ^A-converges to μ in Ω_i for every $i \in I$.*

Proof. The conclusion follows easily from the compactness theorem (Theorem 4.5) and from Theorem 4.10. $\square$

5. Strong Convergence

Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging to a measure $\mu \in \mathcal{M}_0(\Omega)$. Let $f \in H^{-1}(\Omega)$ and let u_h and u be the solutions of problems (4.1) and (4.2). By the definition of γ^A-convergence the sequence (u_h) converges to u weakly in $H_0^1(\Omega)$. In this section we study the strong convergence of the sequence of the gradients (Du_h) in the space $L^p(\Omega, \mathbf{R}^n)$, $1 \leq p \leq 2$. The following theorem proves that (Du_h) converges strongly to Du in $L^p(\Omega, \mathbf{R}^n)$ for every $1 \leq p < 2$.

Theorem 5.1. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging to a measure $\mu \in \mathcal{M}_0(\Omega)$. Let $f \in H^{-1}(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of problems (4.1) and (4.2). Then (u_h) converges to u strongly in $H_0^{1,p}(\Omega)$ for every $1 \le p < 2$.*

Proof. Since A is linear and the solutions of (4.1) depend continuously on the data, uniformly with respect to h (see the estimate (2.6)), it is not restrictive to suppose that $f \in L^\infty(\Omega)$ and $f \ge 0$.

By the definition of γ^A-convergence the sequence (u_h) converges to u weakly in $H_0^1(\Omega)$, and hence (Au_h) converges to Au weakly in $H^{-1}(\Omega)$. By Proposition 2.6 we have $Au_h \le f$, and so $f - Au_h \in H_+^{-1}(\Omega)$, the positive cone of $H^{-1}(\Omega)$. Since $H_+^{-1}(\Omega)$ is compactly imbedded in $H^{-1,p}(\Omega)$ for every $1 \le p < 2$ (see [37]), the sequence (Au_h) converges to Au strongly in $H^{-1,p}(\Omega)$ for every $1 \le p < 2$.

If we apply Meyers' estimate (see [36]) to the operator A^*, we find that there exists a real number $s > 2$ such that the operator $A^*: H_0^{1,q}(\Omega) \to H^{-1,q}(\Omega)$ is an isomorphism for every $2 \le q \le s$. Denote by r the exponent conjugate to s, i.e., $1/r + 1/s = 1$. Then $A: H_0^{1,p}(\Omega) \to H^{-1,p}(\Omega)$ is an isomorphism for every $r \le p \le 2$. Since (Au_h) converges to Au strongly in $H^{-1,p}(\Omega)$ for every $r \le p < 2$, the sequence (u_h) converges to u strongly in $H_0^{1,p}(\Omega)$ for every $r \le p < 2$, and hence for every $1 \le p < 2$. $\square$

Let $f \in L^\infty(\Omega)$ and let u_h and u be the solutions of problems (4.1) and (4.2). By Theorem 5.1 the sequence (Du_h) converges to Du weakly in $L^2(\Omega, \mathbf{R}^n)$ and strongly in $L^p(\Omega, \mathbf{R}^n)$ for every $1 \le p < 2$. To obtain strong convergence in $L^2(\Omega, \mathbf{R}^n)$ we need a corrector term. This is a sequence of Borel functions P_h from $\Omega \times \mathbf{R}$ to $\mathbf{R}^n$, depending on the sequence (μ_h), but independent of f, u, u_h, such that

$$Du_h(x) = Du(x) + P_h(x, u(x)) + R_h(x) \qquad \text{a.e. in } \Omega, \tag{5.1}$$

where (R_h) tends to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$. This condition means that the oscillations of the sequence of the gradients (Du_h) near a point $x \in \Omega$ are determined, up to a term which is small in $L^2(\Omega, \mathbf{R}^n)$, only by the values of the limit function u near x and by the correctors P_h, which depend only on the sequence (μ_h).

Let $w_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $w \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of problems (4.3) and (4.4). The functions $P_h: \Omega \times \mathbf{R} \to \mathbf{R}^n$ are defined by

$$P_h(x, s) = \begin{cases} \dfrac{s}{w(x)} \big(Dw_h(x) - Dw(x) \big), & \text{if } w(x) > 0, \\ 0, & \text{if } w(x) = 0. \end{cases} \tag{5.2}$$

We are now in a position to state the main theorem of this section.

Theorem 5.2. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging to a measure $\mu \in \mathcal{M}_0(\Omega)$, and let (P_h) be the sequence defined by (5.2). Let $f \in L^\infty(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of problems (4.1) and (4.2). Then (5.1) holds, with (R_h) converging to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$.*

Remark 5.3. Let w_0 be the unique function of $H_0^1(\Omega)$ such that $Aw_0 = 1$ in Ω. By the comparison principle (Proposition 2.5) we have $|u_h| \le c\,w_h \le c\,w_0$ and $|u| \le c\,w \le c\,w_0$ q.e. in Ω, with $c = \|f\|_{L^\infty(\Omega)}$. As $w_0 \in L^\infty(\Omega)$ (see [48]), the functions u and w belong to $L^\infty(\Omega)$, and the sequences (u_h) and (w_h) are bounded in $L^\infty(\Omega)$.

To prove Theorem 5.2 we need the following lemmas. For every $\varepsilon > 0$ we set $\Omega_\varepsilon = \{w > \varepsilon\}$.

Lemma 5.4. *Assume that all hypotheses of Theorem 5.2 are satisfied. Let $\varepsilon > 0$ and, for every $h \in \mathbf{N}$, let*

$$r_h^\varepsilon = u_h - \frac{u w_h}{w \vee \varepsilon},$$

where $w_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $w \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ are the solutions of problems (4.3) and (4.4). Then $r_h^\varepsilon \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and (Dr_h^ε) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbf{R}^n)$.

Proof. Since the functions u and $\frac{1}{w\vee\varepsilon}$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, and, in addition, the sequences (u_h) and (w_h) are bounded in $L^\infty(\Omega)$ (Remark 5.3) and converge to u and w weakly in $H_0^1(\Omega)$ (Definition 4.1), we conclude that $r_h^\varepsilon \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and that (r_h^ε) converges to $u - \frac{uw}{w\vee\varepsilon}$ weakly in $H_0^1(\Omega)$. As $u - \frac{uw}{w\vee\varepsilon} = 0$ a.e. in Ω_ε, we obtain that (r_h^ε) converges to 0 strongly in $L^2(\Omega_\varepsilon)$ and (Dr_h^ε) converges to 0 weakly in $L^2(\Omega_\varepsilon, \mathbf{R}^n)$. Let us fix a function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $0 \le \varphi \le 1$ q.e. in Ω, $\varphi = 1$ q.e. in $\Omega_{2\varepsilon}$, and $\varphi = 0$ q.e. in $\Omega \setminus \Omega_\varepsilon$. For instance, we can take $\varphi(x) = \Phi_\varepsilon(w(x))$, where $\Phi_\varepsilon\colon \mathbf{R} \to \mathbf{R}$ is the Lipschitz function defined by $\Phi_\varepsilon(t) = 0$ for $t \le \varepsilon$, $\Phi_\varepsilon(t) = \frac{t}{\varepsilon} - 1$ for $\varepsilon \le t \le 2\varepsilon$, $\Phi_\varepsilon(t) = 1$ for $t \ge 2\varepsilon$. To conclude the proof it is enough to show that

$$\lim_{h\to\infty} \int_\Omega |Dr_h^\varepsilon|^2 \varphi\,dx = 0. \tag{5.3}$$

By the ellipticity condition (2.4) we have

$$\alpha \int_\Omega |Dr_h^\varepsilon|^2 \varphi\,dx + \int_\Omega (r_h^\varepsilon)^2 \varphi\,d\mu_h \le \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j r_h^\varepsilon D_i r_h^\varepsilon\Big)\varphi\,dx + \int_\Omega (r_h^\varepsilon)^2 \varphi\,d\mu_h =$$

$$= \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i r_h^\varepsilon\Big)\varphi\,dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j w_h D_i r_h^\varepsilon\Big)\frac{u\varphi}{w\vee\varepsilon}\,dx -$$

$$- \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j \big(\frac{u}{w\vee\varepsilon}\big) D_i r_h^\varepsilon\Big) w_h \varphi \, dx \; + \int_\Omega u_h r_h^\varepsilon \varphi \, d\mu_h \; - \int_\Omega \frac{u w_h}{w\vee\varepsilon} r_h^\varepsilon \varphi \, d\mu_h \; =$$

$$= \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i (r_h^\varepsilon \varphi)\Big) \, dx + \int_\Omega u_h r_h^\varepsilon \varphi d\mu_h - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j w_h D_i \big(\frac{u r_h^\varepsilon \varphi}{w\vee\varepsilon}\big)\Big) \, dx -$$

$$- \int_\Omega w_h \frac{u r_h^\varepsilon \varphi}{w\vee\varepsilon} \, d\mu_h \; - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i \varphi\Big) r_h^\varepsilon \, dx +$$

$$+ \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j w_h D_i \big(\frac{u\varphi}{w\vee\varepsilon}\big)\Big) r_h^\varepsilon \, dx \; - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j \big(\frac{u}{w\vee\varepsilon}\big) D_i r_h^\varepsilon\Big) w_h \varphi \, dx \, .$$

By (4.1) and (4.3) we obtain

$$\alpha \int_\Omega |D r_h^\varepsilon|^2 \varphi \, dx \; + \int_\Omega (r_h^\varepsilon)^2 \varphi \, d\mu_h \; \leq \int_{\Omega_\varepsilon} f r_h^\varepsilon \varphi \, dx \; - \int_{\Omega_\varepsilon} \frac{u r_h^\varepsilon \varphi}{w\vee\varepsilon} \, dx -$$

$$- \int_{\Omega_\varepsilon} \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i \varphi\Big) r_h^\varepsilon \, dx \; + \int_{\Omega_\varepsilon} \Big(\sum_{i,j=1}^n a_{ij} D_j w_h D_i \big(\frac{u\varphi}{w\vee\varepsilon}\big)\Big) r_h^\varepsilon \, dx -$$

$$- \int_{\Omega_\varepsilon} \Big(\sum_{i,j=1}^n a_{ij} D_j \big(\frac{u}{w\vee\varepsilon}\big) D_i r_h^\varepsilon\Big) w_h \varphi \, dx \, .$$

Since all terms in the right hand side of the previous inequality tend to 0 as $h \to \infty$, (5.3) holds and the proof is complete. $\square$

Lemma 5.5. *Assume that all hypotheses of Theorem 5.2 are satisfied, and let $w \in H_0^1(\Omega) \cap L_\mu^2(\Omega)$ be the solution of problem (4.4). Then*

$$\lim_{\varepsilon\to 0} \limsup_{h\to\infty} \int_{\{w<\varepsilon\}} |D u_h|^2 dx \; = \; 0 \, . \tag{5.4}$$

Proof. For every $\varepsilon > 0$ let $\Phi^\varepsilon : \mathbf{R} \to \mathbf{R}$ be the Lipschitz function defined by $\Phi^\varepsilon(t) = 1$ for $t \leq \varepsilon$, $\Phi^\varepsilon(t) = 2 - \frac{t}{\varepsilon}$ for $\varepsilon \leq t \leq 2\varepsilon$, $\Phi^\varepsilon(t) = 0$ for $t \geq 2\varepsilon$, and let $w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega)$ be the function defined by $w^\varepsilon(x) = \Phi^\varepsilon(w(x))$. As $w^\varepsilon \geq 0$ q.e. in Ω and $w^\varepsilon = 1$ q.e. in $\{w < \varepsilon\}$, by the ellipticity condition (2.4) and by (4.1) we have

$$\alpha \int_{\{w<\varepsilon\}} |D u_h|^2 dx \; + \int_{\{w<\varepsilon\}} (u_h)^2 d\mu_h \; \leq \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i u_h\Big) w^\varepsilon \, dx +$$

$$+ \int_\Omega (u_h)^2 w^\varepsilon d\mu_h \; = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i (u_h w^\varepsilon)\Big) \, dx \; + \int_\Omega (u_h)^2 w^\varepsilon d\mu_h -$$

$$- \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i w^\varepsilon\Big) u_h \, dx \; = \int_\Omega f u_h w^\varepsilon dx \; - \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i w^\varepsilon\Big) u_h \, dx \, .$$

Since, by the definition of γ^A-convergence, (u_h) converges to u weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$, we can take the limit in the last two terms as $h \to \infty$. Therefore we obtain

$$\alpha \limsup_{h\to\infty} \int_{\{w<\varepsilon\}} |Du_h|^2 dx \;\le\; \int_\Omega f u w^\varepsilon dx \;-\; \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i w^\varepsilon\Big) u\, dx\,. \tag{5.5}$$

As (w^ε) is bounded in $L^\infty(\Omega)$ and converges pointwise to the characteristic function of $\{w=0\}$, we have that (uw^ε) converges to 0 strongly in $L^2(\Omega)$ as $\varepsilon \to 0$ (recall that $|u| \le c\,w$ q.e. in Ω by Remark 5.3). Moreover,

$$\int_\Omega |u|^2 |Dw^\varepsilon|^2 dx \;\le\; \frac{c^2}{\varepsilon^2} \int_{\{\varepsilon<w<2\varepsilon\}} w^2 |Dw|^2 dx \;\le\; 4c^2 \int_{\{\varepsilon<w<2\varepsilon\}} |Dw|^2 dx\,,$$

and thus (uDw^ε) converges to 0 strongly in $L^2(\Omega)$. Taking the limit in (5.5) as $\varepsilon \to 0$ we obtain (5.4). □

Proof of Theorem 5.2. Let us fix $\varepsilon > 0$, let $r^\varepsilon_h = u_h - \frac{uw_h}{w\vee\varepsilon}$ as in Lemma 5.4, and let $\Omega_{2\varepsilon} = \{w > 2\varepsilon\}$. Then $R_h = (\frac{w_h}{w}-1)Du - (\frac{w_h}{w}-1)\frac{u}{w}Dw + Dr^\varepsilon_h$ a.e. in $\Omega_{2\varepsilon}$. Since (Dr^ε_h) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbf{R}^n)$ (Lemma 5.4) and, in addition, $(\frac{w_h}{w})$ is bounded in $L^\infty(\Omega_{2\varepsilon})$ and converges to 1 strongly in $L^2(\Omega_{2\varepsilon})$, we conclude that (R_h) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbf{R}^n)$. As $\int_\Omega R_h^2 dx = \int_{\Omega_{2\varepsilon}} R_h^2 dx + \int_{\{w\le 2\varepsilon\}} R_h^2 dx$, it is enough to prove that

$$\lim_{\varepsilon\to 0} \limsup_{h\to\infty} \int_{\{w\le 2\varepsilon\}} R_h^2 dx \;=\; 0\,. \tag{5.6}$$

Since $|u| \le c\,w$ q.e. in Ω (Remark 5.3), we have $|R_h| \le |Du_h - Du| + c|Dw_h - Dw|$ a.e. in Ω. Therefore

$$\begin{aligned}\limsup_{h\to\infty} \int_{\{w\le 2\varepsilon\}} R_h^2 dx \;\le\; 4 \limsup_{h\to\infty} \int_{\{w\le 2\varepsilon\}} |Du_h|^2 dx \;+\; 4 \int_{\{w\le 2\varepsilon\}} |Du|^2 dx \;+\\ +\,4\,c^2 \limsup_{h\to\infty} \int_{\{w\le 2\varepsilon\}} |Dw_h|^2 dx \;+\; 4\,c^2 \int_{\{w\le 2\varepsilon\}} |Dw|^2 dx\end{aligned}$$

for every $\varepsilon > 0$. As $|u| \le c\,w$, we have $Du = Dw = 0$ a.e. in $\{w = 0\}$. Since Lemma 5.5 can be applied to the sequences (u_h) and (w_h), from the previous inequality we obtain (5.6), which concludes the proof of the theorem. □

Lemmas 5.4 and 5.5 enable us to prove the following corrector result in $H^1_0(\Omega)$.

Theorem 5.6. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging to a measure $\mu \in \mathcal{M}_0(\Omega)$, and let $w_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $w \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of problems (4.3) and (4.4). Let $f \in L^\infty(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of problems (4.1) and (4.2). Then for every $\varepsilon > 0$ we have*

$$u_h = \frac{uw_h}{w \vee \varepsilon} + r_h^\varepsilon,$$

with $\lim\limits_{\varepsilon\to 0} \limsup\limits_{h\to\infty} \|r_h^\varepsilon\|_{H_0^1(\Omega)} = 0$.

Proof. Setting $\Omega_{2\varepsilon} = \{w > 2\varepsilon\}$, we have

$$\int_\Omega |Dr_h^\varepsilon|^2 dx = \int_{\Omega_{2\varepsilon}} |Dr_h^\varepsilon|^2 dx + \int_{\{w\leq 2\varepsilon\}} |Dr_h^\varepsilon|^2 dx. \tag{5.7}$$

Since, by Lemma 5.4, (Dr_h^ε) converges to 0 strongly in $L^2(\Omega_{2\varepsilon}, \mathbf{R}^n)$ as $h \to \infty$, we have only to estimate the last term of (5.7). As

$$Dr_h^\varepsilon = Du_h - \frac{u}{w \vee \varepsilon} Dw_h - \frac{w_h}{w \vee \varepsilon} Du + \frac{uw_h}{(w \vee \varepsilon)^2} D(w \vee \varepsilon),$$

and $|u| \leq c\,w$ (Remark 5.3), we have

$$\frac{1}{4}|Dr_h^\varepsilon|^2 \leq |Du_h|^2 + c^2|Dw_h|^2 + \Big(\frac{w_h}{w \vee \varepsilon}\Big)^2|Du|^2 + c^2\Big(\frac{w_h}{w \vee \varepsilon}\Big)^2|Dw|^2.$$

Since (w_h) is bounded in $L^\infty(\Omega)$ and converges to w weakly in $H_0^1(\Omega)$, we obtain

$$\frac{1}{4}\limsup_{h\to\infty} \int_{\{w\leq 2\varepsilon\}} |Dr_h^\varepsilon|^2 dx \leq \limsup_{h\to\infty} \int_{\{w\leq 2\varepsilon\}} |Du_h|^2 dx +$$
$$+ c^2 \limsup_{h\to\infty} \int_{\{w\leq 2\varepsilon\}} |Dw_h|^2 dx + \int_{\{w\leq 2\varepsilon\}} |Du|^2 dx + c^2 \int_{\{w\leq 2\varepsilon\}} |Dw|^2 dx.$$

As $|u| \leq c\,w$, we have $Du = 0$ a.e. in $\{w = 0\}$, and so the last two terms tend to 0 as $\varepsilon \to 0$. The conclusion follows now from Lemma 5.5. □

The case $f \notin L^\infty(\Omega)$ requires a further approximation (see [9]).

Theorem 5.7. *Let (μ_h) be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^A-converging to a measure $\mu \in \mathcal{M}_0(\Omega)$, and let (P_h) be the sequence of correctors defined by (5.2). Let $f \in H^{-1}(\Omega)$ and let $u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of problems (4.1) and (4.2). Finally, let (f^λ) be a sequence in $L^\infty(\Omega)$ converging to f strongly in $H^{-1}(\Omega)$, and let $u^\lambda \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solutions of the problems*

$$\langle Au^\lambda, v\rangle + \int_\Omega u^\lambda v\, d\mu = \int_\Omega f^\lambda v\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega). \tag{5.8}$$

Then $Du_h(x) = Du(x) + P_h(x, u^\lambda(x)) + R_h^\lambda(x)$ *a.e. in* Ω, *with*

$$\lim_{\lambda\to\infty} \limsup_{h\to\infty} \int_\Omega (R_h^\lambda)^2 dx \;=\; 0\,. \tag{5.9}$$

Proof. For every λ and for every h let $u_h^\lambda \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ be the solution of the problem

$$\langle Au_h^\lambda, v\rangle \;+\; \int_\Omega u_h^\lambda v\, d\mu_h \;=\; \int_\Omega f^\lambda v\, dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)\,.$$

By Theorem 5.2 we have $Du_h^\lambda(x) = Du^\lambda(x) + P_h(x, u^\lambda(x)) + S_h^\lambda(x)$ a.e. in Ω, where (S_h^λ) converges to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$ for every λ. As $R_h^\lambda - S_h^\lambda = (Du_h - Du_h^\lambda) - (Du - Du^\lambda)$, from the estimate (2.6) we obtain

$$\|R_h^\lambda\|_{L^2(\Omega,\mathbf{R}^n)} \;\le\; \|S_h^\lambda\|_{L^2(\Omega,\mathbf{R}^n)} \;+\; \frac{2}{\alpha}\|f - f^\lambda\|_{H^{-1}(\Omega)}\,,$$

which implies (5.9). □

Corollary 5.8. *Let* (μ_h) *be a sequence of measures of* $\mathcal{M}_0(\Omega)$ γ^A*-converging to a measure* $\mu \in \mathcal{M}_0(\Omega)$, *and let* $w_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ *and* $w \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ *be the solutions of problems (4.3) and (4.4). Let* $f \in H^{-1}(\Omega)$ *and let* $u_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ *and* $u \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ *be the solutions of problems (4.1) and (4.2). If* (w_h) *converges strongly in* $H_0^1(\Omega)$, *then* (u_h) *converges strongly in* $H_0^1(\Omega)$.

Proof. Let (f^λ) be a sequence in $L^\infty(\Omega)$ converging to f strongly in $H^{-1}(\Omega)$, and, for every λ, let $u^\lambda \in H_0^1(\Omega) \cap L^2_\mu(\Omega)$ be the solution of problem (5.8). By Remark 5.3 each function u^λ/w is bounded on $\{w > 0\}$. Therefore, if (w_h) converges strongly in $H_0^1(\Omega)$, then $\big(P_h(x, u^\lambda(x))\big)$ converges to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$ for every λ, and so the conclusion follows from Theorem 5.7. □

6. The Rôle of the Skew-Symmetric Part of the Operator

Let (a_{ij}^s) and (b_{ij}) be the symmetric and the skew-symmetric part of the matrix (a_{ij}), and let A^s be the operator associated with the matrix (a_{ij}^s) according to (2.3). In this section we shall study the dependence of the γ^A-limit of a sequence (μ_h) on the skew-symmetric part (b_{ij}) of the matrix (a_{ij}). We begin by proving that, if the functions b_{ij} are continuous, then the γ^A-limit depends only on the symmetric part a_{ij}^s.

Theorem 6.1. *Let μ, $\mu_h \in \mathcal{M}_0(\Omega)$. If the functions b_{ij}, $i,j = 1,\dots,n$, are continuous, then (μ_h) γ^A-converges to μ if and only if (μ_h) γ^{A^s}-converges to μ.*

Proof. Since the γ^A-convergence and the γ^{A^s}-convergence are compact (Theorem 4.5), we may assume that (μ_h) γ^{A^s}-converges to a measure μ, and we have only to prove that (μ_h) γ^A-converges to μ.

Suppose that $b_{ij} \in C^1(\Omega)$ for every $i,j = 1,\dots,n$. Then, for every pair of functions u, $v \in H_0^1(\Omega) \cap H^2(\Omega)$, we have

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega \big(\sum_{i,j=1}^n a_{ij}^s D_j u D_i v\big)\,dx + \int_\Omega \big(\sum_{i,j=1}^n b_{ij} D_j u D_i v\big)\,dx = \\ &= \langle A^s u, v\rangle - \int_\Omega \big(\sum_{i,j=1}^n D_i(b_{ij} D_j u)\big) v\,dx \\ &= \langle A^s u, v\rangle - \int_\Omega \big(\sum_{i,j=1}^n D_i b_{ij} D_j u\big) v\,dx\,, \end{aligned} \tag{6.1}$$

where, in the last equality, we have used the fact that (b_{ij}) is skew-symmetric, while $(D_i D_j u)$ is symmetric. By continuity, the same equality holds for every u, $v \in H_0^1(\Omega)$. Therefore the solution $w_h \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)$ of problem (4.3) satisfies

$$\langle A^s w_h, v\rangle + \int_\Omega w_h v\,d\mu_h = \langle f_h, v\rangle \qquad \forall v \in H_0^1(\Omega) \cap L^2_{\mu_h}(\Omega)\,,$$

with

$$f_h = 1 + \sum_{i,j=1}^n D_i b_{ij} D_j w_h\,.$$

By the estimate (2.6) the sequence (w_h) is bounded in $H_0^1(\Omega)$. Passing, if necessary, to a subsequence, we may assume that (w_h) converges weakly in $H_0^1(\Omega)$ to a function w. This implies that (f_h) converges to

$$f = 1 + \sum_{i,j=1}^n D_i b_{ij} D_j w$$

weakly in $L^2(\Omega)$, and hence strongly in $H^{-1}(\Omega)$. Since (μ_h) γ^{A^s}-converges to μ, by Proposition 4.8 the function w is the solution in $H_0^1(\Omega) \cap L^2_\mu(\Omega)$ of the problem

$$\langle A^s w, v\rangle + \int_\Omega wv\,d\mu = \int_\Omega \big(1 + \sum_{i,j=1}^n D_i b_{ij} D_j w\big) v\,dx \qquad \forall v \in H_0^1(\Omega) \cap L^2_\mu(\Omega)\,.$$

By (6.1) w turns out to be the solution in $H_0^1(\Omega) \cap L^2_\mu(\Omega)$ of (4.4), and this implies that (μ_h) γ^A-converges to μ by Theorem 4.3. Since the limit does not depend on the subsequence, the whole sequence (μ_h) γ^A-converges to μ.

Let us consider now the more general hypothesis $b_{ij} \in C^0(\Omega)$. Let (b^ε_{ij}) be a sequence of skew-symmetric matrices of class C^1 converging uniformly to (b_{ij}) as $\varepsilon \to 0$. Let $a^\varepsilon_{ij} = a^s_{ij} + b^\varepsilon_{ij}$ and let A_ε be the corresponding elliptic operators on $H^1(\Omega)$. By the first step of the proof (μ_h) γ^{A_ε}-converges to μ. Therefore, if $w^\varepsilon_h \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$ and $w^\varepsilon \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ are the solutions of the problems

$$\begin{aligned} \langle A_\varepsilon w^\varepsilon_h, v\rangle + \int_\Omega w^\varepsilon_h v \, d\mu_h &= \int_\Omega v \, dx \qquad \forall v \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)\,, \\ \langle A_\varepsilon w^\varepsilon, v\rangle + \int_\Omega w^\varepsilon v \, d\mu &= \int_\Omega v \, dx \qquad \forall v \in H^1_0(\Omega) \cap L^2_{\mu}(\Omega)\,, \end{aligned}$$

then (w^ε_h) converges to w^ε weakly in $H^1_0(\Omega)$ for every $\varepsilon > 0$.

Let us prove that the solutions $w_h \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$ of (4.3) converge weakly in $H^1_0(\Omega)$ to the solution $w \in H^1_0(\Omega) \cap L^2_\mu(\Omega)$ of (4.4). For every $\varepsilon > 0$ we have

$$\|w_h - w\|_{L^2(\Omega)} \le \|w_h - w^\varepsilon_h\|_{L^2(\Omega)} + \|w^\varepsilon_h - w^\varepsilon\|_{L^2(\Omega)} + \|w^\varepsilon - w\|_{L^2(\Omega)}\,. \tag{6.2}$$

We already proved that the second term of the right hand side tends to 0 as $h \to \infty$. Let us estimate the first term. If we choose $w^\varepsilon_h - w_h$ as test functions in the problems solved by w^ε_h and w_h, we obtain

$$\begin{aligned} \langle A_\varepsilon w^\varepsilon_h, w^\varepsilon_h - w_h\rangle + \int_\Omega w^\varepsilon_h (w^\varepsilon_h - w_h)\, d\mu_h &= \int_\Omega (w^\varepsilon_h - w_h)\, dx\,, \\ \langle A w_h, w^\varepsilon_h - w_h\rangle + \int_\Omega w_h (w^\varepsilon_h - w_h)\, d\mu_h &= \int_\Omega (w^\varepsilon_h - w_h)\, dx\,. \end{aligned}$$

By subtracting the second equation from the first one we get

$$\langle A_\varepsilon (w^\varepsilon_h - w_h), w^\varepsilon_h - w_h\rangle + \int_\Omega \sum_{i,j=1}^n (b^\varepsilon_{ij} - b_{ij}) D_j w_h D_i (w^\varepsilon_h - w_h)\, dx + \int_\Omega (w^\varepsilon_h - w_h)^2 \, d\mu_h = 0\,.$$

Then, using the ellipticity assumption (2.4)(that depends only on the symmetric part of the matrix) and the Hölder inequality, we obtain

$$\begin{aligned} \|w^\varepsilon_h - w_h\|^2_{H^1_0(\Omega)} &\le \frac{1}{\alpha} \langle A_\varepsilon (w^\varepsilon_h - w_h), w^\varepsilon_h - w_h\rangle \le \\ &\le \frac{1}{\alpha} \int_\Omega \Big| \sum_{i,j=1}^n (b^\varepsilon_{ij} - b_{ij}) D_j w_h D_i (w^\varepsilon_h - w_h)\Big|\, dx \le \\ &\le \frac{1}{\alpha} \sum_{i,j=1}^n \|b^\varepsilon_{ij} - b_{ij}\|_{L^\infty(\Omega)} \|w_h\|_{H^1_0(\Omega)} \|w^\varepsilon_h - w_h\|_{H^1_0(\Omega)}\,. \end{aligned}$$

Since (b^ε_{ij}) converges uniformly to b_{ij} as $\varepsilon \to 0$, and (w_h) is bounded in $H^1_0(\Omega)$, it follows that $\|w^\varepsilon_h - w_h\|_{H^1_0(\Omega)}$ tends to 0, as $\varepsilon \to 0$, uniformly with respect to h. To prove that $\|w^\varepsilon - w\|_{H^1_0(\Omega)}$ tends to zero we can use the same arguments.

Therefore (6.2) shows that (w_h) converges to w strongly in $L^2(\Omega)$. As (w_h) is bounded in $H^1_0(\Omega)$, we obtain that (w_h) converges to w weakly in $H^1_0(\Omega)$, and, by Theorem 4.3, we conclude that (μ_h) γ^A-converges to μ. $\square$

In the rest of this section we prepare the technical tools for a counterexample (Theorem 6.4) which shows that, if the coefficients of the skew-symmetric part (b_{ij}) of the matrix (a_{ij}) are not continuous, then the γ^A-limit of a sequence (μ_h) of measures of $\mathcal{M}_0(\Omega)$ may depend also on the skew-symmetric part of the matrix, i.e., the γ^A-limit may be different from the γ^{A^s}-limit.

Let us introduce some notion concerning the capacity relative to the (possibly non-symmetric) operator A associated with the matrix (a_{ij}). In particular we are interested in the definition and properties of the capacity with respect the whole space $\mathbf{R}^n$.

In the rest of this section we assume $n \geq 3$. Let $H(\mathbf{R}^n)$ be the space of all functions belonging to $L^{2^*}(\mathbf{R}^n)$, $1/2^* = 1/2 - 1/n$, whose first order distribution derivatives belong to $L^2(\mathbf{R}^n)$. By the Sobolev inequality, it is easy to see that $H(\mathbf{R}^n)$ is a Hilbert space with norm $\|u\|_{H(\mathbf{R}^n)} = \|Du\|_{L^2(\mathbf{R}^n)}$. We assume now that (a_{ij}) is an $n \times n$ matrix of functions of $L^\infty(\mathbf{R}^n)$, satisfying the ellipticity condition (2.4) for a.e. $x \in \mathbf{R}^n$. With a little abuse of notation, A is now the elliptic operator defined by (2.3) for every $u \in H(\mathbf{R}^n)$. Let $a(u,v)$ be the bilinear form defined on $H(\mathbf{R}^n) \times H(\mathbf{R}^n)$ by

$$a(u,v) \;=\; \int_{\mathbf{R}^n} \Big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i v\Big)\, dx\,.$$

Let E be a bounded closed subset of $\mathbf{R}^n$ and let $K = \{v \in H(\mathbf{R}^n) : v \geq 1$ q.e. on $E\}$. By (2.4) we have that the form $a(u,v)$ is coercive on $H(\mathbf{R}^n)$ and hence there exists a unique solution z of the following variational inequality

$$z \in K\,, \qquad a(z, v - z) \geq 0 \quad \forall v \in K\,. \tag{6.3}$$

The *capacity of* E *with respect to* $\mathbf{R}^n$ (relative to the operator A) is defined by

$$\mathrm{cap}^A(E, \mathbf{R}^n) \;=\; a(z,z)\,. \tag{6.4}$$

The function z is called the *capacitary potential of* E *with respect to* $\mathbf{R}^n$.

Let us denote by B_R the closed ball of center 0 and radius R. The corresponding open ball will be denoted by U_R. Given $R_0 > 0$ such that $E \subseteq B_{R_0}$, for every $R > R_0$ we set $K_R = \{v \in H_0^1(U_R) : v \geq 1$ q.e. on $E\}$ and we consider the bilinear form on $H_0^1(U_R) \times H_0^1(U_R)$ defined by

$$a_R(u,v) \;=\; \int_{U_R} \Big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i v\Big)\, dx\,.$$

Then, for every $R > R_0$, there exists a unique solution of the variational inequality

$$z_R \in K_R\,, \qquad a_R(z_R, v - z_R) \geq 0 \quad \forall v \in K_R\,. \tag{6.5}$$

The function z_R is called the *capacitary potential of* E *with respect to* U_R and

$$\mathrm{cap}^A(E, U_R) \,=\, a_R(z_R, z_R) \tag{6.6}$$

is the *capacity of* E *with respect to* U_R (relative to the operator A). For the main properties of cap^A we refer to [48]. In particular, we shall use the following estimate of the capacity relative to the operator A in terms of the harmonic capacity defined in Section 2:

$$k_1 \mathrm{cap}(E, U_R) \,\le\, \mathrm{cap}^A(E, U_R) \,\le\, k_2 \mathrm{cap}(E, U_R)\,, \tag{6.7}$$

where k_1 and k_2 are two positive constants depending only on the ellipticity constant α and on the L^∞ norm of the coefficients a_{ij}.

Our counterexample is based on the following lemma.

Lemma 6.2. *Let* E *be a bounded closed subset of* $\mathbf{R}^n$. *Then*

$$\lim_{R\to\infty} \mathrm{cap}^A(E, U_R) \,=\, \mathrm{cap}^A(E, \mathbf{R}^n)\,, \tag{6.8}$$

and the capacitary potential z *on* $\mathbf{R}^n$ *is the unique solution of the problem*

$$z \in H(\mathbf{R}^n)\,, \qquad \sum_{i,j=1}^n D_i(a_{ij} D_j z) = 0 \quad \textit{in } \mathbf{R}^n \backslash E\,, \qquad z = 1 \textit{ q.e. in } E\,. \tag{6.9}$$

Proof. If z_R is the capacitary potential of E in U_R, we extend it to $\mathbf{R}^n$ by setting $z_R = 0$ in $\mathbf{R}^n \backslash U_R$. By the Sobolev imbedding theorem we have that $z_R \in H(\mathbf{R}^n)$. Using the coerciveness of A, the explicit formula for the harmonic capacity of a ball, and the inequality (6.7) we obtain

$$\|Dz_R\|^2_{L^2(\mathbf{R}^n)} \,\le\, \alpha^{-1} a(z_R, z_R) \,=\, \alpha^{-1}\mathrm{cap}^A(E, U_R) \,\le\, k_2 \alpha^{-1} \mathrm{cap}(B_{R_0}, U_R) \,\le\, C\,,$$

for every $R \ge R_0 + 1$. Thus we may assume, passing, if necessary, to a subsequence, that (z_R) converges weakly to a function $\zeta \in H(\mathbf{R}^n)$. By the lower semicontinuity of $a(v, v)$ and by (6.5), we have

$$\begin{aligned} a(\zeta, \zeta) \,\le\, \liminf_{R\to\infty} a(z_R, z_R) \,=\, \liminf_{R\to\infty} a_R(z_R, z_R) \,\le\, \limsup_{R\to\infty} a_R(z_R, z_R) \,\le \\ \le\, \lim_{R\to\infty} a_R(z_R, v) \,=\, \lim_{R\to\infty} a(z_R, v) \,=\, a(\zeta, v) \end{aligned} \tag{6.10}$$

for every $v \in H(\mathbf{R}^n)$ with compact support in $\mathbf{R}^n$ and with $v \ge 1$ q.e. on E. By a density argument we obtain that ζ is the solution of (6.3), and thus ζ coincides with the capacitary potential z of E in $\mathbf{R}^n$. Taking $v = \zeta = z$ in (6.10), we obtain (6.8).

The characterization of z given by (6.9) follows easily from standard techniques of variational inequalities (see [33], Chapter II). □

Let $\Omega^+ = \{x \in \mathbf{R}^n \,:\, x_n > 0\}$, let $\Omega^- = \{x \in \mathbf{R}^n \,:\, x_n < 0\}$, and let (β_{ij}) be the matrix defined by

$$\beta_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i > j, \\ -1, & \text{if } i < j. \end{cases}$$

To construct the counterexample we consider the matrix (a^0_{ij}) given by

$$a^0_{ij}(x) = \delta_{ij} + b^0_{ij}(x)\,, \tag{6.11}$$

where δ_{ij} is the Kronecker symbol, and $b^0_{ij}(x) = \beta_{ij}$, if $x_n > 0$, while $b^0_{ij}(x) = 0$, if $x_n \le 0$. Note that the skew-symmetric part (b^0_{ij}) of (a^0_{ij}) is discontinuous along the hyperplane $\Gamma = \{x \in \mathbf{R}^n : x_n = 0\}$. We denote by A_0 the elliptic operator associated with (a^0_{ij}).

The following lemma plays a crucial rôle in the counterexample. We recall that B_1 is the closed unit ball of $\mathbf{R}^n$, $n \ge 3$.

Lemma 6.3. *Let (a^0_{ij}) be the matrix defined by (6.11). Then*

$$\mathrm{cap}^{A_0}(B_1, \mathbf{R}^n) \ne \mathrm{cap}(B_1, \mathbf{R}^n)\,, \tag{6.12}$$

where $\mathrm{cap}(B_1, \mathbf{R}^n)$ is the capacity defined by (6.4) relative to the Laplace operator $-\Delta$.

As $A^s_0 = -\Delta$, the previous inequality means that the capacity relative to the operator A_0 is different from the capacity relative to its symmetric part A^s_0.

Proof of Lemma 6.3. Let z be the capacitary potential of B_1 in $\mathbf{R}^n$ relative to the operator A_0, defined as the unique solution of problem (6.3) with $E = B_1$. Let u be the harmonic capacitary potential of B_1 in $\mathbf{R}^n$, i.e., the solution of problem (6.3) corresponding to the Laplace operator $-\Delta$. It is well known that u is characterized as the unique minimum point of the problem

$$\min\{\|Dv\|^2_{L^2(\mathbf{R}^n)} : v \in H(\mathbf{R}^n)\,,\ v \ge 1 \text{ a.e. on } B_1\}\,. \tag{6.13}$$

Suppose, by contradiction, that $\mathrm{cap}^{A_0}(B_1, \mathbf{R}^n) = \mathrm{cap}(B_1, \mathbf{R}^n)$. Then $a_0(z, z) = \|Du\|^2_{L^2(\mathbf{R}^n)}$. Since $a_0(z, z) = \|Dz\|^2_{L^2(\mathbf{R}^n)}$, the function z is a minimum point for the problem (6.13) and hence $z = u$. Therefore, to prove (6.12) it is sufficient to show that $z \ne u$.

Let us define $\tilde\Omega = \mathbf{R}^n \backslash B_1$, $\tilde\Omega^+ = \Omega^+ \backslash B_1$, $\tilde\Omega^- = \Omega^- \backslash B_1$, and $\tilde\Gamma = \Gamma \backslash B_1$. By (6.9), for every $\varphi \in C^\infty_0(\tilde\Omega)$ we have

$$0 = \int_{\tilde\Omega^+} \Big(\sum_{i,j=1}^n a^0_{ij} D_j z D_i \varphi\Big)\, dx + \int_{\tilde\Omega^-} \Big(\sum_{i,j=1}^n a^0_{ij} D_j z D_i \varphi\Big)\, dx =$$

$$= -\int_{\tilde\Gamma} \Big(\sum_{j=1}^n (a^0_{nj} D_j z)^+\Big) \varphi\, d\sigma + \int_{\tilde\Gamma} \Big(\sum_{j=1}^n (a^0_{nj} D_j z)^-\Big) \varphi\, d\sigma - \int_{\mathbf{R}^n \backslash B_1} \varphi \Delta z\, dx\,, \tag{6.14}$$

where $(a^0_{nj}D_jz)^+$ and $(a^0_{nj}D_jz)^-$ denote the limits on Γ of $a^0_{nj}D_jz$ from Ω^+ and Ω^- respectively.

Suppose now, by contradiction, that $z=u$. Since, by (6.9), $\Delta u=0$ on $\mathbf{R}^n\backslash B_1$, by (6.14) we obtain that

$$\int_{\tilde\Gamma}\Big(\sum_{j=1}^n(a^0_{nj}D_ju)^+\Big)\varphi\,d\sigma\;=\;\int_{\tilde\Gamma}\Big(\sum_{j=1}^n(a^0_{nj}D_ju)^-\Big)\varphi\,d\sigma$$

for every $\varphi\in C^\infty_0(\tilde\Omega)$. As $\sum_j(a^0_{nj}D_ju)^+=D_nu+\sum_j\beta_{nj}D_ju$ and $\sum_j(a^0_{nj}D_ju)^-=D_nu$, we have

$$Du{\cdot}\nu=0\qquad\text{q.e. on }\tilde\Gamma\,,\tag{6.15}$$

with $\nu=(\beta_{n1},\beta_{n2},\ldots,\beta_{nn})=(1,1,\ldots,1,0)$. But, using (6.9) with $A=-\Delta$, we find that $u(x)=|x|^{2-n}$ for every $x\in\tilde\Omega$. In particular $Du(x)$ is different from 0 and is parallel to the vector x for every $x\in\tilde\Gamma$. Therefore, (6.15) implies that $x{\cdot}\nu=0$ for every $x\in\tilde\Gamma$, and so we have to conclude that ν is orthogonal to Γ, which is clearly false. This contradiction proves (6.12). □

Let $\Omega=\,]-1,1[^n$, $n\geq3$, and let $\Gamma=\{x\in\Omega\,:\;x_n=0\}$. To give the counterexample for every $h\in\mathbf{N}$ we consider on Γ the periodic lattice, with period $1/h$, composed of the points $x^i_h=i/h=(i_1/h,\ldots,i_{n-1}/h,0)$, with i in the set

$$I_h=\{i=(i_1,\ldots,i_{n-1},0)\,:\;i_j\in\mathbf{Z}\,,\;-h<i_j<h\text{ for }j=1,\ldots,n-1\}\,.$$

Let us fix a constant $\beta>0$. For every $i\in I_h$ let $B^i_{r_h}$ be the closed ball in $\mathbf{R}^n$ with center x^i_h and radius r_h such that

$$r_h^{n-2}h^{n-1}=\beta\,.\tag{6.16}$$

Finally let us define E_h as the union of all closed balls $B^i_{r_h}$ for $i\in I_h$.

We are now in a position to prove the following theorem, which shows that the γ^A-limit of a sequence of measures may depend also on the skew-symmetric part (b_{ij}) of the matrix (a_{ij}), when (b_{ij}) is discontinuous.

Theorem 6.4. *Let E_h be the sets constructed above, let $\mu_h=\infty_{E_h}$ be the measures of $\mathcal{M}_0(\Omega)$ defined by (2.1), let A_0 be the operator associated with the matrix (a^0_{ij}) defined by (6.11), and let μ_0 be the $(n-1)$-dimensional measure on $\Gamma=\{x_n=0\}$. Then (μ_h) γ^{A_0}-converges to $c\,\mu_0$, with $c=\beta\,\mathrm{cap}^{A_0}(B_1,\mathbf{R}^n)$, while (μ_h) $\gamma^{A^s_0}$-converges to $c_s\mu_0$, with $c_s=\beta\,\mathrm{cap}(B_1,\mathbf{R}^n)\neq c$.*

To prove the theorem, we shall use a general result, based on the method introduced in [16]. We recall that the *Kato space* $K^+_n(\Omega)$, $n\geq3$, is the set of all Radon measures μ on Ω such that

$$\lim_{r\to0^+}\,\sup_{x\in\Omega}\int_{\Omega\cap B_r(x)}|y-x|^{2-n}d\mu(y)\;=\;0\,.$$

In particular, the measure μ_0 considered in Theorem 6.4 belongs to $K_n^+(\Omega)$.

For every $i \in \mathbf{Z}^n$ let Q_h^i be the cube with center i/h and side $1/h$, i.e.,

$$Q_h^i = \{x \in \mathbf{R}^n \,:\, (2i_k - 1)/2h \leq x_k < (2i_k + 1)/2h \text{ for } k = 1, \ldots, n\}$$

and let J_h be the set of all indices i such that $Q_h^i \subseteq \Omega$.

Theorem 6.5. *Let $\mu \in K_n^+(\Omega)$. Let (c_h) be a sequence of positive real numbers converging to $c > 0$. For every $i \in J_h$ let A_h^i be the open ball with the same center as Q_h^i and radius $1/2h$, and let E_h^i be the closed ball with the same center such that*

$$\mathrm{cap}^A(E_h^i, A_h^i) = c_h \mu(Q_h^i)\,.$$

Define E_h as the union of all closed balls E_h^i for $i \in J_h$. Then the sequence of measures (∞_{E_h}) γ^A-converges to $c\,\mu$.

Proof. This result can be deduced from [16] and is proved in [21] assuming that A is symmetric and that $c_h = c$ for every h. This proof can be easily adapted to the general case. $\square$

Proof of Theorem 6.4. In order to apply Theorem 6.5, we consider the periodic lattice J_h on Ω. Note that $I_h = \{i \in J_h : i/h \in \Gamma\}$. For every $i \in I_h$ we set $E_h^i = B_{r_h}^i$, if $i \in I_h$, and $E_h^i = \emptyset$, if $i \in J_h \setminus I_h$. Now we apply Theorem 6.5 to the operator A_0 and to the measure μ_0.

Since $a_{ij}^0(\lambda x) = a_{ij}^0(x)$ for every $\lambda > 0$, for every $x \in \mathbf{R}^n$, and for every $i, j = 1, \ldots, n$, it is easy to see that

$$\lambda^{n-2}\mathrm{cap}^{A_0}(B_r, U_R) = \mathrm{cap}^{A_0}(B_{\lambda r}, U_{\lambda R})\,, \tag{6.17}$$

for every $0 < r < R$. Moreover, the capacity relative to A_0 is invariant with respect to translations parallel to the hyperplane $\{x_n = 0\}$. In particular, with notation from Theorem 6.5, $\mathrm{cap}^{A_0}(E_h^i, A_h^i) = \mathrm{cap}^{A_0}(B_{r_h}^i, A_h^i)$ does not depend on $i \in I_h$ and $\mathrm{cap}^{A_0}(E_h^i, A_h^i) = \mathrm{cap}^{A_0}(B_{r_h}, U_{1/2h})$ for every $i \in I_h$, where B_{r_h} and $U_{1/2h}$ denote the closed ball with center 0 and radius r_h and the open ball with center 0 and radius $1/2h$.

As μ_0 is the $(n-1)$-dimensional measure on Γ, from (6.16) and (6.17) we obtain

$$\frac{\mathrm{cap}^{A_0}(E_h^i, A_h^i)}{\mu_0(Q_h^i)} = h^{n-1}\mathrm{cap}^{A_0}(B_{r_h}, U_{1/2h}) = \beta\,\mathrm{cap}^{A_0}(B_1, U_{1/2hr_h})$$

for every $i \in I_h$. Since $\mathrm{cap}^{A_0}(B_1, U_{1/2hr_h})$ tends to $\mathrm{cap}^{A_0}(B_1, \mathbf{R}^n)$ as $h \to \infty$ (Lemma 6.2), Theorem 6.5 implies that (∞_{E_h}) γ^{A_0}-converges to $c\,\mu_0$, where the constant c is given by $c = \beta\,\mathrm{cap}^{A_0}(B_1, \mathbf{R}^n)$. Moreover, if we apply Theorem 6.5 to the case of the operator $A_0^s = -\Delta$, we obtain that (∞_{E_h}) $\gamma^{A_0^s}$-converges to $c_s\mu_0$, with $c_s = \beta\,\mathrm{cap}(B_1, \mathbf{R}^n)$. The fact that $c_s \neq c$ follows from Lemma 6.3. $\square$

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