# HOMOGENIZATIONS OF $L^\infty$ FUNCTIONALS

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**Abstract**

We study, via Γ-convergence, the homogenization in $L^\infty$ of supremal functionals of the form $F_\varepsilon(u) = \operatorname{ess\,sup}_\Omega f\left(\frac{x}{\varepsilon}, Du\right)$. We prove that the homogenized problem is still supremal and its energy density is given by a cell-problem formula.

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# 1 Introduction

The last decade has witnessed a growing interest in variational principles whose energies are not integral and where the relevant quantities does not express a mean property, while the pointwise behaviour of the energy density is important also on very small sets.

An example is the problem of modeling the dielectric breakdown for a composite conductor. In [17] such a model has been derived; in particular, in the case of a composite material made of two homogeneous phases the variational principle that one has to consider is the following

$$\min \left\{ \operatorname*{ess\,sup}_{\Omega} |\lambda(x) Du| \; : \; \fint Du\, dx = \xi \right\} \tag{1.1}$$

where $\lambda(x)$ is a piecewise-constant function taking values $\alpha$ and $\beta$ (the two phases) and the condition that $\fint Du\, dx = \xi$ corresponds to assign the average electric field. The idea is that when this minimum reaches a certain threshold the first-failure dielectric breakdown occurs. A similar model could also be applied to other physical situation like the perfect plasticity problem in the case of anti-plane shear (as formulated, e.g., by [22]).

On the other hand part of the mathematical literature on such problems was originally motivated by the following problem: find the best Lipschitz extension in $\Omega$ of a function $\varphi$ defined on $\partial\Omega$. This can be clearly expressed in the following variational form

$$\inf_{u=\varphi \text{ on } \partial\Omega} \|Du\|_{L^\infty(\Omega)} .$$

Variants of this problem have been extensively studied by many authors using approximations results and the theory of viscosity solutions (see for instance [3], [6], Juutinen's thesis [21] and the references therein).

In general one can consider the following functionals

$$F(u) = \operatorname*{ess\,sup}_{x\in\Omega} f(x, Du(x)) \tag{1.2}$$

where $\Omega \subset \mathbf{R}^N$ and $u \in W^{1,\infty}(\Omega)$. The latter have been recently studied using the direct method of the calculus of variations and a partial theory has been developed parallel to the well established theory for integral functionals. In [1], where they have been baptized *supremal functionals*, a representation theorem has been proved in the case when the functionals do not depend on the gradient, i.e. they are of the form

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x, u(x)) , \tag{1.3}$$

and their semicontinuity has been completely characterized in the natural framework of measurable functions. Subsequently in [23], a relaxation theorem for functionals (1.3) has been proved. The analogous problems for functionals of the form (1.2) were studied in [5] and in [7] requiring a continuous dependence on the $x$ variable, the general case is considered in a forthcoming paper [18]. The question whether this class of functionals is stable under $\Gamma$-convergence in $L^\infty$ arises naturally. The answer is not always clear. In fact it is possible to construct an example of a sequence of supremal functionals whose $\Gamma$-limit can not be represented in the supremal form (see [23]). This problem of studying the $\Gamma$-limit of sequences of supremal functionals has been first approached in [23] and [11] for the class of functionals (1.3) but the case of functionals of the form (1.2) is still open.

In this paper we give a partial answer to this question studying the case of homogenization. Namely, we consider the following sequence of functionals

$$F_\varepsilon(u, \Omega) := \operatorname*{ess\,sup}_{x\in\Omega} f\Big(\frac{x}{\varepsilon}, Du(x)\Big) , \tag{1.4}$$

where $\Omega$ is a bounded subset of $\mathbf{R}^N$, $u \in W^{1,\infty}(\Omega)$ and $f$ is periodic in the second variable. The aim is to replace this highly oscillating functional, as $\varepsilon$ goes to zero, with a simpler functional $F^{\rm hom}$, the *homogenized functional*, which captures the relevant features of the sequence $F_\varepsilon$.

This problem has been considered in [17] in the particular form (1.1) in order to study the macroscopic behavior of a two-phases composite material for the first failure dielectric breakdown model. The authors use an approximation approach: they prove that the approximation of (1.1) by "power-law functionals" used by material scientists and considered by [8] and [6] in the case of the viscosity solutions approach, is indeed a $\Gamma$-limit as $p \to \infty$. More precisely, for any fixed $\varepsilon > 0$, they consider the sequence of functionals

$$F_{p,\varepsilon}(u) := \begin{cases} \left(\displaystyle\int_\Omega \Big|\lambda\Big(\frac{x}{\varepsilon}\Big) Du(x)\Big|^p dx\right)^{1/p} & \text{if } u \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise in } \ L^1(\Omega)\,, \end{cases} \tag{1.5}$$

with $\lambda$ periodic and $0 < \alpha \leq \lambda(x) \leq \beta$, and they prove its $\Gamma$-convergence (with respect to the $L^1$ topology) to the supremal functional

$$F_\varepsilon(u) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} \Big|\lambda\Big(\frac{x}{\varepsilon}\Big) Du(x)\Big| & \text{if } u \in W^{1,\infty}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega)\,. \end{cases} \tag{1.6}$$

One can consider the homogenized functional for (1.5), which describes the macroscopic behavior of the approximating "power-law materials" and by the classical homogenization theory is given by (the power $1/p$ of) an integral functional, $F_p^{\rm hom}$, whose energy density $f_p^{\rm hom}$ is homogeneous and is given by the following cell-problem formula

$$f_p^{\rm hom}(\xi) := \inf \left\{ \int_{(0,1)^N} \lambda^p(x) |Du(x) + \xi|^p dx \ : \ u \in W^{1,p}((0,1)^N)\,,\ \text{1-periodic} \right\}.$$

Thanks to the $\Gamma$-convergence of $F_{p,\varepsilon}$ as $p \to \infty$ one has that the functions $(f_p^{\rm hom})^{1/p}$ converge, as $p \to \infty$, to the function $f^{\rm hom}$ defined by

$$f^{\rm hom}(\xi) := \min \left\{ \operatorname*{ess\,sup}_{(0,1)^N} \lambda(x) |Du(x) + \xi| \ : \ u \in W^{1,\infty}((0,1)^N)\,,\ \text{1-periodic} \right\}. \tag{1.7}$$

In view of the application to the computation of bounds for composite materials, in [17], the authors consider $f^{\rm hom}$ as the "relevant" energy density for the macroscopic behavior of the limit problem $F_\varepsilon$, but they *do not* actually prove that it can be obtained directly by homogenizing it. This will be the main result of our paper in a broader context.

We will prove the homogenization theorem for supremal functionals of the general form (1.4), under very mild assumptions for the function $f(x, \xi)$, and inspired by the previous procedure we will prove that the energy density of the homogenized functional can be represented by means of a cell-problem formula obtained by an approximation technique (see Lemma 3.2). This is possible thanks to the fact that the "power-law" approximation described above holds true for very general $f$ (see [14]). As a consequence we obtain that the homogenization and the power-law approximation commute as summarized by the following diagram

$$
\begin{tikzcd}
F_{p,\varepsilon} \arrow[rr, "\Gamma(L^\infty)", "p \longrightarrow \infty"'] \arrow[dd, "\Gamma(L^p)"', "\varepsilon \to 0"] & & F_\varepsilon \arrow[dd, "\Gamma(L^\infty)", "\varepsilon \to 0"'] \\
& & \\
F_p^{\mathrm{hom}} \arrow[rr, "\Gamma(L^\infty)", "p \longrightarrow \infty"'] & & F^{\mathrm{hom}}
\end{tikzcd}
$$

the down arrow being proved in Theorem 3.3 and the right arrow being proved in Theorem 4.1. A similar approximation argument by "power law" energies has been used in [13] for the homogenization of unbounded functionals in $L^\infty$ of the form described below (see (1.8)).

A second important step in our proof of is the key remark that there is a strict relation between the class of supremal functional and a class of very degenerate functional of the form

$$
G(u) = \int_\Omega 1_{C(x)}(Du)\,dx = \begin{cases} 0 & \text{if } Du(x) \in C(x) \text{ a.e. in } \Omega \\ \\ +\infty & \text{otherwise,} \end{cases} \tag{1.8}
$$

where $C(x)$ is a convex set (see Propositions 2.1 and 2.6 in [17]). In particular the knowledge of the homogenized functional for the latter permits, with a suitable choice of the set $C(x)$, to deduce the $\Gamma$-limsup inequality. The homogenization for functionals of the form (1.8) can be obtained as a particular case of the results by Carbone *et al.* (see [12]) for unbounded integral functionals. We will use their result, however this strategy force us to restrict our study to the class of *convex* subsets $\Omega$ of $\mathbf{R}^N$.

The case of general $\Omega$ is studied under some additional continuity assumptions on $f$. In this case we can use, up to certain extent, a localization strategy for the proof of the $\Gamma$-limsup inequality similar to that used for the homogenization of integral functionals (see e.g. [9]) without applying the result in Theorem 4.3 to the level sets of the function $f$. A key point in the case of integral functionals is the representation of the limit on piecewise-affine functions and then on all functions by a density argument. This is not possible in our case: a major difficulty is that a priori in the limit we cannot neglect sets of zero measure. This is overcome by obtaining the representation of the limit directly on $C^1$ functions by an accurate use of cut-off arguments. We remark that this approach heavily relies on the periodicity assumption; anyhow it shows that, even though the comparison between our problem and the result of Theorem 4.3 is somehow natural being based on pointwise conditions for the gradient, the supremal functionals can be much regular and then less degenerate.

Our result partially overlap in the 1-dimensional case those obtained by Alvarez and Barron [2] using the method of viscosity solutions.

The plan of the paper is the following. In Section 2 we recall the main tools we use for our result. Section 3 will be devoted to the derivation, by approximation, of the cell problem (see Lemma 3.2). In Section 4 we state and prove the homogenization theorem under the assumption $\Omega$ convex (Theorem 4.1). In Section 5 we show that under some additional assumptions on the function $f$ it is possible to prove the homogenization result for a larger class of sets $\Omega$ (Theorem 5.2). Finally in Section 6 we comment the result and we give some examples.

## 2 Formulation of the problem and preliminaries

The aim of this paper is to give an homogenization result for a sequence of *supremal* functionals in $W^{1,\infty}(\Omega)$, i.e. functionals of the form

$$F_\varepsilon(u) = \operatorname*{ess\,sup}_{x\in\Omega} f\left(\frac{x}{\varepsilon}, Du(x)\right), \tag{2.1}$$

where $f(x,\xi)$ is a Borel function 1-periodic in the second variable.

A sufficient condition for the lower semicontinuity in $L^\infty$ for functionals of this type has been proved in [1] and is given by the following two conditions:

(i) (lower semicontinuity) $f(x,\cdot)$ is lower semicontinuous for a.e. $\in \Omega$.

(ii) (level convexity) $f(x,\cdot)$ is level convex for a.e. $x \in \Omega$, i.e. for every $t \in \mathbf{R}$ the level set $\left\{\xi \in \mathbf{R}^N \ : \ f(x,\xi) \le t\right\}$ is convex.

Remark that the level convexity can be equivalently stated as follows: for each $\lambda \in (0,1)$, $\xi_1, \xi_2 \in \mathbf{R}^N$,

$$f(x, \lambda\xi_1 + (1-\lambda)\xi_2) \le f(x,\xi_1) \vee f(x,\xi_2)$$

for a.e. $x \in \Omega$.

Condition (ii) has been proved to be necessary in [5] under a further continuity assumption on $f$.

For the first part of the paper we will require for $f$ the following growth conditions:

(iii) ("standard" growth conditions) there exist two positive constants $C_1$ and $C_2$ such that

$$C_1|\xi| \le f(x,\xi) \le C_2(1+|\xi|)$$

for every $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$.

The growth condition will be highly relaxed in the final homogenization result (see Theorem 4.1).

In order to study the asymptotic behavior of $F_\varepsilon$ as $\varepsilon \to 0$ we will use the notion of $\Gamma$-convergence introduced by De Giorgi. For convenience of the reader let us recall the definition (more details on $\Gamma$-convergence and homogenization theory can be found for instance in [9], [10] and [15]).

We say that a given sequence of functionals $G_\varepsilon$ defined in a metric space $X$, $\Gamma$-*converges* to the functional $G$, as $\varepsilon \to 0$, if the following properties hold

(a) For every $u \in X$ and for every sequence $\{u_\varepsilon\}$ converging to $u$ in $X$ we have

$$G(u) \le \liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon).$$

(b) For every $u \in X$ there exists a sequence $\{u_\varepsilon\}$ (recovering sequence) such that

$$G(u) \ge \limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon).$$

We will refer to (a) as the $\Gamma(X)$-liminf inequality and to (b) as the $\Gamma(X)$-limsup inequality. The former is actually equivalent to

$$G(u) \le \inf\{\liminf_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u \text{ in } X\} := \Gamma(X)\text{-}\liminf_{\varepsilon \to 0} G_\varepsilon\,(u)\,,$$

while the latter gives

$$G(u) \ge \inf\{\limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u \text{ in } X\} := \Gamma(X)\text{-}\limsup_{\varepsilon \to 0} G_\varepsilon\,(u)\,.$$

We will use the notion of $\Gamma$-convergence both for the homogenization in $L^p(\Omega)$, $1 < p \le +\infty$, and for the "power-law" approximation, i.e. as $p \to +\infty$. In the latter the definition above will be applied with $\epsilon = 1/p$.

A key step in order to use an approximation argument is the following result proved in [14].

**Theorem 2.1** *Let $\Omega \subset \mathbf{R}^N$ be a bounded open set and let $f : \Omega \times \mathbf{R}^N \to [0, +\infty]$ satisfy conditions* (i)–(iii)*. For any $p \ge 1$, we define*

$$\mathcal{F}_p(u) := \begin{cases} \left( \displaystyle\int_{(0,1)^N} f^p(x, Du(x))dx \right)^{1/p} & \textit{if} \quad u \in W^{1,p}((0,1)^N) \\ +\infty & \textit{otherwise.} \end{cases}$$

*The family $(\mathcal{F}_p)_{p \ge 1}$ $\Gamma$-converges (as $p$ goes to $+\infty$) to $\mathcal{F} : L^\infty(\Omega) \to [0, +\infty]$*

$$\mathcal{F}(u) := \begin{cases} \operatorname*{ess\,sup}_{x \in (0,1)^N} f(x, Du(x)) & \textit{if} \quad u \in W^{1,\infty}((0,1)^N) \\ +\infty & \textit{otherwise,} \end{cases}$$

*with respect to the topology of the uniform convergence.*

# 3 Power-law approximation and cell-problem

In order to study the $\Gamma$-limit for $F_\varepsilon$ and to give an explicit representation of it, we will follow the strategy of [17]. In view of the approximation result given in Theorem 2.1 let us consider the functional

$$F_{p,\varepsilon}(u) := \left( \int_\Omega f^p\Big(\frac{x}{\varepsilon}, Du(x)\Big) dx \right)^{1/p}. \tag{3.1}$$

For any fixed $1 < p < +\infty$ one can consider the $\Gamma$-limit as $\varepsilon \to 0$, i.e. the homogenized functional $F_p^{\hom}$ of $F_{p,\varepsilon}$. Indeed, the function $f^p(x,\xi)$ satisfies the standard growth conditions in $W^{1,p}$ which permit to characterize the homogenization of $F_{p,\varepsilon}$ as $\varepsilon \to 0$. Namely, for any bounded open set $\Omega$ the sequence $F_{p,\varepsilon}(u)$ $\Gamma(L^p)$-converges to the functional

$$F_p^{\hom}(u) := \left( \int_\Omega f_p^{\hom}(Du(x)) dx \right)^{1/p} \tag{3.2}$$

where the energy density $f_p^{\hom}$ is given by the following cell-problem formula

$$f_p^{\hom}(\xi) := (f^p)^{\hom}(\xi) = \inf \left\{ \int_{(0,1)^N} f^p(x, \xi + Du(x)) dx \ : \ u \in W^{1,p}_{\#}((0,1)^N) \right\} \tag{3.3}$$

(the space $W^{1,p}_{\#}((0,1)^N)$ being defined by

$$W^{1,p}_{\#}((0,1)^N) := \{u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N) : u \text{ is 1-periodic}\}$$

for all $1 < p \leq +\infty$).

Using Theorem 2.1, from (3.3) we can derive a cell-problem formula as $p \to +\infty$ which will be our candidate for the representation of the homogenized functional of $F_\varepsilon$, i.e.

$$f^{\hom}(\xi) = \inf \left\{ \operatorname*{ess\,sup}_{(0,1)^N} f(x, \xi + Du(x)) \ : \ u \in W^{1,\infty}_{\#}((0,1)^N) \right\}. \tag{3.4}$$

**Remark 3.1** If $f$ satisfies conditions (i)–(iii) then for any fixed $\xi \in \mathbf{R}^N$ the functional $\operatorname{ess\,sup}_{(0,1)^N} f(x, \xi + Du(x))$ is lower semicontinuous and coercive in $W^{1,\infty}_{\#}((0,1)^N)$ and thus the infimum in the definition of $f^{\hom}(\xi)$ is achieved. In fact, as a consequence of the Ascoli-Arzela theorem, any minimizing sequence is compact, up to a translation, and, hence up to a subsequence it must converge to a minimum point.

Moreover, it can be easily checked that the function $f^{\hom}(\xi)$ defined by (3.4) is level convex. Indeed, for any fixed $\xi_1, \xi_2 \in \mathbf{R}^N$ there exist $u_1, u_2 \in W^{1,\infty}_{\#}((0,1)^N)$ such that $f^{\hom}(\xi_i) = \operatorname{ess\,sup}_{x\in(0,1)^N} f(x, \xi_i + Du_i(x))$ , $(i = 1\,, 2)$. Thus for every $\lambda \in (0,1)$

$$\begin{aligned} f^{\hom}(\lambda\xi_1 + (1-\lambda)\xi_2) &\leq \operatorname*{ess\,sup}_{x\in(0,1)^N} f(x, \lambda(\xi_1 + Du_1(x)) + (1-\lambda)(\xi_2 + Du_2(x))) \\ &\leq \operatorname*{ess\,sup}_{x\in(0,1)^N} [f(x, \xi_1 + Du_1(x)) \vee f(x, \xi_2 + Du_2(x))] \\ &\leq f^{\hom}(\xi_1) \vee f^{\hom}(\xi_2), \end{aligned}$$

which is equivalent to the level convexity of $f^{\mathrm{hom}}$.

We have the following result.

**Lemma 3.2** *Let $f : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty]$ be a Borel function, 1-periodic in the first variable satisfying conditions* (i)–(iii)*, then*

1. *for every $\xi \in \mathbf{R}^N$*
$$\lim_{p\to\infty} \left(f_p^{\mathrm{hom}}(\xi)\right)^{1/p} = f^{\mathrm{hom}}(\xi), \tag{3.5}$$

2. *for all open bounded set $\Omega \subset \mathbf{R}^N$ and $u \in W^{1,\infty}(\Omega)$,*

$$\lim_{p\to\infty} \left(\int_\Omega f_p^{\mathrm{hom}}(Du(x))dx\right)^{1/p} = \operatorname*{ess\,sup}_{x\in\Omega} f^{\mathrm{hom}}(Du(x)). \tag{3.6}$$

**Proof**.

In order to prove (3.5), let us fix $\xi \in \mathbf{R}^N$ and observe that, by definition, $\left(f_p^{\mathrm{hom}}\right)^{1/p}$ is a non-decreasing sequence such that

$$\left(f_p^{\mathrm{hom}}(\xi)\right)^{1/p} \le f^{\mathrm{hom}}(\xi), \quad \forall p > 1, \tag{3.7}$$

hence the limit as $p \to +\infty$ exists and

$$\lim_{p\to\infty} \left(f_p^{\mathrm{hom}}(\xi)\right)^{1/p} \le f^{\mathrm{hom}}(\xi). \tag{3.8}$$

To prove the reverse inequality, without loss of generality, we may assume that $\lim\limits_{p\to\infty} \left(f_p^{\mathrm{hom}}(\xi)\right)^{1/p} < +\infty$.

Let $\varepsilon > 0$, and $p > 1$, by definition of $f_p^{\mathrm{hom}}(\xi)$ there exists $u_p \in W^{1,p}_{\#}((0,1)^N)$ such that

$$\mathcal{F}_p(u_p + \xi \cdot x) = \left(\int_{(0,1)^N} f^p(x, Du_p(x) + \xi)dx\right)^{1/p} \le \left(f_p^{\mathrm{hom}}(\xi)\right)^{1/p} + \varepsilon.$$

By the coerciveness of $f$, there exists $C > 0$ such that $||Du_p||_{L^q} \le C$ for every $p > q > 1$. Without loss of generality, we may assume that $u_p$ has zero average, thus, by Poincaré-Wirtinger inequality, we conclude that for every $q > 1$ the sequence $(u_p)_{p>q}$ is bounded in $W^{1,q}((0,1)^N)$. Furthermore, there exists $u_\infty \in W^{1,\infty}_{\#}((0,1)^N)$ such that $(u_p)_p$ (up to a subsequence) converges uniformly to $u_\infty$ as $p \to +\infty$. Then, by Theorem 2.1, and the definition of $f^{\mathrm{hom}}$ we get

$$\begin{aligned} f^{\mathrm{hom}}(\xi) &\le \operatorname*{ess\,sup}_{x\in(0,1)^N} f(x, \xi + Du_\infty(x)) \\ &\le \liminf_{p\to\infty} \mathcal{F}_p(u_p + \xi \cdot x) \\ &\le \liminf_{p\to\infty} \left(f_p^{\mathrm{hom}}(\xi)\right)^{1/p} + \varepsilon. \end{aligned}$$

By the arbitrariness of $\varepsilon$, we conclude the proof of (3.5).
In order to prove (3.6) it is enough to obtain

$$\lim_{p\to\infty}\left(\int_\Omega f_p^{\hom}(Du(x))dx\right)^{1/p} \geq \operatorname*{ess\,sup}_{x\in\Omega} f^{\hom}(Du(x))\,,$$

the reverse inequality being an easy consequence of (3.7). Fix $\varepsilon > 0$ and $u \in W^{1,\infty}(\Omega)$. Let us define

$$E_\varepsilon := \left\{x \in \Omega :\ f^{\hom}(Du(x)) > \operatorname*{ess\,sup}_{\Omega} f^{\hom}(Du) - \frac{\varepsilon}{2}\right\},$$

$m_\varepsilon := \mathcal{L}(E_\varepsilon) > 0$ and fix $0 < \delta << m_\varepsilon$. By (3.5) and Egorov's theorem, there exists $F_\delta \subset \Omega$ such that $\mathcal{L}(F_\delta) \leq \delta$ and

$$\lim_{p\to\infty} ||(f_p^{\hom}(Du))^{1/p} - f^{\hom}(Du)||_{L^\infty(\Omega\setminus F_\delta)} = 0.$$

In particular there exists $p_0 = p_0(\varepsilon)$ such that if $p \geq p_0$, then

$$(f_p^{\hom})^{1/p}(Du(x)) - f^{\hom}(Du(x)) \geq -\frac{\varepsilon}{2} \quad \forall x \in \Omega\setminus F_\delta\,.$$

Thanks to our choice of $\delta$, $\mathcal{L}(E_\varepsilon \setminus F_\delta) > 0$ and if $x \in E_\varepsilon \setminus F_\delta$, then

$$\left(f_p^{\hom}\right)^{1/p}(Du(x)) \geq \operatorname*{ess\,sup}_{\Omega} f^{\hom}(Du) - \varepsilon\,, \quad \forall p \geq p_0.$$

Thus the set $E_\varepsilon^p$ defined by

$$E_\varepsilon^p := \left\{x \in \Omega :\ \left(f_p^{\hom}\right)^{1/p}(Du(x)) > \operatorname*{ess\,sup}_{\Omega} f^{\hom}(Du) - \varepsilon\right\}$$

contains $E_\varepsilon \setminus F_\delta$, and hence, for every $p \geq p_0$, has positive measure. Finally we have

$$\begin{aligned}
\left(\int_\Omega f_p^{\hom}(Du(x))\,dx\right)^{1/p} &\geq \left(\int_{E_\varepsilon^p} f_p^{\hom}(Du(x))\,dx\right)^{1/p}\\
&\geq \left(\int_{E_\varepsilon^p} (\operatorname*{ess\,sup}_{\Omega} f^{\hom}(Du) - \varepsilon)^p dx\right)^{1/p}\\
&= \mathcal{L}(E_\varepsilon^p)^{1/p}\big(\operatorname*{ess\,sup}_{x\in\Omega} f^{\hom}(Du(x)) - \varepsilon\big)\\
&\geq \mathcal{L}(E_\varepsilon\setminus F_\delta)^{1/p}\big(\operatorname*{ess\,sup}_{x\in\Omega} f^{\hom}(Du(x)) - \varepsilon\big)
\end{aligned}$$

for every $p \geq p_0$. This implies

$$\lim_{p\to\infty}\left(\int_\Omega f_p^{\hom}(Du(x))dx\right)^{1/p} \geq \operatorname*{ess\,sup}_{x\in\Omega} f^{\hom}(Du(x)) - \varepsilon$$

which, by arbitrariness on $\varepsilon$, gives us (3.6).

□

We now have the means to study the $\Gamma$-limit as $p \to \infty$ of the functionals $F_p^{\hom}$ defined in (3.2).

**Theorem 3.3** *Let $f : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty]$ be a Borel function,* 1*-periodic in the first variable and satisfying conditions* (i)–(iii)*. Let $\Omega$ be a bounded subset of $\mathbf{R}^N$ and, for any $p > 1$, let $F_p^{\hom}$ be defined as in* (3.2)*. Therefore*

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} F_p^{\hom}(u) = F^{\hom}(u) := \sup_{x\in\Omega} f^{\hom}(Du(x))$$

*where $f^{\hom}$ is given by* (3.4)*.*

**Proof.** For every $p > 1$ the functional

$$F_p^{\hom}(u) = \left(\int_\Omega f_p^{\hom}(Du(x))dx\right)^{1/p},$$

being the $\Gamma$-limit of $F_{p,\varepsilon}$ in $L^p$, is lower semicontinuous in $W^{1,p}$ with respect to the $L^p$ topology. In particular it is lower semicontinuous in $W^{1,\infty}$ with respect to the $L^\infty$ topology. Since the sequence $\left(F_p^{\hom}\right)_p$ is increasing and, by Lemma 3.2, converges pointwise to $F^{\hom}$ as $p \to \infty$, this easily implies

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty} F_p^{\hom} = F^{\hom}.$$

$\square$

**Remark 3.4** The homogenization result for integral functionals together with the previous theorem shows that

$$\Gamma(L^\infty)\text{-}\lim_{p\to\infty}\Big(\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0} F_{p,\varepsilon}(u)\Big) = \sup_\Omega f^{\hom}(Du).$$

# 4 Homogenization on convex domains

The present section is devoted to the statement and the proof of the homogenization result in the case of convex domains. More precisely, we will show that the functional $F^{\hom}$ can be obtained directly by taking the limit as $\varepsilon \to 0$ of $F_\varepsilon$. As a consequence the two limits in Remark 3.4 commute.

**Theorem 4.1** *Let $f : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty]$ be a Borel function,* 1*-periodic in the first variable and lower semicontinuous and level convex in the second variable. Assume that $f$ satisfies the following growth condition:*

(H) *(weak growth condition) There exist two functions $\alpha, \beta : [0, +\infty) \to [0, +\infty)$, with $\alpha$ a continuous, increasing function such that $\lim_{t\to\infty} \alpha(t) = +\infty$ and $\beta$ locally bounded, such that*

$$\alpha(|\xi|) \le f(x, \xi) \le \beta(|\xi|) \quad \textit{for every } \xi \in \mathbf{R}^N \ , \quad \textit{a.e. } x \in (0, 1)^N.$$

*Therefore for any convex bounded open set* $\Omega \subset \mathbf{R}^N$*, the sequence* $F_\varepsilon : W^{1,\infty}(\Omega) \to [0, +\infty)$*,*

$$F_\varepsilon(u) = \operatorname*{ess\,sup}_{x\in\Omega} f\Big(\frac{x}{\varepsilon}, Du(x)\Big)\,,$$

$\Gamma(L^\infty)$*-converges to the functional* $F^{\rm hom} : W^{1,\infty}(\Omega) \to [0, +\infty)$ *defined by*

$$F^{\rm hom}(u) := \operatorname*{ess\,sup}_{x\in\Omega} f^{\rm hom}(Du(x))\,, \tag{4.1}$$

*where* $f^{\rm hom} : \mathbf{R}^N \to [0, +\infty)$ *is the level convex function given by the* cell-problem formula

$$f^{\rm hom}(\xi) := \inf\left\{ \operatorname*{ess\,sup}_{x\in(0,1)^N} f(x, \xi + Du(x)) \;:\; u \in W^{1,\infty}_{\#}((0,1)^N) \right\} \tag{4.2}$$

*for all* $\xi \in \mathbf{R}^N$ *.*

Let us remark that the growth conditions (H) is quite general. It for instance include the cases when the function $f$ satisfies non standard growth conditions of the type: $C_1|\xi|^p \le f(x, \xi) \le C_2(1 + |\xi|^q)$, with $q > p$, a.e. $x \in \Omega$. When $f$ satisfies the estimate from below with a function $\alpha$ like in condition (H), we say that $f$ is *uniformly coercive* in the second variable.

We will prove Theorem 4.1 in two steps. The $\Gamma$-liminf inequality (Proposition 4.2) will be obtained as a consequence of the $L^p$ approximation of the supremal functional and of the homogenization result for integral functionals. For the $\Gamma$-limsup inequality (Proposition 4.4) we will use an homogenization result for unbounded integral functionals (see [12]) which we will state in a convenient form in Theorem 4.3. The latter is where the convexity of the set $\Omega$ is needed.

**Proposition 4.2** *Let* $f : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty)$ *be a Borel function,* 1*-periodic in the first variable and lower semicontinuous and level convex in the second variable. Moreover assume that* $f$ *satisfies condition* (H) *of Theorem 4.1. For any bounded open set* $\Omega \subset \mathbf{R}^N$ *and for all* $u \in W^{1,\infty}(\Omega)$ *we have*

$$\Gamma(L^\infty)\text{-}\liminf_{\varepsilon\to 0} F_\varepsilon(u) \ge F^{\rm hom}(u)\,.$$

**Proof.** Clearly it is enough to prove that for every $u_\varepsilon \to u$ in $L^\infty$ as $\varepsilon \to 0$, $u_\varepsilon, u \in W^{1,\infty}(\Omega)$,

$$\operatorname*{ess\,sup}_{x\in\Omega} f^{\rm hom}(Du(x)) \le \liminf_{\varepsilon\to 0} \operatorname*{ess\,sup}_{x\in\Omega} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon(x)\Big)\,. \tag{4.3}$$

Without loss of generality we may assume that $\sup_\varepsilon \operatorname{ess\,sup}_\Omega f(\frac{x}{\varepsilon}, Du_\varepsilon(x)) < +\infty$. We will prove it in three steps:

**Step 1**. First assume that $f$ satisfies the "standard" growth condition (iii), i.e.

$$C_1|\xi| \le f(x, \xi) \le C_2(1 + |\xi|)$$

for every $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$. Under this assumption we have the standard homogenization result for the functional $F_{p,\varepsilon}$ as defined by (3.1). By the fact that $F_{p,\varepsilon}$ $\Gamma$-converges to $F_p^{\hom}$ we have

$$\left( \int_\Omega f_p^{\hom}(Du(x))\,dx \right)^{1/p} \leq \liminf_{\varepsilon \to 0} \left( \int_\Omega f^p\Big(\frac{x}{\varepsilon}, Du_\varepsilon(x)\Big)\,dx \right)^{1/p} \leq \mathcal{L}(\Omega)^{1/p} \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon)$$

for each $p > 1$. Then (4.3) follows taking the limit as $p \to \infty$ and using (3.6).

**Step 2.** Assume now that $f(x, \cdot)$ is locally bounded from above a.e. $x \in \mathbf{R}^N$ and satisfies

$$|\xi| \leq f(x, \xi) \qquad \text{a.e. } x \in \mathbf{R}^N\,,$$

i.e. it satisfies condition (H) with $\alpha(t) = |t|$. For every $M > 0$, let $\phi_M(x, \xi)$ be defined by

$$\phi_M(x, \xi) = f(x, \xi) \wedge \Big[M \vee \frac{1}{2}(1 + |\xi|)\Big]\,.$$

It is easy to check that, if $M > 1$, then $\phi_M(x, \xi)$ satisfies

$$\frac{1}{2}|\xi| \leq \phi_M(x, \xi) \leq M(1 + |\xi|)$$

and it is level convex in the second variable. Indeed, for every $\lambda > 0$ we have

$$\{\Phi_M(x, \xi) \leq \lambda\} = \{f(x, \xi) \leq \lambda\} \cup \Big\{M \vee \frac{1}{2}(1 + |\xi|) \leq \lambda\Big\}$$

and

$$\Big\{M \vee \frac{1}{2}(1 + |\xi|) \leq \lambda\Big\} = \begin{cases} \emptyset & \text{if } \lambda < M \\ \\ \{|\xi| \leq 2\lambda - 1\} & \text{if } \lambda \geq M. \end{cases}$$

By the coerciveness of $f$ we have that $\{f(x, \xi) \leq \lambda\} \subset \{|\xi| \leq \lambda\}$ and then we deduce that

$$\{\Phi_M(x, \xi) \leq \lambda\} = \begin{cases} \{f(x, \xi) \leq \lambda\} & \text{if } \lambda < M \\ \\ \{|\xi| \leq 2\lambda - 1\} & \text{if } \lambda \geq M \end{cases}$$

and hence $\{\Phi_M(x, \xi) \leq \lambda\}$ is convex. Thus applying Step 1, we have

$$\operatorname*{ess\,sup}_{x \in \Omega} \phi_M^{\hom}(Du(x)) \leq \liminf_{\varepsilon \to 0} \operatorname*{ess\,sup}_{x \in \Omega} \phi_M\Big(\frac{x}{\varepsilon}, Du_\varepsilon(x)\Big)\,, \tag{4.4}$$

where

$$\phi_M^{\hom}(\xi) = \inf\Big\{\operatorname*{ess\,sup}_{x \in \Omega} \phi_M(x, \xi + Du(x)) \;:\; u \in W_\#^{1,\infty}((0,1)^N)\Big\}\,.$$

The idea is that for $M$ big enough inequality (4.4) reduces to (4.3). In fact on one hand $\phi_M(x, \xi) \leq f(x, \xi)$ for a.e. $x \in R^N$, and hence

$$\liminf_{\varepsilon \to 0} \operatorname*{ess\,sup}_{x \in \Omega} \phi_M\Big(\frac{x}{\varepsilon}, Du_\varepsilon(x)\Big) \leq \liminf_{\varepsilon \to 0} \operatorname*{ess\,sup}_{x \in \Omega} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon(x)\Big)\,. \tag{4.5}$$

On the other hand let us prove that for $M$ big enough

$$\operatorname*{ess\,sup}_{x\in\Omega} f^{\rm hom}(Du(x)) = \operatorname*{ess\,sup}_{x\in\Omega} \phi_M^{\rm hom}(Du(x))\,. \tag{4.6}$$

Let $M_0 = \sup_{x\in\Omega}|Du(x)|$. By the fact that $f$ is locally bounded, there exists a constant $C_0$, depending on $M_0$, such that for every $M > 0$

$$\phi_M^{\rm hom}(\xi) \le f^{\rm hom}(\xi) \le C_0 \qquad \forall\ \xi\ :\ |\xi| \le M_0\,.$$

Fix $0 < \varepsilon < 1$ and $\xi$, with $|\xi| \le M_0$. For any $M > 0$ there exists a function $u_{M,\xi} \in W^{1,\infty}_{\#}((0,1)^N)$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega} \phi_M(x, Du_{M,\xi}(x)+\xi) < \phi_M^{\rm hom}(\xi) + \varepsilon \le C_0 + \varepsilon\,.$$

By the coerciveness of $\phi_M$ we deduce that $\operatorname{ess\,sup}_\Omega |\xi + Du_{M,\xi}| \le 2(C_0+\varepsilon) < 2C_0+2$ and thus there exists a constant $C_1 > 0$ such that $f(x, \xi + Du_{M,\xi}(x)) \le C_1$ . It is clearly enough to choose $M > C_1$ in order to obtain $f(x, \xi + Du_{M,\xi}(x)) = \phi_M(x, \xi + Du_{M,\xi}(x))$. Then by the definition of $u_{M,\xi}$ we have

$$f^{\rm hom}(\xi) \le \phi_M^{\rm hom}(\xi) + \varepsilon\,,$$

for every $|\xi| \le M_0$ and $M > C_1$, and hence

$$f^{\rm hom}(Du(x)) \le \phi_M^{\rm hom}(Du(x)) \qquad \forall\ x \in \Omega\,.$$

The conclusion follows by taking the supremum in $x$ in the last inequality and by the fact that the reverse inequality is trivially satisfied.
**Step 3.** The general case follows applying the previous step to the function $\alpha^{-1}(f(x,\xi))$.

$\square$

The proof of the $\Gamma$-limsup inequality follows from a repeated application of the homogenization result for unbounded convex integral functionals due to Carbone et al. ([12]), which we state in the following particular case.

**Theorem 4.3** *(Theorem 3.10, [12]) Let $\Omega$ be a convex bounded subset of $\mathbf{R}^N$ and let $C(x)$ be a 1-periodic measurable set function such that $C(x)$ is a closed convex set for any $x \in \Omega$ and there exists $R > 0$ such that*

$$C(x) \subset B_R(0) \qquad a.e.\ x \in \Omega\,. \tag{4.7}$$

*Let $G_\varepsilon : W^{1,\infty}(\Omega) \to [0, +\infty]$ be defined by*

$$G_\varepsilon(u) := \int_\Omega 1_{C(\frac{x}{\varepsilon})}(Du(x))\,dx\,.$$

*Therefore $G_\varepsilon$ $\Gamma(L^\infty)$-converges, as $\varepsilon$ goes to zero, to the homogeneous functional*

$$G^{\rm hom}(u) := \int_\Omega g^{\rm hom}(Du(x))\,dx\,,$$

*where, for* $\xi \in \mathbf{R}^N$,

$$g^{\text{hom}}(\xi) := \inf \left\{ \int_{(0,1)^N} 1_{C(x)}(\xi + Du(x))\, dx \ : \ u \in W^{1,\infty}_{\#}((0,1)^N) \right\}. \tag{4.8}$$

In the following proposition we prove the $\Gamma$-limsup inequality.

**Proposition 4.4** *Let* $f : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty]$ *be a Borel function,* 1*-periodic in the first variable and satisfying conditions* (i), (ii) *and* (H)*. For any convex open bounded set* $\Omega \subset \mathbf{R}^N$ *and* $u \in W^{1,\infty}(\Omega)$*, we have*

$$\Gamma(L^\infty)\text{-}\limsup_{\varepsilon \to 0} F_\varepsilon(u) \le F^{\text{hom}}(u)\,.$$

**Proof.** Fix $\bar{u} \in W^{1,\infty}(\Omega)$ and let $M = \operatorname{ess\,sup}_\Omega f^{\text{hom}}(D\bar{u})$. Our aim is to find a sequence $(u_\varepsilon)_\varepsilon$ such that $u_\varepsilon \to \bar{u}$ in $L^\infty(\Omega)$ as $\varepsilon \to 0$, and $\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \le M$.

For every $x \in \Omega$ let us define $C(x) := \{\xi \in \mathbf{R}^N : \ f(x, \xi) \le M\}$ and $C_\infty := \{\xi \in \mathbf{R}^N : \ f^{\text{hom}}(\xi) \le M\}$. Since $f(x, \cdot)$ is level convex, the set $C(x)$ is convex for a.e. $x \in \Omega$ and, thanks to condition (H) (the coercivity of $f$), it satisfies property (4.7) of Theorem 4.3, with $R = \alpha^{-1}(M)$. Thus, we have that

$$\Gamma(L^\infty)\text{-}\lim_{\varepsilon \to 0} G_\varepsilon(u) = \int_\Omega g^{\text{hom}}(Du(x))\, dx, \tag{4.9}$$

where, for $\xi \in \mathbf{R}^N$, $g^{\text{hom}}(\xi)$ is defined by (4.8).
The key remark is that $g^{\text{hom}}(\xi) = 1_{C_\infty}(\xi)$ for all $\xi \in \mathbf{R}^N$. To prove this it is enough to show that

$$g^{\text{hom}}(\xi) = 0 \Longleftrightarrow 1_{C_\infty}(\xi) = 0$$

and this can be deduced by the definitions of $g^{\text{hom}}$, $C(x)$, $f^{\text{hom}}$ and $C_\infty$. Indeed,

$$\begin{aligned} g^{\text{hom}}(\xi) = 0 \quad &\Longleftrightarrow \quad \exists\, u \in W^{1,\infty}_{\#}((0,1)^N) \ : \ \int_{(0,1)^N} 1_{C(x)}(\xi + Du(x))\, dx = 0, \\ &\Longleftrightarrow \quad \exists\, u \in W^{1,\infty}_{\#}((0,1)^N) \ : \ f(x, \xi + Du(x)) \le M \text{ for a.e. } x \in \Omega, \\ &\Longleftrightarrow \quad \exists\, u \in W^{1,\infty}_{\#}((0,1)^N) : \ \operatorname*{ess\,sup}_{x \in \Omega} f(x, \xi + Du(x)) \le M. \end{aligned}$$

By Remark 3.1 the last condition is equivalent to $f^{\text{hom}}(\xi) \le M$, i.e. $1_{C_\infty}(\xi) = 0$. Thus by (4.9) we have that there exists a sequence $(u_\varepsilon)_\varepsilon \subset W^{1,\infty}(\Omega)$ such that $u_\varepsilon \to \bar{u}$ in $L^\infty(\Omega)$, as $\varepsilon \to 0$, and

$$\limsup_{\varepsilon \to 0} G_\varepsilon(u_\varepsilon) \le \int_\Omega 1_{C_\infty}(D\bar{u}(x)) dx = 0\,.$$

In particular, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \le \varepsilon_0$, $G_\varepsilon(u_\varepsilon) = 0$, i.e.

$$1_{C\left(\frac{x}{\varepsilon}\right)}(Du_\varepsilon(x)) = 0 \qquad \text{a.e. } x \in \Omega,$$

i.e. $\operatorname*{ess\,sup}_{x \in \Omega} f(\frac{x}{\varepsilon}, Du_\varepsilon(x)) \le M$ which implies $F_\varepsilon(u_\varepsilon) \le M$ and then

$$\limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \le F^{\text{hom}}(\bar{u})\,.$$

$\square$

**Proof of Theorem 4.1.** The thesis follows from Proposition 4.2 and Proposition 4.4. $\square$

# 5 Homogenization for continuous $f$ on general domains

In this section we will see that the homogenization result can be proved for general domains if we require some continuity on the function $f$ (see condition (HC)). In this case we can mimic, up to certain extent, the "standard" strategy for the proof of the $\Gamma$-limsup inequality in the homogenization of integral functionals, without applying the result in Theorem 4.3 to the level sets of the function $f$, as we did in the previous section.

We will require for the function $f(x, \xi)$ the following continuity condition:

(HC) For every $M > 0$ there exists a function $\omega_M : [0, +\infty) \to [0, +\infty)$ such that $\omega_M(t) \to 0$ as $t \to 0^+$ and

$$|f(x, \xi) - f(x, \eta)| \leq \omega_M(|\xi - \eta|) \quad \text{a.e. } x \in (0,1)^N, \quad \text{for every } \ \xi\,, \eta \in B_M(0)\,.$$

We will see that under this assumption we can prove the homogenization result for a rich class of domains. Let us recall the definition of a *rich family* of open sets.

**Definition 5.1** *Let $\mathcal{A}$ be the family of the open subsets of $\mathbf{R}^N$. Let $\mathcal{R}$ be a subset of $\mathcal{A}$. We say that $\mathcal{R}$ is rich in $\mathcal{A}$ if for every $\{A_t\}_{t\in\mathbf{R}} \subset \mathcal{A}$ with $A_t \subset\subset A_s$ whenever $t < s$, the set $\{t : \ A_t \notin \mathcal{R}\}$ is at most countable.*

The proof of homogenization result for all the domains in rich family of open sets passes through the representation of the homogenized functional on regular domains. Namely we will first prove the result for all the domains which satisfy one of the following properties:

(C2) $\Omega$ is of class $C^2$, i.e. the boundary of $\Omega$, $\partial\Omega$, is locally the graph of a $C^2$ function.

(S) $\Omega$ is strongly star-shaped.

The main result of this section is the following.

**Theorem 5.2** *Let $f : \mathbf{R}^N \times \mathbf{R}^N \to [0, +\infty)$ be a Borel function, 1-periodic in the first variable and level convex in the second one. Assume that $f$ satisfies the growth condition* (H) *together with the continuity condition* (HC). *Then there exists a rich class $\mathcal{A}' \subset \mathcal{A}$ such that for every $\Omega \in \mathcal{A}'$, $F_\varepsilon(\cdot, \Omega)$ $\Gamma(L^\infty)$-converges to $F^{\hom}(\cdot, \Omega)$. Moreover any bounded open set $\Omega \subset \mathbf{R}^N$ satisfying either* (C2) *or* (S) *belongs to $\mathcal{A}'$.*

**Proof**. With a little abuse of notation we will write $F_\varepsilon(u,\Omega)$ and $F^{\rm hom}(u,\Omega)$ in order to stress the dependence of the functionals on the domain. The proof of the $\Gamma$-liminf inequality is given by Proposition 4.2. Thus it only remains to prove the $\Gamma$-limsup inequality, i.e.

$$F''(u,\Omega) := \Gamma(L^\infty)\text{-}\limsup_{\varepsilon\to 0} F_\varepsilon(u,\Omega) \le F^{\rm hom}(u,\Omega)\,. \tag{5.1}$$

We will prove this upper bound by means of various steps.
**Step 1.** (Affine functions) Let $u=\xi\cdot x$, with $\xi\in\mathbf{R}^N$. By definition of $f^{\rm hom}(\xi)$ there exists a function $v\in W^{1,\infty}_\sharp((0,1)^N)$ such that

$$f^{\rm hom}(\xi) = \operatorname*{ess\,sup}_{x\in(0,1)^N} f(x, Dv(x)+\xi)\,.$$

Let us define $u_\varepsilon(x)=\varepsilon v\big(\frac{x}{\varepsilon}\big)+\xi\cdot x$. Then $Du_\varepsilon(x)=Dv\big(\frac{x}{\varepsilon}\big)+\xi$ and for every $\varepsilon>0$

$$\operatorname*{ess\,sup}_{\Omega} f^{\rm hom}(Du) = f^{\rm hom}(\xi) = \operatorname*{ess\,sup}_{x\in(0,\varepsilon)^N} f\Big(\frac{x}{\varepsilon}, Dv\big(\frac{x}{\varepsilon}\big)+\xi\Big) = \operatorname*{ess\,sup}_{x\in\Omega} f\Big(\frac{x}{\varepsilon}, Du_\varepsilon(x)\Big)\,.$$

Thus $u_\varepsilon$ is a recovery sequence for $u(x)=\xi\cdot x$ and this implies that the $\Gamma$-limsup inequality holds for affine functions and for every open set $\Omega$.
**Step 2.** (A fundamental estimate for the $\Gamma$-limsup) Let $A\subseteq\Omega$ be an open subset of $\Omega$ and let $A_1$, $A_2$ be a polyhedral partition of $A$. Denote by $A_i^\delta$, $i=1,2$, the set $\{x\in\Omega\,:\,\operatorname{dist}(x,A_i)<\delta\}$ and by $S^\delta_{1,2}$ the set $A_1^\delta\cap A_2^\delta\cap A$. Let $\varphi_\delta$ be a cut-off function between $A_1$ and $A_2$ on $S^\delta_{1,2}$, i.e. $\varphi_\delta=1$ on $A_1\setminus S^\delta_{1,2}$, $\varphi_\delta=0$ on $A_2\setminus S^\delta_{1,2}$, $0\le\varphi_\delta\le 1$, $\varphi\in C^1(A)$ and $|D\varphi_\delta|\le\frac{2}{\delta}$. We now prove that for fixed $u_i\in W^{1,\infty}(A_i^\delta)$ there exists $M>0$, $M=M(||u_1-u_2||_{L^\infty(S^\delta_{1,2})},\sup_i F''(u_i,A_i^\delta))$ such that

$$F''(w_\delta,A)\le F''(u_1,A_1^\delta)\vee F''(u_2,A_2^\delta)+\sup_{S^\delta_{1,2}}\omega_M(|(u_1-u_2)||D\varphi_\delta|) \tag{5.2}$$

where $w_\delta=\varphi_\delta u_1+(1-\varphi_\delta)u_2$. Indeed, let $(u_1^\varepsilon)_\varepsilon$ and $(u_2^\varepsilon)_\varepsilon$ be the recovery sequences for $u_1$ and $u_2$ in $A_1^\delta$ and $A_2^\delta$ respectively, i.e. $u_i^\varepsilon$ converges to $u_i$ in $L^\infty(A_i^\delta)$ and

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_i^\varepsilon,A_i^\delta)\le F''(u_i,A_i^\delta)\,.$$

In particular, thanks to condition (H), there exists a positive constant $M_1=\alpha^{-1}(\sup_i F''(u_i,A_i^\delta))$ such that $||Du_i^\varepsilon||_{L^\infty(A_i^\delta)}\le M_1$ for $\varepsilon$ small enough. Let $w_\varepsilon^\delta=\varphi_\delta u_1^\varepsilon+(1-\varphi_\delta)u_2^\varepsilon$. As $\varepsilon$ goes to zero $w_\varepsilon^\delta$ converges to $w_\delta$, then

$$F''(w_\delta,A)\le\liminf_{\varepsilon\to 0}\sup_A f\left(\frac{x}{\varepsilon},Dw_\varepsilon^\delta\right)\,. \tag{5.3}$$

By the level convexity of $f$ and by condition (HC), for $\varepsilon$ small enough we have that

$$\begin{aligned}&\sup_A f\left(\frac{x}{\varepsilon},Dw_\varepsilon^\delta\right)\\ &=\sup_{A_1\setminus S^\delta_{1,2}} f\left(\frac{x}{\varepsilon},Du_1^\varepsilon\right)\vee\sup_{A_2\setminus S^\delta_{1,2}} f\left(\frac{x}{\varepsilon},Du_2^\varepsilon\right)\end{aligned}$$

$$
\begin{aligned}
&\vee \sup_{S^\delta_{1,2}} f\left(\frac{x}{\varepsilon}, \varphi_\delta Du^\varepsilon_1 + (1-\varphi_\delta)Du^\varepsilon_2 + (u^\varepsilon_1 - u^\varepsilon_2)D\varphi_\delta\right) \\
&\le \sup_{A^\delta_1} f\left(\frac{x}{\varepsilon}, Du^\varepsilon_1\right) \vee \sup_{A^\delta_2} f\left(\frac{x}{\varepsilon}, Du^\varepsilon_2\right) + \sup_{S^\delta_{1,2}} \omega_M(|u^\varepsilon_1 - u^\varepsilon_2||D\varphi_\delta|)
\end{aligned}
$$

where $M = 2M_1 + \frac{4}{\delta}||u_1 - u_2||_{L^\infty(S^\delta_{1,2})}$. Since $|u^\varepsilon_1 - u^\varepsilon_2|$ converges uniformly to $|u_1 - u_2|$ in $S^\delta_{1,2}$ as $\varepsilon \to 0$, in view of the continuity of $\omega_M$ and by using (5.3) we get (5.2).

**Step 3.** (Upper bound for $C^1$ functions through piecewise affine approximation) We now prove that the $\Gamma$-limsup inequality (5.1) holds for every (regular) open set $\Omega \subset \mathbf{R}^N$ and for every $u \in C^1(\Omega)$. Fix $u \in C^1(\Omega)$ and fix $n \in \mathbf{N}$. Then there exists a piecewise affine function

$$
u_n(x) = \sum_i (\xi^n_i x + c^n_i)\chi_{A_{n,i}}\,,
$$

where $(A_{n,i})_i$ is a regular triangularization of $\Omega$ of side $l_n$ sufficiently small such that

$$
\|u_n - u\|_{W^{1,\infty}(\Omega)} \le \frac{2}{n} \tag{5.4}
$$

and

$$
|\xi^n_i - \xi^n_j| < \frac{1}{n} \quad \forall i\ ,\ j \quad \text{such that } \overline{A}_{n,i} \cap \overline{A}_{n,j} \neq \emptyset\,. \tag{5.5}
$$

For every $n \in N$ let us fix $\delta = \delta(l_n)$ such that $A^\delta_{n,i} \cap A^\delta_{n,j} \neq \emptyset$ if and only if $\overline{A}_{n,i} \cap \overline{A}_{n,j} \neq \emptyset$. Now we want to apply the estimate (5.2) to the functions $u^n_i = \xi^n_i x + c^n_i$ in the sets $A_{n,i}$ and $A^\delta_{n,i}$. Denote by

$$
S^\delta_{n,i,j} := A^\delta_{n,i} \cap A^\delta_{n,j} \cap A_{n,i} \cap A_{n,j}
$$

for all $i$ and $j$ such that $\overline{A}_{n,i} \cap \overline{A}_{n,j} \neq \emptyset$, otherwise define $S^\delta_{n,i,j} = \emptyset$. Let $\varphi^\delta_{n,i,j}$ be a cut-off function in $S^\delta_{n,i,j}$ and let $w^n_\delta$ be the function obtained by gluing all the functions $u_{n,i}$ through $\varphi^\delta_{n,i,j}$. Note that if we define $M_{i,j} := 2\alpha^{-1}(F''(u^n_i, A^\delta_{n,i}) \vee F''(u^n_j, A^\delta_{n,j})) + \frac{4}{\delta}||u^n_i - u^n_j||_{L^\infty(S^\delta_{n,i,j})}$, then there exists $M \in \mathbf{R}$ such that $M > \sup_{i,j} M_{i,j}$ and it does not depend on $n$. In fact, by Step 1 and by definition of $u_n$, we have that

$$
\begin{aligned}
M_{i,j} &\le 2\alpha^{-1}(f^{\text{hom}}(\xi^n_i) \vee f^{\text{hom}}(\xi^n_j)) + \frac{4}{\delta}\frac{\delta}{n} \\
&\le 2\alpha^{-1}(F^{\text{hom}}(u_n, \Omega)) + \frac{4}{n},
\end{aligned}
$$

which converges to $2\alpha^{-1}(F^{\text{hom}}(u, \Omega))$, thanks to the continuity of $F^{\text{hom}}$. Then, by applying (5.2) and by Step 2, we get

$$
\begin{aligned}
F''(w^n_\delta, \Omega) &\le \bigvee_i F''(u_{n,i}, A^\delta_{n,i}) + C \sup_{i,j} \sup_{S^\delta_{n,i,j}} \omega_M(|u_{n,i} - u_{n,j}||D\varphi^\delta_{i,j}|) \\
&\le \bigvee_i f^{\text{hom}}(\xi^n_i) + C \sup_{i,j} \sup_{S^\delta_{n,i,j}} \omega_M(|\xi^n_i - \xi^n_j|) \\
&\le \operatorname*{ess\,sup}_\Omega f^{\text{hom}}(Du_n) + C\omega_M(\frac{1}{n})
\end{aligned}
$$

where the constant $C$ depends on the dimension of the space (namely $C$ is the maximum number of simplexes that can meet at a point). Finally, since $||w_\delta^n - u||_\infty \leq \frac{1}{n}$, by using the lower semicontinuity of $F''$ and the continuity of $f^{\rm hom}$ we get the $\Gamma$-limsup inequality for $C^1$ functions.

**Step 4.** (The inner regularization on $W^{1,\infty}$ functions) For every $u \in W^{1,\infty}(\Omega)$ we denote by $F''_-(u,\cdot)$ the inner regularization of the increasing set function $F''(u,\cdot)$, i.e. for every open set $E$

$$F''_-(u,E) = \sup\{F''(u,E') \ : \ E' \subset\subset E\}\,.$$

In this step we prove that

$$F''_-(u,\Omega) \leq \operatorname*{ess\,sup}_{\Omega} f^{\rm hom}(Du) \tag{5.6}$$

Since $f^{\rm hom}$ is continuous, then $F^{\rm hom}$ is continuous with respect to the $W^{1,\infty}$ norm. Fix $u \in W^{1,\infty}(\Omega)$. Let $\Omega' \subset\subset \Omega$ and for any $\delta > 0$, let $\varphi_\delta$ be a mollifier, with $\int_\Omega \varphi_\delta\, dx = 1$, and define the function $u_\delta = u * \varphi_\delta$ in $\Omega'$ for every $\delta < {\rm dist}(\Omega', \mathbf{R}^N \setminus \Omega)$. Then $u_\delta \in C^\infty(\Omega')$ and converges to $u$ uniformly in $\Omega'$. Since $Du_\delta(y) = \int_{\mathbf{R}^N} \varphi_\delta(x-y) Du(x)\, dx$ the values of $Du_\delta$ belong to the convex hull of the values assumed by $Du$, then by the step 3 and the level convexity of $f^{\rm hom}$ we have

$$F''_-(u_\delta,\Omega') \leq F''(u_\delta,\Omega') \leq \operatorname*{ess\,sup}_{\Omega'} f^{\rm hom}(Du_\delta) \leq \operatorname*{ess\,sup}_{\Omega} f^{\rm hom}(Du)\,.$$

Taking the limit as $\delta \to 0$ and by using the inner regularity of $F''_-$, we obtain (5.6) for every $u \in W^{1,\infty}(\Omega)$.

**Step 5.** ($C^2$ and star-shaped domains) In this step we will prove that if $\Omega$ satisfies either condition (S) or condition (C2), then

$$F''(u,\Omega) \leq \operatorname*{ess\,sup}_{\Omega} f^{\rm hom}(Du) \tag{5.7}$$

for every $u \in W^{1,\infty}(\Omega)$, i.e. the $\Gamma$-limsup inequality.

Assume first $\Omega$ satisfying (S). The proof in this case is standard. We repeat it here for the sake of completeness. Without any loss of generality we may assume that $\Omega$ is strongly star-shaped with respect to $x_0 = 0$. Then for every $t > 1$ the set $t\Omega = \{tx \ : \ x \in \Omega\}$ strictly contains $\Omega$, i.e. $t\Omega \supset\supset \Omega$. For each $u \in W^{1,\infty}(\Omega)$, let $u_t(x) = tu(x/t)$. Since $Du_t = Du$ by the definition of the inner regularization and Step 4 we have

$$F''(u_t,\Omega) \leq F''_-(u_t,t\Omega) \leq \operatorname*{ess\,sup}_{t\Omega} f^{\rm hom}(Du_t) = \operatorname*{ess\,sup}_{\Omega} f^{\rm hom}(Du)\,.$$

The conclusion follows by taking the limit as $t \to 1$ and using the lower semicontinuity of $F''$.

To conclude this part, let us consider the case of $\Omega$ satisfying condition (C2). The idea is again to construct a "regular" map $\Phi$ which maps $\Omega$ into a set $\Omega'$, with $\Omega \subset\subset \Omega'$.

Let $d(x) = {\rm dist}(x,\Omega) - {\rm dist}(x, \mathbf{R}^n \setminus \Omega)$. Since $\Omega$ is of class $C^2$ there exists $\sigma > 0$ such that the function $d(x)$ is also $C^2$ in $\{x : -\sigma < d(x) < \sigma\}$. Let us denote by

$\Omega_\sigma$ the set $\{x : d(x) < \sigma\}$. Clearly $\Omega_\sigma \supset\supset \Omega$. For any $0 < \eta < 1$ we define the map $\Phi : \Omega \to \Omega_{\sigma\eta}$ by

$$\Phi_\eta(x) = x + \eta(\sigma + d(x))_+ Dd(x)\,.$$

Since $D\Phi_\eta(x) = I + \eta(\sigma + d(x))_+ D^2 d(x) + \eta Dd(x) \otimes Dd(x)\chi_{(-\sigma,+\infty)}(d(x))$, for $\eta$ small enough $\Phi$ is bi-lipschitz. Now fix $u \in W^{1,\infty}(\Omega)$, we can define the function $v_\eta(x) = u(\Phi_\eta^{-1}(x))$ for any $x \in \Omega_{\sigma\eta}$. It is easy to see that $v_\eta \in W^{1,\infty}(\Omega_{\sigma\eta})$ and that it converges to $u$ as $\eta \to 0$ strongly in $W^{1,\infty}(\Omega)$. As above we have

$$\begin{aligned} F''(v_\eta, \Omega) \le F''_-(v_\eta, \Omega_{\sigma\eta}) &\le \operatorname*{ess\,sup}_{x\in\Omega_{\sigma\eta}} f^{\hom}(D\Phi_\eta^{-1}(x)Du(\Phi_\eta^{-1}(x))) \\ &= \operatorname*{ess\,sup}_{x\in\Omega} f^{\hom}\left((D\Phi_\eta(x))^{-1}Du(x)\right)\,. \end{aligned}$$

The conclusion follows taking the limit as $\eta \to 0$ and using the lower semicontinuity of the $\Gamma$-limsup. Then the $\Gamma$-limsup inequality is proved and hence we that

$$F''_-(u, \Omega) = F''(u, \Omega) = \operatorname*{ess\,sup}_{\Omega} f^{\hom}(Du) \quad \forall u \in W^{1,\infty}(\Omega)$$

for every $\Omega$ satisfying (C2) or (S).

**Step 6.** (Sub-supremality of the inner regularization) By the previous step we have that for every $u \in W^{1,\infty}(\Omega)$, the inner regularization of $F''(u, \cdot)$ is *finitely sub-supremal*, i.e.

$$F''_-(u, A \cup B) \le F''_-(u, A) \vee F''_-(u, B)\,. \tag{5.8}$$

Fix $A, B \subseteq \Omega$ and let $A' \subset\subset A$ and $B' \subset\subset B$, with $A'$ and $B'$ satisfying (C2). Thus for every open set $C \subseteq A' \cup B'$ satisfying (C2) we have

$$F''_-(u, C) = \operatorname*{ess\,sup}_{C} f^{\hom}(Du) \le \operatorname*{ess\,sup}_{A'\cup B'} f^{\hom}(Du) \le F''(u, A') \vee F''(u, B')\,.$$

Taking the supremum in $A'$, $B'$ and $C$ we get (5.8). As a consequence, if $\Omega = \bigcup_{i=1}^\infty A_i$, $A_i \in \mathcal{A}$, then

$$F''_-(u, \Omega) = \bigvee_{i=1}^{\infty} F''_-(u, A_i) \tag{5.9}$$

for all $u \in W^{1,\infty}(\Omega)$. In fact, by the inner regularity of $F''_-$, there exists $V_\varepsilon \subset\subset \Omega$ such that $F''_-(u, \Omega) \le F''_-(u, V_\varepsilon) + \varepsilon$. Then there exists a finite subset $J \subset \mathbf{N}$ such that $V_\varepsilon \subset\subset \bigcup_{j\in J} A_j$ and by using the finite sub-supremality, we have that $F''_-(u, \Omega) \le \bigvee_{j\in J} F''_-(u, A_j) + \varepsilon \le \bigvee_{i=1}^\infty F''_-(u, A_i) + \varepsilon$, i.e. the property (5.9).

**Step 7.** Since every open set $\Omega$ can be covered with a countable family of balls $B_n$, thanks to the countable supremality of $F''_-$ and thanks to Step 5 , we have that

$$F''_-(u, \Omega) = \bigvee_n F''_-(u, B_n) = \bigvee_n \operatorname*{ess\,sup}_{B_n} f^{\hom}(Du) = \operatorname*{ess\,sup}_{\Omega} f^{\hom}(Du)$$

for every $u \in W^{1,\infty}(\Omega)$. Now since $F''(u, \Omega)$ is lower semicontinuous in $u$ with respect to the uniform topology and is increasing with respect to $\Omega$, by Proposition

15.15 in [15] we can conclude that it coincides with its inner regularization on a rich family of open sets, and hence we deduce that

$$\{\Omega \in \mathcal{A} : \ F''(u,\Omega) = \operatorname*{ess\,sup}_{\Omega} f^{\hom}(Du) \quad \forall u \in W^{1,\infty}(\Omega)\}$$

is rich in $\mathcal{A}$. □

# 6 Some examples

Using the cell-problem formula we can compute the limit functional in some particular cases. We first show that in the 1-dimensional case it is easy to deduce a necessary condition in order to be a solution to the cell-problem formula.

Using the cell-problem formula we can compute the limit functional in some particular cases. We first show that in the 1-dimensional case it is easy to deduce a necessary condition in order to be a solution to the cell-problem formula.

**Remark 6.1** (Euler equation) Let $a \in L^\infty(\mathbf{R}^2)$ be 1-periodic and let $g : \mathbf{R} \to \mathbf{R}^+$ be coercive, level convex and 1-homogeneus. We consider a function $f : \mathbf{R}^2 \to \mathbf{R}$ of the form $f(x,\xi) = a(x) + g(\xi)$ or of the form $f(x,\xi) = a(x)g(\xi)$. Note that $f(x,0) = \min_{\xi\in\mathbf{R}} f(x,\xi)$ for every $x \in [0,1]$ and $f^{\hom}(\xi) \geq \operatorname{ess\,sup}_{(0,1)} f(x,0)$. In particular $f^{\hom}(0) = \operatorname{ess\,sup}_{(0,1)} f(x,0)$ and $u = 0$ is a solution of the cell-problem formula.

Fix $\xi \in \mathbf{R}$ and let $u \in W^{1,\infty}((0,1))$ be the solution of the corresponding cell-problem formula, i.e. $u(0) = 0$, $u(1) = \xi$ and $\operatorname{ess\,sup}_{(0,1)} f(x,u') = f^{\hom}(\xi)$. If $f^{\hom}(\xi) > \operatorname{ess\,sup}_{(0,1)} f(x,0)$, then $u$ must satisfy

$$f(x,u'(x)) = f^{\hom}(\xi) \qquad \text{a.e.} \quad x \in (0,1)\,. \tag{6.1}$$

In order to show this, note that as a consequence of the level convexity and the continuity of $g$ we have that for every $x \in (0,1)$ and for every $M > f(x,0)$ there exists $a_M(x) < 0 < b_M(x)$ such that

$$C_M(x) := \{f(x,t) \leq M\} = [a_M(x), b_M(x)]$$

and $\{f(x,t) < M\} = (a_M(x), b_M(x))$. Now fix $M = f^{\hom}(\xi)$ and assume by contradiction that there exists a subset $I$ of $(0,1)$, with $\mathcal{L}(I) > 0$, such that $f(x,u'(x)) < M$. Thus $a_M(x) < u'(x) < b_M(x)$ for a.e. $x \in I$. Let $\xi > 0$ (the case $\xi < 0$ is analogous) and define the function $v$ with $v(0) = 0$ and $v'(x) = b_M(x)$ if $x \in I$ and $v'(x) = u'(x)$ otherwise in $(0,1)$. Clearly, since $\mathcal{L}(I) > 0$ and $v' > u'$ in $I$ we have that $v(1) = \xi + \delta$, for some positive $\delta$, and by the choice of $v'$ we also have that $f(x,v') = M$ for every $x \in (0,1)$. We can conclude considering the function $w(x) = \frac{\xi}{\xi+\delta}v(x)$. This is admissible in the computation of $f^{\hom}(\xi)$ and by the 1-homogenity of $g$, the conclusion follows easily. In fact, in the first case we have

$$\operatorname*{ess\,sup}_{\Omega} a(x)g(w') = \frac{\xi}{\xi+\delta} \operatorname*{ess\,sup}_{\Omega} a(x)g(v') = \frac{\xi}{\xi+\delta} f^{\hom}(\xi) < f^{\hom}(\xi)$$

while in the second one

$$\begin{aligned}
\operatorname*{ess\,sup}_{\Omega} \big(a(x) + g(w')\big) &\le \operatorname*{ess\,sup}_{\Omega} \big(\frac{\xi}{\xi+\delta}(a(x) + g(v')) + \frac{\delta}{\xi+\delta} a(x)\big) \\
&= \operatorname*{ess\,sup}_{\Omega} \big(\frac{\xi}{\xi+\delta} f^{\mathrm{hom}}(\xi) + \frac{\delta}{\xi+\delta} a(x)\big) \\
&= \frac{\xi}{\xi+\delta} f^{\mathrm{hom}}(\xi) + \frac{\delta}{\xi+\delta} \operatorname*{ess\,sup}_{\Omega} a(x) < f^{\mathrm{hom}}(\xi).
\end{aligned}$$

Both the previuous inequalities are in contradiction with the definition of $f^{\mathrm{hom}}(\xi)$ and then (6.1) is proved.

We now use the previous remark to explicitly compute some 1-dimensional example.

**Example 6.2** Let $\alpha : \mathbf{R} \to \mathbf{R}$ be a 1-periodic bounded Borel function and let $f(x,\xi) := \alpha(x) + |\xi|$. Upon a translation we can suppose that $\alpha(x) > 0$. Clearly such a function satisfies the conditions in Remark 6.1. Let us denote by $\bar{\alpha} = \operatorname{ess\,sup}_{(0,1)} \alpha(x)$. With fixed $\xi \in \mathbf{R}$, let $u(x)$ be a solution of the corresponding cell-problem formula. If $|\xi| > \bar{\alpha} - \int_0^1 \alpha\, dx$, then

$$f^{\mathrm{hom}}(\xi) = \operatorname*{ess\,sup}_{(0,1)}(\alpha(x) + |u'(x)|) \ge \alpha(x) + |u'(x)| \qquad \text{a.e. } x \in (0,1)$$

and, integrating this inequality, we have $f^{\mathrm{hom}}(\xi) > \bar{\alpha}$. By Remark 6.1, we obtain that $u$ must satisfy

$$|u'(x)| = f^{\mathrm{hom}}(\xi) - \alpha(x)$$

and then $f^{\mathrm{hom}}(\xi) = |\xi| + \int_0^1 \alpha\, dx$. In the case $\xi = \bar{\alpha} - \int_0^1 \alpha\, dx$, testing the cell-problem formula with $u'(x) = \bar{\alpha} - \alpha(x)$, one can check that $f^{\mathrm{hom}}(\xi) = \bar{\alpha}$. Finally it easy to see that

$$f^{\mathrm{hom}}(\xi) = \begin{cases} |\xi| + \int_0^1 \alpha\, dx & \text{if } |\xi| > \bar{\alpha} - \int_0^1 \alpha\, dx \\ \bar{\alpha} & \text{if } |\xi| \le \bar{\alpha} - \int_0^1 \alpha\, dx. \end{cases}$$

**Example 6.3** Let $a : \mathbf{R} \to \mathbf{R}$ be a 1-periodic Borel function, $0 < \alpha \le a(x) \le \beta < +\infty$. Define $f(x,t) := a(x)|t|$ for every $(x,t) \in \mathbf{R}^2$.

The function $f$ satisfies the assumptions in Remark 6.1 and $f(x,0) = 0$ for every $x \in (0,1)$. This implies that $f^{\mathrm{hom}}(\xi) = 0$ if and only if $\xi = 0$. Thus, in order to compute $f^{\mathrm{hom}}(\xi)$, with $\xi \ne 0$, let $u$ be a solution of the corresponding cell-problem formula and by (6.1) we get

$$|u'(x)| = \frac{f^{\mathrm{hom}}(\xi)}{a(x)} \qquad \text{a.e. } x \in (0,1)\,. \tag{6.2}$$

By the minimality of $u$, it is easy to see that it must have constant sign and thus, by simply integrating (6.2), we have

$$f^{\mathrm{hom}}(\xi) = \left(\int_0^1 \frac{1}{a(x)} dx\right)^{-1} |\xi| \qquad \forall\, \xi \in \mathbf{R}\,. \tag{6.3}$$

In the particular case of a two phase mixture, i.e. $a(x) = \alpha\chi_I + \beta(1-\chi_I)$, with $I \subseteq (0,1)$ and $\mathcal{L}(I) = \theta \in (0,1)$, we have

$$f^{\hom}(\xi) = \frac{1}{\frac{\theta}{\alpha} + \frac{(1-\theta)}{\beta}}|\xi| \qquad \forall\, \xi \in \mathbf{R}\,.$$

Already in this simple case of 1-dimensional homogenization, the two examples above shows similarities with different kind of problems. The result in the first example resemble very much a problem of homogenization of Hamilton-Jacobi, while the second example shows a behaviour very similar to that observed in quadratic integral homogenization.

**Remark 6.4** Note that the expression (6.3) for $f^{\hom}$ can also be deduced by using the $L^p$ approximation proved in Lemma 3.2. Indeed using the cell-problem formula for integral homogenization we get

$$(f_p^{\hom})^{1/p}(\xi) = \left(\int_0^1 \big(a(x)\big)^{p/(1-p)}dx\right)^{(1-p)/p}|\xi|$$

and hence we obtain (6.3) taking the limit as $p \to \infty$.

**Remark 6.5** In [11] through a dual formulation is proved that the $\Gamma(L^\infty)$-limit of a sequence of the form

$$\operatorname*{ess\,sup}_{x\in\Omega} f_\varepsilon(x, u(x)) \qquad u \in L^\infty(\Omega) \tag{6.4}$$

is still a supremal functional and an explicit representation formula for the energy density is given in terms of a conjugation argument. In particular in the 1-dimensional case this result can be applied to the sequence

$$F_\varepsilon(u) := \operatorname*{ess\,sup}_{x\in(0,1)} f\Big(\frac{x}{\varepsilon}, u'(x)\Big) \qquad u \in W^{1,\infty}((0,1)) \tag{6.5}$$

and gives an alternative representation formula for $f^{\hom}(\xi)$.

We conclude the paper by recalling that 2-dimensional examples with a two-phase mixture (see [17]) highlight the fact that supremal functionals are very sensitive to conditions imposed on very small sets, differently from integral functionals. In [17] Example 3.9, there has been constructed, for arbitrary $\theta \in (0,1)$, a function $a : (0,1)^2 \to \{\alpha, \beta\}$, with $0 < \alpha < \beta < +\infty$ and $\mathcal{L}(\{x \in (0,1)^2 \;:\; a(x) = \alpha\}) = \theta$, such that the homogenization of

$$\operatorname*{ess\,sup}_{\Omega} a\Big(\frac{x}{\varepsilon}\Big)|Du|$$

only feels the stronger phase, i.e.

$$f^{\hom}(\xi) = \beta|\xi| \qquad \forall\, \xi \in \mathbf{R}^2\,,$$

even if the phase $\beta$ is very small.

## References


[1] E. Acerbi, G. Buttazzo and F. Prinari, The class of functionals which can be represented by a supremum, *J. Convex Anal.* **9** (2002) 225–236.

[2] O. Alvarez and E. N. Barron, Homogenization in $L^\infty$, *J. Differential Equations* **183** (2002), no. 1, 132–164.

[3] G. Aronsson, Extension of Functions satisfying Lipschitz conditions, *Ark. Mat.* **6** (1967) 551–561.

[4] E. N. Barron and R. Jensen, Relaxed minimax control, *SIAM J. Control Optim.* **33** (1995) 1028–1039.

[5] E. N. Barron, R. R. Jensen and C.Y.Wang, Lower Semicontinuity of $L^\infty$ Functionals, *Annales de Inst. Henry Poincare Anal. Non lineaire (4)* **18** (2001) 495–517.

[6] E. N. Barron, R. R. Jensen and C. Y. Wang, The Euler Equation and Absolute Minimizers of $L^\infty$ Functionals, *Arch. Rational Mech. Anal.* (4) **157** (2001) 225–283.

[7] E. N. Barron and W. Liu, Calculus of Variation in $L^\infty$, *Appl. Math. Optim.* (3) **35** (1997) 237–263.

[8] T. Bhattacharya, E. Di Benedetto and J.Manfredi, Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems, *Some topics in nonlinear PDEs (Turin, 1989). Rend. Sem. Mat. Univ. Politec. Torino 1989, Special Issue* (1991) 15–68.

[9] A. Braides and A. Defranceschi, Homogenization of Multiple Integrals, Clarendon Press, Oxford, 1998.

[10] A. Braides, Γ-convergence for beginners, Oxford University Press, Oxford, 2002.

[11] A. Briani and F. Prinari, A representation result for Γ-limit of supremal functionals. *J. Nonlinear Convex Anal.* (4) **2** (2003) 245–268.

[12] L. Carbone, A. Corbo Esposito and R. De Arcangelis, Homogenization of Neumann Problems for Unbounded Integral Functionals, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (2) **2** (1999), 463–491.

[13] T. Champion and L. De Pascale, Homogenization of Dirichlet problems with convex bounded constraints on the gradient, *Z. Anal. Anwend.* (3) **22**, (2003), 591–608.

[14] T. Champion, L. De Pascale and F. Prinari, Γ-convergence and absolute minimizers for supremal functionals, *ESAIM Control Optim. Calc. Var.* to appear.

[15] G. Dal Maso, An introduction to Γ-convergence, Birkhäuser, Boston, 1993.

[16] E. De Giorgi and T. Franzoni, Su un tipo di convergenza variazionale, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **58** (1975) 842–850.

[17] A. Garroni, V. Nesi and M. Ponsiglione, Dielectric Breakdown: optimal bounds, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* (2001) 2317–2335.

[18] A. Garroni, M. Ponsiglione and F. Prinari, In preparation.

[19] R. R. Jensen, Uniqueness of Lipschitz Extensions: Minimizing the Sup Norm of the Gradient. *Arch. Rational Mech. Anal.* **123** (1993) 51–74.

[20] P. Juutinen, Absolutely Minimizing Lipschitz Extensions on a metric space, *Ann. Acad. Sci. Fenn. Math.* **27** (2002) 57–67.

[21] P. Juutinen, Minimization problems for Lipschitz functions via viscosity solutions, *Ann. Acad. Sci. Fenn. Math. Diss.* No. 115, 1998.

[22] R.V. Kohn and T. D. Little, Some model problems of polycrystal plasticity with deficient basic crystals, *SIAM J. Appl. Math.* (1) **59** (1999) 172–197.

[23] F. Prinari, Relaxation and Γ-convergence of supremal functionals. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.*, to appear.