Concentration phenomena for the volume functional in unbounded domains: Identification of concentration points

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Abstract

We study the variational problem

$$S^F_\varepsilon(\Omega) = \frac{1}{\varepsilon^{2^*}} \sup \left\{ \int_\Omega F(u) \; : \; \int_\Omega |\nabla u|^2 \le \varepsilon^2, \; u = 0 \; \text{ on } \; \partial\Omega \right\} ,$$

in possibly unbounded domains $\Omega \subset \mathbf{R}^n$, where $n \ge 3$, $2^* = \frac{2n}{n-2}$ and F satisfies $0 \le F(t) \le \alpha |t|^{2^*}$ and is upper semicontinuous. Extending earlier results for bounded domains we show that (almost) maximizers of $S^F_\varepsilon(\Omega)$ concentrate at a harmonic center, i.e. a minimum point of the Robin function τ_Ω (the regular part of the Green function restricted to the diagonal). Moreover we obtain the asymptotic expansion

$$S^F_\varepsilon(\Omega) = \; S^F \left(1 - \frac{n}{n-2} w_\infty^2 \min_{\overline{\Omega}} \tau_\Omega \, \varepsilon^2 + o(\varepsilon^2) \right)$$

where S^F and w_∞ depend only on F but not on Ω and can be computed from radial maximizers of the corresponding problem in $\mathbf{R}^n$. The crucial point is to find a suitable definition of $\tau_\Omega(\infty)$. Interestingly the correct definition may be different from the lower semicontinuous extension of $\tau_\Omega|_{\overline{\Omega}\setminus\{\infty\}}$ to ∞, at least for $n \ge 5$.

Keywords: variational problem, concentration, Robin function, unbounded domains

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1 Introduction

Let Ω be a domain in $\mathbf{R}^n$, $n \ge 3$. Consider the variational problem

$$\text{(1)} \qquad \sup \left\{ \frac{1}{\varepsilon^{2^*}} \int_\Omega F(u) \; : \; \int_\Omega |\nabla u|^2 \le \varepsilon^2, \; u = 0 \; \text{ on } \; \partial\Omega \right\} ,$$

where the integrand is supposed to satisfy the growth condition

$$0 \;\le\; F(t) \;\le\; \alpha\, |t|^{2^*}$$

for some $\alpha > 0$ and $2^* := \frac{2n}{n-2}$ denotes the critical Sobolev exponent. For smooth integrands every solution of (**??**) satisfies the Euler Lagrange equation

$$\text{(2)} \qquad \begin{aligned} -\Delta u &= \lambda f(u) \;\text{ in }\; \Omega, \\ u &= 0 \;\text{ on }\; \partial\Omega \end{aligned}$$

with $f = F'$ and a large Lagrange multiplier λ. In [**?**] Flucher and Müller studied the asymptotic behaviour of the solutions u_ε of (**??**) as $\varepsilon \to 0$ and they proved (at least for domains of finite volume) that a suitably rescaled sequence of (almost) maximizers u_ε always concentrates at a single point x_0 of $\overline{\Omega}$ (after possible extraction of a subsequence). More precisely

$$\text{(3)} \qquad \frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_{x_0} \qquad \text{and} \qquad \frac{F(u_\varepsilon)}{\varepsilon^{2^*}} \overset{*}{\rightharpoonup} S^F \delta_{x_0}$$

where S^F is a constant depending only on F.

For applications such as Bernoulli free-boundary problem or the plasma problem it is important to know the location of the concentration point. For bounded domains it was shown in [?] that concentration occurs at a harmonic center, i.e. at a minimum point of the Robin function τ_Ω (the regular part of the Green function of Ω restricted to the diagonal). Moreover the supremum $S^F_\varepsilon(\Omega)$ in (??) has the asymptotic expansion

$$S^F_\varepsilon(\Omega) = S^F \left(1 - \frac{n}{n-2} w_\infty^2 \min_{\overline{\Omega}} \tau_\Omega \, \varepsilon^2 + o(\varepsilon^2)\right) .$$

In this paper we extend these results to unbounded domains (see Theorems ?? and ?? below). The crucial point is that in this case concentration may occur at ∞. Thus we need to define τ_Ω also at ∞. This is done in Definition ?? below. The definition ensures that $\tau_\Omega : \overline{\Omega} \to \mathbf{R} \cup \{+\infty\}$ is lower semicontinuous (here and in the following we consider the closure of Ω in $\mathbf{R}^n \cup \{\infty\}$, the one point compactification of $\mathbf{R}^n$). Interestingly $\tau_\Omega(\infty)$ may, however, be strictly lower than the lower semicontinuous extension of $\tau_\Omega|_{\overline{\Omega}\setminus\{\infty\}}$ to ∞ (see Example ??).

The relevance of the critical points of the Robin function for Dirichlet problems that involve the critical Sobolev exponent was first pointed out by Schoen [?] and Bahri [?]. Rey [?] and Han [?] showed that as $p \to 2^*$ the maximum points of the positive solutions of

$$\begin{aligned} \Delta u + u^{p-1} &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

accumulate at a critical point of the Robin function. This has been conjectured by Brézis and Peletier [?]. The simpler proof of [?] applies to all dimensions and shows that the concentration point is a minimum point of the Robin function. Similar results for the Ginzburg-Landau functional have been obtained by Bethuel, Brézis and Hélein [?]. For further discussions on concentration effects and the relevant literature see also [?].

To minimize technicalities we consider mostly the Bernoulli free boundary value problem, i.e. the maximization of volume for given (small) capacity. This corresponds to the integrand $F(t) = \chi_{\{t \geq 1\}}$.

The main technical difficulty for general integrands is that one essentially has to work with the level sets of the maximizer u_∞ of problem

$$S^F = \sup \left\{ \int_{\mathbf{R}^n} F(u) \, : \, \|\nabla u\|_{L^2} \leq 1 \right\} \tag{4}$$

rather then those of the Green function.
Since u_∞ approaches the Green function of $\mathbf{R}^n$ as $|x| \to \infty$ the arguments are similar but technically more involved.
The tools to overcome these technical difficulties, however, are essentially the same as for the bounded domains [?] and we review them briefly in the appendix.

Another subtlety arises in unbounded domains if $F(t)$ has critical growth near the origin. Then maximizing sequences for problem (??) become arbitrarily flat. In this case we need to impose the condition $\tau_\Omega(\infty) > 0$ to assure that maximizing sequences for (??) still concentrate at a single point, after suitable translation. The condition $\tau_\Omega(\infty) > 0$ requires, roughly speaking, that $\mathbf{R}^n \setminus \Omega$ is not to small at ∞ and holds e.g. for cylinders like domains $\Omega = \{(x', x_n) \in \mathbf{R}^n \, : \, |x'| \leq f(x_n)\}$ with f continuous and $\liminf_{t\to\pm\infty} f(t) < +\infty$ (but possibly $\limsup_{t\to\pm\infty} f(t) = +\infty$).
Equivalent conditions and their consequences are also discussed in the appendix.

2 Hypotheses, generalized Sobolev inequality and concentration

Let Ω be an open subset of $\mathbf{R}^n$, $n \geq 3$. By $\overline{\Omega}$ we denote the closure of Ω in $\mathbf{R}^n \cup \{\infty\}$. In particular the closure of an unbounded domain contains the point ∞.

The natural function space for variational problems of the form (??) is the space $D^{1,2}(\Omega)$ defined as the closure of $C^\infty_c(\Omega)$ with respect to the norm

$$\|\nabla v\|_2 = \left(\int_\Omega |\nabla v|^2 \right)^{1/2} .$$

We shall study the behaviour, as $\varepsilon \to 0$, of following variational problem

$$(5) \qquad S^V_\varepsilon(\Omega) \;:=\; \frac{1}{\varepsilon^{2^*}} \sup\left\{|A| \,:\, A \text{ open subset of } \Omega\,,\ \mathrm{cap}_\Omega A \le \varepsilon^2\right\},$$

where $\mathrm{cap}_\Omega A$ denote the harmonic capacity of A with respect to Ω, i.e.

$$(6) \qquad \mathrm{cap}_\Omega A = \inf\left\{\int_\Omega |\nabla u|^2\,dx \,:\, u \in D^{1,2}(\Omega)\,,\ u \ge 1 \text{ a.e. in } \ A\right\}.$$

This infimum is achived by a function u called the capacitary potential of A with respect to Ω. Thus problem (**??**) can be equivalently written as

$$S^V_\varepsilon(\Omega) \;:=\; \frac{1}{\varepsilon^{2^*}} \sup\left\{|\{u \ge 1\}| \,:\, u \in D^{1,2}(\Omega),\ \|\nabla u\|_2 \le \varepsilon\right\},$$

so that it can be seen as a particular case of problem (**??**), when $F(t) = \chi_{\{t\ge 1\}}$.

We require the following very weak assumption for the domain Ω

(7) Ω is a domain in $\mathbf{R}^n$ of dimension $n \ge 3$ with $\Omega \ne \mathbf{R}^n$ in the sense that $\mathrm{cap}_{\mathbf{R}^n}(\mathbf{R}^n \setminus \Omega) > 0$.

Define the generalized Sobolev constant by

$$S^V \;:=\; S^V_1(\mathbf{R}^n).$$

By taking into account that the capacity of a ball of radius r is given by $\mathrm{cap}_{\mathbf{R}^n} B_r = (n-2)|S^{n-1}|r^{n-2}$ we easily compute $S^V = ((n-2)|S^{n-1}|)^{n/(2-n)}$. Since $\mathrm{cap}_{\mathbf{R}^n}(\rho A) = \rho^{n-2}\mathrm{cap}_{\mathbf{R}^n} A$ and $\mathrm{cap}_{\mathbf{R}^n} A \le \mathrm{cap}_\Omega A$ we have $S^V_\varepsilon(\Omega) \le S^V$. A simple scaling argument leads to the isoperimetric inequality for the capacity

$$(8) \qquad |A| \;\le\; S^V\,(\mathrm{cap}_\Omega A)^{2^*/2}$$

Moreover $S^V_\varepsilon(\Omega) \to S^V$ as $\varepsilon \to 0$ (see e.g. [**?**]). By this fact together with the generalized concentration compactness alternative proved in the same paper, one can easily deduce the following concentration result.

Theorem 1 *Let A_ε be a sequence of extremals for problem (**??**), i.e. $\mathrm{cap}_\Omega(A_\varepsilon) = \varepsilon^2$ and $|A_\varepsilon| \to S^V$ as $\varepsilon \to 0$, and let u_ε be the corresponding capacitary potential with respect to Ω. Then there exists $x_0 \in \overline{\Omega}$ such that*

$$(9) \qquad \frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \;\overset{*}{\rightharpoonup}\; \delta_{x_0}, \qquad \frac{\chi_{A_\varepsilon}}{\varepsilon^{2^*}} \;\overset{*}{\rightharpoonup}\; S^V\delta_{x_0}$$

in the sense of measures.

Note that in order to obtain the concentration result it is enough to require that Ω satisfy $\mathrm{cap}_{\mathbf{R}^n}(\mathbf{R}^n \setminus \Omega) > 0$. This assumption essentially excludes only the case $\Omega = \mathbf{R}^n$.

Remark 2 In the result above the concentration at ∞ has to be understood as

$$\int_{\Omega\setminus B_R} \frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \to 1 \quad \text{and} \quad \frac{|A_\varepsilon \setminus B_R|}{\varepsilon^{2^*}} \to S^V \qquad \forall R \ge 0\,.$$

This convergence does not assure a priori that the sets A_ε concentrate at a single point, up a suitable translation. We will see in the sequel (see Proposition **??**) that for the volume functional this result is always true. In the general case of problem (**??**) a further assumption on the set Ω has to be made (see Appendix).

As a consequence of the concentration compactness alternative we have the following lemma.

Lemma 3 ([?], Lemma 13) *Let A_k be a sequence of compact sets such that $|A_k| = |B^1_0|$ and $\mathrm{cap}_{\mathbf{R}^n}(A_k)$ converges to $\mathrm{cap}_{\mathbf{R}^n}(B^1_0)$ as $k \to \infty$. Then, up to a subsequence, there exists a sequence $\{x_k\}$ such that the characteristic function of $A_k - x_k$ converges to the characteristic function of B^1_0 in L^1. Moreover if u_k and u denote the capacitary potential of A_k and B^1_0 respectively, then $u_k(x_k + \cdot)$ converges to u strongly in $D^{1,2}(\mathbf{R}^n)$.*

Proposition 4 *If* $\frac{\mathrm{cap}_{\mathbf{R}^n}(A_\varepsilon)}{|A_\varepsilon|^{\frac{n-2}{n}}} \to S^V$ *and* $|A_\varepsilon| \to 0$, *then there exist* x_ε *and* $r_\varepsilon \to 0$ *such that*

$$\frac{|A_\varepsilon \setminus B(x_\varepsilon, r_\varepsilon)|}{|A_\varepsilon|} \to 0$$

Proof. This result can be obtained as a direct consequence of Lemma **??**, arguing by contradiction. ○

Remark 5 If $\{A_\varepsilon\}$ is a sequence of extremals, then it satisfies (**??**), and therefore satisfies the assumption of Proposition **??**. In particular if (**??**) holds with $x_0 = \infty$, then there exists a sequence $x_\varepsilon \to \infty$ such that

$$\frac{|\nabla u_\varepsilon(\cdot - x_\varepsilon)|^2}{\varepsilon^2} \stackrel{*}{\rightharpoonup} \delta_0, \quad \frac{\chi_{\{A_\varepsilon - x_\varepsilon\}}}{\varepsilon^{2^*}} \stackrel{*}{\rightharpoonup} S^V \delta_0 \tag{10}$$

3 Robin function for unbounded domains

In this section Ω will be an arbitrary open subset of $\mathbf{R}^n$ with $n \geq 3$, which satisfies (**??**). The concentration point x_0 of Theorem **??** will be identified in terms of the Robin function of Ω, i.e. the diagonal of the regular part of the Green function of the Dirichlet problem in Ω for the $-\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$. This function has been considered in the context of concentration phenomena in [**?**] for domains with regular boundary. In [**?**] this definition has been extended to any domain, possibly with irregular boundary, and its main properties have been studied in the case of bounded domains.

In this section we shall summarize the definitions and the results given in [**?**] and we will extend them to the case of unbounded domains. In particular, since the concentration point, for some domains, could be at ∞ we need a good definition of the Robin function at ∞ and a accurate study of its behaviour near ∞.

Let us denote by $K_x(y) = K(|x-y|)$, for every $x, y \in \mathbf{R}^n$, the fundamental solution for $-\Delta$, i.e. $K(r) = c_n r^{2-n}$, with $c_n = ((n-2)|S^{n-1}|)^{-1}$. For every point $x \in \overline{\Omega} \setminus \{\infty\}$, let us define the regular part of the Green function, $H_\Omega(x, \cdot)$, as the solution in the sense of Perron-Wiener-Brelot (PWB) of the following Dirichlet problem

$$\begin{cases} \Delta_y H_\Omega(x,y) = 0 & \text{in } \Omega, \\ H_\Omega(x,y) = K_x(y) & \text{on } \partial\Omega, \end{cases} \tag{11}$$

i.e., $H_\Omega(x, \cdot)$ is the infimum of all superharmonic functions u such that

$$\liminf_{\substack{z \to y \\ z \in \Omega}} u(z) \geq K_x(y)$$

for every $y \in \partial\Omega$ (see [**?**]). If Ω is an external domain, then we require in addition that

$$\liminf_{\substack{z \to \infty \\ z \in \Omega}} u(z) \geq 0\,.$$

Note that the notion of PWB solution is stable under increasing sequences of admissible boundary data. Thus the function $H_\Omega(x,y)$ is well defined also if $x \in \partial\Omega \setminus \{\infty\}$. The Green function of the Dirichlet problem for $-\Delta$ is defined by

$$G_x(y) \quad = \quad K_x(y) - H_\Omega(x,y)\,.$$

The Green function is symmetric in $\Omega \times \Omega$ (see [**?**], Theorem 5.24); hence $H_\Omega(x,y) = H_\Omega(y,x)$ for every $(x,y) \in \Omega \times \Omega$.

If $x \in \Omega$ then the function $H_\Omega(x, \cdot)$ coincides with the weak solution of (**??**) in the sense of $D^{1,2}(\Omega)$.

For every $x \in \Omega \cup \partial\Omega \setminus \{\infty\}$, let us extend the function $H_\Omega(x, \cdot)$ to a superharmonic function $\widetilde{H}_\Omega(x, \cdot)$ defined on all $\mathbf{R}^n$, as follows: for every $y \in \partial\Omega \setminus \{\infty\}$ we set

$$\widetilde{H}_\Omega(x,y) = \liminf_{\substack{z \to y \\ z \in \Omega}} H_\Omega(x,z) \tag{12}$$

and $\widetilde{H}_\Omega(x,y) = K_x(y)$ for every $y \in \mathbf{R}^n \setminus \overline{\Omega}$ (see [?], Theorem 7.7). Finally let us extend $\widetilde{H}_\Omega(x,y)$ to $\mathbf{R}^n \times \mathbf{R}^n$ by setting $\widetilde{H}_\Omega(x,y) = K_x(y)$ for every $x \in \mathbf{R}^n \setminus \overline{\Omega}$. It has been proved in [?], Proposition 8, that for every $y \in \mathbf{R}^n$ the function $x \mapsto \widetilde{H}_\Omega(x,y)$ is superharmonic in $\mathbf{R}^n$ and, moreover, $(x,y) \mapsto \widetilde{H}_\Omega(x,y)$ is lower semicontinuous in $\mathbf{R}^n \times \mathbf{R}^n$.

We are now in a position to recall the definition of the Robin function, the harmonic radius and the harmonic center given in [?] and to extend it to ∞.

Definition 6 (Robin function, harmonic radius, harmonic center) *For every $x \in \Omega \cup \partial\Omega \setminus \{\infty\}$ the leading term of the regular part of the Green function*

$$\tau_\Omega(x) \quad := \quad \widetilde{H}_\Omega(x,x)$$

is called Robin function of Ω at the point x. The harmonic radius of Ω at x is defined by the relation $K(r(x)) = \tau_\Omega(x)$. The Robin function at infinity is defined as

$$\tau_\Omega(\infty) := \lim_{\rho\to 0}\lim_{R\to\infty} \inf_{\substack{x,y\in\mathbf{R}^n \\ |x|\geq R,\, |x-y|\leq\rho}} \widetilde{H}_\Omega(x,y) \tag{13}$$

A minimum point of the Robin function on $\overline{\Omega}$ is called a harmonic center of Ω.

In this way $\tau_\Omega : \overline{\Omega} \subset \mathbf{R}^n \cup \{\infty\} \to \mathbf{R}$ becomes a lower semicontinuous function. Nonetheless $\tau_\Omega(\infty)$ may be strictly below the largest lower semicontinuous extension of τ_Ω, at least for $n \geq 5$ as shown by the example below. A similar phenomenon can arise at other boundary points.

Example 7 We will construct an unbounded domain Ω such that $\tau_\Omega(\infty) < \liminf_{x\to\infty}\tau_\Omega(x)$. It will also provide an example of a set for which the extremals concentrate at ∞. The set Ω will be given by taking the whole space $\mathbf{R}^n$ and subtracting a sequence of small balls that accumulate at ∞. First make a partition of $\mathbf{R}^n$ by considering the annuli $C_k = B_{2^k}(0) \setminus B_{2^{k-1}}(0)$. In each annulus we consider small balls of radius r_k with centers (x_k^i) in a lattice of side d_k. We will choose later two suitable sequences $\{d_k\}$ and $\{r_k\}$ such that $d_k, r_k \to 0$ and $r_k \ll d_k$.

Set $\Omega = \mathbf{R}^n \setminus \cup_k \cup_{x_k^i \in C_k} B_{r_k}(x_k^i)$. Let us denote by u_k^i the capacitary potential of the ball $B_{r_k}(x_k^i)$. Thus $u_k^i(x) = K(|x - x_k^i|)/K(r_k)$. Let us now take any sequence $x_k \to \infty$. To estimate $\tau_\Omega(x_k)$ from below we may assume that $x_k \in C_k$ and that for any k the distance between x_k and the closest ball is of order d_k. Then in particular the Robin function of Ω in the point x_k can be estimated from below by the capacitary potential of such a ball scaled by $K(d_k)$, namely

$$\tau_\Omega(x_k) = \widetilde{H}_\Omega(x_k,x_k) \geq K(d_k)u_k^i(x_k) \approx \frac{K^2(d_k)}{K(r_k)} \approx \frac{r_k^{n-2}}{d_k^{2n-4}}\,. \tag{14}$$

Finally let us fix $0 < \rho < 1$ and let us estimate from above the infimum of $\widetilde{H}_\Omega(x_k,y)$ for $|x_k - y| = \rho$. We will estimate $\widetilde{H}_\Omega(x_k,y)$ by considering separately the contribution of the balls contained in each annulus C_h for $h \neq k$, that of the balls in the annulus $C_k \setminus B_\rho(x_k)$ and finally the contribution of the balls in $B_\rho(x_k)$. Now the capacity of the balls contained in each annulus C_h is of order $r_k^{n-2}2^{hn}/d_h^n$ (i.e., the capacity of a ball times the number of balls). Then the contribution of C_h is given by the total capacity of the balls contained in it multiplied by the fundamental solution computed on the distance between x_k and C_h that we very roughly estimate with 1. Similarly we deal with the balls in $C_k \setminus B_\rho(x_k)$. The contribution of the balls in $B_\rho(x_k)$ can be estimated first considering the contribution of the balls in $B_{\rho/2}(y)$ which gives a term of the form

$$K(\rho/2)r_k^{n-2}\int_0^{\rho/2}\frac{K(s)s^{n-1}}{d_k^n}ds$$

and then the contribution of the balls in $B_\rho(x_k) \setminus B_{\rho/2}(y)$ which similarly can be estimated by

$$K(\rho/2)r_k^{n-2}\int_0^{\rho}\frac{K(s)s^{n-1}}{d_k^n}ds.$$

Then

$$\inf_{|x_k-y|=\rho}\widetilde{H}_\Omega(x_k,y) \leq C\sum_{h\neq k}2^{hn}\frac{r_h^{n-2}}{d_h^n} + CK(\rho)^2 2^{kn}\frac{r_k^{n-2}}{d_k^n} + C\rho^2 K(\rho)\frac{r_k^{n-2}}{d_k^n}\,. \tag{15}$$

Choosing $d_k = 2^{-\alpha k}$, $r_k = 2^{-\beta k}$ and $n > 4$ we easily find values $\beta > \alpha > 0$ such that $\tau_\Omega(\infty) < \infty$ while $\liminf_{x\to\infty} \tau_\Omega(x) = +\infty$. Actually, this construction provides also an example of a set where the concentration occurs at ∞. Indeed a more accurate estimate in (**??**) shows that under the condition $r_k \ll 2^{-kn} d_k^n$ we have $\tau_\Omega(\infty) = 0$.

If $x \in \Omega$, the Green function can be expanded near the singularity as:

$$G_x(y) \quad = \quad K(|y-x|) - \tau_\Omega(x) + O(|y-x|). \tag{16}$$

It has the following properties.

Proposition 8 ([?, ?, ?]) *For fixed $x \in \Omega$ the Dirichlet Green's function G_x satisfies:*

1. *For every $t > 0$ one has*

$$\int_{\{G_x<t\}} |\nabla G_x|^2 \quad = \quad t.$$

2. *As $t \to \infty$ we have $B_x^{r_-} \subset \overline{\{G_x > t\}} \subset B_x^{r_+}$ with $r_\pm = r \pm O(r^n)$ and r defined by $t = K(r) - \tau_\Omega(x)$.*

3. *For every $x \in \overline{\Omega} \setminus \{\infty\}$, with $\tau_\Omega(x) < \infty$, we have*

$$|\{G_x > t\}| \geq |\{K > t + \tau_\Omega(x)\}|$$

Proof. The proof of Part **??** and **??** are recalled in [?], Proposition 12, while Part **??** is proved in [?], Remark 11 as a consequence of Proposition 10. ○

The proposition above implies that for $x \in \Omega$ the capacity of a small ball is asymptotically given by

$$\mathrm{cap}_\Omega(B_x^r) \quad = \quad \frac{1}{K(r) - \tau_\Omega(x) + O(r)} \quad = \quad \mathrm{cap}_{\mathbf{R}^n}(B_0^r) + \mathrm{cap}^2_{\mathbf{R}^n}(B_0^r)\,(\tau_\Omega(x) + O(r)) \tag{17}$$

as $r \to 0$. In the radial case we have

$$\mathrm{cap}_{B_0^R}(B_0^r) \quad = \quad \frac{1}{K(r) - K(R)}. \tag{18}$$

The key point is that an asymptotic expansion similar to (**??**) holds for arbitrary small sets which concentrate at single point. The following estimate for the capacity has been proved in [?], Lemma 16.

Lemma 9 (Asymptotic expansion of capacity) *Let $x_0 \in \Omega \cup \partial\Omega \setminus \{\infty\}$ and let A_k be a sequence of subsets of Ω such that $|A_k| > 0$ and*

$$\frac{1}{|A_k|} \mathcal{X}_{A_k} \overset{*}{\rightharpoonup} \delta_{x_0}.$$

Then

$$\liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} \geq \tau_\Omega(x_0)\,. \tag{19}$$

An important tool in the proof of this result is Proposition **??** below. It provides an approximation of τ_Ω with a sequence of Robin functions obtained approximating Ω with larger domains, and permits to restrict the analysis in Lemma **??** only to interior points.

Fix $x_0 \in \partial\Omega \setminus \{\infty\}$. Let us denote by $\Omega_\rho(x_0)$ the set $\Omega \cup B^\rho_{x_0}$. For any fixed $x \in \Omega \cup \partial\Omega \setminus \{\infty\}$ let $H_{\Omega_\rho(x_0)}(x, \cdot)$ be the PWB solution of the problem

$$\begin{cases} \Delta_y H_{\Omega_\rho(x_0)}(x,y) = 0 & \text{in } \Omega_\rho(x_0), \\ H_{\Omega_\rho(x_0)}(x,y) = K_x(y) & \text{on } \partial\Omega_\rho(x_0) \end{cases} \tag{20}$$

and let $\tau_{\Omega_\rho(x_0)}(x)$ the corresponding Robin function.

Proposition 10 ([?], Proposition 7) *Let $x_0 \in \partial\Omega \setminus \{\infty\}$. Then, for every $x, y \in \mathbf{R}^n$, $H_{\Omega_\rho(x_0)}(x,y)$ converges increasingly to $H_\Omega(x,y)$ as ρ decreases to 0.*

In particular $\tau_{\Omega_\rho(x_0)}(x)$ converges increasingly to $\tau_\Omega(x)$ as $\rho \to 0$, for any $x \in \Omega \cup \partial\Omega \setminus \{\infty\}$ and τ_Ω is lower semicontinuous in $\Omega \cup \partial\Omega \setminus \{\infty\}$.

Our next goal is to establish that a similar approximation result can be proved for $\tau_\Omega(\infty)$.

Proposition 11 *The following equality holds*

$$\tau_\Omega(\infty) = \lim_{\rho\to 0} \lim_{R\to\infty} \inf_{|x|\geq R} \tau_{\Omega\cup B_\rho(x)}(x)\,.$$

In order to prove Proposition **??** we need the following lemma.

Lemma 12 *Let $x \in \mathbf{R}^n$, $\alpha \in (0, \frac{1}{2})$, $\rho \in (0,1)$ and $r = 2\rho^\alpha$. If*

$$\tau_{\Omega\cup B_\rho(x)}(x) < 2^{2-n} K(1) \rho^{-\alpha(n-2)} = K(r) \tag{21}$$

then

$$\inf_{y,z\in B_{2r}(x)} \widetilde{H}_\Omega(y,z) \leq \tau_{\Omega\cup B_\rho(x)}(x) + K(1) 4^{2-n} \rho^{(1-2\alpha)(n-2)}\,. \tag{22}$$

Proof. Let $T = \tau_{\Omega\cup B_\rho(x)}(x)$. By assumption $\widetilde{H}_{\Omega\cup B_\rho(x)}(x,x) = T < K(r)$, with $r = 2\rho^\alpha$. Thus by the superharmonicity of $\widetilde{H}_{\Omega\cup B_\rho(x)}(x,\cdot)$ we get

$$\fint_{\partial B_r(x)} \widetilde{H}_{\Omega\cup B_\rho(x)}(x,z)\,dz \leq \widetilde{H}_{\Omega\cup B_\rho(x)}(x,x) = T \tag{23}$$

Hence there exists a subset S of $\partial B_r(x)$ such that S has positive $(n-1)$-dimensional measure and such that

$$\widetilde{H}_{\Omega\cup B_\rho(x)}(x,z) \leq T \qquad \forall\ z \in S\,. \tag{24}$$

If $z \in \partial B_r(x) \setminus \overline{\Omega \cup B_\rho(x)}$, then by (**??**) $\widetilde{H}_{\Omega\cup B_\rho(x)}(x,z) = K(|x-z|) = K(r) > T$. This is also true if $z \in \partial\Omega \cap \partial B_r(x)$ is a regular boundary point of Ω in the sense of Wiener. Since the set of irregular points of the boundary of an n-dimensional domain has zero capacity, and in particular zero $(n-1)$-dimensional measure, we infer that $S \cap \Omega$ has positive $(n-1)$-dimensional measure. In particular we may fix $z \in \Omega \cap \partial B_r(x)$ such that (**??**) holds.

Again by the superharmonicity of $\widetilde{H}_{\Omega\cup B_\rho(x)}$ we have that

$$\fint_{\partial B_{\frac{r}{2}}(x)} \widetilde{H}_{\Omega\cup B_\rho(x)}(y,z)\,dy \leq \widetilde{H}_{\Omega\cup B_\rho(x)}(x,z) \leq T \tag{25}$$

Thus, as above, we may find $y \in \Omega \cap \partial B_{\frac{r}{2}}(x)$ such that

$$\widetilde{H}_{\Omega\cup B_\rho(x)}(y,z) \leq T \tag{26}$$

Now let $M = \max_{\xi\in\overline{B}_\rho(x)} K(|z-\xi|) \leq K(\frac{r}{2}) = K(1)(\rho^\alpha)^{2-n}$ and consider the function

$$f(\xi) = \widetilde{H}_{\Omega\cup B_\rho(x)}(\xi,z) + M \left(\frac{|\xi - x|}{\rho}\right)^{2-n}$$

then f is superharmonic in $\mathbf{R}^n$ and harmonic in $\Omega \setminus \overline{B}$. Moreover $f(\xi) \geq K(|\xi - z|)$ if $\xi \in \partial(\Omega \setminus \overline{B})$. Hence $H_{\Omega\setminus\overline{B}}(\xi,z) \leq f(\xi)$ for every $\xi \in \Omega \setminus \overline{B}$. Since $y \in \Omega \setminus \overline{B}$ we may take $\xi = y$ and we obtain

$$\begin{aligned} H_\Omega(y,z) \leq H_{\Omega\setminus\overline{B}}(z,y) &\leq \widetilde{H}_{\Omega\cup B_\rho(x)}(y,z) + M\left(\frac{|y-x|}{\rho}\right)^{2-n} \\ &\leq T + K(1)4^{2-n}\rho^{(1-2\alpha)(n-2)}\,, \end{aligned}$$

which concludes the proof. ○

Proof of Proposition ??. Let us first prove that

$$\lim_{\rho\to 0}\lim_{R\to\infty}\inf_{|x|\geq R}\tau_{\Omega\cup B_\rho(x)}(x)\leq \tau_\Omega(\infty)\,.$$

Let $\tau(R,\rho) = \inf_{|x|\geq R}\inf_{|x-y|<\rho}\widetilde{H}_\Omega(x,y)$, and $\tau(\rho) = \lim_{R\to\infty}\tau(R,\rho)$. By definition $\tau_\Omega(\infty) = \lim_{\rho\to 0}\tau(\rho)$. By the harmonicity of $H_{\Omega\cup B_\rho(x)}(x,y)$ we have

$$\tau_{\Omega\cup B_\rho(x)}(x) = \fint_{B_\rho(x)} H_{\Omega\cup B_\rho(x)}(x,y)\,dy \leq \fint_{B_\rho(x)}\widetilde{H}_\Omega(x,y)\,dy\,. \tag{27}$$

Since $\widetilde{H}_\Omega(x,y)\geq \tau(R,\sqrt{\rho})$ for every $|x-y|\leq\sqrt{\rho}$ and $|x|\geq R$, by the Harnack inequality applied to $\widetilde{H}_\Omega(x,y)-\tau(R,\sqrt{\rho})$ in connection with (**??**) we get

$$\tau_{\Omega\cup B_\rho(x)}(x)\leq \tau(R,\sqrt{\rho}) + C\left(\min_{y\in B_\rho(x)}\widetilde{H}_\Omega(x,y)-\tau(R,\sqrt{\rho})\right)\,.$$

By taking the infimum on $|x|\geq R$ we obtain

$$\inf_{|x|\geq R}\tau_{\Omega\cup B_\rho(x)}(x)\leq \tau(R,\sqrt{\rho}) + C\left(\tau(R,\rho)-\tau(R,\sqrt{\rho})\right)\,.$$

and we conclude taking the limit as $R\to\infty$ and then $\rho\to 0$.

Conversely, let $\widetilde{\tau}(\infty) = \lim_{\rho\to 0}\lim_{R\to\infty}\inf_{|x|\geq R}\tau_{\Omega\cup B_\rho(x)}(x)$. Assume that $\widetilde{\tau}(\infty)<\infty$. Then there exist $\rho_k\to 0$ and $x_k\to\infty$ such that

$$\lim_{k\to\infty}\tau_{\Omega\cup B_{\rho_k}(x_k)}(x_k) = \widetilde{\tau}(\infty)\,.$$

In particular assumption (**??**) in Lemma **??** holds for k sufficiently large. Thus, by Lemma **??**, there exist $r_k = 2\rho_k^\alpha\to 0$ and $z_k, y_k\to\infty$, with $|z_k-y_k|\leq 2r_k\to 0$, such that

$$\limsup_{k\to\infty} H_\Omega(y_k,z_k)\leq\widetilde{\tau}(\infty)$$

and then in particular we have

$$\lim_{\rho\to 0}\lim_{R\to\infty}\inf_{\substack{x,y\in\mathbf{R}^n\\ |x|\geq R,\,|x-y|\leq\rho}}\widetilde{H}_\Omega(x,y)\leq\widetilde{\tau}(\infty)\,.$$

Thus $\tau_\Omega(\infty)\leq\widetilde{\tau}(\infty)$, which concludes the proof. ○

As an immediate consequence of Proposition **??** we obtain the following result.

Corollary 13 *For any sequence $\{\rho_k\}$, with $\rho_k>0$ and $\rho_k\to 0$ as $k\to\infty$, there exists a sequence $\{x_k\}$ in $\mathbf{R}^n$ with $x_k\to\infty$ such that*

$$\lim_{k\to\infty}\tau_{\Omega\cup B_{\rho_k}(x_k)}(x_k) = \tau_\Omega(\infty)\,.$$

We now establish a more precise comparison between $\widetilde{H}_\Omega$ and $\tau_\Omega(\infty)$.

Corollary 14 *For any sequence $\{\rho_k\}$, with $\rho_k>0$ and $\rho_k\to 0$ as $k\to\infty$, let $\{x_k\}$ be a sequence in $\mathbf{R}^n$ with $x_k\to\infty$ such that*

$$\lim_{k\to\infty}\inf_{|x_k-y|\leq\rho_k}\widetilde{H}_\Omega(x_k,y) = \tau_\Omega(\infty)\,.$$

Then we also have

$$\lim_{k\to\infty}\tau_{\Omega\cup B_{\rho_k}(x_k)}(x_k) = \tau_\Omega(\infty)\,.$$

Proof. Let us denote $\Omega_k = \Omega\cup B_{\rho_k}(x_k)$. By Proposition **??** we always have that

$$\limsup_{k\to\infty}\tau_{\Omega_k}(x_k)\geq\tau_\Omega(\infty)$$

On the other hand by the harmonicity of $H_{\Omega_k}(x_k, y)$ in $B_{\rho_k(x_k)}$ we have

$$
\begin{aligned}
(28)\qquad \tau_{\Omega_k}(x_k) &= H_{\Omega_k}(x_k, x_k) = \fint_{B_{\rho_k}(x_k)} H_{\Omega_k}(x_k, y)\, dy \le \fint_{B_{\rho_k}(x_k)} \widetilde{H}_\Omega(x_k, y)\, dy = \\
&= \inf_{|x_k - y| \le \sqrt{\rho_k}} \widetilde{H}_\Omega(x_k, y) + \fint_{B_{\rho_k}(x_k)} \left(H_{\Omega_k}(x_k, y) - \inf_{|x_k - z| \le \sqrt{\rho_k}} \widetilde{H}_\Omega(x_k, z) \right) dy\,.
\end{aligned}
$$

By the assumption and the definition of $\tau_\Omega(\infty)$ we have also that

$$
\inf_{|x_k - y| \le \sqrt{\rho_k}} \widetilde{H}_\Omega(x_k, y) = \tau_\Omega(\infty) + o(1)\,.
$$

Thus applying the weak Harnack inequality to the function $H_{\Omega_k}(x_k, y) - \inf_{|x_k - z| \le \sqrt{\rho_k}} \widetilde{H}_\Omega(x_k, z)$, which is superharmonic and positive on $B_{2\rho_k}(x_k)$, we get

$$
\begin{aligned}
\tau_{\Omega_k}(x_k) &\le \inf_{|x_k - y| \le \sqrt{\rho_k}} \widetilde{H}_\Omega(x_k, y) \\
&+ C \left(\inf_{|x_k - z| \le \rho_k} H_{\Omega_k}(x_k, y) - \inf_{|x_k - z| \le \sqrt{\rho_k}} \widetilde{H}_\Omega(x_k, z) \right) = \tau_\Omega(\infty) + o(1)\,.
\end{aligned}
$$

○

We now prove the asymptotic formula for small sets concentrating at ∞.

Lemma 15 *Let A_k be a sequence of sets which concentrates at ∞ in the sense that $|A_k| > 0$ and suppose that there exists a sequence $x_k \to \infty$, such that*

$$
\frac{\chi_{A_k - x_k}}{|A_k|} \overset{*}{\rightharpoonup} \delta_0\,.
$$

Then

$$
(29)\qquad \liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} \ge \tau_\Omega(\infty)\,.
$$

Proof. We may assume $\tau_\Omega(\infty) > 0$ since otherwise there is nothing to show. Note also that the assumptions imply $|A_k| \to 0$. Thus we may suppose that $\mathrm{cap}_\Omega A_k \to 0$ since otherwise the left hand side of (**??**) is ∞. We first assume that $\tau_\Omega(\infty) < +\infty$. Let u_k be the capacitary potential of $A_k - x_k$ and let

$$
\mu_k = -\frac{1}{\mathrm{cap}_\Omega(A_k)} \Delta u_k \llcorner \{\Omega - x_k\}\,.
$$

As in the proof of Lemma 16 in [**?**] we obtain $\mu_k \overset{*}{\rightharpoonup} \delta_0$ and $\|\mu_k\|_{\mathcal{M}(\mathbf{R}^n \setminus B_\rho)} \to 0$ for every $\rho > 0$. We will construct a superharmonic function w_k which satisfies $w_k \ge 1$ on $A_k - x_k$ and we will estimate $\|\Delta w_k\|_{\mathcal{M}}$ to estimate $\mathrm{cap}_{\mathbf{R}^n}(A_k)$.

Fix $\rho > 0$ and let $\mu_k^1 = \mu_k \llcorner B_\rho$, $\mu_k^2 = \mu_k - \mu_k^1 \to 0$ in $\mathcal{M}(\mathbf{R}^n)$. We have

$$
u_k(x) = \mathrm{cap}_\Omega(A_k) \int_{\mathbf{R}^n} G_{\Omega - x_k}(x, y)\, d\mu_k(y)\,,
$$

and define

$$
u_k^i(x) = \mathrm{cap}_\Omega(A_k) \int_{\mathbf{R}^n} G_{\{\Omega - x_k\}}(x, y)\, d\mu_k^i(y)\,,
$$

$$
v_k^i(x) = \mathrm{cap}_\Omega(A_k) \int_{\mathbf{R}^n} K(x - y)\, d\mu_k^i(y)\,,\ i = 1\,, 2\,.
$$

Since $x_k \to \infty$, the definition of $\tau_\Omega(\infty)$ and the convergence of μ_k^2 imply that for every $\delta > 0$ there exist $\rho_0(\delta) > 0$ and $k_0(\delta, \rho)$ such that for all $\rho < \rho_0(\delta)$ and $k \ge k_0$

$$
(30)\qquad \frac{v_k^1 - u_k^1}{\mathrm{cap}_\Omega(A_k)} = \int_\Omega \widetilde{H}_\Omega(x_k + x, x_k + y)\, d\mu_k^1 \ge \tau_\Omega(\infty) - \delta
$$

for every x such that $|x| < 2\rho$.

On the other hand since $\|\mu_k\|_{\mathcal{M}} = 1$ we have

$$u_k^1(x) \le v_k^1(x) \le \mathrm{cap}_\Omega(A_k) K(|x| - \rho) \le \mathrm{cap}_\Omega(A_k) K(\rho) \tag{31}$$

if $|x| \ge 2\rho$. If $u_k(x) \ge 1$ and $|x| \ge 2\rho$ then

$$u_k^2(x) \ge 1 - u_k^1(x) \ge 1 - \mathrm{cap}_\Omega(A_k) K(\rho) \tag{32}$$

Let $\alpha_k = (\tau_\Omega(\infty) - \delta)\mathrm{cap}_\Omega(A_k)$ and $\beta_k = K(\rho)\mathrm{cap}_\Omega(A_k)$, and

$$w_k = \frac{1}{1+\alpha_k} v_k^1 + \frac{1}{1-\beta_k} v_k^2 = \frac{1}{1+\alpha_k} v_k + \left(\frac{1}{1-\beta_k} - \frac{1}{1+\alpha_k} \right) v_k^2 \,. \tag{33}$$

From the first identity, in connection with (**??**) and (**??**) we see that $w_k \ge 1$ on $A_k - x_k$. Indeed this follows immediately from (**??**) for $|x| \ge 2\rho$ since $v_k^2 \ge u_K^2$. For $|x| < 2\rho$ estimate (**??**) and the condition $u_k \ge 1$ on $A_k - x_k$ give

$$w_k \ge \frac{u_k^1 + \alpha_k}{1+\alpha_k} + \frac{1}{1-\beta_k} u_k^2 \ge 1 + \left(\frac{1}{1-\beta_k} - \frac{1}{1+\alpha_k} \right) u_k^2 \ge 1 \,.$$

Now the second identity in (**??**) (in connection with the minimality of the capacitary distribution) yields

$$\mathrm{cap}_{\mathbf{R}^n}(A_k) \le \|\Delta w_k\|_{\mathcal{M}} \le \left[\frac{1}{1+\alpha_k} + \left(\frac{1}{1-\beta_k} - \frac{1}{1+\alpha_k} \right) \|\mu_k^2\| \right] \mathrm{cap}_\Omega(A_k) \,.$$

Taking the limit as $k \to \infty$ and $\rho \to 0$ we easily deduce the assertion for $\tau_\Omega(\infty) < \infty$.

If $\tau_\Omega(\infty) = \infty$ we replace in (**??**) the term $\tau_\Omega(\infty) - \delta$ by $\frac{1}{\delta}$ and proceed as before.

○

In connection with Lemma **??** and the lower semicontinuity of τ_Ω in $\overline{\Omega}$ we deduce immediately the following corollary.

Corollary 16 *Suppose that $\tau_\Omega(\infty) > 0$. Then $\inf_{\overline{\Omega}} \tau_\Omega = \min_{\overline{\Omega}} \tau_\Omega > 0$ and for all sets $A_k \subset \Omega$, with $|A_k| \to 0$*

$$\liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} \ge \min_{\overline{\Omega}} \tau_\Omega \,. \tag{34}$$

4 Localization of concentration points

The main result of this paper is the second order expansion of S_ε^F with respect to ε. It turns out that the second nontrivial term depends on the value of the Robin function at the concentration point. This allows us to identify the concentration point. We say that $\{A_\varepsilon\}$ is a sequence of almost extremals for (**??**) if A_ε is admissible for the definition of $S_\varepsilon^V(\Omega)$ and

$$\frac{|A_\varepsilon|}{\varepsilon^{2^*}} = S_\varepsilon^V(\Omega) + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0.$$

Theorem 17 (Identification of concentration points) *1. If the sequence $\{A_\varepsilon\}$ satisfies $\mathrm{cap}_\Omega A_\varepsilon = \varepsilon^2$ and concentrates at $x \in \overline{\Omega}$ in the sense of Theorem **??** then*

$$|A_\varepsilon| \le \varepsilon^{2^*} S^V \left(1 - \frac{n}{n-2} \tau_\Omega(x)\, \varepsilon^2 + o(\varepsilon^2) \right)$$

as $\varepsilon \to 0$.

2. If $\{A_\varepsilon\}$ is a sequence of almost extremals we have

$$|A_\varepsilon| = \varepsilon^{2^*} S^V \left(1 - \frac{n}{n-2} \min_{\overline{\Omega}} \tau_\Omega\, \varepsilon^2 + o(\varepsilon^2) \right) .$$

3. *In particular a sequence of almost extremals concentrates at a harmonic center, i.e.*

$$\tau(x_0) \quad = \quad \min_{\overline{\Omega}} \tau_\Omega$$

with x_0 as in Theorem ??.

Remark 18 If $\tau_\Omega(x) = \infty$ the inequality in Part 1 is understood as

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\left(\frac{|A_\varepsilon|}{\varepsilon^{2^*}} - S^V\right) = -\infty\,.$$

Proof of Theorem ??. Let us first prove Part 1. In view of Proposition **??** we can apply Lemma **??** if $x \in \overline{\Omega}\setminus\{\infty\}$ or Lemma **??** if $x = \infty$. Taking into account that $\mathrm{cap}_{\mathbf{R}^n} A_\varepsilon^* = (|A_\varepsilon|/S^V)^{2/2^*}$ and $\mathrm{cap}_\Omega A_\varepsilon = \varepsilon^2$ we deduce that

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left[\left(\frac{S^V\varepsilon^{2^*}}{|A_\varepsilon|}\right)^{\frac{2}{2^*}} - 1\right] \geq \tau_\Omega(x)$$

and this proves Part 1 since $\frac{2}{2^*} = \frac{n-2}{n}$.

Since every maximizing sequence concentrates by Theorem **??**, the assertion in Part 1 implies one inequality in Part 2. If $\min_{\overline{\Omega}}\tau_\Omega$ is attained at $x \neq \infty$, then the reverse inequality is an easy consequence of Proposition **??**, Parts **??** and **??**. Indeed if $A_\varepsilon = \{G_x > \varepsilon^{-2}\}$ then $\mathrm{cap}_\Omega A_\varepsilon = \varepsilon^2$ and

$$|A_\varepsilon| \geq |\{K > \frac{1}{\varepsilon^2} + \tau_\Omega(x)\}|\,. \tag{35}$$

Thus computing the right hand side of (**??**) we get the required inequality.

Let us finally consider the case that $\min_{\overline{\Omega}}\tau_\Omega$ is attained only at $\overline{x} = \infty$. In this case we may not apply directly the transplantation argument, but we must apply it to the level sets of the Green function of Ω with singularities in suitable points x_ε approaching ∞. We claim that it is possible to choose $x_\varepsilon \to \infty$ such that

$$|\{G_{x_\varepsilon} > \frac{1}{\varepsilon^2}\}| \geq |\{K > \frac{1}{\varepsilon^2} + \tau_\Omega(\infty) + o(1)\}|\,. \tag{36}$$

This will give us the result as above, taking $A_\varepsilon = \{G_{x_\varepsilon} > \varepsilon^{-2}\}$.

In order to prove (**??**) let $\rho_\varepsilon > 0$ be such that $K(\rho_\varepsilon) = 1/\varepsilon^2$ (i.e. $\rho_\varepsilon = [K(1)\varepsilon^2]^{1/(n-2)}$) and let $R_\varepsilon > 0$ be such that $R_\varepsilon \ll \rho_\varepsilon^2$. By the definition of $\tau_\Omega(\infty)$ we may find a sequence $x_\varepsilon \to 0$ such that

$$\inf_{|x_\varepsilon - y|\leq R_\varepsilon} \widetilde{H}_\Omega(x_\varepsilon, y) = \tau_\Omega(\infty) + o(1) \tag{37}$$

Let $\tau_{\Omega_\varepsilon}$ be the Robin function of the set $\Omega_\varepsilon = \Omega \cup B_{R_\varepsilon}(x_\varepsilon)$. By Corollary **??** we have also

$$\lim_{\varepsilon\to 0}\tau_{\Omega_\varepsilon}(x_\varepsilon) = \tau_\Omega(\infty)\,. \tag{38}$$

By applying the usual transplantation argument (see Proposition **??**, Part **??**) to the Green function of Ω_ε we have

$$|\{G_{\Omega_\varepsilon}(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\}| \geq |\{K(|x_\varepsilon - y|) > \frac{1}{\varepsilon^2} + \tau_\Omega(\infty) + o(1)\}| \tag{39}$$

Thus it remains to prove that

$$|\{G_\Omega(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\}| \geq |\{G_{\Omega_\varepsilon}(x_\varepsilon, y) > \frac{1}{\varepsilon^2} + o(1)\}| + \varepsilon^{2^*} o(\varepsilon^2)\,. \tag{40}$$

This will be done exploiting that far from x_ε the difference $\widetilde{H}_\Omega(x_\varepsilon, y) - \widetilde{H}_{\Omega_\varepsilon}(x_\varepsilon, y)$ is small (see estimate (**??**) below) while close to x_ε the difference between the level sets of G_{Ω_ε} and the levels sets of G_Ω is controlled by the set where $\widetilde{H}_\Omega(x_\varepsilon, \cdot)$ is very big, which is small (see (**??**)).

First we claim that there exists a constant $C > 0$ such that

$$0 \leq \widetilde{H}_\Omega(x_\varepsilon, y) - \widetilde{H}_{\Omega_\varepsilon}(x_\varepsilon, y) \leq C\left(\frac{R_\varepsilon}{\rho_\varepsilon^2}\right)^{n-2} \tag{41}$$

for every $\rho_\varepsilon^2 \le |y - x_\varepsilon| \le \rho_\varepsilon$.

In order to to prove estimate (**??**) let $r_\varepsilon(y)$ be the solution of the following problem

$$\begin{cases} \Delta r_\varepsilon(y) = 0 & \text{in } (\Omega^c \cap B_{R_\varepsilon}(x_\varepsilon))^c, \\ r_\varepsilon(y) = K(|x_\varepsilon - y|) & \text{on } \partial(\Omega^c \cap B_{R_\varepsilon}(x_\varepsilon))^c \text{ and } r_\varepsilon \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{42}$$

It is easy to check that

$$r_\varepsilon(y) \le \widetilde{H}_\Omega(x_\varepsilon, y) \le \widetilde{H}_{\Omega_\varepsilon}(x_\varepsilon, y) + r_\varepsilon(y)\,. \tag{43}$$

Since r_ε is harmonic outside the ball $B_{R_\varepsilon}(x_\varepsilon)$, using a Poisson-type integral representation we get

$$r_\varepsilon(y) = \frac{1}{|S^{n-1}|R_\varepsilon} \int_{\partial B_{R_\varepsilon}(x_\varepsilon)} \frac{|x_\varepsilon - y|^2 - R_\varepsilon^2}{|z - y|^n} r_\varepsilon(z)\, d\mathcal{H}^{n-1}(z)$$

for $|y - x_\varepsilon| > R_\varepsilon$. Thus by (**??**) we have

$$\begin{aligned} \widetilde{H}_\Omega(x_\varepsilon, y) - \widetilde{H}_{\Omega_\varepsilon}(x_\varepsilon, y) &\le C\left(\frac{R_\varepsilon}{\rho_\varepsilon^2}\right)^{n-2} \fint_{\partial B_{R_\varepsilon}(x_\varepsilon)} r_\varepsilon(z)\, d\mathcal{H}^{n-1}(z) \\ &\le C\left(\frac{R_\varepsilon}{\rho_\varepsilon^2}\right)^{n-2} \fint_{\partial B_{R_\varepsilon}(x_\varepsilon)} \widetilde{H}_\Omega(x_\varepsilon, z)\, d\mathcal{H}^{n-1}(z) \end{aligned} \tag{44}$$

for any $\rho_\varepsilon^2 \le |y - x_\varepsilon| \le \rho_\varepsilon$. Finally taking into account the superharmonicity of $\widetilde{H}_\Omega(x_\varepsilon, \cdot)$ and using the weak Harnack inequality we obtain

$$\fint_{\partial B_{R_\varepsilon}(x_\varepsilon)} \widetilde{H}_\Omega(x_\varepsilon, z)\, d\mathcal{H}^{n-1}(z) \le \fint_{B_{R_\varepsilon}(x_\varepsilon)} \widetilde{H}_\Omega(x_\varepsilon, z)\, dz \le C \inf_{|x_\varepsilon - y| < R_\varepsilon} \widetilde{H}_\Omega(x_\varepsilon, y) \le C(\tau_\Omega(\infty) + o(1))$$

which in view of (**??**) gives (**??**).

Since $\{G_\Omega(x_\varepsilon, y) > 1/\varepsilon^2\} \subseteq B_{\rho_\varepsilon}(x_\varepsilon)$, as an immediate consequence of (**??**) we have that

$$|\{G_\Omega(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\} \setminus B_{\rho_\varepsilon^2}(x_\varepsilon)| \ge |\{G_{\Omega_\varepsilon}(x_\varepsilon, y) > \frac{1}{\varepsilon^2} + o(1)\} \setminus B_{\rho_\varepsilon^2}(x_\varepsilon)|\,. \tag{45}$$

Finally it is easy to check that

$$|\{G_\Omega(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\} \cap B_{\rho_\varepsilon^2}(x_\varepsilon)| \ge |\{G_{\Omega_\varepsilon}(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\} \cap B_{\rho_\varepsilon^2}(x_\varepsilon)| + \varepsilon^{2^*} o(\varepsilon^2)\,. \tag{46}$$

Indeed this follows from the fact that, since $K(|x_\varepsilon - y|) \ge 1/\varepsilon^4$ in $B_{\rho_\varepsilon^2}(x_\varepsilon)$, we have

$$\{G_\Omega(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\} \cap B_{\rho_\varepsilon^2}(x_\varepsilon) \subseteq (\{K(|x_\varepsilon - y|) > \frac{1}{\varepsilon^2}\} \setminus \{\widetilde{H}_\Omega(x_\varepsilon, y) > \frac{1}{\varepsilon^4} - \frac{1}{\varepsilon^2}\}) \cap B_{\rho_\varepsilon^2}(x_\varepsilon)\,.$$

Since

$$|\{\widetilde{H}_\Omega(x_\varepsilon, y) > \frac{1}{2\varepsilon^4}\} \cap B_{\rho_\varepsilon^2}(x_\varepsilon)| \le 2\varepsilon^4 \int_{B_{\rho_\varepsilon^2}} \widetilde{H}_\Omega \le C\varepsilon^4 \rho_\varepsilon^{2n} \le C\varepsilon^{22^*+4}$$

we deduce

$$|\{G_\Omega(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\} \cap B_{\rho_\varepsilon^2}(x_\varepsilon)| \ge |\{G_{\Omega_\varepsilon}(x_\varepsilon, y) > \frac{1}{\varepsilon^2}\} \cap B_{\rho_\varepsilon^2}(x_\varepsilon)| - C\varepsilon^{2^*}\varepsilon^{2^*+4}\,.$$

Now estimate (**??**) follows from (**??**) and (**??**). Together with (**??**) this concludes the proof. ○

Appendix: General integrands

We finally consider the general problem

$$S_\varepsilon^F(\Omega) \quad := \quad \frac{1}{\varepsilon^{2^*}} \sup\left\{ \int_\Omega F(u) \,:\, u \in D^{1,2}(\Omega),\ \|\nabla u\|_2 \le \varepsilon \right\},$$

where $0 \le F(t) \le \alpha|t|^{2^*}$, for some $\alpha > 0$, and F is upper semicontinuous. In this general case a further subtlety in unbounded domains may arise if the integrand F has critical growth at the origin, i.e. if $F_0^+ = \limsup_{t\to 0} \frac{F(t)}{t^{2^*}} = \frac{S^F}{S^*}$ (S^F is defined in (**??**) and S^* is the best Sobolev constant, i.e. $\int_\Omega |u|^{2^*} \le S^* \|\nabla u\|_2^{2^*}$). In this case the maximizers of the radial problem in $\mathbf{R}^n$ may become arbitrarily flat (think e.g. of the case $F(t) = \frac{S^F}{S^*} t^{2^*}$, for $t \in [0, \delta]$) and, in order to prove the concentration without the assumption that $|\Omega|$ is finite, we also need an estimate for the capacity of large sets (see Lemma **??** below). Hence, in this case we shall make the additional assumption

$$\tau_\Omega(\infty) > 0$$

which essentially says that $\mathbf{R}^n \setminus \Omega$ is not too small at infinity. An equivalent characterization is the following.

Proposition 19 *The condition $\tau_\Omega(\infty) > 0$ is equivalent to requiring that there exists a constant $C_0 > 0$ such that*

$$H_\Omega(x,y) \ge C_0 \min\{1, |x-y|^{2-n}\} \tag{47}$$

Proof. Clearly, by the definition of $\tau_\Omega(\infty)$, we have that (**??**) implies $\tau_\Omega(\infty) > 0$. To prove the opposite implication we first remark that to have (**??**) satisfied it is enough to know that there exist $\rho_0 > 0$ and $C_0 > 0$ such that

$$\widetilde{H}_\Omega(x,y) \ge C_0 \qquad \forall\ |x-y| \le \rho_0\,. \tag{48}$$

Indeed for any $x \in \Omega$ the function $C_0 K_x(\cdot)/K(\rho_0)$ is harmonic in $\Omega \setminus B_{\rho_0}(x)$ smaller than $H(x,\cdot)$ on $\partial B_{\rho_0}(x)$. Thus by the comparison principle $H(x,y) \ge C_0 K_x(y)/K(\rho_0)$ for any $y \in \Omega \setminus B_{\rho_0}(x)$, which, together with (**??**), gives (**??**). Finally we have that $\tau_\Omega(\infty) > 0$ implies (**??**). Indeed by the definition of $\tau_\Omega(\infty)$ we may find $\rho_0 > 0$ and $R_0 > \rho_0$ such that

$$\widetilde{H}_\Omega(x,y) \ge C_0 \qquad \forall\ |x-y| \le \rho_0 \quad \text{and} \quad |x| > R_0\,.$$

Thus the conclusion follows from the fact that a superharmonic non negative function either is zero or is strictly positive. This implies $H_\Omega(x,y)$ has a strictly positive minimum in $B_{R_0} \times B_{R_0}$ and thus (**??**) (after possibly adjusting the value of C_0). ○

Remark 20 The condition $\tau_\Omega(\infty) > 0$ implies $\min_{\overline{\Omega}} \tau_\Omega > 0$. Indeed by definition $\tau_\Omega(\infty) > 0$ implies $\min_{\{|x|>R\}} \tau_\Omega > \tau_\Omega(\infty)/2$ for some $R > 0$ and then, arguing as above we also have $\min_{\overline{\Omega}} \tau_\Omega > 0$.

We now use the assumption $\tau_\Omega(\infty) > 0$ to prove the counterpart of Lemma **??** for large sets.

Lemma 21 *Assume $\tau_\Omega(\infty) > 0$. Then for any $\rho > 0$ there exists a constant $C_\rho > 0$ such that*

$$\mathrm{cap}_\Omega(A) - \mathrm{cap}_{\mathbf{R}^n}(A^*) \ge C_\rho \mathrm{cap}_{\mathbf{R}^n}(A^*)$$

for every subset A of Ω such that $|A| \ge |B_1|$.

Proof. By a scaling argument we may assume that $\rho = 1$. Moreover we may reduce to the case $|A| = |B_1|$. Indeed for $R \ge 1$ we have

$$H_{\frac{1}{R}\Omega}(x,y) = R^{n-2} H_\Omega(Rx, Ry) \ge C_0 \min\{R^{n-2}, |x-y|^{2-n}\} \ge C_0 \min\{1, |x-y|^{2-n}\}\,,$$

thus if Ω satisfies (**??**) also the rescaled set $\frac{1}{R}\Omega$, with $R \ge 1$, does.

We now proceed by contradiction. Let $A_k \subseteq \Omega_k$ be a sequence such that $|A_k| = |B_1|$ and $\mathrm{cap}_{\Omega_k} A_k \to \mathrm{cap}_{\mathbf{R}^n} B_1$, with Ω_k satisfying (**??**).

Since $\mathrm{cap}_{\mathbf{R}^n} B_1 \le \mathrm{cap}_{\mathbf{R}^n} A_k \le \mathrm{cap}_{\Omega_k} A_k$, we also have that $\mathrm{cap}_{\mathbf{R}^n} A_k \to \mathrm{cap}_{\mathbf{R}^n} B_1$. Thus by Lemma **??** we have that after a translation (note that (**??**) is translation invariant) the characteristic function of A_k converges to the characteristic function of B_1. Let u_k be the capacitary potential of A_k in Ω_k. In particular

$$\begin{cases} -\Delta u_k = \mu_k \ge 0 & \text{in } \Omega_k \\ u_k = 0 & \text{on } \partial\Omega_k, \end{cases}$$

where μ_k is the capacitary distribution, $\int_{\Omega_k} d\mu_k = \mathrm{cap}_{\Omega_k} A_k$ and $\mathrm{supp}\,\mu_k \subseteq \overline{A}_k$.

By Lemma **??** we also have that the sequence u_k converges strongly in $D^{1,2}(\mathbf{R}^n)$ to the capacitary potential u of B_1 in $\mathbf{R}^n$ and that μ_k converges weakly in the sense of measures to the corresponding capacitary distribution μ, with $\operatorname{supp}\mu \subseteq \partial B_1$ and $\int_{\mathbf{R}^n} d\mu = \operatorname{cap}_{\mathbf{R}^n} B_1$.

Since, using the Green function of Ω_k, we have

$$u_k(x) = \int_{\mathbf{R}^n} G_{\Omega_k}(x,y)\,d\mu_k = \int_{\mathbf{R}^n} K_x(y)\,d\mu_k - \int_{\mathbf{R}^n} H_{\Omega_k}(x,y)\,d\mu_k \leq$$
$$\int_{\mathbf{R}^n} K_x(y)\,d\mu_k - C_0 \int_{\mathbf{R}^n} \min\{1, |x-y|^{2-n}\}\,d\mu_k\,,$$

taking $x \in B_1$ and passing to the limit as $k \to \infty$ we get

$$u(x) \leq \int_{\mathbf{R}^n} K_x(y)\,d\mu - C \int_{\partial B_1} d\mu = u(x) - C\operatorname{cap}_{\mathbf{R}^n} B_1$$

which is a contradiction. ○

In the following $S^F := S_1^F(\mathbf{R}^n)$ will denote the generalized Sobolev constant, i.e.

$$\int_{\mathbf{R}^n} F(u)\,dx \leq S^F \left(\int_{\mathbf{R}^n} |\nabla u|^2\,dx \right)^{\frac{2^*}{2}}$$

for every $u \in D^{1,2}$.

Using the previous Lemma we can prove the concentration result without any further assumption, except $\tau_\Omega(\infty) > 0$.

Theorem 22 *Assume $\tau_\Omega(\infty) > 0$. Let $\{u_\varepsilon\}$ be a sequence of maximizing sequence for problem (**??**), i.e. $\varepsilon^{-2^*}\int_\Omega F(u_\varepsilon)\,dx \to S^F$ and $\|\nabla u_\varepsilon\|_2 \leq \varepsilon$. Then*

1. *the sequence $\{u_\varepsilon\}$ concentrates at a single point $x_0 \in \overline{\Omega}$ in the following sense*

$$\text{(49)} \qquad \frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_{x_0}\,, \qquad \frac{F(u_\varepsilon)}{\varepsilon^{2^*}}\,.$$

2. *If $x_0 = \infty$, then there exists a sequence $x_\varepsilon \to \infty$ such that $u_\varepsilon(\cdot - x_\varepsilon)$ concentrates at 0 in the sense of Part 1.*

Sketch of proof. As for the analogous theorem proved in [**?**] (Theorem 3) (under additional assumptions either on F or on Ω), the proof of Part 1 follows by the generalized concentration-compactness alternative proved in [**?**] (Theorem 12), applied to the sequence $v_\varepsilon = u_\varepsilon/\varepsilon$. By this result we know that either v_ε is compact or it concentrates at a single point in the sense of (**??**). To exclude the compactness assume that $v_\varepsilon \to v_0 \neq 0$ and for any $t > 0$ denote $A_{\varepsilon,t} = \{v_\varepsilon > t\}$. Let $\overline{v}_\varepsilon$ be the harmonic extension of v_ε^* outside $A^*_{\varepsilon,t}$, where v_ε^* denote the radial decreasing rearrangement of v_ε and $A^*_{\varepsilon,t} = \{v^*_\varepsilon > t\}$. It is easy to check that

$$\text{(50)} \qquad \int_{\mathbf{R}^n} |\nabla \overline{v}_\varepsilon|^2 \geq 1 - ct^2(\operatorname{cap}_\Omega A_{\varepsilon,t} - \operatorname{cap}_{\mathbf{R}^n} A^*_{\varepsilon,t})\,.$$

Thus the proof is exactly the same as the one given in [**?**] in the case $|\Omega|$ finite, upon noticing that since $v_0 \neq 0$, for t small enough, $\liminf_{\varepsilon\to 0} |A_{\varepsilon,t}| \geq |\{v_0 > t\}| \geq c > 0$ and then by Lemma **??**

$$\operatorname{cap}_\Omega A_{\varepsilon,t} - \operatorname{cap}_{\mathbf{R}^n} A^*_{\varepsilon,t} \geq C > 0$$

The proof of Part 2 can be also obtained by contradiction. We shall give a sketch of it.

If Part 2 does not hold then there exists $\rho_\varepsilon \geq c > 0$ such that

$$\frac{1}{\varepsilon^2}\int_{\mathbf{R}^n\setminus B_{\rho_\varepsilon}} |\nabla u^*_\varepsilon|^2\,dx = \gamma_0 \frac{1}{\varepsilon^2}\int_{\mathbf{R}^n} |\nabla u^*_\varepsilon|^2\,dx$$

where u^*_ε is the radial symmetrization of u_ε. Let $\delta_\varepsilon \to 0$ be such that

$$\delta^2_\varepsilon = S^F - \frac{1}{\varepsilon^2}\int_\Omega |\nabla u_\varepsilon|^2\,dx\,.$$

By the decay estimate for radial maximizing sequences given in [**?**] (Lemma 22), there exists a constant $u_{\varepsilon,\infty}$ such that

$$u_\varepsilon^*(r) \approx u_{\varepsilon,\infty} K(r) \qquad \text{if} \quad 1 \le \frac{r}{\rho_\varepsilon} \le \delta_\varepsilon^{-\frac{2}{n-2}}$$

where

$$c_0^{-1} \varepsilon \rho_\varepsilon^{\frac{n-2}{2}} \le u_{\varepsilon,\infty} \le c_0 \varepsilon \rho_\varepsilon^{\frac{n-2}{2}} .$$

Choose r_ε and t_ε such that $\rho_\varepsilon/r_\varepsilon \to 0$, with $r_\varepsilon/\rho_\varepsilon \ll \delta_\varepsilon^{-2/(n-2)}$, and $t_\varepsilon = u_{\varepsilon,\infty}K(r_\varepsilon)$. Then $r_\varepsilon \ge c > 0$ and $|\{u_\varepsilon^* > t_\varepsilon\}| = |B_{r_\varepsilon}| \ge C > 0$. Let $\overline{u}_\varepsilon$ be the harmonic extension of u_ε^* outside of the set $\{u_\varepsilon^* > t_\varepsilon\}$. Using Lemma **??**, we have

$$\frac{1}{\varepsilon^2}\int_{\mathbf{R}^n} |\nabla \overline{u}_\varepsilon|^2 dx \le 1 - c\frac{1}{\varepsilon^2} t_\varepsilon^2 (\mathrm{cap}_\Omega \{u_\varepsilon > t_\varepsilon\} - \mathrm{cap}_{\mathbf{R}^n}\{u_\varepsilon^* > t_\varepsilon\}) \le 1 - c\left(\frac{\rho_\varepsilon}{r_\varepsilon}\right)^{n-2} .$$

Again by the decay estimates in [**?**] (Lemma 22, formula (31)) we get

$$\frac{1}{\varepsilon^{2^*}} \int_{\{u_\varepsilon^* \le t_\varepsilon\}} F(u_\varepsilon^*)\, dx \le C\left(\frac{\rho_\varepsilon}{r_\varepsilon}\right) .$$

Thus by the generalized Sobolev inequality

$$-\delta_\varepsilon^2 + S^F \le \frac{1}{\varepsilon^{2^*}} \int_{\mathbf{R}^n} F(u_\varepsilon^*)\, dx + \frac{1}{\varepsilon^{2^*}} \int_{\{u_\varepsilon^* \le t_\varepsilon\}} F(u_\varepsilon^*)\, dx \le S^F - C\left(\frac{\rho_\varepsilon}{r_\varepsilon}\right)^{n-2} + C\left(\frac{\rho_\varepsilon}{r_\varepsilon}\right)$$

and then

$$\delta_\varepsilon^2 \left(\frac{r_\varepsilon}{\rho_\varepsilon}\right)^{n-2} \ge c > 0$$

which is a contradiction.

$\bigcirc$

Remark 23 If $\limsup_{t\to 0} F(t)/|t|^{2^*} = F_0^+ < S^F/S^*$, where S^* denotes the best Sobolev constant, the concentration result stated in Theorem **??** is proved in [**?**], Theorem 3, without any further assumption on the domain Ω.

Now for general integrands the concentration point can be identified as in [**?**] by means of an asymptotic expansion of $S_\varepsilon^F(\Omega)$ for any domain which satisfies $\tau_\Omega(\infty) > 0$.

Let $\mathcal{B}^F$ be the class of all radial maximizing sequences for S^F and define

$$w_\infty^2 \quad := \quad \frac{2(n-1)}{n\, S^F} \inf \left\{ \liminf_{k\to\infty} \int_{\mathbf{R}^n} \frac{F(w_k)}{K(|\cdot|)} \; : \; \{w_k\} \in \mathcal{B}^F \right\} .$$

Theorem 24 *Suppose that $0 < w_\infty < \infty$ and $\tau_\Omega(\infty) > 0$ or $F_0^+ < S^F/S^*$.*

1. *If the sequence $\{\widetilde{u}_\varepsilon\} \subset D^{1,2}(\Omega)$ satisfies $\|\nabla \widetilde{u}_\varepsilon\|_2 \le \varepsilon$ and concentrates at $x \in \overline{\Omega}$ in the sense of Theorem **??** then*

$$\int_\Omega F(\widetilde{u}_\varepsilon) \quad \le \quad \varepsilon^{2^*} S^F \left(1 - \frac{n}{n-2} w_\infty^2 \tau(x)\, \varepsilon^2 + o(\varepsilon^2)\right)$$

   as $\varepsilon \to 0$.

2. *For any $\overline{x} \in \overline{\Omega}$ there exist $u_\varepsilon \in D^{1,2}(\Omega)$ such that $\|\nabla u_\varepsilon\|_2 = \varepsilon$ and*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\left[\frac{1}{\varepsilon^{2^*}} \int_\Omega F(u_\varepsilon) - S^F\left(1 - \frac{n}{n-2} w_\infty^2 \tau(\overline{x})\, \varepsilon^2\right)\right] \ge 0 \,. \tag{51}$$

3. *In particular a sequence of almost extremals concentrates at a harmonic center, i.e.*

$$\tau(x_0) \quad = \quad \min_{\overline{\Omega}} \tau_\Omega$$

   *with x_0 as in Theorem **??**.*

Sketch of proof. The proof of Part 1, which in the case of volume functional (Theorem **??**) follows directly from the asymptotic formula for the capacity of small sets, in this case is the most complicated. Nevertheless it is exactly the same proof given in [**?**], Theorem 17 Part 1, for bounded domains, using Lemma **??** instead of Lemma **??**, if the concentration occurs at ∞. Similarly, if $\overline{x} \neq \infty$ Part 2 can be proved using harmonic transplantation exactly as in the case of bounded domains (see [**?**], Theorem 17 Part 1).

Thus we will only consider Part 2 in the case $\overline{x} = \infty$.

Also in this case the main idea is to use transplantation. As for the case of the volume functional the main difficulty is that we must consider a sequence $\{x_\varepsilon\}$ approaching infinity, but in this general case this must be done very carefully. Indeed an additional difficulty lies in the fact that we must estimate all the level sets of the Green function, not only that corresponding to 1.

We will just give the main steps of the proof without any detail.

For any given sequence x_ε we will denote by G_{x_ε} the Green function of Ω with singularity at x_ε, while for any given sequence ρ_ε will denote by $G_{\rho_\varepsilon,x_\varepsilon}$ the Green function of the domain $\Omega \cup B_{\rho_\varepsilon}(x_\varepsilon)$ with singularity at x_ε.

We fix a (radial) maximizer w of S^F in $\mathbf{R}^n$, with optimal decay, i.e., $w(r) = w_\infty K(r)(1+o(r))$ for $r > R_0$. We write $w = \varphi \circ K$ and define $w_\varepsilon(x) = (\varphi \circ K)(\varepsilon^{-\frac{2}{n-2}} x) = (\varphi_\varepsilon \circ K)(x)$, where $\varphi_\varepsilon(t) = \varphi(\varepsilon^2 t)$. Then $\|\nabla w_\varepsilon\|_2 = \varepsilon$ and $\int_{\mathbf{R}^n} F(\varphi_\varepsilon \circ K) = S^F$. The candidate for u_ε is $u_\varepsilon = \varphi_\varepsilon \circ G_{x_\varepsilon}$, for a suitable choice of x_ε.

The usual transplantation arguments give

$$(52) \qquad \frac{1}{\varepsilon^{2^*}} \int_\Omega F(u_\varepsilon) \geq \int_0^\infty C_n (F \circ \varphi)(t) \left(\varepsilon^{-2^*} |\{G_{x_\varepsilon} > \frac{t}{\varepsilon^2}\}| \right)^{2\frac{n-1}{n}}$$

where C_n is the isoperimetric constant.

The main idea, as in the proof of Theorem **??**, is to substitute the Green function G_ε with $G_{\rho_\varepsilon,x_\varepsilon}$ for a suitable choice of ρ_ε and x_ε which permit to approach $\tau_\Omega(\infty)$. To this end fix $\delta > 0$ and denote

$$(53) \qquad \omega_\varepsilon(t) = \frac{|\{G_{\rho_\varepsilon,x_\varepsilon} > \frac{t}{\varepsilon^2} + \delta\} \setminus \{G_{x_\varepsilon} > \frac{t}{\varepsilon^2}\}|}{|\{G_{x_\varepsilon} > \frac{t}{\varepsilon^2}\}|}\,.$$

Using a comparison argument as in the proof of Theorem **??**, formula (**??**), we may estimate ω_ε. In particular it is possible to prove that for any sequence $t_\varepsilon \to \infty$ we can find a sequence of radii $\rho_\varepsilon \to 0$ such that

$$(54) \qquad \lim_{\varepsilon \to 0} \sup_{t \in [0,t_\varepsilon]} \frac{\omega_\varepsilon(t)}{\varepsilon^2} = 0\,.$$

Now let us fix t_ε such that

$$(55) \qquad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{\{K > t_\varepsilon\}} F(\varphi \circ K) = 0$$

then there exists a sequence ρ_ε such that (**??**) holds. Corresponding to this ρ_ε, by Proposition **??**, we may find a sequence x_ε such that $\tau_{\Omega \cup B_{\rho_\varepsilon}(x_\varepsilon)}(x_\varepsilon) \leq \tau_\Omega(\infty) + \delta$. Thus by Part 3 of Proposition **??** we have

$$(56) \qquad |\{G_{\rho_\varepsilon,x_\varepsilon} > \frac{t}{\varepsilon^2} + \delta\}| \geq |\{K > \frac{t}{\varepsilon^2} + \tau_\Omega(\infty) + 2\delta\}|\,.$$

Then using that $G_{\rho_\varepsilon,x_\varepsilon} \leq K$ we obtain by explicit computation

$$(57) \qquad \left(\varepsilon_\varepsilon^{-2^*} |\{G_{x_\varepsilon} > \frac{t}{\varepsilon^2}\}| \right)^{2\frac{n-1}{n}} \geq |\{K > t + \varepsilon^2(\tau_\Omega(\infty) + 2\delta)\}|^{2\frac{n-1}{n}} - C |\{K > t\}|^{2\frac{n-1}{n}} \omega_\varepsilon(t)\,.$$

Finally, let B_ε be the ball of center 0 and radius R_ε such that $K(R_\varepsilon) = \varepsilon^2(\tau_\Omega(\infty) + 2\delta)$, and let $G_{B_\varepsilon} = K - K(R_\varepsilon)$ be the corresponding Green function with pole in 0. By an explicit computation, by changing variables in the integral and taking into account the definition of w_∞, we get

$$(58) \qquad \int_{B_\varepsilon} F(\varphi \circ G_{B_\varepsilon}) = S^F (1 - \frac{n}{n-2} w_\infty^2 (\tau_\Omega(\infty) + 2\delta)\varepsilon^2) + o(\varepsilon^2)\,.$$

Moreover by the radial symmetry of G_{B_ε} and by (**??**) and (**??**) we have

$$\frac{1}{\varepsilon^{2^*}}\left(\frac{1}{\varepsilon^2}\int_\Omega F(u_\varepsilon)-\int_{B_\varepsilon}F(\varphi\circ G_{B_\varepsilon})\right)\geq S^F\sup_{t\in[0,t_\varepsilon]}\frac{\omega_k(t)}{\varepsilon_\varepsilon^2}-C\frac{1}{\varepsilon_k^2}\int_{\{K>t_\varepsilon\}}F(\varphi\circ K)\,. \tag{59}$$

The conclusion follows taking the limit as $\varepsilon\to 0$ and using (**??**), (**??**) and (**??**), and the arbitrariness of δ.

$\bigcirc$

References

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