Dielectric breakdown: optimal bounds

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Abstract

We study dielectric breakdown for composites made of two isotropic phases. We show that Sachs' bound is optimal. This simple example is used to illustrate a variational principle which departs from the traditional one. We also derive the usual variational principle by elementary means without appealing to the technology of convex duality.

1 Introduction

We study a model of (first failure) dielectric breakdown in $\mathbf{R}^n$. We discuss the general setting with an emphasis on the mathematical setting. We actually give two different derivations (both based on a power-law type regularization) and discuss the advantages of each of them. Our choice to treat conductivity is motivated by its simplicity. However, the general setting applies to other problems, including polycrystal plasticity in the context of anti-plane shear or plane stress. In this respect, our analysis follows a recent paper by Kohn and Little [15] on the subject. We begin with an informal description of the model problem which follows closely the notation and the setting given in the latter work.

Consider a body occupies a bounded portion of space Ω which is the union of two non overlapping subdomains Ω_1 and Ω_2 of prescribed volume fractions. Given two positive thresholds M_1 and M_2 which are characteristic of materials one and two respectively, and given $\xi \in \mathbf{R}^n$, we would like to know if there exists a function u, with $⨍ \nabla u \, dx = \xi$, such that the following condition is satisfied at almost every point

$$\begin{array}{ll} |\nabla u(x)| \leq M_1 \quad , & x \in \Omega_1 \, , \\ |\nabla u(x)| \leq M_2 \quad , & x \in \Omega_2 \, . \end{array} \tag{1.1}$$

The physical picture is as follows. The vector ξ represents the average electric field. For any given direction $\widehat{\xi}$, there exists a critical value $e_{\rm cr}$ of the magnitude of the average electric field $\xi = \lambda\widehat{\xi}$ such that if $|\lambda| < e_{\rm cr}$, then the electric field ∇u satisfies (1.1) everywhere and the body behaves as a perfect insulator. Otherwise if $|\lambda| \geq e_{\rm cr}$, then ∇u violates (1.1) in some region and then the body begins to conduct. This is a reasonable way to model the first failure in dielectric breakdown. After this first failure occurs the model is inadequate and other effects have to be taken into account.

Dielectric breakdown is one of the many nonlinear models which are used in the physicist's literature (see [16] and references therein). It is a simplified version of the

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more flexible power-law model which is known to give an adequate description for classes of conductors including ZnO ceramics (see [2] and references therein).

The mathematical justification of the dielectric breakdown model as a limiting case of the power-law model is one of the focus of our paper. Our result is two-fold. First we achieve an efficient mathematical derivation in two different ways, both new to the best of our knowledge. Second we show how the new derivation gives a different point of view on the model which recovers all the information of the traditional approach but, in addition, gives a new variational principle which is quite interesting being less degenerate than the traditional one.

To explain the content of these derivations it is convenient to rewrite (1.1) in a more concise form as follows

$$\nabla u(x) \in K(x) \quad , \quad \text{a.e. } \; x \in \Omega \,, \tag{1.2}$$

where

$$K(x) := \{\eta \in R^2 \;:\; \lambda(x)|\eta| \leq 1\} \tag{1.3}$$

and

$$\lambda(x) = \beta\chi(x) + \alpha(1-\chi(x)) = M_1^{-1}\chi(x) + M_2^{-1}(1-\chi(x))\,. \tag{1.4}$$

Here χ is the characteristic function of the region Ω_β, while α and β are given positive numbers. We will use throughout the usual terminology: the set Ω_β is occupied by "phase β", the arrangement of the two phases is determined by the characteristic function χ and will be called the "microgeometry". In the setting of anti-plane shear Kohn and Little considered a set $K(x)$ which is a rectangle whose orientation varies from point to point. In our case, (in dimension two) the set $K(x)$ is a circle whose radius varies from point to point. In the context of plasticity, the set $K(x)$ is called the yield set. One knows this set for any given phase, i.e. one knows M_1 and M_2. The ultimate goal is to understand the structure of an "effective" yield set, K_{eff} which, roughly speaking, represents the yield set of the given body made of the two phases and which is parametrized by a set of vectors: K_{eff} is the set of all $\xi \in \mathbf{R}^n$ such that there exists a function u, with $⨍ \nabla u\, dx = \xi$, such that $\nabla u \in K(x)$.

We now begin the mathematical formulation. Let $Q = (0,1)^n$ be the unit cube. Fix $0 < \alpha < \beta$, $\theta \in (0,1)$, $\xi \in \mathbf{R}^n$ and a microgeometry χ in Q, with $\int \chi\, dx = \theta$. The problem is to find a function $u = u_\sharp + \langle \xi, x\rangle$ which satisfies (1.1), with $u_\sharp$ a Q-periodic function. The natural function space requires u to have a bounded gradient. Hence we require $u_\sharp \in W^{1,\infty}_\sharp(Q)$ (i.e. $u_\sharp$ is Q-periodic and belongs to $W^{1,\infty}_{\text{loc}}(\mathbf{R}^n)$). In this formulation the analogue of the so-called effective yield set is given by

$$K_{\text{hom}} \equiv \{\xi \in \mathbf{R}^n : (1.1) \text{ holds for some } u = u_\sharp + \langle \xi, x\rangle \text{ with } u_\sharp \in W^{1,\infty}_\sharp(Q)\}\,. \tag{1.5}$$

Now for $p \geq 1$ we consider the corresponding power-law "ancestor problem":

$$g_p^{\text{hom}}(\xi) = \inf_{u \in W^{1,p}_\sharp(Q)} \int_Q g_p(x, \nabla u(x) + \xi)dx\,, \tag{1.6}$$

where

$$g_p(x,\xi) = \frac{1}{p}\lambda^p(x)|\xi|^p\,. \tag{1.7}$$

For $p = 2$, one can interpret $g_2^{\hom}$ as the effective (or homogenized) conductivity associated to the scalar conductivity $\sigma(x) = \lambda^2(x)$. For $p \neq 2$ the material has a nonlinear response and the Euler Lagrange equations associated to (1.6) are

$$\begin{cases} \operatorname{div}[\lambda^p(x)|\nabla u(x) + \xi|^{p-2}(\nabla u(x) + \xi)] \;=\; 0 \quad \text{in } \mathbf{R}^n\,, \\ u \in W^{1,p}_\sharp(Q)\,. \end{cases} \tag{1.8}$$

The heuristic is as follows. As p grows, the function g_p tends to blow up unless the field satisfies (1.2) almost everywhere. It is therefore tempting to consider a formal limit in the following way. One can say that as p tends to infinity, either (1.2) is satisfied at any point, and then the material does not conduct when subject to average field ξ or otherwise at some point in the material (or to be slightly more precise on a set of positive measure) condition (1.2) is violated. In the latter case the material begins to conduct if subject to that particular average field. This leads to a *formal* derivation of the dielectric breakdown model. Following the tradition, we set

$$g_\infty^{\hom}(\xi) = \inf_{u \in W^{1,\infty}_\sharp(Q)} \int_Q g_\infty(x, \nabla u(x) + \xi)\, dx\,, \tag{1.9}$$

where

$$g_\infty(x,\xi) = I_{K(x)}(\xi) := \begin{cases} 0 & \text{if } \lambda(x)|\xi| \le 1 \quad \text{a.e in } Q, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{1.10}$$

and we rewrite the "effective yield" set $K_{\hom}$ as

$$K_{\hom} \equiv \{\xi \in \mathbf{R}^n \;:\; g_\infty^{\hom}(\xi) < \infty\} \equiv \{\xi \in \mathbf{R}^n \;:\; g_\infty^{\hom}(\xi) = 0\}\,. \tag{1.11}$$

The function $g_\infty^{\hom}$ is, in fact, the right substitute of $g_p^{\hom}$ in the case $p = \infty$, but this fact requires some justification. The traditional one uses (among other things) the formalism of convex duality. The reader is referred to the book by Jikov et al. [12] and to [15] for an exposition of this point of view. The papers [4], [5] and [25] are among those studying related problems.

In this paper we will justify the model in a more direct way using the point of view of the De Giorgi's Γ-convergence (see the seminal paper [8] and the monographs [7] and [6]).

We will also describe a different rigorous derivation of the model. We will see that our new derivation recovers all the information delivered by (1.9) and, in our opinion, it also has some advantages. Let us outline our alternative strategy. We consider the new family of functions

$$f_p^{\hom}(\xi) := \inf_{u \in W^{1,p}_\sharp(Q)} \left(\int_Q \lambda^p(x)|\nabla u(x) + \xi|^p dx \right)^{\frac{1}{p}}\,. \tag{1.12}$$

This is a convenient notation. However let us emphasize that we are not claiming that $f_p^{\hom}(\xi)$ is the homogenized energy density associated to the right hand side of (1.12).

Obviously, for any $p \in (1,\infty)$, this function describes the same physics as (1.6). We show that the above family defines a limit function, namely

$$f_\infty^{\hom}(\xi) = \inf_{u \in W^{1,\infty}_\sharp(Q)} \sup_{x \in Q} (\lambda(x)|\nabla u(x) + \xi|)\,. \tag{1.13}$$

With these notations one can check that

$$\xi \in K_{\text{hom}} \Leftrightarrow f_\infty^{\text{hom}}(\xi) \le 1 \tag{1.14}$$

and moreover that

$$\xi \in \partial K_{\text{hom}} \Leftrightarrow f_\infty^{\text{hom}}(\xi) = 1\,. \tag{1.15}$$

Thus our formulation describes in a very direct way K_{hom} exactly like the more traditional one and therefore it is not essentially different form it. An immediate advantage is that it achieves the same goals with more elementary and direct mathematical tools and moreover it characterizes ∂K_{hom} in a very straightforward way through (1.15).

Using the new formulation (1.13) we obtain a number of interesting consequences which are definitely less transparent in the traditional approach. For instance we can deduce the following facts. For fixed microgeometry, $f_p^{\text{hom}}(\xi)$ is a continuous function of p in the interval $(1, \infty]$ (see (2.13)). This provides a useful comparison. Second, among the minimizers of (1.13), there is at least one, let us call it u_∞, which is inherited from the sequence u_p, where the function u_p is the (unique) minimum of the variational principle (1.12). Next, under some smoothness assumptions on g_p, one finds an Euler-Lagrange equation for u_∞. Indeed u_∞ turns out to be a so-called *viscosity solution* of a second order non linear PDE. This is the anisotropic version of the operator Δ_∞ (called the infinite-laplacian) which has been studied among others by Aronsson [1], Jensen [14], T. Bhattacharya, De Benedetto and Manfredi [3]. We will not pursue this interesting direction here. We refer to Sections 1, 2 and 3 of Juutinen's thesis [14] for more details. In looking for a minimal approach to our work one might bring back these consequences to corresponding statements in the language which uses g_p^{hom} and g_∞^{hom}. However, this would be rather artificial since the new approach gives an interesting insight in itself.

Let us now describe the content of each section of the present paper. Section 2 is the most technical of the paper and proves the derivations described above. Sections 3 and 4 address the second focus of the paper. This is a detailed study of the model (1.2), (1.3) and (1.4). More precisely, we study the problem of establishing microgeometry independent bounds for (1.13). To explain the main results of our analysis, we focus first on the upper bounds (Section 3). Our starting point is the well known Sachs' bound. For each ξ, the bound depends on α, β and the volume fraction. One is interested in showing the optimality of this bound. In our new language, this is translated into the following statement: find a vector ξ and a microgeometry, so that the inequality

$$f_\infty^{\text{hom}}(\xi) \le \max(\alpha, \beta)|\xi|$$

holds as an equality.

The optimality, in the sense just mentioned can be called optimality along one direction. This is easy to achieve through a simple laminate microgeometry. A much more complicated issue is to show *simultaneous optimality* with respect to any desired direction. This is our way of saying that the bound is optimal among "isotropic" microgeometries. We achieve this goal by exhibiting a new class of optimal microgeometries. This class has some feature in common with the microgeometry proposed in the context of polycrystalline materials by Kohn and Little. One interesting feature of the class we propose here is its remarkable flexibility.

Our work is strongly connected with the literature concerning attempts to improve Sachs' bound for power-law materials. It is now well understood that improving this elementary bound for non linear material is a fundamentally more difficult problem than improving the equally famous Taylor's bound [31]. The first significant attempt in this direction is due to Talbot and Willis [28] and [29]. They treated three dimensional problems. Their technique is based on previous work by Willis [32], Stein [24] and John and Nirenberg [13]. It was able to give an improvement of Sachs' bound which, at first, seemed disappointingly modest. However our work proves that this improvement ought to be very small for large p and actually it must tend to zero when p tends to infinity, hence giving additional value to their work. Recently Talbot, in the very interesting paper [26], made a further improvement, in particular for $n = 2$. Again similar comments apply.

In Section 4, we turn our attention to microgeometry independent lower bounds. We exhibit a particular example which shows that, in the limit as $\min(\alpha, \beta)$ tends to zero, the lower bound predicts the right scaling law. We conclude, in particular, that in the limit when the ratio of β and α tends to infinity, the composite may behave as perfectly conductor as well as perfectly insulator depending on the type of microgeometry. Section 5 is devoted to the conclusions.

2 The power-law asymptotics

In this section we give two rigorous derivations of the dielectric breakdown model presented in the introduction using the technique of Γ-convergence. This are Propositions 2.1 and 2.6. Proposition 2.5 complements the results, by exploring some direct implications which will be used later in the paper.

This new application of Γ-convergence to material science adds to the many which are already commonly used.

To state our results we need some preliminary definitions. Let $Q = (0, 1)^n$ be the unit cube in $\mathbf{R}^n$, let $0 < \alpha \leq \beta$ be two given real constants and let λ be an $L^\infty(Q)$ function such that

$$\alpha \leq \lambda(x) \leq \beta \qquad \text{a.e. in } \; Q\,.$$

We are interested in the behavior, as p goes to ∞, of the minima of the functionals of the form $\frac{1}{p} \int_Q |\lambda \nabla u|^p dx$ under suitable periodic (or more general) boundary conditions. We shall consider

$$g_p^{\text{hom}}(\xi) = \inf_{u \in W^{1,p}_\sharp(Q)} \frac{1}{p} \int_Q \lambda^p(x) |\nabla u(x) + \xi|^p dx\,, \tag{2.1}$$

where $W^{1,p}_\sharp(Q)$ is the class of all functions in $W^{1,p}(Q)$ which, extended by periodicity, belong to $u \in W^{1,p}_{\text{loc}}(\mathbf{R}^n)$.

The Γ-convergence, result permits, in this case, to describe the behavior of all sequences of "almost minimizers" (as explained in Proposition 2.5 below). Apriori each functional we study is well defined on a different space ($W^{1,p}(Q)$). However, in order to study the Γ-convergence, it is natural to extend our functionals to a single Banach space in which a notion of convergence is given. We recall that such convergence must be weak enough to assure the compactness of any sequence of minimizers and strong enough to assure the lower semicontinuity of the energies. To this aim let G_p be the functional defined in L^1 by

$$G_p(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_Q |\lambda \nabla u|^p dx & \text{if } u \in W^{1,p}(Q), \\ +\infty & \text{otherwise in } L^1(Q). \end{cases} \tag{2.2}$$

We say that a sequence $\{G_p\}$ Γ-converges to the functional G_∞ (with respect to the L^1 topology) if the following two conditions are satisfied:

(i) For every $u \in L^1$ there exists a sequence $\{u_p\}$ (called a recovering sequence), such that $u_p \to u$ in L^1 with

$$\limsup_{p\to\infty} G_p(u_p) \le G_\infty(u)\,.$$

(ii) For every sequence u_p in L^1, with $u_p \to u$ in L^1, we have

$$\liminf_{p\to\infty} G_p(u_p) \ge G_\infty(u)\,.$$

We shall prove the following result

Proposition 2.1 *The family $\{G_p\}$ Γ-converges with respect to the L^1 topology to the functional*

$$G_\infty(u) = \begin{cases} 0 & \text{if } |\lambda \nabla u| \le 1 \text{ a.e. in } Q, \\ +\infty & \text{otherwise in } L^1(Q). \end{cases} \tag{2.3}$$

We postpone the proof of Proposition 2.1.

Remark 2.2 Proposition 2.1 provides (via Proposition 2.5 below), the following consequences. First, for any $\xi \in \mathbf{R}^n$, the convergence of $g_p^{\hom}(\xi)$, defined by (2.1), to $g_\infty^{\hom}(\xi)$ given by

$$g_\infty^{\hom}(\xi) = \inf_{u \in W_\sharp^{1,\infty}(Q)} G_\infty(u + \langle \xi, x\rangle) = \inf_{u \in W_\sharp^{1,\infty}(Q)} \int_Q 1_{K(x)}(\nabla u + \xi)dx\,. \tag{2.4}$$

Second, among the minimizers of (2.4) there is at least one which is a limit (up to subsequences) of the sequence of the minimizers of (1.6).

We shall also consider a slightly different functional defined by

$$F_p(u) = \begin{cases} \left(\displaystyle\int_Q |\lambda \nabla u|^p dx\right)^{\frac{1}{p}} & \text{if } u \in W^{1,p}(Q), \\ +\infty & \text{otherwise in } L^1(Q). \end{cases} \tag{2.5}$$

Clearly, for any given $\xi \in \mathbf{R}^n$ and $p > 1$, v is a minimum for (2.1) if and only if v is a minimum for

$$f_p^{\hom}(\xi) = \inf_{u \in W_\sharp^{1,p}(Q)} F_p(u + \langle \xi, x\rangle)\,. \tag{2.6}$$

Remark 2.3 The main reason to study the Γ-limit of the functionals F_p is that, as proved in Proposition 2.6, this approach will provide us with a new variational principle which is less degenerate than (2.4). In particular this will establish a new characterization of the set K_{hom} (see Remark 2.8).

The Γ-convergence result stated in Proposition 2.1 makes rigorous the statement that "the power-law constraint reduces to a pointwise constraint as p goes to infinity". This crucial point is the content of the following lemma.

Lemma 2.4 *Let u_p be a sequence converging to u in L^1 and such that*

$$\liminf_{p\to\infty} G_p(u_p) < +\infty\,,$$

then $|\lambda\nabla u| \leq 1$ a.e. in Q.

Proof. Without loss of generality we can assume that

$$\liminf_{p\to\infty} G_p(u_p) = \lim_{p\to\infty} G_p(u_p) = C < +\infty\,. \tag{2.7}$$

By Hölder's inequality we have

$$\int_Q |\nabla u_p|^\alpha dx \leq (p\,G_p(u_p))^{\frac{\alpha}{p}} \left(\int_Q \frac{1}{\lambda^\alpha}\,dx\right)^{\frac{p-\alpha}{p}} \qquad \text{for all} \quad \alpha \in [1,p] \tag{2.8}$$

which, together with (2.7), implies in particular that the sequence ∇u_p is bounded in $L^\alpha(Q)$ for every $\alpha \geq 1$. Hence, since u_p converges to u in $L^1(Q)$, ∇u_p converges to ∇u weakly in $L^\alpha(Q)$. For every open subset $B \subseteq Q$, Hölder's inequality yields

$$\int_B |\lambda\nabla u_p|\,dx \leq \left(\int_B |\lambda\nabla u_p|^p dx\right)^{\frac{1}{p}} |B|^{\frac{p-1}{p}}\,.$$

Then, taking the limit as $p\to\infty$, using the weak lower semicontinuity of the norm and (2.7), we have

$$\int_B |\lambda\nabla u|\,dx \leq \liminf_{p\to\infty} \int_B |\lambda\nabla u_p|\,dx \leq \liminf_{p\to\infty} p^{\frac{1}{p}} G_p(u_p)^{\frac{1}{p}} |B|^{\frac{p-1}{p}} = |B| \tag{2.9}$$

for every open set $B \subseteq Q$. Since $\lambda\nabla u$ is an L^1 function, for almost all x_0 in Q

$$\lim_{\rho\to 0} \fint_{B_\rho(x_0)} |\lambda\nabla u|\,dx = |\lambda(x_0)\nabla u(x_0)|\,.$$

The latter in conjunction with (2.9) yields the conclusion. □

Proof of Proposition 2.1. Let us prove first the property (i) of the Γ-convergence. If $G_\infty(u) = +\infty$ the inequality is trivially satisfied by any sequence converging to u. Otherwise, let u be such that $G_\infty(u) = 0$, i.e. $|\lambda\nabla u| \leq 1$ a.e. in Q. Choosing $u_p = u$ we have

$$\limsup_{p\to\infty} G_p(u_p) \leq \limsup_{p\to\infty} \frac{1}{p} = 0 = G_\infty(u)\,.$$

We now check property (ii) . By Lemma 2.4, if $u_p \to u$ and $\liminf_{p\to\infty} G_p(u_p) < +\infty$, then $|\lambda\nabla u| \leq 1$. Hence $G_\infty(u) = 0$ and (ii) follows. □

We now prove the convergence of the minimizers.

Proposition 2.5 *For any $\xi \in \mathbf{R}^n$, $g_p^{\hom}(\xi)$ converges to $g_\infty^{\hom}(\xi)$. Moreover every sequence of minimizers $\{u_{\xi,p}\}$ of problems (2.1), up to a subsequence, converges in L^1 to a minimum u_ξ of problem (2.4).*

Proof. Let us fix $\xi \in \mathbf{R}^n$ and let $u_{\xi,p} \in W^{1,p}_\sharp(Q)$ be a minimizer of (2.1), so that $G_p(u_{\xi,p} + \langle \xi, x\rangle) = g_p^{\hom}(\xi)$.

If $\limsup_{p\to\infty} G_p(u_{\xi,p} + \langle \xi, x\rangle) = +\infty$, then $g_\infty^{\hom}(\xi) = +\infty$. Indeed assume, by contradiction, that there exists $v \in W^{1,\infty}_\sharp(Q)$ such that $G_\infty(v + \langle \xi, x\rangle) = 0$, then $|\lambda(\nabla v + \xi)| \leq 1$ and hence we have

$$\limsup_{p\to\infty} G_p(u_{\xi,p} + \langle \xi, x\rangle) \leq \limsup_{p\to\infty} G_p(v + \langle \xi, x\rangle) = 0\,.$$

Therefore we may assume that

$$\limsup_{p\to\infty} G_p(u_{\xi,p} + \langle \xi, x\rangle) < +\infty\,. \tag{2.10}$$

Up to a translation we have $⨍ u_{\xi,p}\,dx = 0$. Thus, by the equicoerciveness of G_p (see (2.8)), we have that $u_{\xi,p}$ is bounded in $W^{1,\alpha}(Q)$, $\alpha \geq 1$, and, up to a subsequence, it converges to some function u_ξ weakly in $W^{1,\alpha}(Q)$ and strongly in $L^1(Q)$. Finally, by (2.10) and Lemma 2.4, $|\lambda(\nabla u_\xi + \langle \xi, x\rangle)| \leq 1$, i.e. $G_\infty(u_\xi + \langle \xi, x\rangle) = 0$. Moreover $u_\xi \in W^{1,\infty}_\sharp(Q)$ and

$$0 \leq g_\infty^{\hom}(\xi) \leq G_\infty(u_\xi + \langle \xi, x\rangle) = 0\,,$$

hence, $g_\infty^{\hom}(\xi) = 0$. Finally

$$\limsup_{p\to\infty} G_p(u_{\xi,p} + \langle \xi, x\rangle) \leq \limsup_{p\to\infty} G_p(u_\xi + \langle \xi, x\rangle) = 0\,.$$

□

We now establish the new variational principle for the dielectric breakdown problem.

Proposition 2.6 *The sequence $\{F_p\}$, defined by (2.5), Γ-converges with respect to the L^1 topology to F_∞ defined as follows:*

$$F_\infty(u) = \begin{cases} \sup\limits_Q |\lambda \nabla u| & \text{if } u \in W^{1,\infty}(Q) \\ +\infty & \text{otherwise in } L^1(Q). \end{cases} \tag{2.11}$$

Proof. The result is an immediate consequence of the fact that F_p is an increasing sequence of L^1 lower semicontinuous functionals that converges pointwise to F_∞. Indeed the pointwise convergence ensures that condition (i) of the Γ-convergence is satisfied by a recovering sequence $u_p = u$ for every $u \in L^1(Q)$. To obtain condition (ii), let $\{u_p\}$ be a sequence converging to u in L^1. By the monotonicity of $\{F_p\}$ we have

$$F_q(u_p) \leq F_p(u_p) \qquad \forall\, q < p\,,$$

so that, by the lower semicontinuity we get

$$F_q(u) \leq \liminf_{p\to\infty} F_q(u_p) \leq \liminf_{p\to\infty} F_p(u_p)\,.$$

We conclude taking the limit as $q \to \infty$. □

Remark 2.7 Let us define

$$f_\infty(\xi) = \inf_{u\in W^{1,\infty}_\sharp(Q)} F_\infty(u+\langle\xi,x\rangle) = \inf_{u\in W^{1,\infty}_\sharp(Q)} \sup_Q |\lambda(\nabla u+\xi)|\,. \tag{2.12}$$

Arguing as in the proof of Proposition 2.5, we deduce by Proposition 2.6 that

$$\lim_{p\to\infty} f_p^{\mathrm{hom}}(\xi) = f_\infty^{\mathrm{hom}}(\xi)\,. \tag{2.13}$$

Remark 2.8 The function f_∞ "reconstructs" the set K_{hom}. Indeed we have

$$K_{\mathrm{hom}} = \Big\{\xi\in\mathbf{R}^n \;:\; \exists u\in W^{1,\infty}_\sharp(Q) \text{ such that } |\nabla u+\xi|\le 1\Big\}$$

and thus, by (2.12), we get

$$K_{\mathrm{hom}} = \{\xi \;:\; f_\infty(\xi)\le 1\}\,. \tag{2.14}$$

Clearly $f_\infty(\xi)$ is an homogeneous function and, by Lemma 2.9 below, it is also Lipschitz continuous. This implies in particular that

$$\partial K_{\mathrm{hom}} = \{\xi \;:\; f_\infty(\xi) = 1\}\,. \tag{2.15}$$

Lemma 2.9 *The function $f_\infty^{\mathrm{hom}}(\xi)$ is Lipschitz continuous. More precisely*

$$|f_\infty^{\mathrm{hom}}(\xi_1) - f_\infty^{\mathrm{hom}}(\xi_2)| \le \beta|\xi_1-\xi_2| \quad , \qquad \xi_1\,,\ \xi_2\in\mathbf{R}^n\,.$$

Proof. Let us fix $\xi_1,\xi_2\in\mathbf{R}^n$ and let $u_1\in W^{1,\infty}_\sharp(Q)$ be a minimum for problem (2.12) corresponding to the vector ξ_1, i.e. $f_\infty^{\mathrm{hom}}(\xi_1) = \sup(\lambda(x)|\nabla u_1(x)+\xi_1|)$. Thus

$$f_\infty^{\mathrm{hom}}(\xi_2) - f_\infty^{\mathrm{hom}}(\xi_1) \le \sup_x(\lambda(x)|\nabla u_1(x)+\xi_2|) - \sup_x(\lambda(x)|\nabla u_1(x)+\xi_1|) \le \beta|\xi_2-\xi_1|\,.$$

We conclude the proof interchanging the role of ξ_1 and ξ_2. □

3 Upper bounds for f_∞^{hom}

In this section we prove that there exist *isotropic* microgeometries which saturate Sachs' bound. Hence Sachs' bound is optimal for the dielectric breakdown model. From now on $\lambda(x)$ will be given by

$$\lambda(x) = \beta\chi(x) + \alpha(1-\chi(x))\,,$$

where $0<\alpha\le\beta$ and χ is the characteristic function of a measurable subset E_β of the unit cube Q (a microgeometry), i.e. $\chi=\chi_{E_\beta}$. In the sequel we use the notation $f_{\infty,\chi}^{\mathrm{hom}}(\xi)$ when we want to emphasize the dependence of $f_\infty^{\mathrm{hom}}(\xi)$ on the microgeometry.
We will verify that Sachs' bound can be written in the form

$$f_\infty^{\mathrm{hom}}(\xi)\le\beta|\xi|\,. \tag{3.1}$$

We shall first give a sufficient condition for a microgeometry to be optimal for Sachs' bound in one fixed direction $\widehat{\xi}$. Then we show that Sachs' bound is optimal for isotropic microgeometries (a microgeometry is said to be isotropic if $f_\infty^{\mathrm{hom}}(\xi)$ depends only on $|\xi|$).

In contrast, when $p \in [2,\infty)$, it is possible to prove that there exists no isotropic microgeometry satisfying Sachs' bound for the function f_p^{hom}, [21]. However, the result of this section implies that, as p approaches infinity, Sachs' bound becomes nearly optimal for f_p^{hom} and therefore any quantitative improvement of it must be very small as soon as p becomes rather large. We remark that this shows that the results of Talbot and Willis [28] and [29] on the subject are remarkably good.

In the setting of dielectric breakdown (or perfect plasticity), Sachs' bound is given by

$$K_{\mathrm{Sachs}} \subseteq K_{\mathrm{hom}} \tag{3.2}$$

where the set K_{Sachs} is defined by

$$K_{\mathrm{Sachs}} = \{\xi \in \mathbf{R}^n \ : \ \xi \in K(x) \ \text{ for all } x \in Q\}\,, \tag{3.3}$$

so that, in our case

$$K_{\mathrm{Sachs}} = \left\{\xi \in \mathbf{R}^n \ : \ |\xi| \leq \frac{1}{\beta}\right\} =: B_{\frac{1}{\beta}}\,. \tag{3.4}$$

Sachs' bound can be equivalently stated in terms of $f_{\infty,\chi}^{\mathrm{hom}}$, as follows

$$f_{\infty,\chi}^{\mathrm{hom}}(\xi) \leq \beta|\xi| \qquad \forall\, \xi \in \mathbf{R}^n\,. \tag{3.5}$$

To check this, for any $\widehat{\xi} \in \mathbf{R}^n$, with $|\widehat{\xi}| = 1$ (i.e. $\widehat{\xi} \in S^{n-1}$), let us define

$$\lambda(\widehat{\xi}) = \sup\{t \in [0,\infty) \ : \ t\widehat{\xi} \in K_{\mathrm{hom}}\}\,. \tag{3.6}$$

By the homogeneity of $f_{\infty,\chi}^{\mathrm{hom}}(\xi)$ and by (2.14) and (2.15) we have

$$\lambda(\widehat{\xi}) = \frac{1}{f_{\infty,\chi}^{\mathrm{hom}}(\widehat{\xi})} \tag{3.7}$$

for every $\widehat{\xi} \in S^{n-1}$. Now (3.2) is equivalent to saying that

$$t\widehat{\xi} \in K_{\mathrm{hom}} \qquad \forall\, t \in \left[0, \frac{1}{\beta}\right] \quad \text{and} \quad \forall\, \widehat{\xi} \in S^{n-1} \tag{3.8}$$

and this, by definition, implies $\lambda(\widehat{\xi}) \geq \frac{1}{\beta}$, and hence (3.5).

Conversely if (3.5) holds true, then, by (3.7) $\lambda(\widehat{\xi}) \geq \frac{1}{\beta}$ for every $\widehat{\xi} \in S^{n-1}$, and this implies (3.8); hence, by (3.4), we have

$$B_{\frac{1}{\beta}} \subseteq K_{\mathrm{hom}} \qquad \Leftrightarrow \qquad f_{\infty,\chi}^{\mathrm{hom}}(\xi) \leq \beta|\xi| \quad \forall\, \xi \in \mathbf{R}^n\,.$$

Note that $\lambda(\widehat{\xi})$ corresponds to the critical value e_{cr} of the magnitude of the electric field as defined in the introduction.

Definition 3.1 *Sachs' bound (3.5) is optimal along a given direction $\widehat{\xi} \in S^{n-1}$ (or $\widehat{\xi}$-optimal) if*

$$\sup_{\chi} f_{\infty,\chi}^{\mathrm{hom}}(\widehat{\xi}) = \beta\,. \tag{3.9}$$

A microgeometry χ such that

$$f_{\infty,\chi}^{\mathrm{hom}}(\widehat{\xi}) = \beta\,, \tag{3.10}$$

is called a $\widehat{\xi}$-optimal microgeometry.

Remark 3.2 Note that, by the definition of $f^{\rm hom}_{\infty,\chi}(\widehat{\xi})$, the $\widehat{\xi}$-optimality of a microgeometry χ is equivalent to the fact that $u = 0$ is a minimizer for problem (1.13). Moreover since $f^{\rm hom}_{\infty,\chi}(\xi)$ is homogeneous, (3.10) implies that

$$f^{\rm hom}_{\infty,\chi}(\xi) = \beta|\xi|$$

for all vectors ξ proportional to $\widehat{\xi}$.

Let us give a sufficient condition for the optimality of a microgeometry.

Proposition 3.3 *Fix a direction $\widehat{\xi} \in S^{n-1}$. Let χ be a microgeometry and let E_β the set occupied by the phase β ($\chi = \chi_{E_\beta}$). Assume that E_β contains a neighborhood of a closed path γ on the torus ($\mathbf{R}^n/\mathbf{Z}^n$), with γ parallel to $\widehat{\xi}$. Then χ is a $\widehat{\xi}$-optimal microgeometry.*

We postpone the proof.

Example 3.4 The microgeometry in Fig. 1 is optimal along e_1 and e_2.

Remark 3.5 Note that the condition above on the set E_β is equivalent to saying that there exists a closed path γ on the torus and $\delta > 0$ such that $E_\beta \supseteq \{x \in Q \,:\, \mathrm{dist}(x,\gamma) \le \delta\}$. Moreover this condition can be satisfied only if $\widehat{\xi}$ is a "rational" direction, i.e. it is a multiple of $\widehat{\xi}$ belongs to $\mathbf{Z}^n$.

In particular, for any rational direction $\widehat{\xi}$ and any $\theta > 0$, we can construct an $\widehat{\xi}$-optimal microgeometry with $|E_\beta| = \theta$. Indeed, first fix a closed path γ on the torus $\mathbf{R}^n/\mathbf{Z}^n$ with finite length L and parallel to $\widehat{\xi}$ (which is always possible because $\widehat{\xi}$ has a rational direction) and then set $E_\beta = \{x \in Q \,:\, \mathrm{dist}(x,\gamma) \le \delta\}$. Then, by Proposition 3.3, χ_{E_β} is $\widehat{\xi}$-optimal and $|E_\beta| = \theta$ for a suitable choice of δ (see Fig. 2).

Proof of Proposition 3.3. Let $u \in W^{1,\infty}_\sharp(Q)$ be such that $f^{\rm hom}_{\infty,\chi}(\widehat{\xi}) = \|\lambda(\nabla u + \widehat{\xi})\|_\infty$. Assume by contradiction that

$$\|\lambda(\nabla u + \widehat{\xi})\|_\infty < \beta\,,$$

then, in particular, we have

$$\sup_{E_\beta} |\nabla u + \widehat{\xi}| < 1\,.$$

This implies that $\langle \nabla u, \widehat{\xi}\rangle < -\varepsilon$, for some $\varepsilon > 0$.

Let L be the length of γ. Since $u \in W^{1,\infty}_\sharp(Q)$, its restriction to almost every line parallel to $\widehat{\xi}$ is absolutely continuous; thus, since a neighborhood of γ belongs to E_β, we may assume that u is absolutely continuous on γ and therefore the following calculation makes sense

$$0 = \int_\gamma du = \int_0^L \langle \nabla u, \widehat{\xi}\rangle\, ds < -\varepsilon L\,,$$

yielding a contradiction. □

Definition 3.6 *We say that χ is an isotropic microgeometry if $f^{\rm hom}_{\infty,\chi}(\xi) = C_\chi|\xi|$, where C_χ is a positive constant, i.e. if $f^{\rm hom}_{\infty,\chi}(\xi)$ depends only on the norm of ξ.*

In view of (2.14) this definition is equivalent to saying that the corresponding K_{hom} is a ball of radius C_χ^{-1}.

Remark 3.7 Since $f^{\text{hom}}_{\infty,\chi}(\xi)$ is Lipschitz continuous with respect to ξ (see Lemma 2.9), if a microgeometry is optimal along all rational directions, then it is optimal along any direction. In particular it is isotropic.

Remark 3.7 together with Proposition 3.3 will permit us to construct a class of optimal isotropic microgeometry.

Theorem 3.8 *For any fixed* $0 < \theta \leq 1$, *there exist infinitely many optimal isotropic microgeometries with volume fraction* θ.

Proof. Let us fix an arbitrary (countable) family of rational directions $\{\xi_h\}$ which is dense in S^{n-1}. For any such direction ξ_h let us consider a closed path γ_h in $\mathbf{R}^n/\mathbf{Z}^n$ parallel to ξ_h. Set $\gamma = \cup_h \gamma_h$ and note that γ has zero measure. By the continuity of the Lebesgue measure we have that for any $\theta \in (0,1]$ there exists an open set E_β containing γ, with $|E_\beta| = \theta$. By construction the set E_β satisfies the assumptions of Proposition 3.3. Hence, by Remark 3.7, the microgeometry $\chi = \chi_{E_\beta}$ is an optimal isotropic microgeometry. □

Example 3.9 To give an example of a type of geometry that can arise by the construction in the proof of Theorem 3.8, let us consider, for any $h \in \mathbf{N}$, the closed path γ_h as above and let L_h be its length. Fix $\delta \in (0,1)$ and for every $h \in \mathbf{N}$ set

$$E_\beta^{\delta,h} = \left\{ x \in Q \ : \ \operatorname{dist}(x, \gamma_h) < \left(\frac{2^{-h}\delta}{L_h} \right)^{\frac{1}{n}} \right\}.$$

Clearly $|E_\beta^{\delta,h}| \leq 2^{-h}\delta$. Thus the set $E_\beta^\delta = \cup_h E_\beta^{\delta,h}$ is an open set which contains $\gamma = \cup_h \gamma_h$ and, since $0 < |E_\beta^\delta| \leq \delta < 1$, E_β^δ it is non-trivial example of optimal isotropic microgeometry.

Note that, in general, due to its "self intersections" the set E_β^δ has measure strictly less than δ. However, by "fattening" it in an arbitrary way, one can achieve any desired volume fraction greater than $|E_\beta^\delta|$.

4 Lower bound for $p = \infty$

In the present section we give lower bounds for f^{hom}_∞. We will not attempt to give the best possible of such bounds. Our goal is more modest. We give a lower bound which is very easy to compute based on the comparison principle developed by Ponte-Castañeda [22] and P. Suquet [25]. One could probably obtain the same bounds with a different method and, at least in principle, more refined bounds are available. More precisely, the study in [20] suggests that the same result could be attained using the Talbot-Willis method [27] (enlarging upon previous work of Willis [32]). In addition, possibly better bounds could be obtained using the so-called translation method directly at level p.

In principle, even better result could be obtained by applying the reasonings in [19] where the two former methods are combined.

On the other hand, these more refined methods are not always guaranteed to give (strictly) better bounds and in any case the calculations involved are major. The reason our lower bounds are quite satisfactory is that they have the very important feature of having an *asymptotically optimal scaling law.*

Actually we shall see that for $0 < \alpha \le \beta$, our lower bound goes to zero as α goes to zero. This may appear disappointing at first. However, it is easy to prove that it must be so. This is because there exist isotropic microgeometries for which f_∞^{hom} tends to zero as α tends to zero. A more refined question to ask is whether our bounds predict the right scaling. In the following we prove that this is in fact the case.

Indeed, fix the numbers $\beta > \alpha$ and the volume fraction $\theta \in (0,1)$ of the phase β. The main results of the present section are Proposition 4.1 and Proposition 4.5. The former shows that there exists a constant C_L such that *for all* microgeometries χ_{E_β}, with $|E_\beta| = \theta$, one has

$$f_{\infty,\chi}^{\mathrm{hom}}(\xi) \ge \alpha C_L |\xi| \tag{4.1}$$

and therefore

$$\inf_\chi f_{\infty,\chi}^{\mathrm{hom}}(\xi) \ge \alpha C_L |\xi| \,. \tag{4.2}$$

Proposition 4.5 proves that there exists a constant C_U such that

$$\inf_\chi f_{\infty,\chi}^{\mathrm{hom}}(\xi) \le \alpha C_U |\xi| \,. \tag{4.3}$$

The combination of the inequalities (4.2) and (4.3) shows that our lower bound predicts the exact scaling law as α goes to zero. In particular, this shows that as α goes to zero there are microgeometries for which the composite behaves like a perfect insulator. Combining this with the work of Section 3 we see that as the ratio between β and α diverges the composite may behave like a perfect conductor or a perfect insulator depending on the specific microgeometry.

Similar questions emerge in the study of ionic polycrystals. However, the study of these issues requires a different analysis, see [9].

We begin by proving (4.2). More precisely we prove the following proposition.

Proposition 4.1 *(Comparison principle). Given $0 < \alpha \le \beta$, $\theta \in [0,1]$, $n \ge 2$, and given n vectors ξ^i forming an orthonormal basis, let $f_\infty^{\mathrm{hom}}(\xi^i)$ be defined by (2.12). Then for any choice of a microgeometry χ with $⨍ \chi = \theta$, one has*

$$\left(\frac{\sum_{i=1}^n (f_\infty^{\mathrm{hom}}(\xi^i))^2}{n} \right)^{\frac{1}{2}} \ge \alpha \left(\frac{\alpha^2 (n-1)(1-\theta) + \beta^2 (1-\theta+n\theta)}{\alpha^2 (n-1+\theta) + \beta^2 (1-\theta)} \right)^{\frac{1}{2}} . \tag{4.4}$$

A useful and immediate corollary is the following

Corollary 4.2 *If f_∞^{hom} is isotropic, (see Definition 3.6), then*

$$\forall \xi \in \mathbf{R}^n, f_\infty^{\mathrm{hom}}(\xi) \ge \alpha \left(\frac{\alpha^2 (n-1)(1-\theta) + \beta^2 (1-\theta+n\theta)}{\alpha^2 (n-1+\theta) + \beta^2 (1-\theta)} \right)^{\frac{1}{2}} |\xi| \,. \tag{4.5}$$

Clearly (4.5) implies (4.2).

The proof of Proposition 4.1 will follow the plan implemented by Ponte Castañeda, Suquet and others in several papers. See for instance [22], [25], [23] and [20]. The calculation of [15] is also similar in spirit. The only somewhat technical difference is that we assume nothing in terms of isotropy both of the linear and the nonlinear behavior. This is a nice feature since it maybe very difficult to check whether a given microgeometry is indeed isotropic. In this respect our derivation is more efficient. The results in [20] might, in principle be better but we will not use those more complicated bounds. Let us begin focusing on the linear bounds which will be needed to apply the comparison principle. We recall that given $\sigma(x) = \sigma_1(1-\chi) + \sigma_2\chi$ with $0 < \sigma_1 < \sigma_2$, then the effective conductivity σ^* is defined as follows: for any vector $\xi \in \mathbf{R}^n$,

$$\langle \sigma^*\xi, \xi\rangle := \inf_{u \in W^{1,2}_\sharp(Q)} \int_Q \sigma(x)|\nabla u(x) + \xi|^2 dx\,.$$

For given σ_1, σ_2 and volume fraction $\theta = \int \chi$, the set of possible σ^* as χ varies is called the G-closure of the corresponding problem and denoted by G_θ. This set is known in any dimension n. This is a result due to Tartar and Murat [30] and to Lurie and Cherkaev [17]. They proved, in particular the following result

Theorem 4.3 *If σ^* belongs to G_θ, its eigenvalues σ^*_j, $j = 1, 2, \ldots, n$ satisfy the bounds*

$$\begin{aligned} &\sum_{i=1}^n \frac{1}{\sigma^*_i - \sigma_1} \leq \frac{n}{\theta(\sigma_2-\sigma_1)} + \frac{1-\theta}{\theta\sigma_1}, \\ &\sigma^*_j \geq \left(\frac{1-\theta}{\sigma_1} + \frac{\theta}{\sigma_2}\right)^{-1}, \qquad j = 1, 2, \ldots, n\,. \end{aligned} \tag{4.6}$$

We may now begin the proof of Proposition 4.1.

Proof of Proposition 4.1. Set $\sigma = \lambda^2$. For any $\xi \in \mathbf{R}^n$ we have

$$\langle \sigma^*\xi, \xi\rangle := \inf_{u \in W^{1,2}_\sharp(Q)} \int_Q \sigma(x)|\nabla u(x) + \xi|^2 dx\,.$$

In particular for any $u \in W^{1,\infty}_\sharp(Q)$ and any $\xi \in \mathbf{R}^n$ we have

$$\langle \sigma^*\xi, \xi\rangle \leq \|\sigma^{\frac{1}{2}}(\nabla u + \xi)\|^2_\infty\,.$$

Taking the infimum on both sides of the latter inequality over the set of functions $u \in W^{1,\infty}_\sharp(Q)$ we obtain

$$\langle \sigma^*\xi, \xi\rangle \leq \left(f^{\hom}_\infty(\xi)\right)^2 \qquad \forall \xi \in \mathbf{R}^n\,. \tag{4.7}$$

Next we consider n orthonormal vectors ξ^i and apply (4.7) to each of them obtaining

$$\sum_{i=1}^n \left(f^{\hom}_\infty(\xi_i)\right)^2 \geq \text{Trace } \sigma^* \geq \min_{\sigma^* \in G_\theta} \text{Trace } \sigma^*\,. \tag{4.8}$$

Recalling that $\sigma = \lambda^2$, to prove (4.4) it remains to prove that

$$\min_{\sigma^* \in G_\theta} \text{Trace } \sigma^* = n\sigma_1 \frac{\sigma_1(n-1)(1-\theta) + \sigma_2(1-\theta+n\theta)}{\sigma_1(n-1+\theta) + \sigma_2(1-\theta)}\,. \tag{4.9}$$

Indeed, because of a certain convexity property of the set G_θ, one can check that

$$\min_{\sigma^*\in G_\theta} \text{Trace}\,\sigma^* = \min_{\sigma^*\in G_{\theta,\text{isot}}} \text{Trace}\,\sigma^*\,, \tag{4.10}$$

where $G_{\theta,\text{isot}}$ is the intersection of G_θ with the set of matrices proportional to the identity (isotropic). (This calculation, based on a Lagrange multiplier argument is omitted). Therefore using (4.10) and (4.6), (4.8) implies (4.9) and hence (4.4) follows. □

Let us consider an example where we can compute explicitly the function f_∞ in two different directions and that shows a linear behavior in α when α tends to zero.

Example 4.4 For simplicity we shall give a two dimensional geometry for which we compute $f_\infty(\mathrm{e}_i)$, with $i = 1, 2$, $\mathrm{e}_1 = (1, 0)$, and $\mathrm{e}_2 = (0, 1)$. Let E_β be the cube $Q_{\frac{1}{2}}$ of side $\frac{1}{2}$ centered at $(1/2, 1/2)$. For any fixed $0 < \alpha < \beta$, let us consider the piecewise affine function given by

$$u(x_1,x_2) = \begin{cases} \dfrac{\beta-\alpha}{\beta+\alpha}x_1 & \text{if } \ x_1 \le \dfrac{1}{4} \\ -\dfrac{\beta-\alpha}{\beta+\alpha}\left(x_1-\dfrac{1}{2}\right) & \text{if } \ \dfrac{1}{4} \le x_1 \le \dfrac{3}{4} \\ \dfrac{\beta-\alpha}{\beta+\alpha}(x_1-1) & \text{if } \ \dfrac{3}{4} \le x_1 \le 1. \end{cases}$$

Remark that

$$|\lambda(x_1,x_2)(\nabla u(x_1,x_2)+\mathrm{e}_1)| = \|\lambda(\nabla u+\mathrm{e}_1)\|_\infty = \frac{2\alpha\beta}{\beta+\alpha} \qquad \text{a.e. } x=(x_1,x_2)\in R\,,$$

where $R = \{(x_1, x_2)\ :\ 0 \le x_1 \le 1 \text{ and } \frac{1}{4} \le x_2 \le \frac{3}{4}\}$. We shall prove that $f_\infty(\mathrm{e}_1) = \frac{2\alpha\beta}{\beta+\alpha}$. Indeed we have already remarked that

$$f_\infty(\mathrm{e}_1) \le \|\lambda(\nabla u+\mathrm{e}_1)\|_\infty = \frac{2\alpha\beta}{\beta+\alpha}\,. \tag{4.11}$$

To prove the opposite inequality let us arguing by contradiction. Assume that there exists a function $\tilde{u} \in W^{1,\infty}_\sharp(Q)$ such that

$$\|\lambda(\nabla\tilde{u}+\mathrm{e}_1)\|_\infty < \frac{2\alpha\beta}{\beta+\alpha}\,. \tag{4.12}$$

In particular, by (contradiction), for almost every $x = (x_1, x_2) \in E_\beta$

$$\beta(\langle\nabla\tilde{u}(x),\mathrm{e}_1\rangle+1) \le \beta|\nabla\tilde{u}(x)+\mathrm{e}_1| < \frac{2\alpha\beta}{\beta+\alpha}\,,$$

which implies that

$$\langle\nabla\tilde{u}(x),\mathrm{e}_1\rangle < -\frac{\beta-\alpha}{\beta+\alpha}\,. \tag{4.13}$$

Now, by the periodicity of $\tilde{u}$, we get

$$\int_R \langle\nabla\tilde{u}(x),\mathrm{e}_1\rangle\,dx = 0\,,$$

where R is the rectangle defined above, and thus, by (4.13), we have

$$\int_{R\setminus E_\beta} \langle \nabla\tilde{u}(x), \mathrm{e}_1\rangle\, dx = -\int_{E_\beta} \langle \nabla\tilde{u}(x), \mathrm{e}_1\rangle\, dx > |E_\beta|\,\frac{\beta-\alpha}{\beta+\alpha}\,.$$

Since $|E_\beta| = |R\setminus E_\beta|$ we have

$$\sup_{R\setminus E_\beta} \langle \nabla\tilde{u}(x), \mathrm{e}_1\rangle > \frac{\beta-\alpha}{\beta+\alpha}.$$

Hence

$$\|\lambda(\nabla\tilde{u}+\mathrm{e}_1)\|_\infty \geq \sup_{R\setminus E_\beta} \alpha|\nabla\tilde{u}(x)+\mathrm{e}_1| > \alpha\left(\frac{\beta-\alpha}{\beta+\alpha}+1\right) = \frac{2\alpha\beta}{\beta+\alpha}\,,$$

which, together with (4.12), gives a contradiction. Hence $f_\infty^{\mathrm{hom}}(\mathrm{e}_1) = \frac{2\alpha\beta}{\beta+\alpha}$. By the symmetry of this geometry we also get that $f_\infty^{\mathrm{hom}}(\mathrm{e}_2) = \frac{2\alpha\beta}{\beta+\alpha}$.

Let us now go back to the proof of (4.3). We prove the following proposition.

Proposition 4.5 *For any $\theta \in (0,1)$ there exists a microgeometry χ^θ, with $\int \chi^\theta = \theta$, such that for every $\xi \in S^{n-1}$*

$$\sup_{\alpha>0} \frac{f^{\mathrm{hom}}_{\infty,\chi^\theta}(\xi)}{\alpha} \leq C\,, \tag{4.14}$$

where C is a positive constant which depends neither on β nor on ξ. In particular there exists a constant $C_U > 0$ such that

$$\inf_{\int\chi=\theta} f^{\mathrm{hom}}_{\infty,\chi}(\xi) \leq \alpha C_U|\xi|\,.$$

Proof. Fix $\theta \in (0,1)$, $0<\alpha<\beta$, and consider the microgeometry $\chi^\theta = \chi^{Q_\theta}$, where Q_θ denotes the cube of side $\theta^{\frac{1}{n}}$ centered in Q. Let us denote by $d(x)$ the function

$$d(x) = \frac{2}{1-\theta^{\frac{1}{n}}}\,\mathrm{dist}(x,\partial Q)\,,$$

where $\mathrm{dist}(x,\partial Q)$ is the distance between x and ∂Q. Next for any $\xi \in S^{n-1}$, let us define the function

$$u(x) = \begin{cases} -\langle \xi, x\rangle & \text{if } x \in E_\beta = Q_\theta \\ \\ -\langle \xi, x\rangle\, d(x) & \text{if } x \in Q\setminus E_\beta\,. \end{cases}$$

Observe that $u \in W^{1,\infty}_\sharp(Q)$, hence we have

$$\begin{aligned} f_{\infty,\chi}(\xi) &\leq \|\lambda(x)(\nabla u(x)+\xi)\|_\infty = \sup_{Q\setminus E_\beta} \alpha|\nabla u(x)+\xi| \leq \\ &\sup_{Q\setminus E_\beta} \alpha\|\xi - d(x)\xi - \langle\xi,x\rangle\nabla d(x)\| \leq \sup_{Q\setminus E_\beta} \alpha\left((1-d(x)) + |x||\nabla d(x)|\right). \end{aligned} \tag{4.15}$$

Since $|\nabla d(x)| \leq \frac{2}{1-\theta^{\frac{1}{n}}}$ we obtain

$$f^{\mathrm{hom}}_{\infty,\chi}(\xi) \leq \alpha\left(1+\frac{2\sqrt{n}}{1-\theta^{\frac{1}{n}}}\right),$$

which concludes the proof. □

5 Conclusions

We have given two slightly different characterizations of a model of dielectric breakdown. We have described the new characterization and some of its features in the simplest possible example of a mixture of two isotropic phases.

Our new variational principle needs only slight modifications to be extended to other problems. For instance the isotropy of the phases and their number play no role. The same approach should work also in polycrystal plasticity at least for the kind of model presented in [11]. However the issue of bounds and their attainability requires a different analysis, (see [9]).

The obvious disadvantage to depart from a very well known and established way of thinking about the model, is partly compensated by the simplicity of the new approach. The crucial potential advantage of the new formulation is the use of the new variational principle since it is less degenerate than the traditional one.

For the model we have treated, it is known (see [3] and [14]) that, under some smoothness hypotheses, one can derive a sort of Euler Lagrange equation for a variational principle of type (2.12). Unfortunately, the assumptions which are required are still inadequate for problems involving composites. In the smooth case, i.e. when λ is smooth, one finds a second order equation of the form

$$\langle \nabla(\lambda^2 |\nabla u|^2), \lambda^2 \nabla u \rangle = 0\,. \tag{5.1}$$

In particular when $\lambda = 1$, (5.1) reduces to the so called infinite-laplacian, usually denoted by Δ_∞ and defined by

$$\Delta_\infty u := \langle D^2 u \nabla u, \nabla u \rangle\,. \tag{5.2}$$

The mathematics involved is rather subtle since the solutions are to be understood in the weak sense of "viscosity solutions". Nevertheless, we believe that this is a promising issue deserving further investigations. We refer to [14] for a very nice and self-contained introduction to this subject as well as for some of the most recent results (see in particular Corollary 4.33 of [14]).

Our examples in Section 4 make some contact with examples which are present in the literature of the so-called infinite laplacian.

From the point of view of bounds, our results of Section 3 and (2.13) shed new light on the bounds by Talbot and Willis, [28] and [29], showing their considerable efficiency.

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