Concentration of low energy extremals: Identification of concentration points

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Dedicated to Stefan Hildebrandt on the occasion of his 65th birthday

Abstract

We study the variational problem

$$S_\varepsilon^F(\Omega) = \frac{1}{\varepsilon^{2^*}} \sup \left\{ \int_\Omega F(u) \,:\, u \in D^{1,2}(\Omega),\ \|\nabla u\|_2 \le \varepsilon \right\},$$

where $\Omega \subset \mathbf{R}^n$, $n \ge 3$, is a bounded domain, $2^* = \frac{2n}{n-2}$ and F satisfies $0 \le F(t) \le \alpha t^{2^*}$ and is upper semicontinuous. We show that to second order in ε the value $S_\varepsilon^F(\Omega)$ only depends on two ingredients. The geometry of Ω enters through the Robin function τ_Ω (the regular part of the Green's function) and F enters through a quantity w_∞ which is computed from (radial) maximizers of the problem in $\mathbf{R}^n$. The asymptotic expansion becomes

$$S_\varepsilon^F(\Omega) = \varepsilon^{2^*} S^F \left(1 - \frac{n}{n-2} w_\infty^2 \min_{\overline{\Omega}} \tau_\Omega \, \varepsilon^2 + o(\varepsilon^2)\right).$$

Using this we deduce that a subsequence of (almost) maximizers of $S_\varepsilon^F(\Omega)$ must concentrate at a harmonic center of Ω, i.e., $\frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_{x_0}$, where $x_0 \in \overline{\Omega}$ is a minimum point of τ_Ω.

Keywords: variational problem, concentration, Robin function

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1 Introduction

Let Ω be a domain in $\mathbf{R}^n$, $n \ge 3$. We continue the investigation of the variational problem

$$\sup \left\{ \int_\Omega F(u) \,:\, \int_\Omega |\nabla u|^2 \le \varepsilon^2,\ u = 0 \text{ on } \partial\Omega \right\} \tag{1}$$

started in [9]. We are interested in the asymptotic behaviour of the solutions u_ε of (1) as $\varepsilon \to 0$. The integrand is supposed to satisfy the growth condition

$$0 \le F(t) \le \alpha |t|^{2^*}$$

where $2^* := \frac{2n}{n-2}$ denotes the critical Sobolev exponent. For smooth integrands every solution of (1) satisfies the Euler Lagrange equation

$$\begin{aligned} -\Delta u &= \lambda f(u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2}$$

with $f = F'$ and a large Lagrange multiplier λ. In [9] it is shown that as $\varepsilon \to 0$ the sequence $\{u_\varepsilon\}$ concentrates at a single point $x_0 \in \overline{\Omega}$. For small ε the major part of the energy is concentrated in the vicinity of this point. For applications like the Bernoulli free-boundary problem or the plasma problem it is important to know the location of the concentration point.

In this paper we show that the concentration point is a minimum point of the Robin function (the regular part of the Green's function with equal arguments), see Theorem 17. In particular the concentration point does not depend on the integrand. The proof relies mostly on two ingredients. The first is a sharp decay estimate for almost maximizers (see Lemma 22 in Section 7). The corresponding result for exact maximizers was first obtained in [8]. The second ingredient is an approximation formula for the capacity of small sets. We show in particular that this formula requires no regularity conditions on Ω, if one defines the Green's function and the Robin function in the appropriate way (see Section 2, in particular Definition 6). Another subtle point is that we allow discontinuous integrals F in order to include e.g. Bernoulli's problem (maximization of volume for given relative capacity, i.e. $F = \chi_{[1,\infty)}$). Therefore we cannot use the usual form of the Euler Lagrange equations. Instead we use the weak Euler Lagrange equation obtained by variation of the independent variable [8]. It involves F but no derivatives of F.

The relevance of the critical points of the Robin function for Dirichlet problems that involve the critical Sobolev exponent was first pointed out by Schoen [17] and Bahri [1]. Rey [16] and Han [13] showed that as $p \to 2^*$ the maximum points of the positive solutions of

$$\begin{aligned} \Delta u + u^{p-1} &= 0 \ \text{ in } \ \Omega, \\ u &= 0 \ \text{ on } \ \partial\Omega \end{aligned}$$

accumulate at a critical point of the Robin function. This has been conjectured by Brézis and Peletier [4]. The simpler proof of [11] applies to all dimensions and shows that the concentration point is a minimum point of the Robin function. Similar results for the Ginzburg-Landau functional have been obtained by Bethuel, Brézis and Hélein [3]. The influence of the Robin function on the location of concentration points is weaker than that of any kind of anisotropy. For instance the solutions of

$$\sup\left\{\int_\Omega G(\cdot)F(u) \,:\, \int_\Omega |\nabla u|^2 \le \varepsilon^2,\ u = 0 \ \text{ on } \ \partial\Omega\right\}$$

concentrate at a maximum point of G and not at a minimum point of the Robin function. Those of

$$\sup\left\{\int_\Omega F(u) \,:\, \int_\Omega \nabla u \cdot A(\cdot)\nabla u \le \varepsilon^2,\ u = 0 \ \text{ on } \ \partial\Omega\right\}$$

concentrate at a minimum point of $\det A$ [6]. For further references see also [7].

2 Hypotheses and generalized Sobolev inequality

Let Ω be an open subset of $\mathbf{R}^n$, $n \ge 3$. By $\overline{\Omega}$ we denote the closure of Ω in $\mathbf{R}^n \cup \{\infty\}$. In particular the closure of an unbounded domain contains the point ∞.

The natural function space for variational problems of the form (1) is $D^{1,2}(\Omega)$ defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|\nabla v\|_2 \;=\; \left(\int_\Omega |\nabla v|^2\right)^{1/2}.$$

The results of this paper require one or more of the following hypotheses.

(Ω) Ω is a domain in $\mathbf{R}^n$ of dimension $n \ge 3$ with $\Omega \ne \mathbf{R}^n$ in the sense $\operatorname{cap}_{\mathbf{R}^n}(\mathbf{R}^n \setminus \Omega) > 0$. Moreover Ω is not an exterior domain, i.e. $\infty \in \overline{\mathbf{R}^n \setminus \Omega}$.

(F) The integrand F satisfies the growth condition $0 \le F(t) \le \alpha\,|t|^{2^*}$ for some constant α. It is upper semicontinuous and $F \not\equiv 0$ in the L^1 sense.

(F^+) $\max(F_0^+, F_\infty^+) < S^F/S^*$ with each term as defined below.

As in [9] we set

$$\begin{aligned} F_0^+ \;&:=\; \limsup_{t\to 0}\frac{F(t)}{|t|^{2^*}}, \quad F_\infty^+ \;:=\; \limsup_{|t|\to\infty}\frac{F(t)}{|t|^{2^*}}, \\ S_\varepsilon^F(\Omega) \;&:=\; \frac{1}{\varepsilon^{2^*}}\sup\left\{\int_\Omega F(u) \,:\, u \in D^{1,2}(\Omega),\ \|\nabla u\|_2 \le \varepsilon\right\}, \end{aligned}$$

and we define the generalized Sobolev constant by

$$S^F \;:=\; S_1^F(\mathbf{R}^n).$$

For the critical power $F(t) = |t|^{2^*}$ we denote by $S^* := S^F$ the best Sobolev constant. A simple scaling argument leads to the generalized Sobolev inequality

$$(3) \qquad \int_\Omega F(u) \quad \leq \quad S^F \|\nabla u\|_2^{2^*} \quad \text{for} \quad u \in D^{1,2}(\Omega)$$

In fact, the rescaled function $u^s(x) := u(x/s)$, with $s := \|\nabla u\|_2^{-\frac{2}{n-2}}$, satisfies $\|\nabla u^s\|_2 = 1$ and

$$(4) \qquad \int_{s\Omega} F(u^s) \quad = \quad \|\nabla u\|_2^{-2^*} \int_\Omega F(u).$$

By the generalized Sobolev inequality we know that $S^F_\varepsilon(\Omega) \leq S^F$. Moreover $S^F_\varepsilon(\Omega) \to S^F$ as $\varepsilon \to 0$. For the critical power $F(t) = |t|^{2^*}$ we have $S^F = S^F_\varepsilon(\Omega)$ for every ε. But typically $S^F_\varepsilon(\Omega)$ decreases as ε increases (see Theorem 17 below). An extremal for the generalized Sobolev constant or entire extremal is a function $w \in D^{1,2}(\mathbf{R}^n)$ with $\|\nabla w\|_2 = 1$ and $\int_{\mathbf{R}^n} F(w) = S^F$.

We say that $\{u_\varepsilon\}$ is a sequence of almost extremals for (1) if u_ε is admissible for the definition of $S^F_\varepsilon(\Omega)$ and

$$\frac{\int_\Omega F(u_\varepsilon)}{\varepsilon^{2^*}} = S^F_\varepsilon(\Omega) + o(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0.$$

3 Concentration and asymptotic shape of low energy extrem als

The main results of previous papers [9, 8] are summarized in the following two theorems.

THEOREM 1 ([9]) *Suppose (Ω) and (F). Suppose in addition that one of the following assumptions holds: (a) $F_0^+ < S^F/S^*$ or (b) $F_0^- = F_0^+$ or (c) Ω has finite volume. Then*

1. *If $\{u_\varepsilon\}$ satisfies $\|\nabla u_\varepsilon\| \leq \varepsilon$ and $\varepsilon^{-2^*}\int F(u_\varepsilon) \to S^F$ as $\varepsilon \to 0$, then a subsequence of $\{u_\varepsilon\}$ concentrates at a single point $x_0 \in \overline{\Omega}$, i.e.*

$$\frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \stackrel{*}{\rightharpoonup} \delta_{x_0}, \qquad \frac{F(u_\varepsilon)}{\varepsilon^{2^*}} \stackrel{*}{\rightharpoonup} S^F \delta_{x_0}$$

in the sense of measures.

If in addiction (F^+) holds then

2. *For every $\varepsilon > 0$ the variational problem (1) has a solution u_ε.*

3. *There are points $x_\varepsilon \to x_0$ such that a subsequence of the rescaled functions*

$$w_\varepsilon(y) \quad := \quad u_\varepsilon\left(x_\varepsilon + \varepsilon^{\frac{2}{n-2}} y\right)$$

tend to an extremal for S^F, i.e. $w_\varepsilon \to w$ in $D^{1,2}(\mathbf{R}^n)$, $\|\nabla w\|_2 = 1$, and $\int_{\mathbf{R}^n} F(w) = S^F$.

Concerning entire extremals we have the following additional information. Let

$$K(r) = \frac{1}{(n-2)|S^{n-1}|r^{n-2}}$$

denote the fundamental solution of $-\Delta$.

THEOREM 2 ([8]) *Assume (F) and let w be an extremal for S^F. Then:*

1. *Either $w > 0$ or $w < 0$.*

2. *There is a ball $B^{r_0}_{x_0}$ such that w agrees with the Schwarz symmetrization w^* outside this ball.*

3. *If we assume $w > 0$ and $x_0 = 0$, then the function $r \mapsto w(r)$ is strictly decreasing on (r_0, ∞) and*

$$w(r) = W_\infty K(r)\left(1 + O(r^{-2})\right), \tag{5}$$
$$w'(r) = W_\infty K'(r)\left(1 + O(r^{-2})\right) \tag{6}$$

for $r \to \infty$, where

$$W_\infty^2 = \frac{2(n-1)}{n\,S^F}\int_{\mathbf{R}^n}\frac{F(w)}{K(|\cdot|)}.$$

4. *In particular $w(r) \le c\,r^{2-n}$, $F(w(r)) \le c\,r^{-2n}$, and*

$$\int_{\mathbf{R}^n\setminus B_0^R}|\nabla w|^2 \le c\,R^{2-n}, \qquad \int_{\mathbf{R}^n\setminus B_0^R}F(w) \le c\,R^{-n}$$

for every $R > 0$.

5. *If F is non-decreasing on $\mathbf{R}^+$ and non-increasing on $\mathbf{R}^-$ then $\overline{B_0^{r_0}} = \{w = \max w\}$.*

In the following we denote by $\mathcal{B}^F$ the class of maximizing sequences for S^F consisting of radial functions.

THEOREM 3 *Suppose (F) and set*

$$w_\infty^2 := \frac{2(n-1)}{n\,S^F}\inf\left\{\liminf_{k\to\infty}\int_{\mathbf{R}^n}\frac{F(w_k)}{K(|\cdot|)} \;:\; \{w_k\}\in\mathcal{B}^F\right\}.$$

Then

1. *$w_\infty = 0$ if and only if some sequence in $\mathcal{B}^F$ concentrates at 0.*
2. *$0 < w_\infty < \infty$ if and only if no sequence in $\mathcal{B}^F$ concentrates at 0 and an extremal for S^F exists.*
3. *$w_\infty = \infty$ if and only if every sequence in $\mathcal{B}^F$ concentrates at ∞.*

Proof. Consider an arbitrary sequence $\{w_k\}\in\mathcal{B}^F$. By the generalized concentration-compactness alternative [9, Theorem 9] exactly one of the following three possibilities can occur (after suitable extraction of a subsequence):

A) Concentration at the origin: $|\nabla w_k|^2 \overset{*}{\rightharpoonup} \delta_0$, $F(w_k) \overset{*}{\rightharpoonup} S^F\delta_0$,

B) Compactness: $w_k \to w$ in $D^{1,2}(\mathbf{R}^n)$, $F(w_k)\to F(w)$ in $L^1(\mathbf{R}^n)$,

C) Concentration at infinity: $|\nabla w_k|^2 \overset{*}{\rightharpoonup} \delta_\infty$, $F(w_k) \overset{*}{\rightharpoonup} S^F\delta_\infty$.

If $w_\infty = 0$ then

$$\int_{\mathbf{R}^n}\frac{F(w_k)}{K(|\cdot|)} \to 0$$

for some $\{w_k\}\in\mathcal{B}^F$. This excludes B and C because $S^F > 0$. Conversely if there is a maximizing sequence which concentrates at the origin we choose a radial cut-off function η supported in B_0^r with $\eta(0) = 1$. After suitable scaling as in (4) with $s_k \to 1$ the sequence $\{(\eta w_k)^{s_k}\}$ is in $\mathcal{B}^F$. Thus

$$w_\infty^2 \le \frac{1}{K(r)}\int_{B_0^r}F(w_k) + o(1)$$

which tends to 0 as $r\to 0$. This proves 1. of Theorem 3.

If an extremal function for S^F exists then $w_\infty < \infty$ by Theorem 2. If $\{w_k\}\in\mathcal{B}^F$ concentrates at infinity then

$$\int_{\mathbf{R}^n}\frac{F(w_k)}{K(|\cdot|)} \ge \frac{1}{K(R)}\int_{\mathbf{R}^n\setminus B_0^R}F(w_k) \to \frac{S^F}{K(R)}$$

which tends to infinity as $R\to\infty$. Finally $w_\infty = \infty$ excludes A and B. ○

Remark 4 If $0 < w_\infty < \infty$, by Theorems 1 and 2 we deduce that $w_\infty^2 \le W_\infty^2$.

As a consequence of Theorems 2 and 3 we obtain the following compactness criterion.

COROLLARY 5 *If (F) and (F^+) holds then S^F admits a radial extremal w, $0 < w_\infty < \infty$, and $w(r)/K(r) \to w_\infty$ as $r\to\infty$.*

4 Robin function and harmonic centers

In this section Ω will be an arbitrary open subset of $\mathbf{R}^n$ with $n \geq 3$, which satisfy (Ω). To simplify the notation we will make the convention that in this section ∞ is not considered a boundary point.

The concentration point x_0 of Theorem 1 will be identified in terms of the Robin function of Ω, i.e. the diagonal of the regular part of the Green's function of the Dirichlet problem in Ω for the Laplace operator. This function has been considered in [2] in the case of domains with regular boundary. In the following we shall give a definition which extends the one of [2] and holds for any domain, possibly with irregular boundary, and we shall study its basic properties.

Let us denote by $K_x(y) = K(|x-y|)$, for every $x, y \in \mathbf{R}^n$, the fundamental solution for the negative Laplacian. For every point $x \in \Omega \cup \partial\Omega$, let us define the regular part of the Green's function, $H_\Omega(x, \cdot)$, as the solution in the sense of Perron-Wiener-Brelot (PWB) of the following Dirichlet problem

$$\text{(7)} \qquad \begin{cases} \Delta_y H_\Omega(x,y) = 0 & \text{in } \Omega, \\ H_\Omega(x,y) = K_x(y) & \text{on } \partial\Omega, \end{cases}$$

i.e., $H_\Omega(x, \cdot)$ is the infimum of all superharmonic functions u such that

$$\liminf_{\substack{z \to y \\ z \in \Omega}} u(z) \geq K_x(y)$$

for every $y \in \partial\Omega \cup \{\infty\}$ (see [14]).

Note that the notion of PWB solution is stable under increasing sequences of resolutive boundary data. Thus the function $H_\Omega(x, y)$ is well defined also if $x \in \partial\Omega$. The Green's function of the Dirichlet problem for the Laplacian is defined by

$$G_x(y) \quad = \quad K_x(y) - H_\Omega(x,y)\,.$$

The Green's function is symmetric in $\Omega \times \Omega$ (see [14], Theorem 5.24), hence $H_\Omega(x, y) = H_\Omega(y, x)$ for every $(x, y) \in \Omega \times \Omega$.

If $x \in \Omega$ the function $H_\Omega(x, \cdot)$ coincides with the weak solution of (7) in the sense of $D^{1,2}(\Omega)$ and the Green's functions agree with the solution in the sense of Stampacchia of the problem

$$\begin{cases} -\Delta_y G_x(y) = \delta_x & \text{in } \Omega, \\ G_x(y) = 0 & \text{on } \partial\Omega \end{cases}$$

(see [15]). In general, given a measure μ of bounded variation, we say that a function $u \in L^1(\Omega)$ is a solution in the sense of Stampacchia of the equation $-\Delta u = \mu$, vanishing at $\partial\Omega$ if it satisfies

$$\text{(8)} \qquad \int_\Omega u\Phi dx = \int_\Omega G(\Phi) d\mu$$

for every $\Phi \in C^0(\Omega)$, where $G(\Phi)$ is the solution vanishing on $\partial\Omega$ of the equation $-\Delta G(\Phi) = \Phi$. The solution in the sense of Stampacchia is unique and belongs (for bounded domains) to the space $W_0^{1,p}(\Omega)$ for every $p < \frac{n}{n-1}$. For more general domains the truncations $(-t) \vee (t \wedge u)$ belong to $D^{1,2}(\Omega)$ and $\int_{\{|u|<t\}} |\nabla u|^2 \leq t\|\mu\|$. After a short calculation this yields weak L^p bounds $\|\nabla u\|_{\frac{n}{n-1},\infty} + \|u\|_{\frac{n}{n-2},\infty} \leq C\|\mu\|$, i.e. $|\{|\nabla u| > t\}| t^{\frac{n}{n-1}} \leq C\|\mu\|$ and $|\{|u| > t\}| t^{\frac{n}{n-2}} \leq C\|\mu\|$.

Moreover this notion is stable with respect to the weak convergences of measures, that is if μ_k is a sequence of measures of bounded variation such that supp$\mu_k \subset K \subset \Omega$, for a fixed compact set K, and $\mu_k \overset{*}{\rightharpoonup} \mu$ then the corresponding solutions converge to the solution of (8) in $W^{1,p}_{loc}(\Omega)$ for every $p < \frac{n}{n-1}$.

For every $x \in \Omega \cup \partial\Omega$, let us extend the function $H_\Omega(x, \cdot)$ to a superharmonic function $\widetilde{H}_\Omega(x, \cdot)$ defined on all $\mathbf{R}^n$, as follows: for every $y \in \partial\Omega$ we set

$$\text{(9)} \qquad \widetilde{H}_\Omega(x,y) = \liminf_{\substack{z \to y \\ z \in \Omega}} H_\Omega(x,z)$$

and $\widetilde{H}_\Omega(x,y) = K_x(y)$ for every $y \in \mathbf{R}^n \setminus \overline{\Omega}$ (see [14], Theorem 7.7). Finally let us extend $\widetilde{H}_\Omega(x,y)$ to $\mathbf{R}^n \times \mathbf{R}^n$ by setting $\widetilde{H}_\Omega(x,y) = K_x(y)$ for every $x \in \mathbf{R}^n \setminus \overline{\Omega}$.

In the following definition we extend to $\partial\Omega$ the usual notions of Robin function, harmonic radius and harmonic center.

Definition 6 (Robin function, harmonic radius, harmonic center) *For every $x \in \Omega \cup \partial\Omega$ the leading term of the regular part of the Green's function*

$$\tau_\Omega(x) \quad := \quad \widetilde{H}_\Omega(x,x)$$

is called Robin function of Ω at the point x. The harmonic radius of Ω at x is defined by the relation $K(r(x)) = \tau_\Omega(x)$. A minimum point of the Robin function on $\Omega \cup \partial\Omega$ is called a harmonic center of Ω.

The harmonic radius of the ball B_0^R is

$$r(x) \quad = \quad R - \frac{|x|^2}{R}.$$

In particular the harmonic center of a ball is its geometric center and the maximum of the harmonic radius is the radius of the ball.

At every boundary point satisfying the Wiener regularity condition the Robin function tends to $+\infty$. Thus a bounded domain with regular boundary has at least one harmonic center on Ω.

We will prove in Proposition 7 that $\tau_\Omega(x)$ is lower semicontinuous on $\Omega \cup \partial\Omega$. Nevertheless it is possible to show with an explicit example that this extension of the Robin function to all $\Omega \cup \partial\Omega$ does not agree with its lower semicontinuous envelope of $\tau_\Omega\big|_\Omega$ on $\Omega \cup \partial\Omega$ (at least for $n \geq 5$). For further discussion on the relation between Wiener regularity and the condition $\tau_\Omega(x) = +\infty$ for $x \in \partial\Omega$ see the appendix.

From the lower semicontinuity of τ_Ω we conclude that every bounded domain, possibly with irregular boundary, has at least one harmonic center.

Fix $x_0 \in \partial\Omega$. Let us denote by $\Omega_\rho(x_0)$ the set $\Omega \cup B_{x_0}^\rho$. For any fixed $x \in \Omega \cup \partial\Omega$ let $H_{\Omega_\rho(x_0)}(x,\cdot)$ be the PWB solution of the problem

$$(10) \qquad \begin{cases} \Delta_y H_{\Omega_\rho(x_0)}(x,y) = 0 & \text{in } \Omega_\rho(x_0), \\ H_{\Omega_\rho(x_0)}(x,y) = K_x(y) & \text{on } \partial\Omega_\rho(x_0) \end{cases}$$

and let $\tau_{\Omega_\rho(x_0)}(x)$ the corresponding Robin function.

PROPOSITION 7 *Let $x_0 \in \partial\Omega$. Then, for every $x, y \in \mathbf{R}^n$, $H_{\Omega_\rho(x_0)}(x,y)$ converges increasingly to $H_\Omega(x,y)$ as ρ decreases to 0.*

In particular $\tau_{\Omega_\rho(x_0)}(x)$ converges increasingly to $\tau_\Omega(x)$ as $\rho \to 0$, for any $x \in \Omega \cup \partial\Omega$ and τ_Ω is lower semicontinuous in $\Omega \cup \partial\Omega$.

Proof. Let $x_0 \in \partial\Omega$. Let us fix $x \in \Omega \cup \partial\Omega$ and let $H_{\Omega_\rho(x_0)}(x,y)$ be the solution of problem (10). By the definition of PWB solution we have that $H_{\Omega_\rho(x_0)}(x,y) \leq K_x(y)$ for every $y \in \Omega_\rho(x_0)$ and then, by a comparison argument $H_{\Omega_\rho(x_0)}$ is decreasing with respect to ρ. Thus $H_{\Omega_\rho(x_0)}(x,\cdot)$ converges increasingly, as ρ decreases to 0, pointwise in Ω to the PWB solution of a Dirichlet problem with boundary value which coincides with $K_x(y)$ at least on $\partial\Omega \setminus (\mathcal{Z} \cup \{x_0\})$, where $\mathcal{Z}$ is the set of irregular points of $\partial\Omega$. Since the capacity of $\mathcal{Z}$ is zero, this implies that $H_{\Omega_\rho(x_0)}(x,y)$ converges, as $\rho \to 0$, to $H_\Omega(x,y)$ for every $x \in \Omega \cup \partial\Omega$ and $y \in \Omega$.

Let us denote by $\widetilde{H}_{\Omega_\rho(x_0)}(x,\cdot)$ the superharmonic extension to $\mathbf{R}^n$ of $H_{\Omega_\rho(x_0)}(x,\cdot)$ obtained as above. Clearly $\widetilde{H}_{\Omega_\rho(x_0)}(x,\cdot)$ is also decreasing with respect to ρ. Thus it converges, as $\rho \to 0$, to some function $H^*(\cdot)$ and this function is superharmonic on $\mathbf{R}^n$. We already proved that $H^*(y) = H_\Omega(x,y)$ if $y \in \Omega$. Therefore $H^*(y) = \widetilde{H}_\Omega(x,y)$ for every $y \in \mathbf{R}^n \setminus \mathcal{Z}$ and $y \neq x$. Then by the uniqueness of the superharmonic extension on a set of capacity zero we obtain that $H^*(y) = \widetilde{H}_\Omega(x,y)$ for every $y \in \mathbf{R}^n$.

Hence, in particular, $\tau_{\Omega_\rho(x_0)}(x)$ converges to $\tau_\Omega(x)$, as $\rho \to 0$, for every $x \in \Omega \cup \partial\Omega$.

Finally, since $\tau_{\Omega_\rho(x_0)}$ are lower semicontinuous in x_0 so is τ_Ω as an increasing limit of those functions. Indeed if $x_j \to x_0$, then

$$\liminf_{j\to\infty} \tau_\Omega(x_j) \geq \liminf_{j\to\infty} \tau_{\Omega_\rho(x_0)}(x_j) \geq \tau_{\Omega_\rho(x_0)}(x_0)\,.$$

The conclusion follows taking the limit as $\rho \to 0$. ○

PROPOSITION 8 *For every $y \in \mathbf{R}^n$ the function $x \mapsto \widetilde{H}_\Omega(x,y)$ is superharmonic in $\mathbf{R}^n$. Moreover, $(x,y) \mapsto \widetilde{H}_\Omega(x,y)$ is lower semicontinuous in $\mathbf{R}^n \times \mathbf{R}^n$.*

Proof. Let $y \in \mathbf{R}^n$. The function $\widetilde{H}_\Omega(\cdot, y)$ is superharmonic if and only if it is lower semicontinuous and

$$\widetilde{H}_\Omega(x,y) \geq \fint_{B_x^s} \widetilde{H}_\Omega(t,y)\,dt \tag{11}$$

for every $x \in \mathbf{R}^n$ and $s > 0$.

If $y \in \mathbf{R}^n \setminus \overline{\Omega}$ then $\widetilde{H}_\Omega(x,y) = K_x(y)$ for every $x \in \mathbf{R}^n$ and hence is clearly superharmonic.

If $y \in \Omega$ and $x \in \mathbf{R}^n \setminus \overline{\Omega}$ or $x \in \Omega$, then $\widetilde{H}_\Omega(x,y)$ agrees with $K_x(y)$ or $H_\Omega(x,y)$, respectively, and those are superharmonic functions.

To check the superharmonicity in the remaining cases let us fix $x_0, y_0 \in \Omega \cup \partial\Omega$ and let us prove the lower semicontinuity of $\widetilde{H}_\Omega(\cdot, y_0)$ in x_0.

For any $x \in \mathbf{R}^n$, let $H_{\rho,r}(x,\cdot)$ be the PWB solution of the following problem

$$\begin{cases} \Delta_y H_{\rho,r}(x,y) = 0 & \text{in } \Omega \cup B_{x_0}^\rho \cup B_{y_0}^r, \\ H_{\rho,r}(x,y) = K_x(y) & \text{on } \partial(\Omega \cup B_{x_0}^\rho \cup B_{y_0}^r) \end{cases} \tag{12}$$

and let $\widetilde{H}_{\rho,r}(x,\cdot)$ be its superharmonic extension to $\mathbf{R}^n$ as above. By Proposition 7, $\widetilde{H}_{\rho,r}(x,y)$ converges increasingly to $\widetilde{H}_\Omega(x,y)$ as ρ and r decrease to 0. In particular $\widetilde{H}_{\rho,r}(x,y_0)$ are lower semicontinuous in x_0 for any $r, \rho > 0$ and so is $\widetilde{H}_\Omega(x,y_0)$.

It remains to prove condition (11) for every $x_0, y_0 \in \Omega \cup \partial\Omega$, with either $x_0 \in \partial\Omega$ or $y_0 \in \partial\Omega$. By the symmetry of the Green's function and the fact that, for any $x \in \mathbf{R}^n$, $\widetilde{H}_{\rho,r}(x,\cdot)$ is superharmonic we have

$$\begin{aligned} \widetilde{H}_{\rho,r}(x_0,y_0) = H_{\rho,r}(x_0,y_0) = H_{\rho,r}(y_0,x_0) &\geq \fint_{B_{x_0}^s} \widetilde{H}_{\rho,r}(y_0,t)\,dt \\ = \fint_{B_{x_0}^s \setminus \mathcal{Z}} \widetilde{H}_{\rho,r}(y_0,t)\,dt &= \fint_{B_{x_0}^s} \widetilde{H}_{\rho,r}(t,y_0)\,dt\,, \end{aligned} \tag{13}$$

where for the last equality we used that, up to a set of capacity zero, $\widetilde{H}_{\rho,r}(y_0,\cdot)$ agrees with a symmetric function. We conclude, by (13), taking the supremum in ρ and r and using the monotone convergence of $\widetilde{H}_{\rho,r}(t,y_0)$ to $\widetilde{H}_\Omega(t,y_0)$.

Finally, using the superharmonicity of $\widetilde{H}_\Omega(x,y)$ in x and y we get its lower semicontinuity in (x,y). Indeed

$$\begin{aligned} \liminf_{\substack{t \to x\ z \to y \\ t,z \in \Omega}} \widetilde{H}_\Omega(t,z) &\geq \liminf_{\substack{t \to x \\ z \to y}} \fint_{B_s(t)} \fint_{B_l(z)} \widetilde{H}_\Omega(\xi,\eta)\,d\xi\,d\eta \\ = \lim_{\substack{t \to x \\ z \to y}} \fint_{B_s(t)} \fint_{B_l(z)} \widetilde{H}_\Omega(\xi,\eta)\,d\xi\,d\eta &= \fint_{B_s(x)} \fint_{B_l(y)} \widetilde{H}_\Omega(\xi,\eta)\,d\xi\,d\eta\,. \end{aligned}$$

Taking the supremum in s and l, using the superharmonicity of $\widetilde{H}_\Omega$ we get

$$\liminf_{\substack{t \to x\ z \to y \\ t,z \in \Omega}} \widetilde{H}_\Omega(t,z) \geq \widetilde{H}_\Omega(x,y)\,.$$

○

In the following example we construct a bounded domain where the harmonic center is on the boundary.

Example 9 Let $\Omega_0 = B_0^1$ and let τ_{Ω_0} be the corresponding Robin function. The harmonic center for Ω_0 is 0 and τ_{Ω_0} is strictly convex. The idea is to construct a sequence of small balls centered in the first axis, with radii which go to zero, in a way that the set obtained from Ω_0 by subtracting a finite number of them has its unique harmonic center in the same axis.

Let us fix real positive number $0 < x_1 < 1$, let us denote $r_1 = |x_1|/2$ and let $\varepsilon_1 > 0$ be such that $0 < \varepsilon_1 < \min_{\Omega_0 \setminus B_0^{r_1}} \tau_{\Omega_0} - \min_{B_0^{r_1}} \tau_{\Omega_0}$. With a little abuse of notation we shall denote by x_k the points of coordinates $(x_k, 0, \ldots, 0)$, with $x_k \in \mathbf{R}$. Fix $0 < \alpha < 1/2$. Let $\rho_1 > 0$ and denote $\Omega_1 = \Omega_0 \setminus B_{x_1}^{\rho_1}$. It is easy to check that τ_{Ω_1} converges uniformly to τ_{Ω_0} in $\Omega_0 \setminus B_{x_1}^{\rho_1^\alpha}$ and the same is true for the derivatives. Thus we can choose ρ_1 small enough such that τ_{Ω_1} is strictly convex on $\Omega_0 \setminus B_{x_1}^{\rho_1^\alpha}$, $B_{x_1}^{\rho_1^\alpha} \cap B_0^{r_1} = \emptyset$ and we have

$$\tau_{\Omega_0}(x) \le \tau_{\Omega_1}(x) \le \tau_{\Omega_0}(x) + \frac{\varepsilon_1}{2} \qquad \forall x \in \Omega_0 \setminus B_{x_1}^{\rho_1^\alpha}.$$

This implies that the harmonic center, x_1^0, of Ω_1 is unique, belongs to $B_0^{r_1}$, and, arguing by symmetry, belongs to the first axis. Let us denote it by $x_1^0 = (x_1^0, 0, \ldots, 0)$.

By induction we can construct four sequences $\{x_n\}$, $\{\rho_n\}$, $\{x_n^0\}$, and $\{\varepsilon_n\}$ such that, with the notation $\Omega_n = \Omega_0 \setminus (\cup_{i=1}^n B_{x_i}^{\rho_i})$ and $r_n = |x_n - x_{n-1}^0|/2$, we have

1) $0 < \varepsilon_n < \min_{\Omega_{n-1} \setminus B_{x_{n-1}^0}^{r_n}} \tau_{\Omega_{n-1}} - \tau_{\Omega_{n-1}}(x_{n-1}^0)$;

2) $B_{x_n}^{\rho_n^\alpha} \cap (B_{x_{n-1}}^{\rho_{n-1}} \cup B_{x_{n-1}^0}^{r_n}) = \emptyset$;

3) $\tau_{\Omega_{n-1}}(x) \le \tau_{\Omega_n}(x) \le \tau_{\Omega_{n-1}}(x) + \dfrac{\varepsilon_n}{2}$ for every $x \in \Omega_{n-1} \setminus B_{x_n}^{\rho_n^\alpha}$;

4) $\tau_{\Omega_n}(x)$ is strictly convex in $\Omega_{n-1} \setminus B_{x_n}^{\rho_n^\alpha}$.

Moreover $x_n = (x_{n-1} - x_{n-1}^0)/2$, $\{x_n\}$ is decreasing and $x_n^0 = (x_n^0, \ldots, 0)$ is the unique harmonic center of τ_{Ω_n}. Clearly the sequence $\{x_n\}$ converges to some $\overline{x}$. Hence $\{x_n^0\}$ converges to $\overline{x}$.

Finally, by Proposition 7, $\tau_{\Omega_n}(x)$ converges to $\tau_{\Omega_\infty}(x)$ for every $x \in \overline{\Omega}_\infty$, with $\Omega_\infty = \Omega_0 \setminus (\cup_{i=1}^\infty B_{x_i}^{\rho_i})$. Moreover, since $\{\tau_{\Omega_n}\}$ is an increasing sequence, $\overline{x}$ is the harmonic center of Ω_∞ and by construction belongs to the boundary of Ω_∞.

PROPOSITION 10 *Let Ω^* be the ball of radius R_Ω centered in zero and such that $|\Omega^*| = |\Omega|$. Then $r(x) \le R_\Omega$ for every $x \in \Omega \cup \partial\Omega$.*

Proof. If $x \in \Omega$ the inequality is proved in [2], Corollary 14. If $x \in \partial\Omega$, we apply the result of [2] to the set $\Omega_\rho = \Omega \cup B_x^\rho$ and we obtain $r_{\Omega_\rho}(x) \le R_{\Omega_\rho}$ for every $\rho > 0$, where $r_{\Omega_\rho}(x)$ is the harmonic radius of Ω_ρ in x. Thus $K(r_{\Omega_\rho}(x)) \ge K(R_{\Omega_\rho})$, that is $\tau_{\Omega_\rho}(x) \ge K(R_{\Omega_\rho})$. As $\rho \to 0$ the radius R_{Ω_ρ} converges to R_Ω and, by Proposition 7, $\tau_{\Omega_\rho}(x)$ converges to $\tau_\Omega(x)$. This concludes the proof.

○

Remark 11 An equivalent formulation of the previous assertion is

$$(14) \qquad |\Omega| \ge |B^{r(x)}| = |\{K > \tau_\Omega(x)\}| \qquad \forall \in \overline{\Omega}.$$

In the case $\tau_\Omega(x) < +\infty$, we have $x \in \{G_x > t\} := \Omega_x^t$. If we apply (14) to Ω_x^t and observe that

$$G_{\Omega_x^t, x} = G_x - t$$

whence $\tau_{\Omega_x^t}(x) = t + \tau_\Omega(x)$, we obtain

$$|\{G_x > t\}| \ge |\{K > t + \tau_\Omega(x)\}|.$$

A simple comparison argument shows that $r(x) \ge \mathrm{dist}(x, \partial\Omega)$. If $x \in \Omega$, near the singularity the Green's function can be expanded as:

$$(15) \qquad G_x(y) = K(|y - x|) - \tau_\Omega(x) + O(|y - x|).$$

It has the following properties.

PROPOSITION 12 ([2, 11]) *For fixed $x \in \Omega$ the Dirichlet Green's function G_x satisfies:*

1. *For every $t > 0$ one has*

$$\int_{\{G_x<t\}} |\nabla G_x|^2 \quad = \quad t.$$

2. *As $t \to \infty$ we have $B_x^{r-} \subset \overline{\{G_x > t\}} \subset B_x^{r+}$ with $r_\pm = r \pm O(r^n)$ and r defined by $t = K(r) - \tau_\Omega(x)$.*

Proof. The proof of 1 follows by the fact that $G_x \wedge t$ belongs to $D^{1,2}$ and coincides with t in a neighborhood of x. By an approximation argument one can show that it is possible to take $G_x \wedge t$ as a test function for $-\Delta G_x = \delta_x$ which yields immediately the result. Assertion 2 follows by the expansion (15). ○

This implies that the capacity of a small ball is asymptotically given by

$$\text{(16)} \qquad \mathrm{cap}_\Omega(B_x^r) \quad = \quad \frac{1}{K(r) - \tau_\Omega(x) + O(r)} \quad = \quad \mathrm{cap}_{\mathbf{R}^n}(B_0^r) + \mathrm{cap}^2_{\mathbf{R}^n}(B_0^r)\,(\tau_\Omega(x) + O(r))$$

as $r \to 0$. In the radial case

$$\text{(17)} \qquad \mathrm{cap}_{B_0^R}(B_0^r) \quad = \quad \frac{1}{K(r) - K(R)}.$$

LEMMA 13 *Let A_k be a sequence of compact sets such that $|A_k| = |B_0^1|$ and $\mathrm{cap}_{\mathbf{R}^n}(A_k)$ converges to $\mathrm{cap}_{\mathbf{R}^n}(B_0^1)$ as $k \to \infty$. Then, up to a subsequence, there exists a sequence $\{x_k\}$ such that $A_k - x_k$ converges to B_0^1 in L^1. Moreover if u_k and u denote the capacitary potential of A_k and B_0^1 respectively, then $u_k(x_k + \cdot)$ converges to u strongly in $D^{1,2}(\mathbf{R}^n)$.*

Proof. Up to a subsequence we have that u_k converges weakly in $D^{1,2}(\mathbf{R}^n)$. Using the concentration compactness alternative, we can exclude splitting by the fact that u_k is a maximizing sequence for the volume functional. Since $|\{u_k \geq 1\}| = |A_k| = |B_0^1|$ vanishing and concentration are not possible.

Hence there exists a sequence x_k in $\mathbf{R}^n$ such that $u_k(\cdot + x_k)$ is compact in $D^{1,2}(\mathbf{R}^n)$, then up to a subsequence it converges to some function u strongly in $D^{1,2}(\mathbf{R}^n)$ and so in $L^{2^*}(\mathbf{R}^n)$. This implies that $|\{u > 1-\eta\}| \geq \liminf |\{u_k \geq 1\}| = |B_0^1|$ for every $\eta > 0$ and hence, since $\int_{\mathbf{R}^n} |\nabla u|^2 = \mathrm{cap}_{\mathbf{R}^n}(B_0^1)$, $\{u \geq 1\}$ is a ball of radius 1 and u is its capacitary potential. Clearly the sequence x_k can be chosen in a way that $\{u \geq 1\} = B_0^1$.

○

As consequence of this lemma we have the following proposition which state essentially that if the capacity of a set A approach the capacity of its symmetrization, then A is almost a ball.

PROPOSITION 14 *There exist $\omega : \mathbf{R}_+ \to \mathbf{R}$, with $\lim_{\delta\to 0^+} \omega(\delta) = 0$ with the following property. Let A be a subset of $\mathbf{R}^n$ with positive measure and define $\rho > 0$ by $|A| = |B_0^\rho|$. Suppose that*

$$\frac{\mathrm{cap}_{\mathbf{R}^n}(A)}{\mathrm{cap}_{\mathbf{R}^n}(B_0^\rho)} \leq 1 + \delta\,.$$

Then there exist $y \in \mathbf{R}^n$ such that

$$\frac{|A \Delta B_y^\rho|}{|B_y^\rho|} \leq \omega(\delta)\,.$$

Proof. Without loss of generality we can assume that $\rho = 1$. Suppose by contradiction that for every $y \in \mathbf{R}^n$ there exist $\omega_0(y)$ and $|A_k|$, with $|A_k| = |B_0^1|$, such that $\mathrm{cap}_{\mathbf{R}^n}(A_k) \to \mathrm{cap}_{\mathbf{R}^n}(B_0^1)$ and

$$\text{(18)} \qquad \inf_k |A_k \Delta B_y^1| \geq \omega_0(y)$$

Then Lemma 13 gives a contradiction. ○

Remark 15 Let $A_k \subset \Omega$ is a sequence of compact sets with $|A_k| \to 0$ and

$$\text{(19)} \qquad \limsup_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} < +\infty\,.$$

Since $|A_k| \to 0$, as $k \to \infty$, we have that $\mathrm{cap}_{\mathbf{R}^n}(A_k^*) \to 0$, as $k \to \infty$. So that, by (19) we have that

$$\lim_{k\to\infty} \frac{\mathrm{cap}_\Omega(A_k)}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} = 1$$

Then, as $\mathrm{cap}_{\mathbf{R}^n}(A_k) \le \mathrm{cap}_\Omega(A_k)$ and $\mathrm{cap}_{\mathbf{R}^n}(A_k^*) \le \mathrm{cap}_{\mathbf{R}^n}(A_k)$, we get

$$\lim_{k\to\infty} \frac{\mathrm{cap}_\Omega(A_k)}{\mathrm{cap}_{\mathbf{R}^n}(A_k)} = 1 \tag{20}$$

By a rescaling argument applying Lemma 13 we have that the sequence of the capacitary potentials of the sets $A_k/|A_k|^{\frac{1}{n}}$ is compact up to a translation. Thus, denoting by u_k the capacitary potential of A_k in Ω, it is easy to check that there exists a point $\overline{x} \in \overline{\Omega}$ such that

$$\frac{|\nabla u_k|^2}{\mathrm{cap}_\Omega(A_k)} \stackrel{*}{\rightharpoonup} \delta_{\overline{x}},$$

in the sense of measures.

With the following lemma we obtain an asymptotic expansion for the capacity of concentrating sets in terms of the Robin function.

LEMMA 16 (Asymptotic expansion of capacity)
(i) Let $x_0 \in \Omega \cup \partial\Omega$ and let A_k be a sequence of subsets of Ω such that $|A_k| > 0$ and

$$\frac{1}{|A_k|}\mathcal{X}_{A_k} \stackrel{*}{\rightharpoonup} \delta_{x_0}.$$

Then

$$\liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} \ge \tau_\Omega(x_0)\,. \tag{21}$$

(ii) Suppose now that Ω bounded and let $A_k \subseteq \Omega$, with $|A_k| > 0$ and $|A_k| \to 0$ then

$$\liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} \ge \min_{\overline{\Omega}} \tau_\Omega\,.$$

Proof. Let us prove (21) first in the case $x_0 \in \Omega$.

By an approximation argument it is not restrictive to assume that the A_k are compact. Moreover we may assume that the liminf in (21) is a limit. By the assumption of concentration $|A_k| \to 0$ as $k \to \infty$ and we may assume that (19) holds true, otherwise the result is trivial. Thus by Remark 15 we have that there exists a point $\overline{x} \in \overline{\Omega}$ such that

$$\frac{|\nabla u_k|^2}{\mathrm{cap}_\Omega(A_k)} \stackrel{*}{\rightharpoonup} \delta_{\overline{x}}, \tag{22}$$

in the sense of measures, where u_k is the capacitary potential of A_k in Ω. In the case (i), since A_k concentrates at x_0, we can obtain that $\overline{x} = x_0$. We shall prove that the rescaled functions

$$v_k = \frac{u_k}{\mathrm{cap}_\Omega(A_k)}$$

converge to the Green's function. Let us denote by μ_k the capacitary distribution of A_k, i.e., the non negative Radon measure with support in $\overline{A}_k$ such that $-\Delta u_k = \mu_k$ in the sense of $H^{-1}(\Omega)$. Let us prove that $\lambda_k = \mu_k/\mathrm{cap}_\Omega(A_k)$ converges to δ_{x_0} in the weak sense of measures. Test $-\Delta v_k = \lambda_k$ with φu_k, for $\varphi \in C_0^\infty(\Omega)$, and use that $u_k = 1$ on $A - k$ to obtain

$$\int_\Omega \varphi\, d\lambda_k = \int_\Omega \varphi \frac{|\nabla u_k|^2}{\mathrm{cap}_\Omega(A_k)}\, dx + \int_\Omega \nabla\varphi \frac{u_k \nabla u_k}{\mathrm{cap}_\Omega(A_k)}\, dx = T_1^k + T_2^k. \tag{23}$$

By (22) T_1^k converges to $\varphi(x_0)$, while T_2^k converges to zero. Indeed by Hölder inequality we have, for every $\rho > 0$,

$$T_2^k \le \Big(\int_{\Omega\setminus B_{x_0}^\rho} \frac{|\nabla u_k|^2}{\mathrm{cap}_\Omega(A_k)}\Big)^{\frac{1}{2}} \Big(\int_{\Omega\setminus B_{x_0}^\rho} \frac{{u_k}^{2^*}}{\mathrm{cap}_\Omega(A_k)^{\frac{2^*}{2}}}\Big)^{\frac{1}{2^*}} \Big(\int_{\Omega\setminus B_{x_0}^\rho} |\nabla\varphi|^n\Big)^{\frac{1}{n}} + C\rho\,.$$

By the arbitrariness of ρ we obtain that T_2^k converges to zero. Hence $\lambda_k \overset{*}{\rightharpoonup} \delta_{x_0}$. As discussed above this implies that v_k converges strongly in $W^{1,p}_{loc}(\Omega)$ ($W^{1,p}_0(\Omega)$ for bounded domains), for every $p < \frac{n}{n-1}$ to the Green's function G_{x_0}, i.e., the solution in the sense of Stampacchia of $-\Delta G_{x_0} = \delta_{x_0}$. Moreover for every $t > 0$ we have that $\int_\Omega |\nabla(v_k \wedge t)|^2 = \int_{\{v_k<t\}} |\nabla v_k|^2 \le t$. In view of Proposition 12 1 this implies that $v_k \wedge t$ converges to $G_{x_0} \wedge t$ strongly in $D^{1,2}(\Omega)$.

Let us prove now that

(24) $$\lim_{k\to\infty} \mathrm{cap}_{\mathbf{R}^n}(\{v_k > t\}) = \mathrm{cap}_{\mathbf{R}^n}(\{G_{x_0} > t\})$$

for a.e. $t \in \mathbf{R}$. Indeed for a given $\delta > 0$, denote $D_k^\delta = \{|v_k \wedge 2t - G_{x_0} \wedge 2t| > \delta\}$. Since

$$\mathrm{cap}_{\mathbf{R}^n}(D_k^\delta) \le C \frac{\|\nabla(v_k \wedge 2t) - \nabla(G_{x_0} \wedge 2t)\|_2^2}{\delta^2}$$

we have that $\mathrm{cap}_{\mathbf{R}^n}(D_k^\delta)$ converges to zero as $k \to \infty$. Therefore, since $\{v_k > t\} \subseteq \{G_{x_0} > t - \delta\} \cup D_k^\delta$ and $\{v_k > t\} \cup D_k^\delta \supseteq \{G_{x_0} > t + \delta\}$,

$$\mathrm{cap}_{\mathbf{R}^n}(\{G_{x_0} > t+\delta\}) - \mathrm{cap}_{\mathbf{R}^n}(D_k^\delta) \le \mathrm{cap}_{\mathbf{R}^n}(\{v_k > t\}) \le \mathrm{cap}_{\mathbf{R}^n}(\{G_{x_0} > t-\delta\}) + \mathrm{cap}_{\mathbf{R}^n}(D_k^\delta).$$

The conclusion follows since the monotone function $t \mapsto \mathrm{cap}_{\mathbf{R}^n}(\{G_{x_0} > t\})$ is continuous for a.e. t.

To conclude the proof let us fix $s > 0$ and set $B_k = \{v_k > s\} = \{u_k > s\,\mathrm{cap}_\Omega(A_k)\}$. Since B_k is a level set of the capacitary potential of A_k we have

$$\frac{1}{\mathrm{cap}_\Omega(A_k)} = \frac{1}{\mathrm{cap}_{B_k}(A_k)} + \frac{1}{\mathrm{cap}_\Omega(B_k)} = \frac{1 - s\,\mathrm{cap}_\Omega(A_k)}{\mathrm{cap}_\Omega(A_k)} + s\,.$$

Indeed taking $u_k \wedge t$ and $(u_k \vee t) - t$ as test functions in $-\Delta u_k = \mu_k$, we obtain $t^2 \mathrm{cap}_\Omega(B_k) = \int_{\{u_k<t\}} |\nabla u_k|^2 = t\,\mathrm{cap}_\Omega(A_k)$ and $(1-t)^2 \mathrm{cap}_{B_k}(A_k) = \int_{\{u_k>t\}} |\nabla u_k|^2 = (1-t)\mathrm{cap}_\Omega(A_k)$. Moreover

$$\frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} \ge \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k)} \ge \frac{1}{\mathrm{cap}_{B_k}(A_k)} + \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(B_k)}\,.$$

So that, taking into account (24), we obtain

$$\liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} \ge \liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(B_k)} - \frac{1}{\mathrm{cap}_\Omega(B_k)} \ge \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(\{G_{x_0} > s\})} - s$$

for a.e. $s \in \mathbf{R}$. Proposition 12 (2), implies that

$$\frac{1}{\mathrm{cap}_{\mathbf{R}^n}(\{G_{x_0} > s\})} \ge K(r_+(s) + O(r)) \ge (s + \tau_\Omega(x_0))(1 + O(s^{\frac{n-1}{2-n}}))$$

and taking the limit $s \to \infty$, we obtain assertion (i) for $x_0 \in \Omega$.

If $x_0 \in \partial\Omega$, by the previous step applied to $\Omega_\rho(x_0) = \Omega \cup B_\rho(x_0)$, with $\rho > 0$, we obtain

(25) $$\liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_{\Omega_\rho(x_0)}(A_k)} \ge \tau_{\Omega_\rho(x_0)}(x_0)\,.$$

Since $\Omega \subset \Omega_\rho(x_0)$ implies $\mathrm{cap}_{\Omega_\rho(x_0)}(A_k) \le \mathrm{cap}_\Omega(A_k)$, we obtain (21) taking the limit as $\rho \to 0$ and using Proposition 7.

In order to treat case (ii) it is enough to note that, from (22), we can proceed as in the previous case assuming that $\overline{x} \in \Omega$ and obtain

$$\liminf_{k\to\infty} \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(A_k^*)} - \frac{1}{\mathrm{cap}_\Omega(A_k)} \ge \tau_\Omega(\overline{x}) \ge \min_{\overline{\Omega}} \tau_\Omega\,.$$

We recover the general case $\overline{x} \in \overline{\Omega}$ arguing as above. ○

5 Localization of concentration points

The main result of this paper is the second order expansion of S_ε^F with respect to ε. It turns out that the second nontrivial term depends on the value of the Robin function at the concentration point. This allows us to identify the concentration point. We recall the definition of w_∞ given in Theorem 3

$$w_\infty^2 \quad := \quad \frac{2(n-1)}{n\, S^F} \inf \left\{ \liminf_{k\to\infty} \int_{\mathbf{R}^n} \frac{F(w_k)}{K(|\cdot|)} \; : \; \{w_k\} \in \mathcal{B}^F \right\}.$$

THEOREM 17 (Identification of concentration points) *Assume that Ω is bounded and (F). Let w_∞ be defined as in Theorem 3 and suppose that $0 < w_\infty < \infty$.*

1. *If the sequence $\{\widetilde{u}_\varepsilon\} \subset D^{1,2}(\Omega)$ satisfies $\|\nabla \widetilde{u}_\varepsilon\|_2 \le \varepsilon$ and concentrates at x in the sense of Theorem 1 then*

$$\int_\Omega F(\widetilde{u}_\varepsilon) \quad \le \quad \varepsilon^{2^*} S^F \left(1 - \frac{n}{n-2} w_\infty^2 \tau(x)\, \varepsilon^2 + o(\varepsilon^2) \right)$$

   as $\varepsilon \to 0$.

2. *If $\{u_\varepsilon\}$ is a sequence of almost extremals we have*

$$\int_\Omega F(u_\varepsilon) \quad = \quad \varepsilon^{2^*} S^F \left(1 - \frac{n}{n-2} w_\infty^2 \min_{\overline{\Omega}} \tau_\Omega\, \varepsilon^2 + o(\varepsilon^2) \right).$$

3. *In particular a sequence of almost extremals concentrates at a harmonic center, i.e.*

$$\tau(x_0) \quad = \quad \min_{\overline{\Omega}} \tau_\Omega$$

   with x_0 as in Theorem 1.

If $w_\infty = 0$ then $S_\varepsilon^F(\Omega) = S^F$ for every $\varepsilon > 0$. Conversely, if $S_\varepsilon^F(\Omega) = S^F - o(\varepsilon^2)$ and $\min_{\overline{\Omega}} \tau_\Omega > 0$ then $w_\infty = 0$.

Remark 18 By Proposition 10 we have

$$\min_{\overline{\Omega}} \tau_\Omega \ge \min_{\overline{\Omega}^*} \tau_{\Omega^*} = \frac{1}{\mathrm{cap}_{\mathbf{R}^n}(\Omega^*)}$$

so that the assumption Ω bounded implies $\min_{\overline{\Omega}} \tau_\Omega > 0$. The result stated in Theorem 3 is still true if we replace the assumption (Ω) with a weaker assumption which allows also unbounded domains and still implies $\min_{\overline{\Omega}} \tau_\Omega > 0$.

Theorem 17 can be extended to unbounded domains. In this case one extends the Robin function to ∞ by

$$\tau_\Omega(\infty) := \lim_{\rho\to 0} \lim_{R\to\infty} \inf_{\substack{x,y \in \mathbf{R}^n \\ |x| \ge R,\, |x-y| \le \rho}} \widetilde{H}_\Omega(x,y) \tag{26}$$

and one requires that $\tau_\Omega(\infty) > 0$. This is the precise statement of the condition that the complement of Ω is not too small near ∞.

Note that with the above definition τ_Ω is lower semicontinuous at ∞. One crucial technical ingredient in the proof for unbounded domains is the following counterpart of Proposition 7, namely

$$\tau_\Omega(\infty) = \lim_{\rho\to 0} \lim_{R\to\infty} \inf_{|x|\ge R} \tau_{\Omega \cup B_\rho(x)}(x).$$

Details of this argument will appear elsewhere.

We apply Theorem 17 to two examples already studied in [9].

Example 19 [Volume functional] For

$$F(t) \quad := \quad \begin{cases} 0 & (t < 1), \\ 1 & (t \geq 1), \end{cases}$$

we have $\int_\Omega F(u) = |\{u \geq 1\}|$. The corresponding Euler Lagrange equation is Bernoulli's free-boundary problem [10]. Up to translation entire extremals are given by

$$w(r) \quad = \quad \begin{cases} K(r) & (r > R), \\ 1 & (r \leq R) \end{cases}$$

with R such that

$$\int_{\mathbf{R}^n} |\nabla w|^2 \quad = \quad \mathrm{cap}_{\mathbf{R}^n}(B_0^R) \quad = \quad \frac{1}{K(R)} \quad = 1,$$

$$S^F \quad = \quad \int_{\mathbf{R}^n} F(w) \quad = \quad \left|B_0^R\right|.$$

In particular $w_\infty = 1$ by Theorem 2. Application of Theorem 17 yields

$$\sup\left\{|A| \,:\, \mathrm{cap}_\Omega(A) \leq \varepsilon^2\right\} \quad = \quad \frac{\varepsilon^{2^*}}{(n(n-2))^{\frac{n}{n-2}}\,|B_0^1|^{\frac{2}{n-2}}}\left(1 - \frac{n}{n-2}\min_{\overline{\Omega}} \tau_\Omega\, \varepsilon^2 + o(\varepsilon^2)\right).$$

The corresponding formula in two dimensions is quite different. Already the leading term depends on the geometry, namely on the value of the harmonic radius ρ at the concentration point:

$$\sup\left\{|A| \,:\, \mathrm{cap}_\Omega(A) \leq \varepsilon^2\right\} \quad = \quad \frac{\pi \max_\Omega \rho^2}{\exp\left(4\pi/\varepsilon^2\right)}(1 + o(1))$$

as shown in [5].

Example 20 [Plasma problem] As a further illustration of Theorem 17 we consider $F(t) = (t-1)_+^2$ in three dimensions. See [12] for the physical context. Up to translation entire extremals are given by

$$w(r) \quad = \quad \begin{cases} \frac{R}{r} & (r > R), \\ 1 + \frac{R}{\pi r}\sin\left(\frac{\pi r}{R}\right) & (r \leq R) \end{cases}$$

with $R = (6\pi)^{-1}$. This leads to $w_\infty = 2/3$ and $S^F = \int_{\mathbf{R}^n} F(w) = (108\pi^4)^{-1}$. By Theorem 17 we have:

$$\int_\Omega (u_\varepsilon - 1)_+^2 \quad = \quad \frac{\varepsilon^6}{108\,\pi^4}\left(1 - \frac{4}{3}\min_{\overline{\Omega}} \tau_\Omega\ \varepsilon^2 + o(\varepsilon^2)\right).$$

6 Lower bound

We begin with a short overview of the proof. We first establish a lower bound for $S_\varepsilon^F(\Omega)$ by the usual transplantation argument (Step 1, this section).

To prove the upper bounds we essentially use two facts about sequences u_ε which are almost optimal, i.e., optimal up to $O(\varepsilon^2)$. First u_ε behaves like the Green's function away from the concentration point (at least after symmetrization, Step 2). This observation will allow us to exploit capacity estimates like (21). Secondly the rescaled functions $w_\varepsilon(x) = u_\varepsilon(x_\varepsilon + \varepsilon^{\frac{2}{n-2}} x)$ converge in $D^{1,2}(\mathbf{R}^n)$ to a maximizer w for S^F. This is easily established under the additional assumption (F^+) which restricts the behavior of F near 0 and near ∞. Without this assumption the situation is more subtle as the example $F(t) = |t|^{2^*}$ demonstrates. In this case $S_\varepsilon^F(\Omega) = S^F$ and maximizing sequences concentrates on a scale much shorter than $\varepsilon^{\frac{2}{n-2}}$.

In Step 2 below we show that the assumption $w_\infty > 0$, which rules out concentration of maximizing sequences for S^F, prevents such behavior. We also use the decay and capacity estimates to show that the length scale of the concentrating sequence u_ε can not be much larger that $\varepsilon^{\frac{2}{n-2}}$. Once the length scale $\varepsilon^{\frac{2}{n-2}}$ is established, a routine application of concentration compactness yields compactness of w_ε (Step 3).

The upper bound then essentially follows from the capacity estimates. One subtle point is that we need to show that all relevant level sets (including those whose volume goes to infinity in the rescaled variables) concentrate at the same point (Step 4). Another subtlety is that it is not known whether the maximizers for S^F are unique or whether they at least have all the same decay rate W_∞ (see Theorem 2). We show that those w that arise as limits of the rescaled sequence w_ε must have the optimal decay rate $W_\infty = w_\infty$. The desired upper bounds and the identification of concentration points follow easily (Steps 5-7).

To prove the lower bound we recall that by Theorems 2 and 3 there exists a radial maximizer w for S^F and we may assume $w > 0$ and that the limit $W_\infty := \lim_{r\to\infty} w(r)/K(r)$ exists. The lowest possible value for W_∞ is w_∞ and we consider a corresponding maximizer w.

Step 1

$$S^F_\varepsilon(\Omega) \geq \quad S^F\left(1 - \frac{n}{n-2} w_\infty^2 \min_{\overline{\Omega}} \tau_\Omega\, \varepsilon^2 + o(\varepsilon^2)\right)$$

as $\varepsilon \to 0$.

Proof. Let $z \in \overline{\Omega}$ be a harmonic center for Ω. In particular $\tau_\Omega(z) < +\infty$ and the harmonic radius $r(z)$ is strictly positive. For a radial function $U \in D^{1,2}(B_0^{r(z)})$ we define its harmonic transplantation to (Ω, z) as follows (see [2]). Let $G_{B,0}$ denote the Green's function of $B_0^{r(z)}$ with pole at zero, write U as $\varphi \circ G_{B,0}$ and let $u := \varphi \circ G_z \in D^{1,2}(\Omega)$. It is easy to see that this transformation preserve the Dirichlet integral. Moreover the following inequality holds

$$\int_\Omega F(u) \geq \int_{B_0^{r(z)}} F(U)\,. \tag{27}$$

This inequality is proved in [2] in the case that $z \notin \partial\Omega$. The general case $z \in \overline{\Omega}$, $\tau_\Omega(z) < +\infty$, follows by the coarea formula and the estimate of the level sets of the Green's function given by Remark 11.

Therefore by (27) we have $S^F_\varepsilon(\Omega) \geq S^F_\varepsilon(B_0^{r(z)})$. Now we set

$$r_\varepsilon \quad := \quad \varepsilon^{-\frac{2}{n-1}} r(z), \qquad R_\varepsilon \quad := \quad \varepsilon^{-\frac{2}{n-2}} r(z).$$

and we define the comparison functions $W_\varepsilon \in D^{1,2}(B_0^{R_\varepsilon})$ by $W_\varepsilon = w$ in $B_0^{r_\varepsilon}$ and $\Delta W_\varepsilon = 0$ in $B_0^{R_\varepsilon} \setminus B_0^{r_\varepsilon}$. Since $\int_{\mathbf{R}^n \setminus B_0^r} F(w) = O(r^{-n})$ (Theorem 2) we have

$$\int_{B_0^{R_\varepsilon}} F(W_\varepsilon) \quad \geq \quad \int_{\mathbf{R}^n} F(w) - \int_{\mathbf{R}^n \setminus B_0^{r_\varepsilon}} F(w) \quad = \quad S^F - O(r_\varepsilon^{-n}) \quad = \quad S^F - o(\varepsilon^2).$$

To establish the assertion errors of this order can be ignored. Using the formula for the capacity of balls (17) we can estimate the Dirichlet integral from above by

$$\begin{aligned}
\|\nabla W_\varepsilon\|_2^2 \quad &= \quad 1 - \int_{\mathbf{R}^n \setminus B_0^{r_\varepsilon}} |\nabla w|^2 + \int_{B_0^{R_\varepsilon} \setminus B_0^{r_\varepsilon}} |\nabla W_\varepsilon|^2 \\
&\leq \quad 1 - w(r_\varepsilon)^2 \left(\mathrm{cap}_{\mathbf{R}^n}(B_0^{r_\varepsilon}) - \mathrm{cap}_{B_0^{R_\varepsilon}}(B_0^{r_\varepsilon})\right) \\
&= \quad 1 - w(r_\varepsilon)^2 \left(K(r_\varepsilon)^{-1} - (K(r_\varepsilon) - K(R_\varepsilon))^{-1}\right) \\
&= \quad 1 + \left(\frac{w(r_\varepsilon)}{K(r_\varepsilon)}\right)^2 K(R_\varepsilon)(1 + o(1)) \\
&= \quad 1 + w_\infty^2 \tau_\Omega(z)\varepsilon^2 + o(\varepsilon^2)
\end{aligned}$$

since $K(R_\varepsilon) = \varepsilon^2 \tau_\Omega(z)$ (Definition 6). Moreover the ratio $K(R_\varepsilon)/K(r_\varepsilon)$ tends to zero. We scale W_ε with $s \leq 1$ such that $\|\nabla W_\varepsilon^s\|_2 = 1$. By the scaling property (4) and the above estimate we obtain

$$\begin{aligned}
S^F_\varepsilon(B_0^{r(z)}) \quad &\geq \quad S^F_\varepsilon(B_0^{sr(z)}) \quad \geq \quad \int_{B_0^{sR_\varepsilon}} F(W_\varepsilon^s) \\
&= \quad \|\nabla W_\varepsilon\|_2^{-\frac{2n}{n-2}} \int_{B_0^{R_\varepsilon}} F(W_\varepsilon) \quad \geq \quad S^F\left(1 - \frac{n}{n-2} w_\infty^2 \tau_\Omega(z)\varepsilon^2 + o(\varepsilon^2)\right).
\end{aligned}$$

Since $\tau_\Omega(z) = \min_{\overline{\Omega}} \tau_\Omega$ this concludes the proof. ○

Remark 21 This in particular implies that $S^F_\varepsilon(\Omega) \geq S^F - C\varepsilon^2$. From this we can deduce that, if $\{u_\varepsilon\}$ is a sequence of almost maximizers, then

$$\frac{1}{\varepsilon^{2^*}}\int_\Omega F(u_\varepsilon) \geq S^F - C\varepsilon^2 \,.$$

It is easy to see that for each $z \in \overline{\Omega}$, with $\tau_\Omega(z) < \infty$, the above construction yields a sequence which concentrates at z, satisfies $\|\nabla u_\varepsilon\| = \varepsilon$ and

$$\frac{1}{\varepsilon^{2^*}}\int_\Omega F(u_\varepsilon) \geq S^F\left(1 - \frac{n}{n-2}w_\infty^2\tau_\Omega(z)\varepsilon^2 + o(\varepsilon^2)\right) \,.$$

7 Upper bound and identification of concentration points

Let $D^{1,2}_*$ be the set of all function u in $D^{1,2}(\mathbf{R}^n)$ such that $u \geq 0$, u is radially symmetric and decreasing. For sake of simplicity let us assume that condition (F) is satisfied with $\alpha = 1$.

LEMMA 22 (Decay estimates for radial low energy extremals) *For $c > 0$ there exists positive constants c_0, γ_0, with $0 < \gamma_0 < 1$, and ε_0 that only depend on the dimension n, on S^F and c with the following properties.*

If (F) holds (with $\alpha = 1$) and $0 < \varepsilon < \varepsilon_0$ and if $u \in D^{1,2}_*$ satisfies

$$\int_{\mathbf{R}^n} F(u) \quad\geq\quad (S^F - c\varepsilon^2)\|\nabla u\|_2^{2^*}\,, \tag{28}$$

then there exist $\rho > 0$ and $u_\infty > 0$, such that

$$u(r) \quad\leq\quad c_0\rho^{\frac{n-2}{2}}\|\nabla u\|_2\,K(r) \quad \textit{for} \quad 1 \leq \frac{r}{\rho} \leq \varepsilon^{-\frac{2}{n-2}}, \tag{29}$$

$$\int_{\mathbf{R}^n\setminus B_0^r} |\nabla u|^2 \quad\leq\quad c_0\rho^{n-2}\|\nabla u\|_2^2\,K(r) \quad \textit{for} \quad 1 \leq \frac{r}{\rho} \leq \varepsilon^{-\frac{2}{n-2}}, \tag{30}$$

$$\int_{\mathbf{R}^n\setminus B_0^r} |u|^{2^*} \quad\leq\quad S^* c_0^{\frac{2^*}{2}}\rho^n\|\nabla u\|_2^{2^*}\,K(r)^{\frac{2^*}{2}} \quad \textit{for} \quad 1 \leq \frac{r}{\rho} \leq \varepsilon^{-\frac{2}{n-2}}, \tag{31}$$

$$|u(r) - u_\infty K(r)| \quad\leq\quad c_0\rho^{\frac{n-2}{2}}\|\nabla u\|_2\,K(r)\left(\frac{\rho}{r} + \varepsilon\left(\frac{r}{\rho}\right)^{\frac{n-2}{2}}\right) \quad \textit{for} \quad 1 \leq \frac{r}{\rho} \leq \varepsilon^{-\frac{2}{n-2}}, \tag{32}$$

and ρ is characterized as the greatest radius which satisfies the condition

$$\int_{\mathbf{R}^n\setminus B_0^\rho} |\nabla u|^2 = \gamma_0\|\nabla u\|_2^2. \tag{33}$$

Moreover we have

$$c_0^{-1} \leq \frac{u_\infty}{\rho^{\frac{n-2}{2}}\|\nabla u\|_2} \leq c_0\,. \tag{34}$$

Proof. It is sufficient to prove the assertion under the assumptions $\rho = 1$ and $\|\nabla u\|_2 = 1$. Indeed if u satisfies (28) so does $u_\rho(y) = u(\rho y)$. Thus $\widetilde{u} = u_\rho/\|\nabla u_\rho\|_2$ satisfies (28) if F is replaced by $\widetilde{F}(t) = F(\|\nabla u_\rho\|_2 t)/\|\nabla u_\rho\|_2^{2^*}$. Thus the assertions for u follows from those for $\widetilde{u}$ by unscaling.

Let us denote now

$$\gamma(R) \quad:=\quad \int_{\mathbf{R}^n\setminus B_0^R} |\nabla u|^2\,.$$

We compare u to the function $U \in D^{1,2}(\mathbf{R}^n)$ defined by $U = u$ in B_0^R and $\Delta U = 0$ in $\mathbf{R}^n \setminus B_0^R$. Then

$$\int_{\mathbf{R}^n} |\nabla U|^2 \quad=\quad \int_{B_0^R} |\nabla u|^2 + u(R)^2\mathrm{cap}_{\mathbf{R}^n}(B_0^R) \quad=\quad 1 - \gamma(R) + \frac{u(R)^2}{K(R)}.$$

By (28) and application of the generalized Sobolev inequality to the function $\min(u, u(R))$ we obtain

$$S^F(1 - C\varepsilon^2) \quad\leq\quad \int_{\mathbf{R}^n} F(u) \quad\leq\quad \int_{\mathbf{R}^n} F(U) + S^F\gamma_\varepsilon(R)^{\frac{n}{n-2}}.$$

Therefore

$$\int_{\mathbf{R}^n} F(U) \quad \geq \quad S^F \left(1 - c\varepsilon^2 - \gamma(R)^{\frac{n}{n-2}}\right).$$

On the other hand by the generalized Sobolev inequality we have

$$\int_{\mathbf{R}^n} F(U) \quad \leq \quad S^F \left(1 - \gamma(R) + \frac{u(R)^2}{K(R)}\right)^{\frac{n}{n-2}}$$

and then, since $u(R)^2/K(R) \leq \gamma(R) \leq \gamma_0 < 1$, we get

$$\int_{\mathbf{R}^n} F(U) \quad \leq \quad S^F \left(1 - C\gamma(R) + C\frac{u(R)^2}{K(R)}\right).$$

Combining the upper and the lower bound we obtain

$$\gamma(R) \quad \leq \quad \frac{u(R)^2}{K(R)} + C\gamma(R)^{\frac{n}{n-2}} + c\varepsilon^2. \tag{35}$$

In absence of the last two terms u would be harmonic outside B_0^R and $u(2R) - 2^{2-n}u(R) = 0$. We will see that this difference is small if (35) holds. By the Cauchy-Schwarz inequality and the formula for $K(r)$ we can estimate

$$\begin{aligned}
\gamma(R) - \gamma(2R) \quad &= \quad \left|S^{n-1}\right| \int_R^{2R} r^{n-1} \left|u'\right|^2 \\
&\geq \quad \left|S^{n-1}\right| \frac{\left(\int_R^{2R} u'\right)^2}{\left(\int_R^{2R} r^{1-n}\right)} \quad \geq \quad \frac{\left(u(R) - u(2R)\right)^2}{\left(K(R) - K(2R)\right)}.
\end{aligned}$$

Together with $\gamma(2R) \geq u^2(2R)/K(2R)$ (cf. (17)) and the estimate $(1-x)^2/(1-\lambda) + x^2/\lambda \geq 1 + c(\lambda)(x-\lambda)^2$ we obtain

$$\begin{aligned}
\gamma(R) \quad &\geq \quad \frac{u(R)^2}{K(R)} \left(\frac{\left(1 - u(2R)/u(R)\right)^2}{(1 - 2^{2-n})} + \frac{\left(u(2R)/u(R)\right)^2}{2^{2-n}}\right) \\
&\geq \quad \frac{u(R)^2}{K(R)} \left(1 + C\left(\frac{u(2R)}{u(R)} - 2^{2-n}\right)^2\right)
\end{aligned}$$

with $C > 0$. Combining the above with (35) we deduce

$$\left(\frac{u(2R)}{u(R)} - 2^{2-n}\right)^2 \quad \leq \quad C\frac{K(R)}{u(R)^2}\left(\gamma(R)^{\frac{n}{n-2}} + \varepsilon^2\right) \tag{36}$$

after cancellation of the leading term.

Let us estimate now the decay of u by iteration. Let $a_i := \frac{u(2^i)}{K(2^i)}$. We claim that there exists $\alpha \in (0, \frac{n-2}{n})$ and a positive constant c_2 such that

$$a_i \leq c_2 2^{i\alpha}, \quad \text{if} \quad 2^i \leq \varepsilon^{-\frac{2}{n-2}}. \tag{37}$$

For $i = 0$ the assertion follow from the estimate $u^2(1) \leq \gamma_0 K(1)$. To proceed by induction we distinguish two cases. Since u is decreasing, if $a_i \leq 2^{2-n}c_2$ then $a_{i+1} \leq c_2$. If $a_i \geq 2^{2-n}c_2$ then

$$\frac{K(2^i)}{u(2^i)^2}\varepsilon^2 \leq a_i^{-2} K(2^i)^{-1}\varepsilon^2 \leq c\, c_2^{-2}.$$

Multiplying (35) by $\frac{K(2^i)}{u(2^i)^2}\gamma(2^i)^{\frac{n}{n-2}}$ and taking into account that for sufficiently small γ_0 the term $c\gamma(R)^{\frac{n}{n-2}}$ can be absorbed into the left hand side we deduce that

$$\frac{K(2^i)}{u(2^i)^2}\gamma(2^i)^{\frac{n}{n-2}} \leq \gamma_0^{\frac{2}{n-2}}(C + c_2^{-2}).$$

Thus (36) yields with a sufficiently large choice of c_2 and a sufficiently small choice of γ_0

$$\frac{a_{i+1}}{a_i} \le 2^{n-2}(2^{2-n} + C\gamma_0^{\frac{1}{n-2}} + C\, c_2^{-1}) \le 2^\alpha$$

and hence (37) is proved.

Since u is decreasing we deduce from (37) and (35) that

$$\begin{aligned} u(R) &\le CR^{2-n+\alpha}, \\ \gamma(R) &\le CR^{2-n+2\alpha} + C\varepsilon^2 \le CR^{2-n+2\alpha}, \\ \gamma(R)^{\frac{n}{n-2}} &\le CR^{-n+2^*\alpha} \end{aligned}$$

for $1 \le R \le \varepsilon^{\frac{-2}{n-2}}$. Multiplying (36) by $fracu(R)^2K(2R)$ we obtain

$$\left|\frac{u(2R)}{K(2R)} - \frac{u(R)}{K(R)}\right| \le C(R^{\frac{2^*\alpha-2}{2}} + \varepsilon R^{\frac{n-2}{2}}) \tag{38}$$

Since $2^*\alpha = \frac{2n}{n-2}\alpha < 2$, we obtain by iteration of this estimate

$$\frac{u(2^i)}{K(2^i)} - \frac{u(1)}{K(1)} \le c(1 + \varepsilon(2^i)^{\frac{n-2}{2}}) \le C\,.$$

This implies (29), and (30) follows from (35). In view of (29) and (30) the estimates immediately preceding (38) hold with $\alpha = 0$ and (38) becomes

$$\left|\frac{u(2R)}{K(2R)} - \frac{u(R)}{K(R)}\right| \le c(R^{-1} + \varepsilon R^{\frac{n-2}{2}}). \tag{39}$$

If we fix j such that $2^j \le \varepsilon^{-\frac{2}{n-2}} \le 2^{j+1}$ and we define $u_\infty = u(2^{[j/2]})/K(2^{[j/2]})$, then iterative application of (39) with $R = 2^i$, $i = [j/2], \dots j$ yields

$$|u(2^i) - u_\infty| \le c(2^{-[j/2]} + \varepsilon 2^{\frac{n-2}{2}i}) \le c\varepsilon 2^{\frac{n-2}{2}i},$$

while application for $i = [j/2], \dots 1$ gives $|u(2^i) - u_\infty| \le c2^{-i}$. Combining these two estimates we obtain (32) for every $R = 2^i \le \varepsilon^{-\frac{2}{n-2}}$.

To deduce (32) for all $r \in [1, \varepsilon^{-\frac{2}{n-2}}]$ it suffices to observe that similarly to (36) one can derive the estimate

$$\left(\frac{u(\lambda R)}{u(R)} - \lambda^{2-n}\right)^2 \quad \le \quad c\frac{K(R)}{u(R)^2}\left(\gamma(R)^{\frac{n}{n-2}} + o(\varepsilon^2)\right)$$

for all $\lambda \in [2,4]$.

It only remains to prove (34). The upper bound follows from (29) and (32), applied with $r = \rho$.

Suppose that the lower bound was false. Then there exists ε_k converging to zero and a function $u_k = u_{\varepsilon_k} \in D^{1,2}_*$ that satisfy (28) to (33) with u, u_∞ and ε replaced by u_k, $u_{k,\infty}$ and ε_k and with $\|\nabla u_k\|_2 = 1$, $\rho_k = \rho = 1$. Moreover $u_{k,\infty} \to 0$. Since the u_k are non negative radially decreasing it follows from the Sobolev embedding theorem that, up to a subsequence, u_k converge to some u_0 uniformly on $[1, +\infty)$. This implies, together with (32), that

$$u_0(r) \le c_0 K(r) r^{-1} \tag{40}$$

for every $r \ge 1$. It follows from (35) and the choice of γ_0 that

$$\gamma_k(R) \le C\left(\frac{u_k^2(R)}{K(R)} + \varepsilon_k^2\right) \tag{41}$$

and thus

$$\lim_{k\to\infty} \gamma_k(R) \le C\frac{u_0^2(R)}{K(R)} \le CR^{-n}\,, \tag{42}$$

uniformly for $R \in [1,\infty)$. Thus (38) applied to u_k in combination with (42) and the uniform convergence of u_k implies

$$\left|\frac{u_0(2R)}{K(2R)} - \frac{u_0(R)}{K(R)}\right| \le cK(R)^{\frac{1}{n-2}}\left(\frac{u_0(R)}{K(R)}\right)^{\frac{n}{n-2}}\,.$$

If we let $a_i = \frac{u_0(2^i)}{K(2^i)}$, then we have

$$|a_{i+1} - a_i| \le C2^{-i} a_i^{\frac{n}{n-2}} \tag{43}$$

and, in view of (40), $a_i \le C2^{-i}$. It follows that $a_i = 0$ for every i. Indeed let $\delta > 0$ be sufficiently small and let i_0 be the largest integer such that $a_{i_0} \ge \delta$. Then, for every $l \ge i_0$ such that $C2^{-l} < 1$, (43) implies

$$a_l \le \sum_{j=l+1}^{\infty} C2^{-j} a_j^{\frac{n}{n-2}} \le C2^{-l} \delta^{\frac{n}{n-2}} < \delta$$

which is a contradiction. Hence $u_0 = 0$ on $[1, +\infty)$ and thus by (41)

$$\int_{\mathbf{R}^n \setminus B_1(0)} |\nabla u_k|^2 dx = \gamma_k(1) \to 0 \,.$$

This contradicts the definition of ρ_k. Thus the lower bound in (34) must hold and the proof of Lemma 22 is finished.

○

We now begin with the proof of Theorem 17 (1).

Let $\{u_\varepsilon\} \subseteq D^{1,2}(\Omega)$ be a sequence with $\|\nabla u_\varepsilon\| \le \varepsilon$ which concentrates at $x_0 \in \overline{\Omega}$, i.e. $\frac{|\nabla u_\varepsilon|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_0$. We may assume

$$\int_{\mathbf{R}^n} F(u_\varepsilon) \quad \ge \quad (S^F - C\varepsilon^2) \|\nabla u_\varepsilon\|_2^{2^*}$$

(for suitable large C) since otherwise the assertion is trivial. In particular the Schwarz symmetrizations $\{u_\varepsilon^*\}$ as well as the rescaled sequence $w_\varepsilon^*(r) = u_\varepsilon^*(\varepsilon^{\frac{2}{n-2}} r)$ satisfy the assumption of Lemma 22. We obtain that there exist ρ_ε and $u_{\infty,\varepsilon}$ such that $u_\varepsilon^*(r) = u_{\infty,\varepsilon} K(r)(1 + o(1))$ for $\rho_\varepsilon \ll r \ll \rho_\varepsilon \varepsilon^{-2/n-2}$ and ρ_ε is characterized by

$$\rho_\varepsilon = \sup \left\{ \rho > 0 \,:\, \int_{B_0^\rho} \frac{|\nabla u_\varepsilon^*|^2}{\varepsilon^2} = (1 - \gamma_0) \frac{\|\nabla u_\varepsilon^*\|_2^2}{\varepsilon^2} \right\} \tag{44}$$

with $0 < \gamma_0 < 1$.

Step 2 *We have* $c\,\varepsilon^{\frac{2}{n-2}} \le \rho_\varepsilon \le C\,\varepsilon^{\frac{2}{n-2}}$, $c > 0$.

Proof. Let us first prove that $\rho_\varepsilon \to 0$. Since $\frac{|u_\varepsilon|^{2^*}}{\varepsilon^{2^*}}$ concentrates at x_0 one easily checks that $\frac{|u_\varepsilon^*|^{2^*}}{\varepsilon^{2^*}}$ concentrates at zero. Thus $\varepsilon^{-2^*} F(u_\varepsilon^*) \le \alpha \frac{|u_\varepsilon^*|^{2^*}}{\varepsilon^{2^*}}$ concentrates at zero, in fact $\varepsilon^{-2^*} F(u_\varepsilon^*) \overset{*}{\rightharpoonup} S^F \delta_0$, since $\int_{\mathbf{R}^n} \varepsilon^{-2^*} F(u_\varepsilon^*) = \int_\Omega \varepsilon^{-2^*} F(u_\varepsilon) \to S^F$. Hence part 2 of Theorem 12 of [9], applied to u_ε^* shows that

$$\frac{|\nabla u_\varepsilon^*|^2}{\varepsilon^2} \overset{*}{\rightharpoonup} \delta_0, \tag{45}$$

since otherwise $u_\varepsilon^*/\varepsilon \to v_0$ in L^{2^*} which contradicts concentration of $u_\varepsilon^*/\varepsilon$. Now (45) and the definition of ρ_ε imply $\rho_\varepsilon \to 0$.

Suppose now that $\rho_\varepsilon \ll \varepsilon^{\frac{2}{n-2}}$, i.e. $R_\varepsilon := \rho_\varepsilon \varepsilon^{-\frac{2}{n-2}} \to 0$. By (30) we have, for every $r_\varepsilon \to 0$ such that $R_\varepsilon \ll r_\varepsilon \ll R_\varepsilon \varepsilon^{-\frac{2}{n-2}}$

$$\int_{\mathbf{R}^n \setminus B_0^{r_\varepsilon}} |\nabla w_\varepsilon^*|^2 \le C \left(\frac{R_\varepsilon}{r_\varepsilon} \right)^{n-2} \to 0$$

and then $|\nabla w_\varepsilon^*|^2$ concentrates at zero, which is in contradiction with Theorem 3.

It remains to prove that $\rho_\varepsilon \le C\varepsilon^{\frac{2}{n-2}}$. Let $r_\varepsilon \to 0$ such that $r_\varepsilon \varepsilon^{\frac{2}{n-2}} / \rho_\varepsilon \to 0$ and $\rho_\varepsilon / r_\varepsilon \to 0$ (for instance we can choose $r_\varepsilon = \rho_\varepsilon^{\frac{1}{n}}$). Define $A_\varepsilon = \{u_\varepsilon > u_{\infty,\varepsilon} K(r_\varepsilon)\}$ and $A_\varepsilon^* = \widetilde{B_0^{r_\varepsilon}}$. By (32), for every r, with $\rho_\varepsilon \le r \le \rho_\varepsilon \varepsilon^{-\frac{2}{n-2}}$, we get

$$|u_\varepsilon^*(r) - u_{\infty,\varepsilon} K(r)| \le c_0 \rho_\varepsilon^{\frac{n-2}{2}} \|\nabla u_\varepsilon^*\|_2 K(r) \left(\frac{\rho_\varepsilon}{r} + \varepsilon \left(\frac{r}{\rho_\varepsilon} \right)^{\frac{n-2}{2}} \right) \,,$$

and by (34) and the fact that $\rho_\varepsilon \geq C\varepsilon^{\frac{2}{n-2}}$ we obtain

$$\left|\frac{u_\varepsilon^*(r)}{u_{\infty,\varepsilon}K(r)} - 1\right| \leq C\left(\frac{\rho_\varepsilon}{r} + \varepsilon\left(\frac{r}{\rho_\varepsilon}\right)^{\frac{n-2}{2}}\right) \leq C\left(\frac{\rho_\varepsilon}{r} + r^{\frac{n-2}{2}}\right).$$

Since $\rho_\varepsilon \ll r_\varepsilon \ll \rho_\varepsilon \varepsilon^{-\frac{2}{n-2}}$ and $r_\varepsilon \to 0$ we have

$$\left|\frac{u_\varepsilon^*(r_\varepsilon)}{u_{\infty,\varepsilon}K(r_\varepsilon)} - 1\right| \leq C\left(\frac{\rho_\varepsilon}{r_\varepsilon} + r_\varepsilon^{\frac{n-2}{2}}\right) = o(1).$$

As, by definition, $u_\varepsilon^*(\widetilde{r}_\varepsilon) = u_{\infty,\varepsilon}K(r_\varepsilon)$ we obtain that $\widetilde{r}_\varepsilon / r_\varepsilon \to 1$, and in particular $|A_\varepsilon| = |B_0^{\widetilde{r_\varepsilon}}| \to 0$.

Since A_ε is a superlevel set of u_ε we have

$$\int_{A_\varepsilon^*} |\nabla u_\varepsilon^*|^2 \leq \int_{A_\varepsilon} |\nabla u_\varepsilon|^2 .$$

Let $U_\varepsilon : \mathbf{R}^n \to \mathbf{R}$ denote the harmonic extension of u_ε^* outside of A_ε^*. Taking into account Lemma 16 (ii), (34) and the identities $\mathrm{cap}_{\mathbf{R}^n}(A_\varepsilon^*) = \mathrm{cap}_{\mathbf{R}^n}(B_0^{\widetilde{r_\varepsilon}}) = \frac{1}{K(\widetilde{r}_\varepsilon)}$ we deduce that

$$\begin{aligned}
\frac{1}{\varepsilon^2}\int_{\mathbf{R}^n} |\nabla U_\varepsilon|^2 &= \frac{1}{\varepsilon^2}\int_{A_\varepsilon} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbf{R}^n \setminus A_\varepsilon^*} |\nabla U_\varepsilon|^2 \\
&\leq 1 - \frac{1}{\varepsilon^2}\int_{\Omega\setminus A_\varepsilon} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbf{R}^n \setminus A_\varepsilon^*} |\nabla U_\varepsilon|^2 \\
&\leq 1 - \frac{1}{\varepsilon^2} u_{\infty,\varepsilon}^2 K(r_\varepsilon)^2 [\mathrm{cap}_\Omega(A_\varepsilon) - \mathrm{cap}_{\mathbf{R}^n}(A_\varepsilon^*)] \qquad (46) \\
&\leq 1 - \frac{1}{\varepsilon^2} u_{\infty,\varepsilon}^2 K(r_\varepsilon)^2 K(\widetilde{r}_\varepsilon)^{-2} \min_{\overline{\Omega}} \tau_\Omega \\
&\leq 1 - C\rho_\varepsilon^{n-2} \min_{\overline{\Omega}} \tau_\Omega (1 + o(1)).
\end{aligned}$$

On the other hand, by the lower bound (Step 1), by (31) and by the fact that $\widetilde{r}_\varepsilon \approx r_\varepsilon$ we get

$$\begin{aligned}
\frac{1}{\varepsilon^{2^*}}\int_{\mathbf{R}^n} F(U_\varepsilon) &\geq \frac{1}{\varepsilon^{2^*}}\int_{A_\varepsilon} F(u_\varepsilon) \\
&= \frac{1}{\varepsilon^{2^*}}\int_{\Omega} F(u_\varepsilon) - \frac{1}{\varepsilon^{2^*}}\int_{\Omega\setminus A_\varepsilon} F(u_\varepsilon) \\
&\geq S^F(1 - C\varepsilon^2) - \frac{1}{\varepsilon^{2^*}}\int_{\mathbf{R}^n \setminus B_0^{\widetilde{r_\varepsilon}}} F(u_\varepsilon^*) \\
&\geq S^F(1 - C\varepsilon^2) - C\left(\frac{\rho_\varepsilon}{r_\varepsilon}\right)^n .
\end{aligned}$$

So that if $r_\varepsilon = \rho_\varepsilon^{1/n}$ we have

$$\frac{1}{\varepsilon^{2^*}}\int_{\mathbf{R}^n} F(U_\varepsilon) \geq S^F(1 - C\varepsilon^2 - C\rho_\varepsilon^{n-1}).$$

Combining this with the upper bound for $\|\nabla U_\varepsilon\|$ and the generalized Sobolev inequality we deduce

$$(1 - C\varepsilon^2 - C\rho_\varepsilon^{n-1}) \leq (1 - C\rho_\varepsilon^{n-2} \min_{\overline{\Omega}} \tau_\Omega)^{\frac{2^*}{2}}$$

and then

$$C\rho_\varepsilon^{n-2}(\min_{\overline{\Omega}} \tau_\Omega - C\rho_\varepsilon) \leq C\varepsilon^2$$

which implies $\rho_\varepsilon \leq C\varepsilon^{\frac{2}{n-2}}$. ○

Step 3 (Convergence of rescaled maximizing sequences) *There exists a sequence* $x_\varepsilon \to x_0$ *such that the rescaled functions* $w_\varepsilon(z) = u_\varepsilon(x_\varepsilon + \varepsilon^{\frac{2}{n-2}} z)$ *converge strongly in* $D^{1,2}(\mathbf{R}^n)$ *to some function* w*, which is an extremals for* S^F*.*

Proof. This is a standard application of the concentration compactness alternative. Let $v_\varepsilon(x) = u_\varepsilon(\varepsilon^{\frac{2}{n-2}} x)$. Concentration is excluded since concentration of v_ε implies concentration of the symmetrized sequence v_ε^* which would imply $w_\infty = 0$ (see Theorem 3), contradicting the hypothesis. Splitting is excluded by the strict convexity of the function $\lambda \to \lambda^{2^*/2}$ and the fact that v_ε is maximizing for S^F. Finally vanishing is excluded by the estimate

$$(47) \qquad |\{v_\varepsilon > \delta\}| \geq C\delta^{\frac{n}{n-2}} \qquad \forall \varepsilon > 0\,, \qquad \varepsilon^2 \ll \delta \ll 1\,.$$

This estimate follows from (32), (34) and the fact that $\rho_\varepsilon \approx \varepsilon^{\frac{2}{n-2}}$ which yields $u_\varepsilon^*(r) \approx u_{\infty,\varepsilon} K(r) \approx \varepsilon^2 K(r) \approx (\varepsilon^{\frac{2}{n-2}} r)^{2-n}$ for $\rho_\varepsilon \ll r \ll \rho_\varepsilon \varepsilon^{-\frac{2}{n-2}}$, whence $v_\varepsilon^*(R) \approx R^{2-n}$ for $1 \ll R \ll \varepsilon^{-\frac{2}{n-2}}$.

To see that (47) exclude vanishing first note that vanishing of $|\nabla v_\varepsilon|^2$ implies vanishing of $|v_\varepsilon|^{2^*}$. Indeed using the n-harmonic capacity potential $\varphi_R^r(x) = \frac{\log(|x-a|/R)}{\log(r/R)}$ extended by 1 in B_a^r and by 0 in B_a^R we get the estimates

$$\begin{aligned} \left(\frac{1}{S^*} \int_{B_a^r} |v_\varepsilon|^{2^*} \right)^{\frac{2}{2^*}} &\leq \int_{B_a^R} |\nabla(\varphi_R^r v_\varepsilon)|^2 \\ &\leq \int_{B_a^R} |\nabla v_\varepsilon|^2 + \omega(r/R) \int_{\mathbf{R}^n} |\nabla v_\varepsilon|^2 \end{aligned}$$

with $\omega(t) \to 0$ as $t \to 0$ (see [9], Lemma 8 for the details). Now consider a cover of $\mathbf{R}^n$ by the translated unit cubes $Q_z = z + (0,1)^n$, $z \in \mathbf{Z}^n$, and let $\lambda_z^\varepsilon = |\{v_\varepsilon > \delta\} \cap Q_z|$, $\mu_z^\varepsilon = |\{v_\varepsilon > \delta/2\} \cap Q_z|$. Vanishing of $|v_\varepsilon|^{2^*}$ implies that $\sup_{z \in \mathbf{Z}^n} (\lambda_z^\varepsilon + \mu_z^\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular the function $(v_\varepsilon - \frac{\delta}{2})_+$ vanishes on a set of volume fraction $1 - \mu_z^\varepsilon \geq 1/2$ on each cube Q_z, provided $\varepsilon < \varepsilon_0$. Hence a suitable version of the Poincaré inequality (see e.g. [9], Lemma 28) yields

$$(\lambda_z^\varepsilon)^{\frac{2}{2^*}} \left(\frac{\delta}{2}\right)^2 \leq \left(\int_{Q_z} \left(v_\varepsilon - \frac{\delta}{2}\right)_+^{2^*} \right)^{\frac{2}{2^*}} \leq C \int_{Q_z} |\nabla v_\varepsilon|^2\,.$$

Since $\sum_z \lambda_z^\varepsilon = |\{v_\varepsilon > \delta\}| \geq C(\delta) > 0$ we deduce

$$\int_{\mathbf{R}^n} |\nabla v_\varepsilon|^2 \geq (\max_z \lambda_z^\varepsilon)^{1-\frac{2}{2^*}} C(\delta) \to \infty$$

as $\varepsilon \to 0$, which yields a contradiction.

Thus by the concentration compactness alternative, there exists a sequence $\{a_\varepsilon\} \subset \mathbf{R}^n$ such that (a subsequence of) $v_\varepsilon(\cdot + a_\varepsilon)$ is compact in $D^{1,2}(\mathbf{R}^n)$ and taking $x_\varepsilon = \varepsilon^{\frac{2}{n-2}} a_\varepsilon$ we obtain the assertion. ○

Step 4 (Concentration of level sets) *There exists* $\eta_0 > 0$ *such that the following holds. If* u_ε *concentrates at* $x_0 \in \overline{\Omega}$ *and* $t_\varepsilon/\varepsilon^2 \to \infty$ *and* $t_\varepsilon \leq \eta_0$*, then* $\{u_\varepsilon > t_\varepsilon\}$ *concentrates at* x_0*.*

Proof. Let w_ε as in the previous step. Fix $t > 0$ and let $\rho_{t,\varepsilon}$ such that $|B_0^{\rho_{t,\varepsilon}}| = |\{w_\varepsilon > t\}|$. By (32) and Step 2 we deduce that

$$(48) \qquad |\{w_\varepsilon > t\}| \approx t^{-\frac{n}{n-2}}\,, \qquad \rho_{t,\varepsilon} \approx t^{-\frac{1}{n-2}}$$

for every t such that $\varepsilon^2 \ll t \ll 1$. Let us prove that

$$(49) \qquad \frac{\mathrm{cap}_{\mathbf{R}^n}(\{w_\varepsilon > t\})}{\mathrm{cap}_{\mathbf{R}^n}(B_0^{\rho_{t,\varepsilon}})} \leq 1 + C\left(\frac{\varepsilon^2}{t} + t^{\frac{2}{n-2}}\right).$$

Indeed let $\widetilde{w}_\varepsilon$ be the harmonic extension of w_ε^* outside of $B_0^{\rho_{t,\varepsilon}}$. Then by (31)

$$\begin{aligned} \int_{\mathbf{R}^n} F(\widetilde{w}_\varepsilon) &\geq \int_{B_0^{\rho_{t,\varepsilon}}} F(w_\varepsilon^*) \geq S^F - C\varepsilon^2 - C \int_{\mathbf{R}^n \setminus B_0^{\rho_{t,\varepsilon}}} |w_\varepsilon|^{2^*} \\ &\geq S^F - C\varepsilon^2 - C\rho_{t,\varepsilon}^{-n} \end{aligned}$$

and

$$\begin{aligned}\int_{\mathbf{R}^n} |\nabla \widetilde{w}_\varepsilon|^2 &= \int_{B_0^{\rho_{t,\varepsilon}}} |\nabla \widetilde{w}_\varepsilon|^2 + t^2 \mathrm{cap}_{\mathbf{R}^n}(B_0^{\rho_{t,\varepsilon}}) \\ &\le \int_{\{w_\varepsilon > t\}} |\nabla w_\varepsilon|^2 + t^2 \rho_{t,\varepsilon}^{n-2} \mathrm{cap}_{\mathbf{R}^n}(B_0^1) \\ &\le 1 - \int_{\{w_\varepsilon < t\}} |\nabla w_\varepsilon|^2 + t^2 \rho_{t,\varepsilon}^{n-2} \mathrm{cap}_{\mathbf{R}^n}(B_0^1)\,.\end{aligned}$$

The combination of these two estimates with the generalized Sobolev inequality gives

$$(50) \qquad \int_{\{w_\varepsilon < t\}} |\nabla w_\varepsilon|^2 - t^2 \rho_{t,\varepsilon}^{n-2} \mathrm{cap}_{\mathbf{R}^n}(B_0^1) \le C(\varepsilon^2 + \rho_{t,\varepsilon}^{-n}) \le C(\varepsilon^2 + t t^{\frac{2}{n-2}})\,.$$

Since

$$\mathrm{cap}_{\mathbf{R}^n}(\{w_\varepsilon > t\}) \le \frac{1}{t^2} \int_{\{w_\varepsilon < t\}} |\nabla w_\varepsilon|^2\,,$$

we obtain (49) from (50) and (48). Now (49) may be rewritten as

$$(51) \qquad \frac{\mathrm{cap}_{\mathbf{R}^n}(\{w_\varepsilon > t\})}{\mathrm{cap}_{\mathbf{R}^n}(B_0^{\rho_{t,\varepsilon}})} \le 1 + \delta(\eta_0) \qquad \text{if} \quad \frac{1}{\eta_0} \le t \le \eta_0\,.$$

Applying Proposition 14 we get that there exist $z_{t,\varepsilon}$ such that

$$(52) \qquad \frac{|\{w_\varepsilon > t\} \Delta B(z_{t,\varepsilon}, \rho_{t,\varepsilon})|}{|B(z_{t,\varepsilon}, \rho_{t,\varepsilon})|} \le \delta'(\eta_0)\,.$$

Let us prove now that if $\varepsilon^2/\eta_0 \le t < t' \le \eta_0$, then

$$(53) \qquad |z_{t,\varepsilon} - z_{t',\varepsilon}| \le C \rho_{t,\varepsilon}$$

Suppose first that $t \ge t'/2$. By (32) we have

$$\frac{|\{w_\varepsilon > t\} \cap \{w_\varepsilon > t'\}|}{|\{w_\varepsilon > t\}|} = \frac{|\{w_\varepsilon > t'\}|}{|\{w_\varepsilon > t\}|} \ge c_0 > 0\,.$$

Combining this with (52) we obtain

$$\frac{|B(z_{t,\varepsilon}, \rho_{t,\varepsilon}) \cap B(z_{t',\varepsilon}, \rho_{t',\varepsilon})|}{|B(z_{t,\varepsilon}, \rho_{t,\varepsilon})|} \ge c_0 - 2\delta'(\eta_0) > 0$$

if δ' is sufficiently small (which can be achieved by choosing η_0 sufficiently small). Hence

$$(54) \qquad |z_{t,\varepsilon} - z_{t',\varepsilon}| \le \rho_{t,\varepsilon} + \rho_{t',\varepsilon} \le 2\rho_{t,\varepsilon}$$

and the assertion is proved under the additional assumption $t \ge t'/2$. To obtain the general case let $j \in I\!N$ be such that $2^{-j}t' \ge t \ge 2^{-j-1}t'$, define $t_i = 2^{-i}t'$ for $i = 0, \ldots, j$ and $t_{j+1} = t$, and apply (54) to t_i and t_{i+1}. Since $ct^{-\frac{1}{n-2}} \le \rho_{t,\varepsilon} \le Ct^{-\frac{1}{n-2}}$, summation over i leads to a geometric series and (53) follows.

Now we know that w_ε converges to a maximizer w for S^F and by Theorem 2 we have that there exists $r_0 > 0$ such that $w = w^*$ on $\mathbf{R}^n \setminus B_0^{r_0}$ and $w^*(r)$ is strictly decreasing for $r \ge r_0$. Thus choosing η_0 so small enough such that $\eta_0 < w^*(r_0)$, we have

$$(55) \qquad |\{w_\varepsilon > \eta_0\} \Delta \{w > \eta_0\}| \to 0$$

and $\{w > \eta_0\} = B_0^r$. Hence $z_{\eta_0,\varepsilon} \to 0$. Therefore by (53) we deduce

$$\limsup_{\varepsilon \to 0} \frac{|z_{t_\varepsilon,\varepsilon}|}{\rho_{t_\varepsilon,\varepsilon}} \le C \qquad \text{if} \quad \frac{\varepsilon^2}{\eta_0} \le t_\varepsilon \le \eta_0\,.$$

Now by (52)

$$\frac{|\{u_\varepsilon > t_\varepsilon\} \Delta B(x_\varepsilon + \varepsilon^{\frac{2}{n-2}} z_{t_\varepsilon,\varepsilon}, \varepsilon^{\frac{2}{n-2}} \rho_{t_\varepsilon,\varepsilon})|}{|\{u_\varepsilon > t_\varepsilon\}|} \to 0\,.$$

Finally the assumption $t_\varepsilon/\varepsilon^2 \to \infty$ implies that $\varepsilon^{\frac{2}{n-2}} \rho_{t_\varepsilon,\varepsilon} \to 0$ and $x_\varepsilon + \varepsilon^{\frac{2}{n-2}} z_{t_\varepsilon,\varepsilon} \to x_0$. Hence $\{u_\varepsilon > t_\varepsilon\}$ concentrates at x_0. ○

Step 5 (Upper bound for sequences concentrating at x_0) *Let w be the limit of (a subsequence of) w_ε as above and recall that*

$$W_\infty^2 \quad = \quad \frac{2(n-1)}{n\,S^F}\int_{\mathbf{R}^n}\frac{F(w)}{K(|\cdot|)}$$

(see Theorem 2). Then

$$S_\varepsilon^F(\Omega) \quad \leq \quad S^F\left(1-\frac{n}{n-2}W_\infty^2\tau_\Omega(x_0)\varepsilon^2+o(\varepsilon^2)\right)$$

as $\varepsilon\to 0$.

Proof. For u_ε as above and U_ε as in Step 2 we have

$$\frac{1}{\varepsilon^{2^*}}\int_{\mathbf{R}^n}F(U_\varepsilon)\geq\frac{1}{\varepsilon^{2^*}}\int_\Omega F(u_\varepsilon)-\frac{1}{\varepsilon^{2^*}}\int_{\Omega\setminus A_\varepsilon}F(u_\varepsilon)\geq\frac{1}{\varepsilon^{2^*}}\int_\Omega F(u_\varepsilon)-o(\varepsilon^2)$$

where we used $\rho_\varepsilon\approx\varepsilon^{\frac{2}{n-2}}$. Moreover by (46)

$$\frac{1}{\varepsilon^2}\int_{\mathbf{R}^n}|\nabla U_\varepsilon|^2\leq 1-\frac{1}{\varepsilon^2}u_{\infty,\varepsilon}^2K(r_\varepsilon)^2[\mathrm{cap}_\Omega(A_\varepsilon)-\mathrm{cap}_{\mathbf{R}^n}(A_\varepsilon)]$$

where $A_\varepsilon=\{u_\varepsilon>u_{\infty,\varepsilon}K(r_\varepsilon)\}$, $r_\varepsilon=\varepsilon^{\frac{2}{n-2}}\rho_\varepsilon$. We know that w_ε converges to a maximizer w for S^F and by Theorem 2 we have $w=w^*$ on $\mathbf{R}^n\setminus B^{r_0}$ and $w^*(R)=W_\infty K(R)(1+o(1))$ as $R\to\infty$. We claim that $u_{\infty,\varepsilon}/\varepsilon^2$ converges to W_∞ as $\varepsilon\to\infty$. Indeed for $\varepsilon^2\ll t\ll 1$ as in (55), $|\{w_\varepsilon^*>t\}|=|\{w_\varepsilon>t\}|\to|\{w>t\}|$ and (32) yields, with $\rho_\varepsilon\approx\varepsilon^{\frac{2}{n-2}}$ and $r=\varepsilon^{\frac{2}{n-2}}R$,

$$\left|w_\varepsilon^*(R)-u_{\infty,\varepsilon}K(\varepsilon^{\frac{2}{n-2}}R)\right|\leq c_0\varepsilon^2\,K(\varepsilon^{\frac{2}{n-2}}R)\left(\frac{\rho_\varepsilon}{\varepsilon^{\frac{2}{n-2}}R}+\varepsilon\left(\frac{\varepsilon^{\frac{2}{n-2}}R}{\rho_\varepsilon}\right)^{\frac{n-2}{2}}\right)\quad\text{for}\quad\rho_\varepsilon\leq\varepsilon^{\frac{2}{n-2}}R\leq\varepsilon^{\frac{-2}{n-2}}\rho_\varepsilon \tag{56}$$

or

$$\left|w_\varepsilon^*(R)-\frac{u_{\infty,\varepsilon}}{\varepsilon^2}K(R)\right|\leq c_0\,K(R)\left(\frac{1}{R}+\varepsilon R^{\frac{n-2}{2}}\right)\quad\text{for}\quad 1\ll R\ll\varepsilon^{\frac{-2}{n-2}}.$$

Thus

$$|\{w_\varepsilon^*>t\}|=\widetilde{c}\left(\frac{u_{\infty,\varepsilon}}{\varepsilon^2}\right)^{\frac{n}{n-2}}t^{-\frac{n}{n-2}}\left(1+O(t+\frac{\varepsilon^2}{t})\right),$$

where $\widetilde{c}$ is a universal constant. Similarly the asymptotic behaviour of w yields

$$|\{w>t\}|=\widetilde{c}\,W_\infty^{\frac{n}{n-2}}t^{-\frac{n}{n-2}}\,(1+o(1))$$

as $t\to\infty$, with the same constant $\widetilde{c}$. Thus (55) proves the claim by taking $\varepsilon\to 0$ and $t\to\infty$.

Moreover, by Step 4, we have that A_ε concentrates at x_0 and hence, by Lemma 16 (i) we obtain

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathbf{R}^n}|\nabla U_\varepsilon|^2 &\leq 1-W_\infty^2K(r_\varepsilon)^2\mathrm{cap}_{\mathbf{R}^n}(A_\varepsilon^*)^2\tau_\Omega(x_0)\varepsilon^2(1+o(1))\\ &= 1-W_\infty^2\tau_\Omega(x_0)\varepsilon^2+o(\varepsilon^2)\,.\end{aligned}$$

Using the generalized Sobolev inequality (3) we conclude

$$\frac{1}{\varepsilon^{2^*}}\int_{\mathbf{R}^n}F(u_\varepsilon)\quad\leq\quad\frac{1}{\varepsilon^{2^*}}\int_{\mathbf{R}^n}F(U_\varepsilon)+o(\varepsilon^2)\quad\leq\quad S^F\left(1-W_\infty^2\tau_\Omega(x_0)\varepsilon^2+o(\varepsilon^2)\right)^{\frac{n}{n-2}}+o(\varepsilon^2) \tag{57}$$

which proves part 1 of Theorem 17. ○

Step 6 (Asymptotic expansion for maximizing sequences)

By Theorem 1 every maximizing sequence $\{u_\varepsilon\}$ concentrates at some point $x_0\in\overline{\Omega}$. Thus by part 1 of Theorem 17

$$\frac{1}{\varepsilon^{2^*}}\int_{\mathbf{R}^n}F(u_\varepsilon)\leq\quad S^F\left(1-\frac{n}{n-2}W_\infty^2\tau_\Omega(x_0)\varepsilon^2+o(\varepsilon^2)\right).$$

In view of the lower bound established in Step 1 and the inequality $W_\infty\geq w_\infty$ (see Remark 4), equality holds and we must have $W_\infty=w_\infty$, i.e., the rescaled sequences w_ε can only converge to those maximizers of S^F which attain the optimal value of W_∞.

Step 7 (Identification of concentration points)

From part 2 and the estimate (57) which holds for all sequences concentrating at x_0, it follows immediately that maximizing sequences must concentrate at a minimum of τ_Ω.

Appendix: Regularity Criterion for the Robin function

We saw that if the set Ω is regular in the sense of Wiener then the Robin function τ_Ω is $+\infty$ on the boundary. This is a kind of regularity for the function τ_Ω; indeed it assures that it attains its minimum at an interior point of Ω. It is well known that a point $x_0 \in \partial\Omega$ is regular for the Dirichlet problem if and only if the following property holds true

(W)
$$W(x_0,R) = \int_0^R \frac{\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0}))}{\rho^{n-2}}\frac{d\rho}{\rho} = +\infty$$
for some $R > 0$.

By this criterion we deduce that a boundary point is regular if the complement of Ω around this point is not too small. We will see that to assure that τ_Ω is infinity at a boundary point the complement of Ω can be even "smaller" than what is prescribed by (W). Indeed we will prove the following result.

THEOREM 23 *Let $x_0 \in \partial\Omega$. Then $\tau_\Omega(x_0)$ is finite if and only if*
$$I(x_0,R) = \int_0^R \frac{\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0}))}{\rho^{2n-4}}\frac{d\rho}{\rho} < +\infty$$
for some $R > 0$.

LEMMA 24 *Let $f(\rho)$ be an integrable function and fix $R > 0$. Then there exist R_0 and R_1 in $(R/2, R)$ such that*
$$\frac{1}{R_0}\sum_{i=0}^{\infty} f(R_0 2^{-i}) \le \int_0^R \frac{f(\rho)}{\rho} d\rho \le \frac{1}{R_1}\sum_{i=0}^{\infty} f(R_1 2^{-i}).$$
If f is continuous, then $R_0 = R_1$.

Proof. Let define $F(\lambda) = \sum_{i\ge 1}\frac{f(\lambda 2^{-i})}{\lambda}$ with $\lambda \in (R/2, R)$. The result follows immediately by the fact that
$$\int_{\frac{R}{2}}^R F(\lambda)d\lambda = \int_0^R \frac{f(\rho)}{\rho}d\rho.$$
○

Let $x_0 \in \partial\Omega$ and let $\rho > 0$. Let $r_\rho(x_0,\cdot)$ be the solution of the following Dirichlet problem

(58)
$$\begin{cases} -\Delta r_\rho(x_0,\cdot) = 0 & \text{in } \mathbf{R}^n \setminus (\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0})) \\ r_\rho(x_0,y) = K(|x_0-y|) & \text{if } y \in \partial(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0})) \\ r_\rho(x_0,y) \to 0 & \text{as } y \to \infty. \end{cases}$$

We shall consider the function $r_\rho(x_0,\cdot)$ extended to $\mathbf{R}^n$ by setting $r_\rho(x_0,y) = K(|x_0-y|)$ if $y \in \Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0})$ and by lower semicontinuity on $\partial\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0})$.

LEMMA 25 *Let $r_\rho(x_0,\cdot)$ be the solution of (58), then*

(59)
$$K(2)^2\frac{\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0}))}{\rho^{2n-4}} \le r_\rho(x_0,x_0) \le K(1)^2\frac{\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0}))}{\rho^{2n-4}}.$$

Proof. Let u_ρ be the capacitary potential of $\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0})$ in $\mathbf{R}^n$ and let μ_ρ be its capacitary distribution. We have

(60)
$$K(2\rho)u_\rho(x) \le r_\rho(x_0,x) \le K(\rho)u_\rho(x).$$

By the fact that

(61)
$$u_\rho(x) = \int_{\partial(\Omega^c\cap(B^{2\rho}_{x_0}\setminus B^{\rho}_{x_0}))} K(|x-y|)d\mu_\rho$$

we get
$$K(2\rho)\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0})) \le u_\rho(x_0) \le K(\rho)\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0}))$$
which concludes the proof together with (60). ○

We denote by H_r the regular part of the Green's functions of $\Omega \cup B^r_{x_0}$, with $r > 0$, and let τ_r be the corresponding Robin function.

Proof of Theorem 23. We will prove Theorem 23 by means of an upper bound and a lower bound of $\tau_\Omega(x_0)$ in terms of $I(x_0, R)$.

Step 1. (the upper bound) *Let $x_0 \in \partial\Omega$ and let us fix $R > 0$. Then*

$$\tau_\Omega(x_0) \le K(1)^2 R I(x_0, R) + \tau_R(x_0)\,. \tag{62}$$

Let $r_\rho(x_0, \cdot)$ be the solution of problem (58). Since $r_\rho(x_0,\cdot)+H_{2\rho}(x_0,\cdot)$ is harmonic in the set $\Omega\cup B^\rho_{x_0}$ and greater than $K(|x_0 - \cdot|)$ on its boundary, for every $\rho > 0$ we have $H_\rho(x_0,\cdot) \le r_\rho(x_0,\cdot) + H_{2\rho}(x_0,\cdot)$ in $\Omega \cup B^\rho_{x_0}$, so that in particular

$$\tau_\rho(x_0) \le r_\rho(x_0, x_0) + \tau_{2\rho}(x_0) \tag{63}$$

By iteration and taking into account that $\tau_\rho(x_0)$ converges to $\tau_\Omega(x_0)$ as $\rho \to 0$, we get that for any fixed $\rho > 0$

$$\tau_\Omega(x_0) \le \sum_{i=0}^{\infty} r_{2^{-i}\rho}(x_0, x_0) + \tau_{2\rho}(x_0)\,. \tag{64}$$

By Lemma 25 we have

$$\tau_\Omega(x_0) \le K(1)^2 \sum_{i=0}^{\infty} \frac{\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2^{-i+1}\rho}_{x_0} \setminus B^{2^{-i}\rho}_{x_0}))}{(2^{-i}\rho)^{2n-4}} + \tau_{2\rho}(x_0)\,. \tag{65}$$

The conclusion follows by applying Lemma 24 to the function $f(\rho) = \mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap (B^{2\rho}_{x_0} \setminus B^{\rho}_{x_0}))\rho^{4-2n}$, using (65) with $\rho = R_1$ (where R_1 is given by Lemma 24), and that $\tau_{2R_1} \le \tau_R$.

Step 2. (the lower bound) *Let x_0 be a boundary point such that $W(R, x_0) < +\infty$ for some R (and then for all), then*

$$I(R, x_0) \le \frac{C}{R}(2R\,W(2R, x_0) + 1)\tau_\Omega(x_0)\,, \tag{66}$$

where C is a positive constant depending only on n.

Let us denote by $C_i(r)$ the set $B^{2^{-i+1}r}_{x_0} \setminus B^{2^{-i}r}_{x_0}$. For any $h \in I\!N$, let $S^h_r(x) = \sum_{i=0}^h r_{2^{-i}r}(x_0, x)$ and let $S_r(x) = \sum_{i\ge 0} r_{2^{-i}r}(x_0, x)$. We shall estimate the function $S_r(x)$ on $\partial\Omega$. More precisely we will prove that

$$S^h_r(x) \le S_r(x) \le C\,K(1)(2R\,W(2R, x_0) + 1)K(|x - x_0|) \qquad \forall\, x \in \partial(\Omega^c \cap B^{2r}_{x_0}) \tag{67}$$

for every $r < R$. If (67) is true, since $S^h_r(x)$ is harmonic in $\mathbf{R}^n \setminus (\Omega^c \cap (B^{2r}_{x_0} \setminus B^{2^{-h}r}_{x_0}))$ and in particular in $\Omega \cup B^{2^{-h}r}_{x_0}$, the same estimate holds on $\partial(\Omega \cup B^{2^{-h}r}_{x_0})$. This implies in particular

$$S^h_r(x) \le C\,K(1)(2R\,W(2R, x_0) + 1)H_{2^{-h}r}(x_0, x) \qquad \forall\; h \in I\!N \qquad \forall\; x \in \Omega\,.$$

Thus, since by Proposition 7 $H_{2^{-n}r}(x_0, x)$ converges to $\widetilde{H}_\Omega(x_0, x)$ as $n \to \infty$, we have

$$S_r(x_0) \le C\,K(1)(2R\,W(2R, x_0) + 1)\tau_\Omega(x_0) \qquad \forall\; r < R\,.$$

We conclude the proof using Lemma 25 and Lemma 24 as in the previous step.

It remains to prove (67). Let $\rho > 0$. Let us fix $x \in \partial C_k(\rho)$; so that in particular $2^{-k}\rho \le |x - x_0| \le 2^{-k+1}\rho$. Let $i \in I\!N$ be such that $i > k + 1$, using (60) and the integral representation (61), we have

$$\begin{aligned} r_{2^{-i}\rho}(x_0, x) &\le K(2^{-i}\rho)K(2^{-k+1}\rho - 2^{-i}\rho)\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap C_i(\rho)) \\ &\le 2^{n-2}K(1)^2(2^{-k}\rho)^{2-n}(2^{-i}\rho)^{2-n}\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap C_i(\rho)) \qquad (68) \\ &\le 2^{n-2}K(1)K(|x - x_0|)\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap C_i(\rho))\,. \end{aligned}$$

Similarly we can estimate $r_{2^{-i}\rho}(x_0, x)$ for $i < k-1$ and we obtain that there exists a constant C depending only on n such that

$$S_\rho(x) \leq C\,K(1)K(|x-x_0|)\left(\sum_{i=0}^{\infty}(2^{-i}\rho)^{2-n}\mathrm{cap}_{\mathbf{R}^n}(\Omega^c \cap C_i(\rho)) + 1\right).$$

Now, by Lemma 24, for any $r < R$ there exists $r_1 \in (r, 2r)$ such that

$$S_{r_1}(x) \leq C\,K(1)K(|x-x_0|)\,(r_1 W(r_1, x_0) + 1) \leq C\,K(1)K(|x-x_0|)\,(2RW(2R, x_0) + 1)\ .$$

Finally we get (67) taking into account that $S_\rho(x)$ is increasing in ρ.

○

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