ON THE NON-LOCAL APPROXIMATION OF FREE-DISCONTINUITY PROBLEMS

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Abstract. A class of free-discontinuity problems is approximated in the sense of Γ-convergence by a sequence of non-local integral functionals.

1. Introduction. The Mumford-Shah functional has been introduced in [6] to model some problems in image segmentation. A weak version of this functional, from the study of which it is possible to obtain "classical" solutions, is

$$F(u) = \int_\Omega |\nabla u|^2 \, dx + 2\lambda \, \mathcal{H}^{n-1}(S(u)), \qquad u \in SBV(\Omega)\,, \tag{1.1}$$

where $S(u)$ is the set of discontinuity points of u, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $SBV(\Omega)$ is the space of special functions of bounded variation on the open set $\Omega \subset \mathbf{R}^n$ (see De Giorgi and Ambrosio

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[4]). Variational approximations of functionals of this type, depending on a volume and a surface energy, by functionals defined on spaces of Sobolev functions are motivated by computational and evolution problems.

While the lack of convexity of F forbids its approximation by simple local functionals, in [1] it has been shown that the Mumford-Shah functional (1.1) is the variational limit as $\varepsilon \to 0^+$ of the non-local functionals

$$F_\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega f(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u|^2 \, dy) \, dx, \qquad u \in H^1(\Omega)\,, \tag{1.2}$$

under the assumptions that f is continuous, increasing, and

$$\lim_{t \to 0^+} \frac{f(t)}{t} = 1, \qquad \lim_{t \to +\infty} f(t) = \lambda. \tag{1.3}$$

A simple relaxation argument, which is straightforward in the one-dimensional case, shows that we can drop all hypotheses on f, considering in its place the lower semicontinuous increasing envelope $\overline{f}$ (*i.e.,* the greatest lower semicontinuous and increasing function not greater than f), and obtain the same result. Furthermore, from a comparison argument it is clear that if $\lim_{t \to +\infty} f(t) = +\infty$ then we obtain in the limit just the Dirichlet integral.

Purpose of this work is to show that more general surface energies can be obtained by non-local approximation as above, considering functionals of the form

$$F_\varepsilon(u) = \frac{1}{\varepsilon} \int_\Omega f_\varepsilon(\varepsilon \fint_{B_\varepsilon(x)} |\nabla u|^2 \, dy) \, dx, \qquad u \in H^1(\Omega), \tag{1.4}$$

with varying integrands f_ε. More precisely, we show that lower semicontinuous functionals of the form

$$F(u) = \int_\Omega |\nabla u|^2 \, dx + \int_{S(u)} \varphi(u^+(x) - u^-(x)) \, d\mathcal{H}^{n-1}\,, \quad u \in SBV(\Omega)\,, \tag{1.5}$$

where $u^\pm$ are the traces of u on both sides of $S(u)$, can be obtained as Γ-limits of functionals (1.4), with φ computable from (f_ε). The examples of non-trivial φ in Section 3 justify the generality of recent integral representation results (see [2]).

In order to avoid the technicalities of the higher dimensional case, for which we refer to [1], we treat only the one-dimensional case, which contains all the original features of the approximation procedure.

2. Limits of non-local functionals. We recall that a family $(F_\varepsilon)_{\varepsilon>0}$ of functionals $F_\varepsilon : L^1_{loc}(\mathbf{R}) \to [0,+\infty]$ is said to Γ-converge to a functional $F : L^1_{loc}(\mathbf{R}) \to [0,+\infty]$ (with respect to the L^1_{loc}-convergence) as $\varepsilon \to 0^+$ if for every $u \in L^1_{loc}(\mathbf{R})$ and for every sequence (ε_j) of positive numbers decreasing to 0, the two conditions hold:

(i) (*lower semicontinuity inequality*) for all sequences (u_j) converging to u in $L^1_{loc}(\mathbf{R})$ we have $F(u) \le \liminf_j F_{\varepsilon_j}(u_j)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u in $L^1_{loc}(\mathbf{R})$ such that $F(u) \ge \limsup_j F_{\varepsilon_j}(u_j)$.

We refer to [3] for an exposition of the main properties of Γ-convergence, and to [1] for the relevance of such an approximation in the framework of free-discontinuity problems.

We recall moreover that the space $SBV_{loc}(\mathbf{R})$ is defined as the space of functions $u \in L^1_{loc}(\mathbf{R})$ whose distributional derivative Du can be written as $Du = f\,\mathcal{L}_1 + \sum_{t\in S(u)} a_t\delta_t$ (here $\mathcal{L}_1$ denotes the Lebesgue measure, and δ_t the Dirac mass at t) for some $f \in L^1(\mathbf{R})$, a (at most countable) set $S(u) \subset \mathbf{R}$ and a sequence of real numbers $(a_t)_{t\in S(u)}$ with $\sum_t |a_t| < +\infty$. It is easy to see that for such functions the left hand-side and right hand-side approximate limits $u^-(t)$, $u^+(t)$ exist at every point, that in the notation above $S(u) = \{t \in \mathbf{R} :\ u^-(t) \ne u^+(t)\}$, and that $a_t = u^+(t) - u^-(t)$. This notation describes a particular case of a SBV-functions space as introduced by De Giorgi and Ambrosio [4].

In what follows, fixed a non-decreasing function $g : [0,+\infty) \to [0,+\infty)$ with

$$g(0) = 0, \qquad \inf\{g(x) : x > 0\} =: c_g > 0\,, \tag{2.1}$$

we will consider a new function φ defined by

$$\varphi(z) = \inf\Big\{\int_{-\infty}^{+\infty} g\Big(\int_{x-1}^{x+1} |u'(t)|^2\,dt\Big)\,dx : u(-\infty) = 0,\ u(+\infty) = z\Big\} \tag{2.2}$$

where the infimum is taken over all functions in $H^1_{loc}(\mathbf{R})$. The meaning of the conditions at $\pm\infty$ is understood as the existence of the corresponding limits. Actually, it suffices to consider functions u with derivatives of compact support. In fact, if we define

$$w_u(t) := \int_{t-1}^{t+1} |u'(x)|^2\,dx, \qquad G(u) := \int_{-\infty}^{+\infty} g(w_u(t))\,dt, \tag{2.3}$$

and $G(u) < +\infty$ then $|\mathrm{supp}\, w_u| \le c_g^{-1}\, G(u)$, and also $|\mathrm{supp}\, u'| \le c_g^{-1}\, G(u) - 2$. Note that $\mathrm{supp}\, u'$ cannot contain more than $|\mathrm{supp}\, w_u|/2$ points of mutual distance greater than 2, so that it is not restrictive to assume, up to eliminating some bounded intervals where u is constant, that

$$\mathrm{supp}\, w_u \subseteq [0, c_g^{-1}\varphi(z) + 1]\,. \tag{2.4}$$

Hence, we can suppose that $u(x) = 0$ for $x < 1$, and $u(x) = z$ for $x > c_g^{-1}\varphi(z)$.

Note moreover, considering as test function $u_z(t) = \operatorname{sign} z\,\Big((0 \vee |z|t) \wedge |z|\Big)$, that

$$\varphi(z) \leq \int_{-\infty}^{+\infty} g\big(\int_{x-1}^{x+1} |u_z'(t)|^2\,dt\big)\,dx \leq 3\,g(z^2)\,,$$

so that we can assume that

$$\operatorname{supp} w_u \subseteq [0, 3c_g^{-1}\,g(z^2) + 1], \tag{2.5}$$

in place of (2.4).

The following theorem is the main result of the paper. It provides a construction of a family (F_ε) Γ-converging to the functional (1.5).

Theorem 2.1. *Let $g : [0, +\infty) \to [0, +\infty)$ be a lower semicontinuous increasing function satisfying* (2.1)*. Define*

$$f_\varepsilon(\xi) = \begin{cases} \xi & \text{if } 0 \leq \xi \leq c_g \\ g(\varepsilon\xi) & \text{if } \xi > c_g, \end{cases}$$

and

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} f_\varepsilon(\int_{x-\varepsilon}^{x+\varepsilon} |u'(t)|^2 dt)\,dx, \qquad u \in H^1_{loc}(\mathbf{R})$$

(extended to $+\infty$ on $L^1_{loc}(\mathbf{R}) \setminus H^1_{loc}(\mathbf{R})$). Then the Γ-limit in $L^1_{loc}(\mathbf{R})$ of F_ε as $\varepsilon \to 0^+$ exists, and it equals

$$F_\varphi(u) := 2\int_{-\infty}^{+\infty} |u'|^2\,dx + \sum_{x \in S(u)} \varphi(u^+(x) - u^-(x)), \qquad u \in SBV_{loc}(\mathbf{R}),$$

(extended to $+\infty$ on $L^1_{loc}(\mathbf{R}) \setminus SBV_{loc}(\mathbf{R})$) where φ is defined by (2.2)*.*

Proof. Let (ε_j) decrease to 0, and $u_j \to u$ in $L^1_{loc}(\mathbf{R})$. With a slight abuse of notation we set $F_j := F_{\varepsilon_j}$. We have to check that

$$F_\varphi(u) \leq \liminf_j F_j(u_j). \tag{2.6}$$

For each $j \in \mathbf{N}$, define $\psi_j(x) = f_{\varepsilon_j}\Big(\int_{x-\varepsilon_j}^{x+\varepsilon_j} |u_j'(t)|^2\,dt\Big)$, so that

$$F_j(u_j) = \int_{-\infty}^{+\infty} \frac{1}{\varepsilon_j}\psi_j(x)\,dx = \sum_{k \in \mathbf{Z}} \int_{2k\varepsilon_j - \varepsilon_j}^{2k\varepsilon_j + \varepsilon_j} \frac{1}{\varepsilon_j}\psi_j(x)\,dx = \frac{1}{\varepsilon_j}\int_{-\varepsilon_j}^{\varepsilon_j} \Psi_j(x)\,dx,$$

where $\Psi_j(x) := \sum_{k\in\mathbf{Z}} \psi_j(x+2k\varepsilon_j)$. By the Mean Value Theorem, we can find $t_j \in (-\varepsilon_j, \varepsilon_j)$ such that

$$F_j(u_j) \geq 2\Psi_j(t_j) = 2\sum_{k\in\mathbf{Z}} f_{\varepsilon_j}\Big(\int_{(2k-1)\varepsilon_j}^{(2k+1)\varepsilon_j} |u_j'(t-t_j)|^2\,dt\Big).$$

Since the translation by t_j does not affect neither the value $F_j(u_j)$ nor the limit u, we can suppose $t_j = 0$ for all j. Hence

$$F_j(u_j) \geq 2\sum_{k\in\mathbf{Z}} f_{\varepsilon_j}\Big(\int_{(2k-1)\varepsilon_j}^{(2k+1)\varepsilon_j} |u_j'(t)|^2\,dt\Big).$$

Define

$$\xi_j(k) := \int_{(2k-1)\varepsilon_j}^{(2k+1)\varepsilon_j} |u_j'(t)|^2\,dt, \qquad k \in \mathbf{Z},$$

$$G_j := \{k \in \mathbf{Z} :\ 2\xi_j(k) \leq c_g\}, \qquad B_j := \{k \in \mathbf{Z} :\ 2\xi_j(k) > c_g\}.$$

Note that for all $x \in \mathbf{R} \setminus (2\varepsilon_j B_j + [-2\varepsilon_j, 2\varepsilon_j])$ we have

$$\int_{x-\varepsilon_j}^{x+\varepsilon_j} |u_j'(t)|^2\,dt \leq c_g.$$

Moreover, $f_{\varepsilon_j}(\xi_j(k)) \geq c_g/2$ for all $k \in B_j$, so that

$$\#(B_j) \leq \frac{2}{c_g}\sum_{k\in B_j} f_{\varepsilon_j}(\xi_j(k)) \leq \frac{1}{c_g}\sup_j F_j(u_j).$$

Hence, we can suppose that $\#(B_j) = N$, independent of j. If we write $B_j = \{k_j^1, \ldots, k_j^N\}$ with $k_j^1 < k_j^2 < \ldots < k_j^N$, then we can suppose that there exist indices i_1, i_2 such that $\varepsilon_j k_j^i \to -\infty$ for $i < i_1$, $\varepsilon_j k_j^i \to +\infty$ for $i > i_2$, and $2\varepsilon_j k_j^i \to x^i \in \mathbf{R}$ for $i_1 \leq i \leq i_2$. Let $S = \{x^i :\ i_1 \leq i \leq i_2\}$ (which is not empty if $i_1 \leq i_2$). Since

$$\begin{aligned} F_j(u_j) &\geq \frac{1}{\varepsilon_j}\int_{\mathbf{R}\setminus(2\varepsilon_j B_j+[-2\varepsilon_j,2\varepsilon_j])}\int_{x-\varepsilon_j}^{x+\varepsilon_j} |u_j'(t)|^2\,dt\,dx \\ &\geq 2\int_{\mathbf{R}\setminus(2\varepsilon_j B_j+[-2\varepsilon_j,2\varepsilon_j])} |u_j'(t)|^2\,dt \end{aligned}$$

(the last inequality is obtained by changing the integration order), we have that fixed $\eta > 0$ there exists $j(\eta)$ such that the sequence $(u_j)_{j\geq j(\eta)}$ is equibounded in $H^1([-\frac{1}{\eta},\frac{1}{\eta}]\setminus(S+[-\eta,\eta]))$, and

$$\int_{[-\frac{1}{\eta},\frac{1}{\eta}]\setminus(S+[-\eta,\eta])}|u_j'|^2\,dt\leq\sum_{k\in G_j}\int_{(2k-1)\varepsilon_j}^{(2k+1)\varepsilon_j}|u_j'|^2\,dt$$

for $j\geq j(\eta)$. It follows that $u\in SBV_{loc}(\mathbf{R})$, $S(u)\subset S$, and

$$\int_{-\infty}^{+\infty}|u'|^2\,dt\leq\liminf_j\sum_{k\in G_j}\int_{(2k-1)\varepsilon_j}^{(2k+1)\varepsilon_j}|u_j'|^2\,dt\,.$$

By the local nature of the arguments in the proofs below it will not be restrictive to suppose $x^1=\ldots=x^N=0$. Moreover, we can suppose that u_j is constant in $[(2k_j^1-3)\varepsilon_j,(2k_j^1-1)\varepsilon_j]$ and in $[(2k_j^N+1)\varepsilon_j,(2k_j^N+3)\varepsilon_j]$. This is not restrictive, up to substituting u_j by a function v_j constant on these intervals, with $v_j'=u_j'$ elsewhere, and coinciding with u_j for $t<(2k_j^1-3)\varepsilon_j$. Clearly $F_j(v_j)\leq F_j(u_j)$, and still $v_j\to u$ since

$$\|u_j-v_j\|_\infty\leq\int_{-\infty}^{+\infty}|u_j'-v_j'|\,dt\leq c_g\sqrt{\varepsilon_j}$$

(using Hölder's inequality). We can split $F_j(u_j)$ into three integrals, and we have then

$$\begin{aligned}F_j(u_j)\geq 2\int_{-\infty}^{(2k_j^1-2)\varepsilon_j}|u_j'|^2\,dt+2\int_{(2k_j^N+2)\varepsilon_j}^{-\infty}|u_j'|^2\,dt\\+\frac{1}{\varepsilon_j}\int_{(2k_j^1-2)\varepsilon_j}^{(2k_j^N+2)\varepsilon_j}f_{\varepsilon_j}\Big(\int_{x-\varepsilon_j}^{x+\varepsilon_j}|u_j'|^2\,dt\Big)\,dx.\end{aligned}$$

As for the last term, we can suppose that u_j is monotone on the interval $[(2k_j^1-2)\varepsilon_j,(2k_j^N+2)\varepsilon_j]$ and constant on the intervals $[(2k-1)\varepsilon_j,(2k+1)\varepsilon_j]$ with $k\in G_j$, $k_j^1<k<k_j^N$, up to substituting u_j by a function v_j enjoying these properties which makes the value $F_j(v_j)$ decrease and does not affect the limit u, using the same argument as above. Hence, we can also suppose that $k_j^{i+1}-k_j^i\leq 2$ for $i=1,\ldots,N-1$ so that $k_j^N-k_j^1\leq 2N+2$. Finally, by a translation argument it is not restrictive to suppose that $u_j(0)=0=u^-(0)$ for all j (we tacitly use the continuous representatives of Sobolev functions throughout the paper).

Define for all $K > 0$, $z \in \mathbf{R}$

$$\varphi_\varepsilon^K(z) = \inf\Big\{\int_0^K f_\varepsilon\Big(\frac{1}{\varepsilon}\int_{x-1}^{x+1} |v'|^2\,dt\Big)\,dx : v(0) = 0,\ v(K) = z\Big\}. \tag{2.7}$$

By the lower semicontinuity of f_ε the function φ_ε^K is itself lower semicontinuous. Note that $f_{\varepsilon_j}(\cdot/\varepsilon_j)$ converges increasingly to g; hence, $\varphi_{\varepsilon_j}^K$ converges increasingly to

$$\varphi^K(z) = \inf\Big\{\int_0^K g\Big(\int_{x-1}^{x+1} |v'|^2\,dt\Big)\,dx : v(0) = 0,\ v(K) = z\Big\}.$$

It is easy to check that $\varphi^K(z) = \varphi(z)$ for K large, and that $\varphi(z) \geq 2c_g$ if $z \neq 0$.

Let $K \geq 4N + 4$ be fixed. Define now

$$v_j(x) = \begin{cases} u_j(x + (2k_j^1 - 2)\varepsilon_j) & \text{if } x \leq 0 \\ u_j(x + (2k_j^N + 2)\varepsilon_j) & \text{if } x > 0. \end{cases}$$

Note that

$$\frac{1}{\varepsilon_j}\int_{(2k_j^1-2)\varepsilon_j}^{(2k_j^N+2)\varepsilon_j} f_{\varepsilon_j}\Big(\int_{x-\varepsilon_j}^{x+\varepsilon_j} |u_j'|^2\,dt\Big)\,dx \geq \varphi_{\varepsilon_j}^K(v_j^+(0) - v_j^-(0)),$$

using $v(t) = u_j(t/\varepsilon_j)$ as test function in (2.7), and performing a change of variables. Moreover, since we suppose $N \geq 1$ (otherwise there is nothing to prove), we have

$$\frac{1}{\varepsilon_j}\int_{(2k_j^1-2)\varepsilon_j}^{(2k_j^N+2)\varepsilon_j} f_{\varepsilon_j}\Big(\int_{x-\varepsilon_j}^{x+\varepsilon_j} |u_j'|^2\,dt\Big)\,dx \geq 2\,g\Big(\int_{2k_j^1\varepsilon_j-\varepsilon_j}^{2k_j^1\varepsilon_j+\varepsilon_j} |u_j'|^2\,dt\Big) \geq 2c_g,$$

so that

$$F_j(u_j) \geq 2\int_{-\infty}^{+\infty} |v_j'|^2\,dx + \varphi_{\varepsilon_j}^K(v_j^+(0) - v_j^-(0)) \vee 2c_g.$$

Note that $v_j \to u$ in L^1_{loc}, and $v_j^+(0) - v_j^-(0) \to u^+(0) - u^-(0)$.

Since φ_j^K is a sequence of lower semicontinuous functions converging increasingly to φ we have $\varphi(z) \leq \liminf_j \varphi_{\varepsilon_j}^K(z_j)$ for all $z_j \to z$. Hence,

$$\begin{aligned}\liminf_j F_j(u_j) &\geq \liminf_j 2\int_{-\infty}^{+\infty} |v_j'|^2\,dt + \liminf_j \varphi_{\varepsilon_j}^K(v_j^+(0) - v_j^-(0)) \\ &\geq 2\int_{-\infty}^{+\infty} |u'|^2\,dt + \varphi(u^+(0) - u^-(0)) = F_\varphi(u),\end{aligned}$$

that is (2.6).

It remains now to find a recovery sequence for $F_\varphi(u)$ when $u \in SBV_{loc}(\mathbf{R})$, $u' \in L^2(\mathbf{R})$ and $\#(S(u)) < +\infty$. It is not restrictive to suppose $S(u) = \{0\}$ since all the arguments are local. Moreover, we can suppose $u^-(0) = 0$, $u^+(0) = z$. Fix K such that $\varphi(z) = \varphi^K(z)$. For all $\varepsilon > 0$ there exists $\tilde{u}_\varepsilon \in H^1_{loc}(\mathbf{R})$ such that $\tilde{u}_\varepsilon(x) = 0$, for $x \leq 0$, $\tilde{u}_\varepsilon(x) = z$, for $x \geq K\varepsilon$, and

$$\frac{1}{\varepsilon}\int_{-\infty}^{+\infty} f_\varepsilon\Big(\int_{x-\varepsilon}^{x+\varepsilon} |\tilde{u}_\varepsilon|^2\, dt\Big)\, dx = \varphi^K_\varepsilon(z).$$

Define then

$$u_\varepsilon(x) = \begin{cases} u(x+2\varepsilon) & \text{if } x \leq -2\varepsilon \\ \tilde{u}_\varepsilon(x) & \text{if } -2\varepsilon < x < K\varepsilon + 2\varepsilon \\ u(x-2\varepsilon-K\varepsilon) & \text{if } x \geq K\varepsilon + 2\varepsilon. \end{cases}$$

We have $u_\varepsilon \to u$, and $\lim_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) = 2\int_{-\infty}^{+\infty} |u'|^2\, dt + \lim_{\varepsilon\to 0^+} \varphi^K_\varepsilon(z) = F_\varphi(u)$, as required. □

Remark 2.2. (i) It is clear from the proof that in the statement of Theorem 2.1 we can define f_ε also as

$$f_\varepsilon(\xi) = \begin{cases} \xi & \text{if } 0 \leq \xi \leq C \\ g(\varepsilon\xi) & \text{if } \xi > C, \end{cases}$$

for any $C \leq c_g$;

(ii) In the one-dimensional case, we can recover Theorem 3.1 of [1] as a corollary of our Theorem 2.1. In fact, let f be a function as in the Introduction, and let $\tilde{f}(t) = 2f(t/2)$. The functionals F_ε in (1.2) can be rewritten as

$$F_\varepsilon(u) = \frac{1}{2}\int_{-\infty}^{+\infty} \tilde{f}\Big(\int_{x-\varepsilon}^{x+\varepsilon} |u'|^2\, dt\Big)\, dx.$$

For fixed $\eta > 0$ let $0 < C \leq 1$ satisfy

$$(1-\eta)t \leq \tilde{f}(t) \leq (1+\eta)t \text{ if } 0 \leq t \leq C,$$

$$2(\lambda - \eta) \leq \tilde{f}(t) \text{ if } t \geq 1/C.$$

Define, for $\sigma > 0$

$$g^\sigma(t) = \begin{cases} 0 & \text{if } t = 0 \\ (1-\eta)C & \text{if } 0 < t \leq \sigma \\ 2(\lambda-\eta) & \text{if } t > \sigma, \end{cases} \qquad g^+(t) = \begin{cases} 0 & \text{if } t = 0 \\ 2\lambda & \text{if } t > 0, \end{cases}$$

$$f^\sigma_\varepsilon(t) = \begin{cases} (1-\eta)t & \text{if } 0 \le t \le C \\ g^\sigma(\varepsilon t) & \text{if } t > C, \end{cases} \qquad f^+_\varepsilon(t) = \begin{cases} (1+\eta)t & \text{if } 0 \le t \le C \\ 2(1+\eta)\lambda & \text{if } t > C, \end{cases}$$

We have, if $\varepsilon < C/\sigma$, $f^\sigma_\varepsilon \le \tilde f \le f^+_\varepsilon$. We can apply Theorem 2.1 to

$$F^+_\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} \frac{1}{(1+\eta)} f^+_\varepsilon\Big(\int_{x-\varepsilon}^{x+\varepsilon} |u'(t)|^2 dt\Big)\, dx,$$

(using the observation (i) above) and

$$F^\sigma_\varepsilon(t) = \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} \frac{1}{(1-\eta)} f^\sigma_\varepsilon\Big(\int_{x-\varepsilon}^{x+\varepsilon} |u'(t)|^2 dt\Big)\, dx,$$

obtaining jump energy densities φ^+ and φ^σ_η, respectively. Since

$$\begin{aligned} 4\lambda = \varphi^+(z) &= \inf\Big\{\int_{-\infty}^{+\infty} g^+\Big(\int_{x-1}^{x+1} |u'(t)|^2\, dt\Big)\, dx : u(-\infty) = 0,\ u(+\infty) = z\Big\} \\ &= \sup_{\eta>0}\sup_{\sigma>0}\inf\Big\{\int_{-\infty}^{+\infty} g^\sigma\Big(\int_{x-1}^{x+1} |u'(t)|^2\, dt\Big)\, dx : u(-\infty) = 0,\ u(+\infty) = z\Big\} \\ &= \sup_{\eta>0}\sup_{\sigma>0} \varphi^\sigma_\eta(z) \end{aligned}$$

and

$$\frac{(1-\eta)}{2} F^\sigma_\varepsilon \le F_\varepsilon \le \frac{(1+\eta)}{2} F^+_\varepsilon,$$

we get by comparison the functional in (1.1) as the Γ-limit of the functionals F_ε.

3. Estimates for the jump energy density. The jump energy density in the Γ-limit does not seem to be explicitly computable in general. We are able to give some estimates and an explicit calculation in some particular cases. We begin by computing φ when g is affine on $(0, +\infty)$.

Proposition 3.1. *If*

$$g(t) = \begin{cases} at + b & \textit{if } t > 0 \\ 0 & \textit{if } t = 0 \end{cases}$$

then

$$\varphi(z) = \begin{cases} 2\sqrt{2ab}|z| + 2b & \textit{if } z \ne 0 \\ 0 & \textit{if } z = 0. \end{cases}$$

Proof. Let $z \neq 0$ and let $w(t) = \int_{t-1}^{t+1} |u'|^2 \, dx$, where $u(x)$ is a function such that $u(-\infty) = 0$ and $u(+\infty) = z$. We can assume that the support of w is bounded and that $\operatorname{supp} w = [0, T]$, with $T \geq 2$. Thus

$$\int_{-\infty}^{+\infty} g(w(t)) \, dt = a \int_{-\infty}^{+\infty} \int_{t-1}^{t+1} |u'|^2 \, dx \, dt + bT = 2a \int_{1}^{T-1} |u'|^2 \, dx + bT \, .$$

Then we have that

$$\begin{aligned} \varphi(z) &= \min_{T\geq 2} \inf \Big\{ 2a \int_{1}^{T-1} |u'|^2 \, dx + bT \; : \; u(1) = 0 \, , \; u(T-1) = z \Big\} \\ &= \min_{T\geq 2} \{ \frac{2a}{T-2} z^2 + bT \} = 2\sqrt{2ab}|z| + 2b \, , \end{aligned}$$

as required. □

Proposition 3.2. *Let $\vartheta : \mathbf{R} \to \mathbf{R}$ be a subadditive lower semicontinuous function, increasing on $[0, +\infty)$, with $\vartheta(0) = 0$. Let $g(t) = \frac{1}{2}\vartheta(\sqrt{2t})$ for $t \geq 0$. Then $\varphi(z) \geq \vartheta(|z|)$.*

Proof. It suffices to consider $z > 0$. Let u be an increasing H^1_{loc}-function with $u(x) = 0$ for $x \leq 0$, $u(x) = z$ for $x \geq T$. Using the same "discretization argument" as in the first part of the proof of Theorem 2.1 (with $\varepsilon_j = 1$), we can suppose that

$$\int_{-\infty}^{+\infty} g(\int_{x-1}^{x+1} |u'|^2 \, dt) \, dx \geq 2 \sum_{k=0}^{[T/2]+1} g(\int_{2k-1}^{2k+1} |u'|^2 \, dt).$$

Now, clearly we have

$$\int_{2k-1}^{2k+1} |u'|^2 \, dt \geq \frac{(u(2k+1) - u(2k-1))^2}{2},$$

so that

$$g(\int_{2k-1}^{2k+1} |u'|^2 \, dt) = \frac{1}{2} \vartheta\Big(\sqrt{2 \int_{2k-1}^{2k+1} |u'|^2 \, dt}\Big) \geq \frac{1}{2} \vartheta(u(2k+1) - u(2k-1)).$$

Hence, by the subadditivity of ϑ and the fact that u is increasing,

$$\begin{aligned} \int_{-\infty}^{+\infty} g(\int_{x-1}^{x+1} |u'|^2 \, dt) \, dx &\geq \sum_{k=0}^{[T/2]+1} \vartheta(u(2k+1) - u(2k-1)) \\ &\geq \vartheta\Big(\sum_{k=0}^{[T/2]+1} \big(u(2k+1) - u(2k-1)\big)\Big) = \vartheta(z) \, , \end{aligned}$$

as desired. □

Remark 3.3. Let ϑ be any increasing positive function and let g be defined as in Proposition 3.2. Then $\varphi(z) \le 2\vartheta(|z|)$. To show this, it suffices to take $u(t) = \operatorname{sign} z\,\big((0 \vee |z|t/2) \wedge |z|\big)$ as test function in the definition of φ. If ϑ is concave, the same test function gives $\varphi(z) \le 2\vartheta(2|z|/3)$, after applying Jensen's inequality.

From Proposition 3.2 and Remark 3.3 we obtain that we can construct functions φ with prescribed growth, as precised in the following proposition.

Proposition 3.4. *Let $\vartheta : \mathbf{R} \to \mathbf{R}$ be a subadditive lower semicontinuous function, increasing on $[0, +\infty)$, with $\vartheta(0) = 0$. Let $g(t) = \frac{1}{2}\vartheta(\sqrt{2t})$ for $t \ge 0$. Then $\vartheta(|z|) \le \varphi(z) \le 2\,\vartheta(|z|)$.*

Remark 3.5. From Propositions 3.2 and 3.1 we can compute φ in the case $g(z) = \sqrt{|z|}$. In fact, from Proposition 3.2 applied with $\vartheta(t) = \sqrt{2}\,t$ we get $\varphi(z) \ge \sqrt{2}\,|z|$. On the other hand, we have for all $\alpha > 0$ $g(z) \le \frac{1}{2\sqrt{\alpha}} z + \frac{\sqrt{\alpha}}{2}$. Hence, using Proposition 3.1, $\varphi(z) \le \sqrt{2}\,|z| + \sqrt{\alpha}$. By the arbitrariness of $\alpha > 0$ we get $\varphi(z) \le \sqrt{2}\,|z|$. The same argument applies to $g(z) = c \wedge \sqrt{|z|}$, obtaining $\varphi(z) = 2c \wedge \sqrt{2}|z|$.

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