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1 Setting, motivation, and main result

The course relies on joint work with A. Gloria, S. Neukamm, and D. Marahrens. In particular, we refer to the three preprints, which are available on my web page: [3] requires the least machinery and will be closest to the course, [1] gives an extensive introduction next a couple of quantitative results, and [2] uses both to give a full error estimate.

The first definition introduces discrete differential operators on the lattice $\mathbb{Z}^d$. Vertices are typically called x, y, and z. Edges, that we think of as unoriented, are typically called b and e. If x and y are neighbors, we write $[x, y]$ for the connected edge. The unit vectors are denoted by e_i, $i = 1, \cdots, d$.

Definition 1. *For a field $u(x)$, $x \in \mathbb{Z}^d$, we define the field $\nabla u(b)$, where b runs through all edges, as*

$$\nabla u(b) = u(x + e_i) - u(x) \quad \text{if}\, b = [x, x + e_i].$$

For a field $g(b)$, where b runs through all edges b, we define the field $\nabla^* g(x)$, $x \in \mathbb{Z}^d$, as

$$\nabla^* g(x) = \sum_{i=1}^{d} (g(b_{i,-}) - g(b_{i,+})) \quad \text{if}\, b_{i,+} = [x, x + e_i] \text{ and } b_{i,-} = [x - e_i, x].$$

We think of ∇u as the gradient field of u and of $\nabla^* g$ as the (negative) divergence of g. The operator ∇^* is the ℓ^2-transpose of ∇ in the sense that for compactly supported u the following integration by parts formula holds:

$$\sum_b g(b) \nabla u(b) = \sum_x u(x)(\nabla^* g)(x).$$

We fix a number $0 < \lambda \leq 1$.

Definition 2. *We denote by Ω the space of all $a(b)$, where b runs through all edges, with the property*

$$\forall \text{ edges } b \quad \lambda \leq a(b) \leq 1.$$

For $a \in \Omega$, we will be interested in the operator $\nabla^* a \nabla$, that is defined on fields $u(x)$, $x \in \mathbb{Z}^d$, and can be thought of as a discrete elliptic operator. In this sense, we think of $a \in \Omega$ as a coefficient field $a(b)$, where b runs through all edges.

Each a describes a network of resistors, where two neighboring sites x and y are connected by a resistor of conductivity $a(b = [x, y])$, that is, of resistivity $\frac{1}{a(b)}$. Within this interpretation, a field $u(x)$, $x \in \mathbb{Z}^d$, can be interpreted as describing the potential $u(x)$ at a site x; and $j(b) = a(b)\nabla u(b)$ the ensuing current along the edge $b = [x, x + e_i]$. This current is stationary provided $\nabla^* j = 0$.

We now consider an ensemble of coefficients

Definition 3. *Endow Ω with the product topology. Denote by $\langle\cdot\rangle$ a probability measure on Ω. Note that $\mathbb{Z}^d$ acts on Ω by shift: For a "shift vector" $z \in \mathbb{Z}^d$, $a(\cdot + z)$ denotes the shifted $a \in \Omega$.*

We call $\langle\cdot\rangle$ stationary if for every $z \in \mathbb{Z}^d$, a and its shifted version $a(\cdot + z)$ have the same distribution.

We call $\langle\cdot\rangle$ ergodic if any measurable function $\zeta(a)$ that is shift invariant, that is, has the property $\zeta(a) = \zeta(a(\cdot + z))$ for all $z \in \mathbb{Z}$, is constant almost surely.

Told **Theorem 1.** *[Kozlov, Papanicolaou & Varadhan '79]*
Suppose $\langle\cdot\rangle$ is stationary and ergodic. Then there exists a symmetric $d \times d$ matrix a_{hom} with $\lambda \mathrm{id} \leq a_{hom} \leq \mathrm{id}$ and the following property: For any bounded and compactly supported function $\hat{f}(\hat{x})$, $\hat{x} \in \mathbb{R}^d$, consider

$$f_\epsilon(x) = \epsilon^2 \hat{f}(\epsilon x).$$

For $a \in \Omega$ consider the decaying solution $u_\epsilon(a; x) \overset{short}{=} u_\epsilon(x)$ of

$$\nabla^* a \nabla u_\epsilon = f_\epsilon.$$

Then we have

$$\lim_{\epsilon \downarrow 0} \left(\epsilon^d \sum_{x \in \mathbb{Z}^d} |u_\epsilon(a;x) - \hat{u}(\epsilon x)|^2 \right)^{\frac{1}{2}} = 0 \quad \text{for } \langle\cdot\rangle\text{-a. e. } a \in \Omega,$$

where $\hat{u}(\hat{x})$, $\hat{x} \in \mathbb{R}^d$, *is the decaying solution of the homogeneous (continuum) elliptic equation*

$$-\hat{\nabla} \cdot a_{hom} \hat{\nabla} \hat{u} = \hat{f}.$$

Can we obtain an error estimate? Here: At least for the *random part of the error*, that is, for the fluctuations $u_\epsilon(x) - \langle u_\epsilon(x) \rangle$.

T'

Theorem 2. *Suppose* $\langle\cdot\rangle$ *is stationary and satisfies the Logarithmic Sobolev Inequality with constant* $\rho > 0$. *For any bounded and compactly supported function* $\hat{f}(\hat{x})$, $\hat{x} \in \mathbb{R}^d$, *consider*

$$f_\epsilon(x) = \epsilon^2 \hat{f}(\epsilon x).$$

For $a \in \Omega$ *consider the decaying solution* $u_\epsilon(a;x) \overset{short}{=} u_\epsilon(x)$ *of*

$$\nabla^* a \nabla u_\epsilon = f_\epsilon.$$

i) Suppose $d > 2$. *For all* $2 \leq p < \infty$ *and* $r < \infty$ *it holds*

$$\left\langle \left(\epsilon^d \sum_{x \in \mathbb{Z}^d} |u_\epsilon(x) - \langle u_\epsilon(x) \rangle|^p \right)^{\frac{r}{p}} \right\rangle^{\frac{1}{r}} \leq C(d, \lambda, \rho, r, p, \hat{f})\epsilon.$$

ii) Suppose $d \geq 2$. *For all bounded and compactly supported function* $\hat{g}(\hat{x})$, $\hat{x} \in \mathbb{R}^d$, *and* $r < \infty$ *it holds*

$$\left\langle \left(\epsilon^d \sum_{x \in \mathbb{Z}^d} (u_\epsilon(x) - \langle u_\epsilon(x) \rangle) \hat{g}(\epsilon x) \right)^r \right\rangle^{\frac{1}{r}} \leq C(d, \lambda, \rho, r, \hat{g}, \hat{f})\epsilon^{\frac{d}{2}}.$$

[Yurinskii '86] first algebraic, but suboptimal estimates.
[Conlon & Naddaf '00] case of small ellipticity contrast, that is, $1 - \lambda \ll 1$ (more precisely $1 - \lambda \leq \frac{1}{C(d)}$).
[Gloria '12] i) suboptimal for $d > 3$

2 The Logarithmic Sobolev Inequality

Definition 4. *We say that* $\langle\cdot\rangle$ *satisfies LSI with constant* $\rho > 0$ *if for all* $\zeta(a) \geq 0$ *we have*

$$\langle\zeta \ln \frac{\zeta}{\langle\zeta\rangle}\rangle \;\leq\; \frac{1}{2\rho}\langle\frac{1}{\zeta}|\partial\zeta|^2_{\ell^2}\rangle.$$

Here $|\partial\zeta|^2_{\ell^2} := \sum_b (\frac{\partial\zeta}{\partial a(b)})^2$.

The following lemma and remark provide a first interpretation of LSI:

L2.1bis **Lemma 1.** *The definition can also be reformulated as: For all* $\zeta(a)$ *we have*

$$\langle\zeta^2 \ln\zeta^2\rangle - \langle\zeta^2\rangle \ln\langle\zeta^2\rangle \;\leq\; \frac{2}{\rho}\langle|\partial\zeta|^2_{\ell^2}\rangle. \tag{1}$$ L2.a

Suppose that $\langle\cdot\rangle$ *satisfies LSI with constant* $\rho > 0$. *Then* $\langle\cdot\rangle$ *has a Spectral Gap (SG) with constant* ρ*, meaning that for all* $\zeta(a)$ *it holds*

$$\rho\langle(\zeta - \langle\zeta\rangle)^2\rangle \;\leq\; \langle|\partial\zeta|^2_{\ell^2}\rangle. \tag{2}$$ L2.b

Formulation (1) motivates the name of *Sobolev* inequality: Like the traditional Sobolev inequality, it encodes a gain in integrability, which however is just *logarithmic*. Inequality (2) is a Poincaré inequality with mean value zero, it is called Spectral Gap, because ρ is a lower bound on the spectral gap of the operator $\sum_b \frac{\partial}{\partial a(b)}^* \frac{\partial}{\partial a(b)}$, where $\frac{\partial}{\partial a(b)}^*$ denotes the $\langle\cdot\rangle$-adjoint of $\frac{\partial}{\partial a(b)}$. This operator is the generator of a stochastic process for a called Glauber dynamics. Hence, analytically speaking, LSI is a combination of a Poincaré inequality (cf. (2)) and a Sobolev inequality.

R1.1 **Remark 1.** *Suppose that* $\langle\cdot\rangle$ *is stationary and satisfies SG with constant* $\rho > 0$. *Let* S *be a finite subset of edges and consider*

$$\zeta(a) = |S|^{-\frac{1}{2}} \sum_{b\in S} (a(b) - \langle a\rangle).$$

Then

$$\langle\zeta^2\rangle \leq \frac{1}{\rho}.$$

In this sense, SG encodes the scaling of the Central Limit Theorem. In fact, by "concentration of measure", that follows from LSI via Herbst's argument, ζ has even Gaussian moments.

There are many criteria for LSI. The probably most relevant for us is based on the tensorization principle, due to Gross, and thus applies to independently and identically distributed coefficient fields $\{a(b)\}$.

L2.2 **Lemma 2.** *[Gross '75] Let $\langle\cdot\rangle_0$ be a probability measure on $[\lambda, 1]$ that satisfies LSI with constant ρ in the sense that such that for all $\zeta(a) > 0$, $a \in [\lambda, 1]$, we have*

$$\langle\zeta \ln \frac{\zeta}{\langle\zeta\rangle_0}\rangle_0 \leq \frac{1}{2\rho}\langle\frac{1}{\zeta}|\frac{d\zeta}{da}|^2\rangle_0.$$

Let $\langle\cdot\rangle$ denote the corresponding product measure on $\{a(e)\}_{e edges}$. Then $\langle\cdot\rangle$ satisfies LSI with constant ρ.

The most elementary example of a single-edge distribution that satisfies LSI is the uniform distribution.

R1.2 **Remark 2.** *Let $\langle\cdot\rangle_0$ denote the uniform distribution on $[\lambda, 1]$. Then $\langle\cdot\rangle_0$ satisfies LSI with constant $\rho = \frac{1}{2(1-\lambda)^2}$ in the sense that for all $\zeta(a) > 0$, $a \in [\lambda, 1]$, we have*

$$\langle\zeta \ln \frac{\zeta}{\langle\zeta\rangle_0}\rangle_0 \leq \frac{1}{2\rho}\langle\frac{1}{\zeta}|\frac{d\zeta}{da}|^2\rangle_0.$$

LSI also holds in non-iid situations. Morally speaking, the dependencies between $a(b_1)$ and $a(b_0)$ should decay sufficiently fast in the distance between the edges b_1 and b_0. Loosely speaking, the correlations should be integrable as the following remark hints at.

R1.3 **Remark 3.** *Let $\langle\cdot\rangle$ satisfy LSI with constant ρ. For $\{\phi(x)\}_{x\in\mathbb{Z}^d} \in \ell^1$ consider the probability measure $\langle\cdot\rangle'$ that describes the distribution of a' defined through convolution with ϕ, i. e. $a'(b) = \sum_{x\in\mathbb{Z}^d}\phi(x)a(x+b)$, under $\langle\cdot\rangle$. Then $\langle\cdot\rangle$ satisfy LSI with constant $\frac{\rho}{|\phi|_{\ell^1}}$.*

The only way we will use LSI is through the following lemma

L2.3 **Lemma 3.** *Let $\langle\cdot\rangle$ satisfy LSI with constant ρ. Then for any random variable $\zeta(a)$, any integrability exponent $p < \infty$ and any $\delta > 0$ we have*

$$\langle|\zeta|^{2p}\rangle \leq C(\rho, p, \delta)\langle|\zeta|\rangle^{2p} + \delta\langle|\partial\zeta|_{\ell^2}^{2p}\rangle.$$

If a random variable satisfies $\langle|\partial\zeta|_{\ell^2}^{2p}\rangle \lesssim \langle|\zeta|^{2p}\rangle$, as will be the case in our application, then the preceding lemma yields the *reverse Jensen inequality* $\langle|\zeta|^{2p}\rangle \lesssim \langle|\zeta|\rangle^{2p}$.

L2.1bis

Proof of Lemma 1

Let $\zeta(a)$ be given. By an approximation argument, we may assume that ζ is bounded. W. l. o. g. we may assume $|\zeta| \leq 1$ and $\langle\zeta\rangle = 0$. For $0 < \epsilon < 1$ apply LSI with ζ replaced by $1 + \epsilon\zeta > 0$:

$$\langle(1+\epsilon\zeta)\ln(1+\epsilon\zeta)\rangle \leq \frac{1}{2\rho}\langle\frac{1}{1+\epsilon\zeta}\epsilon^2|\partial\zeta|_{\ell^2}^2\rangle. \tag{3}$$

L2.1.1bis

We note that

$$\begin{aligned} &\ln(1+\epsilon\zeta) = \epsilon\zeta - \tfrac{1}{2}(\epsilon\zeta)^2 + O(\epsilon^3),\\ \text{thus}\quad &(1+\epsilon\zeta)\ln(1+\epsilon\zeta) = \epsilon\zeta + \tfrac{1}{2}(\epsilon\zeta)^2 + O(\epsilon^3),\\ \text{thus}\quad &\langle(1+\epsilon\zeta)\ln(1+\epsilon\zeta)\rangle = \tfrac{1}{2}\epsilon^2\langle\zeta^2\rangle + O(\epsilon^3). \end{aligned}$$

Hence if we divide (3) [L2.1.1bis] by ϵ^2 and send ϵ to zero, we obtain as desired

$$\frac{1}{2}\langle\zeta^2\rangle \leq \frac{1}{2\rho}\langle|\partial\zeta|_{\ell^2}^2\rangle.$$

L2.2bis

Proof of Lemma 5

Select an enumeration $\{e_n\}_{n\in\mathbb{N}}$ of all edges. Let $\langle\cdot\rangle_n$ denote the operation of taking the expectation of $a(e_n)$ according to the (identical) distribution $\langle\cdot\rangle_0$. Define $\{\zeta_n(a)\}_{n\in\mathbb{N}_0}$ recursively by successively integrating out: $\zeta_0 := \zeta$ and

$$\zeta_n = \langle\zeta_{n-1}\rangle_n.$$

By an approximation argument, we may assume that ζ only depends on *finitely* many of the arguments $\{a(b)\}_{b \text{ edges}}$, say, only on $\{a(e_n)\}_{n=1,\cdots,N}$ for some $N \in \mathbb{N}$. Hence in particular

$$\zeta_N = \langle\zeta\rangle.$$

Therefore, we can write the l. h. s. as a finite telescoping sum:

$$\langle\zeta\ln\frac{\zeta}{\langle\zeta\rangle}\rangle = \langle\zeta\ln\zeta\rangle - \langle\zeta\rangle\ln\langle\zeta\rangle = \sum_{n=1}^{N}(\langle\zeta_{n-1}\ln\zeta_{n-1}\rangle - \langle\zeta_n\ln\zeta_n\rangle).$$

Using $\langle\cdot\rangle = \langle\langle\cdot\rangle_n\rangle$ and the definition of ζ_n we have

$$\langle\zeta_{n-1}\ln\zeta_{n-1}\rangle - \langle\zeta_n\ln\zeta_n\rangle = \langle\langle\zeta_{n-1}\ln\zeta_{n-1}\rangle_n - \langle\zeta_{n-1}\rangle_n\ln\langle\zeta_{n-1}\rangle_n\rangle.$$

We now use our assumption that the 1-d distribution $\langle\cdot\rangle_n = \langle\cdot\rangle_0$ of $a(e_n)$ satisfies LSI, which we apply to the function ζ_{n-1}:

$$\langle\zeta_{n-1}\ln\zeta_{n-1}\rangle_n - \langle\zeta_{n-1}\rangle_n\ln\langle\zeta_{n-1}\rangle_n = \langle\zeta_{n-1}\ln\frac{\zeta_{n-1}}{\langle\zeta_{n-1}\rangle_n}\rangle_n \leq \frac{1}{2\rho}\langle\frac{1}{\zeta_{n-1}}(\frac{\partial\zeta_{n-1}}{\partial a(e_n)})^2\rangle_n,$$

so that we obtain

$$\langle\zeta_{n-1}\ln\zeta_{n-1}\rangle - \langle\zeta_n\ln\zeta_n\rangle \leq \frac{1}{2\rho}\langle\frac{1}{\zeta_{n-1}}(\frac{\partial\zeta_{n-1}}{\partial a(e_n)})^2\rangle.$$

We note that $\zeta_{n-1} = \langle\zeta\rangle_{<n}$, where $\langle\cdot\rangle_{<n}$ denotes the expectation w. r. t. $\{a(e_k)\}_{k=1,\cdots,n-1}$. Hence we obtain by Cauchy-Schwarz w. r. t. $\langle\cdot\rangle_{<n}$

$$\frac{1}{\zeta_{n-1}}(\frac{\partial\zeta_{n-1}}{\partial a(e_n)})^2 = \frac{1}{\langle\zeta\rangle_{<n}}(\langle\frac{\partial\zeta}{\partial a(e_n)}\rangle_{<n})^2 \leq \langle\frac{1}{\zeta}(\frac{\partial\zeta}{\partial a(e_n)})^2\rangle_{<n},$$

so that, using $\langle\langle\cdot\rangle_{<n}\rangle = \langle\cdot\rangle$,

$$\langle\frac{1}{\zeta_{n-1}}(\frac{\partial\zeta_{n-1}}{\partial a(e_n)})^2\rangle \leq \langle\frac{1}{\zeta}(\frac{\partial\zeta}{\partial a(e_n)})^2\rangle.$$

Argument for Remark R1.1 1

By definition $\langle\zeta\rangle = 0$ and

$$\frac{\partial\zeta}{\partial a(b)} = \left\{\begin{array}{ll} |S|^{-\frac{1}{2}} & \text{if } b\in S \\ 0 & \text{else}\end{array}\right\},$$

so that $|\frac{\partial\zeta}{\partial a(b)}|^2_{\ell^2} = 1$.

Argument for Remark R1.2 2

W. l. o. g. we may assume that $\langle\zeta\rangle_0 = 1$ so that we have for the l. h. s.

$$\langle\zeta\ln\frac{\zeta}{\langle\zeta\rangle_0}\rangle_0 = \langle\zeta\ln\zeta - \zeta + 1\rangle_0.$$

Since $\zeta\ln\zeta - \zeta + 1 \leq (\zeta-1)^2$ we have

$$\langle\zeta\ln\zeta - \zeta + 1\rangle_0 \leq \sup_{a\in[\lambda,1]}(\zeta-1)^2.$$

Thanks to our 1-d situation and the uniform distribution, we have

$$\sup_{a\in[\lambda,1]}|\zeta(a)-1| \leq \sup_{a,a'\in[\lambda,1]}|\zeta(a)-\zeta(a')| \leq \int_\lambda^1 |\frac{d\zeta}{da}|da = (1-\lambda)\langle|\frac{d\zeta}{da}|\rangle_0.$$

Finally, we have by Cauchy-Schwarz

$$\langle|\frac{d\zeta}{da}|\rangle_0^2 \leq \langle\frac{1}{\zeta}|\frac{d\zeta}{da}|^2\rangle_0\langle\zeta\rangle_0 = \langle\frac{1}{\zeta}|\frac{d\zeta}{da}|^2\rangle_0.$$

Argument for Remark R1.3 3

The convolution with ϕ defines a linear map $T\colon a \mapsto a'$ on the space of coefficient fields. By definition, $\langle\cdot\rangle'$ is the push forward of $\langle\cdot\rangle$ under T. This means that for some given $\zeta(a')$ we have

$$\langle\zeta\rangle' = \langle\zeta\circ T\rangle.$$

By our assumption on $\langle\cdot\rangle$ we thus have

$$\langle\zeta\ln\frac{\zeta}{\langle\zeta\rangle'}\rangle' = \langle\zeta\circ T\ln\frac{\zeta\circ T}{\langle\zeta\circ T\rangle}\rangle \leq \frac{1}{2\rho}\langle\frac{1}{\zeta\circ T}|\partial(\zeta\circ T)|_{\ell^2}^2\rangle.$$

We note that by the chain rule $\partial(\zeta\circ T) = T^*(\partial'\zeta)\circ T$, where T^* denotes the ℓ^2-transpose of T and ∂' denotes the ℓ^2-gradient w. r. t. the a'-variable. Hence we have

$$\langle\frac{1}{\zeta\circ T}|\partial(\zeta\circ T)|_{\ell^2}^2\rangle \leq |T^*|_{\mathcal{L}(\ell^2,\ell^2)}\langle\frac{1}{\zeta\circ T}|(\partial'\zeta)\circ T)|_{\ell^2}^2\rangle = |T^*|_{\mathcal{L}(\ell^2,\ell^2)}\langle\frac{1}{\zeta}|\partial'\zeta|_{\ell^2}^2\rangle',$$

where $|T^*|_{\mathcal{L}(\ell^2,\ell_2)}$ denotes the operator norm of T^*. Since T^* is the convolution with $x\mapsto\phi(-x)$, its operator norm is indeed estimated by $|\phi|_{\ell^1}$.

Proof of Lemma L2.3 3

Let $p<\infty$ and $\delta>0$ be given. At first, we think of $\zeta(a)$ as being strictly positive. We start with the elementary real-variable estimate

$$\zeta \leq C_0(p,\delta)\zeta^{\frac{1}{2p}} + \delta(\zeta\ln\zeta - \zeta + 1),$$

(note that the last term on the r. h. s. is non-negative and vanishes only for $\zeta=1$). Taking the expectation yields

$$\langle\zeta\rangle \leq C_0(p,\delta)\langle\zeta^{\frac{1}{2p}}\rangle + \delta\langle\zeta\ln\zeta - \zeta + 1\rangle.$$

By Young's inequality we have

$$C_0(p,\delta)\langle\zeta^{\frac{1}{2p}}\rangle \le C_1(p,\delta)\langle\zeta^{\frac{1}{2p}}\rangle^{2p} + \frac{1}{2}.$$

Combining the last two estimates yields

$$\langle\zeta\rangle \le C_1(p,\delta)\langle\zeta^{\frac{1}{2p}}\rangle^{2p} + \frac{1}{2} + \delta\langle\zeta\ln\zeta - \zeta + 1\rangle.$$

In case of $\langle\zeta\rangle = 1$, this estimate can be rewritten as

$$\frac{1}{2}\langle\zeta\rangle \le C_1(p,\delta)\langle\zeta^{\frac{1}{2p}}\rangle^{2p} + \delta\langle\zeta\ln\zeta\rangle.$$

We use the latter estimate with ζ replaced by $\frac{\zeta}{\langle\zeta\rangle}$ and rearrange:

$$\frac{1}{2}\langle\zeta\rangle \le C_1(p,\delta)\langle\zeta^{\frac{1}{2p}}\rangle^{2p} + \delta\langle\zeta\ln\frac{\zeta}{\langle\zeta\rangle}\rangle.$$

We now are in the position to insert LSI:

$$\frac{1}{2}\langle\zeta\rangle \le C_1(p,\delta)\langle\zeta^{\frac{1}{2p}}\rangle^{2p} + \frac{\delta}{2\rho}\langle\frac{1}{\zeta}|\partial\zeta|_{\ell^2}^2\rangle.$$

We use the last estimate with ζ replaced by $|\zeta|^{2p}$:

$$\frac{1}{2}\langle|\zeta|^{2p}\rangle \le C_1(p,\delta)\langle|\zeta|\rangle^{2p} + \frac{\delta}{2\rho}\langle\frac{1}{|\zeta|^{2p}}|\partial|\zeta|^{2p}|_{\ell^2}^2\rangle.$$

Appealing to the chain rule in form of $|\partial|\zeta|^{2p}|_{\ell^2} = 2p|\zeta|^{2p-1}|\partial\zeta|_{\ell^{2p}}$, this turn into

$$\frac{1}{2}\langle|\zeta|^{2p}\rangle \le C_1(p,\delta)\langle|\zeta|\rangle^{2p} + 4p^2\frac{\delta}{2\rho}\langle|\zeta|^{2p-2}|\partial\zeta|_{\ell^2}^2\rangle.$$

Using Young's inequality in form of

$$4p^2\frac{\delta}{2\rho}|\zeta|^{2p-2}|\partial\zeta|_{\ell^2}^2 \le \frac{1}{4}|\zeta|^{2p} + C_2(p,\rho)\delta^p|\partial\zeta|_{\ell^2}^{2p}$$

yields

$$\frac{1}{4}\langle|\zeta|^{2p}\rangle \le C_1(p,\delta)\langle|\zeta|\rangle^{2p} + C_2(\rho,p)\delta^p\langle|\partial\zeta|_{\ell^2}^{2p}\rangle.$$

It remains to rename $4C_2(\rho,p)\delta^p$ with δ.

3 Quenched estimates on Green's function

Let us start by motivating why the Green function comes up. Recall that for some right hand side $f(x)$, we are interested in the solution $u(a;x)$ of

$$\nabla^* a \nabla u = f. \tag{4}$$ m.1

More precisely, we are interested in estimating the fluctuations $u(x) - \langle u(x) \rangle$. Focussing on the variance $\langle (u(x) - \langle u(x) \rangle)^2 \rangle$, we are lead to use SG, cf. Lemma 1:

$$\langle (u(x) - \langle u(x) \rangle)^2 \rangle \le \frac{1}{\rho} \langle \sum_b (\frac{\partial u(x)}{\partial a(e)})^2 \rangle.$$

Hence we are lead to consider the partial derivative $\frac{\partial u(x)}{\partial a(e)}$, which measures how sensitively the value $u(x)$ of the solution u of (4) evaluated at site x depends on the value $a(e)$ of the coefficient field a at edge e. This sensitivity can be expressed in terms of the elliptic Green function, cf. Definition 5 below. Indeed, applying $\frac{\partial}{\partial a(e)}$ to (4), we obtain

$$\nabla^* a \nabla \frac{\partial u}{\partial a(e)} + \nabla^* \delta(\cdot, e) \nabla u(e) = 0,$$

where $\delta(b,e)$ is the discrete Dirac distribution on edges, i. e. $\delta(e,e) = 1$ and $\delta(b,e) = 0$ for $b \neq e$. Hence we obtain the representation

$$\frac{\partial u(x)}{\partial a(e)} = -\nabla G(x,e) \nabla u(e).$$

Hence in order to carry out this program, we need decay estimates on the *gradient* of the elliptic Green's function $G(a;x,y)$. In fact, it will be important to get estimates that depend on $a \in \Omega$ only through λ. This automatically leads to the theory of Nash, De Giorgi and Moser. For our purposes, it will be most convenient to use its parabolic, i. e. Nash, version.

m.2 **Definition 5.** *Let* $\delta(x) = \left\{ \begin{array}{cc} 1 & \textit{for } x = 0 \\ 0 & \textit{else} \end{array} \right\}$ *denotes the (discrete) Dirac distribution.*

The parabolic Green's function $G(a;t,x,y) \overset{short}{=} G(t,x,y)$ *is defined as follows: For any* $a \in \Omega$, $y \in \mathbb{Z}^d$, $[0,\infty) \times \mathbb{Z}^d \ni (t,x) \mapsto G(a;t,x,y)$ *is differentiable in* t*, decaying in* x *and satisfies*

$$\begin{aligned} \partial_t G(t,x,y) + (\nabla^* a \nabla G(t,\cdot,y))(x) &= 0 \quad \textit{for } t \ge 0, \\ G(t=0,x,y) &= \delta(x-y). \end{aligned}$$

The elliptic Green's function $G(a;x,y) \overset{short}{=} G(x,y)$ *is defined as follows: For any* $a \in \Omega$, $y \in \mathbb{Z}^d$, $\mathbb{Z}^d \ni x \mapsto G(a;x,y)$ *is the decaying solution of*

$$(\nabla^* a \nabla G(\cdot, y))(x) = \delta(x-y). \qquad (5)$$ P.5

The following lemma establishes decay rates for the parabolic Green function in time and space. They are optimal in the sense that the constant-coefficient Green function would not decay at a better rate. They are "quenched" in the sense that they do not depend on $a \in \Omega$. However, in particular the gradient bound is not pointwise in space, but expresses spatial decay only in an ℓ^2-sense.

L2.1 **Lemma 4.** *[Nash '59] It holds for all* $a \in \Omega$, $\alpha < \infty$, $y \in \mathbb{Z}^d$, *and* $t \geq 0$

$$\begin{aligned} \sum_x \Big(\frac{|x-y|^2}{t+1} + 1\Big)^\alpha G^2(t,x,y) &\leq C(d,\lambda,\alpha)(t+1)^{-\frac{d}{2}}, \\ \sum_b \Big(\frac{|x(b)-y|^2}{t+1} + 1\Big)^\alpha (\nabla G)^2(t,b,y) &\leq C(d,\lambda,\alpha)(t+1)^{-\frac{d}{2}-1}, \end{aligned} \qquad (6)$$

where $x(b)$ *denotes the coordinate of the midpoint of the edge* b.

The following lemma establishes spatial decay rates for the elliptic Green function. As for Lemma 4 (L2.1), both estimates are quenched but non-pointwise. The first estimate is optimal in the sense that even for the constant-coefficient Green function, the weight exponent α can not be increased. The second estimate is not-optimal in this sense, but cannot be improved as a quenched estimate.

L2.2bis **Lemma 5.** *i) For all* $a \in \Omega$, $\alpha < \frac{d}{2} - 1$ *(α is allowed to be negative),* $y \in \mathbb{Z}^d$ *we have*

$$\sum_b (|x(b)-y|+1)^{2\alpha} (\nabla G)^2(b,y) \leq C(d,\lambda,\alpha). \qquad (7)$$ L2.2.1

ii) There exists an $\alpha > \frac{d}{2} - 1$ *only depending on* d *and* λ *such that for all edges* e *we have*

$$\sum_b (|x(b)-x(e)|+1)^{2\alpha} (\nabla\nabla G)^2(b,e) \leq C(d,\lambda), \qquad (8)$$ L2.2.3

where $\nabla\nabla G(b,e)$ *denotes the mixed derivative.*

Proof of Lemma L2.1 4

W. l. o. g. we may assume that $y = 0$; we write $G(t,x) \overset{\text{short}}{=} G(t,x,0)$. We will show the statement of the lemma in form of

$$\begin{aligned} \sum_x (|x|^\alpha + 1) G^2(t,x) &\leq C(d,\lambda,\alpha)(t+1)^{-\frac{d}{2}+\frac{\alpha}{2}}, \\ \sum_b (|x(b)|^\alpha + 1)(\nabla G)^2(t,b) &\leq C(d,\lambda,\alpha)(t+1)^{-\frac{d}{2}+\frac{\alpha}{2}-1}. \end{aligned}$$

By Hölder's inequality, it is enough to consider either $\alpha = 0$ or $\alpha > d+2$.

Step 1. The interpolation estimate

$$\left(\sum_x u^2\right)^{\frac{1}{2}} \leq C(d) \left(\sum_b (\nabla u)^2\right)^{\frac{1}{2}\frac{d}{d+2}} \left(\sum_x |u|\right)^{\frac{2}{d+2}},$$

which we use in form of

$$\sum_b (\nabla u)^2 \geq \frac{1}{C(d)} \left(\sum_x u^2\right)^{\frac{d+2}{d}} \qquad \text{for } \sum_x |u| = 1.$$

Here, we only give the argument for $d > 2$. The interpolation estimate is a combination of the discrete Sobolev estimate (inverse exponents related by $\frac{1}{2} - \frac{1}{d} = \frac{d-2}{2d}$)

$$\left(\sum_x |u|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{2d}} \leq C(d) \left(\sum_b (\nabla u)^2\right)^{\frac{1}{2}}$$

and Hölder's inequality (inverse exponents are related by $\frac{1}{2} = \theta\frac{d-2}{2d} + (1-\theta)\frac{1}{1}$ with $\theta = \frac{d}{d+2}$)

$$\left(\sum_x u^2\right)^{\frac{1}{2}} \leq C(d) \left(\sum_x |u|^{\frac{2d}{d-2}}\right)^{\frac{d-2}{2d}\frac{d}{d+2}} \left(\sum_x |u|\right)^{\frac{2}{d+2}}.$$

The discrete version of the Sobolev inequality can be derived from the continuum one by identifying u with a finite element function on a triangulation subordinate to the lattice $\mathbb{Z}^d$.

Step 2. Unweighted estimates of G and unweighted, time-integrated estimates of ∇G. We claim that

$$\sum_x G(t,x)^2 \;\leq\; C(d,\lambda)(t+1)^{-\frac{d}{2}}, \tag{9}$$

$$\int_t^\infty \sum_b a(b)(\nabla G)^2(t',b)dt' \;\leq\; C(d,\lambda)(t+1)^{-\frac{d}{2}}. \tag{10}$$

We need the three ingredients

1) From the maximum principle: $\forall\, x \in \mathbb{Z}^d \quad G(t=0,\cdot) \geq 0$ implies $\forall\, x \in \mathbb{Z}^d \quad G(t,\cdot) \geq 0$.

2) From the equation $\frac{d}{dt}\sum_x G(t,x) = 0$, so that $\sum_x G(t=0,x) = 1$ yields $\sum_x G(t,x,0) = 1$. Together with 1), we get $\sum_x |G(t,x)| = 1$.

3) From the equation $\frac{d}{dt}\sum \frac{1}{2}G^2(t,x) = -\sum_b a(b)(\nabla G)^2(t,b) \leq -\lambda \sum_b (\nabla G)^2(t,b)$, where we used $a(b) \geq \lambda$. In particular, (10) follows from (9).

Applying Step 1 to $u(x) = G(t,x)$ we obtain the differential inequality

$$\frac{d}{dt}\sum_x G^2(t,x) \leq -\frac{1}{C(d,\lambda)}\left(\sum_x G^2(t,x)\right)^{\frac{d+2}{d}},$$

which can be rewritten as

$$\frac{d}{dt}\left(\sum_x G^2(t,x)\right)^{-\frac{2}{d}} \geq \frac{1}{C(d,\lambda)}$$

and thus yields

$$\sum_x G(t,x)^2 \leq C(d,\lambda)t^{-\frac{d}{2}}.$$

Moreover, thanks to discreteness, we have

$$\sum_x G(t,x)^2 \leq \left(\sum_x |G(t,x)|\right)^2 = 1,$$

which we use for $t \leq 1$.

Step 3. Unweighted estimates on ∇G. We claim that

$$\sum_b (\nabla G)^2(t,b) \le C(d,\lambda)(t+1)^{-\frac{d}{2}-1}. \tag{11}$$ L21.3

Indeed, because of

$$\frac{d}{dt}\sum_b a(b)(\nabla G)^2(t,b) = -\sum_x (\nabla^* a\nabla G)^2(t,x) \le 0,$$

which implies

$$\lambda\sum_b (\nabla G)^2(t,b) \le \sum_b a(b)(\nabla G)^2(t,b) \le 2t^{-1}\int_{\frac{t}{2}}^t \sum_b a(b)(\nabla G)^2(t',b)dt',$$

we may upgrade the time-integrated estimate (10) in Step 2. This establishes (11) for $t \ge 1$. The range $t \le 1$ follows from discreteness:

$$\left(\sum_b (\nabla G)^2(t,b)\right)^{\frac{1}{2}} \le 2d\left(\sum_x G^2(t,x)\right)^{\frac{1}{2}} \le 2d\sum_x |G(t,x)| = 2d.$$

Step 4. Weighted estimates on G. We claim that for any $d < \alpha < \infty$

$$\sum_x (|x|^\alpha + 1)G^2(t,x) \quad\le\quad C(d,\lambda,\alpha)(t+1)^{\frac{d}{2}+\frac{\alpha}{2}}. \tag{12}$$

We start by using the equation in form of

$$\frac{d}{dt}\frac{1}{2}\sum_x (|x|^\alpha+1)G(t,x)^2 = -\sum_b a(b)\nabla G(t,b)\nabla[(|\cdot|^\alpha+1)G(t,\cdot)](b).$$

We now need the following discrete analog on of Leibniz' rule

$$\nabla(vu) = \overline{v}\nabla u + \overline{u}\nabla v,$$

where we introduced the notation $\overline{u}(b) = \frac{1}{2}(u(x)+u(y))$ if the edge b connects the vertices x and y.

$$\begin{aligned}
&\nabla G(t,b)\nabla[(|\cdot|^\alpha+1)G(t,\cdot)](b)\\
&= \overline{|\cdot|^\alpha+1}(b)(\nabla G)^2(t,b) + \nabla G(t,b)\overline{G(t,\cdot)}(b)(\nabla|\cdot|^\alpha)(b)\\
&\ge -\frac{1}{4}\frac{(G(t,\cdot)\nabla|\cdot|^\alpha)^2(b)}{\overline{|\cdot|^\alpha+1}(b)}\\
&\ge -C(\alpha)\overline{(|\cdot|^{\alpha-2}+1)G^2(t,\cdot)}(b).
\end{aligned}$$

Together with $\lambda \leq a(b) \leq 1$, and Hölder's inequality (where we use $\alpha > d \geq 2$) this yields the differential inequality

$$
\begin{aligned}
&\frac{d}{dt}\sum_x (|x|^\alpha + 1)G(t,x)^2 \\
&\leq \; C(\alpha)\sum_x (|x|^{\alpha-2} + 1)G^2(t,x) \\
&\leq \; C(\alpha)\left(\sum_x G^2(t,x)\right)^{\frac{2}{\alpha}}\left(\sum_x (|x|^\alpha + 1)G^2(t,x)\right)^{1-\frac{2}{\alpha}}.
\end{aligned}
$$

We rewrite the differential inequality as

$$
\frac{d}{dt}\left(\sum_x (|x|^\alpha + 1)G(t,x)^2\right)^{\frac{2}{\alpha}} \;\leq\; C(\alpha)\left(\sum_x G^2(t,x)\right)^{\frac{2}{\alpha}},
$$

so that we may plug in the result of Step 3:

$$
\frac{d}{dt}\left(\sum_x (|x|^\alpha + 1)G(t,x)^2\right)^{\frac{2}{\alpha}} \;\leq\; C(d,\lambda,\alpha)(t+1)^{-\frac{d}{\alpha}}.
$$

We integrate this differential inequality over $(0,t)$; since $-\frac{d}{\alpha} > -1$ and $\sum_x (|x|^\alpha + 1)G^2(t=0,x) = 1$, this yields as desired (12).

Step 5. Weighted estimates on ∇G. We claim that for any $d+2 < \alpha < \infty$

$$
\sum_b (|x(b)|^\alpha + 1)(\nabla G)^2(t,b) \;\leq\; C(d,\lambda,\alpha)(t+1)^{-\frac{d}{2}-1+\frac{\alpha}{2}}.
$$

We start by using the equation in form of

$$
\frac{d}{dt}\frac{1}{2}\sum_b \overline{|\cdot|^\alpha + 1}\, a(\nabla G(t,\cdot))^2 = -\sum_x (\nabla^* a\nabla G(t,\cdot))(\nabla^*(\overline{|\cdot|^\alpha + 1}\, a\nabla G(t,\cdot)).
$$

We note that by duality, the discrete Leibniz rule for the divergence assumes the form

$$
\nabla^*(\overline{v}g) = v\nabla^* g - \overline{g\nabla v}^*,
$$

where $\overline{g}^*$ denotes the dual operation to $\overline{v}$; it is given by d times the arithmetic mean of g over the adjacent edges to x. We obtain

$$
\begin{aligned}
&(\nabla^* a\nabla G(t,\cdot))(x)(\nabla^*(\overline{|\cdot|^\alpha+1}\, a\nabla G(t,\cdot))(x) \\
&= (|x|^\alpha+1)(\nabla^* a\nabla G(t,\cdot))^2(x) - (\nabla^* a\nabla G(t,\cdot))(x)\overline{(\nabla|\cdot|^\alpha)\, a\nabla G(t,\cdot)}^*(x) \\
&\geq -\frac{1}{4}\frac{1}{|x|^\alpha+1}\overline{(\nabla|\cdot|^\alpha)\, a\nabla G(t,\cdot)}^{*2}(x) \\
&\geq -C(d,\alpha)\overline{\overline{|\cdot|^\alpha+1}^{\frac{\alpha-2}{\alpha}}\, a(\nabla G(t,\cdot))^2}^*(x).
\end{aligned}
$$

Hence we obtain with help of Step 3

$$
\begin{aligned}
&\frac{d}{dt}\sum_b \overline{|\cdot|^\alpha+1}\, a(\nabla G(t,\cdot))^2 \\
&\leq C(d,\alpha)\sum_b \overline{|\cdot|^\alpha+1}^{\frac{\alpha-2}{\alpha}}\, a(\nabla G(t,\cdot))^2 \\
&\leq C(d,\alpha)\left(\sum_b (\nabla G)^2(t,b)\right)^{\frac{2}{\alpha}}\left(\sum_b \overline{|\cdot|^\alpha+1} a(\nabla G(t,\cdot))^2\right)^{1-\frac{2}{\alpha}} \\
&\leq C(d,\lambda,\alpha)(t+1)^{-\frac{d}{\alpha}-\frac{2}{\alpha}}\left(\sum_b \overline{|\cdot|^\alpha+1}\, a(\nabla G(t,\cdot))^2\right)^{1-\frac{2}{\alpha}},
\end{aligned}
$$

that is,

$$
\frac{d}{dt}\left(\sum_b \overline{|\cdot|^\alpha+1}\, a(\nabla G(t,\cdot))^2\right)^{\frac{2}{\alpha}} \leq C(d,\lambda,\alpha)(t+1)^{-\frac{d}{\alpha}-\frac{2}{\alpha}}.
$$

Since by assumption $-\frac{d}{\alpha}-\frac{2}{\alpha} > -1$ and $\sum_b \overline{|\cdot|^\alpha+1}\, a(\nabla G(t=0,\cdot))^2 \leq C(d,\alpha)$, we obtain upon integration as desired

$$
\begin{aligned}
\lambda\sum_b(|x(b)|^\alpha+1)(\nabla G(t,b))^2 &\leq \sum_b \overline{|\cdot|^\alpha+1}\, a(\nabla G(t,\cdot))^2 \\
&\leq C(d,\lambda,\alpha)(t+1)^{-\frac{d}{2}-1+\frac{\alpha}{2}}.
\end{aligned}
$$

Proof of Lemma L2.2 2

We only give the argument in case of $d > 2$, the case of $d = 2$ is more subtle. We start with (7); since $d > 2$ we may restrict ourselves to $\alpha \geq 0$. We use Lemma 4, more precisely (6), in form of

$$\left(\sum_b (|x(b) - y| + 1)^{2\alpha} (\nabla G)^2(t, b, y) \right)^{\frac{1}{2}} \leq C(d, \lambda, \alpha)(t + 1)^{-\frac{d}{4} - \frac{1}{2} + \frac{\alpha}{2}}.$$

Because of $G(x, y) = \int_0^\infty G(t, x, y)dt$, this yields by the triangle inequality w. r. t. ℓ_b^2

$$\left(\sum_b (|x(b) - y| + 1)^{2\alpha} (\nabla G)^2(b, y) \right)^{\frac{1}{2}} \leq C(d, \lambda, \alpha) \int_0^\infty (t + 1)^{-\frac{d}{4} - \frac{1}{2} + \frac{\alpha}{2}} dt.$$

The time integral is finite for $\alpha < \frac{d}{2} - 1$.

We now turn to (8) and only give the (easy) argument for $\alpha < \frac{d}{2} - 1$ (which is all we later need to treat the case $d > 2$): Writing $e = [y_-, y_+]$, we see that (8) follows from (7) (with $y = y_\pm$) via the triangle inequality w. r. t. ℓ_b^2.

4 Annealed estimates on Green's function and proof of main result

In this section, we bring the concepts from statistical mechanics (LSI, SG) together with the tools from elliptic/parabolic regularity theory. The first result only relies on the latter and establishes *optimal* spatial decay of the elliptic Green function. It is superior to Lemma 5 in the sense that the estimate is *pointwise* in space. But, as opposed to Lemma 5, it is not "quenched" (i. e. uniform in $a \in \Omega$), but only "annealed" in the sense that the pointwise quantities are only optimally estimated after taking the expectation.

L4.1 **Lemma 6.** *[Delmotte & Deuschel '05] Let $\langle \cdot \rangle$ be stationary. Then we have for all $y \in \mathbb{Z}^d$ and edges b and e:*

$$\begin{aligned} \langle |\nabla G(b, y)| \rangle &\leq C(d, \lambda)(|x(b) - y| + 1)^{1-d}, \\ \langle |\nabla\nabla G(b, e)| \rangle &\leq C(d, \lambda)(|x(b) - x(e)| + 1)^{-d}. \end{aligned} \tag{13}$$

The main role of the assumption of LSI for $\langle \cdot \rangle$ is to upgrade the annealed bounds of Lemma 6 to higher stochastic moments — in fact, to any stochastic moment $2p$. This way, we almost get a pointwise quenched estimate.

P **Proposition 1.** *Let $\langle\cdot\rangle$ be stationary and satisfy LSI with constant $\rho > 0$. Then we have for all $p < \infty$, $y \in \mathbb{Z}^d$ and edges b and e:*

$$\langle |\nabla G(b,y)|^{2p}\rangle^{\frac{1}{2p}} \leq C(d,\lambda)(|x(b)-y|+1)^{1-d}, \qquad (14)$$

$$\langle |\nabla\nabla G(b,e)|^{2p}\rangle^{\frac{1}{2p}} \leq C(d,\lambda)(|x(b)-x(e)|+1)^{-d}. \qquad (15)$$

Theorem T' 2 is an immediate consequence of the ϵ-free version of the following Theorem, which itself is an easy consequence of the annealed Green's function estimates of Proposition P 1.

T **Theorem 3.** *Suppose $\langle\cdot\rangle$ is stationary and satisfies the Logarithmic Sobolev Inequality with constant $\rho > 0$. For any compactly supported function $f(x)$, $x \in \mathbb{Z}^d$ and for $a \in \Omega$ consider the decaying solution $u(a;x) \overset{short}{=} u(x)$ of*

$$\nabla^* a \nabla u = f.$$

i) Suppose $d > 2$. For all $2 \leq p < \infty$ and $r < \infty$ it holds

$$\left\langle \left(\sum_x |u(x) - \langle u(x)\rangle|^p \right)^{\frac{r}{p}} \right\rangle^{\frac{1}{r}} \leq C(d,\lambda,\rho,r,p) \left(\sum_x |f(x)|^q \right)^{\frac{1}{q}},$$

where q is related to p via $\frac{1}{q} = \frac{1}{p} + \frac{1}{d}$.

ii) Suppose $d \geq 2$. For all compactly supported function $g(x)$, $x \in \mathbb{Z}^d$ and $r < \infty$ it holds

$$\left\langle \left(\sum_x (u(x) - \langle u(x)\rangle) g(x) \right)^{r} \right\rangle^{\frac{1}{r}} \leq C(d,\lambda,\rho,r,p) \left(\sum_x |g(x)|^p \right)^{\frac{1}{p}} \left(\sum_x |f(x)|^q \right)^{\frac{1}{q}}, \quad \text{T.1}$$

provided $1 < p, q < \infty$ are related by $\frac{1}{p} + \frac{1}{q} = \frac{2}{d} + \frac{1}{2}$.

Proof of Lemma L4.1 6
We will just give the argument for (L4.1.1 13), the argument for the first estimate is similar.

Step 1. Representation formula

$$\nabla\nabla G(b,e) = \int_0^\infty \sum_x \nabla G(\frac{t}{2}, b, x)\nabla G(\frac{t}{2}, e, x) dt$$

This follows from

- the relation between parabolic and elliptic Green's function

$$G(y,z) = \int_0^\infty G(t,y,z)dt,$$

- the semi group property of the parabolic Green's function

$$G(t,y,z) = \sum_x G(\frac{t}{2}, y, x)G(\frac{t}{2}, x, z),$$

- and the symmetry of the parabolic Green's function

$$G(\frac{t}{2}, x, z) = G(\frac{t}{2}, z, x).$$

It remains to take the gradient w. r. t. the variables y and z.

Step 2. From Step 1 we obtain the inequality

$$\begin{aligned} &|\nabla\nabla G(b,e)| \\ \leq\ & \int_0^\infty (\frac{|x(b)-x(e)|^2}{t+1}+1)^{-\frac{\alpha}{2}} \left(\sum_x (\frac{2|x(b)-x|^2}{t+1}+1)^\alpha (\nabla G)^2(\frac{t}{2}, b, x)\right)^{\frac{1}{2}} \\ & \times \left(\sum_x (\frac{2|x(e)-x|^2}{t+1}+1)^\alpha (\nabla G)^2(\frac{t}{2}, e, x)\right)^{\frac{1}{2}} dt. \end{aligned}$$

We start from the triangle inequality $|x(b)-x(e)| \leq |x(b)-x| + |x(e)-x|$ and its square $\frac{|x(b)-x(e)|^2}{t+1} \leq \frac{2|x(b)-x|^2}{t+1} + \frac{2|x(e)-x|^2}{t+1}$, which we use in form of

$$1 \leq (\frac{|x(b)-x(e)|^2}{t+1}+1)^{-\frac{\alpha}{2}}(\frac{2|x(b)-x|^2}{t+1}+1)^{\frac{\alpha}{2}}(\frac{2|x(e)-x|^2}{t+1}+1)^{\frac{\alpha}{2}}.$$

Hence we obtain from Step 1:

$$|\nabla\nabla G(b,e)| \leq \int_0^\infty (\frac{|x(b)-x(e)|^2}{t+1}+1)^{-\frac{\alpha}{2}}$$
$$\sum_x (\frac{2|x(b)-x|^2}{t+1}+1)^{\frac{\alpha}{2}} \nabla G(\frac{t}{2},b,x)(\frac{2|x(e)-x|^2}{t+1}+1)^{\frac{\alpha}{2}}\nabla G(\frac{t}{2},e,x)dt.$$

It remains to apply Cauchy-Schwarz in x.

Step 3. Using stationarity, we upgrade Step 2 to

$$\langle|\nabla\nabla G(b,e)|\rangle$$
$$\leq \left\langle \int_0^\infty (\frac{|x(b)-x(e)|^2}{t+1}+1)^{-\frac{\alpha}{2}} \sum_{b'} (\frac{2|x(b')|^2}{t+1}+1)^{\alpha}(\nabla G)^2(\frac{t}{2},b',0)dt \right\rangle.$$

Indeed, taking the expectation of Step 2 and using Cauchy Schwarz w. r. t. $\langle\cdot\rangle$, we obtain

$$\langle|\nabla\nabla G(b,e)|\rangle$$
$$\leq \int_0^\infty (\frac{|x(b)-x(e)|^2}{t+1}+1)^{-\frac{\alpha}{2}} \left(\sum_x (\frac{2|x(b)-x|^2}{t+1}+1)^{\alpha}\langle(\nabla G)^2(\frac{t}{2},b,x)\rangle\right)^{\frac{1}{2}}$$
$$\times \left(\sum_x (\frac{2|x(e)-x|^2}{t+1}+1)^{\alpha}\langle(\nabla G)^2(\frac{t}{2},e,x)\rangle\right)^{\frac{1}{2}} dt.$$

By stationarity of $\langle\cdot\rangle$ and G, we have

$$\langle(\nabla G)^2(a;\frac{t}{2},b,x)\rangle = \langle(\nabla G)^2(a(\cdot+x);\frac{t}{2},b,x)\rangle = \langle(\nabla G)^2(a;\frac{t}{2},b-x,0)\rangle,$$

so that by a change of variables

$$\sum_x (\frac{2|x(b)-x|^2}{t+1}+1)^{\alpha}\langle(\nabla G)^2(\frac{t}{2},b,x)\rangle \leq \sum_{b'} (\frac{2|x(b')|^2}{t+1}+1)^{\alpha}\langle(\nabla G)^2(\frac{t}{2},b',0)\rangle.$$

Step 4. Using Lemma 4, Step 3 allows to conclude. Indeed, by (6), Step 3 yields

$$\langle|\nabla\nabla G(b,e)|\rangle \;\leq\; C(d,\lambda,\alpha)\int_0^\infty (\frac{(|x(b)-x(e)|+1)^2}{t+1}+1)^{-\frac{\alpha}{2}}(t+1)^{-\frac{d}{2}-1}dt.$$

By the change of variables $\hat{t} = \frac{t+1}{(|x(b)-x(e)|+1)^2}$, this turns into

$$\langle |\nabla\nabla G(b,e)| \rangle \;\; \leq \;\; C(d,\lambda,\alpha)(|x(b)-x(e)|+1)^{-d} \int_0^\infty \frac{\hat{t}^{\frac{\alpha}{2}-\frac{d}{2}-1}}{(\hat{t}+1)^{\frac{\alpha}{2}}} d\hat{t}.$$

The last integral is finite for $\alpha > d$.

Proof of Proposition 1.
We only prove (15). The proof of (14) relies on (15) and uses similar arguments.
Step 1. Formula for partial derivatives

$$\begin{aligned} \frac{\partial}{\partial a(e)} G(x,x') &= -\nabla G(x,e) \nabla G(x',e), && (16) \\ \frac{\partial}{\partial a(e)} \nabla G(b,x') &= -\nabla\nabla G(b,e) \nabla G(x',e), \\ \frac{\partial}{\partial a(e)} \nabla G(b,b') &= -\nabla\nabla G(b,e) \nabla G(b',e). && (17) \end{aligned}$$

Taking the partial derivative of (5) w. r. t. $a(e)$, we obtain

$$\nabla^* a \nabla \frac{\partial G}{\partial a(e)}(\cdot,x') + \nabla^*(\delta(\cdot - e)\nabla G(\cdot,x')) = 0,$$

which we rewrite as

$$\nabla^* a \nabla \frac{\partial G}{\partial a(e)}(\cdot,x') = -\nabla^*(\delta(\cdot - e)\nabla G(e,x')).$$

Hence we obtain the Green's function representation

$$\begin{aligned} \frac{\partial}{\partial a(e)} G(x,x') &= -\sum_y G(x,y) \nabla^*(\delta(\cdot - e)\nabla G(e,x'))(y) \\ &= -\sum_b \nabla G(x,b)\delta(b-e)\nabla G(e,x') = -\nabla G(x,e)\nabla G(e,x'). \end{aligned}$$

Because of the symmetry of G, this yields (16). The two other identities are obtained upon differentiating (16).

Step 2. There exists $p_0(d,\lambda) < \infty$ such that for any $p \geq p_0(d,\lambda)$:

$$\begin{aligned}&(|x(b)-x(b')|+1)^{2pd}\langle|\partial\nabla\nabla G(b,b')|_{\ell^2}^{2p}\rangle\\ \leq\;& C(d,\lambda,p)\sup_{b'',b'''}(|x(b'')-x(b''')|+1)^{2pd}\langle|\nabla\nabla G(b'',b''')|^{2p}\rangle.\end{aligned}$$

We start from (17) in Step 1

$$\begin{aligned}|\partial G(b,b')|_{\ell^2}^2 &= \sum_e\Big(\frac{\partial}{\partial a(e)}\nabla\nabla G(b,b')\Big)^2\\ &\leq \sum_{e:|x(e)-x(b)|\leq|x(e)-x(b')|}(\nabla\nabla G(b,e))^2(\nabla\nabla G(b',e))^2\\ &\quad+\sum_{e:|x(e)-x(b')|\leq|x(e)-x(b)|}(\nabla\nabla G(b',e))^2(\nabla\nabla G(b,e))^2\\ &=: |\partial_+G(b,b')|_{\ell^2}^2+|\partial_-G(b,b')|_{\ell^2}^2.\end{aligned}$$

In the sequel, we only treat the term $|\partial_+G(b,b')|_{\ell^2}^2$, since the second term $|\partial_+G(b,b')|_{\ell^2}^2$ is estimated the same way, just exchanging the roles of b and b'. We now apply Hölder's inequality in e with dual exponents p and q, where we smuggle in a weight:

$$\begin{aligned}&|\partial_+\nabla\nabla G(b,b'))|_{\ell^2}^{2p}\\ &\leq \left(\sum_e|(|x(b)-x(e)|+1)^{\alpha}\nabla\nabla G(b,e)|^{2q}\right)^{p-1}\\ &\quad\times\sum_{e:|x(e)-x(b)|\leq|x(e)-x(b')|}|(|x(b)-x(e)|+1)^{-\alpha}\nabla\nabla G(b',e)|^{2p}. \qquad (18)\end{aligned}$$

Using discreteness, we have for the first factor

$$\begin{aligned}&\sum_e|(|x(b)-x(e)|+1)^{\alpha}\nabla\nabla G(b,e)|^{2q}\\ &\leq \left(\sum_e((|x(b)-x(e)|+1)^{\alpha}\nabla\nabla G(b,e))^2\right)^q \leq C(d,\lambda,p)\end{aligned}$$

by the second estimate in Lemma 5 for some $\alpha>0$ only depending on d and λ. Hence (18) turns into

$$\begin{aligned}&|\partial_+\nabla\nabla G(b,b')|_{\ell^2}^{2p}\\ &\leq C(d,\lambda,p)\sum_{e:|x(e)-x(b)|\leq|x(e)-x(b')|}(|x(b)-x(e)|+1)^{-2p\alpha}|\nabla\nabla G(b',e)|^{2p}.\end{aligned}$$

We take the expectation

$$
\begin{aligned}
&\langle|\partial_+\nabla\nabla G(b,b')|_{\ell^2}^{2p}\rangle \\
&\quad\le\; C(d,\lambda,p)\sum_{e:|x(e)-x(b)|\le|x(e)-x(b')|}(|x(b)-x(e)|+1)^{-2p\alpha}\langle|\nabla\nabla G(b',e)|^{2p}\rangle \\
&\quad\le\; C(d,\lambda,p)\sum_{e:|x(e)-x(b)|\le|x(e)-x(b')|}(|x(b)-x(e)|+1)^{-2p\alpha}(|x(b')-x(e)|+1)^{-2pd} \\
&\qquad\times\sup_{b''}(|x(b')-x(b'')|+1)^{2pd}\langle|\nabla\nabla G(b',b'')|^{2p}\rangle.
\end{aligned}
$$

We conclude by noting that because $|x(e)-x(b)|\le|x(e)-x(b')|$ implies $|x(b')-x(e)|\ge\frac{1}{2}|x(b)-x(b')|$ we have for the first factor

$$
\begin{aligned}
&\sum_{e:|x(e)-x(b')|\le|x(e)-x(b)|}(|x(b)-x(e)|+1)^{-2p\alpha}(|x(b')-x(e)|+1)^{-2pd} \\
&\quad\le\; (\frac{1}{2}|x(b)-x(b')|+1)^{-2pd}\sum_{e}(|x(b)-x(e)|+1)^{-2p\alpha},
\end{aligned}
$$

and that the last sum converges provided $p>\frac{d}{2\alpha}$.

Step 3. Conclusion for $\nabla\nabla G$, that is: For $p\ge p_0(d,\lambda)$ we have

$$
\sup_{b}(|x(b)-x(b')|+1)^{2pd}\langle|\nabla\nabla G(b,b')|^{2p}\rangle\le C(d,\lambda,\rho,p).
$$

Indeed, we apply Lemma L2.3 3 to $\zeta=G(b.b')$:

$$
\langle|\nabla\nabla G(b,b')|^{2p}\rangle\;\le\;C(\rho,p,\delta)\langle|\nabla\nabla G(b,b')|\rangle^{2p}+\delta\langle|\partial\nabla\nabla G(b,b')|_{\ell^2}^{2p}\rangle.
$$

We apply Lemma L4.1 6 to the first r. h. s. term and obtain

$$
\begin{aligned}
&\langle|\nabla\nabla G(b,b')|^{2p}\rangle \\
&\quad\le\; C(d,\lambda,\rho,p,\delta)(|x(b)-x(b')|+1)^{-2pd}+\delta\langle|\partial\nabla\nabla G(b,b')|_{\ell^2}^{2p}\rangle.
\end{aligned}
$$

We multiply by $(|x(b)-x(b')|+1)^{2pd}$ and apply Step 2 to the second r. h. s. term and obtain

$$
\begin{aligned}
&(|x(b)-x(b')|+1)^{2pd}\langle|\nabla\nabla G(b,b')|^{2p}\rangle \\
&\quad\le\; C(d,\lambda,\rho,p,\delta)+C(d,\lambda,p)\delta\sup_{b'',b'''}(|x(b'')-x(b''')|+1)^{2pd}\langle|\nabla\nabla G(b'',b''')|^{2p}\rangle
\end{aligned}
$$

and take the sup over (b,b'):

$$\begin{aligned}&\sup_{b,b'}(|x(b)-x(b')|+1)^{2pd}\langle|\nabla\nabla G(b,b')|^{2p}\rangle\\ &\quad\le\; C(d,\lambda,\rho,p,\delta)+C(d,\lambda,p)\delta\sup_{b,b'}(|x(b)-x(b')|+1)^{2pd}\langle|\nabla\nabla G(b,b')|^{2p}\rangle.\end{aligned}$$

For $\delta=\delta(d,\lambda,p)$ we may buckle.

Proof of Theorem 3.
Here, we only prove (16) for $r=2$.
Step 1. Application of SG. We claim that

$$\begin{aligned}&\left\langle\left(\sum_x(u-\langle u\rangle)g\right)^2\right\rangle\\ &\quad\le\; \frac{1}{\rho}\sum_e\sum_{x,x'}\sum_{y,y'}\langle|\nabla G(e,x)|^4\rangle^{\frac14}\langle|\nabla G(e,x')|^4\rangle^{\frac14}\langle|\nabla G(e,y)|^4\rangle^{\frac14}\langle|\nabla G(e,y')|^4\rangle^{\frac14}\\ &\qquad\times|g(x)g(x')f(y)f(y')|. \qquad (19)\end{aligned}$$

Indeed, we apply Lemma 1 to $\zeta=\sum_x ug$. Since g does not depend on a, we obtain by expanding the square

$$\left\langle\left(\sum_x(u-\langle u\rangle)g\right)^2\right\rangle\le\frac{1}{\rho}\sum_e\sum_{x,x'}\langle\frac{\partial u(x)}{\partial a(e)}\frac{\partial u(x')}{\partial a(e)}\rangle g(x)g(x'). \qquad (20)$$ T.2

We now represent $u(x)$ with help of the elliptic Green's function; since f does not depend on a, we obtain

$$\frac{\partial u(x)}{\partial a(e)}=\sum_y\frac{\partial G}{\partial a(e)}(x,y)f(y)=-\sum_y\nabla G(e,x)\nabla G(e,y)f(y),$$

where we used the formula in Step 1 of the proof of Proposition 1. Inserting this into (20) yields

$$\begin{aligned}&\left\langle\left(\sum_x(u-\langle u\rangle)g\right)^2\right\rangle\\ &\quad\le\; \frac{1}{\rho}\sum_e\sum_{x,x'}\sum_{y,y'}\langle\nabla G(e,x)\nabla G(e,x')\nabla G(e,y)\nabla G(e,y')\rangle^{\frac14}\\ &\qquad\times g(x)g(x')f(y)f(y').\end{aligned}$$

An application of Hölder's inequality w. r. t. $\langle\cdot\rangle$ yields (19).

Step 2. Application of Proposition 1. We claim that

$$\begin{aligned}&\left\langle\left(\sum_x (u-\langle u\rangle)g\right)^2\right\rangle \\ &\quad\le\; C(d,\lambda,\rho)\sum_e\left(\sum_x |x(e)-x|^{1-d}|f(x)|\right)^2\left(\sum_y |x(e)-y|^{1-d}|g(y)|\right)^2 \qquad (21)\end{aligned}$$

Indeed, inserting the estimate of Proposition 1 for $p=4$ into Step 1, one obtains

$$\begin{aligned}&\left\langle\left(\sum_x (u-\langle u\rangle)g\right)^2\right\rangle \\ &\quad\le\; C(d,\lambda,\rho)\sum_e\sum_{x,x'}\sum_{y,y'}|x(e)-x|^{1-d}|x(e)-x'|^{1-d} \\ &\qquad\times|x(e)-y|^{1-d}|x(e)-y'|^{1-d}|g(x)g(x')f(y)f(y')| \\ &\quad=\;\sum_e\left(\sum_x |x(e)-x|^{1-d}|f(x)|\right)^2\left(\sum_y |x(e)-y|^{1-d}|g(y)|\right)^2.\end{aligned}$$

Step 3. Conclusion by Hardy-Littlewood-Sobolev's inequality. Hölder's inequality applied to the last estimate yields

$$\begin{aligned}&\left\langle\left(\sum_x (u-\langle u\rangle)g\right)^2\right\rangle \\ &\quad\le\; C(d,\lambda,\rho)\left(\sum_e\left(\sum_x |x(e)-x|^{1-d}|f(x)|\right)^{\tilde q}\right)^{\frac{2}{\tilde q}}\left(\sum_e\left(\sum_y |x(e)-y|^{1-d}|g(y)|\right)^{\tilde p}\right)^{\frac{2}{\tilde p}},\end{aligned}$$

provided $\frac{1}{\tilde p}+\frac{1}{\tilde q}=\frac{1}{2}$. Estimate (21) now follows from an application of the (discrete version of the) Hardy-Littlewood-Sobolev inequality, that is,

$$\left(\sum_e\left(\sum_x |x(e)-x|^{1-d}f(x)\right)^{\tilde q}\right)^{\frac{1}{\tilde q}}\le C(d,q)\left(\sum_x |f(x)|^q\right)^{\frac{1}{q}},$$

which holds for $\frac{1}{\tilde q}+\frac{1}{d}=\frac{1}{q}$.

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