HOPF ALGEBRAS WITH TRACE AND CLEBSCH-GORDAN COEFFICIENTS

CORRADO DE CONCINI

Abstract. In this lecture I shall report on some joint work with Procesi, Reshetikhin and Rosso [1].

1. Recollections and the problem

Let G be a simply connected semisimple algebraic group over $\mathbb{C}$. $\mathfrak{g}$ =Lie G.
$T \subset G$ a maximal torus, $\mathfrak{t}$ =Lie T, R the root system, R_+ positive roots, $\Delta = \{\alpha_1, \dots, \alpha_n\}$ the simple roots, $A = (a_{i,j})$, the Cartan matrix, D =diag$(d_1, \dots, d_n)$ the diagonal matrix with DA symmetric. $B \supset T \subset B^-$ the corresponding Borel subgroup and its opposite.

Fix ℓ odd (and prime with 3 if there are G_2 components). Let ε be a primitive ℓ root of 1.

The quantized enveloping algebra is the $\mathbb{C}$-algebra $U_\varepsilon(\mathfrak{g})$ generated by elements $\{E_1, \dots E_n\}$, $\{F_1, \dots F_n\}$, K_λ, $\lambda \in X^*(T)$ the character group of T, with relations:

$$K_\lambda K_\mu = K_{\lambda+\mu}$$

$$K_\lambda E_i K_\lambda^{-1} = \varepsilon^{\langle \check{\alpha}_i, \lambda\rangle} E_i$$

$$K_\lambda F_i K_\lambda^{-1} = \varepsilon^{-\langle \check{\alpha}_i, \lambda\rangle} F_i$$

$$E_i F_j = F_j E_i \quad i \neq j, \quad [E_i, F_i] = \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{\varepsilon_i - \varepsilon_i^{-1}} \quad \varepsilon_i = \varepsilon^{d_i}$$

$$\sum \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{\varepsilon_i} E_i^{1-k-a_{ij}} E_j \, E_i^k = 0 \,, \quad i \neq j$$

$$\sum \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{\varepsilon_i} F_i^{1-k-a_{ij}} F_j \, F_i^k = 0 \,, \quad i \neq j$$

$\begin{bmatrix} m \\ h \end{bmatrix}_\varepsilon = \frac{[m]_\varepsilon!}{[m-h]_\varepsilon! [h]_\varepsilon!}$, $[h]_\varepsilon! = [h]_\varepsilon \dots [2]_\varepsilon [1]_\varepsilon$ and $[h]_\varepsilon = \frac{\varepsilon^h - \varepsilon^{-h}}{\varepsilon - \varepsilon^{-1}}$. Δ acts on generators by

$$\Delta K_\mu = K_\mu \otimes K_\mu, \quad \Delta E_i = E_i \otimes 1 + K_{\alpha_i} \otimes E_i,$$

$$\Delta F_i = F_i \otimes K_{\alpha_i}^{-1} + 1 \otimes F_i$$

The main peculiarity of being at roots of unity is that $U_\varepsilon(\mathfrak{g})$ has a very big center Z.

Indeed, $U_\varepsilon(\mathfrak{g})$ is a finite Z-module. Z is the coordinate ring of an algebraic variety X of dimension equal to dim $\mathfrak{g}$. So:

1) Every irreducible $U_\varepsilon(\mathfrak{g})$-module is finite dimensional.

2) If $\widehat{U_\varepsilon(\mathfrak{g})}$ denotes the set of irreducible $U_\varepsilon(\mathfrak{g})$-modules, taking central characters we get a surjective map

$$\gamma : \widehat{U_\varepsilon(\mathfrak{g})} \to X.$$

Problem 1. Given two irreducible $U_\varepsilon(\mathfrak{g})$-modules, V, W, describe the composition factors of $V \otimes W$ and their multiplicities.

Consider the sub Hopf algebra $U_\varepsilon(\mathfrak{b}) \subset U_\varepsilon(\mathfrak{g})$ generated by the K_λ and $E_1, \dots E_n$.

Problem 2. Given an irreducible $U_\varepsilon(\mathfrak{g})$-module, V "decompose" its restriction to $U_\varepsilon(\mathfrak{b})$.

Notice that as Problem 2, also Problem 1 is a problem of branching. In fact it consists of decomposing the restriction of the irreducible $U_\varepsilon(\mathfrak{g}) \otimes U_\varepsilon(\mathfrak{g})$-module $V \otimes W$ to the subalgebra $\Delta(U_\varepsilon(\mathfrak{g}))$.

To explain our results recall that Z contains a sub Hopf algebra Z_0 such that $U_\varepsilon(\mathfrak{g})$ is a free Z_0 module of rank $\ell^{\dim \mathfrak{g}}$. Z_0 is the coordinate ring of an algebraic group H (the Poisson dual of G) which is the kernel of the homomorphism $p : B \times B^- \to T$ defined by $p((b, b')) = p_+(b)p_-(b')^{-1}$ where $p_\pm$ are the quotients modulo the unipotent radicals. The inclusion $Z_0 \subset Z$ gives a map $\pi : X \to H$.

Theorem 1. *There is a non empty Zariski open set $\mathcal{V} \subset H \times H$ such that if V and W are two irreducible representations of $U_\varepsilon(\mathfrak{g})$, such that if $(h_1, h_2) = (\pi\gamma(V), \pi\gamma(W)) \in \mathcal{V}$, as a $U_\varepsilon(\mathfrak{g})$ module,*

$$V \otimes W = \oplus_{U \in (\pi\gamma)^{-1}(h_1 h_2)} U^m$$

with $m = \ell^{|R_+| - \mathrm{rk}\,\mathfrak{g}}$.

As for problem 2, $Z_0^+ = Z_0 \cap U_\varepsilon(\mathfrak{b})$ is a sub Hopf algebra of $U_\varepsilon(\mathfrak{b})$. Z_0^+ is the coordinate ring of B^- and the inclusion $Z_0^+ \subset Z_0$ induces the homomorphism $\mu : H \to B^-$ given by the projection on the second factor. Again, taking central character we get a map

$$\gamma' : \widehat{U_\varepsilon(\mathfrak{b})} \to B^-.$$

Theorem 2. *There is a non empty Zariski open set $\mathcal{V} \subset H$ such that if V is an irreducible representations of $U_\varepsilon(\mathfrak{g})$, such that if $h = \pi\gamma(V) \in \mathcal{V}$, as a $U_\varepsilon(\mathfrak{b})$ module,*

$$V = \oplus_{U \in (\gamma')^{-1}(\mu(h))} U^m$$

with $m = \ell^{(|R_+| - s)/2}$ where s is the number of orbits of $-w_0$ on the set of simple roots.

The main content of these theorems is that in both cases, at least generically, the multiplicities are uniformly distributed among the various irreducible components.

2. **Cayley-Hamilton algebras.**

The results we have just stated follow from a careful study of the centers of the algebras involved, by applying the theory of Cayley-Hamilton algebras which we now briefly recall.

A trace on a algebra R over a field k (here $=\mathbb{C}$) is a 1-ary operation $t : R \to R$ such that

(1) t is $k-$linear. (This implies that $t(R)$ is an $k-$subspace)
(2) $t(a)b = b\,t(a), \quad \forall a, b \in R$. (This implies that $t(R)$ is in the center of R),

(3) $t(ab) = t(ba), \quad \forall a, b \in R$. (This implies that t is 0 on the space of commutators $[R, R]$,)
(4) $t(t(a)b) = t(a)t(b), \quad \forall a, b \in R$. (This implies that $t(R)$ is an subalgebra, called the trace algebra, and that t is also $t(R)-$linear).

The most famous example is that of the algebra M_n of $n \times n$ matrices with the usual trace tr.

Algebras with trace form a category. Given (R_1, t_1), (R_2, t_2) a morphism ϕ from (R_1, t_1) to (R_2, t_2) is an algebra homomorphism such that $\phi t_1(a) = t_2\phi(a)$ for each $x \in R_1$.

A morphism $\phi : (R, t) \to (M_n, tr)$ is called a representation.

(M_n, tr) have the following extra properties:

(1) $tr(1) = n$
(2) Given $A \in M_n$, set $p_a^{(n)}(x) = \det\,(xI - A)$, the characteristic polynomial, then we get the Cayley-Hamilton identity

$$p_a^{(n)}(a) = 0$$

Remark that $p_a^{(n)}(x)$ has coefficients which are universal polynomials in $tr(a), \ldots, tr(a^n)$. For example for $n = 2$,

$$p_a^{(2)}(x) = x^2 - tr(a)x + \frac{1}{2}(tr(a)^2 - tr(a^2)).$$

Thus given (R, t) we can consider $p_a^{(n)}(x)$ for every $a \in R$ as a polynomial with coefficients in the trace algebra and define

Definition 3. *An algebra with trace (A, t) is a n-Cayley Hamilton algebra (or briefly an n-CH algebra) if*
1) $t(1) = n$
2) $p_a^{(n)}(a) = 0$ *for all* $a \in R$.

Remark that if $\rho : (R, t) \to (M_r, tr)$ is a representation of a n C-H algebra, then $r = n$ since

$$r = tr(1) = tr(\rho(1)) = \rho(t(1)) = \rho(n) = n.$$

If (R, t) n-CH algebra and h is a positive integer, (R, ht) is a hn-CH algebra and if V is a (n dimensional) (R, t)-module, $V^{\oplus h}$ is a (R, ht)-module.

The category of C-H algebras clearly contains free objects. The free n-CH algebra $F_n\langle x_1, \ldots, x_m\rangle$ on m generators is the algebra generated freely by the x_i's and by traces of monomials modulo the ideal generated by evaluating the n^{th} Cayley Hamilton identity.

3. **Representations of Cayley-Hamilton algebras.**

The following results describes the free n-CH algebra $F_n\langle x_1, \ldots, x_m\rangle$. Consider the algebra $C_n(\xi_1, \ldots, \xi_m)$ of $GL(n)$-equivariant polynomial maps of matrices $M_n(k)^m \to M_n(k)$, $\xi_i(A_1, \ldots, A_m) = A_i$, with trace $t(\psi) = tr \cdot \psi$. We have a universal map $F_n\langle x_1, \ldots, x_m\rangle \to C_n(\xi_1, \ldots, \xi_m)$

Theorem 4. *(*Procesi*)*
(1) *The universal map*

$$F_n\langle x_1, \ldots, x_m\rangle \xrightarrow{i} C_n(\xi_1, \ldots, \xi_m)$$

is an isomorphism.

(2) *The trace algebra $T_n(\xi_1, \ldots, \xi_m)$ of the algebra $C_n(\xi_1, \ldots, \xi_m)$ is the algebra of invariants of $m-$tuples of matrices and, if $m > 1$, it is the center of $C_n(\xi_1, \ldots, \xi_m)$.*

(3) *$C_n(\xi_1, \ldots, \xi_m)$ is a finite $T_n(\xi_1, \ldots, \xi_m)$ module.*

The above theorem tells us in particular that if (R, t) is a n-CH algebra which is finitely generated also its trace algebra A is finitely generated and R is a finite A-module.

It follows that every irreducible representation of R (as an algebra) is finite dimensional.

A is a ring of functions on an affine algebraic variety X, so if $\widehat{R}$ denotes the set of irreducible representations of R, we get, by taking central characters, a surjective map

$$\gamma : \widehat{R} \to X.$$

The points of X itself are a "moduli space" for a class of representations. Indeed

Theorem 5. *Given a point $x \in X$, there is a unique semisimple, trace compatible, representation W_x of the form*

$$W_x = \oplus_{V \in \gamma^{-1}(x)} V^{\oplus m_V},$$

and each semisimple, trace compatible, representation of R is of this form.

In fact there is a Zariski open set (possibly empty), called the unramified locus, in X such that if $m_x \subset A$ is the maximal ideal in A, $R/m_x R$ is semisimple and so W_x is the unique representation "lying over x" which is compatible with the trace.

If A is the center of R and R satisfies suitable conditions, for example it is a domain, there is a non empty Zariski open set for which W_x is irreducible.

In general however, the determination of $\gamma^{-1}(x)$ and of the multiplicities m_V is a very hard problem.

If we consider (R, ht), with h a positive integer, then the semisimple, trace compatible, representation corresponding to $x \in X$ with respect to the new trace is just $W_x^{\oplus h}$.

4. **Reduced trace**

In our example of quantized enveloping algebra we have no trace, so if we want to use the results about the CH algebras we need to find ways to canonically introduce a trace.

Here is a way of doing this under some assumptions.

A prime algebra R is an algebra in which the product of two non-zero ideals is non-zero. Let R be a prime algebra, assume that A is a subalgebra of the center and R is an $A-$module of finite type. Then:

1) A is an integral domain.

2) R is a torsion free module.

So if F is the field of fractions of A, then $R \subset R \otimes_A F$ and $S := R \otimes_A F$ is a finite dimensional simple algebra isomorphic to $M_k(D)$ with D a finite dimensional division ring.

Denote by Z the center of S and of D. Then $\dim_Z D = h^2$.

If $\overline{Z}$ is an algebraic closure of Z, $M_k(D) \otimes_Z \overline{Z} = M_{hk}(\overline{Z})$. Setting $p := [Z : F] = \dim_F Z$ we also have

$$S \otimes_F \overline{Z} = M_k(D) \otimes_F \overline{Z} = M_{hk}(\overline{Z})^{\oplus p}$$

The number $n := hkp$ is called *degree* of S over F. Consider the $F-$linear operator $a^L : S \to S$, $a^L(b) := ab$. We define the reduced trace

$$t_{S/F}(a) = \frac{1}{hk} tr(a^L).$$

This theorem tells us that, provided A is integrally closed, we have constructed a nice trace:

Theorem 6. *If $S = R \otimes_A F$ as before and A is integrally closed we have that the reduced trace $t_{S/F}$ maps R into A, so we will denote by $t_{R/A}$ the induced trace. The algebras R and S with their reduced trace are $n-$Cayley Hamilton algebras.*

Even more important for us is the fact that, in the situation above, the nice trace is essentially unique. Indeed,

Theorem 7. *Under the same circumstances of Theorem 6, if $\tau : R \to A$ is any trace for which R is an $m-$Cayley Hamilton algebra then there is a positive integer r for which:*

$$m = rn, \qquad \tau = r\, t_{R/A}$$

Notice that in particular, this implies that if x is a point in X and W_x is the unique semisimple representation above x, compatible with $t_{R/A}$ then $W_x^{\oplus r}$ is the unique semisimple representation above x compatible with τ.

5. Compatibility

Since our final goal is to analyze the restriction of certain representation of an algebra R to a subalgebra, what I am going to explain now is how the reduced trace of R relative to a subring A restricts to a subalgebra.

Assume that we have two prime algebras $R_1 \subset R_2$ with two central subrings $A_1 \subset R_1$, $A_2 \subset R_2$, $A_1 \subset A_2$ such that R_1 is a finite A_1 module, R_2 is a finite A_2 module and A_1, A_2 are integrally closed. We have the reduced traces t_{R_1/A_1} and t_{R_2/A_2} for which both algebras are Cayley-Hamilton. We need a criterion ensuring that the restriction of t_{R_2/A_2} to R_1 is a multiple of t_{R_1/A_1} (in this case we say that they are compatible). The criterion is the following,

Theorem 8. *Denote by Z_1 the center of R_1. If the algebra $Z_1 \otimes_{A_1} A_2$ is a domain there is a positive integer r such that*

$$r\, t_{R_1/A_1} = t_{R_2/A_2} \quad \text{on} \quad R_1.$$

6. Back to quantum algebras

In this final section I am going to explain how the theory of Cayley Hamilton algebras can be applied to the quantum group situation described in Section 1.

As a first step we need to show that quantized enveloping algebras have reduced traces. This is not hard. One knows that both $U_\varepsilon(\mathfrak{g})$ and $U_\varepsilon(\mathfrak{b})$ are domains and hence prime and that their centers Z, Z' are integrally closed. Z_0 and Z_0^+ are clearly integrally closed, so we have reduced traces

$$t_{U_\varepsilon(\mathfrak{g})/Z_0} \quad t_{U_\varepsilon(\mathfrak{g})/Z'} \quad t_{U_\varepsilon(\mathfrak{g})^{\otimes 2}/Z^{\otimes 2}} \quad t_{U_\varepsilon(\mathfrak{b})/Z_0^+}.$$

Next we have to compare them.

We start with Problem 1 and the inclusion $\Delta(U_\varepsilon(\mathfrak{g})) \subset U_\varepsilon(\mathfrak{g}) \otimes U_\varepsilon(\mathfrak{g})$. Unfortunately we have that $\Delta(Z)$ is not contained in $Z \otimes Z$. On the other hand the fact that Z_0 is a sub Hopf algebra means that $\Delta(Z_0) \subset Z_0 \otimes Z_0 \subset Z \otimes Z$. So, we may try to compare $t_{U_\varepsilon(\mathfrak{g})^{\otimes 2}/Z^{\otimes 2}}$ with $t_{U_\varepsilon(\mathfrak{g})/Z_0}$.

In order to use Theorem 8 we need to show that the variety $\tilde{X}$ defined as the fiber product

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\phi"] \arrow[d, "q"'] & & X \arrow[d, "\pi"'] \\ X \times X \arrow[r, "\pi\times\pi"] & H \times H \arrow[r, "m"] & H. \end{tikzcd}$$

with $m : H \times H \to H$ the group multiplication, is irreducible.

Let us recall that X is obtained as the fiber product

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[d, "\pi"'] & & T/W \arrow[d, "\ell"'] \\ H \arrow[r, "\rho"] & G \arrow[r, "\sigma"] & G//G \cong T/W. \end{tikzcd}$$

Where for $(h, k) \in H$, $\rho((h, k)) = hk^{-1}$, σ is the quotient modulo the adjoint action, W is the Weyl group and the map ℓ is the map induced by the homomorphism $t \to t^\ell$ of the torus T on W-invariants.

Putting the two diagrams together we obtain $\tilde{X}$ as the fiber product

$$\begin{tikzcd} \tilde{X} \arrow[r, "p\phi"] \arrow[d, "q"'] & T/W \arrow[d, "\ell"'] \\ X \times X \arrow[r, "\sigma\cdot\rho\cdot m\cdot\pi\times\pi"] & T/W. \end{tikzcd}$$

From this one proves:

Step 1. $\tilde{X}$ is a complete intersection over $X \times X$ which is Cohen-Macaulay so it is Cohen-Macaulay.

Step 2. $\tilde{X}$ is non singular in codimension 1 so in particular it is reduced and normal.

Step 3. $\tilde{X}$ is connected. This is the hardest step and one needs to use a variation of Steinberg section for the quotient $G \to G//G$ ($G//G$ denoted the categorical quotient defined using invariant polynomial functions).

Using the irreducibility of $\tilde{X}$ we deduce that the restriction to $\Delta(U_\varepsilon(\mathfrak{g}))$ of $t_{U_\varepsilon(\mathfrak{g})^{\otimes 2}/Z^{\otimes 2}}$ is $mt_{U_\varepsilon(\mathfrak{g})/Z_0}$ with $m = \ell^{|R_+|-\mathrm{rk}\,\mathfrak{g}}$ and Theorem 1 follows.

The open set $\mathcal{V} \subset H \times H$ consists of the set of pairs $((h_1, k_1), (h_2, k_2))$ with $h_1k_1^{-1}$, $h_2k_2^{-1}$, $h_1h_2k_2^{-1}k_1^{-1}$ all regular semisimple.

The solution to Problem 2 is quite similar with the extra difficulty that one has to compute the center of $U_\varepsilon(\mathfrak{b})$ first since this has not been determined before. We have seen that Z_0^+ is the coordinate ring of the group B^-. For a dominant weight λ, consider the irreducible G-module V_λ of highest weight λ. Take a highest weight vector $v \in V_\lambda$, a highest weight vector ϕ and a lowest vector ψ in V_λ^*. Define the function $f_\lambda(b) = \phi(bv)\psi(bv)$ for $b \in B^-$. These functions generate a subring $\mathcal{Z}$ in Z_0^+, isomorphic to the polynomial ring $\mathbb{C}[f_{\omega_1}, \dots, f_{\omega_r}]$, ω_i the fundamental weights.

Embed $\mathcal{Z}$ into the polynomial ring

$$\mathcal{Z}' = \mathbb{C}[f_{\omega_1}^{1/\ell}, \dots, f_{\omega_r}^{1/\ell}].$$

$\mathcal{Z}$ is the ring of invariants in $\mathcal{Z}'$ under an obvious action of the group $\mathbb{Z}/\ell\mathbb{Z}$ which we can consider as the group T_ℓ of ℓ-torsion points in T. Set $\Gamma = \{t \in T_\ell |\ t^{w_0} = t^{-1}\}$. and $Z_1^+ = (\mathcal{Z}')^\Gamma$.

Theorem 9. *The center of the algebra $U_\varepsilon(\mathfrak{b})$ is isomorphic to $Z^+ = Z_0^+ \otimes_{\mathcal{Z}} Z_1^+$.*

Once we know the center of $U_\varepsilon(\mathfrak{b})$, in order to solve Problem 2 we need to compare $t_{U_\varepsilon(\mathfrak{g})/Z}$ and $t_{U_\varepsilon(\mathfrak{b})/Z_0^+}$. In order to use Theorem 8 we need to show that the fiber product $\tilde{Y}$

$$\begin{tikzcd} \tilde{Y} \arrow[rr] \arrow[d, "q"'] & & Y \arrow[d, "\ell"] \\ X \arrow[r, "\pi"] & H \arrow[r, "p_-"] & B^- \end{tikzcd}$$

with Y=Spec Z^+, is a irreducible variety. This is done as in the previous case. After these facts have been established the proof of Theorem 2 follows rather easily.

7. Final remarks and problems

The technique explained can be used in other cases. One example is the so called quantized function algebra $\mathbb{F}_\varepsilon[G]$. In this case the role played by H is played by G and we have a projection $\pi : \widehat{F_\varepsilon[G]} \to G$.

Theorem 10. *There is a non empty Zariski open set $\mathcal{V} \subset G \times G$ such that if V and W are two irreducible representations of $\mathbb{F}_\varepsilon[G]$, such that if $(g_1, g_2) = (\pi(V), \pi(W)) \in \mathcal{V}$, as a $\mathbb{F}_\varepsilon[G]$ module,*

$$V \otimes W = \oplus_{U \in (\pi)^{-1}(g_1 g_2)} U^m$$

with $m = \ell^{|R_+| - \mathrm{rk}\,\mathfrak{g}}$.

References

[1] De Concini, C.; Procesi, C.; Reshetikhin, N.; Rosso, M. *Hopf algebras with trace and representations.* Invent. Math. 161 (2005), no. 1, 1–44.