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The MOD 2 Cohomology of the Orthogonal Groups over a Finite Field

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INTRODUCTION

The purpose of this paper is to generalize the results of Quillen [10] about the cohomology of (the classifying space of) the general linear groups over a finite field to the orthogonal case.

In this paper we will restrict ourselves to the study of the cohomology with mod 2 coefficients of (the classifying space of) the orthogonal groups.

While in [5] we gave a computation of the cohomology ring of the split orthogonal groups in the case of the ground field k having $q = 4m + 1$ elements, in this paper we also cover the $q = 4m + 3$ and the nonsplit cases.

The computation of the mod p cohomology with p odd and different from the characteristic of k, is basically simpler. It had been announced by Quillen in his Nice talk [8], as a consequence of his study of the etale homotopy types of algebraic varietes. He also announced partial results for the mod 2 case. The details have never appeared. The proof that we give here for the mod 2 case applies, with no essential modifications, essentially by substituting the Stiefel–Whitney classes with the mod p Pontrjagin classes.

In [12] Fiedorowicz and Priddy have announced the computation of the homology of the orthogonal groups over k.

The proof that we give follows the general lines of the one given by Quillen for the general linear case. There are in our case some obstacles which did not appear in Quillen's proof, expecially considering the fact that for a finite group the first KO-theory group $KO^{-1}(BG)$ is not necessarily zero.

This problem does not arise in mod p computations when p is odd, therefore making the computations in this case considerably simpler.

We now give a summary of parts of the paper. In Section 1 we define a diagram that we call the fundamental diagram, which is the starting point for all of our work. This diagram contains a Cartesian square in which the space $\widetilde{FO}\psi^q$ in the upper left corner is the real analog of Quillen's $F\psi^q$. The other piece of the diagram is obtained by using results about the Brauer lifting of a modular representation in an orthogonal setting. Section 2 is devoted to the computation of the mod 2 cohomology of $\widetilde{FO}\psi^q$. It contains five different sub-

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sections. In the first subsection we give a rough computation of $H^*(\widetilde{FO\psi^q}, Z_2)$. In the second we use the fundamental diagram to define some elements u_i in $H^*(\widetilde{FO\psi^q}, Z_2)$ which will be fundamental in the sequel. Unfortunately their definition depends on choices made in the definition of the fundamental diagram. In the third subsection we consider the u_i's relative to a particular choice and we compute a multiplicative formula for them (the corollary to Proposition 3). In the fourth we find an explicit base for $H^*(\widetilde{FO\psi^q}, Z_2)$ (Theorem 1). Finally in the fifth we compute the value of u_i^2 (Theorem 2) and this together with the results of the fourth section give us the complete structure of the algebra $H^*(\widetilde{FO\psi^q}, Z_2)$ (Theorem 3). In the third Section we study the homology of $\widetilde{FO\psi^q}$.

In the last Section we apply our results on $\widetilde{FO\psi^q}$ plus the fact that the mod 2 cohomology of the orthogonal groups over finite fields of odd characteristic is detected by suitable subgroup to give our main result (Theorem 6).

Finally in the Appendix we give the proofs of two results which are well known but do not seem to have ever been published.

1. The Fundamental Diagram

In this section we want to establish the existence of the diagram

$$\begin{array}{ccccccc} & & & \widetilde{FO\psi^q} & \xrightarrow{\gamma'} & BO' \\ & & \nearrow^{\pi} & \downarrow & & \downarrow{\scriptstyle \Delta'} \\ BQ & \xrightarrow[j]{} & BO(k) & \xrightarrow[\pi']{} & BO & \xrightarrow[(\sigma, id)']{} & B \end{array}$$

with the following properties: BO and BO' are two homotopy equivalent spaces representing the functor $\widetilde{KO}$ on compact spaces. $BO(k)$ (resp. BQ) is the classifying space of the group $O(k)$ (resp. Q) obtained as the infinite union of the orthogonal groups $O_n(k)$ (resp. their subgroups of diagonal matrices $Q(n)$) of the vector space k^n with bilinear form $\sum_i x_i y_i$ under the inclusions $O_n(k) \subset O_{n+1}(k)$ (resp. $Q(n) \subset Q(n+1)$) induced by the inclusion of k^n in k^{n+1} as the subspace with the last coordinate equal to zero. The map j is the map induced by the union of the inclusions $\bar{j}_n : Q(n) \subset O_n(k)$. The square,

$$\begin{array}{ccc} FO & \xrightarrow{\gamma'} & BO' \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle \Delta'} \\ BO & \xrightarrow[(\sigma, id)']{} & B \end{array}$$

is Cartesian, and its vertical lines are fibrations with fiber ΩBSO, BSO being the universal double covering of BO. The other spaces and maps will be defined in the sequel.

We shall begin with the Cartesian square. Let BO be a fixed classifying space, for example, the infinite real Grassmanian, for $\widetilde{KO}$. We have (see the Appendix)

LEMMA A1. *Let $N((\widetilde{KO})^n, \widetilde{KO})$ denote the set of natural transformations $(\widetilde{KO})^n \to \widetilde{KO}$. Then,*

$$N((\widetilde{KO})^n, \widetilde{KO}) = [BO^n, BO].$$

Now let q be an odd integer and let,

$$\sigma\colon BO \to BO$$

represent the Adams operation ψ^q in $\widetilde{KO}$.

We define the homotopy theoretical fixpoint set of ψ^q as the fiber product

$$\begin{array}{ccc} FO\psi^q & \xrightarrow{\ \gamma\ } & BO^I \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Delta} \\ BO & \xrightarrow[(\sigma, id)]{} & BO \times BO \end{array}$$

where Δ is the map which sends each path to its endpoints.

We want to define a slightly different space from $FO\psi^q$ which will be more useful for our purposes.

It is well known that $H^*(BO, Z_2) \simeq Z_2[w_1, w_2, \ldots]$ where the w_i's are the universal Stiefel–Whitney classes and so, by the Kunneth formula, we have, $H^*(BO \times BO, Z_2) = Z_2[w_1', w_2'', w_2', \ldots]$ with $w_i'^{('')} = p_{1(2)}^*(w_i)$, where p_1 (resp. p_2) denote the projection onto the first (resp. the second) factor.

Now let us define B to be the total space of the double covering of $BO \times BO$ associated to the element $w_1' + w_1'' \in H^1(BO \times BO, Z_2)$:

Immediately it can be seen that

$$H^*(B, Z_2) = H^*(BO \times BO, Z_2)/(w_1' + w_1'').$$

Now consider the map $BO \xrightarrow[(\sigma, id)]{} BO^2$. Since q is odd we have that σ^* is equal to the identity in mod 2 cohomology, so, we have $(\sigma, id)^*(w_1' + w_1'') = 0$.

This implies that there exists $(\sigma, id)'\colon BO \to B$ such that the diagram

$$\begin{array}{ccc} & & B \\ & {\scriptstyle(\sigma, id)'}\nearrow & \downarrow{\scriptstyle f} \\ BO & \xrightarrow[(\sigma, id)]{} & BO \times BO \end{array}$$

commutes.

Now let us consider the maps $BO \times BO \xrightarrow[d]{} BO$ representing the difference operation in $\widetilde{KO}$. Fixing a base point $b \in BO$, we can define d, using the homotopy extension theorem, in such a way that $d(x, x) = b$ and $d(x, b) = d(b, x) = x$, $\forall x \in BO$.

If we define $m\colon BO^I \to BO^I \times_{BO}\{b\}$ to be the map which sends the path p to the path $t \to d(p(t), p(1))$ which joins $d\Delta(p) = d(p(0), p(1))$ to the base point, we get a diagram

$$\begin{array}{ccccc} FO\psi^q & \xrightarrow{\gamma} & BO^I & \xrightarrow{m} & BO^I \times_{BO}\{b\} \\ {\scriptstyle\varphi}\downarrow & & \downarrow & & \downarrow{\scriptstyle h} \\ BO & \xrightarrow[(id,\sigma)]{} & BO \times BO & \xrightarrow[d]{} & BO \end{array}$$

which is commutative and in which all the vertical lines are fibrations with the same fiber ΩBO. So BO^I is homotopy equivalent to $(BO^I \times_{BO}\{b\}) \times_{BO} (BO \times BO)$, and we identify BO^I with this space.

Now let us consider the universal double covering of BO

$$\begin{array}{c} BSO \\ \downarrow{\scriptstyle k} \\ BO \end{array}$$

We have that, since the map $d\Delta$ is nullhomotopic, $(d\Delta)^*(w_1) = 0$, this means $d^*(w_1) \in \operatorname{Ker} \Delta^* = \{0, w_1' + w_1''\}$ since Δ is homotopic to the diagonal.

So $(df)^*(w_1) = 0$ and so there exists d' such that the following diagram

$$\begin{array}{ccc} B & \xrightarrow{d'} & BSO \\ \downarrow & & \downarrow{\scriptstyle k} \\ BO \times BO & \xrightarrow[d]{} & BO \end{array}$$

commutes.

Now consider the fiber product

$$\begin{array}{ccc} Z & \xrightarrow{b} & BSO^I \times_{BSO}\{b'\} \\ {\scriptstyle \Delta''}\downarrow & & \downarrow{\scriptstyle \bar{h}} \\ B & \longrightarrow & BSO \end{array}$$

where b' is chosen such that $k(b') = b$.

PROPOSITION 1. *Z is homotopy equivalent to BO^I.*

Proof. If we consider the two diagrams

$$\begin{array}{ccc} & & BSO^I \times_{BSO}\{b'\} \\ & & \big\downarrow \bar{h} \\ B & \xrightarrow[d']{} & BSO \end{array}$$

and

$$\begin{array}{ccc} & & BO^I \times_{BO}\{b\} \\ & & \big\downarrow h \\ BO \times BO & \xrightarrow[d]{} & BO \end{array}$$

using k and d' we can easily define a map from the first to the second, so a map a: $Z \to BO^I$ is defined.

Now, since the map $BO^I \xrightarrow[\Delta]{} BO^2$ is homotopic to the diagonal, it clearly lifts to a map $BO^I \xrightarrow[\Delta']{} B$.

In order to have a map $f: BO^I \to BSO \times_{BSO}\{b'\}$ such that the diagram

$$\begin{array}{ccc} BO^I & \xrightarrow{s} & BSO^I \times_{BSO}\{b'\} \\ \Delta' \big\downarrow & & \big\downarrow \bar{h} \\ B & \xrightarrow[d']{} & BSO \end{array}$$

commutes, we have to prove that $d'\Delta'$ is nullhomotopic. But now let us choose an homotopy preserving the base points

$$BO^I \times I \xrightarrow[H]{} BO$$

between $d\Delta$ and the constant map.

The obstruction for lifting such a homotopy to a homotopy between $d'\Delta'$ and the constant map lies in $H^1(BO^I \times I, BO^I \times \{0\} \cup BO^I \times \{1\} \cup \{b\} \times I, Z_2) = H^0(BO^I, b, Z_2) = 0$.

So $d'\Delta'$ is homotopic to the constant map and we can lift it to $BSO^I \times_{BSO}\{b\}$, thus proving the existence of f and getting a map τ: $BO^I \to Z$.

Now it is clear that $a\tau$: $BO^I \to Z \to BO^I$ is equal to the identity of BO^I. Vice versa, for τa: $Z \to BO^I \to Z$, we get $\Delta'\tau a \sim \Delta'$ by a homotopy T because both are liftings of the same map $h^2\Delta'$ and, reasoning as before, we have that the obstruction for these maps to be homotopic lies in $H^1(Z \times I, Z \times \{0\} \cup Z \times \{1\} \cup \{b''\} \times I) = 0$, where $b'' \in Z$ is a base point and all the maps are chosen to be base-point preserving.

Again if $\bar{T}$ is the homotopy $d'T$ then $\bar{T}$ is clearly nullhomotopic as a map $\bar{T}$: $Z \times I \to BSO$ so it lifts to a $\bar{T}'$: $Z \times I \to BSO \times_{BSO} I\{b\}$.

It follows that, using the homotopy extension theorem, we can define $\bar{T}'$ in such a way that $h\bar{T}'/Z \times \{0\} = h\tau a$ and $h\bar{T}'/Z \times \{1\} = h$. This implies that using the universal properties of fiber product we can define a homotopy $\bar{T}\colon Z \times I \to Z$ such that $\bar{T}/Z \times \{0\} = \tau a$ and $\bar{T}/Z \times \{1\} = id$, thus proving the proposition. Q.E.D.

We now define $\widetilde{FO\psi^q}$ to be the fiber product

$$\begin{array}{ccc} \widetilde{FO\psi^q} & \xrightarrow{\gamma'} & Z \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \Delta''} \\ BO & \xrightarrow[(\sigma, id)']{} & B. \end{array}$$

Note. In view of Proposition 1 we shall denote Z by BO' and the fibration Δ'' by Δ'.

We now pass to the remaining part of the fundamental diagram. It is well known [6, 9] that, given a finite group G, the Brauer lifting allows us to associate to any n-dimensional orthogonal representation of G over the algebraic closure of k, $\bar{k}$, a well-defined element in $[BG, BO]$.

Further, if the representation under consideration is obtained by extension of scalars, from a representation over k, the induced homotopy class in $[BG, BO]$ lies in $[BG, BO]^{\psi^q}$, i.e., the invariants under the action of the Adams operation ψ^q. Now let, for every positive integer s, $k_{(s)} \supset k$ be the extension of k with q^s elements, then $\bar{k} = \bigcup_s k_{(s)}$ under the canonical inclusions. Such inclusions induce inclusions $O_n(k_{(s)}) \subset O_n(k_{(s+1)})$; further, in the same fashion as for $O_n(k)$ we have inclusions $O_n(k_{(s)}) \subset O_{n+1}(k_{(s)})$. We define

$$O(\bar{k}) = \bigcup_{(n,s)} O_n(k_{(s)})$$

under the above inclusions.

We have (see the Appendix)

Lemma A2. $[BO(\bar{k}), BO] = \varprojlim_{(n,s)} [BO_n(k_{(s)}), BO]$.

If we put $O(k_{(s)}) = \bigcup_n O_n(k_{(s)})$, we get the

Corollary. $[BO(k_{(s)}), BO] = \varprojlim_n [BO_n(k_{(s)}), BO]$.

If we consider the canonical n-dimensional orthogonal representation over $k_{(s)}$ of $O_n(k_{(s)})$ we have already shown how to associate to such a representation an element of $[BO_n(k_{(s)}), BO]$, let us call it $\pi_n^{(s)}$.

Further, if we consider the inclusion $O_n(k_{(s)})\ O_{n'}(k_{(s')})$ for $n \leqslant n'$, $s \leqslant s'$, we can associate to this inclusion an element in $[BO_n(k_{(s)}), BO_{n'}(k_{(s')})]$, let us call it $\pi_{(n,n')}^{(s,s')}$. It follows immediately, by compatibility, that we have

$$\pi_{(n)}^{(s)} = \pi_{(n,n')}^{(s,s')} \pi_{(n')}^{(s')}$$

as elements of $[BO_n(k_{(s)}), BO]$.

LEMMA 1. (i) *The sequence* $\{\pi^{(s)}_{(n)}\}_{(n,s)}$ *defines a unique element* $\pi \in [BO(\bar{k}), BO]$.

(ii) *The sequence* $\pi^{(s)}_{(n)}$ *defines a unique element* $\pi_{(s)} \in [BO(k_{(s)}), BO]^{\psi^q}$ *for each s.*

Proof. (i) is clear by Lemma A2.

(ii) follows from the corollary and the fact that $\pi^{(s)}_{(n)} \in [BO_n(k_{(s)}), BO)^{\psi^{q^s}}$ for each n. Q.E.D.

Note. It is clear by unicity that if $\pi^{(s)} \in [BO(k_{(s)}), BO(\bar{k})]$ denotes the element associated to the inclusion $O(k_{(s)}) \subset O(\bar{k})$, we have $\pi_{(s)} = \pi^{(s)}\pi$.

In particular Lemma 1 gives us an element $\pi' = \pi_{(1)} \in [BO(k), BO]$.

Thus, in order to complete the construction of the fundamental diagram we only have to define a map $\bar{\pi}: BO(k) \to FO\psi^q$, satisfying the relation $\varphi'\bar{\pi} = \pi'$. In order to do so, let us consider the composite map

$$BO(k) \xrightarrow[\pi']{} BO \xrightarrow[(\sigma, id)']{} B \xrightarrow[d']{} BSO$$

then this map can be proved to be nullhomotopic by reasoning as in the proof of Proposition 1. So, $d'(\sigma, id)'\pi'$ lifts to $BSO^I \times_{BSO} \{b\}$ thus defining a map $\bar{\pi}: BO(k) \to \widetilde{FO}\psi^q$, and completing the construction of the fundamental diagram.

Remark. Two liftings of the map π' are not necessarily homotopic so that the homotopy class of $\bar{\pi}$ is not uniquely defined.

2. THE COHOMOLOGY OF $\widetilde{FO}\psi^q$

2.1. *A First Computation of* $H^*(\widetilde{FO}\psi^q, Z_2)$

From now on, given any space X, $H^*(X)$ will denote the mod 2 cohomology of X.

PROPOSITION 2. *For a suitable filtration of the ring* $H^*(\widetilde{FO}\psi^q)$ *we have,*

$$\text{gr } H^*(\widetilde{FO}\psi^q) = Z_2[w_1, w_2, \ldots] \otimes \Lambda[u_2, u_3, \ldots]$$

with $\deg(w_i) = i$ *and* $\deg(u_i) = i - 1$.

In particular the Poincaré series of $H^*(\widetilde{FO}\psi^q)$ *is*

$$\prod_{i=1}^{\infty} (1 + t^i)/(1 - t^i).$$

Proof. We consider the square

$$\text{(a)} \qquad \begin{array}{ccc} \widetilde{FO\psi^q} & \xrightarrow{\gamma'} & BO' \\ {\scriptstyle \varphi'}\big\downarrow & & \big\downarrow{\scriptstyle \Delta'} \\ BO & \xrightarrow[(\sigma,id)']{} & B \end{array}$$

of the preceding paragraph.

In order to apply the result in [9] asserting that, given a fiber square

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & Y \end{array}$$

where the vertical lines are fibrations, and Y is simply connected, there exists a spectral sequence $\{E_r\} \Rightarrow H^*(X \times_Y Z)$ such that $E_2 \cong \mathrm{Tor}^{H^*(Y)}(H^*(Z), H^*(X))$, we should have B simply connected; but it is easy to see that the proof in [9] goes over verbatim in the weaker hypothesis that the fibration $X \to Y$ is orientable, i.e., if the action of $\pi_1(X)$ over the homology of the fiber is trivial.

The fibration $BO^I \xrightarrow{\Delta} B$ is clearly orientable since it is induced by the fibration $BSO^I \times_{BSO}\{b'\} \to BSO$ which has a simply connected base space.

The above discussion implies that we have an Eilenberg–Moore spectral sequence $\{E_r\} \Rightarrow H^*(\widetilde{FO\psi^q})$ with

$$E_2^{s,*} \cong \mathrm{Tor}_{-s}^{H^*(B)}(H^*(BO), H^*(BO')).$$

We have $H^*(B) = Z_2[w_1, w_2', w_2'', \ldots]$ with $w_1 = f^*(w_1') = f^*(w_1'')$ and $w'^{('')} = f^*(w'^{('')})$ for each $i \geqslant 2$.

Since q is odd we have already noted that σ^* acts as the identity in cohomology and since Δ (resp. $(\sigma, id)'$) is a lifting to B of Δ (resp. (id, σ)), we have $(\sigma, id)'^*(w_i') = (\sigma, id)'^*(w_i'') = \Delta'(w_i') = \Delta'(w_i'') = w_i$ for $i \geqslant 2$ and $(\sigma, id)'^*(w_1) = \Delta'^*(_1) = w_1$.

This means that $(\sigma, id)'^*$ and Δ'^* define the same $H^*(B)$ module structure on the two isomorphic groups $H^*(BO)$ and $H^*(BO')$, and that they are both equal, as $H^*(B)$ modules, to the module $H^*(B)/I$; where I is the ideal generated by $w_i' + w_i''$ for $i \geqslant 2$.

Now let A_1 and A_2 be the two subrings of $H^*(B)$ generated, respectively, by

$$w_1' + w_i'' \qquad \text{for} \quad i \geqslant 2,$$

$$w_1 \text{ and } w_1' \qquad \text{for} \quad i \geqslant 2.$$

We have

$$H^*(B) = A_1 \otimes A_2,$$

$$H^*(B)/I = A_2.$$

Then, by the Kunneth formula [5], we have

$$E_2 = \mathrm{Tor}^{A_1 \otimes A_2}(A_2, A_2) = \mathrm{Tor}^{A_1}(Z_2, Z_2) \otimes A_2 .$$

Since A_1 is a polynomial algebra with generators in degrees 2, 3,..., we have [5]

$$\mathrm{Tor}^{A_1}(Z_2, Z_2) = \Lambda[u_2, u_3, \ldots]$$

with $\deg(u_i) = i - 1$.

This implies

$$E_2 = Z_2[w_1, w_2, \ldots] \otimes \Lambda[u_2, u_3, \ldots]$$

with $w_i \in E_2^{0,i}$ and $u_i \in E_2^{-1,i-1}$. Since E_2 is generated be elements in $E_2^{0,*}$ and $E_2^{-1,*}$, and on these the differentials are all zero, we get $E_2 = E_\infty$ and hence the result. Q.E.D.

2.2. *The Elements u_i*

Let us now take under consideration the fundamental diagram of Section 1:

$$\begin{array}{ccccccc} & & & & \widetilde{FO\psi^q} & \xrightarrow{\gamma'} & BO' \\ & & & \overset{\bar\pi}{\nearrow} & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\Delta'} \\ BQ & \xrightarrow{j} & BO(k) & \xrightarrow{\pi'} & BO & \xrightarrow{(\sigma, id)'} & B. \end{array}$$

We have that, since the symmetric group on n letters acts on $Q(n)$ by permuting the entries on the diagonal, there is an action of the infinite symmetric group Σ_∞ on Q inducing an action on $H^*(BQ)$.

LEMMA 2. (1) *The homomorphism φ: $H^*(BO) \to H^*(\widetilde{FO\psi^q})$ is injective.* (2) *The homomorphism $(\bar\pi j)^*$ maps $H^*(\widetilde{FO\psi^q})$ onto $H^*(BQ)^{\Sigma_\infty}$, the invariants under the action of Σ_∞.*

Proof. It is easily seen that the map$(\pi' j)^*$ maps $H^*(BO)$ onto $H^*(BQ)^{\Sigma_\infty}$ injectively. Indeed, if we consider the element in $[BQ(n), BO]$ induced by the Brauer lifting of the inclusion of $Q(n)$ in $O_n(k)$, it is immediately seen that such an element is the same as the one induced by the inclusion of $Q(n)$ in O_n as the subgroup of diagonal matrices. Thus, passing to Q, our assertion follows from well-known facts on the cohomology of BO.

Since $\pi' j = \varphi' \bar\pi j$, this gives the first part of the lemma and also proves that $\mathrm{Im}(\bar\pi j)^* \supset H^*(BQ)$.

But now, since, for each n, the normalizer of $Q(n)$ in $O_n(k)$ acts by conjugation on $Q(n)$ by permuting the entries on the diagonal, it is clear that $\mathrm{Im}\, j^* \subset H^*(BQ)^{\Sigma_\infty}$, thus implying that also $\mathrm{Im}(\bar\pi j)^* \subset H^*(BQ)^{\Sigma_\infty}$ and proving the lemma. Q.E.D.

Now if we consider the diagram

$$\begin{array}{ccc} FO & \xrightarrow{\gamma'} & BO' \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \Delta'} \\ BO & \xrightarrow[(\sigma, id)']{} & B \end{array}$$

we can define, by diagram chasing (see [10] for details), a homomorphism (Γ': $\varphi \to \Delta'$ denotes the couple of maps (γ', $(\sigma id)'$)):

$$\begin{array}{c} D_{\Gamma'}\colon \ker(H^n(B) \longrightarrow H^n(BO) \otimes H^n(BO')) \\ \downarrow \\ H^{n-1}(\widetilde{FO\psi^q})/\varphi^*(H^{n-1}(BO)) + \gamma'^*(H^{n-1}(BO')), \end{array}$$

which is an $H^*(B)$-module homomorphism and is such that, if $u \in \ker(\sigma, id)'^*$ and $v \in \ker(\Delta')^*$, $D_{\Gamma'}(uv) = 0$. Further, since Δ'^* and $(\sigma, id)'^*$ are onto, we have

$$\frac{H^*(\widetilde{FO\psi^q})}{\varphi'(H^*(BO)) + \gamma^*(H^*(BO'))} = \frac{H^*(\widetilde{FO\psi^q})}{\varphi'(H^*(BO))}.$$

Now, let us take for each $t \geqslant 2$, the elements $w_t' + w_t'' \in H^t(B)$. Since such elements lie in the kernel of $(\Delta'^*, (\sigma, id)'^*)$, we can consider $D_{\Gamma'}(w_t' + w_t'') = \tilde{u}_t \in H^{t-1}(\widetilde{FO\psi^q})/\varphi^*(H^{t-1}(BO))$.

It is clear by Lemma 6 that there is only one element in the lateral class u_t which is in the kernel of $(\bar{\pi} j)^*$.

So we can give the following

DEFINITION. For each t the elements ${}_{\bar{\pi}}u_t \in H^{t-1}(\widetilde{FO\psi^q})$, $t \geqslant 2$, are defined as the unique elements in the lateral classes $\tilde{u}_t$ such that $(\bar{\pi} j_t)^* \, ({}_{\bar{\pi}}u_t) = 0$.

Remarks. (1) By putting a subscript $\bar{\pi}$ under u_t we want to emphasize the fact that the construction of the $\{{}_{\bar{\pi}}u_t\}$ depends on the choice of $\bar{\pi}$.

(2) We have defined the $\{{}_{\bar{\pi}}u_t\}$ in $H^*(\widetilde{FO\psi^q})$ only when q is the order of a finite field of odd characteristic (i.e., $q = p^a$ for some odd prime p).

The case of any odd integer can be treated in the same way since the role of $O(k)$ in the above discussion is irrelevant, because we could have studied directly the elements in $[BQ(n), \widetilde{FO\psi^q}]$, which again are not uniquely defined, that arise in any case from $j_n \in [BQ(n), BO]$, j_n depending only by the diagonal representation of $Q(n)$ in $O(n)$.

2.3 *Multiplicative formulas*

Let $O_n(\bar{k})$ be the nth orthogonal group of the vector space $\bar{k}^n$ with bilinear form $\sum_{i=1}^n x_i y_i$.

If $k_{(s)}$ are defined, for each s, as in Section 4 we clearly have $O_n(\bar{k}) = \bigcup_{s=1}^{\infty} O_n(k_{(s)})$.

Let $x \to x^q$ be the qth Frobenius automorphism in $\bar{k}$, and let $\bar{F}_n : O_n(\bar{k})\circlearrowleft$ be the automorphism of $O_n(\bar{k})$ defined by

$$\bar{F}_n(a_{ij}) = (a_{ij}^q)$$

where $(a_{ij}) = A$ denotes an $n \times n$ matrix in $O_n(\bar{k})$.

If $G_n \subset O_n(\bar{k}) \times O_n(\bar{k})$ is the kernel of the homomorphism d: $O_n(\bar{k}) \times O_n(\bar{k}) \to \{-1, 1\}$ defined as $d(A, B) = \det A \det B$, let $\Delta_n : O_n(\bar{k}) \to G_n$ be the homomorphism defined as $\bar{\Delta}_n(A) = (A, A)$, and let $\bar{F}_n' : O_n(\bar{k}) \to G_n$ be the homomorphism defined as $\bar{F}_n'(A) = (\bar{F}_n(A), A)$.

Now let us consider a map $\Delta_n : BO_n(\bar{k}) \to BG_n$ (resp. $F_n : BO_n(\bar{k}) \to BG_n$) representing the element in $[BO_n(\bar{k}), BG_n]$ associated to Δ_n (resp. $\bar{F}_n'$). Further, since Δ_n is an inclusion let us choose Δ_n to be a fibration with fiber $G_n/O_n(\bar{k})$.

We define X_n to be the fiber product

$$\begin{array}{ccc} X_n & \xrightarrow{\gamma_n} & BO_n(\bar{k}) \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Delta_n} \\ BO_n(\bar{k}) & \xrightarrow[F_n]{} & BG_n . \end{array}$$

PROPOSITION 3. *$\pi_i(X_n) = 0$ if $i \neq 1$ $\pi_1(X_n) = O_n(k)$. Thus X_n is a classifying space for $O_n(k)$.*

Proof. Let us take base points in $BO_n(\bar{k})$ and BG_n so that F_n and Δ_n are based maps (this is possible since we can vary F_n up to homotopy).

It follows that also φ_n and γ_n can be considered as base-point preserving maps.

Now since Δ_n is a fibration we have that the map δ: $\pi_1(BG_n) \to \pi_0(G_{n/O_n(\bar{k})})$ is just the map which assigns to an element $(A, B) \in G_n = \pi_1(BG_n)$ its left lateral class modulo $O_n(\bar{k})$.

But, given an element $(A, B) \in G_n$, we have that

$$\delta(A, B) = \delta(AB^{-1}, 1).$$

So δ factors through the map $\boldsymbol{\delta}$: $G_n \to SO_n(\bar{k})$, which assigns to each $(A, B) \in G_n$ the element $AB^{-1} \in SO_n(\bar{k})$ and the map $\underline{\boldsymbol{\delta}}$: $SO_n(\bar{k}) \to G_{n/O_n(\bar{k})}$ which assigns to each $A \in SO_n(\bar{k})$ the lateral class $[(A, 1)] \in G_{n/O_n(\bar{k})}$.

The map $\underline{\boldsymbol{\delta}}$ is clearly bijective.

Now let us consider the map of homotopy exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(X_n) & \longrightarrow & \pi_1(BO_n(\bar{k})) & \longrightarrow & \pi_0(G_{n/O_n(\bar{k})}) & \longrightarrow & \pi_0(X) \longrightarrow 0 \\ \downarrow & & \downarrow{\scriptstyle \gamma_{n\neq}} & & \downarrow{\scriptstyle F_{n\neq}} & & \downarrow{\scriptstyle id} & & \downarrow \\ 0 & \longrightarrow & \pi_1(BO_n(\bar{k})) & \longrightarrow & \pi_1(BG_n) & \longrightarrow & \pi_0(G_{n/O_n(\bar{k}),id}) & \longrightarrow & 0 \end{array} .$$

We have $F_{n\#} = \bar{F}_n : \pi_1(BO_n(\bar{k})) \simeq O_n(\bar{k}) \to \pi_1(BG_n) \simeq G_n$. Since $\varphi_{n\#}$ is injective, we have that $\pi_1(X)$ is isomorphic to the subgroup of $O_n(\bar{k})$ which is mapped by $\bar{F}_n$ into the kernel of δ. But this is exactly the subgroup of matrices $A \in O_n(\bar{k})$ such that $F_n(A) = A$, i.e., the subgroup of the matrices with entries in k, $O_n(k)$.

So we have proved $\pi_1(X_n) \simeq O_n(k)$.

Since we have $\delta' = \delta F_{n\#}$ and since $\delta = \underline{\boldsymbol{\delta}}\boldsymbol{\delta}$ and $\underline{\boldsymbol{\delta}}$ is bijective it is sufficient to prove that $\boldsymbol{\delta} F_{n\#} : O_n(\bar{k}) \to SO_n(\bar{k})$ is onto.

We have

$$\boldsymbol{\delta} F_{n\#}(A) = A^F A^{-1}$$

and $\delta_n F_n$ is onto since, by the Lang isomorphism [6], the restriction of $\boldsymbol{\delta} F_{n\#}$ to $SO_n(\bar{k})$ is onto.

So we have proved $\pi_0(X_n) = 0$.

Now $\pi_i(X_n) = 0$ for $i \geqslant 2$ follows from the homotopy exact sequence of the fibration

$$\begin{array}{c} G_{n/O_N(\bar{k})} \\ \downarrow \\ X_n \\ \downarrow \\ BO_n(\bar{k}) \end{array}$$

since $\pi_i(G_{n/O_n(\bar{k})}) = 0$ for $i \geqslant 1$ and $\pi_i(BO_n(\bar{k})) = 0$ for $i \geqslant 2$. Q.E.D.

Since X_n is a classifying space for $O_n(k)$ we shall denote it by $BO_n(k)$.

Now let us consider the groups $O(\bar{k})$, $O(k)$ which have already been defined, and $G = \bigcup_n G_n$. Clearly $O(\bar{k}) = O_n(\bar{k})$. Since the F_n's are compatible we can define a homomorphism $\tilde{F}: O(\bar{k}) \to O(\bar{k})$ by taking $\tilde{F} = \bigcup_n \bar{F}_n$ and also a homomorphism $\bar{F}': O(\bar{k}) \to G$ which is the union of the $\{\bar{F}_n'\}$.

Similarly we can define the homomorphism $\bar{\Delta} = \bigcup_n \bar{\Delta}_n : O(\bar{k}) \to G$. Now let us denote by $\Delta: BO(\bar{k}) \to BG$ the fibration induced by $\bar{\Delta}$ with fiber $G/O(\bar{k})$, and let $F: BO(\bar{k}) \to BG$ be a map in the homotopy class of $[BO(\bar{k}), BG]$ induced by $\bar{F}'$.

We define X to be the fiber product

$$\begin{array}{ccc} X & \xrightarrow{\nu} & BO(\bar{k}) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\Delta} \\ BO(\bar{k}) & \xrightarrow[F]{} & BG \end{array}.$$

It follows immediately from Proposition 4, by passing to the limit that X is a classifying space for $O(k)$. In view of this we shall denote X by $BO(k)$.

Now let us consider the element $\pi \in [BO(k), BO]$.
We have

THEOREM [7].

$$H^*(BO(\bar{k})) \simeq Z_2[\bar{w}_1, \bar{w}_2, \ldots] \qquad \textit{where} \quad \bar{w}_i = \pi^*(w_i).$$

Using this theorem we get that

$$H^*(BG) \simeq H^*(BO(\bar{k}) \times BO(\bar{k}))/(\bar{w}_1' + \bar{w}_1'')$$

with $\bar{w}'^{('')} = \mathrm{pr}_{1(2)}(\bar{w}_i)$, where pr_i is the projection of $BO(\bar{k}) \times BO(\bar{k})$ on the ith factor $(i = 1, 2)$.

Now let us consider the element $\eta \in [BG, BO \times BO]$ defined as $\bar{\eta} = (\pi \times \pi)\alpha$ where $\alpha \in [BG, BO(\bar{k}) \times BO(\bar{k})]$ denotes the element associated to the inclusion of G into $O(\bar{k}) \times O(\bar{k})$.

If we take a map $\bar{e}\colon BG \to BO \times BO$ in the homotopy class $\bar{\eta}$ it is trivial from the above that there exists a map $e\colon BG \to B$ such that the diagram

$$\begin{array}{ccc} & & B \\ & {}^{e}\nearrow & \downarrow f \\ BG & \xrightarrow[\bar{e}]{} & BO \times BO \end{array}$$

where f denotes, as in Section 1, the double covering, commutes.

It also follows, since the Brauer lifting is additive, that, if we consider the composite map $e\Delta\colon BO(\bar{k}) \to B$, then there exists a map $a\colon BO(\bar{k}) \to BO'$ such that the following diagram

$$\begin{array}{ccc} & & BO' \\ & {}^{a}\nearrow & \downarrow \Delta' \\ BO(\bar{k}) & \xrightarrow[e]{} & B \end{array}$$

Δ' being the fibration with fiber ΩBSO, commutes. Thus we can define the following commutative diagram

$$\begin{array}{ccccc} BO(k) & \xrightarrow{\gamma} & BO(\bar{k}) & \xrightarrow{a} & BO' \\ \varphi\downarrow & & \downarrow\Delta & & \downarrow\Delta' \\ BO(\bar{k}) & \xrightarrow[F]{} & BG & \xrightarrow[e]{} & B. \end{array} \qquad (*)$$

Now, easy computations and Lemma A2 provide

$$[eF] = [(\sigma, id)'\pi]$$

as elements of $[BO(\bar{k}), B]$. Thus we can choose a homotopy $H_t : BO(\bar{k}) \times I \to B$ such that $H_0 = eF$ and $H_1 = (\sigma, id)'\pi$ where $\tilde{\pi}$ is a representative for the class $\pi \in [BO(\bar{k}), BO]$.

If we apply H_t it follows, by the covering homotopy theorem that there exists a homotopy $H_t' \colon BO(k) \times I \to BO'$ such that $H_0' = a\gamma$ and H_t' covers H_t for each $t \in I$.

At the end of these homotopies the diagram $(*)$ will be transformed into the diagram

$$\begin{array}{ccccc} BO(k) & \xrightarrow{\pi} & \widetilde{FO\psi^q} & \xrightarrow{\gamma'} & BO' \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi'}\downarrow & & \downarrow \\ BO(\bar{k}) & \xrightarrow[\tilde{\pi}]{} & BO & \xrightarrow[(\sigma, id)']{} & B \end{array}$$

in fact it follows immediately by the universal property of fiber product that H_1' factors through γ'.

It follows from Lemma 1 and the note under it, that using the notation of Section 1, $[\varphi'\tilde{\tilde{\pi}}] = [\hat{\pi}\varphi] = \pi'$.

So we have that we can define the elements ${}_{[\tilde{\pi}]}u_i$ $H^{i-1}(\widetilde{FO\psi^q})$, for each $i \geqslant 2$.

Note. Since from now on we shall consider only the elements ${}_{\bar{\pi}}u_i$ with $\bar{\pi} = \tilde{\tilde{\pi}}$ we shall put ${}_{[\tilde{\pi}]}u_i = u_i$.

Now let us take up the notation of Section 5, we have

LEMMA 3. (i) *The homomorphism* $\varphi^* \colon H^*(BO(\bar{k})) \to H^*(BO(k))$ *is into.*

(ii) *The homomorphism* $j_n^* \colon H^*(BO(k)) \to H^*(BQ)$ *maps* $H^*(BO)$ *onto* $H^*(BQ)^{\Sigma_\infty}$.

Proof. By the theorem, the proof proceeds exactly as the proof of Lemma 2.

Now, by reasoning as in number 2 we can define, for each $t \geqslant 2$, the element $\bar{u}_t \in H^{t-1}(BO(k))$ as the unique elements in the lateral class $D_\Gamma(\bar{w}'_{t+1} + \bar{w}''_{t+1})$ such that $j^*(\bar{u}_t) = 0$, where we put $\Gamma \colon \varphi \to \Delta$ equal to the couple of maps(γ, F).

Since from the construction of $\tilde{\pi}$ and from the fact that $D_{\Gamma(\Gamma')}$ clearly depends on the homotopy class of $\Gamma(\Gamma')$, it follows that $D_\Gamma e^* = \tilde{\tilde{\pi}}^* D_{\Gamma'}$ as maps from $\ker((\sigma, id)'^*, \Delta'^*)$ to $\operatorname{Coker}(\varphi^*)$.

LEMMA 4. $\tilde{\tilde{\pi}}^*(u_t) = \bar{u}_t$.

Proof. The lemma is an immediate consequence of the definition of the u_t's and $\bar{u}_t$'s and of the relation $D_\Gamma e^* = \tilde{\tilde{\pi}}^* D_{\Gamma'}$. Q.E.D.

Now let us consider the homomorphism $m \colon O(\bar{k}) \times O(\bar{k}) \to O(\bar{k})$ defined as the union of the direct sum homomorphism $m_{(n,t)} : O_n(\bar{k}) \times O_t(\bar{k}) \to O_{n+t}(\bar{k})$. By the definition of G_n we have that, if we consider the restriction $v_{(n,t)}$ of the homomorphism $m_{(n,t)} \times m_{(n,t)} : (O_n(\bar{k}) \times O_t(\bar{k}))^2 \to (O_{n+t}(\bar{k}))^2$ to the subgroup $G_n \times G_t$ we get $\operatorname{Im} v_{(n,t)} \subset G_{n+t}$.

Further, it can be immediately verified that the following diagram

$$\begin{array}{ccc} O_n(\bar{k}) \times O_t(\bar{k}) & \xrightarrow{m_{(n,t)}} & O_{n+t}(\bar{k}) \\ \downarrow{\scriptstyle F_n' \times F_t'} & & \downarrow{\scriptstyle F'_{n+t}} \\ G_n \times G_t & \xrightarrow[v_{(n,t)}]{} & G_{n+t} \end{array}$$

commutes.

So this implies that the diagram

$$\begin{array}{ccc} O(\bar{k}) \times O(\bar{k}) & \xrightarrow{m} & O(\bar{k}) \\ \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F'} \\ G \times G & \xrightarrow[v]{} & G \end{array}$$

where v: $G \times G \to G$ is defined as the union of the $v_{(n,t)}$'s, commutes.

By taking representatives for the homotopy classes of maps induced by the homomorphisms in the above diagram, we get a diagram

$$(\tau_1) \qquad \begin{array}{ccc} BO(\bar{k}) \times BO(\bar{k}) & \xrightarrow{\tilde{m}} & BO(\bar{k}) \\ \downarrow{\scriptstyle F \times F} & & \downarrow{\scriptstyle F} \\ BG \times BG & \xrightarrow[\tilde{v}]{} & BG \end{array}$$

which is commutative up to homotopy.

Similarly, we get the homotopy commutative diagram

$$(\tau_2) \qquad \begin{array}{ccc} BO(\bar{k}) \times BO(\bar{k}) & \xrightarrow{\tilde{m}} & BO(\bar{k}) \\ \downarrow{\scriptstyle \Delta \times \Delta} & & \downarrow \\ BG \times BG & \xrightarrow[\tilde{v}]{} & BG. \end{array}$$

Since in this case we have chosen Δ to be a fibration we can make (τ_2) into a commutative diagram by the covering homotopy theorem. So, from now on we fix $\tilde{m}$ and $\tilde{v}$ in such a way that (τ_2) is commutative.

Now let us consider the diagram

$$(\Omega) \qquad \begin{array}{ccccc} BO(k) \times BO(k) & \longrightarrow & BO(\bar{k}) \times BO(\bar{k}) & \xrightarrow{\tilde{m}} & BO(\bar{k}) \\ \downarrow{\scriptstyle \varphi \times \varphi} & & \downarrow{\scriptstyle \Delta \times \Delta} & & \downarrow{\scriptstyle \Delta} \\ BO(\bar{k}) \times BO(\bar{k}) & \xrightarrow[F \times F]{} & BG \times BG & \xrightarrow[\tilde{v}]{} & BG \end{array}$$

which is commutative by the above discussion; and let us choose a homotopy

$H_t : BO(\bar{k}) \times BO(\bar{k}) \times I \to BG$ such that $H_0 = \tilde{v}(F \times F)$ and $H_1 = F\tilde{m}$. By the covering homotopy theorem there exists a homotopy $L_t : BO(k) \times BO(k) \times I \to BO(\bar{k})$ covering H_t. So, at the end of these homotopies, the above diagram will be transformed in the commutative diagram

$$(\Omega_2) \qquad \begin{array}{ccccc} BO(k) \times BO(k) & \xrightarrow{\mu} & BO(k) & \xrightarrow{\gamma} & BO(\bar{k}) \\ \downarrow{\scriptstyle \varphi \times \varphi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \Delta} \\ BO(\bar{k}) \times BO(\bar{k}) & \xrightarrow[m]{} & BO(\bar{k}) & \xrightarrow[F]{} & BG \end{array}$$

where $L_1 = \gamma\mu$ by the universal property of fiber product.

Lemma 5. *μ: $BO(k) \times BO(k) \to BO(k)$ represents the homomorphism defined as the union of the direct sum homomorphism $\mu_{(n,t)}$; $O_n(k) \times O_t(k) \to O_{n+t}(k)$.*

Proof. Since $\varphi\mu = \tilde{m}(\varphi \times \varphi)$ and we have seen that φ represents the inclusion $O(k) \subset O(\bar{k})$ we must have that μ must represent the restriction to $O(k) \times O(k)$ of the homomorphism m, thus proving the lemma. Q.E.D.

Let us return to the diagrams (Ω_1) and (Ω_2). Since, as we have already noticed, the homomorphism D_Γ depends only by the homotopy class of Γ, we have

$$\mu^* D_\Gamma = D_{\Gamma \times \Gamma} \tilde{v}^*. \qquad (*)$$

Now let us consider the canonical projections of the square

$$\begin{array}{ccc} BO(k) \times BO(k) & \longrightarrow & BO(\bar{k}) \times BO(\bar{k}) \\ \downarrow{\scriptstyle \varphi \times \varphi} & & \downarrow{\scriptstyle \Delta \times \Delta} \\ BO(\bar{k}) \times BO(\bar{k}) & \xrightarrow[F \times F]{} & BG \times BG \end{array}$$

onto the square

$$\begin{array}{ccc} BO(k) & \xrightarrow{\gamma} & BO(\bar{k}) \\ \downarrow & & \downarrow \\ BO(\bar{k}) & \xrightarrow[F]{} & BG. \end{array}$$

If we denote by $x \otimes 1$ (resp. $1 \otimes x$) the image of an element of $H^*(X)$, X is any space in the above square, in $H^*(X \times X)$ under the cohomology homomorphism induced by the first (resp. the second) projection, we get, by the functoriality of D, that $D_{\Gamma^2}(y \otimes 1) = D_\Gamma(y) \otimes 1$ for $y \in H^*(BG)$, and similarly for $1 \otimes y$.

LEMMA 6.

$$
\begin{aligned}
D_{\Gamma^2}((\bar{w}_1 \otimes \bar{w}_j)' &+ (\bar{w}_1 \otimes \bar{w}_j'')) \\
&= (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes (\varphi^* \bar{w}_j) + (\varphi^* \bar{w}_i) \otimes (D_\Gamma(\bar{w}_j' + \bar{w}_j'')) \quad \text{for } i, j \geqslant 2, \\
&= (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes (\varphi^* \bar{w}_1) \quad \text{for } j = 1, i \geqslant 2, \\
&= 0 \quad \text{for } j = i = 1.
\end{aligned}
$$

Proof. From what we have noticed above, it follows

$$D_{\Gamma^2}((\bar{w}_i' + \bar{w}_i'') \otimes 1) = (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes 1$$

and similarly for $1 \otimes (\bar{w}_i' + \bar{w}_i'')$.

Since

$$(\bar{w}_i \otimes \bar{w}_j)' + (\bar{w}_i \otimes \bar{w}_j)'' = (\bar{w}_i' + \bar{w}_i'') \otimes \bar{w}_j' + \bar{w}_i'' \otimes (\bar{w}_j' + \bar{w}_j'') \quad \text{for} \quad i, j \geqslant 2,$$

we have by the properties of D_Γ :

$$
\begin{aligned}
D_{\Gamma^2}((\bar{w}_i \otimes \bar{w}_j)' &+ (\bar{w}_i \otimes \bar{w}_j)'') \\
&= (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes \bar{w}_j' + \bar{w}_i'' \otimes (D_\Gamma(\bar{w}_j' + \bar{w}_j'')) \\
&= (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes (\varphi^* \bar{w}_j) + (\varphi^* \bar{w}_i) \otimes (D_\Gamma(\bar{w}_j' + \bar{w}_j' + \bar{w}'')).
\end{aligned}
$$

Now suppose $j = 1$, $i \geqslant 2$.

Then $\bar{w}_1' = \bar{w}_1'' = \bar{w}_1$. So, $(\bar{w}_i \otimes \bar{w}_1)' + (\bar{w}_i \otimes \bar{w}_1) = (\bar{w}_i' + \bar{w}_i'') \otimes \bar{w}_1$, then by the properties of D, $D_{\Gamma^2}((\bar{w}_i' + \bar{w}_i'') \otimes \bar{w}_1) = (D_\Gamma(\bar{w}_i' + \bar{w}_i'')) \otimes \varphi^*(\bar{w}_1)$.

Finally if $i = j = 1$,

$$(\bar{w}_1 \otimes \bar{w}_1)' = (\bar{w}_1 \otimes \bar{w}_1)''$$

and so the proof of the lemma is complete. Q.E.D.

If we consider the square

$$
\begin{array}{ccc}
BO(k) \times BO(k) & \xrightarrow{m} & BO(k) \\
\Big\downarrow{\scriptstyle \pi \times \pi} & & \Big\downarrow{\scriptstyle \pi} \\
BO \times BO & \xrightarrow[s]{} & BO
\end{array}
$$

we have that the additivity of the Brauer lifting plus Lemma A2 imply its homotopy commutativity.

This gives

LEMMA 7. $m^*(\bar{w}_i) = \sum_{k+j=i} \bar{w}_k' \otimes \bar{w}_j''$.

Proof. By the known multiplicative formulas for Stiefel–Witney classes we have

$$s^*(w_i) = \sum_{k+j=i} w_k \otimes w_j ,$$

and by the above diagram

$$\tilde{m}^*(\bar{w}_i) = \tilde{m}^*(\pi^*(w_i)) = (\pi \times \pi)^*(s^*(w_i)).$$

So, we have

$$\tilde{m}^*(\bar{w}_i) = (\pi \times \pi)^* \left(\sum_{k+j=i} w_k \otimes w_j \right) = \sum_{k+j=i} \bar{w}_k \otimes \bar{w}_j .$$

We are now ready to prove

PROPOSITION 4.

$$\mu^*(\bar{u}_i) = \sum_{a+b=i} \bar{u}_a \otimes (\varphi^* \bar{w}_b) + (\varphi^* \bar{w}_a) \otimes \bar{u}_b$$

for each $i \geqslant 2$, *where we put* $u_1 = u = 0$.

Proof. If we consider the image of $\mu^*(\bar{u}_i)$ modulo $\mathrm{Im}(\varphi \times \varphi)^*$ we get

$$\mu^*(\bar{u}_i) = \mu^*(D_\Gamma(\bar{w}_i{}' + \bar{w}_i'')) = D_{\Gamma^2}(\tilde{v}^*(\bar{w}_i{}' + \bar{w}_i''))$$

by (*).

But,

$$\begin{aligned} & D_{\Gamma^2}(\tilde{v}^*(\bar{w}_i{}' + \bar{w}_i'')) \\ & \quad = D_{\Gamma^2}\left(\sum_{a+b=i} (\bar{w}_a \otimes \bar{w}_b)' \right) + \left(\sum_{a=b=i} (\bar{w}_a \otimes \bar{w}_b)'' \right) \\ & \quad = D_{\Gamma^2}\left(\sum_{a+b=i} (\bar{w}_a \otimes \bar{w}_b)' + (\bar{w}_a \otimes \bar{w}_b)'' \right) \\ & \quad = \sum_{a+b=i} ((D_\Gamma(\bar{w}_a{}' + \bar{w}_a'')) \otimes (\varphi^* \bar{w}_b) + (\varphi^* \bar{w}_a) \otimes (D_\Gamma(\bar{w}_b{}' + \bar{w}_b''))), \end{aligned}$$

by Lemma 7, with $D_\Gamma(\bar{w}_a{}' + \bar{w}_a'') = 0$ when $a = 0, 1$.

Now by the definition of $\bar{u}_i$ it follows that $\mu^*(\bar{u}_i)$ must be the only element in the lateral class $D_{\Gamma^2}(\tilde{v}^*(\bar{w}_i{}' + \bar{w}_i''))$ which lies in $\mathrm{Ker}(j_i \times j_i)^*$; in fact it is clear that $\tilde{v}(j \times j)$ is homotopic to j. But this element is clearly just

$$\sum_{a+b=i} (u_a \otimes (\varphi^* \bar{w}_b) + u_b \otimes (\varphi^* \bar{w}_a)) \qquad \text{with} \quad u = u_1 = 0,$$

thus proving the proposition. Q.E.D.

COROLLARY.

$$(\tilde{\bar{\pi}}v)^*(u_i) = \sum_{a+b=i} (\bar{u}_a \otimes (\varphi^*\bar{w}_b) + u_b \otimes (\varphi^*\bar{w}_a)).$$

2.4. *An Explicit Base for* $H^*(\widetilde{FO}\psi^q)$

From now on we put $\bar{w}_i = \varphi^*(\bar{w}_i)$ and $w_i = \varphi'^*(w_i)$, for each i.

So we can write the multiplicative formulas of the preceding paragraph as

$$\mu^*(\bar{w}_i) = \sum_{a+b=i} w_a \otimes w_b ,$$

$$\mu^*(\bar{u}_i) = \sum_{a+b=1} (\bar{u}_a \otimes \bar{w}_b + \bar{w}_a \otimes \bar{u}_b),$$

with $u_1 = u = 0$.

Now let us introduce indeterminates t, s with $s^2 = 0$. If we put

$$\bar{w}_{ts} = 1 + \sum_{i\geqslant 1} \bar{w}_i t^i + \bar{u}_i t^{i+1} s \qquad (u_1 = 0)$$

we can rewrite our multiplicative formulas as

$$\mu^*(\bar{w}_{ts}) = \bar{w}_{ts} \otimes \bar{w}_{ts} .$$

Let us consider now the group $O_2(k)$.

It is easy to see that this group is a diedral group with $2(q-1)$ elements if $q = 4n+1$, $2(q+1)$ elements if $q = 4n+3$ and it is known [9] that

$$H^*(O_2(k)) \cong Z_2[x_1, x_2, 1]/(1^2 + 1x_1)$$

with $\deg x_1 = \deg 1 = 1$ and $\deg x_2 = 2$, and with $\varphi_2(\bar{w}_i) = x_i$, $i = 1, 2$.

PROPOSITION 5. *If* $f \in [BO_2(k), BO(k)]$ *is the homotopy class associated to the canonical inclusion of* $O_2(k)$ *in* $O(k)$ *then*:

(i) *If* A *is the subalgebra of* $H^*(O_2(k))$ *generated by* x_1, x_2, $f^*(\bar{u}_2)$, *we have* $A = H^*(O_2(k))$. *In particular* $f^*(u_2) = 0$.

(ii) $f^*(\bar{w}_i) = f^*(\bar{u}_i) = 0$, *for* $i \geqslant 3$.

Proof. (i) Let us consider the two squares

$$\begin{array}{ccc} BO(k) & \xrightarrow{\ \gamma\ } & BO(\bar{k}) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\Delta} \\ BO(\bar{k}) & \xrightarrow[F]{} & BG \end{array}$$

and

$$\begin{array}{ccc} BSO_2(k) & \xrightarrow{\bar{\gamma}} & BSO(\bar{k}) \\ {\scriptstyle \bar{\varphi}_2}\downarrow & & \downarrow{\scriptstyle \bar{\Delta}_2} \\ BSO_2(\bar{k}) & \xrightarrow[(\bar{F},id)]{} & BSO_2(\bar{k}) \times BSO_2(\bar{k}) \end{array}$$

where the second is defined in exactly the same way as the corresponding square for $O_2(k)$, $\bar{F}$ denotes a map induced by the homomorphism F: $SO_2(\bar{k}) \to SO_2(\bar{k})$ defined using the Frobenius homomorphism.

By using the same methods of the preceding section, it is easy to see that, if $\tilde{f} \in [BSO_2(k), BO(k)]$ denotes the homotopy class corresponding to the canonical inclusion of $SO_2(k)$ in $O(k)$ and $\bar{\bar{f}} \in [BSO_2(\bar{k}) \times BSO_2(\bar{k}), BG]$ denotes the homotopy class corresponding to the canonical inclusion of $SO_2(\bar{k}) \times SO_2(\bar{k})$ in G, we have

$$\tilde{f}^* D_\Gamma = D_{\Gamma_2} \bar{\bar{f}}^*$$

where D_{Γ_2} is defined for $\Gamma_2 = (\gamma_2, (\bar{F}, id))$ as D_Γ for Γ.

Now it is known $SO_2(\bar{k}) = \bar{k}^*$ and, since $\bar{k}^*$ is a union of an expanding sequence of finite cyclic groups of order prime to char $\bar{k}$ and since the relevant Bocksteins are all zero $H^*(SO_2(\bar{k}), C) \simeq C[x]$ with $\deg x = 2$, where C is any finite cyclic group of order prime to char $\bar{k}$. In particular if $C = Z_2$, it follows immediately from Quillen's theorem that, if $\bar{f} \in [BSO_2(\bar{k}), BO(\bar{k})]$ is the homotopy class induced by the canonical inclusion of $SO_2(\bar{k})$ in $O(\bar{k})$, then $x = \bar{f}(\bar{w}_2)$. Now we separate two cases. If $q = 4m + 1$ let us take coefficients in $Z/h(q-1)$, where h is an even integer prime to char $\bar{k}$, and let us consider the map of exact sequences

$$\begin{array}{ccccc} \cdots \to H^1(BSO_2(\bar{k}), Z/h(q-1)) & \xrightarrow{\delta} & H^2(\Delta_2, Z/h(q-1)) & \xrightarrow{\tau} & H^2((BSO_2(\bar{k}))^2, Z/h(q-1)) \\ \downarrow{\scriptstyle \bar{\gamma}_2^*} & & \downarrow{\scriptstyle \Gamma_2} & & \downarrow{\scriptstyle (F,id)^*} \\ \cdots \to H^1(BSO_2(k), Z/h(q-1)) & \xrightarrow{\delta} & H^2(\varphi_2, Z/h(q-1)) & \xrightarrow{\tau'} & H^2(BSO_2(\bar{k}), Z/h(q-1)). \end{array}$$

We have that, if we put $x'^{(\prime\prime)} = \mathrm{pr}_{1(2)}(x)$ where pr_i ($i = 1, 2$) denotes the ith canonical projection $BSO_2(\bar{k}) \times BSO_2(\bar{k}) \to BSO_2(\bar{k})$, $\Delta_2(x' - x'') = 0$. This implies that there is an element $z \in H^2(\Delta_2, Z/h(q-1))$ such that $\tau(z) = x' - x''$.

Now let us consider $\Gamma_2{}^*(z) = z'$. Since, if we consider the homomorphism $Z/h(q-1) \to Z_2$ which sends one to one and the corresponding homomorphism $\mathcal{O}$: $H^2(BSO_2(\bar{k}), Z/h(q-1)) \to H^2(BSO_2(\bar{k}), Z_2)$, we get that $\mathcal{O}(x) = \bar{f}^*(w_2)$, so by the definition of D and the fact that $\tilde{f}_\Gamma D = D_{\Gamma_2} \bar{\bar{f}}^*$ we get that, in order to prove that $\tilde{f}^*(u_2) \neq 0$, it is sufficient to prove that there is no element $\bar{z} \in H^2(\varphi_2, Z/h(q-1))$ such that $2\bar{z} = z'$. Now, since $4/q - 1$ it is easily seen that $SO_2(k) \simeq k^* = Z/(q-1)$.

It follows from the universal coefficients exact sequence that $H^2(SO_2(k), Z/h(q-1)) \cong Z/(q-1)$, and we can choose $\bar{\varphi}_2{}^*(x)$ as a generator. Since $(F, id)^*(x' - x'') = (q-1)x$ we have $\tau'(z') = (q-1)x$ so if we suppose that there exists $\bar{z}$ such that $2\bar{z} = z'$ we have $\tau'(\bar{z}) = ((q-1)/2)x$, or $\tau'(\bar{z}) = (((q-1)/2) + h((q-1)/2))x$.

But, by exactness $\bar{\varphi}^{*\prime}(\bar{z}) = 0 = ((q-1)/2)\, \varphi_2(x)$ which is absurd since $\bar{\varphi}_2{}^*(x)$ is a generator of $H^2(BSO(k), Z/h(q-1)) = Z/q-1$.

If $q = 4m+3$ the proof goes in exactly the same way, since $SO_2(k) \cong Z/(q+1)$, by taking $h = q+1$.

Now, if we consider the homotopy class $h \in [BSO_2(h), BO_2(k)]$, induced by the canonical inclusion, we clearly get

$$\tilde{f} = fh$$

and since we have proved $\tilde{f}^*(u_2) \neq 0$ while it is known $\tilde{f}^*(\bar{w}_1) = 0$, we have that $f^*(u_2) \neq f^*(w_1)$. So, by the structure of $H^*(BO_2(k))$ we have that $f^*(u_2) = 1$ or $f^*(u_2) = 1 + f^*(w_1)$. In either case it can be immediately seen that $f^*(w_1)$, $f^*(w_2)$, and $f^*(u_2)$ generate the whole $H^*(BO_2(k))$.

So (i) is proved and (ii) follows immediately from the relation

$$\tilde{f}^*D = D_{\Gamma(2)}\tilde{f}^*$$

and the theorem of Quillen. Q.E.D.

Remark. Given a finite group G and an orthogonal representation $\mathscr{H}$ of G over k, if $\tilde{\mathscr{H}} \in [BG, BO(k)]$ corresponds to $\mathscr{H}$, we can consider the elements $\tilde{\mathscr{H}}^*(w_i)$, $\tilde{\mathscr{H}}^*(u_i)$ as characteristic classes for the representation $\mathscr{H}$ and the class

$$w_{ts}(\mathscr{H}) = 1 + \sum_{i \geqslant 1} \tilde{\mathscr{H}}^*(w_i)\, t^i + \tilde{\mathscr{H}}^*(u_i)\, t^{i-1}s$$

as a total cohomology characteristic class for $\mathscr{H}$.

With these notations, Proposition 7 asserts that if $\mathscr{H}$ is the canonical two-dimensional representation of $O_2(k)$, then

$$w_{ts}(\mathscr{H}) = 1 + f^*(\bar{w}_1)t + f^*(\bar{w}_2)t^2 + f^*(u_2)ts,$$

and the coefficients of the nonconstant terms of the above polynomial in t and s, generate $H^*(BO_2(k))$.

We have the following:

THEOREM 1. *The monomials*

$$w_1^{\alpha_1} w_2^{\alpha_2} \cdots u_2^{\beta_1} u_3^{\beta_2} \cdots$$

where $\alpha_1 \geqslant 0$, $0 \leqslant \beta_1 \leqslant 1$, $\forall i$ *and* $\alpha_i = \beta_i$ *for all but a finite number of* i*'s, form a basis for the algebra* $H(\widetilde{FO}\psi^q)$.

Proof. Let us consider the map

$$\tilde{h}_n\colon \underbrace{BO_2(k) \times \cdots \times BO_2(k)}_{n \text{ times}} \to BO(k)$$

defined by induction in the following way:

$$\tilde{h}_1 = f \qquad \tilde{h}_n = \mu(\tilde{h}_{n-1} \times \tilde{h}_1).$$

We put $\tilde{\tilde{\pi}}\tilde{h}_n = h_n$.

Now let us define a homomorphism F from

$$H^*(O_2(k)) \cong \frac{Z_2[h_1(\bar{w}_1), (\bar{w}_2), h_1{}^*(u_2)]}{(h_1{}^*(u_2)^2 + h_1(u_2)\, h_1{}^*(w_1))} = A$$

to the algebra

$$Z_2[x', x'', y]/(y^2 + y) = B$$

by $F(h_1{}^*(w_1)) = x' + x''$, $F(h_1{}^*(w_2)) = x'x''$, $F(h_1{}^*(u_2)) = (x' + x'')\,y$. It is clear that F is injective.

Consider the homomorphism:

$$F^n\colon \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}} \to \underbrace{B \otimes \cdots \otimes B}_{n \text{ times}} = \frac{Z_2[x_1{}', x_1'', \ldots, x_n{}', x_n'', y_1, \ldots, y_n]}{(y_1{}^2 + y_1, \ldots, y_n{}^2 + y_n)}.$$

Since

$$H^*(\underbrace{O_2(k) \times \cdots \times O_2(k)}_{n \text{ times}}) = \underbrace{A \otimes \cdots \otimes A}_{n \text{ times}},$$

by Kunneth's formula, we immediately get from the corollary to Proposition 4, Proposition 5, and the definition of F^n that

$$F^n h_n{}^*(w_i) = \sigma_i$$

where σ_i denotes the ith elementary symmetric function in $(x_1{}', x_1'', \ldots, x_n{}', x_n'')$ for $i \leqslant 2n$; $F^n h_n{}^*(w_i) = 0$ for $i \geqslant 2n$; and also

$$F^n h_n{}^*(u_i) = \sum_{k=1}^{n} \sigma_{i-2}(x_1{}', x_1'', \ldots, \hat{x}_k{}', \hat{x}_k'', \ldots, x_n{}', x_n'')(x_k{}' + x_k'')\, y_k, \qquad \text{for} \quad i \leqslant 2n,$$

$$F^n h_n{}^*(u_i) = 0, \qquad \text{for} \quad i > 2n.$$

Now we want to prove that the elements $F^n h_n{}^*(w_1^{\alpha_1} \cdots w_{2n}^{\alpha_n} u_1^{\beta_1} \cdots u_{2n}^{\beta_{2n}})$ with $\alpha_1, \ldots, \alpha_{2n} \geqslant 0$, $0 \leqslant \beta_1, \ldots, \beta_n \leqslant 1$ are independent in

$$\underbrace{B \otimes \cdots \otimes B}_{n \text{ times}} = B^{\otimes n}.$$

It is readily seen that we can consider $B^{\otimes n}$ as a quotient of the algebra

$$Z_2[x_1', x_1'', \ldots, x_n', x_n'', y_1', y_1'', \ldots, y_n', y_n'']/(y_1'^2 + y_1', y_1''^2 + y_1'', \ldots, y_n''^2 + y_n'')$$

over the ideal generated by the elements $y_1' + y_1'', \ldots, y_n' + y_n''$. Let us call q the quotient homomorphism.

LEMMA 7. *The following identity holds*

$$F^{\otimes n} h_n^*(u_i) = q\Bigg(\sum_{s=1}^{n} (\sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s', x_s'', \ldots, x_n'')\, y_s' + \sigma_{i-1}x_1', x_1'', \ldots, \hat{x}_s'', \ldots, x_n'')\, y_s''\Bigg).$$

Proof. We can write

$$(\alpha) \qquad \sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s', \ldots, x_n'') = x_s''\sigma_{i-2}(x_1', x_1'', \ldots, \hat{x}_s', \hat{x}_s'', \ldots, x_n'') + \sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s', \hat{x}_s'', \ldots, x_n'').$$

We have

$$q(\sigma_{i-1}(x_1', x'', \ldots, \hat{x}_s', \ldots, x_n'')\, y_s' + \sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s'', \ldots, x_n'')\, y_s'')$$
$$= [\sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s', \ldots, x_n'') + \sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s'', \ldots, x_n)\, y_s].$$

Introducing the relations (α) we get, for each $s \leqslant n$:

$$\sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s', \ldots, x_n'') + \sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s'', \ldots, x_n'')$$
$$= (x_s' + x_s'')(\sigma_{i-2}(x_1', x_1'', \ldots, \hat{x}_s', \hat{x}_s'', \ldots, x_n'')$$

which proves the lemma. Q.E.D.

Now let us put for $2 \leqslant i \leqslant 2n$,

$$v_i = \sum_{s=1}^{n} (\sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s', \ldots, x_n'')\, y_s' + \sigma_{i-1}(x_1', x_1'', \ldots, \hat{x}_s'', \ldots, x_n'')\, y_s''),$$

and

$$v_1 = y_1' + y_1'' + \cdots + y_n' + y_n''.$$

LEMMA 8. *The monomials*

$$v_1^{\beta} \cdots v_{2n}^{\beta} \qquad 0 \leqslant \beta_1, \ldots, \beta_{2n} \leqslant 1$$

are linearly independent over $Z_2(x_1', x_1'', \ldots, x_n', x_n'')$, *the field of fractions of* $Z_2[x_1', x_1'', \ldots, x_n', x_n'']$.

Proof. Suppose we have an expression $\sum a_I v_I = 0$ where $a_I \in Z_2(x_1', x_1'', ..., x_n', x_n'')$ and $v_I = v_1 \cdots v_i$ for some subset $I = (i_1, ..., i_k) \subset (1, ..., 2n)$.

Suppose that for some of the I's, $a_I \neq 0$ and let $\bar{I}$ be a set of maximal order among those. We can suppose $a_{\bar{I}} = 1$.

Let J be the complement of $\bar{I}$ in $(1, ..., 2n)$.

We have

$$\left(\sum_I a_I v_I\right) v_j = 0.$$

But now, by maximality, only the term $a_{\bar{I}} v_{\bar{I}} v_J$ can contain a monomial of type $b\, y_1' y_1'' \cdots y_n''$. So we must have $b\, y_1' y_1'' \cdots y_n'' = 0$.

Since $a_{\bar{I}} = 1$ we have that b is equal to the coefficient of $y_1' y_1'' \cdots y_n''$ in $v_1 \cdots v_{2n}$. So b comes to the equal to the determinant of the Jacobian matrix:

$$\begin{vmatrix} 1 \cdots\cdots\cdots\cdots\cdots\cdots 1 \\ \sigma_1(\hat{x}_1', ..., x_n'') \cdots\cdots \sigma_1(x_1', ..., \hat{x}_n'') \\ \\ \sigma_{2n-1}(\hat{x}_1', ..., x_n'') \cdots \sigma_{2n-1}(x_1', ..., \hat{x}_n'') \end{vmatrix}$$

which is different from zero by the algebraic independence of the elementary symmetric functions. So, also $b\, y_1' y_1'' \cdots y_n'' \neq 0$ and this implies that $a_{\bar{I}} = 0$ thus giving a contradiction. Q.E.D.

Now for any two-by-two partition p of the set $(y_1', ..., y_n'')$, let us consider the corresponding algebra Q_p given by taking the quotient of the algebra

$$\frac{Z_2(x_1', ..., x_n'')[y_1', ..., y_n'']}{(y_1'^2 + y_1', ..., y_n''^2 + y_n'')} = \tilde{R}$$

obtained by identifying, two by two, the elements coupled in the partition p.

Let us take the vector space over $\tilde{K} = Z_2(x_1', ..., x_n'')$ given by $\oplus_{p \in T} Q_p$ where T is the set of two-by-two partitions of $(1, ..., 2n)$, and $G\colon R \to \oplus_{p \in T} Q_p$ the vector space homomorphism which is the quotient defined above on each factor.

We want to prove $\dim(\operatorname{Ker} G) = 2^{n-1}$.

In order to do so let us prove the following.

LEMMA 9, *Let K be any field and*

$$R = K[y_1, ..., y_{2n}]/(y_1^2 + y_1, ..., y_{2n}^2 + y_{2n}).$$

Let us consider, for each element p of the set T of two by two partitions of the set $(1, ..., 2n)$, the quotient Q_p defined as above. And let $G\colon R \to \oplus_{p \in T} Q_p$ also be defined as above. Then, $\dim(\operatorname{Im} G) \geqslant 2^{n-1}$.

Proof. Let R' be the subalgebra of R generated by $y_1, \dots, y_{2n-1}$. It will be sufficient to prove $R' \cap \mathrm{Ker}(G) = 0$.

Now suppose $G(\sum_I a_I y_I) = 0$, where $a_I \in K$ and $y_I = y_{i_1} \cdots y_{i_k}$ with with $I = (i_1, \dots, i_k) \subset (1, \dots, 2n-1)$. Clearly $a_\phi = 0$; so we can make induction on the order of I and suppose $a_I = 0$ for $|I| < m$.

Consider any element $a_I y_{i_1} \cdots y_{i_m}$ and suppose m to be even.

Now take any partition p containing the couples $(i_1, i_2), \dots, (i_{m-1}, i_m)$ and consider the image of $y_{i_1} \cdots y_{i_m}$ in Q_p. It is clear that there is no set J with $|J| \geqslant |I|$ such that y_J and y_I are mapped to the same element in Q_p, so this implies $a_I = 0$.

If I is odd, consider any p containing $(i_2, i_3), \dots, (i_{m-1}, i_m)(i_1, 2n)$ and also in this case one proves readily that $a_I = 0$. Q.E.D.

If we go back to $\tilde{K}$, then Lemmas 8 and 9 imply that a basis for Ker(G) is given by the elements

$$v_1 v_2^{\beta_2} \cdots v_{2n}^{\beta_{2n}} \qquad 0 \leqslant \beta_2, \dots, \beta_{2n} \leqslant 1.$$

Now let us restrict ourselves to the subring $\bar{R} \subset R$ generated by the elementary symmetric functions $\sigma_i(x_1', \dots, x_n'')$ and by the v_i's.

It is clear that an element $x \in \bar{R} \cap \mathrm{Ker}(G)$ if and only if $x \in \mathrm{Ker}(G_p) \cap \bar{R}$ where G_p denotes the quotient $\tilde{R} \to Q_p$ relative to any partition $p \in T$. If we consider the partition $\bar{p}$: $(y_1', y_1''), \dots, (y_n', y_n'')$ the above implies that the elements $G_{\bar{p}}(v_2^{\beta_2}, \dots, v_{2n}^{\beta_{2n}})$, $0 \leqslant \beta_2, \dots, \beta_{2n} \leqslant 1$ are linearly independent over $Z_2(\sigma_1, \dots, \sigma_{2n})$ with $\sigma_i = \sigma_i(x_1', x_1'', \dots, x_n'')$.

In particular the elements

$$\sigma_1^{\alpha_1} \cdots \sigma_{2n}^{\alpha_{2n}} G_{\bar{p}}(v_2^{\beta_2}) \cdots G_{\bar{p}}(v_{2n}^{\beta_{2n}})$$

$\alpha_1, \dots, \alpha_{2n} \geqslant 0$, $0 \leqslant \beta_2, \dots, \beta_{2n} \leqslant 1$, are linearly independent over Z_2. Since we know that $\sigma_i = F^{\otimes n} h_n^*(w_i)$ and, by Lemma 11, $G_{\bar{p}}(v_j) = F^{\otimes n} h_n^*(u_j)$, for $j \geqslant 2$, we have that the monomials:

$$w_1^{\alpha_1} \cdots w_{2n}^{\alpha_{2n}} u_2^{\beta_2} \cdots u_{2n}^{\beta_{2n}} \qquad \alpha_1, \dots, \alpha_{2n} \geqslant 0, \quad 0 \leqslant \beta_2, \dots, \beta_{2n} \leqslant 1$$

are linearly independent in $H^*(\widetilde{FO\psi^q})$.

Applying this for larger and larger n we get that the w_i's and u_i's generate a subalgebra of $H^*(\widetilde{FO\psi^q})$ with Poincaré series

$$\frac{(1+t)(1+t^2)\cdots}{(1-t)(1-t^2)\cdots}$$

but this, by Proposition 2, is just the Poincaré series of $H^*(FO\psi^q)$ and the theorem follows. Q.E.D.

2.5. *The Algebra* $H^*(\widetilde{FO\psi^q})$

Given a group G, we say that a family $\{N_i\}_{i\in I}$ of subgroups of G detects the (mod 2) cohomology of G when, if we consider the elements $j_{N_i} \in [BN_i, BG]$ for each $i \in I$ associated to the inclusions of the N_i's in G, the homomorphism $\prod_{i\in I} j_N{}^*: H^*(BG) \to \prod_{i\in I} H^*(BN_i)$ is injective.

It is known [7] that the cohomology of $O_2(k)$ is detected by its family of maximal elementary Abelian 2-subgroups.

Since there are just two conjugacy classes of maximal elementary Abelian 2-subgroups, one of which contains the subgroup of diagonal matrices $Q(2)$, by taking a representative V for the class not containing $Q(2)$, we have that the cohomology of $O_2(k)$ is detected by $Q(2)$ and V (both $Q(2)$ and V have rank 2).

By the definition of u_2 we have $j^*_{Q(2)}(h_1{}^*(u_2)) = 0$, so we must have $j_V{}^*(h_1{}^*(u_2)) \neq 0$. Since the center C of $Q_2(k)$ has order 2, by maximality C is contained in both $Q(2)$ and V. Let us take polynomial generators x, y (resp. $\bar{x}, \bar{y}$), for $H^*(BQ(2))$ (resp. $H^*(BV)$) with the property that the kernel of the homomorphism $H^*(BQ(2)) \to H^*(BC)$ (resp. $H^*(BV) \to H^*(BC)$) induced by inclusion, is the ideal $(x + y)$ (resp. $(\bar{x} + \bar{y})$).

We get:

$$(*)\qquad \begin{aligned} j^*_{Q(2)}(h_1{}^*(w_1)) &= x + y, \\ j^*_{Q(2)}(h_1{}^*(w_2)) &= xy, \\ j_V(h_1{}^*(w_2)) &= \bar{x}\bar{y}, \\ j_V{}^*(h_1{}^*(w_1)) = j_V{}^*(h_1{}^*(u_2)) &= \bar{x} + \bar{y}. \end{aligned}$$

This follows for the w_i's because the two subgroups $Q(2)$ and V are conjugate in $O_2(k)$ and for u_2 by the definition of $\bar{x}$ and $\bar{y}$ and by the fact that $C = Q(2) \cap V$.

It follows from the above properties that the cohomology of

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{n \text{ times}}$$

is detected by the subgroups of type $E_1 \times \cdots \times E_n$ where each E_i can be equal to $Q(2)$ or V.

Since the proof of Theorem 1 implies that the homomorphism

$$h_n{}^*: H^i(\widetilde{FO\psi^q}) \to H^i(\underbrace{BO_2(k) \times \cdots \times BO_2(k)}_{n \text{ times}})$$

is injective for $i \leqslant 2n - 1$, we have that the homomorphism

$$\left(\bigoplus (j_{E_1} \times \cdots \times j_{E_n})^*\right) h_n{}^*: H^i(\widetilde{FO\psi^q}) \to \bigoplus H^i(BE_1 \times \cdots \times BE_n),$$

where the sum is taken over the number of different subgroups of type $E_1 \times \cdots \times E_n$, is injective for $i \leqslant 2n - 1$.

By definition

$$h_n = \tilde{\tilde{\pi}} \tilde{h}_n ,$$

$\tilde{h}_n$ being induced by the canonical inclusion of

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{n \text{ times}}$$

in $O(k)$. Since in $O(k)$ any two subgroups $E_1 \times \cdots \times E_n$ and $E_1' \times \cdots \times E_n'$ with the same number of E_i's and E_i''s equal to $Q(2)$, are conjugate, we get that the homomorphism

$$\lambda_n{}^m: \left(\bigoplus_{m=0}^{n} (j_{Q(2)_1} \times \cdots \times j_{Q(2)_m} \times j_{V_1} \times \cdots \times j_{V_{n-m}})^* \right) h_n{}^*: H^i(FO\psi^q)$$

$$\to \bigoplus_{m=0}^{n} H^i(BQ(2)_1 \times \cdots \times BQ(2)_m \times BV_1 \times \cdots \times BV_{n-m})$$

is injective for $i \leqslant 2n - 1$.

THEOREM 2. *In* $H^*(\widetilde{FO\psi^q})$

$$u_k{}^2 = \sum_{\substack{a+b=2k-1 \\ b \geqslant 2}} w_a u_b \qquad \text{for each} \quad k \geqslant 2.$$

Proof. It follows from the above discussion that it is sufficient to prove, for any fixed $n \geqslant k$

$$\lambda_n{}^m(u_k{}^2) = \lambda_n{}^m \left(\sum_{\substack{a+b=2k-1 \\ b \geqslant 2}} w_a u_b \right)$$

for each $0 \leqslant m \leqslant n$.

Let us fix such an n and let us put for simplicity $\lambda_n{}^m(w_i) = w_i$ and $\lambda_n{}^m(u_i) = u_i$. First of all suppose $m = n$. Then, by the definition of the u_i's, we have

$$0 = \lambda_n{}^n(u_k{}^2) = \lambda_n{}^n \left(\sum_{\substack{a+b=2k-1 \\ b \geqslant 2}} w_a u_b \right).$$

Now suppose $m = 0$. We have the following relations:

if g is odd

$$\lambda_n{}^0(u_g) = 0$$

if g is even

$$\lambda_n{}^0(u_g) = w_{g-1} \qquad \text{for} \quad g \leqslant 2n,$$

$$\lambda_n{}^0(u_g) = 0 \qquad \text{for} \quad g > 2n.$$

To prove this, let us make induction on n, for $n = 1$ the above relations follow from Proposition 4 and the relations (*). Suppose they are true for $n - 1$ and let us put $\lambda_{n-1}^{0}(w_i) = w_i'$, $\lambda_{n-1}^{0}(u_i) = u_i'$, $\lambda_1{}^0(w_i) = w_i''$, $\lambda_1{}^0(u_i) = u_i''$.

Using the multiplicative relations and the induction hypothesis we have

$$\lambda_n{}^0(w_{t,s}) = \left(1 + \sum_{i=0}^{2(n-1)} w_i't^i + \sum_{\substack{j=1\\ j=\text{odd}}}^{2(n-1)-1} w_j't^js\right)(1 + w_1''t + w_2''t^2 + w_1''ts).$$

This implies if g is odd and $g \leqslant 2n - 1$

$$u_g = w_{g-2}'w_1'' + w_{g-2}'w_{1=0}''$$

if $g \geqslant 2n + 1$

$$u_g = 0.$$

If g is even and $g \leqslant 2n$

$$u_g = w_{g-2}'w_1'' + w_{g-3}'w_2{}' + w_{g-1}' = w_{g-1}$$

if $g > 2n$

$$u_g = 0,$$

so the above relations are proved.

They imply, if k is odd,

$$\lambda_n{}^0\left(\sum_{a+b=2k-1} w_a u_b\right) = \sum_{e+f=2k-2}^{f=\text{odd}} w_e w_f = 0 = \lambda_n{}^0(u_k{}^2),$$

if k is even

$$\lambda_n{}^0\left(\sum_{a+b=2k-1} w_a u_b\right) = \sum_{e+f=2k-2}^{f=\text{odd}} w_e w_f = w_{k-1}^2 = \lambda_n{}^0(u_k{}^2).$$

Finally suppose $0 < m < n$. Let us put $w_j' = \lambda_n{}^m(w_j)$ and $w_j'' = \lambda_{n-m}^{0}(w_j)$. The above relations and the multiplicative formulas imply

$$\lambda_n{}^m\left(\sum_{a+b=2k-1} w_a u_b\right) = \sum_{a+b=2k-1} w_a\left(\sum_{e+f=b-1}^{f=\text{odd}} w_e'w_f''\right)$$

$$= \sum_{a+b=2k-1}\left(\left(\sum_{u+v=a} w_u'w_v''\right)\left(\sum_{e+f=b-1}^{f=\text{odd}} w_e'w_f''\right)\right).$$

Take any 4-ple (e, f, u, v) with f odd, $(e, f) \neq (u, v)$, $e + f + u + v = 2k - 2$. For this 4-ple we get the element

$$w_e'w_f''w_u'w_v''$$

in the above sum.

We separate two cases:

(1) If v is odd we get four 4-ple

$$(e, f, u, v), (u, v, e, f), (e, v, u, f), (u, f, e, v)$$

which gives the same element in the above sum (clearly if $e = u$ or $f = v$ the four 4-ple reduce to two).

(2) If u is even we get two 4-ple

$$(e, f, u, v), (u, f, e, v)$$

which gives the same element in the above sum.

Now it is clear that in either case the elements associated to those 4-ple cancel two by two.

So, we are left with the case $e = u, f = v$.

This implies

$$\lambda_n{}^m\left(\sum_{a+b=2k-1} w_a u_b\right) = \left(\sum_{h+s=2k-2}^{e=\text{odd}} w_h{}' w_s''\right)^2 = \lambda_n{}^m(u_k{}^2)$$

where the second equality follows from the multiplicative relations.

Thus

$$\lambda_n{}^m\left(\sum_{a+b=2k-1} w_a u_b\right) = \lambda_n{}^m(u_k{}^2) \qquad \text{for each} \quad 0 \leqslant m \leqslant n$$

and the theorem is proved. Q.E.D.

Combining Theorem 1 with Theorem 2 we get

THEOREM 3. *$H^*(\widetilde{FO\psi^q})$ as a algebra over Z_2 has generators $w_1, w_2, \ldots;$ $u_2, u_3, \ldots$ with* $\deg w_i = i$, $\deg u_k = k - 2$, *and relations*

$$u_k{}^2 = \sum_{\substack{a+b=2k-1 \\ b\geqslant 2}} w_a u_n \, .$$

Remark. Just by using diedral groups and a multiplicative relation which can be easily defined for $H^*(\widetilde{FO\psi^q})$ one could prove similar results to Theorems 1, 2, and 3 without the restrictions $q = |\, k \,|$.

3. The Homology of $FO\psi^q$

In this section we suppose that k is a field with q elements.

Let Q' and V' be two proper subgroups of the groups $Q(2)$ and V considered in the preceding paragraph, which are both different from C. Since both Q'

and V' are elementary Abelian 2-subgroups of rank 1, $H_i(Q') \cong H_i(V') \cong Z_2$ for each $i \geqslant 0$, where by H_i we denote the ith homology group with coefficients in Z_2.

Let $\bar{\xi}_i$ (resp. $\bar{\eta}_i$) the unique nonzero element in $H_i(Q')$ (resp. $H_i(V')$) for $i \geqslant 1$.

Let $R = M_1 \times \cdots \times M_n$ be any subgroup of

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{n \text{ times}}$$

which is the product of copies of Q' and V'.

For each R we get the homomorphism

$$(h_n j_R)_* \, H_*(BR) \to H_*(\widetilde{FO}\psi^b).$$

We put $\xi_i = (h_n j_{Q'})_* \bar{\xi}_i$ and $\eta_i = (h_n j_{V'})_* \bar{\eta}_i$.

Now let τ, $h \in H_*(\widetilde{FO}\psi^q)$ be such that $\tau = (h_{n'}, j_{R'})(\bar{\tau})$ and $h = (h_{n''} f_{R''})_*$ $(\bar{h})$ for two subgroups R' and R'' of the type described above. We can define $\tau h = (h_{n'+n''} j_{R' \times R''})_* \, (\tau \otimes h)$ by using the Kunneth formula.

THEOREM 4. *$H_*(\widetilde{FO}\psi^q)$ has a basis formed by the monomials*

$$\xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \eta_1^{\beta_1} \eta^{\beta_2} \cdots$$

with $\alpha_i \geqslant 0$, $0 \leqslant \beta_i \leqslant 1$ and all but a finite number of α_i's and β_i's equal to zero. Furthermore, $(\eta_i + \xi_i)^2 = 0$.

Proof. Let $t_1, \ldots, t_N$; $s_1, \ldots, s_N$ be indeterminates with $s_j^2 = 0$ for each $1 \leqslant j \leqslant n$. We define the homomorphism

$$T_N: H_*(\widetilde{FO}\psi^q) \to Z_2[t_1, \ldots, t_N] \otimes \Lambda[s_1, \ldots, s_N]$$

by

$$T_N(z) = \left\langle z, \prod_{j=1}^{N} w_{t_j s_j} \right\rangle$$

where by $\langle \ \rangle$ we mean the canonical pairing between homology and cohomology.

Now let

$$\tilde{\xi}_i = I_N(\xi_i) \quad \text{and} \quad \tilde{\eta}_i = T_N(\eta_i).$$

The multiplicative relations and the definition of Q' and V' clearly imply that, if $x \in H^1(BQ')$ (resp. $y \in H^1(BV')$) is the one-dimensional polynomial generator of $H^*(BQ')$ (resp. $H^*(BV')$),

$$\tilde{\xi}_i = \left\langle \xi_i, \prod_{j=1}^{N} (1 + x t_j) \right\rangle$$

and

$$\tilde{\eta}_i = \left\langle \bar{\eta}_i , \prod_{j=1}^{N} (1 + y(t_j + s_j)) \right\rangle,$$

and that, given two elements $\tau, h \in H_*(\widetilde{FO}\psi^q)$ for which λh is defined

$$T_N(\tau h) = T_N(\tau)\, T_N(h).$$

The above relations give

$$\tilde{\xi}_i = \sigma_i(t_1 ,..., t_N),$$
$$\tilde{\eta}_i = \sigma_i((t_1 + s_1),..., (t_N + s_N)),$$

where by σ_i we mean the elementary symmetric function of the variables in brackets.

We also have

$$\tilde{\xi}_i + \tilde{\eta}_i = \sigma_i(s_1 ,..., s_N) + \sum_{h=1}^{N} \sigma_{i-1}(s_1 ,..., \hat{s}_h ,..., s_n)_{t_h}$$
$$+ \cdots + \sum_{h=1}^{N} s_h \sigma_{i-1}(t_1 ,..., \hat{t}_h ,..., t_N).$$

So

$$T_N(\xi_i + \eta_i)^2 = (\tilde{\xi}_i + \tilde{\eta}_i)^2 = 0. \qquad (*)$$

Finally we can filter $Z_2[t_1 ,..., t_N] \otimes \Lambda[s_1 ,..., s_N]$ by powers of the ideal $(s_1 ,..., s_N)$; then under this filtration, the leading term of $\tilde{\xi}_i + \tilde{\xi}_i$ is

$$\sum_{h=1}^{N} s_h(\sigma_{i-1}(t_1 ,..., \hat{t}_h ,..., t_N)).$$

If we consider $Z_2[t_1 ,..., t_N] \otimes \Lambda[s_1 ,..., s_N]$ as a De Rham complex with $dt_i = s_i$ we get that

$$\sum_{h=1}^{N} s_h(\sigma_{i-1}(t_1 ,..., \hat{t}_h ,..., t_N)) = d\sigma_i(t_1 ,..., t_N).$$

We apply the following:

LEMMA 10 [10]. *The ring homomorphism $Z_2[\sigma_1 ,..., \sigma_N] \otimes \Lambda[d\sigma_1 ,..., d\sigma_N] \to Z_2[x_1 ,..., x_N] \otimes \Lambda[dx_1 ,..., dx_N]$ defined in the obvious way is injective.*

We clearly get from the above lemma that the monomials $\xi_1^{\alpha_1} \cdots \xi_N^{\alpha_N} \tilde{\eta}_1^{\beta_1} \cdots \tilde{\eta}_N^{\beta_N}$ with $\alpha_i \geqslant 0$, $0 \leqslant \beta_i \leqslant 1$ are linearly independent. Thus by applying this result for larger and larger N together with Proposition 2, we get the first part of the theorem.

The second follows from $(*)$ and the fact that $T_N/H_i(\widetilde{FO}\psi^q)$ is injective for $i \leqslant N$. Q.E.D.

Remarks. (1) The same remark at the end of Section 2 is valid in the case of this theorem.

(2) Lemma 14 is essentially Lemma 12.

(3) It arises from the proof of Theorem 3 that we can define a ring structure on $H_*(\widetilde{FO}\psi^q)$. With this ring structure $H^*(\widetilde{FO}\psi^q) = Z_2[\xi_1, \xi_2, ...] \otimes \Lambda[\xi_1 + \eta_1 + \xi_2 + \eta_2, ...]$.

4. The Cohomology of the Orthogonal Groups over k

Now let us consider, for each n, the subspace V_n of k^{n+1} which has as a basis the vectors $v, v_3, ..., v_n$, where $v_3, ..., v_n$ are the last $n - 2$ vectors of the canonical basis of k^{n+1} and $v = av_1 + bv_2$, v_1, v_2 being the first two vectors in the canonical basis of k^{n+1} and $a^2 + b^2$ being equal to a nonsquare. If we give V_n the bilinear form induced by the usual bilinear form on k^n, we clearly get an inclusion of $O_n(V_v) = \bar{O}_n(k)$ in $O_{n+1}(k)$, where $O_n(V_n)$ denotes the orthogonal group of V_n with respect to such a bilinear form.

Further, it is easily seen that we can define direct sum homomorphisms $\bar{O}_n(k) \times \bar{O}_m(k) \to O_{m+n}(k)$, $\bar{O}_n(k) \times O_m(k) \to \bar{O}_{m+n}(k)$, and $O_n(k) \times \bar{O}_m(k) \to \bar{O}_{n+m}(k)$ for each m, n, and that they can be defined to be compatible with the direct sum homomorphisms considered above and with inclusions, thus implying that the multiplicative formulas are valid also for the homotopy maps on classifying spaces induced by those homomorphisms.

Remarks. (1) We recall, see [4], that, give any n-dimensional vector space V over k together with a nonsingular bilinear form φ on V we have $O(V, \varphi) = O_n(k)$ or $O(V,) = \bar{O}_n(k)$. Further $O_n(k) = \bar{O}_n(k)$ for n odd.

(2) Instead of the usual distinction between the split and nonsplit cases, we have made a different one for practical reasons which will appear in what follows. In any case our distinction coincides with the usual one when k has $4m + 1$ elements.

Let us consider now the group $\oplus_{r\geqslant 1} H_*(BO_r(k)) \oplus \oplus_{r\geqslant 1} H_*(B\bar{O}_r(k))$; the direct sum homomorphisms induce a multiplication on such a group which is readily seen to be associative and commutative.

Let $\mathscr{E}$ be the generator of $H_0(BO_1(k))$ (resp. $\mathscr{E}'$ that of $H_0(B\bar{O}_1(k))$), then $\mathscr{E}^r$ will be the generator of $H_0(BO_r(k))$, $\mathscr{E}'^r = \mathscr{E}^r$ for r even and $\mathscr{E}'\mathscr{E}^{r-1}$ will be the generator of $H_0(B\bar{O}_r(k))$.

We have:

THEOREM 5. *If, for each* n, $\mathcal{O}_n \in [BO_n(k), BO(k)]$ (*resp.* $\mathcal{O}_n \in [B\bar{O}_n(k), BO(k)]$) *is the homotopy class induced by the canonical inclusion of* $O_n(k)$ *in* $O(k)$ (*resp. to the canonical inclusion* $\bar{O}_n(k) \subset O_{n+1}(k) \subset O(k)$), *then the homomorphisms*:

$$(\tilde{\bar{\pi}}\mathcal{O}_n)_*: H_*(BO_n(k)) \to H_*(\widetilde{FO}\psi^q)$$

$$(\textit{resp. } (\tilde{\bar{\pi}}\bar{\mathcal{O}}_n)_m: H_*(BO_n(k)) \to H_*(\widetilde{FO}\psi^q))$$

are injective.

Proof. By their definitions we can choose Q' to be $O_1(k)$ and V' to be $\bar{O}_1(k)$ under their canonical inclusions in $O_2(k)$. Thus, if we consider the elements

$$\xi_i \in H_i(BO_1(k)), \qquad i \geqslant 1,$$

$$\eta_i \in H_i(B\bar{O}_1(k)), \qquad i \geqslant 1,$$

we have, since it is clear that $\tilde{\bar{\pi}}\mathcal{O}_2 = h_1$ and since, if j (resp. $\bar{j}$) denotes the homotopy class associated to the inclusion $\bar{O}_1(k) \subset O_2(k)$ (resp. $\bar{O}_1(k) \subset O_2(k)$), $\mathcal{O}_2 j = \mathcal{O}_1$ (resp. $\mathcal{O}_2 \bar{j} = \mathcal{O}_1$), $\mathcal{O}_1$ (resp. $\bar{\mathcal{O}}_1$) takes the ξ_i's (resp. the η_i's), into the elements denoted by the same name in $H_*(FO\psi^q)$.

It also follows from the multiplicative relations that each monomial in the ξ_i's and η_i's in $\oplus_{r\geqslant 1} H_*(BO_r(k)) \oplus \oplus_{r\geqslant 1} H_*(B\bar{O}_r(k))$ goes into the corresponding monomial in the ξ_i's and η_i's, in $H_*(FO\psi^q)$.

In order to prove the theorem we need some lemmas.

LEMMA 11. *The cohomology of* $O_n(k)$ *and of* $\bar{O}_n(k)$ *is detected by their elementary Abelian 2-subgroups.*

The proof of this lemma in all the cases not covered by Theorem 4.3 in [9] is exactly as the proof of that lemma, using the appropriate numerical relation in [4].

LEMMA 12. *In* $n = 2m + e$ $(e = 0, 1)$, *then the cohomology of* $O_n(k)$ *is detected by the subgroup which is the image of*

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{m\text{-times}} \times O_1(k)^e$$

under its canonical inclusion. Similarly for $n = 2m$, *the cohomology of* $O_n(k)$ *is detected by*

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{m\text{-times}} \times \bar{O}_2(k)$$

and for $n = 2m + 1$ *by*

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{m\text{-times}} \times \bar{O}_1(k).$$

Proof. By Lemma 11, it is sufficient to prove that each elementary Abelian 2-subgroup of $O_n(k)$ (resp. $\bar{O}_n(k)$) is conjugate to a subgroup of the groups in which we want to detect cohomology.

Since given such a subgroup $A \subset O_n(k)$, we can consider k^n as an orthogonal n-dimensional representation of A, it is sufficient to prove that any orthogonal representation of A can be decomposed as a sum of one-dimensional orthogonal representations.

Since for one-dimensional representations this is trivial we suppose, by induction, that any m-dimensional representation of A can be written as a sum of one-dimensional representations for $m < n$.

Let us consider an n-dimensional orthogonal representation W of A, and let L be an irreducible invariant subspace for this representation. Since the exponent of A divides $q - 1$, L is of dimension 1. We divide two cases:

(1) If L is not an isotropic subspace, then $W = L \oplus L^{\perp}$ where $L^{\perp}$ is the space orthogonal to L, and by applying induction for $L^{\perp}$, W can be written as a sum of one-dimensional representations.

(2) If L is an isotropic subspace, then, by choosing an invariant subspace which is complementary to L (this exists because the order of A is prime to the characteristic of k), we write W as a direct sum of a hyperbolic orthogonal representation and an $(n - 2)$-dimensional representation. Thus we only have to prove the lemma for a two-dimensional representation in case we get an invariant subspace which is isotropic. Suppose we are in this case, let us take the other isotropic subspace. This too must be an invariant subspace since the representation is orthogonal. But each element of A has order two so that it must act on each isotropic subspace as 1 or -1, and orthogonality implies that it must act in the same fascion on both, thus implying the lemma. Q.E.D.

We are now ready to prove Theorem 5.

Since $\tilde{\pi}\mathcal{O}_2 = h_2$ it follows from Proposition 5 that $(\tilde{\pi}\mathcal{O}_2)_*$ is injective so, by Theorem 4 we get

$$\xi_i^2 = \eta_i^2 \qquad i \geqslant 1.$$

Theorem 4 also implies that the elements $\xi_j\xi_h$, $0 \leqslant j \leqslant h$, and $\eta_i\eta_k$, $0 \leqslant i < k$, where we put $\xi_0 = \mathscr{E}$, and $\eta_0 = \mathscr{E}'$, are linearly independent, thus they generate a submodule of $H_*(BO_2(k))$ with Poincaré series $(1 + t)/(1 - t)(1 - t^2)$. But, by the known structure of $H^*(BO_2(k))$, this is just the Poincaré series of $H_*(BO_2(k))$.

Thus the above elements form a basis for $H_*(BO_2(k))$.

By reasoning in a similar way we can see that $H_*(B\bar{O}_2(k))$ has a basis formed by the monomials $\xi_i\eta_j$, $i \geqslant 0, j \geqslant 0$.

Now we have that if G_n (resp. $\bar{G}_n$) are the subgroups detecting the cohomology of $O_n(k)$ (resp. $\bar{O}_n(k)$) found in Lemma 12 the homomorphisms

$$\delta_{n*} : H_*(BG_n) \to H_*(BO_n(k)),$$

resp.

$$\bar{\delta}_{n*} : H_*(B\bar{G}_n) \to H_*(B\bar{O}_n(k)),$$

induced by inclusion are onto. Thus using the fact that $\delta_1{}^2 = \mathscr{E}_i{}^2$ we get that the elements

$$\mathscr{E}^{\alpha_0}\xi_{i_1}^{\alpha_1} \cdots \xi_{i_s}^{\alpha_s}\mathscr{E}'^{\beta_0}\eta_{j_1}^{\beta_1} \cdots \eta_{j_t}^{\beta_t} \qquad \begin{array}{l} i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s, \\ j_2 < j_2 \cdots < j_t, \\ \alpha_0 \geqslant 0,\ 0 \leqslant \beta_0 \leqslant 1, \end{array}$$

form a set of generators over Z_2 for $\oplus_{r\geqslant 1} H(BO_r(k)) \oplus \oplus_{r\geqslant 1} H(B\bar{O}_r(k))$.

Now, if for such a monomial A we define

$$\deg(A) = \sum_1 \alpha_1 + \sum_i \beta_i$$

we have that $A \in H_*(BO_r(k))$ if and only if $\deg(A) = r$ and the number of η_j's is even, and $A \in H_*(B\bar{O}_r(k))$ if and only if $\deg(A) = r$ and the number of η_j's is odd; so, in order to prove our theorem, it is sufficient to prove that the monomials of a fixed degree are mapped by $(\tilde{\pi}\mathscr{O}_n)_*$ (resp. $(\tilde{\bar{\pi}}\mathscr{O}_n)_*$) to independent monomials. We have from the above,

$$(\tilde{\pi}\mathscr{O}_n)_*(\mathscr{E}^{\alpha_0}\xi_{i_1}^{\alpha_1} \cdots \xi_{i_s}^{\alpha_s}\mathscr{E}'^{\beta_0}\eta_{j_2} \cdots \eta_{i_t}) = \xi_{i_1}^{\alpha_1} \cdots \xi_{i_s}^{\alpha_s}\eta_{j_i} \cdots \eta_{i_t}$$

(with t even if $\beta_0 = 0$, t odd if $\beta_0 = 1$), and similarly for $(\tilde{\bar{\pi}}\mathscr{O}_n)_*$ which by Theorem 4 clearly implies that the monomials contained in $H_*(BO_r(k))$ (resp. in $H_*(B\bar{O}_r(k))$) are mapped to independent monomials by $(\tilde{\bar{\pi}}\mathscr{O}_n)_*$ (resp. $(\tilde{\bar{\pi}}\mathscr{O}_n)_*$).

Q.E.D.

THEOREM 6. *$H^*(BO_n(k))$ is generated as an algebra by elements $\bar{w}_1, \ldots, \bar{w}_n$; $\bar{u}_2, \ldots, \bar{u}_n$, with $\deg(\bar{w}_i) = i$, $\deg(\bar{u}_i) = i - 1$, subject to the following relations*

$$\bar{u}_i{}^2 = \sum_{a+b=2i-1} \bar{w}_a\bar{u}_b, \qquad \textit{where } \bar{w}_0 = 1; \quad \bar{w}_n = \bar{u}_{n+1}.$$

$H^*(B\bar{O}_n(k))$ is generated as an algebra by elements $\bar{w}_1, \ldots, \bar{w}_n$; $\bar{u}_2, \ldots, \bar{u}_{n+1}$, with $\deg(\bar{w}_i) = i$, $\deg(\bar{u}_i) = i - 1$ subject to the following relations

$$\bar{u}_i{}^2 = \sum_{a+b=2i-1} \bar{w}_a\bar{u}_b, \qquad \textit{where} \quad \bar{w}_0 = 1; \quad \bar{w}_n = \bar{u}_{n+1}.$$

Proof. It follows by Theorem 5 that the homomorphism $(\tilde{\bar{\pi}}\mathscr{O}_n)^*: H^*(FO(k)) \to$

$H^*(BO_n(k))$ is onto for each $n-1$; and we known, by Lemma 16 that, if $n = 2m + e$ ($e = 0, 1$), the homomorphism

$$\delta_n^*\colon H^*(BO_n(k)) \to H^*(\underbrace{BO_2(k) \times \cdots \times BO_2(k)}_{m\text{-times}} \times BO_1(k)^e)$$

induced by inclusion, is into.

If $n = 2m$, then $(\tilde{\bar{\pi}}\mathcal{O}_n\delta_n) = h_n$, so the theorem follows from the proof of Theorem 1 and Theorem 2 by taking $\bar{w}_1 = (\tilde{\bar{\pi}}\mathcal{O}_n)^*(w_i)$ and $\bar{u}_i = (\tilde{\bar{\pi}}\mathcal{O}_n)^*(u_i)$.

If $n = 2m + 1$, we have that, by definition the inclusion of

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{m\text{-times}} \times O_1(k)$$

in $O_n(k)$ is obtained by composing the inclusion of

$$\underbrace{O_2(k) \times \cdots \times O_2(k)}_{m\text{-times}} \times O_1(k),$$

in $O_{n-1}(k) \times (O_1(k))$ with the direct sum homomorphism $O_{n-1}(k) \times O_1(k) \to O_n(k)$; thus by the multiplicative relations and the result for $O_{n-1}(k)$, we get

$$(\tilde{\bar{\pi}}\mathcal{O}_n\delta_n)^*(w_{ts}) = \left(1 + \sum_{i=1}^{n-1} \bar{w}_i't^i + \sum_{j=2}^{n-1} \bar{u}_j't^{j-1}s\right) \otimes (1 + xt)$$

where $\bar{w}_i' = h_{n-1}^*(w_i)$, $\bar{u}_j' = h_{n-1}^*(u_j)$, and $x \in H^1(BO_1(k))$ is the one-dimensional polynomial generator of $H^*(BO_1(k))$.

Thus we get $(\tilde{\bar{\pi}}\mathcal{O}_n\delta_n)^*(w_i) = 0$ and $= (\tilde{\bar{\pi}}\mathcal{O}_n\delta_n)^*(u_i) = 0$ for $i > n$.

This means that the ideal generated by $w_{n+1}, w_{n+2}, \ldots, u_{n+1}, u_{n+2}, \ldots$ lies in the kernel of $(\tilde{\bar{\pi}}\mathcal{O}_n)^*$.

Since by the proof of Theorem 5 the Poincaré series of $H^*(BO_n(k))$ is

$$\frac{(1+t)\cdots(1+t^{n-1})}{(1-t)\cdots(1-t^n)}$$

and since also the algebra

$$H^*(\widetilde{FO}\psi^q)/(w_{n+1}, w_{n+2}, \ldots, u_{n+1}, u_{n+2}, \ldots)$$

has this Poincaré series, thus the theorem follows also for $n = 2m + 1$, because of $(\tilde{\bar{\pi}}\mathcal{O}_n)^*$ being onto by putting $\bar{w}_i = (\tilde{\bar{\pi}}\mathcal{O}_n)^*(w_i)$ and $\bar{u}_i = (\tilde{\bar{\pi}}\mathcal{O}_n)^*u_i$.

Finally, the case of $\bar{O}_n(k)$ is much similar, so we leave the details to the reader. Q.E.D.

Remark. Theorem 5 clearly implies that given an orthogonal representation φ of a finite group G over k, the class $u_i(\varphi)$ can be interpreted as obstructions for φ to be of the type split, if $q = 4m + 1$ or $q = 4m + 3$ and $n = 4s$, nonsplit in the remaining cases.

APPENDIX

Proof of Lemma A1. If we take the Grassmanian model, then $(BO)^n = \varprojlim_{m,s} (G_{m,s})^n$, where $G_{m,s}$ denotes the real Grassmanian of m-dimensional subspaces of a vector space of dimension $m + s$.

Then, if we consider the Milnor exact sequence

$$0 \to R^1 \varprojlim_{m,s} KO^{-1}((G_{m,s})^n) \to [BO^n, BO] \to \varprojlim_{m,s} \widetilde{KO}((G_{m,s})^n) \to 0$$

where R^1 denotes the first derived functor of $\varprojlim$, we must have in order to prove the lemma, $R^1 \varprojlim_{m,s} KO^{-1}((G_{m,s})^n) = 0$.

Now the real completion theorem [2] implies that the inverse system $KO^{-1}((G_{m,s})^n)$ is isomorphic as a pro-object to the inverse system

$$\left\{\frac{RO((O_m)^n)/R((O_m)^n)}{(I((O_m)^n))^s(RO((O_m)^n)/R((O_m)^n))}\right\}_{m,s}$$

where, for any group G, $RO(G)$ (resp. $R(G)$) denotes the real (resp. complex) representation ring of G and $I(G)$ denotes the real augmentation ideal in $RO(G)$.

It follows that, if we fix m, the inverse system $KO^{-1}((G_{m,s})^n)$ satisfied the Mittag–Leffler condition.

If we make m vary, we notice that it follows from the representation theory of O_m [1], that, if m is odd, the restriction map $RO((O_h)^n) \to RO((O_m)^n)$, for $h \geqslant m$, is onto and this easily implies that the entire system $KO^{-1}((G_{m,s})^n)$ satisfies the Mittag–Leffler condition, which implies, [2],

$$R^1 \varprojlim_{m,s} KO^{-1}((G_{m,s})^n) = 0$$

thus proving the lemma. Q.E.D.

Proof of Lemma A2. We consider the Milnor construction for the classifying space of a topological group G, we have

$$BG = BG^{(m)}, \qquad \text{with} \quad BG^{(m)} = \underbrace{G * G * G * \cdots * GG}_{m \text{ times}}.$$

and $*$ denotes the join operation.

Now by the definition of $BO(\bar{k})$ we have $BO(\bar{k}) = \bigcup_{m,n,s} B_{n,s}^{(m)}$ with $B_{n,s}^{(m)} = (BO_n(k_{(s)}))^{(m)}$.

The Milnor exact sequence in this case gives us the exact sequence

$$0 \to R^1 \varprojlim_{m,n,s} [B_{n,s}^{(m)}, \Omega BO] \to [BO(\bar{k}), BO] \to \varprojlim_{m,n,s} [B_{n,s}^{(m)}, BO]$$
$$\to 0.$$

So, in order to prove the proposition we have to show

$$(1)\quad R^1 \varprojlim_{m,n,s} [B_{n,s}^{(m)}, \Omega BO] = 0,$$

$$(2)\quad \varprojlim_{m} [B_{n,s}^{(m)}, BO] = [BO_n(k_{(s)}), BO].$$

But (1) follows because, if we fix a couple (n, s) we have [2] that the inverse system $\{[B_{n,s}^{(m)}, \Omega BO]\}_m$ with only m varying, is isomorphic as a pro-object to the inverse system

$$\left\{ \frac{RO(O_n(k_{(s)}))/R(O_n(k_{(s)})}{(I(O_n(k_{(s)}))^m(RO(O_n(k_{(s)})/R(O_n(k_{(s)}))} \right\}_m . \qquad (*)$$

(we use the notations in [2]), and we have that this inverse system consists of finite groups. So the entire inverse system $\{[B_{n,s}^{(m)}, BO]\}$ is isomorphic to an inverse system of finite groups.

(2) follows from the Milnor exact sequence

$$0 \to R^1 \varprojlim_m [B_{n,s}^{(m)}, \Omega BO] \to [BO_n(k_{(s)}), BO]$$
$$\to \varprojlim_m B_{n,s}^{(m)}, BO \to 0$$

for each couple (n, s), using the isomorphism between the system $\{[B_{n,s}^{(m)}, \Omega BO]\}_m$ and the system $(*)$. Q.E.D.

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