Representations of Quantum Groups at Roots of 1

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Dedicated to Jacques Dixmier on his 65th birthday

Introduction.

Quantum group $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{a})$ is a certain (Hopf algebra) deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{a})$ of a complex simple finite–dimensional Lie algebra $\mathfrak{a}$, introduced by Drinfeld [3], [4] and Jimbo [5] in their study of the quantum Yang–Baxter equation.

An important problem is to describe finite–dimensional representations of the algebra $\mathcal{U}_q$. When q is *generic*, i.e. it is not a root of 1, then the (finite–dimensional) representation theory of $\mathcal{U}_q$ is essentially the same as that of $\mathcal{U}(\mathfrak{a})$, namely representations of $\mathcal{U}_q$ are deformations of representations of $\mathcal{U}(\mathfrak{a})$, so that the latter are obtained as $q \to 1$ [12], [18]. Our contribution to the generic case is Proposition 1.9 giving a formula for the determinant of the contravariant form on a Verma module over $\mathcal{U}_q$, and Proposition 2.2 giving an explicit description of the center of $\mathcal{U}_q$. Proposition 2.2 is derived by a method developed in [7]. Proposition 1.9 implies in a usual way the description of irreducible subquotients of Verma modules over $\mathcal{U}_q$; this description was obtained by a different method in [1]. A result, somewhat weaker than Proposition 2.2b was previously obtained by a different method in [19].

When $q = \varepsilon$, a primitive ℓ–th root of 1, the situation changes dramatically. We study this case in §3. Our first key observation is that $\mathcal{U}_\varepsilon$ contains a large "standard" central subalgebra Z_0, so that $\mathcal{U}_\varepsilon$ is finite–dimensional over its center Z_ε. Since $\mathcal{U}_\varepsilon$ has no zero divisors, the algebra $Q(\mathcal{U}_\varepsilon) := Q(Z_\varepsilon) \otimes_{Z_\varepsilon} \mathcal{U}_\varepsilon$ is a division algebra of dimension m^2 over the field of fractions $Q(Z_\varepsilon)$ of Z_ε. Since, moreover, $\mathcal{U}_\varepsilon$ is integrally closed (i.e. is a maximal order in $Q(\mathcal{U}_\varepsilon)$)(Theorem 1.8), Z_ε is integrally closed as well and we may use the theory of finite–dimensional associative algebras. Denoting by Rep $\mathcal{U}_\varepsilon$ the set of equivalence classes of finite–dimensional irreducible representations of $\mathcal{U}_\varepsilon$, we obtain a sequence of

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canonical surjective maps: Rep $\mathcal{U}_\varepsilon \xrightarrow{X}$ Spec $Z_\varepsilon \xrightarrow{\tau}$ Spec Z_0. Outside the discriminant $\mathcal{D} \subset$ Spec Z_ε the map X is bijective and $X^{-1}(\chi)$ for $\chi \in$ Spec $Z_\varepsilon \backslash \mathcal{D}$ is a representation of dimension m. Dimensions of all other irreducible representations are less than m. The map τ is finite. In the case when ℓ is odd we find an explicit formula for m and for the degree of τ:

$$(0.1) \qquad m = \ell^N, \ \deg \tau = \ell^n,$$

where $N = (\dim \mathfrak{a} - \text{rank } \mathfrak{a})/2$ and $n = \text{rank } \mathfrak{a}$.

The proof of the above formulas consists of two components, both of which, we think, are of independent interest. First, it is the study of a family of "diagonal" modules, which for generic values of parameters are irreducible of dimension ℓ^N (for odd ℓ). Secondly, we introduce a (infinite-dimensional) group G of automorphisms of the algebra $\mathcal{U}_\varepsilon$ (or rather of its analytic completion $\hat{\mathcal{U}}_\varepsilon$). We show that the group G acts on Spec Z_0 by analytic transformations and that the G-orbit of the set $H_0 \subset$ Spec Z_0 corresponding to irreducible diagonal modules contains an open (in metric topology) subset of Spec Z_0. This proves (0.1).

We call the action of G on Spec Z_0 the quantized coadjoint action for the following reasons. As an algebraic variety, Spec Z_0 is isomorphic to $\mathbb{C}^{\dim \mathfrak{a}}$ with n subspaces of codimension 1 removed. The group G has 2^n fixed points in Spec Z_0. In a subsequent paper [2] (written jointly with Procesi)we show that the action of G in the tangent space to a fixed point is precisely the coadjoint action on $\mathfrak{a}$. We hope that this will lead us to the proof of conjectures on the quantized coadjoint action stated in §5. We find it quite remarkable that the quantized coadjoint action is independent of ℓ.

In §4 we study in detail the case of $\mathcal{U}_q(s\ell_2)$.

The most interesting representations of the algebra $\mathcal{U}_\varepsilon$ are the "restricted" representations, i.e. those which correspond to the fixed points of G in Spec Z_0. A remarkable conjecture on the structure of these representations has been proposed recently by Lusztig [14]. We hope that our work will contribute to progress in the solution of this conjecture by some deformation arguments.

Our work was greatly inspired and motivated by the works [24], [21], [20], [23] and [9] on representation theory of simple classical type Lie algebras in characteristic p. We conjecture that the reduction mod p of an irreducible representation of $\mathcal{U}_\varepsilon$, where ε is a primitive p-th root of 1, remains irreducible (for "restricted" representations this has been conjectured by Lusztig [14]). Moreover, the whole "quantum" picture described above descends nicely to the Lie algebra picture in characteristic p. Both theories should undoubtedly benefit from the interaction between these two pictures.

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§1. Algebras $\mathcal{U}$ and $\mathcal{U}_\varepsilon$ and Verma modules.

1.1. Let q be an indeterminate and let $\mathcal{A} = \mathbf{C}[q, q^{-1}]$, with the quotient field $\mathbf{C}(q)$. For $n \in \mathbf{Z}$ and $d \in \mathbf{N}$, let $[n]_d = (q^{dn} - q^{-dn})/(q^d - q^{-d}) \in \mathcal{A}$. As usual, we define

$$[n]_d! = [n]_d[n-1]_d \cdots [1]_d,$$

and the Gaussian binomial coefficients

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = [n]_d[n-1]_d \ldots [n-j+1]_d/[j]_d! \text{ for } j \in \mathbf{N}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix}_d = 1.$$

We shall omit the subscript d when $d = 1$. Here and further we let $\mathbf{N} = \{1, 2, \ldots\}$, $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$.

One knows that $\begin{bmatrix} n \\ j \end{bmatrix}_d \in \mathcal{A}$; this follows from the Gauss binomial formula

$$\prod_{j=0}^{m-1} (1 + q^{2j}x) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} q^{j(m-1)} x^j. \tag{1.1.1}$$

1.2. Fix an $n \times n$ indecomposable matrix (a_{ij}) with integer entries such that $a_{ii} = 2$ and $a_{ij} \leq 0$ for $i \neq j$, and a vector $(d_1, \ldots, d_n)$ with relatively prime entries d_i such that the matrix $(d_i a_{ij})$ is symmetric and positive definite. Note that (a_{ij}) is a Cartan matrix of a simple finite-dimensional Lie algebra. Let $q_i = q^{d_i}$.

Following Drinfeld [3], [4] and Jimbo [6], we consider the $\mathbf{C}(q)$-algebra $\mathcal{U}$ defined by the generators E_i, F_i, K_i, K_i^{-1} $(1 \leq i \leq n)$ and the relations (1.2.1)-(1.2.5):

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{1.2.1}$$

(1.2.2) $$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j,\ K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j,$$

(1.2.3) $$E_iF_j - F_jE_i = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1}),$$

(1.2.4) $$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s}E_jE_i^s = 0 \text{ if } i \neq j,$$

(1.2.5) $$\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s}F_jF^si = 0 \text{ if } i \neq j.$$

$\mathcal{U}$ is a Hopf algebra, called the *quantum group* associated to the matrix (a_{ij}), with comultiplication Δ, antipode S and counit ε defined by

(1.2.6) $$\Delta E_i = E_i\otimes 1 + K_i\otimes E_i, \Delta F_i = F_i\otimes K_i^{-1} + 1\otimes F_i, \Delta K_i = K_i\otimes K_i$$

(1.2.7) $$SE_i = -K_i^{-1}E_i, SF_i = -F_iK_i, SK_i = K_i^{-1},$$

(1.2.8) $$\varepsilon E_i = 0, \varepsilon F_i = 0, \varepsilon K_i = 1.$$

Introduce the $\mathbb{C}$–algebra anti–automorphism ω and the $\mathbb{C}$–algebra automorphism φ of $\mathcal{U}$ by

(1.2.9) $$\omega E_i = F_i, \omega F_i = E_i, \omega K_i = K_i^{-1}, \omega q = q^{-1},$$

(1.2.10) $$\varphi E_i = F_i, \varphi F_i = E_i, \varphi K_i = K_i, \varphi q = q^{-1}.$$

Finally, introduce the following elements of $\mathcal{U}$:

$$[K_i; n] = (K_iq^n - K_i^{-1}q^{-n})/(q_i - q_i^{-1}).$$

1.3. One has the following useful relation (due to Kac, see [12]):
(1.3.1)

$$[E_i^m, F_i^s] = \sum_{j=1}^{\min(m,s)} \begin{bmatrix} m \\ j \end{bmatrix}_{d_i} \begin{bmatrix} s \\ j \end{bmatrix}_{d_i} [j]_{d_i}!F_i^{s-j} \prod_{r=j-m-s+1}^{2j-m-s} [K_i; d_ir]E_i^{m-j}$$

This formula is proved in two steps. First, one shows by induction on s that [6]

(1.3.2) $$[E_i, F_i^s] = [s]_{d_i} F_i^{s-1}[K_i, d_i(1-s)],$$

and then proves (1.3.1) by induction on m for arbitrary s. Other useful special cases of (1.3.1) are:

(1.3.3) $$[E_i^s, F_i] = [s]_{d_i}[K_i; d_i(1-s)]E_i^{s-1},$$

(1.3.4) $$[E_i^s/[s]_{d_i}!, F_i^s] = \prod_{j=-s+1}^{0} [K_i; d_i j] + [s]A_i(q),$$

where $A_i(q) \in \mathcal{U}$ has no poles except for $q = 0$ and $q = \varepsilon$, where $\varepsilon^{2j} = 1$ for $j \in \mathbf{Z}$, $|j| < s$.

Let $\mathcal{U}^+$, $\mathcal{U}^-$ and $\mathcal{U}^0$ be the subalgebras of $\mathcal{U}$ generated by the E_i, the F_i, and the K_i, K_i^{-1} $(i = 1, \ldots, n)$ respectively. It follows from (1.2.1)–(1.2.5) and (1.3.1) that $\mathcal{U} = \mathcal{U}^-\mathcal{U}^0\mathcal{U}^+$. Using the comultiplication [18], it is easy to deduce that, moreover, multiplication defines a $\mathbf{C}(q)$–vector space isomorphism

(1.3.5) $$\mathcal{U} = \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+.$$

1.4. Let P be a free abelian group with basis ω_i, $i = 1, \ldots, n$. Define the following elements:

$$\rho = \sum_{i=1}^{n} \omega_i, \quad \alpha_j = \sum_{i=1}^{n} a_{ij}\omega_i \quad (j = 1, \ldots, n).$$

Let $Q = \sum_i \mathbf{Z}\alpha_i$, $Q_+ = \sum_i \mathbf{Z}_+\alpha_i$. For $\beta = \sum_i k_i\alpha_i \in Q$ let $\mathrm{ht}\beta = \sum_i k_i$. Introduce a partial ordering on P by $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$.

Define a bilinear pairing $P \times Q \to \mathbf{Z}$ by

(1.4.1) $$(\omega_i|\alpha_j) = \delta_{ij} d_i.$$

Then $(\alpha_i|\alpha_j) = d_i a_{ij}$, so that we get a symmetric $\mathbf{Z}$–valued bilinear form on Q such that $(\alpha|\alpha) \in 2\mathbf{Z}$.

Define automorphisms r_i of P by $r_i\omega_j = \omega_j - \delta_{ij}\alpha_i$ $(i, j = 1, \ldots, n)$. Then $r_i\alpha_j = \alpha_j - a_{ij}\alpha_i$. Let W be the (finite) subgroup of $GL(P)$ generated by $r_1, \ldots, r_n$. Then Q is W-invariant and the pairing $P \times Q \to \mathbf{Z}$ is W-invariant. Let

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}, \ R = W\Pi, \ R^+ = R \cap Q.$$

Then, of course, R is a root system corresponding to the Cartan matrix (a_{ij}), W is its Weyl group, R^+ a set of positive roots, etc.

For $\beta = \sum_i n_i\alpha_i \in Q$ we let $K_\beta = \prod_i K_i^{n_i}$. If $\beta = w\alpha_i \in R$ for some $w \in W$, $\alpha_i \in \Pi$, we let:

$$d_\beta = d_i, \ q_\beta = q_i \text{ and } [K_\beta; n] = (K_\beta q^n - K_\beta^{-1} q^{-n})/(q_\beta - q_\beta^{-1}).$$

1.5. Let $U_\mathcal{A}$ be the $\mathcal{A}$-subalgebra of U generated by the elements E_i, F_i, K_i, K_i^{-1}, $[K_i; 0]$ $(i = 1, \ldots, n)$. Let $\mathcal{U}_\mathcal{A}^+$ (resp. $\mathcal{U}_\mathcal{A}^-$) be the $\mathcal{A}$-subalgebra of $\mathcal{U}_\mathcal{A}$ generated by the E_i (resp. F_i) and $\mathcal{U}_\mathcal{A}^0$ that generated by the K_i and $[K_i; 0]$. Note that relations (1.2.1), (1.2.2), (1.2.4), (1.2.5) together with

(1.5.1) $$E_iF_j - F_jE_i = \delta_{ij}[K_i; 0],$$

(1.5.2) $$(q_i - q_i^{-1})[K_i; 0] = K_i - K_i^{-1},$$

are defining relations of the algebra $\mathcal{U}_\mathcal{A}$. Note also that $\mathcal{U}_\mathcal{A}$ is a Hopf algebra with Δ, S and ε defined by (1.2.6)–(1.2.8) and

(1.5.3)
$$\Delta[K_i; 0] = [K_i; 0] \otimes K + K^{-1} \otimes [K_i; 0], \ S[K_i; 0] = -[K_i; 0], \ \varepsilon[K_i; 0] = 0.$$

Finally, note that the elements $[K_i; d_in]$ lie in $\mathcal{U}_\mathcal{A}$ since

(1.5.4) $$[K_i; d_in] = [K_i; 0]q_i^{-n} + K_i[n]_{d_i}.$$

Note also the following useful formulas:

(1.5.5)
$$[K_i; d_in]E_j = E_j[K_i; d_i(n + a_{ij})], \ [K_i; d_in]F_j = F_j[K_i; d_i(n - a_{ij})].$$

Given $\varepsilon \in \mathbf{C}^\times$, we may consider the "specialization" $\mathcal{U}_\varepsilon = \mathcal{U}_\mathcal{A}/(q - \varepsilon)\mathcal{U}_\mathcal{A}$. If $\varepsilon^{2d_i} \neq 1$ for all $i = 1, \ldots, n$, then $\mathcal{U}_\varepsilon$ is an algebra over $\mathbf{C}$ on generators E_i, F_i, K_i, K_i^{-1} and defining relations (1.2.1)–(1.2.5), in which $q = \varepsilon$. We denote by $\mathcal{U}_\varepsilon^+$, $\mathcal{U}_\varepsilon^-$, $\mathcal{U}_\varepsilon^0$ the images of $\mathcal{U}_\mathcal{A}^+$, $\mathcal{U}_\mathcal{A}^-$ and $\mathcal{U}_\mathcal{A}^0$ in $\mathcal{U}_\varepsilon$.

Especially important is the "limiting" specialization $\mathcal{U}_1$.

PROPOSITION 1.5. *$\mathcal{U}_1$ is an associative algebra over* $\mathbf{C}$ *on generators* E_i, F_i, K_i *and* $H_i(= [K_i;0])(i = 1,\dots,n)$ *and the following defining relations:*

(1.5.6) $$[E_i, F_j] = \delta_{ij} H_i,\ [H_i, E_j] = a_{ij} K_i E_j,\ [H_i, F_j] = -a_{ij} K_i F_j$$

(1.5.7) $$K_i \text{ are central elements and } K_i^2 = 1,$$

(1.5.8) $$\underbrace{[E_i, \dots [E_i}_{1-a_{ij} times}, E_j] = 0 = \underbrace{[F_i, \dots [F_i}_{1-a_{ij}\ \text{times}}, F_j]] \text{ if } i \neq j.$$

In particular, $\mathcal{U}_1/(K_i - 1; i = 1,\dots,n)$ *is isomorphic to* $U(\mathfrak{g}(a_{ij}))$, *the universal enveloping algebra of the simple Lie algebra* $\mathfrak{g}(a_{ij})$ *over* $\mathbf{C}$ *associated to the Cartan matrix* (a_{ij}).

PROOF: (1.5.6) follows from (1.5.1), (1.5.4) and (1.5.5); (1.5.7) follows from (1.2.2) and (1.5.2); (1.5.8) follows from (1.2.4) and (1.2.5). □

REMARK 1.5. Note that $\mathcal{U}_1^+ := \mathcal{U}_{\mathcal{A}}^+/(q-1)\mathcal{U}_{\mathcal{A}}^+$ is an algebra on generators E_i $(i = 1,\dots,n)$ and defining relations (1.5.8) for E_i. Hence, being an enveloping algebra of a Lie algebra, $\mathcal{U}_1^+$ has no zero divisors. Hence $\mathcal{U}_{\mathcal{A}}^+$ and $\mathcal{U}^+$ (and similarly $\mathcal{U}_{\mathcal{A}}^-$ and $\mathcal{U}^-$) have no zero divisors. It follows from (1.3.3) that $\mathcal{U}_{\mathcal{A}}$ and $\mathcal{U}$ have no zero divisors. This proof works for an arbitrary symmetrizable generalized Cartan matrix (a_{ij}).

1.6. Given $s \in \mathbf{N}$, we shall often write $E_i^{(s)}$ and $F_i^{(s)}$ for $E_i^s/[s]_{d_i}!$ and $F_i^s/[s]_{d_i}!$, respectively. Following Lusztig [13], introduce the following automorphisms T_i $(i = 1,\dots,n)$ of the algebra $\mathcal{U}$:

(1.6.1)
$$T_i E_i = -F_i K_i,\ T_i E_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)} \text{ if } i \neq j,$$

(1.6.2)
$$T_i F_i = -K_i^{-1} E_i,\ T_i F_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} \text{ if } i \neq j,$$

(1.6.3)
$$T_i K_j = K_j K_i^{-a_{ij}}.$$

Note that

$$T_i\omega = \omega T_i, \tag{1.6.4}$$

$$T_i^{-1} = \varphi T_i \varphi^{-1}. \tag{1.6.5}$$

PROPOSITION 1.6. *[13], [14].*

(a) *Let $w \in W$ and let $r_{i_1} \dots r_{i_k}$ be a reduced expression of w. Then the automorphism $T_w = T_{i_1} \dots T_{i_k}$ of U is independent of the choice of the reduced expression of w.*

(b) *Suppose that $\beta = w\alpha_i \in R^+$, $\alpha_i \in \Pi$. Then $T_w E_i \in U^+$.* □

Proposition 1.6(a) shows that the T_i define a representation of the braid group of W in the group $\mathrm{Aut}_{\mathbf{C}(q)}\mathcal{U}$. Note that when restricted to $\mathcal{U}^0$ this descends to an action of W (given by (1.6.3)), so that $T_w K_\beta = K_{w\beta}$.

REMARK 1.6. Recall that for a Hopf algebra $\mathcal{U}$ one defines the adjoint representation $x \longmapsto \mathrm{ad}x$ in $\mathcal{U}$ by $(\mathrm{ad}x)u = \sum_i a_i u S(b_i)$, where $\Delta x = \sum_i a_i \otimes b_i$. One checks that

$$T_i E_j = (\mathrm{ad} - E_i^{(-a_{ij})})E_j \text{ if } i \neq j.$$

1.7. Fix a reduced expression $r_{i_1} r_{i_2} \dots r_{i_N}$ of the longest element of W. This gives us an ordering of the set of positive roots R^+:

$$\beta_1 = \alpha_{i_1},\ \beta_2 = r_{i_1}\alpha_{i_2},\ \beta_N = r_{i_1} \dots r_{i_{N-1}}\alpha_{i_N}. \tag{1.7.1}$$

Introduce *root vectors* [13], [1]:

$$E_{\beta_s} = T_{i_1} \dots T_{i_{s-1}} E_{i_s},\ F_{\beta_s} = T_{i_1} \dots T_{i_{s-1}} F_{i_s}\ (= \omega E_{\beta_s}).$$

For $k = (k_1, \dots, k_N) \in \mathbf{Z}_+^N$ let $E^k = E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N}$, $F^k = \omega E^k$.

For $k, r \in \mathbf{Z}_+^N$ and $u \in U^0$ define the monomial $M_{k,r;u} = F^k u E^r$. Define the height of this monomial by $\mathrm{ht}(M_{k,r;u}) = \prod_i (k_i + r_i)\mathrm{ht}\beta_i \in \mathbf{Z}_+$, and its degree by

$$d(M_{k,r;u}) = (k_N, k_{N-1}, \dots, k_1, r_1, \dots, r_N, \mathrm{ht}(M_{k,r;u})) \in \mathbf{Z}_+^{2N+1}.$$

We shall view $\mathbf{Z}_+^{2N+1}$ as a totally ordered semigroup with the lexicographical order $<$ such that $u_1 < u_2 < \dots < u_{2N+1}$, where $u_i = (\delta_{i,1}, \dots, \delta_{i,2N+1})$ is the standard basis.

The following important lemma was first stated in [11]. Soibelman kindly sent us a letter with its proof. Also Reshetikhin informed us that he and Kirillov independently discovered this result. Finally, Lusztig pointed out that the proof of the lemma at least for some particular choices of the reduced expression can be easily derived from the quiver approach [17] to quantum groups.

LEMMA 1.7 [11]. *For $i < j$ one has:*

$$E_{\beta_j}E_{\beta_i} - q^{(\beta_i|\beta_j)}E_{\beta_i}E_{\beta_j} = \sum_{k\in\mathbb{Z}_+^N} \rho_k E^k,$$

where $\rho_k \in \mathcal{A}$ and $\rho_k = 0$ unless $d(E^k) < d(E_{\beta_i}E_{\beta_j})$. □

Recall that, given a commutative totally ordered semigroup S, an S-filtration of an algebra A is a collection of subspaces $A^{(s)}$, $s \in S$, such that $\bigcup_s A^{(s)} = A$, $A^{(s)} \subset A^{(s')}$ if $s < s'$, and $A^{(s)}A^{(s')} \subset A^{(s+s')}$. The associated graded algebra is $\mathrm{Gr}A = \oplus_{s\in S}\mathrm{Gr}_s A$, where $\mathrm{Gr}_s A = A^{(s)}/\sum_{s'<s} A^{(s')}$, with the usual multiplication. We shall always assume that every subset of S has a minimal element. Then we may define a degree $d(a) \in S$ of $a \in A$ as the minimal s such that $a \in A^{(s)}$. We have a linear isomorphism $A \xrightarrow{\sim} \mathrm{Gr}A$ denoted by $a \longmapsto \overline{a}$, where $\overline{a} = a \mod \sum_{s<d(a)} A^{(s)} \in \mathrm{Gr}_{d(a)}A$.

Given $s \in \mathbb{Z}_+^{2N+1}$, denote by $\mathcal{U}^{(s)}$ the linear span over $\mathbb{C}(q)$ of the monomials $M_{k,r;u}$ such that $d(M_{k,r;u}) \leq s$. We define $\mathcal{U}_\varepsilon^{(s)} \subset \mathcal{U}_\varepsilon$ similarly.

PROPOSITION 1.7.

(a) *The $\mathcal{U}^{(s)}$, $s \in \mathbb{Z}_+^{2N+1}$, form a filtration of $\mathcal{U}$ (similarly for $\mathcal{U}_\varepsilon$).*

(b) *[13] Elements E^k, $k \in \mathbb{Z}_+^N$, form a basis of $\mathcal{U}^+$ (resp. $\mathcal{U}_\varepsilon^+$) over $\mathbb{C}(q)$ (resp. $\mathbb{C}$).*

(c) *Elements $F^k K_1^{m_1} \ldots K_n^{m_n} E^r$, where $k, r \in \mathbb{Z}_+^N$, $(m_1, \ldots, m_n) \in \mathbb{Z}^n$, form a basis of $\mathcal{U}$ over $\mathbb{C}(q)$ (resp. a basis of $\mathcal{U}_\varepsilon$ over $\mathbb{C}$, provided that $\varepsilon^{2d_i} \neq 1$ for $i = 1, \ldots, n$).*

(d) *The associated graded algebra $\mathrm{Gr}\mathcal{U}$ (resp. $\mathrm{Gr}\mathcal{U}_\varepsilon$ provided that $\varepsilon^{2d_i} \neq 1$, $i = 1, \ldots, n$) is an associative algebra over $\mathbb{C}(q)$ (resp. $\mathbb{C}$) on generators E_α, F_α ($\alpha \in R_+$), $K_i^{\pm 1}$ ($i = 1, \ldots, n$) subject to the following relations:*

$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = 1,\ E_\alpha F_\beta = F_\beta E_\alpha,$$
$$K_iE_\alpha = q^{(\alpha|\alpha_i)}E_\alpha K_i,\ K_iF_\alpha = q^{-(\alpha|\alpha_i)}F_\alpha K_i,$$
$$E_\alpha E_\beta = q^{(\alpha|\beta)}E_\beta E_\alpha, F_\alpha F_\beta = q^{(\alpha|\beta)}F_\beta F_\alpha \text{ if } \alpha > \beta,$$

(resp. same relations with $q = \varepsilon$).

PROOF: It follows from (1.2.3) that $E_\alpha F_\beta = F_\beta E_\alpha$ + (linear combination of the $M_{k,r;u}$ of degree less than $d(E_\alpha F_\beta)$. This together with Lemma 1.7 imply (a) and the relations in (d). The fact that the E^k span

$\mathcal{U}^+$ (resp. $\mathcal{U}_\varepsilon^+$) follows also from Lemma 1.7. Their linear independence is proved as in [13]. This proves (b) and (c). (d) now follows from (c). □

REMARK 1.7. (a) Let $\beta = w\alpha_i$. Applying T_w to both sides of (1.3.1) and using (1.5.4), we see that (1.3.1) holds if we replace E_i, F_i and K_i by E_β, F_β and K_β.

(b) For different presentations $\beta = w\alpha_i$ the E_β may be not proportional. For example we have:

$$E_{ij} := T_j E_i = -E_j E_i + q_j^{-1} E_i E_j \text{ if } a_{ji} = -1. \tag{1.7.2}$$

1.8. An algebra $\mathcal{P}$ over $\mathbf{C}$ on generators $x_1, \dots, x_k$ and defining relations $x_i x_j = \lambda_{ij} x_j x_i$ for $i > j$ where $\lambda_{ij} \in \mathbf{C}^\times$, $i, j = 1, \dots, k$, is called a quasipolynomial algebra. One may introduce a $\mathbf{N}^k$-gradation in $\mathcal{P}$ by letting $\deg x_i = (\delta_{i1}, \dots, \delta_{ik})$. It is clear that $\mathcal{P}$ is spanned over $\mathbf{C}$ by monomials $x_1^{r_1} \dots x_k^{r_k}$. Moreover, they form a basis of $\mathcal{P}$. This follows by looking at the representation π of $\mathcal{P}$ in the polynomial algebra $\mathbf{C}[t_1, \dots, t_k]$ defined by

$$\pi(x_i) t_1^{m_1} \dots t_k^{m_k} = \left(\prod_{j=1}^{i-1} \lambda_{ij}^{m_j}\right) t_1^{m_1} \dots t_{i-1}^{m_{i-1}} t_i^{m_i+1} t_{i+1}^{m_{i+1}} \dots t_k^{m_k}.$$

As usual, this implies that $\mathcal{P}$ has no zero divisors. Furthermore, let A be a commutative algebra on generators $K_1, \dots, K_s$ with no zero divisors. Let $\mathcal{P}_A = A \otimes_{\mathbf{C}} \mathcal{P}$ with multiplication given by $K_i x_j = \mu_{ij} x_j K_i$, $\mu_{ij} \in \mathbf{C}^\times$. We may extend the $\mathbf{N}^k$-gradation from $\mathcal{P}$ to $\mathcal{P}_A$ by letting $\deg K_i = 0$. This gives $\mathcal{P}_A$ a structure of a free left A-module with basis $\underline{x}^m := x_1^{m_1} \dots x_k^{m_k}$, $m \in \mathbf{Z}_+^k$. It follows that $\mathcal{P}_A$ has no zero divisors as well. Hence $\mathrm{Gr}\mathcal{U}$ and $\mathrm{Gr}\mathcal{U}_\varepsilon$ have no zero divisors by Proposition 1.7d, and we deduce the following corollary.

COROLLARY 1.8. *The algebras $\mathcal{U}$ and $\mathcal{U}_\varepsilon$ have no zero divisors.* □

Let A be an algebra with no zero divisors, let Z be the center of A and $Q(Z)$ the quotient field of Z, and let $Q(A) = Q(Z) \otimes_Z A$. We shall call the algebra A *integrally closed* if for any subring B of $Q(A)$ such that $A \subset B \subset z^{-1}A$ for some $z \in Z$, $z \neq 0$, we have $B = A$. It is clear that for a commutative algebra A this definition of an integrally closed algebra implies the usual one.

PROPOSITION 1.8. *Let A be an integrally closed algebra with no zero divisors. Let C be an S-filtered algebra such that $C_0 = A$ and $\mathrm{Gr}\, C = \mathcal{P}_A$. Assume that each generator x_i of $\mathcal{P} \subset \mathcal{P}_A$ has a preimage $\tilde{x}_i$ in C such that $\tilde{x}_i^\ell$ lies in the center Z of C. Then the algebra C is integrally closed.*

PROOF: Let $z \in Z$, $z \neq 0$, and let B be a subalgebra of $z^{-1}C$ containing C. Let $\varphi \in B$, so that $y := z\varphi \in C$. We can write:

$$y = a\underline{\tilde{x}}^h + \text{lower degree terms, where } a \in A,\ h \in \mathbf{Z}_+^k,$$

$$z = b\underline{\tilde{x}}^r + \text{lower degree terms, where } b \in A,\ r \in \mathbf{Z}_+^k.$$

Pick $N \in \mathbf{Z}$ such that $N\ell \geq h_i$ for each i and set

$$t = (N\ell - h_1, \dots, N\ell - h_k) \in \mathbf{Z}_+^k.$$

Since $\underline{x}^h \underline{x}^t = \lambda \underline{x}^{h+t}$, $\lambda \in \mathbf{C}^\times$, we get:

$$Y := \lambda^{-1} y \underline{\tilde{x}}^t = a\underline{\tilde{x}}^{h+t} + \text{lower degree terms} = \lambda^{-1} z\varphi\underline{\tilde{x}}^t.$$

Letting $\nu = \lambda^{-1}\varphi\underline{\tilde{x}}^t \in B$, we get $Y = z\nu$. For each $m \in \mathbf{N}$ we have:

$$Y^m = z^{m-1}(z\nu^m) \in z^{m-1}C. \tag{1.8.1}$$

Since, by the assumption, $\underline{\tilde{x}}^{h+t} \in Z$, we get:

$$Y^m = a^m \underline{\tilde{x}}^{m(h+t)} + \text{lower degree terms.} \tag{1.8.2}$$

But $z^j = z^{j-1}(b\underline{\tilde{x}}^r + \text{lower degree terms}) = bz^{j-1}\underline{\tilde{x}}^r + \text{lower degree terms}$. Hence we have by induction:

$$z^{m-1} = \lambda_m b^{m-1} \underline{\tilde{x}}^{(m-1)r} + \text{lower degree terms, where } \lambda_m \in \mathbf{C}^\times. \tag{1.8.3}$$

Since $\mathrm{Gr}_s C$ is a free A-module of rank 1 generated by $\underline{x}^s$, $s \in \mathbf{Z}_+^k$, we deduce from (1.8.1), (1.8.2) and (1.8.3) that b^{m-1} divides a^m for all $m \in \mathbf{N}$. It follows that the subalgebra of $Q(A)$ generated by A and a/b is contained in $b^{-1}A$. Since A is integrally closed, we deduce that b divides a.

Note now that $y^m = z^{m-1}(z\varphi^m)$, where $z\varphi^m \in C$. It follows that $mh_i \geq (m-1)r_i$ for all $m \in \mathbf{N}$ and all $i = 1, \dots, k$. Hence $h_i \geq r_i$ for all i.

Suppose now that $B \neq C$ and choose $\varphi \in B \backslash C$ such that $y = z\varphi \in C$ has minimal possible degree. Let $c = a/b$, $\underline{s} = \underline{h} - \underline{r}$ and consider the element

$$\varphi' = \varphi - \mu^{-1} c \underline{\tilde{x}}^{\underline{s}},$$

where $\mu \in \mathbf{C}^\times$ is taken from $\underline{x}^h = \mu \underline{x}^r \underline{x}^s$. But $\varphi' \in B \backslash C$ and $d(z\varphi') = d(y - \mu^{-1} cz\underline{\tilde{x}}^s) < \deg y$, a contradiction. □

An immediate corollary of Propositions 1.8 and 1.7d is

THEOREM 1.8. *The algebra $\mathcal{U}_\varepsilon$ is integrally closed.* □

1.9. Let Q_2^* be the group of all homomorphisms of Q to the group $\{\pm 1\}$. Given $\lambda \in P$ and $\delta \in Q_2^*$, define the Verma module of type δ, $M^\delta(\lambda)$ over $\mathcal{U}$ (which is a vector space over $\mathbf{C}(q)$) as the (unique) $\mathcal{U}$-module having a vector v_λ such that

$$\mathcal{U}^+ v_\lambda = 0,\ K_i v_\lambda = \delta(\alpha_i) q^{(\lambda|\alpha_i)} v_\lambda,$$

and the vectors

$$F^k v_\lambda \ (k \in \mathbf{Z}_+^N)$$

form a basis of $M^\delta(\lambda)$.

Such a module exists due to Proposition 1.7c. Any quotient V of $M^\delta(\lambda)$, called a highest weight $\mathcal{U}$-module, admits a weight space decomposition.

$$V = \oplus_{\eta \in Q_+} V_\eta, \text{ where } V_\eta = \{v \in V | K_i v = \delta(\alpha_i) q^{(\lambda - \eta|\alpha_i)} v\}.$$

We denote the (unique) irreducible quotient of $M^\delta(\lambda)$ by $L^\delta(\lambda) = M^\delta(\lambda)/J^\delta(\lambda)$.

Given a highest weight module V over the algebra $\mathcal{U}$, let v_λ be its highest weight vector and consider its $\mathcal{U}_\mathcal{A}$-submodule (which is an $\mathcal{A}$-module) $V_\mathcal{A} = \mathcal{U}_\mathcal{A} v_\lambda$. This gives rise to a $\mathcal{U}_1$-module $V_1 = V_\mathcal{A}/(q-1)V_\mathcal{A}$. The K_i act on V_1 as $\delta(\alpha_i)$, hence by Proposition 1.5 the module V_1 becomes a highest weight module with highest weight vector v_λ over $U(\mathfrak{g}(a_{ij}))$, the enveloping algebra of the simple finite-dimensional Lie algebra $\mathfrak{g}(a_{ij})$ over $\mathbf{C}$ associated to the Cartan matrix (a_{ij}).

LEMMA 1.9. *[12]. Let V be a Verma module of type δ or an irreducible highest weight module with highest weight λ over $\mathcal{U}$ and let $\eta \in Q_+$. Then*

$$\dim_{\mathbf{C}(q)} V_\eta = \dim_{\mathbf{C}} (V_1)_\eta. \quad \square$$

Given $a = a(q) \in \mathbf{C}(q)$, let $\bar{a} = a(q^{-1})$. A $\mathbf{C}$-bilinear form H on a vector space V over $\mathbf{C}(q)$ with values in $\mathbf{C}(q)$ is called Hermitian if

$$\begin{aligned} H(au, v) &= \bar{a} H(u, v),\ H(u, av) = a H(u, v), \text{and} \\ H(u, v) &= \overline{H(v, u)},\ a \in \mathbf{C}(q),\ u, v \in V. \end{aligned} \tag{1.9.1}$$

The $\mathcal{U}$-module $M^\delta(\lambda)$ carries a unique Hermitian form H, called the contravariant Hermitian form, such that

(1.9.2)

$$H(v_\lambda, v_\lambda) = 1 \text{ and } H(gu, v) = H(u, \omega(g) v) \text{ for } g \in \mathcal{U},\ u, v \in M^\delta(\lambda).$$

We have: $H(M^\delta(\lambda)_\mu, M^\delta(\lambda)_\nu) = 0$ if $\mu \neq \nu$, and Ker $H = J^\delta(\lambda)$. Denote by H_η the restriction of H to $M^\delta(\lambda)_\eta$, $\eta \in Q_+$, and let $\det^\delta_\eta(\lambda)$ denote the determinant of the matrix of H_η in the basis consisting of elements $F^k v_\lambda$, $k \in \text{Par}\,(\eta)$. Here for $\eta \in Q$ we denote by Par (η) the set of all $k \in \mathbf{Z}_+^N$ such that $\sum_i k_i\beta_i = \eta$. Also, for given $\delta \in Q_2^*$, we view K_β as a function on P defined by $K_\beta(\lambda) = \delta(\beta)q^{(\lambda|\beta)}$; any $\varphi \in \mathcal{U}^0$ thereby becomes a function on P with values in $\mathbf{C}(q)$, which we denote by $\varphi^\delta(\lambda)$.

Given $\beta \in R^+$ and $r \in \mathbf{N}$, let $T_{r\beta} = \{\lambda \in P | 2(\lambda+\rho|\beta) = r(\beta|\beta)\}$, and let $T^0_{r\beta} = \{\lambda \in T_{r\beta} | 2(\lambda+\rho|\gamma) \neq m(\gamma|\gamma)$ for all $\gamma \in R^+\backslash\{\beta\}$ and $m \in \mathbf{N}$ such that $m\gamma < r\beta\}$. Note that if $\varphi \in \mathcal{U}^0$ and φ vanishes on $T^0_{r\beta}$ then φ vanishes on $T_{r\beta}$.

PROPOSITION 1.9.
(a) $\det_\eta = \prod_{\beta\in R^+}\prod_{m\in\mathbf{N}}([m]_{d_\beta}[K_\beta;(\rho|\beta) - \frac{m}{2}(\beta|\beta)])^{|\text{Par}(\eta-m\beta)|}$.
(b) *If* $\lambda \in T^0_{r\beta}$, $r \in \mathbf{N}, \beta \in R^+$, *then* $M^\delta(\lambda)$ *contains a submodule isomorphic to* $M^\delta(\lambda - r\beta)$.

PROOF: Let $\lambda \in T_{r\beta}$, $r \in \mathbf{N}$, $\beta \in R^+$. By Shapovalov's formula [22] for the determinant of the contravariant form of the $U(\mathfrak{g}(A))$–module $M(\lambda)$ it follows that $\dim_\mathbf{C}(L^\delta(\lambda)_1)_{r\beta} < \dim_\mathbf{C}(M^\delta(\lambda)_1)_{r\beta}$. Hence by Lemma 1.9 we have: $\dim_{\mathbf{C}(q)} L^\delta(\lambda)_{r\beta} < \dim_{\mathbf{C}(q)} M^\delta(\lambda)_{r\beta}$. It follows that $\det_{r\beta}$ is divisible by $[K_\beta;(\rho|\beta) - \frac{r}{2}(\beta|\beta)]$ and that there exists a non–zero vector $v \in M^\delta(\lambda)_{r\beta} \cap J^\delta(\lambda)$, such that $\mathcal{U}^+ v = 0$, provided that $\lambda \in T^0_{r\beta}$. Since $\mathcal{U}^-$ has no zero divisors it follows that v generates a submodule of $M^\delta(\lambda)$ isomorphic to $M^\delta(\lambda - r\beta)$ which lies in $J^\delta(\lambda)$, proving (b). Hence $\det_\eta$ is divisible by $[K_\beta;(\rho|\beta) - \frac{r}{2}(\beta|\beta)]^{|\text{Par}(\eta-r\beta)|}$. Thus, $\det_\eta$ is divisble by the right–hand side of the formula in question.

To complete the proof of (a) we calculate the leading term of $\det_\eta$. Using formula (1.3.1) it is easy to see (as in [22]) that it equals to

$$\prod_{\beta\in R^+}\prod_{m\in\mathbf{N}}([m]_{d_\beta}!K_\beta^m)^{|\text{Par}(\eta-m\beta)|-|\text{Par}(\eta-(m+1)\beta)|}$$

which proves (a). □

Let $\varepsilon \in \mathbf{C}^\times$. Given a homomorphism $\lambda : \mathcal{U}^0_\varepsilon \to \mathbf{C}$, we define the Verma module $M_\varepsilon(\lambda)$ over $\mathcal{U}_\varepsilon$ as a vector space over $\mathbf{C}$ with an action of $\mathcal{U}_\varepsilon$ having a vector v_λ such that $\mathcal{U}^+_\varepsilon v_\lambda = 0$, $uv_\lambda = \lambda(u)v_\lambda$ for $u \in \mathcal{U}^0_\varepsilon$ and the vectors $F^k v_\lambda$ ($k \in \mathbf{Z}^N_+$) form a basis of $M_\varepsilon(\lambda)$. Such a module exists due to Proposition 1.7c. The module $M_\varepsilon(\lambda)$ admits a Q_+–gradation $M_\varepsilon(\lambda) =$

$\oplus_{\eta\in Q_+} M_\varepsilon(\lambda)_\eta$, where $M_\varepsilon(\lambda)_\eta$ is the linear span over $\mathbf{C}$ of the $F^k v_\lambda$ with $k \in \text{Par}(\eta)$. Note that $M_\varepsilon(\lambda)_\eta \subset \{v \in M_\varepsilon(\lambda) | K_i v = \lambda(K_i)\varepsilon^{-(\eta|\alpha_i)} v\}$, and that this inclusion is an equality if ε is not a root of 1. Note also that $M^1(\lambda)/(q-\varepsilon)M^1(\lambda) = M_\varepsilon(\lambda_\varepsilon)$, where $\lambda_\varepsilon(K_i) = \varepsilon^{(\lambda|\alpha_i)}$.

Let now $|\varepsilon| = 1$. The contravariant Hermitian form H on $M_\varepsilon(\lambda)$ is defined as above (see (1.9.1) and (1.9.2)) (H is then a usual Hermitian form). We define similarly $\det_{\eta,\varepsilon}(\lambda)(\eta \in Q_+)$ as the determinant of H on $M_\varepsilon(\lambda)_\eta$ in the basis $F^k v_\lambda$, $k \in \text{Par}(\eta)$. Proposition 1.9a gives us immediately the following formula (we assume that $\varepsilon^{2d_i} \neq 1$ for all i):

(1.9.3)

$$\det_{\eta,\varepsilon}(\lambda) = \prod_{\beta\in R+}\prod_{m\in\mathbf{N}} \left(\frac{\varepsilon^{md_\beta} - \varepsilon^{-md_\beta}}{(\varepsilon^{d_\beta} - \varepsilon^{-d_\beta})^2}\right)^{|\text{Par}(\eta - m\beta)|}$$
$$\times \left(\lambda(K_\beta)\varepsilon^{(\rho|\beta) - \frac{m}{2}(\beta|\beta)} - \lambda(K_\beta)^{-1}\varepsilon^{-(\rho|\beta)+\frac{m}{2}(\beta|\beta)}\right)^{|\text{Par}(\eta-m\beta)|}.$$

We define the $\mathcal{U}_\varepsilon$-module $L_\varepsilon(\lambda)$ as the (unique) quotient of $M_\varepsilon(\lambda)$ by the maximal Q_+-graded submodule $J_\varepsilon(\lambda)$; this module is irreducible.

Formula (1.9.3) implies the following

COROLLARY 1.9. *Let $\lambda(K_i) = \varepsilon^{(\hat\lambda|\alpha_i)}$, $i = 1, \dots, n$, where $\hat\lambda \in \mathbf{C} \otimes_{\mathbf{Z}} P$ and ε is not a root of 1. Then $M_\varepsilon(\lambda)$ is an irreducible $\mathcal{U}_\varepsilon$-module if and only if $2(\hat\lambda + \rho|\beta) \neq m(\beta|\beta)$ for all $m \in \mathbf{N}$, $\beta \in R^+$.* □

REMARK 1.9. Using the usual argument (see [5] or [8]) one can show that Proposition 1.9b holds for all $\lambda \in T_{r\beta}$ and derive the usual conditions for inclusions of Verma modules over $\mathcal{U}$ and occurence of $L^\delta(\mu)$ in $M^\delta(\lambda)$ (cf. [1]). One can also prove similar results for the $\mathcal{U}_\varepsilon$-modules $M_\varepsilon(\lambda)$ provided that ε is not a root of 1.

1.10. Let $a_{ij} = -a$. Introduce the following notation:

$$E_{j\underbrace{i\dots i}_{a\text{ times}}} = T_i E_j \text{ if } a = 1, 2 \text{ or } 3,$$
$$E_{ji} = (T_i T_j)^{a-1} E_i \text{ if } a = 2 \text{ or } 3,$$
$$E_{iij} = T_j T_i E_j \text{ if } a = 2,\ E_{iij} = T_j T_i T_j E_i \text{ if } a = 3,$$
$$E_{jii} = T_i T_j E_i \text{ and } E_{iiij} = (T_j T_i)^2 E_j \text{ if } a = 3.$$

Then one has the following important identities:

$$E_i^{(s)} E_j = \sum_{k=0}^{a} q_i^{(a-k)s+k} E_{\underbrace{i\dots i}_{k\text{ times}}j} E_i^{(s-k)}, \tag{1.10.1}$$

(1.10.2) $$E_j E_i^{(s)} = \sum_{k=0}^{a} q_i^{(a-k)s+k} E_i^{(s-k)} E_{j\underbrace{i\ldots i}_{k \text{ times}}},$$

(1.10.3) $$E_i^s E_j^{(s)} = q_j^{s^2} E_j^{(s)} E_i^s + q_j^s E_{ij}^s + [s] A_{ij}(q), \text{ if } a_{ji} = -1,$$

where $A_{ij}(q) \in \mathcal{U}$ has no poles except for $q = 0$ and $q = \varepsilon$, where $\varepsilon^{2j} = 1$ for $j \in \mathbf{Z}$, $|j| < s$.

All these formulas can be deduced directly from [13] or proved in a similar way (by induction).

§2. The center of $\mathcal{U}$.

2.1. Let Z be the center of $\mathcal{U}$ (i.e. the set of elements commuting with all elements of $\mathcal{U}$). Since Z, in particular, commutes with $\mathcal{U}^0$, it is clear (in view of Proposition 1.7c), that any element of Z is of the form

(2.1.1) $$z = \sum_{\eta \in Q_+} \sum_{k,r \in \mathrm{Par}(\eta)} F^k \varphi_{k,r} E^r, \text{ where } \varphi_{k,r} \in \mathcal{U}^0.$$

As usual, the map $z \longmapsto \varphi_{0,0}$ is a homomorphism $h: Z \to \mathcal{U}^0$, called the Harish-Chandra homomorphism.

Fix an element $z \in Z$ of the form (2.1.1). It is clear that $z \in Z$ if and only if z acts as a scalar equal to $\varphi_{0,0}(\lambda)$ on each Verma module $M^\delta(\lambda)$, $\lambda \in P$, $\delta \in Q_2^*$. Denote by ϕ_η the matrix $(\varphi_{k,m})_{k,m \in \mathrm{Par}(\eta)}$. We will compute the matrix ϕ_η by induction on $\eta \in Q_+$. Denote by $G_\gamma^\delta(\lambda)$ the matrix of the operator $\sum_{k,r \in \mathrm{Par}(\gamma)} F^k \varphi_{k,r} E^r$ on $M^\delta(\lambda)_\eta$ in the basis $F^s v_\lambda$ ($s \in \mathrm{Par}(\eta)$). By the inductive assumption, we know G_γ for $\gamma < \eta$. We obviously have

(2.1.2) $$G_\eta = \phi_\eta H_\eta.$$

Hence the fact that z acts as a scalar $\varphi_{00}^\delta(\lambda)$ on $M^\delta(\lambda)_\eta$ can be written as follows:

(2.1.3) $$\phi_\eta H_\eta + \sum_{\gamma < \eta} G_\gamma = \varphi_{0,0} I.$$

This gives us an effective way for calculating the coefficients $\varphi_{k,m}$ of the series (2.1.1), given $\varphi_{0,0}$. By construction, the series z commutes with the F_i and K_i. Since z acts as a scalar also on the dual to a Verma module, we see that z commutes also with the E_i, and hence with $\mathcal{U}$. Denote by S the set of all products of elements $[K_\beta; (\rho|\beta) - \frac{m}{2}(\beta|\beta)], \beta \in R^+$, $m \in \mathbb{N}$. In view of Proposition 1.9a, we have:

$$\varphi_{k,m} \in S^{-1}\mathcal{U}^0. \tag{2.1.4}$$

Note also that we have proved that h is an injective homomorphism.

2.2. Define the homomorphism $\gamma : \mathcal{U}^0 \to \mathcal{U}^0$ by $\gamma K_i = q_i K_i$. Then we have:

$$(\gamma\varphi)(\lambda) = \varphi(\lambda + \rho),\ \lambda \in P. \tag{2.2.1}$$

Given $\varphi \in \mathcal{U}^0$, denote by z_φ the series (2.1.1) with $\varphi_{00} = \gamma(\varphi)$ constructed above.

Note that the group Q_2^* acts on $\mathcal{U}^0$ by

$$\delta K_\beta = \delta(\beta) K_\beta,\ \delta \in Q_2^*,\ \beta \in Q.$$

We thus have an action of the group $W \ltimes Q_2^*$ on $\mathcal{U}^0$. Denote by $\tilde{W}$ the subgroup of this group generated by all subgroups $\sigma W \sigma^{-1}$, $\sigma \in Q_2^*$. Note that in the simply laced case, Q_2^* is canonically isomorphic to $P/2P$, and denoting by $\bar{Q}$ the image of Q, we have: $\tilde{W} = W \ltimes \bar{Q}$.

PROPOSITION 2.2. **(a)** *Given $\varphi_{0,0} \in \mathcal{U}^0$, all coefficients $\varphi_{k,m}$ of the corresponding series z lie in $\mathcal{U}^0$ if and only if*

$$\gamma^{-1}(\varphi_{0,0}) \in \mathcal{U}^{0\tilde{W}} := \{\psi \in \mathcal{U}^0 | w\psi = \psi, w \in \tilde{W}\}. \tag{2.2.2}$$

Moreover, we then have:

$$\deg \varphi_{k,m} + \min(|k|, |m|) \leq \deg \varphi_{0,0}. \tag{2.2.3}$$

(b) *We have:*

$$\gamma^{-1} \circ h :\ Z \xrightarrow{\sim} \mathcal{U}^{0\tilde{W}},$$

and the map $\varphi \longmapsto z_\varphi$ establishes the inverse isomorphism.

PROOF: It is easy to see that condition (2.2.2) is equivalent to

$$\lambda \in T_{r\beta} \text{ implies } \varphi_{00}^\delta(\lambda - r\beta) = \varphi_{00}^\delta(\lambda) \tag{2.2.4}$$

for every $\delta \in Q_2^*$, $r \in \mathbf{N}$ and $\beta \in R^+$.

Suppose now that all coefficients $\varphi_{k,m}$ of the series z lie in $\mathcal{U}^0$. Then z is defined on every Verma module $M^\delta(\lambda)$ and acts on it as the scalar $\varphi_{00}^\delta(\lambda)$. But by Proposition 1.9b, $M^\delta(\lambda) \supset M^\delta(\lambda - r\beta)$ provided that $\lambda \in T_{r\beta}^0$, hence $\varphi_{0,0}^\delta(\lambda - r\beta) = \varphi_{0,0}^\delta(\lambda)$ if $\lambda \in T_{r\beta}^0$. It follows that $\varphi_{0,0}^\delta(\lambda - r\beta) = \varphi_{0,0}^\delta(\lambda)$ for all $\lambda \in T_{r\beta}$, hence (2.2.2) is necessary.

Conversely, suppose that (2.2.2) holds. As has been mentioned above, (2.2.4) holds for $\beta \in R^+$ and $r \in \mathbf{N}$. We shall prove by induction on η that the denominator of $\varphi_{k,m}$ with $k, m \in \operatorname{Par}(\eta)$ does not contain factors $[K_\beta; (\rho|\beta) - \frac{r}{2}(\beta|\beta)]$. In view of (2.1.4), this will imply the sufficiency of (2.2.2) for the $\varphi_{k,m}$ to lie in $\mathcal{U}^0$, and also will complete the proof of (b).

Let $T_{r\beta,\eta}^0 = \{\lambda \in T_{r\beta}^0 | 2(\lambda + \rho|\gamma) \neq m(\gamma|\gamma)$ for all $\gamma \in R^+\backslash\{\beta\}$ and $m \in \mathbf{N}$ such that $m\gamma \leq \eta$. Let $\lambda \in T_{r\beta,\eta}^0$; then, by Proposition 1.9, $M^\delta(\lambda) \supset M^\delta(\lambda - r\beta)$ and, moreover,

$$M^\delta(\lambda)_\eta \cap J^\delta(\lambda) = M^\delta(\lambda - r\beta)_{\eta - r\beta}. \tag{2.2.5}$$

It is clear that z coincides on $V := M^\delta(\lambda - r\beta)_{\eta - r\beta}$ with the operator $\sum_{\gamma<\eta} \sum_{k,m\in\operatorname{Par}\gamma} F^k \varphi_{k,m} E^m$ (and is defined on this subspace by the inductive assumption). On the other hand, z acts on V as the scalar $\varphi^\delta(\lambda - r\beta)$, which is $\varphi_{0,0}^\delta(\lambda)$ by assumption (2.2.4). Hence the matrix $B^\delta(\lambda) := \varphi_{0,0}^\delta(\lambda) I - \sum_{\gamma<\eta} G_\eta^\delta(\lambda)$ is zero on V. Thus, by (2.13) and Proposition 1.7a) we have an equality of matrix-valued functions on V:

$$\phi_\eta^\delta(\lambda) H_\eta^\delta(\lambda) = B^\delta(\lambda), \ \lambda \in P,$$

such that $\operatorname{Ker} H_\eta^\delta(\lambda) \subseteq \operatorname{Ker} B^\delta(\lambda)$ (by (2.2.5)) for $\lambda \in T_{r\beta,\eta}^0$, hence for all $\lambda \in T_{r\beta}$, and $\dim \operatorname{Ker} H_\eta^\delta(\lambda) =$ multiplicity of zero of $\det H_\eta^\delta(\lambda)$ for $\lambda \in T_{r\beta,\eta}^0$. Hence, using [7, Lemma 2], we see that the matrix ϕ_η has a removable singularity in the hyperplane $T_{r\beta}$, as desired.

Inequality (2.2.3) is clear by looking at degrees in (2.1.3), where we let $\deg K_i = 1$, $i = 1, \dots, n$. □

2.3. Let now $\varepsilon \in \mathbf{C}^\times$ be not a root of 1. Then for every $\varphi \in \mathcal{U}_\varepsilon^{0\tilde{W}}$ we denote by $z_{\varphi,\varepsilon}$ the element z_φ in which q is replaced by ε. This is an element of the center Z_ε of $\mathcal{U}_\varepsilon$. Defining the Harish–Chandra homomorphism $h_\varepsilon : Z_\varepsilon \to \mathcal{U}_\varepsilon^0$ as in 2.1 and $\gamma : \mathcal{U}_\varepsilon^0 \to \mathcal{U}_\varepsilon^0$ by $\gamma_\varepsilon K_i = \varepsilon^{d_i} K_i$ we obtain from Proposition 2.2 that $\gamma_\varepsilon^{-1} \circ h_\varepsilon : Z_\varepsilon \xrightarrow{\sim} \mathcal{U}_\varepsilon^{0\tilde{W}}$ is an isomorphism, the map $\varphi \longmapsto z_{\varphi,\varepsilon}$ being the inverse homomorphism.

§3. The center of $\mathcal{U}_\varepsilon$ and the group G, when ε is a root of 1.

3.1. Let ε be a primitive ℓ-th root of 1. Let $\ell' = \ell$ if ℓ is odd and $\ell' = \frac{1}{2}\ell$ if ℓ is even. We shall assume that

$$\ell' > d := \max_i\{d_i\} \tag{3.1.1}$$

Then $\mathcal{U}_\varepsilon$ is the algebra over $\mathbf{C}$ on generators E_i, F_i, K_i and K_i^{-1} and defining relations (1.2.1)–(1.2.5), where $q = \varepsilon$.

Lemma 3.1. *The following relations hold in $\mathcal{U}_\varepsilon$:*

$$E_\alpha^{\ell'} E_\beta = \varepsilon^{-(\alpha|\beta)\ell'} E_\beta E_\alpha^{\ell'},\ F_\alpha^{\ell'} F_\beta = \varepsilon^{(\alpha|\beta)\ell'} F_\beta F_\alpha^{\ell'}, \tag{3.1.2}$$

$$E_\alpha^{\ell'} F_\beta = F_\beta E_\alpha^{\ell'},\ F_\alpha^{\ell'} E_\beta = E_\beta F_\alpha^{\ell'}, \tag{3.1.3}$$

$$K_\beta^{\ell'} E_\alpha = \varepsilon^{(\alpha|\beta)\ell'} E_\alpha K_\beta^{\ell'},\ K_\beta^{\ell'} F_\alpha = \varepsilon^{-(\alpha|\beta)\ell'} F_\alpha K_\beta^{\ell'}. \tag{3.1.4}$$

Proof: It suffices to check the relations (3.1.2) and (3.1.3) for $\alpha = \alpha_i$ and $\beta = \alpha_j$ since these relations for arbitrary $\alpha, \beta \in R^+$ follow by using automorphisms T_k; (3.1.4) follows immediately from (1.2.2). If $\alpha = \alpha_i$ and $\beta = \alpha_j$, formula (3.1.3) follows from (1.3.2), formula (3.1.2) for the E's follows from (1.10.1) and for the F's it follows by applying the automorphism φ. □

Denote by Z_ε the center of $\mathcal{U}_\varepsilon$.

Corollary 3.1. **(a)** *All elements E_α^ℓ, F_α^ℓ ($\alpha \in R^+$) and K_β^ℓ ($\beta \in Q$) lie in Z_ε.*

(b) *If ℓ is even, then all elements $E_\alpha^{\ell'}$, $F_\alpha^{\ell'}$ and K_β^ℓ lie in Z_ε iff A is of type B_n ($n \geq 1$).* □

3.2. Consider the Verma module $M_\varepsilon(\lambda)$ over the algebra $\mathcal{U}_\varepsilon$. By formula (3.1.3), we have:

$$\mathcal{U}_\varepsilon^+ F_\alpha^{\ell'} v_\lambda = 0,\ \alpha \in R^+.$$

It follows that $F_\alpha^{\ell'} v_\lambda \in J_\varepsilon(\lambda)$. Let $\overline{M}_\varepsilon(\lambda)$ denote the quotient of $M_\varepsilon(\lambda)$ by the $\mathcal{U}_\varepsilon$–submodule generators by all the vectors $F_\alpha^{\ell'} v_\lambda$, $\alpha \in R^+$. We call $\overline{M}_\varepsilon(\lambda)$ a *diagonal $\mathcal{U}_\varepsilon$–module.* Proposition 1.7b and formulas (3.1.2)–(3.1.4) imply

PROPOSITION 3.2. *Vectors*

(3.2.1) $$F_{\beta_1}^{m_1} \dots F_{\beta_N}^{m_N} v_\lambda, \text{ where } m_i \in \mathbf{Z}_+,\ m_i < \ell',$$

form a basis over $\mathbf{C}$ *of the space* $\overline{M}_\varepsilon(\lambda)$. □

It is clear that $\overline{M}_\varepsilon(\lambda)$ is irreducible if and only if $\overline{M}_\varepsilon(\lambda)_\eta$, $\eta \neq 0$, contains no singular vectors, i.e. no non-zero vectors v such that $E_i v = 0$, $i = 1, \dots, n$. Let $\hat{\lambda} \in \mathbf{C} \otimes_{\mathbf{Z}} P$ be such that $\lambda(K_\beta) = \varepsilon^{(\hat{\lambda}|\beta)}$. From formula (1.9.3) and Proposition 3.2 we obtain

THEOREM 3.2. *The* $\mathcal{U}_\varepsilon$*-module* $\overline{M}_\varepsilon(\lambda)$ *is irreducible if and only if*

(3.2.2) $$\varepsilon^{2(\hat{\lambda}+\rho|\beta)-m(\beta|\beta)} \neq 1 \text{ for all } \beta \in R^+ \text{ and } m \in \mathbf{Z}_+ \text{ such that } m < \ell'.$$ □

COROLLARY 3.2. (a) *If* $\lambda(K_\beta^\ell)^2 \neq 1$ *for all* $\beta \in R^+$, *then the* $\mathcal{U}_\varepsilon$*-module* $\overline{M}_\varepsilon(\lambda)$ *is irreducible.*

(b) *The* $\mathcal{U}_\varepsilon$*-module* $\overline{M}_\varepsilon(-\rho)$, *where* $(-\rho)(K_\beta) = \varepsilon^{-(\rho|\beta)}$, *is irreducible provided that* $\varepsilon^{2md_i} \neq 1$ *for* $m \in \mathbf{Z}_+$ *such that* $1 \leq m < \ell'$ *and* $i = 1, \dots, n$ *(this condition always holds in the simply laced case or if* ℓ *is odd).* □

3.3. For each $\alpha \in R^+$, let $x_\alpha = E_\alpha^\ell$ and $y_\alpha = F_\alpha^\ell$; for each $\beta \in Q$, let $z_\beta = K_\beta^\ell$. We shall often write x_i and y_i for x_{α_i} and y_{α_i}. Denote by Z_0^+ (resp. Z_0^-) the subalgebra of Z_ε generated by the x_α (resp. y_α), $\alpha \in R^+$, and denote by Z_0^0 the subalgebra generated by the z_β, $\beta \in Q$. Let Z_0 be the subalgebra of Z_ε generated by the subalgebras Z_0^+, Z_0^- and Z_0^0. Then, by Proposition 1.6b, $Z_0^\pm \subset \mathcal{U}_\varepsilon^\pm$ and hence

$$Z_0 = Z_0^+ \otimes Z_0^0 \otimes Z_0^- .$$

It is clear that Z_0^0 is the algebra of Laurent polynomials in z_i and z_i^{-1} ($i = 1, \dots, n$), where $z_i = z_{\alpha_i}$.

Denote by Par_ℓ the set of all $k \in \mathbf{Z}_+^N$ such that $0 \leq k_i < \ell$ ($i = 1, \dots, N$), and for $\eta \in Q$ denote by $\mathrm{Par}_\ell(\eta)$ the set of all $k \in \mathrm{Par}_\ell$ such that $\sum_i k_i \beta_i = \eta$.

We obtain from Proposition 1.7c the following

COROLLARY 3.3. (a) *The algebra Z_0^+ (resp. Z_0^-) is a polynomial algebra in the x_α (resp. y_α), $\alpha \in R^+$.*

(b) *The algebra $\mathcal{U}_\varepsilon$ (resp. $\mathcal{U}_\varepsilon^-$, or $\mathcal{U}_\varepsilon^+$) is a free module over Z_0 (resp. Z_0^-) with basis $F^k K_1^{m_1} \dots K_n^{m_n} E^r$ (resp. F^k, or E^k), where $k, r \in \mathrm{Par}_\ell$, $0 \le m_i < \ell$.* □

Given a homomorphism $\lambda : \mathcal{U}_\varepsilon^0 \to \mathbf{C}$ and a homomorphism $\nu : Z_0^- \to \mathbf{C}$ we construct the associated *triangular $\mathcal{U}_\varepsilon$-module* $\overline{M}_\varepsilon(\lambda, \nu)$ as the quotient of $\mathcal{U}_\varepsilon$ by the left ideal generated by the elements E_i, $K_i - \lambda(K_i)$, $K_i^{-1} - \lambda(K_i)^{-1}$, $y_\alpha - \nu(y_\alpha)$ ($i = 1, \dots, n$, $\alpha \in R^+$). Denote by v_λ the image of 1 in $\overline{M}_\varepsilon(\lambda, \nu)$. It is clear from Corollary 3.3b, that vectors

$$F_{\beta_1}^{k_1} \dots F_{\beta_N}^{k_N} v_\lambda, \text{ where } (k_1, \dots, k_N) \in \mathrm{Par}_\ell, \tag{3.3.1}$$

form a basis of $\overline{M}_\varepsilon(\lambda, \nu)$. Denoting by $\overline{M}_\varepsilon(\lambda, \nu)_\eta$ the linear span of vectors (3.3.1) with $(k_1, \dots, k_N) \in \mathrm{Par}_\ell(\eta)$, we obtain a Q_+-gradation $\overline{M}_\varepsilon(\lambda, \nu) = \oplus_\eta \overline{M}_\varepsilon(\lambda, \nu)_\eta$, consistent with the action of $\mathcal{U}_\varepsilon^+$ (but not $\mathcal{U}_\varepsilon^-$). It follows that if the only vector in $\overline{M}_\varepsilon(\lambda, \nu)_\eta$, $\eta \neq 0$, killed by $\mathcal{U}^+$ is zero, then $\overline{M}_\varepsilon(\lambda, \nu)$ is irreducible.

Using formula (1.9.3) we deduce from the above remarks

LEMMA 3.3. *If ℓ is odd and λ satisfies (3.2.2), then the $\mathcal{U}_\varepsilon$-module $\overline{M}_\varepsilon(\lambda, \nu)$ is irreducible.* □

REMARK 3.3. (a) The converse to the last statement is false (in contrast to diagonal modules).

(b) $\overline{M}_\varepsilon(\lambda) = \overline{M}_\varepsilon(\lambda, 0)$ if ℓ is odd.

We use triangular modules to prove the following

PROPOSITION 3.3. (a) *Let $\alpha = w'\alpha_{i'} \in R^+$ and let $E'_\alpha = T_{w'} E_i$, $F'_\alpha = \omega E'_\alpha$. Then $x'_\alpha = E'^{\ell}_\alpha$ lies in Z_0^+ and $y'_\alpha = F'^{\ell}_\alpha$ lies in Z_0^-.*

(b) *$\mathcal{U}_\varepsilon^\pm \cap Z_\varepsilon = Z_0^\pm$.*

(c) *The subalgebra Z_0 is invariant with respect to the automorphisms T_i.*

PROOF: It suffices to prove a) for y'_α, as the case of x'_α follows by applying ω. Recall that by Proposition 1.6b, $y'_\alpha \in \mathcal{U}_\varepsilon^-$. Hence by Corollary 3.3b, we have:

$$y'_\alpha = \sum_{k \in \mathrm{Par}_\ell} a_k F^k, \text{ where } a_k \in Z_0^-. \tag{3.3.2}$$

Choose λ such that $\overline{M}_\varepsilon(\lambda,\nu)$ is irreducible for all ν (see Lemma 3.3). Since, by Schur's lemma, all elements of Z_0 act as scalars on $\overline{M}_\varepsilon(\lambda,\nu)$, applying both sides of (3.3.2) to v_λ we obtain:

$$\nu(y'_\alpha)I = \sum_{k\in \mathrm{Par}_\ell} \nu(a_k)F^k.$$

By (3.3.1) it follows that $y'_\alpha = a_0 \in Z_0^-$, proving (a). The proof of (b) is similar.

Finally, we prove (c). We have from the definition (1.6.1)–(1.6.3):

$$T_i x_i = (-1)^\ell z_i y_i,\ T_i y_i = (-1)^\ell z_i^{-1} x_i, \tag{3.3.3}$$

$$T_i z_\beta = z_{r_i\beta}. \tag{3.3.4}$$

It is known that for every i there exists a reduced expression $r_{i_1} r_{i_2} \dots r_{i_N}$ of the longest element of W such that $i_1 = i$. Then we have:

$$R^+ = \{\alpha_{i_1}, r_{i_1}\alpha_{i_2}, r_{i_1}r_{i_2}\alpha_{i_3}, \dots, r_{i_1} \dots, r_{i_{n-1}}\alpha_{i_n}\},$$

and

$$R^+\backslash\{\alpha_i\} = \{\alpha_{i_2}, r_{i_2}\alpha_{i_3}, \dots, r_{i_2} \dots r_{i_{n-1}}\alpha_{i_n}\}.$$

Since $E_{i_2}^\ell, (T_{i_2}E_{i_3})^\ell, \dots, (T_{i_2} \dots T_{i_{n-1}}E_{i_n})^\ell$ lie in Z_0^+, we obtain, by Proposition 1.6b that their images under T_i also lie in Z_0^+. □

3.4. Let from now on ℓ be an odd integer greater than d (see (3.1.1)). For each $\alpha \in R^+$ define derivations e_α, f_α and $k'_{\pm\alpha}$ of the algebra $\mathcal{U}$ as follows:

(3.4.1)

$$e_\alpha(u) = [E_\alpha^\ell/[\ell]_{d_\alpha}!, u],\ f_\alpha(u) = [F_\alpha^\ell/[\ell]_{d_\alpha}!, u],\ k'_{\pm\alpha}(u) = [K_{\pm\alpha}^\ell/[\ell]_{d_\alpha}!, u].$$

Let $e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$, $k'_i = k'_{\alpha_i}$. Note that if $\alpha = w\alpha_i$, then

$$e_\alpha = T_w e_i T_w^{-1},\ f_\alpha = T_w f_i T_w^{-1}. \tag{3.4.2}$$

A remarkable fact is that the derivations e_α, f_α and $k'_{\pm\alpha}$ can be pushed down to $\mathcal{U}_\varepsilon$, where ε is a primitive ℓ–th root of 1, ℓ odd (if ℓ is even one should replace ℓ by ℓ' in (3.4.1)). For $k'_{\pm\alpha}$ this is clear. Due to (3.4.2) it suffices to check this for the e_i and f_i. In the latter case this is clear from the formulas (1.3.2), (1.3.3), and (1.10.1) and the formula obtained from

(1.10.1) by applying ω. In fact, using them we obtain explicit formulas for the derivations e_i, f_i and $k_{\pm\alpha} = c_\alpha^{-\ell} k'_{\pm\alpha}$. Here and further we let

$$c_\alpha = \varepsilon^{d_\alpha} - \varepsilon^{-d_\alpha} \text{ for } \alpha \in R^+, \ c_j = c_{\alpha_j}.$$

We have (cf. [13] where analogous derivations are considered in the "restricted" case):

$$e_i(F_j) = \frac{\delta_{ij} c_i^{\ell-2}}{\ell}(K_i \varepsilon^{d_i} - K_i^{-1}\varepsilon^{-d_i})E_i^{\ell-1}, \tag{3.4.3}$$

$$e_i(K_j^{\pm 1}) = \mp \frac{(\alpha_i|\alpha_j)c_i^\ell}{(\alpha_i|\alpha_i)\ell} E_i^\ell K_j^{\pm 1}, \tag{3.4.4}$$

$$e_i(E_j) = \frac{1}{2}\sum_{s=1}^{-a_{ij}} \begin{bmatrix} -a_{ij} \\ s \end{bmatrix} (E_i^{(s)} E_j E_i^{(\ell-s)} - E_i^{(\ell-s)} E_j E_i^{(s)}), \tag{3.4.5}$$

$$k_\beta(E_\alpha) = \frac{(\alpha|\beta)}{\ell(\alpha|\alpha)} z_\beta E_\alpha. \tag{3.4.6}$$

(We have specialized $q = \varepsilon$ in (3.4.5)). Indeed (3.4.3, 4 and 6) are easy using (1.3.2), (1.3.3) and the formula

$$[\ell - 1]! = \ell/(\varepsilon - \varepsilon^{-1})^{\ell-1}.$$

Formula (3.4.5) is obtained by taking the difference of (1.10.1) and (1.10.2) and specializing $q = \varepsilon$. Formulas for the f_i follow immediately from (3.4.3)–(3.4.5) by applying the automorphism φ:

$$\varphi e_i \varphi^{-1} = f_i, \ \varphi E_i = F_i, \ \varphi K_i = K_i, \ \varphi\varepsilon = \varepsilon^{-1}.$$

Denote by $\hat{\mathfrak{g}}$ (resp. $\hat{\mathfrak{g}}^+$, or $\hat{\mathfrak{g}}^-$) the subalgebra over Z_0 of the Lie algebra of all derivations of the algebra $\mathcal{U}_\varepsilon$ generated by all the e_α, $k_{\pm\alpha}$ and f_α (resp. e_α, or f_α), $\alpha \in R^+$.

LEMMA 3.4. *The Lie algebra $\hat{\mathfrak{g}}$ is normalized by the T_i.*

PROOF: This follows from (3.4.2) and

$$T_i e_i T_i^{-1} = -z_i f_i - y_i k'_i. \quad \square \tag{3.4.7}$$

PROPOSITION 3.4. *The subalgebra Z_0 is invariant with respect to $\hat{\mathfrak{g}}$.*

PROOF: Trivially, $\hat{\mathfrak{g}} Z_\epsilon \subset Z_\epsilon$ and $\hat{\mathfrak{g}}^\pm Z_0^\pm \subset \mathcal{U}_\epsilon^\pm$. Hence, by Proposition 3.3b:

(3.4.8) $$\hat{\mathfrak{g}}^\pm Z_0^\pm \subset Z_0^\pm .$$

Furthermore, we have by (3.4.4) and (3.4.6):

(3.4.9) $$e_i(z_j^{\pm 1}) = \mp \frac{d_j a_{ji} c_i^\ell}{2d_i} z_j^{\pm 1} x_i .$$

(3.4.10) $$k_\beta(x_\alpha) = \frac{(\alpha|\beta)}{(\beta|\beta)} x_\alpha z_\beta .$$

Now we show that

(3.4.11) $$e_\alpha(y_\beta) \in Z_0 .$$

Indeed, if $\alpha = \beta$, (3.4.11) follows from

(3.4.12) $$e_\alpha(y_\alpha) = (z_\alpha - z_\alpha^{-1}) c_\alpha^{-\ell} .$$

The latter formula is clear if $\alpha = \alpha_i$, by (1.3.4); for general $\alpha \in R^+$ it is deduced by applying T_w. Finally, if $\alpha \neq \beta$, then, using (3.4.2), we may assume that $\alpha = \alpha_i$. We have:

$$T_i e_i(y_\beta) = T_i e_i T_i^{-1} T_i(y_\beta) = -(z_i f_i + y_i k_i') T_i(y_\beta)$$

by (3.4.7). Since $T_i(y_\beta) \in Z_0^-$ and, by (3.4.8), $f_i Z_0^- \subset Z_0^-$, the proof of (3.4.11) is completed. The inclusion $f_\alpha(x_\beta) \in Z_0$ follows from (3.4.11) by applying ω. □

For $\gamma \in R^+$ we shall often let $e_{-\gamma} = f_\gamma$, $x_{-\gamma} = y_\gamma$. Proposition 3.4 implies the following

COROLLARY 3.4. *We have for $\alpha, \beta \in R$:*

$$e_\alpha(x_\beta)|_{x_\gamma = 0, \gamma \in R} = 0 \text{ if } \alpha + \beta \neq 0, \ e_\alpha(z_\beta)|_{x_\gamma = 0, \gamma \in R} = 0. \quad \square$$

The following useful formula follows from (3.4.9) by applying the T_i:

$$e_\alpha(z_\beta) = -\frac{(\alpha|\beta)c_\alpha^\ell}{(\alpha|\alpha)} z_\beta x_\alpha, \quad \alpha \in R, \beta \in Q. \tag{3.4.13}$$

We deduce immediately from (1.10.3) one more formula (in which we put $x_{ij} = E_{ij}^\ell$):

$$\frac{d_i}{d_j}\left(\frac{c_j}{c_i}\right)^\ell e_i(x_j) = -e_j(x_i) = \frac{1}{2}c_j^\ell x_i x_j + x_{ij}, \text{ if } a_{ji} = -1. \tag{3.4.14}$$

REMARK 3.4. Note that in coordinates $\overline{x}_\alpha = c_\alpha^\ell x_\alpha$ and z_i all our formulas for the derivations e_α, f_α and $k_{\pm\alpha}$ are independent of ℓ. We can show that indeed the action of the $e_\alpha (\alpha \in R)$ on Z_0 is independent of ℓ in these coordinates [2] (for the k_α this is clear by (3.4.10)).

3.5. Denote by $\hat{Z}_0$ the algebra of all formal power series in the x_α $(\alpha \in R)$, z_i and z_i^{-1} $(i = 1, \dots, n)$ which converge to a holomorphic function for all complex values of the x_α and all non-zero complex values of the z_i. Let $\hat{\mathcal{U}}_\varepsilon = \hat{Z}_0 \otimes_{Z_0} \mathcal{U}_\varepsilon$, $\hat{Z}_\varepsilon = \hat{Z}_0 \otimes_{Z_0} Z_\varepsilon$. It is clear by (3.4.10) that the series $\exp tk_\alpha$, $t \in \mathbb{C}$, converges to an automorphism of $\hat{\mathcal{U}}_\varepsilon$.

PROPOSITION 3.5. *The series* $\exp te_\alpha$, $t \in \mathbb{C}$, $\alpha \in R$, *converges to an automorphism (over* $\mathbb{C}$*) of the algebra* $\hat{\mathcal{U}}_\varepsilon$.

PROOF: Due to the automorphisms T_i it suffices to show that the series $\exp te_i$ when applied to the K_β, F_j and E_j converges to an element of the algebra $\hat{\mathcal{U}}_\varepsilon$. We obtain from (3.4.4):

$$(\exp te_i)K_\beta = e^{-((\alpha_i|\beta)c_i^\ell/(\alpha_i|\alpha_i)\ell)tx_i} K_\beta. \tag{3.5.1}$$

Furthermore, using (3.4.3), we obtain:

$$e_i^n(F_i) = (-1)^{n-1}c_i^{-2}(c_i^\ell/\ell)^n x_i^{n-1}(\varepsilon^{d_i}K_i + (-1)^n\varepsilon^{-d_i}K_i^{-1})E_i^{\ell-1},$$

It follows that:

(3.5.2)

$$(\exp te_i)F_j = F_j - \delta_{ij}\left(\frac{e^{-t(c_i^\ell/\ell)x_i} - 1}{c_i^2 x_i}\varepsilon^{d_i}K_i + \frac{e^{t(c_i^\ell/\ell)x_i} - 1}{c_i^2 x_i}\varepsilon^{-d_i}K_i^{-1}\right)E_i^{\ell-1}.$$

Finally, define endomorphisms ψ_s of $\mathcal{U}_\varepsilon$ for $s = 1, \dots, \ell - 1$ by: $\psi_s(u) = E_i^s u E_i^{\ell-s}$. Then (3.4.5) can be rewritten as follows:

$$e_i(E_j) = \frac{1}{2} \sum_{s=1}^{-a_{ij}} \begin{bmatrix} -a_{ij} \\ s \end{bmatrix} ([s]_{d_i}![\ell - s]_{d_i}!)^{-1}(\psi_s - \psi_{\ell-s})E_j$$

Since ψ_i and ψ_j commute, it suffices to show that $(\exp t\psi_s)u$ converges to an element of $\hat{\mathcal{U}}_\varepsilon$ if $u \in \hat{\mathcal{U}}_\varepsilon$. We have:

$$(\exp t\psi_s)u = \sum_{r=0}^{\ell-1} t^r E_i^{rs} u E_i^{r(\ell-s)} \sum_{m=0}^{\infty} \frac{(tx_i)^{m\ell}}{(m\ell + r)!}. \qquad \square$$

One deduces from (3.4.12 and 13) the following formulas for the action on $\hat{Z}_0$:

(3.5.3) $$(\exp te_i)y_j = y_j - \delta_{ij} c_i^{-2\ell} \left(z_i \frac{e^{-c_i^\ell tx_i} - 1}{x_i} + z_i^{-1} \frac{e^{c_i^\ell tx_i} - 1}{x_i}\right),$$

(3.5.4) $$(\exp te_i)z_\beta = e^{-((\alpha_i/\beta)c_i^\ell/(\alpha_i/\alpha_i)tx_i} z_\beta .$$

3.6. Recall that we have: $\mathcal{U}_\varepsilon \supset Z_\varepsilon \supset Z_0$, where Z_ε is the center of the algebra $\mathcal{U}_\varepsilon$, $Z_0 = \mathbb{C}[x_\alpha(\alpha \in R); z_i, z_i^{-1} \ (i = 1, \dots, n)]$, and that $\mathcal{U}_\varepsilon$ is a free module over Z_0 of rank ℓ^{2N+n} (Corollary 3.3b).

Since $\mathcal{U}_\varepsilon$ is a finitely generated module over Z_0, it follows [see e.g. [15, (3.5)]) that Z_ε is as well, hence Z_ε is integral over Z_0 and it is a finitely generated algebra.

As usual, for a finitely generated algebra R over $\mathbb{C}$, we denote by Spec R the affine algebraic variety of algebra homomorphisms $R \to \mathbb{C}$. For example, Spec $Z_0 \simeq \mathbb{C}^{2N} \times \mathbb{C}^{\times n}$; denote by H the (n-dimensional) subvariety of Spec Z_0 consisting of the points such that $x_\alpha = 0$ for all $\alpha \in R$ and by H_0 the subset of H consisting of the points such that $z_\alpha^2 \neq 1$ for all $\alpha \in R$.

Denote by G the subgroup of the automorphism group of the algebra $\hat{\mathcal{U}}_\varepsilon$ generated by all automorphisms $\exp te_\alpha$ and $\exp tk_\alpha$, $\alpha \in R$. This group leaves invariant $\hat{Z}_\varepsilon$ and $\hat{Z}_0$ (by Proposition 3.4), hence acts by holomorphic transformations on the algebraic varieties Spec Z_ε and Spec Z_0.

REMARK 3.6. The action of G on Spec Z_0 may be viewed as a "quantization" of the coadjoint action, H being a "quantization" of a Cartan subalgebra and H_0 being the set of regular elements. Note also, that by Proposition 3.3c, the T_i act on Spec Z_0. This action leaves H invariant and induces the action of W on H. Note finally that, by Remark 3.4, the action of G on Spec Z_0 in coordinates $\overline{x}_\alpha$, z_i is independent of ℓ.

LEMMA 3.6. *The set GH_0 contains a non-empty open (in metric topology) subset of Spec Z_0.*

PROOF: Consider the (holomorphic) map $\Psi : \mathbf{C}^{2N} \times H_0 \to \text{Spec } Z_0$ given by $\Psi((t_\alpha)_{\alpha\in R}, u) = (\prod_{\alpha\in R} \exp t_\alpha e_\alpha)u$. It follows from formula (3.4.10) and Corollary 3.4 that the differential $d\Psi$ is non-degenerate at all points $(0, u)$, $u \in H_0$. The lemma follows. □

3.7. Denote by Rep $\mathcal{U}_\varepsilon$ the set of all (non-zero) irreducible finite-dimensional representations (up to equivalence) of the algebra $\mathcal{U}_\varepsilon$ over $\mathbf{C}$. Then we have a sequence of canonical maps:

$$\text{Rep } \mathcal{U}_\varepsilon \xrightarrow{X} \text{Spec } Z_\varepsilon \xrightarrow{\tau} \text{Spec } Z_0. \tag{3.7.1}$$

The map τ is induced by the inclusion $Z_0 \subset Z_\varepsilon$. Since Z_ε is integral over Z_0, τ is a finite and hence surjective morphism. The map X is defined as follows. By Schur's lemma, for $\pi \in \text{Rep } \mathcal{U}_\varepsilon$ we have

$$\pi(u) = \chi_\pi(u)I, \quad u \in Z_\varepsilon, \quad \text{where } \chi_\pi(u) \in \mathbf{C},$$

and we let $X(\pi) = \chi_\pi$.

Given $\chi \in \text{Spec } Z_\varepsilon$, let I^χ denote the ideal of $\mathcal{U}_\varepsilon$ generated by the kernel of the homomorphism $\chi : Z_\varepsilon \to \mathbf{C}$. Let $\mathcal{U}_\varepsilon^\chi = \mathcal{U}_\varepsilon / I^\chi$. Since $\mathcal{U}_\varepsilon$ has no zero divisors (Corollary 1.8b), it is a torsion free (finitely generated) module over Z_ε, hence (see e.g. [15, (3.10)]) $\mathcal{U}_\varepsilon^\chi \neq 0$. Hence the map X is surjective.

Since $\mathcal{U}_\varepsilon$ has no zero divisors, the same is true for Z_ε and we can consider its quotient field $Q(Z_\varepsilon)$. Let $Q(\mathcal{U}_\varepsilon) = Q(Z_\varepsilon) \otimes_{Z_\varepsilon} \mathcal{U}_\varepsilon$; this algebra is finite-dimensional over the field $Q(Z_\varepsilon)$ and has no zero divisors. Hence $Q(\mathcal{U}_\varepsilon)$ is a division algebra over its center $Q(Z_\varepsilon)$, hence it is the quotient algebra of $\mathcal{U}_\varepsilon$. It follows that $\mathcal{U}_\varepsilon$ is a $\mathbf{C}$-subalgebra of a full matrix algebra $\text{Mat}_m(\mathsf{F})$ over some finite extension F of $Q(Z_\varepsilon)$, such that $\mathsf{F} \otimes_{Q(Z_\varepsilon)} \mathcal{U}_\varepsilon = \text{Mat}_m(\mathsf{F})$, for some $m \in \mathbf{N}$ (see e.g. [16]). Then m is called the *degree* of $\mathcal{U}_\varepsilon$. Hence we may consider the characteristic polynomial of $x \in \mathcal{U}_\varepsilon$:

$$\det(\lambda - x) = \lambda^m - (\text{tr } x)\lambda^{m-1} + \ldots + (-1)^m \det x,$$

its norm $\mathcal{N}(x) = \det x$ and the bilinear form $(x, y) = \operatorname{tr} xy$.

Being the center of an integrally closed algebra $\mathcal{U}_\varepsilon$ (see Theorem 1.8), Z_ε is integrally closed. Hence all coefficients of the characteristic polynomial of $x \in \mathcal{U}_\varepsilon$ lie in Z_ε. The determinants $\det((u_i, u_j)_{i,j=1}^{m^2})$, where $u_1, \dots, u_{m^2} \in \mathcal{U}_\varepsilon$, generate a non-zero ideal of Z_ε called the discriminant ideal of $\mathcal{U}_\varepsilon$. The set $\mathcal{D} \subset \operatorname{Spec} Z_\varepsilon$ of zeros of the discriminant ideal is a well-defined closed (proper) subvariety of $\operatorname{Spec} Z_\varepsilon$, called the *discriminant subvariety*. The following facts are special cases of well-known general results:

LEMMA 3.7. (a) *$\mathcal{U}_\varepsilon^\chi$ is isomorphic to $\operatorname{Mat}_m(\mathbb{C})$ if and only if $\chi \notin \mathcal{D}$.*

(b) *If $\chi \in \mathcal{D}$, then $\dim_{\mathbb{C}} \mathcal{U}_\varepsilon^\chi \geq m^2$, but the dimension of every irreducible representation of $\mathcal{U}_\varepsilon^\chi$ is less than m.*

PROOF: Let $\varphi : Z_\varepsilon \to \mathbb{C}$ be a point of $\operatorname{Spec} Z_\varepsilon$ outside $\mathcal{D}$. Then there exist elements $u_1, \dots, u_{m^2}$ in $\mathcal{U}_\varepsilon$ such that $\varphi(\det(u_i, u_j)) \neq 0$. We have: $D := \det((u_i, u_j)) \in Z_\varepsilon$; let $d = \varphi(D)$. Consider the algebra $A := \mathcal{U}_\varepsilon \otimes_\varphi \mathbb{C}$. We claim that the elements $\overline{u}_i := u_i \otimes 1$ form a basis of A over $\mathbb{C}$. Indeed, the u_i being linearly independent over $Q(Z_\varepsilon)$, form a basis of $Q(\mathcal{U}_\varepsilon)$ over $Q(Z_\varepsilon)$. Let $\{u^i\}$ be the dual basis with respect to the trace form. Then $Du^i \in \mathcal{U}_\varepsilon$ for each i. For $u \in \mathcal{U}_\varepsilon$ we have:

$$Du = \sum_i (Du, u^i) u_i = \sum_i (u, Du^i) u_i.$$

Passing to A we get:

$$d\overline{u} = \sum_i \varphi((u, Du^i)) \overline{u}_i.$$

Thus, the $\overline{u}_i$ span A over $\mathbb{C}$ and $\det((\overline{u}_i, \overline{u}_j)) \neq 0$. Hence A is a m^2–dimensional semisimple algebra over $\mathbb{C}$. On the other hand, each element of $\mathcal{U}_\varepsilon$ satisfies a polynomial of degree m with coefficients in Z_ε, hence each element of A satisfies a polynomial of degree m with complex ccefficients. It follows that $A = \operatorname{Mat}_m(\mathbb{C})$ (since $m^2 = n_1^2 + n_2^2 + \dots$, $n_i \geq 0$ and $n_1 + n_2 + \dots \leq m$ imply that $m = n_i$ for some i).

Let now $\rho : \mathcal{U}_\varepsilon \to \operatorname{Mat}_m(\mathbb{C})$ be an irreducible representation and let $\varphi = X(\rho) : Z_\varepsilon \to \mathbb{C}$ be the corresponding element of $\operatorname{Spec} Z_\varepsilon$. Let $u_1, \dots, u_{m^2}$ be elements of $\mathcal{U}_\varepsilon$ such that $\rho(u_i)$ form a basis of $\operatorname{Mat}_m(\mathbb{C})$ over $\mathbb{C}$, and let $D = \det((u_i, u_j))$. Then $\varphi(D) = \det(\operatorname{tr} \rho(u_i)\rho(u_j)) \neq 0$, hence $\varphi \notin \mathcal{D}$.

In order to prove (b) notice that since each element of $\mathcal{U}_\varepsilon$ satisfies a polynomial of degree m with coefficients in Z_ε, every irreducible representation of $\mathcal{U}_\varepsilon$ has dimension $\leq m$. This and (a) give the second part of (b), while the first part follows from the fact that $\mathcal{U}_\varepsilon$ is a torsion free rank m^2 module over Z_ε. □

3.8. Consider a finite-dimensional irreducible representation π of $\mathcal{U}_\varepsilon$ in a complex vector space V. Since Z_0 acts by scalar operators on V, π extends in an obvious way to $\hat{\mathcal{U}}_\varepsilon$. Given $g \in G$, denote by π^g the "twisted" (irreducible) representation of $\mathcal{U}_\varepsilon$ in V defined by

$$\pi^g(u) = \pi(gu), \ u \in \mathcal{U}_\varepsilon.$$

Note that

$$\chi_{\pi^g} = \chi_\pi \circ g, \ g \in G.$$

We call the representation π *diagonalizable* (resp. *triangulizable*) if there exists $g \in G$ such that π^g is a diagonal (resp. triangular) representation.

Denote by Ω the set of all $\lambda \in \operatorname{Spec} Z_0$ such that $(\tau \cdot X)^{-1}\lambda$ consists of ℓ^n irreducible diagonalizable representations of (maximal) dimension ℓ^N. Note that by Corollary 3.2a it follows $H_0 \subset \Omega$. An immediate corollary of Lemma 3.6 and Corollary 3.2a is

LEMMA 3.8. *Ω contains a non-empty open (in metric topology) subset of Spec Z_0.* □

Since, due to Corollary 3.3b:

$$\dim_{Q(Z_0)} Q(\mathcal{U}_\varepsilon) = \ell^{2N+n}, \tag{3.8.1}$$

Lemma 3.8 gives us immediately

$$\dim_{Q(Z_0)} Q(Z_\varepsilon) = \ell^n, \tag{3.8.2}$$

$$m^2 := \dim_{Q(Z_\varepsilon)} Q(\mathcal{U}_\varepsilon) = \ell^{2N}. \tag{3.8.3}$$

The following theorem summarizes the obtained results.

THEOREM 3.8. *Let ℓ be an odd integer, $\ell > \max_i\{d_i\}$ and let ε be a primitive ℓ-th root of 1.*

(a) *Spec Z_ε is a normal affine algebraic variety and the map τ : Spec $Z_\varepsilon \to$ Spec Z_0 is finite (surjective) of degree ℓ^n.*

(b) *If $\chi \in \operatorname{Spec} Z_\varepsilon \backslash \mathcal{D}$ (where the discriminant $\mathcal{D}$ is a closed proper subvariety), then $X^{-1}(\chi)$ consists of a single irreducible representation of $\mathcal{U}_\varepsilon$ of dimension ℓ^N. If $\chi \in \mathcal{D}$, then $X^{-1}(\chi)$ consists of a finite number of irreducible representations of $\mathcal{U}_\varepsilon$ of dimension less than ℓ^N.* □

REMARK 3.8. Let the Cartan matrix A be of type B_n, $n \geq 1$. Let ℓ be a positive even integer, and let, as before, $\ell' = \ell/2$. Then elements $E_\alpha^{\ell'}, F_\alpha^{\ell'}$ ($\alpha \in R^+$) and $K_\beta^{\ell'}$ ($\beta \in Q$) lie in the center Z_ε of the algebra $\mathcal{U}_\varepsilon$ (by Lemma 3.1). Denote by Z_0 the subalgebra generated by all these elements. Then all the results proved above still hold if ℓ is replaced by ℓ'. In particular:

$$\dim_{Q(Z_\varepsilon)} Q(\mathcal{U}_\varepsilon) = \ell'^{2N}, \ \dim_{Q(Z_0)} Q(Z_\varepsilon) = \ell'^{n}. \tag{3.8.4}$$

3.9. Given $\varphi \in \mathcal{U}_\varepsilon^{0\tilde{W}}$, let $\varphi_{0,0} = \varphi(\varepsilon^{d_1}K_1, \dots, \varepsilon^{d_n}K_n)$. Furthermore, for each pair, $k, r \in \mathrm{Par}_\ell$ construct elements $\varphi_{k,r;\varepsilon}$ by formula (2.1.3), in which q is replaced by ε. By formula (1.9.3) and the argument proving Proposition 2.2 (in which Verma modules are replaced by diagonal modules), we see that $\varphi_{k,r;\varepsilon} \in \mathcal{U}_\varepsilon^0$. We let

$$z_{\varphi,\varepsilon} = \sum_{k,r\in \mathrm{Par}_\ell} F^k \varphi_{k,r;\varepsilon} E^r \in \mathcal{U}_\varepsilon.$$

LEMMA 3.9. $z_{\varphi,\varepsilon} \in Z_\varepsilon$.

PROOF: Note that the intersection of kernels of all triangular modules is $M := \sum_{\alpha\in R^+} x_\alpha \mathcal{U}_\varepsilon$. Since, by the construction, $z_{\varphi,\varepsilon}$ acts on each triangular module as a scalar, we obtain: $[z_{\varphi,\varepsilon}, F_i] \in M$. It follows, using Corollary 3.3b, and commutation relations (1.2.1)–(1.2.3), that $[z_{\varphi,\varepsilon}, F_i] = 0$. Since $\omega(z_{\varphi,\varepsilon}) = z_{\varphi,\varepsilon}$ (see §2.1), we obtain that also $[z_{\varphi,\varepsilon}, E_i] = 0$. □

Thus, we have an injective homomorphism $\mathcal{U}_\varepsilon^{0\tilde{W}} \to Z_\varepsilon$ defined by $\varphi \longmapsto z_{\varphi,\varepsilon}$.

§4. The example of $\mathcal{U}$ and $\mathcal{U}_\varepsilon$ of type A_1.

4.1. We consider here the simplest example, that of the quantum algebra $\mathcal{U}$, associated to the matrix (2), which first appeared in the work of Kulish-Reshetikhin and Sklyanin. This is a $\mathbb{Q}(q)$–algebra on generators E, F, K and K^{-1} and defining relations

$$KK^{-1} = 1, \ KEK^{-1} = q^2E, \ KFK^{-1} = q^{-2}F, \tag{4.1.1}$$

(4.1.2) $$EF - FE = (K - K^{-1})/(q - q^{-1}).$$

The center Z of $\mathcal{U}$ contains the well-known element

(4.1.3) $$c = \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2} + FE.$$

Note that $c = z_\varphi$, where $\varphi = (K + K^{-1})/(q - q^{-1})^2$ (see § 2.2). Since φ generates $\mathcal{U}^{0\tilde{W}}$, we deduce that c generates Z.

It is well-known that all finite-dimensional irreducible representations over $\mathbf{Q}(q)$ of $\mathcal{U}$ are equivalent to representations $\pi_n^{\pm}$, $n \in \mathbf{Z}_+$, of dimension $n+1$, which in some basis $v_0, \dots, v_n$ are given as follows (we let $v_{-1} = 0 = v_{n+1}$):

(4.1.4)
$$\pi_n^{\pm}(K)v_j = \pm q^{n-2j}v_j,\ \pi_n^{\pm}(E)v_j = \pm[n-j+1]v_{j-1},\ \pi_n^{\pm}(F)v_j = [j+1]v_{j+1}.$$

These facts still hold for the specialization of q to any complex number different from 0 and a root of 1.

4.2. Let now $\ell > 2$ be an integer, and let, as before $\ell' = \ell$ if ℓ is odd and $\ell' = \ell/2$ if ℓ is even. Let ε be a primitive ℓ'th root of 1, and let $\varepsilon' = \varepsilon$ if ℓ is odd and $= \varepsilon^2$ if ℓ is even. Denote by $\mathcal{U}_\varepsilon$ the algebra over $\mathbf{C}$ on generators E, F, K and K^{-1} and defining relations (4.1.4), (4.1.2) where q is replaced by ε. Let Z_0 be the subalgebra of the center Z_ε generated by $x = E^{\ell'}$, $y = F^{\ell'}$, $z = K^{\ell'}$ and z^{-1}. We have by (3.8.2), (3.8.3) and (3.8.4):

(4.2.1) $$\dim_{Q(Z_\varepsilon)} Q(\mathcal{U}_\varepsilon) = \ell'^2,\ \dim_{Q(Z_0)} Q(Z_\varepsilon) = \ell'.$$

Note that the norm of an element $f(K) \in \mathcal{U}^0$ over $Q(Z_\varepsilon)$ can be calculated as follows:

(4.2.3) $$\mathcal{N}(f(K)) = \prod_{j=0}^{\ell'-1} f(\varepsilon'^j K).$$

We also have:

(4.2.4) $$\mathcal{N}(E) = (-1)^{\ell'+1}x,\ \mathcal{N}(F) = (-1)^{\ell'+1}y.$$

Denote again by c the element of Z_ε given by formula (4.1.3) where q is replaced by ε. Taking norm of both sides of the equality

$$c - \frac{K\varepsilon + K^{-1}\varepsilon^{-1}}{(\varepsilon - \varepsilon^{-1})^2} = FE,$$

we obtain:

(4.2.5) $$\prod_{j=0}^{\ell'-1}\left(c-\frac{K\varepsilon\varepsilon'^{j}+K^{-1}\varepsilon^{-1}\varepsilon'^{-j}}{(\varepsilon-\varepsilon^{-1})^2}\right)=xy.$$

The left-hand side of (4.2.5) is of the form:

$$c^{\ell'}+a_1c^{\ell'-1}+\dots+a_{\ell'-1}c+a,$$

where, clearly, $a_i\in\mathbf{C}$ for $i=1,\dots,\ell'-1$, and

$$a=(-1)^{\ell}(\varepsilon-\varepsilon^{-1})^{-2\ell'}(z+z^{-1}).$$

Hence we can rewrite (4.2.5) in either of the following two forms, where $c_j^{\pm}\in\mathbf{C}$:

(4.2.6±) $$\prod_{j=0}^{\ell'-1}(c-c_j^{\pm})=xy+(-1)^{\ell+1}\frac{z+z^{-1}\pm 2}{(\varepsilon-\varepsilon^{-1})^{2\ell'}}.$$

In order to calculate the constants $c_j^{\pm}(\varepsilon)$, consider the representations $\pi_{n,\varepsilon}^{\pm}$ of $\mathcal{U}_\varepsilon$ over $\mathbf{C}$ defined as follows. If ℓ is odd (resp. even), $\pi_{n,\varepsilon}^{\pm}$ (resp. $\pi_{n,\varepsilon}^{(-)^n}$) is defined for each $n=0,1,\dots,\ell-1$ by formulas (4.1.4) for $\pi_n^{\pm}$ (resp. π_n^{+}), where q is replaced by ε and $0\le j\le n$ (resp. $0\le j\le n$ if $n<\ell'$ and $0\le j\le n-\ell'$ if $n\ge\ell'$).

The proof of the following lemma is standard.

LEMMA 4.2. (a) *Representations $\pi_{n,\varepsilon}^{\pm}$ form a complete non-redundant list of all finite-dimensional irreducible representations of $\mathcal{U}_\varepsilon$ over $\mathbf{C}$ such that $x=y=0$, $z^2=1$.*

(b) $\pi_{n,\varepsilon}^{\pm}(c)=(\pm1)^{\ell}c_n(\varepsilon)$, *where*

(4.2.7) $$c_n(\varepsilon)=\frac{\varepsilon^{n+1}+\varepsilon^{-n-1}}{(\varepsilon-\varepsilon^{-1})^2}.\quad\square$$

Comparing Lemma 4.2 with (4.2.6±), we obtain an explicit form of (4.2.6±):

(4.2.8±)

$$\prod_{j=0}^{\ell-1}(c\pm c_j(\varepsilon))=xy+\frac{z+z^{-1}\pm 2}{(\varepsilon-\varepsilon^{-1})^{2\ell}}\quad\text{if } \ell \text{ is odd,}$$

$$\prod_{\substack{j=0\\ j\ \text{even}\\ \text{(odd)}}}^{\ell-1}(c-c_j(\varepsilon))=xy-\frac{z+z^{-1}\mp 2}{(\varepsilon-\varepsilon^{-1})^{\ell}}\quad\text{if } \ell \text{ is even.}$$

Finally, Z_ε is generated by Z_0 and Z_1. Indeed, if $z \in Z_\varepsilon$, then $z = \sum_{n\geq 0} z_n^\pm$, where $z_n^+ = \sum_{k\geq 0} F^k \varphi_k E^{k+n}, z_n^- = \sum_{k\geq 0} F^{k+n}\psi_k E^k$, $\varphi_k, \psi_k \in \mathcal{U}_\varepsilon^0$ and $z_n^\pm \in Z_\varepsilon$. Since $z_n^\pm$ commute with K, $z_n^\pm = 0$ unless n is a multiple of ℓ'. Hence Z_ε is generated by x, y and central elements of the form $\sum_{k\geq 0} F^k \varphi_k E^k = \sum_{k\geq \ell'} F^k \varphi_k E^k +$(element from Z_1). Noting that Z_1 is generated by c, we are done. (An alternative proof: due to Section 5.3 it suffices to show that Spec Z_ε' is normal, which is the case since this is a hypersurface, non-singular in codimension 1.)

We can state the results obtained in the A_1 case in the following form:

THEOREM 4.2. *Let $\mathcal{U}_\varepsilon$ be the quantum group of type A_1 at a primitive ℓ'th root of unity ε, $\ell > 2$. Let $\ell' = \ell$ (resp. $\ell/2$) if ℓ is odd (resp. even). Then:*

(a) *The center Z_ε of $\mathcal{U}_\varepsilon$ is generated by the elements x, y, z, z^{-1} and c with defining relations $zz^{-1} = 1$ and (4.2.8$_+$) (or (4.2.8$_-$)).*

(b) *Z_ε is integral over the subalgebra $Z_0 = \mathbb{C}[x, y, z][z^{-1}]$.*

(c) $\dim_{Q(Z_\varepsilon)} Q(\mathcal{U}_\varepsilon) = \ell'^2$, $\dim_{Q(Z_0)} Q(Z_\varepsilon) = \ell'$.

(d) *Spec Z_ε is a 3-dimensional normal affine algebraic variety with the following singular points given in coordinates (x, y, z, c):*

$$a_j^\pm = (0, 0, \pm 1, \pm c_j(\varepsilon)) \text{ for } j = 0, 1, \dots, \ell - 2 \text{ if } \ell \text{ is odd},$$
$$a_j = (0, 0, (-1)^j, c_j(\varepsilon)) \text{ for } j = 0, 1, \dots, \ell - 2,\ j \neq \ell' - 1 \text{ if } \ell \text{ is even.}$$

(e) *The map X : Rep $\mathcal{U}_\varepsilon \to$ Spec Z_ε is surjective and $X^{-1}(a)$ is a single representation of dimension ℓ' if and only if a is a non-singular point of Spec Z_ε. Furthermore $X^{-1}(a_j^\pm)$ consists of two representations $\pi_{j,\varepsilon}^\pm$ and $\pi_{\ell-j-2,\varepsilon}^\pm$ of dimensions $j+1$ and $\ell-j-1$ if ℓ is odd, and $X^{-1}(a_j)$ consists of two representations $\pi_{j,\varepsilon}^{(-)^j}$ and $\pi_{\ell-j-2,\varepsilon}^{(-)^j}$ of dimensions $j+1$ and $\ell' - j - 1$ (resp. $j - \ell' + 1$ and $\ell - j - 1$) if $j < \ell'$ (resp. $j \geq \ell'$) if ℓ is even.* □

REMARK 4.2. **(a)** Since $e(c) = 0 = f(c)$, it follows from (4.2.8) that the polynomial

$$P = (-1)^{\ell+1}(\varepsilon - \varepsilon^{-1})^{2\ell'} xy + z + z^{-1}$$

is fixed by G. Moreover, Z_0^G is generated by P. Denote by $\mathcal{O}_a$ the hypersurface $P = a$, $a \in \mathbb{C}$, in Spec Z_0. It is easy to show that $\mathcal{O}_a$ is a G-orbit if $a \neq \pm 2$, that points $(0, 0, \pm 1)$ are fixed and the complements to them in $\mathcal{O}_{\pm 2}$ are G-orbits. Hence every irreducible representations of $\mathcal{U}_\varepsilon$ of type A_1 is triangulizable, and up to the action of G, the only non-diagonalizable representations are those corresponding to the points $(0, 1, \pm 1)$ of Spec Z_0.

(b) Every ℓ'-dimensional irreducible representation of $\mathcal{U}_\varepsilon$ of type A_1 can be written in some basis $v_0, v_1, \ldots, v_{\ell'-1}$ in the following form, for some $\lambda \in \mathbf{C}^\times$, $a, b \in \mathbf{C}$:

$$Kv_j = \lambda\varepsilon^{-2j}v_j, \quad Fv_j = v_{j+1} \ (j = 0, \ldots, \ell' - 2), \quad Fv_{\ell'-1} = bv_0,$$
$$Ev_j = \Big(\frac{(\lambda\varepsilon^{1-j} - \lambda^{-1}\varepsilon^{j-1})(\varepsilon^j - \varepsilon^{-j})}{(\varepsilon - \varepsilon^{-1})^2} + ab\Big)v_{j-1} \ (j = 1, \ldots \ell' - 1),$$
$$Ev_0 = av_{\ell'-1}.$$

Indeed, we let v_0 be an eigenvector of K, and let $v_j = F^j v_0$ for $j = 1, \ldots, \ell' - 1$.

§5. Open problems.

5.1. The case of even ℓ seems to be more difficult than the case of odd ℓ. First, Z_0 probably should be replaced by Z_0', the intersection of Z_ε with the subalgebra generated by the $K_\beta^{\ell'}$ $(\beta \in Q)$, $E_\alpha^{\ell'}$ and $F_\alpha^{\ell'}$ $(\alpha \in R^+)$, which is a more complicated algebra. Second, the set of diagonalizable modules does not contain an open subset in general.

The only case when these difficulties can be easily resolved is the case of the matrix (a_{ij}) of type B_n, $n \geq 1$, as explained by Remark 3.8.

Let s denote the cardinality of the set $\{\beta \in R^+ | (\beta|Q) \subset 2\mathbf{Z}\}$.

Conjecture 5.1. $m = \ell^N/2^s$ if ℓ is even.

5.2.*Conjecture 5.2.* **(a)** $\Omega = \text{Spec } Z_0 \backslash \tau(\mathcal{D})$.

(b) $\tau(\mathcal{D}) = \overline{G(H\backslash H_0)}$.

(c) $\mathcal{D}$ is the set of singular points of Spec Z_ε.

5.3. Denote by Z_1 the image of the homomorphism $\mathcal{U}_\varepsilon^{0\tilde{W}} \to Z_\varepsilon$ constructed in Section 3.9 and by Z_ε' the subalgebra of Z_ε generated by Z_0 and Z_1. It is clear that $\dim_{Q(Z_0)} Q(Z_\varepsilon') \geq \ell^n$. It follows from (3.7.3) that $Q(Z_\varepsilon') = Q(Z_\varepsilon)$. Since Z_ε' is integral over Z_0, hence over Z_ε, we conclude that the embedding $Z_\varepsilon' \subset Z_\varepsilon$ induces the normalization map:

Spec $Z_\varepsilon \to$ Spec Z'_ε. Note also that using diagonal modules, it is easy to show that

$$Z_\varepsilon \subset Z'_\varepsilon + \sum_{\alpha \in R} x_\alpha \mathcal{U}_\varepsilon .$$

Conjecture 5.3. $Z'_\varepsilon = Z_\varepsilon$.
This is checked in the A_1 case in § 4.

5.4. Let $\lambda \in H_0$, so that the diagonal representation π_λ of $\mathcal{U}_\varepsilon$ in $\overline{M}(\lambda)$ is irreducible. Then

$$\pi_\lambda^{T_i} = \pi_{r_i . \lambda}, \tag{5.4.1}$$

where $(r_i.\lambda)(K_\beta) = \varepsilon^{-(\beta|\alpha_i)}\lambda(r_i K_\beta)$. This follows from two easy facts: v_λ is a unique up to a constant factor singular vector of π_λ, and $F_i^{\ell-1}v_\lambda$ is a singular vector of $\pi_\lambda^{T_i}$. It follows from (3.8.1), that $\pi_\lambda(T_i z)v_\lambda = \pi_\lambda^{T_i}(z)v_\lambda = \pi_{r_i.\lambda}(z)v_\lambda = \pi_\lambda(z)v_\lambda$ for $z \in Z_1$, and any $\lambda \in H$. Thus,

$$T_i z = z \text{ mod } \sum_{\alpha \in R} x_\alpha Z_\varepsilon \text{ if } z \in Z_1. \tag{5.4.2}$$

Conjecture 5.4 (a) All elements of Z_1 are fixed by the T_i.
(b) $Z_1 = \mathcal{U}_\varepsilon^G$.

5.5. We propose below the following hypothetical picture for the action of G on Spec Z_0, similar to the coadjoint action of a simple algebraic group (see [10] in characteristic 0 and [9] in characteristic p).

Denote by $N(H)$ (resp. $C(H)$) the subgroup of those elements of G that leave H invariant (resp. pointwise fixed, and let $W_0 = N(H)/C(H)$.

Conjecture 5.5.1 (a) Closed G–orbits are precisely those which intersect H.

(b) The intersection of a closed G–orbit with H is a W_0–orbit.

(c) W_0 is a finite group.

(d) The restriction homomorphism from Spec Z_0 to H induces an isomorphism of algebras of invariants:

$$Z_0^G \simeq \mathbb{C}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]^{W_0}.$$

It is easy to show that the fixed points of G are $\chi \in$ Spec Z_0 such that $\chi(x_\alpha) = 0$ $(\alpha \in R)$ and $\chi(z_i)^2 = 1$ $(i = 1, \dots, n)$. We call a G–orbit *nilpotent* if its closure contains a fixed point of G.

Conjecture 5.5.2. There are finitely many nilpotent orbits.

Conjecture 5.5.3. Any element of Spec Z_0 can be transformed by G to an element χ such that $\chi(y_\alpha) = 0$, and $\chi(z_\alpha)^2 \neq 1 \Longrightarrow \chi(x_\alpha) = 0$ ($\alpha \in R^+$). In particular every irreducible finite dimensional representation of $\mathcal{U}_\epsilon$ is triangulizable.

Given χ as in Conjecture 5.5.3, we define χ_s, the *semisimple part* of χ by $\chi_s(z_\alpha) = \chi(z_\alpha)$, $\chi_s(x_\alpha) = 0$ ($\alpha \in R$).

Conjecture 5.5.4. (a) The semisimple part χ_s of an element $\chi \in$ Spec Z_0 is well-defined up to a G-conjugacy.

(b) The closure of the G-orbit of χ contains χ_s.

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