COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 35, n° 1 (1977), p. 65-82.
<http://www.numdam.org/item?id=CM_1977__35_1_65_0>

COMPOSITIO MATHEMATICA, Vol. 35, Fasc. 1, 1977, pag. 65–82
Noordhoff International Publishing
Printed in the Netherlands

BORDISM WITH CODIMENSION-ONE SINGULARITIES

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0. Introduction

Throughout the first part of this introduction we shall place ourselves in the differentiable category.

Thus 'manifold' will mean differentiable manifold.

Let l be an integer ≥ 2 and G a group of permutations on l-letters. Define a *G-manifold* of dimension n to be a pair M, S_M, where $M - S_M$ is an oriented manifold of dimension n, S_M is a (possibly empty) oriented manifold of dimension $n-1$ and a typical neighbourhood of S_M in M is the mapping cylinder on an l-sheeted covering of S_M having G as a structure group.

There is a bordism group $\hat{\Omega}_n^G$ of G-manifolds of dimension n, constructed in the usual way.

The calculation of this group is an interesting but difficult problem. However one can try to compute various bordism groups obtained from $\hat{\Omega}_n^G$ by imposing restrictive conditions on the covering π.

For instance, suppose we require that π be a trivial covering, i.e. $G = (1)$. Then the concept of G-manifold coincides with that of Z_l-manifold introduced by Sullivan in [4]. The bordism group of Z_l-manifolds can be proved to be equal to the bordism group of oriented manifolds with coefficients Z_l, denoted Ω_n (point, Z_l) (see [1] for details and generalisations). Therefore in this case the computation is possible using the universal-coefficient formula.

One other particular case of interest is obtained by requiring that π be a 2-sheeted covering, $G = Z_2$. The associated bordism group is easily proved to be isomorphic to the usual bordism group of unoriented manifolds.

In §4 we give a complete computation of $\hat{\Omega}_n^G$ in the case where l is an odd prime p and π is a principal covering. For this group we continue to use the notation $\hat{\Omega}_n^{Z_p}$. We show that $\hat{\Omega}_n^{Z_p}$ is isomorphic to the reduced bordism group of Z_p-equivariant oriented manifolds of dimension $n+1$. The proof, which works in a more general context, is based on transversality constructions which take place in a classifying space for Z_p.

The concept of G-manifold makes equally sense if we do not start from differentiable manifolds but from the cycles of any connected generalised homology theory (see [1]) $U_*(-)$, so that one can form an associated bordism group $\hat{U}_n^G$. Moreover one can construct bordism groups $\{\hat{U}_n^G(X)\}$ (X a topological space) using maps of G-manifolds into X in the usual way.

In §1 we establish a long exact sequence which generalizes in a natural way the usual Bockstein sequence for Z_l coefficients.

In §2 we take into consideration the bordism group, still denoted $\hat{U}_n^G(X)$, of those G-manifolds in X for which π is principal. Under the assumption that $U_n(X)$ is free of l-torsion we establish a short exact sequence relating $\hat{U}_n^G(X)$ to the bordism group of G-invariant manifolds of dimension $n-1$. This exact sequence, together with the results of §4, yields another short exact sequence (theorem 4.5) which is the Z_p-analogue (p odd prime) of a short exact sequence established by Wall ([6]) in the Z_2-case.

Finally, in §5 we set $l=p$ odd prime and prove that the theory $\hat{\Omega}_n^{S_p}(-)$ does not represent homology with Z_p-coefficients. Here S_p is the full symmetric group on p letters. Note that, if $l=2$, then $\hat{\Omega}_n^{S_2}(-)$ is just unoriented differentiable bordism and therefore a famous theorem of Thom's ([5]) ensures that $\hat{\Omega}_n^{S_2}(-)$ represents Z_2-homology. The fact that $\hat{\Omega}_n^{S_p}(-)$ does not map onto Z_p-homology is interesting because in [3] Rourke has proved that the multiplicative closure of $\hat{\Omega}_n^{S_p}(-)$ does represent Z_p-homology.

1. A Long Exact Sequence

Let U be an oriented geometric homology theory. The cycles of U, endowed with a well-defined orientation will be called U-manifolds. Let G be a group of permutations on l letters.

A covering $\pi\colon \bar{M} \to M$ of order l is a *(U, G)-covering* of dimension n if

(1) $\bar{M}$ and M are U-manifolds of dimension n (possibly with boundary) and π is orientation preserving

(2) G is a structural group for π.

$\pi|\colon \partial\bar{M} \to \partial M$ is the *boundary* of the (U, G)-covering π, written $\partial\pi$; it is clearly a (U, G)-covering of dimension $n-1$. Then there is an obvious notion of *bordism* of (U, G)-coverings and the bordism classes can be added by disjoint union. Thus, for each integer $n \geq 0$, we obtain an abelian group U_n^G, the *group of bordism classes of* (U, G)*-coverings of dimension n.*

We now give the basic definition of this paper.

A $\hat{U}^G$*-manifold of dimension n* is a pair P, S_P such that:

(1) $P - S_P$ is a (possibly non-compact) U-manifold of dimension n

(2) S_P is a (possibly empty) compact U-manifold of dimension $n-1$

(3) there exists a homeomorphism

$$h: (N, S_p) \to (Z_\pi, S_P)$$

where:

N is a neighbourhood of S_P in P

π: $\bar{S}_P \to S_P$ is a (U, G)-covering

Z_π is the mapping cylinder of π.

Moreover we require that $h|S_P$ is the identity and $h|N - S_P \to Z_\pi - S_P$ is an isomorphism of U-manifolds.

Note: S_P stands for '*singularity of P*'. Of course if $S_P = \emptyset$, then $N = Z_\pi = \emptyset$.

There is an obvious notion of boundary of a $\hat{U}^G$-manifold of dimension n, the boundary being itself a $\hat{U}^G$-manifold of dimension $n-1$. Therefore the *group*, $\hat{U}_n^G$, of bordism classes of $\hat{U}^G$-manifolds of dimension n is formed in the usual way.

For the sake of simplicity and when no confusion is possible we shall make use of the following notation and terminology: 'manifold' will stand for U-manifold, 'covering' for (U, G)-covering without boundary, 'relative covering' for covering with boundary; we shall write $\hat{U}$ instead of $\hat{U}^G$ and refer to a $\hat{U}$-manifold also as a 'manifold with singularities'. Finally [ ] will denote 'bordism class' in the appropriate category (U, U^G or $\hat{U}$).

Now let X be a topological space. Groups $U_n^G(X)$ and $\hat{U}_n(X)$ can be defined in the usual way by mapping coverings and $\hat{U}$-manifolds of dimension n respectively into X. Therefore the elements of $U_n^G(X)$ are bordism classes $[\pi, \bar{f}, f]$ where $\pi: \bar{M} \to M$ is a covering of dimension n, f is a continuous map and $\bar{f} = f\pi$. The elements of $\hat{U}_n(X)$ are bordism classes $[P, g]$ where P is a $\hat{U}$-manifold of dimension n and g is a continous map $P \to X$. In the following, for the sake of clarity of exposition and when no confusion is possible, we shall omit to mention the various maps into X.

We want to set up homomorphisms connecting the groups $\{U_n(X)\}$, $\{U_n^G(X)\}$ and $\{\hat{U}_n(X)\}$.

The *singularity map* $\sigma\colon \hat{U}_n(X) \to U^G_{n-1}(X)$ assigns to each $\hat{U}$-manifold P, S_P the covering $\pi\colon \bar{S}_P \to S_P$ mentioned in the definition of P; π is mapped into X by restricting the given map $P \to X$. It is immediately seen that this correspondence yields a well-defined homomorphism of abelian groups.

The *forgetful map* $\phi\colon U^G_n(X) \to U_n(X)$ is defined by $\phi(\pi) = \bar{M}$ for each covering $\pi\colon \bar{M} \to M$ mapped into X. The *natural map* $\psi\colon U_n(X) \to \hat{U}_n(X)$ is defined to be the identity on representatives.

The *augmentation* $\epsilon_n\colon U^G_n(X) \to U_n(X)$ is defined by $\epsilon_n[\pi, \bar{f}, f] = [M, f]$ for $\pi\colon \bar{M} \to M$.

We set $\bar{U}^G_n(X) = \operatorname{Ker} \epsilon_n$.

1.1 THEOREM: *The long sequence*

$$\cdots \to U_n(X) \xrightarrow{\psi} \hat{U}_n(X) \xrightarrow{\sigma} U^G_{n-1}(X) \xrightarrow{\phi} U_{n-1}(X) \to \cdots \qquad \text{(L)}$$

is exact.

PROOF: As the space X does not play any essential role in the proof, we shall assume, for the sake of simplicity, $X = \text{point}$.

(1) $\sigma\psi = 0$. This is obvious since, for each manifold M, $\psi(M)$ does not have any singularities.

(2) $\phi\sigma = 0$. If P, S_P is a $\hat{U}$-manifold, then $\phi\sigma(P)$ bounds a manifold isomorphic to the closure of $P - N$ (see the definition of $\hat{U}$-manifold).

(3) $\psi\phi = 0$. Let $\pi\colon \bar{M} \to M$ be a covering. Then $\psi\phi(\pi)$ is the boundary of the $\hat{U}$-manifold Z_π, M.

(4) $\operatorname{Ker} \sigma \subseteq \operatorname{Im} \psi$. Let $[P] \in \operatorname{Ker} \sigma$. Then the covering $\pi\colon \bar{S}_P \to S_P$ is a boundary, i.e. there exists a relative covering $\pi'\colon \bar{W} \to W$ such that $\partial\pi' = \pi$. Form $P \times [0, 1]$ and glue the mapping cylinder $Z_{\pi'}$, to $P \times 1$ by means of the isomorphism h given in the definition of $\hat{U}$-manifold. The resulting object provides a bordism between $P \cong P \cong 0$ and a $\hat{U}$-manifold without singularities are required.

(5) $\operatorname{Ker} \phi \subseteq \operatorname{Im} \sigma$. If $[\pi\colon \bar{M} \to M]$ is in $\operatorname{Ker} \phi$, then there exists a manifold Q with $\bar{M} = \partial Q$. The $\hat{U}$-manifold P obtained by glueing Q to Z_π along $\bar{M}$ is such that $\sigma(P) = \pi$.

(6) $\operatorname{Ker} \psi \subseteq \operatorname{Im} \phi$. If $[V] \in \operatorname{Ker} \psi$, there exists a manifold with singularities P such that $\partial P \cong V$. Then the manifold $Q = \text{closure of } P - N$ provides a bordism between V and a covering.

Thus exactness is proved $\square$

Now suppose that G is the trivial group 1: a $(U, 1)$-covering is, up to isomorphism, a product $M \times \underline{l} \xrightarrow{\text{projection}} M$ with $\underline{l}$ = discrete set $\{1, 2, \ldots, l\}$. Then the augmentation $\epsilon_n\colon U_n^1(X) \to U_n(X)$, which associates to each trivial covering its base, defines an isomorphism of abelian groups. Moreover, in the case $G = 1$, a manifold with singularities is nothing else than a 'manifold with Z_l-coefficients' as defined in [4] and [1] and the group $\hat{U}_n^1(X)$ coincides, by definition, with $U_n(X; Z_l)$.

The following proposition, the proof of which is trivial, explains how the sequence (L) may be viewed as a generalisation of the Bockstein sequence of $U_n(X; Z_l)$ (see [4] and [1]).

1.2 Proposition: *There is an isomorphism of long exact sequences given by*:

$$\begin{array}{lccccccccc}
(L) \cdots \to & U_n(X) & \xrightarrow{\psi} & \hat{U}_n(X) & \xrightarrow{\sigma} & U_{n-1}^1(X) & \xrightarrow{\phi} & U_{n-1}(X) & \to \cdots \\
& \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \epsilon_{n-1}} & & \downarrow{\scriptstyle id} & \\
\text{Bockstein} \cdots \to & U_n(X) & \xrightarrow{\psi'} & U_n(X; Z_l) & \xrightarrow{\sigma'} & U_{n-1}(X) & \xrightarrow{l_*} & U_{n-1}(X) & \to \cdots
\end{array}$$

where ψ' is the natural map, σ' is restriction to singularities and l_* is 'multiplication by l' □

2. A Short Exact Sequence

From now on we shall restrict our attention to those (U, G)-coverings $\pi\colon \bar{M} \to M$ which satisfy a further condition, namely:

(3) π is a principal covering (i.e. a fibre bundle with fibre G and action given by multiplication).

Of course in this case $l = |G|$ = order of G.

The definitions of the bordism groups of principal (U, G)-coverings and the associated bordism groups of manifolds with singularities are formulated in exactly the same way as their analogues in the unrestricted case and we continue to use the notation of §1.

Clearly propositions 1.1 and 1.2 are not affected by condition (3).

Let BG be a classifying space for the group G. The following theorem is important because it interprets the group U_n^G as the group $U_n(BG)$. Its proof, in the case where X = point and U is oriented differentiable bordism, can be found in [2]. The extension to the general case is trivial.

2.1 Theorem: *Let $[\pi, \bar{f}, f] \in U_n^G(X)$ and $\varphi\colon M \to BG$ be a classify-*

ing map for π. The correspondence $[\pi, \bar{f}, f] \rightsquigarrow [M, g]$, where $g: M \to X \times BG$ is given by $g(z) = (f(z), \varphi(z))$, defines an isomorphism between $U_n^G(X)$ and $U_n(X \times BG)$ $\square$

It follows from 2.1 that commutativity holds in the diagram

$$\begin{tikzcd} U_n^G(X) \arrow[d, phantom, "\cong" rotate=90] \arrow[dr, "\epsilon_n"] & \\ U_n(X \times BG) \arrow[r] & U_n(X) \end{tikzcd}$$

where the vertical arrow is the isomorphism of 2.1 and the horizontal arrow is induced by the projection $X \times BG \to X$. From this commutativity and from exactness, applied to the pair $(X \times BG, X)$, it follows that $\bar{U}_n^G(X)$ can be identified with $U_n(X \times BG, X)$.

From now on X will be assumed to be a CW complex.

NOTATION: For each n, U_n will stand for U_n (point).

2.2 PROPOSITION: *For each element $x \in \bar{U}_n^G(X)$ there exists a power l^t with $l^t x = 0$ (recall that $l = |G|$).*

PROOF: Consider the spectral sequence $\{E_{p,q}^r\}$ associated with $U_n(X \times BG, X)$. In this spectral sequence $E_{p,q}^2 = H_p(X \times BG, X; U_q)$, where $H(-)$ denotes singular homology, and there is a filtration

$$0 \subseteq J_{0,n} \subseteq J_{1,n-1} \subseteq \cdots \subseteq J_{p,q} \subseteq \cdots \subseteq J_{n,0} = \bar{U}_n^G(X).$$

By the universal-coefficient theorem

$$H_p(X \times BG, X; U_q) \cong H_p(X \times BG, X) \otimes U_q \oplus \\ + \operatorname{Tor}(H_{p-1}(X \times BG, X), U_q).$$

By the Kunneth formula

$$H_p(X \times BG, X) \cong \sum_{i=1}^{p} H_{p-i}(X) \otimes H_i(BG) \oplus \sum_{i=0}^{p-1} \operatorname{Tor}(H_{p-i-1}(X), H_i(BG)).$$

Now it is known that, if $z \in \tilde{H}(BG, \mathcal{A})$, $\mathcal{A}$ any abelian group, then $lz = 0$. From this and the Kunneth formula follows that each element $y \in H_p(X \times BG, X)$ is such that $ly = 0$. Similarly for $H_{p-1}(X \times BG, X)$. Therefore the universal-coefficient formula above ensures that for every element $u \in H_p(X \times BG, X, U_q)$ one has $lu = 0$. We deduce that

$|E^2_{p,q}|$ divides some power of l. Because $E^{r+1}_{p,q} = H(E^r_{p,q})$ we obtain inductively that $|E^\infty_{p,q}|$ divides a power of l. From the relationship $E^\infty_{p,q} \cong J_{p,q}/J_{p-1,q+1}$ we deduce that also $|J_{p,q}|$ divides a power of l. Therefore induction on p brings us to the conclusion that $|\bar{U}^G_n(X)|$ is a divisor of some power l^t. In particular, for each $x \in \bar{U}^G_n(X)$ one has $l^t x = 0$, which proves the theorem □

2.3 THEOREM: *If $U_n(X)$ has no l-torsion, we have $\bar{U}^G_n(X) = \operatorname{Ker} \phi$, where $\phi\colon U^G_n(X) \to U_n(X)$ is the forgetful map.*

PROOF: Consider the commutative diagram

$$\begin{tikzcd}
U_n(X) \arrow[d, "i"] \arrow[dr, "l_*"] \arrow[dd, bend right=60, "id"'] & \\
U^G_n(X) \arrow[r, "\phi"] \arrow[d, "\epsilon_n = \epsilon"] & U_n(X) \\
U_n(X) &
\end{tikzcd}$$

where $i[M] = [G \times M \xrightarrow{\text{projection}} M]$.

We must show that $\operatorname{Ker} \phi = \operatorname{Ker} \epsilon$.

$\operatorname{Ker} \epsilon \subseteq \operatorname{Ker} \phi$. Let $x \in \operatorname{Ker} \epsilon$. By theorem 2.2 there exists $t > 0$ with $l^t x = 0$. Since $U_n(X)$ has no l-torsion, $\phi(x) = 0$ necessarily.

$\operatorname{Ker} \phi \subseteq \operatorname{Ker} \epsilon$. Let $x \in \operatorname{Ker} \phi$. Because $\epsilon i = id$, x can be written uniquely as $y + z$ with $y = i(y')$, $y' \in U_n(X)$, and $z \in \operatorname{Ker} \epsilon$. We want to show that $y = 0$. Suppose $y \neq 0$; then $y' \neq 0$ and $\phi(y) = \phi i(y') = l_*(y') \neq 0$ because l_* is injective by the hypothesis that $U_n(X)$ is free of l-torsion. On the other hand $z \in \ker \epsilon \subseteq \operatorname{Ker} \phi$ implies $0 = \phi(x) = \phi(y+z) = \phi(y) + \phi(z) = \phi(y)$ which is a contradiction. It follows that $y = 0$ and the theorem is proved □

2.4 THEOREM: *If $U_n(X)$ has no l-torsion, one has $(\bar{M}, \bar{f}] = l[M, f]$, as elements of $U_n(X)$, for each $[\pi, \bar{f}, f] \in U^G_n(X)$.*

PROOF: Consider the diagram

$$\begin{tikzcd}
U_n(X) \arrow[d, "i"] \arrow[dr, "l_*"] & \\
U^G_n(X) \arrow[r, "\phi"] \arrow[d, "\epsilon"] & U_n(X) \\
U_n(X) \arrow[ur, "l_*"'] &
\end{tikzcd}$$

In order to prove the theorem we need to show that the lower triangle is commutative.

Let $y \in U_n^G(X)$; y can be written uniquely as $u+v$ with $u = i(u')$, $u' \in U_n(X)$ and $v \in \operatorname{Ker} \epsilon$. Then we have

$$l_*\epsilon(y) = l_*\epsilon(u+v) = l_*\epsilon(u) = l_*\epsilon i(u') = l_*u'.$$

On the other hand, $v \in \operatorname{Ker} \phi$ by theorem 2.3 and therefore

$$\phi(y) = \phi(u) + \phi(v) = \phi(u) = \phi i(u') = l_*u'.$$

Thus we have proved that $\phi(y) = l_*\epsilon(y)$ as required □

Note that the proof of 2.4 uses only the inclusion $\ker \epsilon \subseteq \operatorname{Ker} \phi$.

We now come to the main theorem of this section.

2.5 THEOREM: *If $U_n(X)$ has no l-torsion, there is a short exact sequence*:

$$0 \to U_n(X; Z_l) \xrightarrow{\ j\ } \hat{U}_n(X) \xrightarrow{\ \sigma\ } \bar{U}_{n-1}^G(X) \to 0$$

where j is the identity on representatives.

PROOF: Consider the map of exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & U_n(X) \arrow[r, "l*"] \arrow[d, "i"] & U_n(X) \arrow[r, "\psi'"] \arrow[d, equal, "id"] & U_n(X; Z_l) \arrow[r, "\sigma'"] \arrow[d, "j"] & U_{n-1}(X) \arrow[r, "l*"] \arrow[d, "i"] & U_{n-1}(X) \arrow[r, "\psi'"] \arrow[d, equal, "id"] & \cdots \\
\cdots \arrow[r] & U_n^G(X) \arrow[r, "\phi"] & U_n(X) \arrow[r, "\psi"] & \hat{U}_n(X) \arrow[r, "\sigma"] & U_{n-1}^G(X) \arrow[r, "\phi"] & U_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

By 2.3 we have $\operatorname{Im} \sigma = \operatorname{Ker} \phi = \bar{U}_{n-1}^G(X)$.

Therefore, in order to establish the theorem, we only need to prove that j is an isomorphism of $U_n(X; Z_l)$ onto $\operatorname{Ker} \sigma$.

(1) $j(U_n(X; Z_l)) \subseteq \operatorname{Ker} \sigma$. Because l_* is injective, ψ' is onto and therefore, given $x \in U_n(X; Z_l)$, there exists $y \in U_n(X)$ with $\psi'(y) = x$. It follows that

$$\sigma j(x) = \sigma j \psi'(y) = \psi\sigma(y) = 0 \quad \text{so that } j(x) \in \operatorname{Ker} \sigma.$$

(2) $\operatorname{Ker} \sigma \subseteq j(U_n(X, Z_l))$. Given $x \in \operatorname{Ker} \sigma$, there exists $y \in U_n(X)$ with $\psi(y) = x$; then $x' = \psi'(y)$ is such that $j(x') = x$.

(3) j is injective. Let $x \in \operatorname{Ker} j$. Choose $y \in U_n(X)$ such that $\psi'(y) = x$. Because $\psi(y) = 0$ there exists $z \in U_n^G(X)$ with $\phi(z) = y$. On the other hand theorem 2.4 implies that $\operatorname{Im} \phi = \operatorname{Im} l_*$; therefore there

exists $w \in U_n(X)$ such that $l_*(w) = y$. But then

$$x = \psi'(y) = \psi' l_*(w) = 0$$

and everything is proved □

2.6 Corollary (Generalised Rohlin sequence): *If $U_n(X)$ has no l-torsion, there is an exact sequence*

$$U_n(X) \xrightarrow{l_*} U_n(X) \xrightarrow{\psi} \hat{U}_n(X).$$

Proof: The result follows from theorem 1.1, the Bockstein sequence, theorem 2.5 and the commutative diagram

$$\begin{tikzcd} U_n(X) \arrow[r, "l_*"] & U_n(X) \arrow[r, "\psi"] \arrow[dr, "\psi'"'] & \hat{U}_n(X) \\ & & U_n(X; Z_l). \arrow[u, "j"'] \end{tikzcd}$$

□

2.7 Corollary: *Suppose that the theory $U(-)$ is 'oriented differentiable bordism $\Omega(-)$'; $l = |G|$ is an odd number, X is a space whose integral homology is finitely generated and has no odd torsion. Then there is an exact sequence*

$$0 \to \Omega_n(X; Z_l) \xrightarrow{j} \hat{\Omega}_n(X) \xrightarrow{\sigma} \bar{\Omega}^G_{n-1}(X) \to 0$$

and an exact Rohlin sequence

$$\Omega_n(X) \xrightarrow{l_*} \Omega_n(X) \xrightarrow{\psi} \hat{\Omega}_n(X).$$

Proof: By Conner and Floyd [2] (15.2)

$$\Omega_n(X) \cong \sum_{p+q=n} H_p(X; \Omega_q).$$

Therefore, since Ω_q has no odd torsion, $\Omega_n(X)$ has no l-torsion. The result follows from 2.5 and 2.6 □

Before ending this section we give a proposition containing two sufficient conditions for a space X to have an l-torsion-free $U_n(X)$, assuming that U_n(point) is l-torsion-free.

2.8 Proposition: *Suppose U_n is free of l-torsion.*

(1) *If* $\tilde{H}(X; Z)$ *is a torsion group and the torsion is prime to* l, *then* $U_n(X)$ *is also such a group.*
(2) *If* $H(X; Z)$ *has no torsion and the natural map* $e: U_n(X) \to H(X; U_0)$ *is onto, then* $U_n(X)$ *has no* l*-torsion.*

PROOF: Consider the spectral sequence

$$E^2_{p,q} = H_p(X; U_q) \Rightarrow U_{p+q}(X)$$

In the case (1) the universal-coefficient formula implies that $E^2_{p,q}$ is a torsion group, the torsion being prime to l. Therefore the E^∞-term and the limit are also such groups.

In case (2) the universal-coefficient formula implies that $E^2_{p,q} = H_p(X) \otimes U_q$. This, together with the hypothesis on e, implies by a standard argument that the spectral sequence collapses.

The result then follows □

3. Brief Review of Geometric Cohomology Theory

In this section we briefly recall the theory of mock bundles developed in [1].

These objects give a geometric picture, based essentially on trasversality constructions, of the cohomological theory $U^*(-)$ dual to the theory $U_*(-)$ which we are considering. We shall make much use of the language of mock bundles in the following section.

Here we simply outline the main features of mock bundles and the reader is referred to [1] for the details.

Let K be an oriented ball complex.

A (U, q)*-mock bundle* ξ^q/K with base K and total space E_ξ consists of a PL projection $p_\xi: E_\xi \to |K|$ such that, for each $\sigma \in K$, $p_\xi^{-1}(\sigma)$ is a U-manifold of dimension $q + \dim \sigma$ with boundary $p^{-1}(\dot{\sigma})$; $\xi(\sigma) = p^{-1}(\sigma)$ is the *block* over σ. We also require $\epsilon(\xi(\tau), \xi(\sigma)) = \epsilon(\tau, \sigma)$ where τ is a codimension-one face of σ and $\epsilon(-,-)$ denotes the usual incidence number.

If $U(-)$ is not an oriented theory (i.e. the cycles of U are not oriented) then the reference to the incidence numbers is to be omitted.

The empty set is regarded as a manifold of any dimension; thus q may be negative.

Mock bundles ξ, η/K are *isomorphic*, written $\xi \cong \eta$, if there exists a

PL isomorphism $h: E_\xi \to E_\eta$ which restricts to an isomorphism of U-manifolds $h: \xi(\sigma) \to \eta(\sigma)$ for each $\sigma \in K$.

Given ξ/K and $L \subset K$, the *restriction* $\xi|L$ is defined by $E(\xi|L) = p_\xi^{-1}(L)$ and $p_{(\xi|L)} = p_\xi|E(\xi|L)$.

A *(U, q)-mock bundle over* (K, L) is a (U, q)-mock bundle over K which is empty over L.

$\xi_0/(K, L)$ is *cobordant* to $\xi_1/(K, L)$ if there exists a (U, q)-mock bundle $\eta/(K \times I,\ L \times I)$ such that $\eta|K \times \{0\} \cong \xi_0$ and $\eta|K \times \{1\}$ is obtained from ξ_1 by reversing the orientation in each block and each ball of the base. If $U(-)$ is not oriented, $\eta|K \times \{1\} = \xi_1$. Cobordism is an equivalence relation and $[\xi]$ stands for "equivalence class of ξ".

Define $T^q(K, L)$ as the set of cobordism classes of (U, q)-mock bundles over K, L. It is easy to see $T^q(K, L)$ is an abelian group under disjoint union. The zero-element is the empty mock bundle.

3.1 THEOREM: *Let M be a PL manifold of dimension m, which is oriented with respect to the theory $U(-)$; ξ/K a (U, q)-mock bundle over a triangulation of M. Then E_ξ is a U-manifold of dimension $(m+q)$ with boundary $p_\xi^{-1}(\partial M)$.*

The proof of this theorem can be found in [1]. □

In the following, for the sake of simplicity, we shall be deliberately careless about orientation in most of the definitions.

Let ξ^q/K' be a (U, q)-mock bundle and K' a subdivision of K. Then, by 3.1, $p_\xi: E_\xi \to |K|$ is a (U, q)-mock bundle called the *amalgamation* of ξ and written $\mathrm{am}(\xi)$.

One has an abelian-group homomorphism

$$\mathrm{am}: T^q(K') \to T^q(K).$$

3.2 PROPOSITION: *Let $f: |K| \to |L|$ be a simplicial map and $E_\xi \xrightarrow{p_\xi} |L|$ a (U, q)-mock bundle. Then the pull-back $f^{\#}(\xi)$*

$$\begin{tikzcd} E(f^{\#}(\xi)) \arrow[r, dashed] \arrow[d, dashed] & E_\xi \arrow[d, "p_\xi"] \\ |K| \arrow[r, "f"] & |L| \end{tikzcd}$$

is also a (U, q)-mock bundle.

3.3 SUBDIVISION THEOREM: *Given ξ/K and K' a subdivision of K, there exists ξ'/K' such that $\mathrm{am}(\xi') \cong \xi$.*

The proof can be found in [1], II, §2 □

There is also a relative version of the subdivision theorem, which gives a homomorphism

$$sd\colon T^q(K) \to T^q(K').$$

3.4 THEOREM:
(a) *am and sd are inverse isomorphisms*
(b) *the isomorphism-type of $T^q(K)$ depends only on the polyhedron $|K|$.* $\square$

The extension of the above theorems to the case of pairs K, L is straightforward.

We now look at the functoriality of $T^q(-)$.

Let $f\colon P - Q$ be a continous map. We want to define $f^*\colon T^q(Q) \to T^q(P)$. Using the PL approximation theorem we assume that f is a PL map (up to a homotopy). Let K, L be triangulations of P, Q respectively with $f\colon |K| \to |L|$ simplicial and let ξ/L be a (U, q)-mock bundle. We set $f^*[\xi] = [f^{\#}(\xi)]$ (see 3.2). It is not difficult to check that f^* is well-defined, functorial, and depends only on the homotopy class of f.

Also maps between pairs of polyhedra induce well-defined homomorphisms in the same fashion so that we obtain a homotopy functor $T^q(-;-)$ defined on the category of pairs of polyhedra.

Let ξ^q/K and η^r/E_ξ be (U, q)-mock bundles such that the blocks of ξ are PL manifolds oriented with respect to the theory U. Subdivide η so as to obtain that the blocks of ξ are subcomplexes. Then, given $\sigma \in K$, we have that $E(\eta|\xi(\sigma))$ is a U-manifold by 3.1. Then there is a $(U, q+r)$-mock bundle

$$E_\eta \xrightarrow{p_\xi \cdot p_\eta} |K|$$

This gives a *transgression-homomorphism*

$$(p_\xi)_! T^r(E_\xi) \to T^{q+r}(|K|).$$

Let M^m be a closed PL manifold oriented with respect to the theory $U(-)$; ξ^q/M a (U, q)-mock bundle.

Then $p_\xi\colon E_\xi \to M$ represents an element of $U_{m+q}(M)$ by 3.1. Thus we have a function

$$D\colon T^q(M) \to T_{m+q}(M)$$
$$D[\xi] = [p_\xi].$$

3.5 Poincaré Duality Theorem: *D is an isomorphism.*

For the proof see [1], II, §3.

3.6 Alexander–Whitehead Duality Theorem: *Let M^m be a PL manifold oriented with respect to the theory U; $X \subset M$ a subpolyhedron such that M is obtained from X by attaching one m-cell e^m. Then there is an isomorphism*

$$\varphi : T^q(X) \cong T_{m+q}(M, \hat{e})$$

where $\hat{e}$ is the center of e^m.

Sketch of Proof: We shall only define the homomorphism φ. Let $R \subset M$ be the closure of the complement of an oriented m-disk B centered at $\hat{e}$. We define φ to be the composition

$$\begin{tikzcd} T^q(X) \arrow[drr, "\varphi"'] & T^q(R) \arrow[l, "i^*"'] \arrow[r, "a"] & T_{m+q}(R, \partial B) \arrow[d, "i_*"] \\ & & T_{m+q}(M, B) \arrow[d, phantom, "\wr\| \text{ homotopy}"] \\ & & T_{m+q}(M, \hat{e}) \end{tikzcd}$$

where the i's are the injections and a is the map that regards a (U, q)-mock bundle on R as a relative U-manifold in R, ∂B.

Observe that a is the relative version of the Poincaré-duality map.

It is easy to check that φ is an isomorphism □

Terminology: In the following section we shall refer to φ as the process of 'expanding' a mock bundle on X over the regular neighbourhood R.

Finally, the duality isomorphism allows us to state the following

3.7 Theorem: *The functor $\{T^q(-)\}_q$ is naturally equivalent to $\{U^{-q}(-)\}_q$ where $U^*(-)$ is the cohomological theory dual to $U_*(-)$* □

4. The Z_p-case

In this section we examine closely the theory of $\hat{U}^{Z_p}$-manifolds, p an odd prime.

We recall that a classifying space for Z_p may be constructed as follows. Given an integer $A > 0$, consider the sphere

$$S^{2A-1} = \{(z_0, \ldots, z_{A-1}) \in \mathbb{C}^A | \Sigma |z_i|^2 = 1\}$$

where the z_i's are complex coordinates, and define $T\colon S^{2A-1} \to S^{2A-1}$ by

$$T(z_0, \ldots, z_{A-1}) = (e^{2\pi i/p} z_0, \ldots, e^{2\pi i/p} z_{A-1})$$

Then T determines a free action of Z_p on S^{2A-1}; the orbit-space of this action is the lens space of dimension $2A-1$ and is denoted by $L_{2A-1} = S^{2A-1}/T$.

The orbit of $(z_0, \ldots, z_{A-1})$ will be written $[z_0, \ldots, z_{A-1}]$.

Now Z_p acts on the union $E(Z_p) = \cup_{A \geq 1} S^{2A-1}$ by means of T; this action of T on $E(Z_p)$ makes $E(Z_p)$ a universal space for Z_p and $BZ_p = E(Z_p)/T$ a classifying space for Z_p. Moreover BZ_p is a CW complex whose $2A-1$ skeleton is the lens space L_{2A-1}.

Let Q_{2A} be the *CW* complex obtained from L_{2A-1} by attaching one $2A$-cell e^{2A} via the projection-map $q\colon S^{2A-1} \to L_{2A-1}$. It is known that L_{2A+1} is obtained from Q_{2A} by adjoining one $(2A+1)$-cell e^{2A+1}.

From now on we shall make the assumption that the lens spaces L_{2A-1} are endowed with a fixed orientation with respect to the theory U.

4.1 THEOREM: *Fix integers n, $A \geq 0$ with $n < 2A$. There is an isomorphism between $\hat{U}_n = \hat{U}_n(point)$ and $U^{2A-n}(Q_{2A})$.*

PROOF: Let P, S_p be a $\hat{U}$-manifold of dimension n. Let $D^{2A} \subset e^{2A}$ be a disk centered at the origin of e^{2A}. Then $Q_{2A} - \mathring{D}^{2A}$ ($\mathring{D}$ = interior of D) can be identified to the mapping cylinder of the projection $q\colon S^{2A-1} \to L_{2A-1}$. Therefore, since $n < 2A$, there is a (classifying) map

$$\gamma\colon (N, S_p) \to (Q_{2A} - \mathring{D}^{2A}, L_{2A-1})$$

preserving the mapping-cylinder structures on N and $Q_{2A} - \mathring{D}^{2A}$. Here N is the (regular) neighbourhood of S_p mentioned in the definition of P.

Now γ can be extended to a map of $P - N$ into $\mathring{D}^{2A}$ using the connectivity of S^{2A-1}.

This construction gives a map $\theta\colon P, S_p \to Q_{2A}, L_{2A-1}$ which is unique up to homotopy. Moreover, by choosing suitable ball decompositions of P and Q_{2A}, θ becomes and $(n-2A)$-mock bundle on Q_{2A}.

It is not difficult to check that the correspondence $[P] \rightsquigarrow [\theta]$ defines a homomorphism

$$\alpha: \hat{U}_n \to U^{2A-n}(Q_{2A}).$$

That α is an isomorphism follows immediately from the fact that, because L_{2A-1} is oriented, an $(n-2a)$-mock bundle on Q_{2A} can be amalgamated to give a $\hat{U}$-manifold of dimension n. $\square$

4.2 THEOREM: *There is an isomorphism λ between $\hat{U}_n$ and the reduced bordism group $\tilde{U}_{n+1}(BZ_p)$.*

PROOF: Let A be such that $n < 2A$. By 4.1 $\hat{U}_n$ can be identified with $U^{2A-n}(Q_{2A})$. Now we apply the duality theorem 3.6 with $M = L_{2A+1}$, $X = Q_{2A}$, $R =$ 'L_{2A+1} minus an oriented disk centered at the origin of e^{2A+1}' and we obtain an isomorphism λ_{2A+1}: $\hat{U}_n \to \tilde{U}_{n+1}(L_{2A+1})$. Because $n < 2A$ the natural inclusion j^{2A+1}: $L_{2A+1} \subset BZ_p$ induces an isomorphism j_*^{2A+1}: $\tilde{U}_{n+1}(L_{2A+1}) \to \tilde{U}_{n+1}(BZ_p)$.

We now prove that the composition $j_*^{2A+1}\lambda_A$ does not depend on A. In order to clarify the geometry of the following argument we first remark that the proof of 3.6 shows that the isomorphism λ_{2A+1}: $U_n \to \tilde{U}_{n+1}(L_{2A+1})$ is given by expanding a $\hat{U}$-manifold, thought of as a mock bundle on Q_{2A}, over the regular neighbourhood R. The inverse isomorphism λ_{2A+1}^{-1} is given by first regarding an $(n+1)$-manifold V in L_{2A+1} as a mock bundle on L_{2A+1} (this is essentially Poincaré Duality) and then taking its restriction to Q_{2A} (this is transversality with respect to Q_{2A}).

In $S^{2A+1} = \{(z_0, \ldots, z_A) | \Sigma |z_i|^2 = 1\}$ consider the subspace $\bar{Q}_{2A} = \{(z_0, \ldots, z_{A-1}, \; t\, e^{2\pi ik/p}) | t \in R$ and $|z_0|^2 + \cdots + |z_{A-1}|^2 + t^2 = 1;\; k = 0, \ldots, p-1\}$.

Clearly $\bar{Q}_{2A}$ is invariant under the Z_p-action and its orbit space is Q_{2A}. Let μ_{2A}: $Q_{2A} \subset L_{2A+1}$ be the inclusion induced by $\bar{Q}_{2A} \subset S^{2A+1}$.

Let μ_{2A+1}: $L_{2A+1} \subset Q_{2A+2}$ be the inclusion

$$[z_0, \ldots, z_A] \rightsquigarrow [z_0, \ldots, z_A, 0] \in Q_{2A+2}.$$

Let i_{2A}: $Q_{2A} \hookrightarrow Q_{2A+2}$ be the inclusion

$$[z_0, \ldots, z_{A-1}, t\, e^{2\pi ik/p}] \rightsquigarrow [0, z_0, \ldots, z_{A-1}, t\, e^{2\pi ik/p}].$$

It is easy to see that i_{2A} can be regarded as the projection of a mock bundle.

Finally, let ν^{2A+1}: $L_{2A+1} \hookrightarrow L_{2A+3}$ be the inclusion $[z_0, \ldots, z_A] \rightsquigarrow$

$[z_0, \ldots, z_A, 0]$ and $\bar{\nu}^{2A+1}$: $L_{2A+1} \hookrightarrow L_{2A+3}$ the inclusion

$$[z_0, \ldots, z_A] \rightsquigarrow [0, z_0, \ldots, z_A].$$

Now consider the commutative diagram

$$\begin{tikzcd}
Q_{2A} \arrow[r, hook, "\mu_{2A}"] & L_{2A+1} \arrow[d, hook, "\mu_{2A+1}"'] \arrow[dr, "\nu^{2A+1}"] \arrow[drr, "j^{2A+1}"] & & \\
& Q_{2A+2} \arrow[r, hook, "\mu_{2A+2}"'] & L_{2A+3} \arrow[r, hook, "j^{2A+3}"'] & BZ_p
\end{tikzcd}$$

Let $[P] \in \hat{U}_n$ and $[\theta]$ be the corresponding element of $U^{2A-n}(Q_{2A})$ (theorem 4.1). From definitions and commutativity we have

$$j_*^{2A+1}\lambda_{2A+1}[P] = j_*^{2A+3}\nu_*^{2A+1}D\mu_{2A}^*[\theta]$$

where D stands for Poincaré dual.

Consider also the commutative diagram

$$\begin{tikzcd}
Q_{2A} \arrow[r, hook, "i_{2A}"'] \arrow[drr, hook, "\mu_{2A}"'] & Q_{2A+2} \arrow[r, hook, "\mu_{2A+2}"'] & L_{2A+3} \arrow[r, hook, "j^{2A+3}"'] & BZ_p \\
& & L_{2A+1} \arrow[u, "\bar{\nu}^{2A+1}"'] &
\end{tikzcd}$$

We have

$$\begin{aligned} j_*^{2A+3}\lambda_{2A+3}[P] &= j_*^{2A+3}D\mu_{2A+2}^*(i_{2A})_![\theta] \\ &= j_*^{2A+3}\bar{\nu}_*^{2A+1}D\mu_{2A}^*[\theta]. \end{aligned}$$

Because ν^{2A+1} and $\bar{\nu}^{2A+1}$ are clearly homotopic maps, we have proved that

$$j_*^{2A+1}\lambda_{2A+1} = j_*^{2A+3}\lambda_{2A+3}.$$

Thus we obtain a well-defined isomorphism $\lambda: \hat{U}_n \to \tilde{U}_{n+1}(BZ_p)$ for each n $\square$

Note added in proof. If the theory U is multiplicative then λ extends to an isomorphism of U_* modules λ_*: $\hat{U}_* \cong \tilde{U}_*(BZ_p)$ (here $U_* = \Sigma_n U_n$; $\hat{U}_* = \Sigma_n \hat{U}_n$; $\tilde{U}_*(BZ_p) = \Sigma_n \tilde{U}_n(BZ_p)$).

The above theorem reduces the calculation of $\hat{U}_n^{Z_p}$ to that of $\tilde{U}_{n+1}(BZ_p)$. For instance, in the case where $U(-)$ is oriented differentiable bordism $\Omega(-)$ this calculation has been performed by Conner and Floyd and we refer the reader to [2] (36.5) for a complete statement of the results.

Theorem 4.2 deals with the case $X = \text{point}$.

However one can prove the corresponding result for a general CW complex X (theorem 4.4. below) using the direct construction of λ outlined in the proof of 4.2.

We leave the details to the reader.

4.4 THEOREM: *There is an isomorphism*

$$\lambda_X: \hat{U}_n(X) \to U_{n+1}(BZ_p \times X, X)$$

which reduces to λ for X = point. □

4.5 THEOREM: *Suppose $U_n(X)$ has no p-torsion for each n. Let $\hat{S}$: $\hat{U}_n(X) \to \hat{U}_{n-2}(X)$ be the composition*

$$\hat{U}_n(X) \xrightarrow{\sigma} \bar{U}^{Z_p}_{n-1}(X) \overset{2.1}{\cong} U_{n-1}(BZ_p \times X, X) \xrightarrow{\lambda_X^{-1}} \hat{U}_{n-2}(X).$$

Then the sequence

$$0 \to U_n(X; Z_p) \xrightarrow{j} U_n(X) \xrightarrow{\hat{S}} \hat{U}_{n-2}(X) \to 0$$

is exact.

PROOF: The theorem follows immediately from 2.5 and 4.4. □

5. Representing homology cycles

Let X be a topological space and φ: $\hat{U}^{Z_p}_*(X) \to H_*(X; Z_p)$ be the edge-homomorphism of the spectral sequence associated with the theory $\hat{U}^{Z_p}(-)$.

Recall that φ is obtained by assigning to a $\hat{U}^{Z_p}$-manifold M the chain formed by the sum of the top dimensional simplexes of any triangulation of M.

Consider the square

$$\begin{array}{ccc} \hat{U}^{Z_p}(X) & \xrightarrow{\sigma} & U^{Z_p}(X) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\gamma} \\ H(X; Z_p) & \xrightarrow{\beta} & H(X) \end{array}$$

where γ is the composition of the augmentation ϵ: $U^{Z_p}(X) \to U(X)$ and the usual Thom map. The map β is homology Bockstein.

The following lemma is elementary and we leave the proof to the reader.

5.1 LEMMA: *The above square is commutative* □

Now we are ready to prove the following

5.2 THEOREM: *If γ is not onto for $X = \underbrace{BZ_p \times \cdots \times BZ_p}_{n\ times}$ then also φ is not onto (for the same X).*

PROOF: Let us suppose that φ is onto. Then, since each element of $H^*(X)$ has order p we obtain, by the exactness of the Bockstein sequence, that β is onto; by lemma 5.1 γ has to be onto while we have assumed γ not onto.

The theorem is proved □

If we take $U(-)$ to be oriented differentiable bordism $\Omega(-)$ then the Thom example $Hz(BZ_p \times BZ_p)$ (see [5]) together with our theorem shows that $\hat{\Omega}^{Z_p}$ does not represent mod p homology.

REMARK: The same argument applies also in the stronger case in which, instead of the theory $\hat{U}^{Z_p}(-)$, we consider the theory where we allow (U, S_p) coverings of order p, S_p being the full symmetric group of permutations on p letters.

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(Oblatum 15-I-1976)

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